

Feedback Control and Classification of
Generalized Linear Systems

by

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OF GENERALIZED LINEAR SYSTEMS

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ABSTRACT: We present a unified theory of control synthesis for generalized linear (i.e. descriptor) systems using constant-ratio proportional and derivative (CRPD) feedback. Our framework includes the theory of static state feedback and output feedback for regular state-space systems as a special case. The main elements of this theory include (1) a covering of the space of all systems, both regular and singular, by a family of open and dense subsets indexed by the unit circle; (2) a group of transformations which may be viewed as symmetries of the cover; (3) an admissible class of feedback transformations on each subset which is specifically adapted to that subset. We obtain a general procedure of control synthesis of CRPD feedback for generalized linear systems which uses the symmetry transformations to systematically reduce each synthesis problem to an ordinary static state feedback (or output feedback) synthesis problem for a corresponding regular system. We apply this approach to obtain natural generalizations of the Disturbance Decoupling Theorem, the Pole Assignment Theorem, and the Brunovsky Classification Theorem.

I. INTRODUCTION

In recent years, there has been considerable interest in generalized linear systems-i.e. state-space models of the form

$$E \dot{x}(t) = A x(t) + B u(t)$$

with the matrix E possibly singular. We represent this system by the matrix triple (E,A,B) and refer to it as a regular system if E is nonsingular and as a singular system if E is singular. There are extensive applications of singular systems in areas which include large-scale systems, singularly perturbed systems, circuit theory, and economic models [1] [2] [3] [4]. There has been considerable success in extending many results to singular systems. These include controllability and observability [5] [6] [26], pole placement by state feedback [7] [8], optimal regulation [9] [10] [21], and singular control [11]. (This list of references is intended to be indicative rather than comprehensive.)

For many applications, it is useful to have a theory of control which treats regular systems and singular systems together in a unified way, rather than separately. One such application is singular perturbations. In singular perturbations, singular systems represent idealized models which are obtained by neglecting various small parameters in more complex models. For example, a singular system may be obtained by neglecting the inductance in a model for an armature-controlled DC-motor [27]. In these as well as in various other applications, the singular system represents an approximation for a "nearby" system which itself may or may not be singular. Consequently, it is important to develop control techniques which are robust in the sense that they are effective when it is only known that the actual system is in a suitable neighborhood of a given singular system, and may

or may not itself be singular. In other words, the techniques should work well on open sets in the space of all systems, both regular and singular.

Many of the existing techniques for the control of singular systems depend on the Weierstrass decomposition of the system into finite-frequency and infinite-frequency subsystems. (See e.g. [28] [7].) Since the dimension of the finite-frequency subsystem is equal to the degree of $\det(sE-A)$, these techniques generally require that this degree be known precisely. However, $\deg \det(sE-A)$ is not constant on any open neighborhood of a singular system. Consequently, control methods based on the Weierstrass decomposition may encounter difficulties in applications of the type described in the preceding paragraph.

Given the goal of developing control methods which work well on open sets in the space of systems, it might be tempting to try to develop techniques which work well for all systems, both regular and singular. For example, one may attempt to extend directly to all systems the well-developed theory of state feedback for regular systems. However, a common situation in mathematics is the existence of nice results which apply to almost all systems -- i.e. to a generic subset. To extend the results to the remaining non-generic systems may be very difficult, and the results obtained may be extremely complicated and pathological. For example, if static state feedback is applied to a singular system, the resulting closed-loop system may fail to have unique solutions. This is a pathology which is never encountered when state feedback is applied to regular systems.

In our approach, we view the set of regular systems as an open and dense subset of the space of generalized systems, and we view control tools such as static state feedback and static output feedback as techniques appropriate for this subset. We do not attempt to apply these techniques directly to the complement of this open and dense subset-i.e. to the singular systems. Instead, as the first element of the theory, we define a family of open and dense subsets (Σ_θ) each of which is isomorphic to the space of regular systems, and with the property that this family covers the space of all generalized systems. Σ_θ consists of those systems which do not have an eigenvalue at the point $\text{ctn}\theta$ on the extended real axis. In particular, Σ_0 is the set of regular systems-i.e. those systems which do not have an eigenvalue at infinity. Since each subset Σ_θ is open, a given generalized system is an interior point of each subset to which it belongs. Thus, if a system (E,A,B) belongs to Σ_θ , the control techniques to be developed for Σ_θ apply not only to (E,A,B) , but to all those systems in an open neighborhood of (E,A,B) .

The second element of the theory is a symmetry group of the covering (Σ_θ) . This consists of a group of transformations (R_ϕ) with the property that R_ϕ maps Σ_θ isomorphically onto $\Sigma_{\theta+\phi}$.

The third element of the theory is an admissible feedback on each subset Σ_θ which is specifically adapted to that subset. On the subset Σ_θ , we consider feedback of the form

$$u = F(\cos\theta x - \sin\theta \dot{x}) + v$$

as well as the analogous output feedback. Thus, the parameter θ which

indexes the subset Σ_θ specifies the constant ratio of the state to derivative in the feedback. We refer to this as constant-ratio proportional and derivative (CRPD) feedback. If $\theta=0$, this feedback is ordinary static state feedback. Hence, the theory of state feedback for regular systems is included as a special case of the theory we present.

For each fixed value of θ , we will see that the theory of the feedback $u = F(\cos\theta x - \sin\theta \dot{x}) + v$ for the open and dense subset Σ_θ is completely analogous to the theory of ordinary state feedback for the subset Σ_0 of regular systems. Thus, instead of attempting to extend the theory of state feedback from the subset Σ_0 of regular systems to the space of all generalized systems, we cover the space of generalized systems with a family of subsets (Σ_θ) each isomorphic to Σ_0 , and define on Σ_θ a type of feedback which is "natural" for Σ_θ in the same way that ordinary state feedback is "natural" for Σ_0 .

It turns out that there is a very special "intertwining" relationship between the symmetry transformations (R_ϕ) and the CRPD feedback transformations. By exploiting this property, we obtain a general procedure of control synthesis of CRPD feedback for generalized linear systems which systematically reduces each synthesis problem to an ordinary static state feedback (or output feedback) synthesis problem for a corresponding regular system. We emphasize that the regular system is not obtained by the Weierstrass decomposition of the system, but rather by an appropriate symmetry transformation R_ϕ , which may be viewed as a system rotation.

We illustrate the effectiveness of this approach by deriving direct generalizations of three major results in the theory of static state feedback for regular systems. These are the Disturbance Decoupling Theorem [12] [13] [14], the Pole Assignment Theorem [15], and the Brunovsky Classification Theorem [16]. In addition to its theoretical usefulness,

the transformation of feedback compensation problems for generalized systems to feedback compensation problems for regular systems is of interest from the computational viewpoint since numerical techniques for finding compensators for regular state-space systems are much more fully developed than for generalized linear systems.

Proportional and derivative feedback of the form $u = F_1 x + F_2 \dot{x} + v$ has been applied to singular systems by Langenhop [29] and by Mukundan-Dayawansa [30]. The specialized form of proportional and derivative feedback which we refer to as CRPD feedback was introduced as a design tool for singular systems by Christodoulou [31], where it was applied to input-output decoupling, and independently by Zhou [17] (also Zhou-Shayman-Tarn [18]), where it was applied to a variety of control synthesis problems.

II. OPEN COVERING OF THE SPACE OF GENERALIZED SYSTEMS

For the most part, we consider time-invariant generalized linear systems of the form

$$E \dot{x}(t) = A x(t) + B u(t) \quad (1)$$

where E , A , and B are real matrices of dimensions $n \times n$, $n \times n$, and $n \times m$ respectively, and E may or may not be singular. When appropriate, we will augment (1) with a disturbance input and/or an output equation. We represent the system (1) by the matrix triple (E, A, B) . If E is nonsingular, (1) is equivalent (via left multiplication by E^{-1}) to the ordinary state-space system

$$\dot{x}(t) = E^{-1}A x(t) + E^{-1}B u(t) \quad (2)$$

which we represent by the matrix triple $(I, E^{-1}A, E^{-1}B)$ or by the matrix pair $(E^{-1}A, E^{-1}B)$.

Let $\hat{\Sigma}(n, m)$ denote the space of all matrix triples $(E, A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$. Let $\check{\Sigma}(n, m)$ denote the open and dense subset of $\hat{\Sigma}(n, m)$ defined by

$$\Sigma(n,m) \triangleq \widehat{\{(E,A,B) \in \Sigma(n,m) : \det(sE-A) \not\equiv 0\}}.$$

The condition that the polynomial $\det(sE-A)$ is not identically zero guarantees uniqueness for the solutions of (1). In the literature, the systems belonging to $\Sigma(n,m)$ are generally referred to as regular systems. However, we will reserve the word "regular" to refer to a generalized linear system (E,A,B) for which E is nonsingular. We refer to the systems in $\Sigma(n,m)$ as the admissible systems, and to the condition $\det(sE-A) \not\equiv 0$ as the admissibility assumption. If E is singular, we refer to (E,A,B) as a singular system.

We now define a covering of the space $\Sigma(n,m)$ of admissible systems. For each $\theta \in \mathbb{R}$, let $\Sigma_\theta(n,m)$ denote the subset of $\Sigma(n,m)$ given by

$$\Sigma_\theta(n,m) \triangleq \{(E,A,B) \in \Sigma(n,m) : \det(\cos\theta E - \sin\theta A) \neq 0\}.$$

The proof of the following result is a direct consequence of this definition.

Proposition 1:

- (a) $\Sigma_\theta(n,m)$ is an open and dense subset of $\Sigma(n,m)$.
- (b) $\Sigma_{\theta+\pi}(n,m) = \Sigma_\theta(n,m)$.
- (c) $\Sigma(n,m) = \bigcup_{\theta \in [0, \pi)} \Sigma_\theta(n,m)$.

Remark 1: By virtue of the periodicity $\Sigma_{\theta+\pi}(n,m) = \Sigma_\theta(n,m)$, it is natural to regard $\{\Sigma_\theta(n,m) : \theta \in [0, \pi)\}$ as a covering of $\Sigma(n,m)$ by open and dense subsets indexed by the points on a circle.

Remark 2: If $\theta=0$, $\Sigma_0(n,m)$ consists of those triples (E,A,B) for which E is nonsingular-i.e. the regular systems. Thus, the regular systems constitute one of the open and dense subsets in the covering $\{\Sigma_\theta(n,m)\}$.

III. SYMMETRY GROUP OF THE COVERING

Next, we define a group of symmetries of the cover $\{\Sigma_\theta(n,m) : \theta \in [0, \pi)\}$ -transformations which map these subsets into each other. For each $\phi \in \mathbb{R}$, define a mapping $R_\phi : \Sigma(n,m) \rightarrow \Sigma(n,m)$ by

$$R_\phi(E, A, B) \stackrel{\Delta}{=} (\cos\phi E + \sin\phi A, -\sin\phi E + \cos\phi A, B).$$

If $(\hat{E}, \hat{A}, B) \stackrel{\Delta}{=} R_\phi(E, A, B)$, then

$$\begin{bmatrix} \hat{E} \\ \hat{A} \end{bmatrix} = \begin{bmatrix} \cos\phi I & \sin\phi I \\ -\sin\phi I & \cos\phi I \end{bmatrix} \begin{bmatrix} E \\ A \end{bmatrix}. \quad (3)$$

Thus, (\hat{E}, \hat{A}) is obtained from (E, A) via rotation by an angle ϕ . The following result is an immediate consequence of (3).

Proposition 2:

- (a) R_0 is the identity transformation on $\hat{\Sigma}(n, m)$.
- (b) $R_{\phi_1} \circ R_{\phi_2} = R_{\phi_1 + \phi_2}$.

Proposition 2 implies that the map of $\mathbb{R} \times \hat{\Sigma}(n, m)$ into $\hat{\Sigma}(n, m)$ given by $(\phi, (E, A, B)) \rightarrow R_\phi(E, A, B)$ defines a group action of the additive group of real numbers on the manifold $\hat{\Sigma}(n, m)$.

The following result describes the relationship of the transformations (R_ϕ) to the covering $(\hat{\Sigma}_\phi(n, m))$ of the space $\hat{\Sigma}(n, m)$ of admissible systems. The proof is straightforward.

Proposition 3:

- (a) $R_\phi(\hat{\Sigma}(n, m)) = \hat{\Sigma}(n, m)$.
- (b) $R_\phi(\hat{\Sigma}_\theta(n, m)) = \hat{\Sigma}_{\theta+\phi}(n, m)$.

Remark 3: Since $R_\phi^{-1} = R_{-\phi}$, it follows from Proposition 3 that the subsets $(\hat{\Sigma}_\theta(n, m))$ are mutually isomorphic. Since $\hat{\Sigma}_0(n, m)$ is the space of all regular systems, each set $\hat{\Sigma}_\theta(n, m)$ is isomorphic to the space of regular systems.

We now examine how the system eigenvalues transform when the system undergoes a rotation. It will be useful to recall the identification of the extended complex plane, $\mathbb{C} \cup \{\infty\}$, with the complex projective space $\mathbb{C}P(1)$. Define an equivalence relation \sim on $\mathbb{C}^2 - \{(0, 0)\}$ whereby $(s_1, s_2) \sim (s_1, s_2)$ if and only if there exists a nonzero complex number λ such that

$(\hat{s}_1, \hat{s}_2) = \lambda(s_1, s_2)$. (Equivalently, $\mathbb{CP}(1)$ can be regarded as the set of all lines through the origin in \mathbb{C}^2 .) If $(s_1, s_2) \in \mathbb{C}^2 - \{(0,0)\}$, we denote by $[(s_1, s_2)]$ the corresponding element of $\mathbb{CP}(1)$ -i.e. the equivalence class containing (s_1, s_2) . We refer to (s_1, s_2) as the homogeneous coordinates of $[(s_1, s_2)]$. If $(s_1, s_2) \sim (\hat{s}_1, \hat{s}_2)$, then $s_1/s_2 = \hat{s}_1/\hat{s}_2$. Consequently, we can identify $\mathbb{CP}(1)$ with the extended complex plane via the map $[(s_1, s_2)] \rightarrow s_1/s_2$. If $s_2=0$, then $[(s_1, s_2)]$ is identified with the point at infinity in the complex plane.

Let (E, A, B) be an admissible system-i.e. $(E, A, B) \in \Sigma(n, m)$. We say that $[(s_1, s_2)] \in \mathbb{CP}(1)$ is a system eigenvalue of (E, A, B) if and only if $\det(s_1 E - s_2 A) = 0$. Note that if $[(s_1, s_2)] = [(\hat{s}_1, \hat{s}_2)]$, then $\det(s_1 E - s_2 A) = 0$ if and only if $\det(\hat{s}_1 E - \hat{s}_2 A) = 0$. Thus, the system eigenvalues are well-defined.

Remark 4: Since $[(s_1, s_2)]$ is identified with the extended complex number $\alpha \triangleq s_1/s_2$, we will also refer to α as a system eigenvalue provided $\det(s_1 E - s_2 A) = 0$. If $s_2 \neq 0$, then α is a (finite) complex number and is a system eigenvalue if and only if $\det(\alpha E - A) = 0$. This coincides with the usual definition of a finite eigenvalue of a generalized linear system. If $s_2 = 0$, then $\alpha = \infty$ and is a system eigenvalue if and only if $\det E = 0$. We will define the multiplicity of the eigenvalue $\alpha = \infty$ to be $n - \deg \det(sE - A)$. Thus, the multiplicity of $\alpha = \infty$ is equal to the dimension of the infinite-frequency subsystem in the Weierstrass decomposition, and does not depend on the Jordan structure of the nilpotent operator associated with that subsystem.

The preceding definition of eigenvalues at infinity is unusual in that it does not distinguish between dynamic infinite-frequency (i.e. impulsive) modes and nondynamic constraints. It is more common to define the multiplicity of $\alpha = \infty$ to be the number of independent impulsive modes, namely $\text{rank } E - \deg \det(sE - A)$. (See e.g. [1].) By this definition, the system has a total number of eigenvalues (both finite and infinite) equal to $\text{rank } E$. In contrast,

by our definition, the system has n eigenvalues, independent of rank E .

The usual definition of infinite eigenvalues is well-suited for many purposes. Our unusual definition of infinite eigenvalues has two sources of motivation. For applications which require the type of robustness discussed in Section I, it is useful to base the control techniques on data which depends continuously on the system parameters. Since rank E is discontinuous at any singular system, the usual definition of infinite eigenvalues has the consequence that the total number of system eigenvalues is discontinuous. The second motivation stems from our use of system rotations in control synthesis. The behavior of the eigenvalues under system rotation is crucial to this approach. It is important that the total number of system eigenvalues be invariant under system rotation. Since rank E is not invariant under rotation, the usual definition of eigenvalues at infinity does not yield this property.

Remark 5: It follows immediately from the definition of $\Sigma_{\theta}(n,m)$ that a system (E,A,B) belongs to $\Sigma_{\theta}(n,m)$ if and only if it has no system eigenvalue at the point $[(\cos\theta, \sin\theta)]$ of $\mathbb{C}P(1)$, or equivalently at the point $\cot\theta$ of the extended complex plane. In particular, $\Sigma_0(n,m)$ (the set of all regular systems) consists of those systems which have no eigenvalues at infinity.

The following result shows how the eigenvalues of a generalized linear system change under a system rotation. If the system is rotated by ϕ , the homogeneous coordinates of an eigenvalue undergo a rotation of $-\phi$. If the eigenvalue is regarded as an extended complex number, then it undergoes a linear fractional transformation.

Proposition 4: Let $(E,A,B) \in \sum(n,m)$, and let $(\hat{E},\hat{A},\hat{B}) \triangleq R_\phi(E,A,B)$. Then $[(s_1, s_2)]$ is an eigenvalue of (E,A,B) if and only if $[(\hat{s}_1, \hat{s}_2)]$ is an eigenvalue of $(\hat{E},\hat{A},\hat{B})$, where

$$\begin{bmatrix} \hat{s}_1 \\ \hat{s}_2 \end{bmatrix} \triangleq \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \quad (4)$$

Equivalently, the extended complex number α is an eigenvalue of (E,A,B) if and only if the extended complex number $\hat{\alpha}$ is an eigenvalue of $(\hat{E},\hat{A},\hat{B})$, where

$$\hat{\alpha} \triangleq \frac{(\cos\phi)\alpha - \sin\phi}{(\sin\phi)\alpha + \cos\phi} \quad (5)$$

Proof: Using the definition of R_ϕ , it is trivial to verify that

$$\hat{s}_1 \hat{E} - \hat{s}_2 \hat{A} = s_1 E - s_2 A, \quad (6)$$

which establishes the first assertion. The second assertion follows from the first by setting $\alpha \triangleq s_1/s_2$ and $\hat{\alpha} \triangleq \hat{s}_1/\hat{s}_2$. ■

Let (E,A,B) be an admissible system, and let $R(E,A,B)$ denote its controllable subspace. $R(E,A,B)$ consists of those states in \mathbb{R}^n which are reachable in positive time from the initial state $x(0^-) \triangleq 0$. If $R(E,A,B) = \mathbb{R}^n$, then (E,A,B) is called controllable [5]. The following result was proved by Cobb [19]. (For a simplified proof, see also [17], [18]). If P is a linear transformation on \mathbb{R}^n and S is a subspace of \mathbb{R}^n , $\langle P|S \rangle$ denotes the subspace $S + P(S) + \dots + P^{n-1}(S)$ -i.e. the smallest P -invariant subspace containing S .

Lemma 1 [19]: If $(E,A,B) \in \sum(n,m)$ and α is a real number satisfying $\det(\alpha E - A) \neq 0$, then

$$R(E,A,B) = \langle (\alpha E - A)^{-1} E | \text{Im}(\alpha E - A)^{-1} B \rangle.$$

Corollary 1: The generalized system (E,A,B) is controllable if and only if the regular system $(I, (\alpha E - A)^{-1} E, (\alpha E - A)^{-1} B)$ is controllable.

Using Lemma 1, we can prove the following result.

Lemma 2: Let $(E, A, B) \in \sum_{\theta}(n, m)$, and let $(\hat{E}, \hat{A}, B) \triangleq R_{-\theta}(E, A, B)$. Then

$$R(E, A, B) = R(\hat{E}, \hat{A}, B).$$

Proof: If $\sin\theta=0$, the assertion holds trivially, so we may assume $\sin\theta \neq 0$.

Since (\hat{E}, \hat{A}, B) is a regular system, we have

$$R(\hat{E}, \hat{A}, B) = \langle \hat{E}^{-1} \hat{A} | \text{Im } \hat{E}^{-1} B \rangle. \quad (7)$$

Let $\alpha \triangleq \cos\theta/\sin\theta$. Since $(E, A, B) \in \sum_{\theta}(n, m)$, $\det(\alpha E - A) \neq 0$. By Lemma 1,

$$R(E, A, B) = \langle (\alpha E - A)^{-1} E | \text{Im}(\alpha E - A)^{-1} B \rangle. \quad (8)$$

It is trivial to verify that

$$\text{Im } \hat{E}^{-1} B = \text{Im } (\alpha E - A)^{-1} B. \quad (9)$$

Letting $P \triangleq (\alpha E - A)^{-1} E$ and $Q \triangleq \hat{E}^{-1} \hat{A}$, it is straightforward to show that

$$\begin{aligned} Q &= -\alpha I + (1 + \alpha^2)P \\ P &= \frac{\alpha}{1 + \alpha^2} I + \frac{1}{1 + \alpha^2} Q. \end{aligned} \quad (10)$$

Thus, P and Q have the same invariant subspaces. Consequently, it

follows from (9) that

$$\langle Q | \text{Im } \hat{E}^{-1} B \rangle = \langle P | \text{Im}(\alpha E - A)^{-1} B \rangle. \quad (11)$$

Then the assertion of the lemma follows from (7), (8), and (11). ■

Corollary 2: $(E, A, B) \in \sum_{\theta}(n, m)$ is controllable if and only if the regular system $R_{-\theta}(E, A, B)$ is controllable.

Remark 6: Let $(E, A, B) \in \sum_{\theta}(n, m)$, and let $(\hat{E}, \hat{A}, B) \triangleq R_{-\theta}(E, A, B) = (\cos\theta E - \sin\theta A, \sin\theta E + \cos\theta A, B)$. Since the regular system (\hat{E}, \hat{A}, B) is controllable if and only if

$$\text{rank}[\hat{E}^{-1} B, (\hat{E}^{-1} \hat{A}) \hat{E}^{-1} B, \dots, (\hat{E}^{-1} \hat{A})^{n-1} \hat{E}^{-1} B] = n, \quad (12)$$

it follows from Corollary 2 that the generalized linear system (E, A, B) is controllable if and only if the rank condition (12) holds. If $\theta=0$, then

(12) specializes to give the usual rank test for the controllability of a regular system.

We now show that the controllable subspace is invariant under system rotation.

Proposition 5: Let $(E, A, B) \in \Sigma_{\theta}(n, m)$, and let $(\hat{E}, \hat{A}, \hat{B}) \stackrel{\Delta}{=} R_{\phi}(E, A, B) \in \Sigma_{\theta+\phi}(n, m)$. Then

$$R(E, A, B) = R(\hat{E}, \hat{A}, \hat{B}).$$

Proof: Applying Lemma 2 twice, we have $R(\hat{E}, \hat{A}, \hat{B}) = R(R_{-(\theta+\phi)}(\hat{E}, \hat{A}, \hat{B})) = R(R_{-\theta}(R_{-\phi}(\hat{E}, \hat{A}, \hat{B}))) = R(R_{-\theta}(E, A, B)) = R(E, A, B)$. ■

Corollary 3: Let $(E, A, B) \in \Sigma(n, m)$. Then (E, A, B) is controllable if and only if $R_{\phi}(E, A, B)$ is controllable.

We now consider the state equation (1) together with a linear output equation

$$y(t) = C x(t) \quad (13)$$

where C is a real $p \times n$ matrix. We represent the system (1), (13) by the matrix quadruple (E, A, B, C) . In place of $\hat{\Sigma}(n, m)$, $\Sigma(n, m)$, and $\Sigma_{\theta}(n, m)$, we define sets $\hat{\Gamma}(n, m, p)$, $\Gamma(n, m, p)$, and $\Gamma_{\theta}(n, m, p)$ as follows: Let $\hat{\Gamma}(n, m, p)$ denote the space of all matrix quadruples $(E, A, B, C) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n}$. Let $\Gamma(n, m, p)$ denote the open and dense subset of $\hat{\Gamma}(n, m, p)$ defined by

$$\Gamma(n, m, p) \stackrel{\Delta}{=} \{(E, A, B, C) \in \hat{\Gamma}(n, m, p) : \det(sE - A) \neq 0\}.$$

Let $\Gamma_{\theta}(n, m, p)$ denote the subset of $\Gamma(n, m, p)$ given by

$$\Gamma_{\theta}(n, m, p) \stackrel{\Delta}{=} \{(E, A, B, C) \in \Gamma(n, m, p) : \det(\cos\theta E - \sin\theta A) \neq 0\}.$$

In analogy to Proposition 1, $\{\Gamma_\theta(n,m,p) : \theta \in [0,\pi)\}$ is a cover of $\Gamma(n,m,p)$ by open and dense subsets. Abusing notation slightly, we use the symbol R_ϕ to denote the mapping of $\hat{\Gamma}(n,m,p)$ given by

$$R_\phi(E,A,B,C) \triangleq (\cos\phi E + \sin\phi A, -\sin\phi E + \cos\phi A, B, C).$$

With this redefinition of R_ϕ , analogues of Propositions 2 and 3 are immediately obtained by replacing $\hat{\Sigma}(n,m)$, $\Sigma(n,m)$, and $\Sigma_\theta(n,m)$ with $\hat{\Gamma}(n,m,p)$, $\Gamma(n,m,p)$, and $\Gamma_\theta(n,m,p)$ respectively.

We now examine the relationship between system rotation and observability. We adopt the definitions of observability and unobservable subspace suggested by Cobb [6]. Let (E,A,B,C) be an admissible system, and let $N(E,A,B,C)$ denote its unobservable subspace. $N(E,A,B,C)$ consists of those states x_0 such that if $x(0^-) = x_0$, then $y(0^-) = 0$ and the free response of the system is identically zero on $[0, \infty)$. (E,A,B,C) is observable if $N(E,A,B,C) = 0$, or equivalently, if knowledge of $y(0^-)$ together with the input and output on $[0, \infty)$ is sufficient to determine $x(0^-)$.

The following result was proved in [17], [18].

Lemma 3: If $(E,A,B,C) \in \Gamma(n,m,p)$ and α is a real number satisfying $\det(\alpha E - A) \neq 0$, then

$$N(E,A,B,C) = \bigcap_{i=0}^{n-1} \ker C[(\alpha E - A)^{-1}E]^i.$$

Corollary 4: The generalized system (E,A,B,C) is observable if and only if the regular system $(I, (\alpha E - A)^{-1}E, (\alpha E - A)^{-1}B, C)$ is observable.

Using Lemma 3, we can prove the following result.

Lemma 4: Let $(E,A,B,C) \in \Gamma_\theta(n,m,p)$, and let $(\hat{E}, \hat{A}, B, C) \triangleq R_{-\theta}(E,A,B,C)$. Then

$$N(E,A,B,C) = N(\hat{E}, \hat{A}, B, C).$$

Proof: If $\sin\theta = 0$, the assertion holds trivially, so we may assume $\sin\theta \neq 0$.

Since (\hat{E}, \hat{A}, B, C) is a regular system, we have

$$N(\hat{E}, \hat{A}, B, C) = \bigcap_{i=0}^{n-1} \ker C(\hat{E}^{-1}\hat{A})^i, \quad (14)$$

the largest $\hat{E}^{-1}\hat{A}$ -invariant subspace contained in $\ker C$. Let $\alpha \triangleq \cos\theta/\sin\theta$. Since $(E, A, B, C) \in \Gamma_\theta(n, m, p)$, $\det(\alpha E - A) \neq 0$. By Lemma 3,

$$N(E, A, B, C) = \bigcap_{i=0}^{n-1} \ker C[(\alpha E - A)^{-1}E]^i, \quad (15)$$

the largest $(\alpha E - A)^{-1}E$ -invariant subspace contained in $\ker C$. From the proof of Lemma 2, we know that $\hat{E}^{-1}\hat{A}$ and $(\alpha E - A)^{-1}E$ have the same invariant subspaces. Thus, it follows from (14), (15) that $N(E, A, B, C) = N(\hat{E}, \hat{A}, B, C)$. ■

Corollary 5: $(E, A, B, C) \in \Gamma_\theta(n, m, p)$ is observable if and only if the regular system $R_{-\theta}(E, A, B, C)$ is observable.

Remark 7: Let $(E, A, B, C) \in \Gamma_\theta(n, m, p)$, and let $(\hat{E}, \hat{A}, B, C) \triangleq R_{-\theta}(E, A, B, C) = (\cos\theta E - \sin\theta A, \sin\theta E + \cos\theta A, B, C)$. Since the regular system (\hat{E}, \hat{A}, B, C) is observable if and only if

$$\text{rank} \begin{bmatrix} C \\ C(\hat{E}^{-1}\hat{A}) \\ \vdots \\ C(\hat{E}^{-1}\hat{A})^{n-1} \end{bmatrix} = n, \quad (16)$$

it follows from Corollary 5 that the generalized linear system (E, A, B, C) is observable if and only if the rank condition (16) holds. If $\theta=0$, (16) specializes to the usual rank test for the observability of a regular system.

We now show that the unobservable subspace is invariant under system rotation.

Proposition 6: Let $(E, A, B, C) \in \Gamma_\theta(n, m, p)$, and let $(\hat{E}, \hat{A}, B, C) \triangleq R_\phi(E, A, B, C) \in \Gamma_{\theta+\phi}(n, m, p)$. Then

$$N(E, A, B, C) = N(\hat{E}, \hat{A}, B, C).$$

Proof: Applying Lemma 4 twice, we have $N(\hat{E}, \hat{A}, B, C) = N(R_{-(\theta+\phi)}(\hat{E}, \hat{A}, B, C)) = N(R_{-\theta}(E, A, B, C)) = N(E, A, B, C)$. ■

Corollary 6: Let $(E, A, B, C) \in \Gamma(n, m, p)$. Then (E, A, B, C) is observable if and only if $R_{\phi}(E, A, B, C)$ is observable.

IV. CONSTANT-RATIO PROPORTIONAL AND DERIVATIVE FEEDBACK

Without question, static state feedback (and static output feedback) are extremely useful tools in control design for regular linear systems. Various results in the theory of control by state feedback have been extended to singular systems in recent years. Cobb [7] has investigated the effects of applying state feedback $u = Fx + v$ to the system (E, A, B) , and has studied pole placement problems. He has shown that under certain conditions, state feedback can be used to eliminate impulses in the system. State feedback has also been used by Pandolfi [20] in connection with stabilization.

In spite of these and other useful results for the control of singular systems using state feedback, there is evidence that state feedback is less natural as a tool for singular systems than it is as a tool for regular systems. For example, if (E, A, B) is a regular system (i.e. E nonsingular) to which the state feedback $u = Fx + v$ is applied, the resulting closed loop system is $(E, A+BF, B)$, which is again a regular system. In particular, the closed-loop system is an admissible system-i.e. $\det(sE - (A+BF))$ not identically zero. On the other hand, if the open loop system (E, A, B) is admissible but singular, the closed loop system $(E, A+BF, B)$ may fail to be admissible. Thus, admissibility of regular

systems is preserved by state feedback, but admissibility of singular systems is not preserved.

Recall that $\{\Sigma_\theta(n,m) : \theta \in [0, \pi)\}$ is a covering of the space $\Sigma(n,m)$ of admissible generalized linear systems by open and dense subsets. Our approach is to define for each subset $\Sigma_\theta(n,m)$ an allowable class of feedback transformations which is natural for $\Sigma_\theta(n,m)$ in the same way that state feedback transformations are natural for the subset $\Sigma_0(n,m)$ of regular systems. Thus, the type of feedback which may be applied to an admissible system (E,A,B) depends on which of the subsets $\{\Sigma_\theta(n,m)\}$ the system (E,A,B) belongs.

Specifically, we will allow feedback of the form

$$u = F(\cos\theta x - \sin\theta \dot{x}) + v \quad (17)$$

to be applied to the systems belonging to the subset $\Sigma_\theta(n,m)$. In (17), θ is fixed, while the $m \times n$ gain matrix F is arbitrary, and v represents a new external input. The fixed parameter θ specifies the ratio of state to derivative in the feedback law. Consequently, we refer to (17) as constant-ratio proportional and derivative (CRPD) state feedback.

Remark 8: In the case where $\theta=0$, the subset $\Sigma_\theta(n,m)$ is precisely the set $\Sigma_0(n,m)$ of regular systems, while the feedback (17) is ordinary state feedback. Thus, the theory we present includes the theory of state feedback for regular systems as a special case. Note however that if (E,A,B) is a singular system, then $(E,A,B) \notin \Sigma_0(n,m)$, so ordinary state feedback is not an allowable transformation. Thus, the CRPD feedback applied to a singular system will always contain some contribution of the derivative.

Remark 9: Although we focus primarily on CRPD state feedback, it is equally natural to consider output feedback from this viewpoint. The CRPD state feedback (17) is replaced by the CRPD output feedback

$$u = F(\cos\theta y - \sin\theta \dot{y}) + v \quad (18)$$

which is applied to those systems (E, A, B, C) belonging to the subset $\Gamma_\theta(n, m, p)$.

Remark 10: In certain applications, derivatives of states or outputs can be measured directly, without requiring differentiation. Examples include rate gyros and accelerometers [32]. In applications where this is not the case, CRPD feedback can be regarded as an idealized compensation which can be approximated by a proper compensator obtained by adding a high-frequency pole. This is analogous to the use of lead compensation to approximate pure proportional and derivative feedback in classical control [33].

The following result shows that just as the set $\Sigma_0(n, m)$ of regular systems is invariant under state feedback, the set $\Sigma_\theta(n, m)$ is invariant under the CRPD feedback (17).

Proposition 7: Let $(E, A, B) \in \Sigma_\theta(n, m)$, and let $(\hat{E}, \hat{A}, \hat{B})$ denote the closed loop system resulting from the CRPD feedback (17). Then $(\hat{E}, \hat{A}, \hat{B}) \in \Sigma_\theta(n, m)$.

Proof: $(\hat{E}, \hat{A}, \hat{B}) = (E + \sin\theta BF, A + \cos\theta BF, B)$, so $\cos\theta \hat{E} - \sin\theta \hat{A} = \cos\theta E - \sin\theta A$. Hence, $(\hat{E}, \hat{A}, \hat{B}) \in \Sigma_\theta(n, m)$ if and only if $(E, A, B) \in \Sigma_\theta(n, m)$. ■

We have defined a different class of allowable feedback transformations for each subset in the covering $\{\Sigma_\theta(n, m) : \theta \in [0, \pi)\}$. Since the rotation map R_ϕ is an isomorphism of $\Sigma_\theta(n, m)$ onto $\Sigma_{\theta+\phi}(n, m)$ (Proposition 3 and Remark 3), one might hope that R_ϕ would relate the allowable feedback transformations on $\Sigma_{\theta+\phi}(n, m)$ with those on $\Sigma_\theta(n, m)$. The result which follows shows that this is indeed the case. It is a fundamental result which provides the basis for a systematic procedure of control synthesis for generalized linear systems by CRPD feedback. Let $g_\theta(F) : \Sigma_\theta(n, m) \rightarrow \Sigma_\theta(n, m)$

denote the transformation on $\Sigma_\theta(n,m)$ induced by the feedback law (17). In other words,

$$g_\theta(F)(E,A,B) \triangleq (E + \sin\theta BF, A + \cos\theta BF, B). \quad (19)$$

Fundamental Lemma of CRPD Feedback: The following is a commutative diagram:

$$\begin{array}{ccc} \Sigma_\theta(n,m) & \xrightarrow{g_\theta(F)} & \Sigma_\theta(n,m) \\ R_\phi \downarrow & & \downarrow R_\phi \\ \Sigma_{\theta+\phi}(n,m) & \xrightarrow{g_{\theta+\phi}(F)} & \Sigma_{\theta+\phi}(n,m) \end{array}$$

I.e. $R_\phi \circ g_\theta(F) = g_{\theta+\phi}(F) \circ R_\phi$.

Proof: By direct verification. ■

The following corollary will prove crucial.

Corollary 7: The following is a commutative diagram:

$$\begin{array}{ccc} \Sigma_\theta(n,m) & \xrightarrow{g_\theta(F)} & \Sigma_\theta(n,m) \\ R_{-\theta} \downarrow & & \downarrow R_{-\theta} \\ \Sigma_0(n,m) & \xrightarrow{g_0(F)} & \Sigma_0(n,m) \end{array}$$

Remark 11: An analogous result holds for CRPD output feedback. Let $\hat{g}_\theta(F)$ denote the transformation on $\Gamma_\theta(n,m,p)$ induced by the output feedback law (18), where F is a $p \times m$ matrix. In other words,

$$\hat{g}_\theta(F)(E,A,B,C) \triangleq (E + \sin\theta BFC, A + \cos\theta BFC, B, C). \quad (20)$$

Then $R_\phi \circ \hat{g}_\theta(F) = \hat{g}_{\theta+\phi}(F) \circ R_\phi$.

Corollary 7 is of critical importance because it permits us to deduce properties of generalized systems under CRPD feedback from known properties of regular systems under ordinary state feedback. To illustrate this, we prove the invariance of the controllable subspace under CRPD state feedback,

and the invariance of both the controllable subspace and the unobservable subspace under CRPD output feedback.

Proposition 8:

(a) If $(E, A, B) \in \Sigma_{\theta}(n, m)$ and $(\hat{E}, \hat{A}, B) \stackrel{\Delta}{=} g_{\theta}(F)(E, A, B)$, then $R(E, A, B) = R(\hat{E}, \hat{A}, B)$.

(b) If $(E, A, B, C) \in \Gamma_{\theta}(n, m, p)$ and $(\bar{E}, \bar{A}, B, C) \stackrel{\Delta}{=} \hat{g}_{\theta}(F)(E, A, B, C)$, then $R(E, A, B) = R(\bar{E}, \bar{A}, B)$ and $N(E, A, B, C) = N(\bar{E}, \bar{A}, B, C)$.

Proof: (a) We have $R(\hat{E}, \hat{A}, B) = R(R_{-\theta}(\hat{E}, \hat{A}, B)) = R(g_0(F) \circ R_{-\theta}(E, A, B)) = R(R_{-\theta}(E, A, B)) = R(E, A, B)$, where we have used successively Proposition 5, the Fundamental Lemma (Corollary 7), the invariance of the controllable subspace of a regular system under ordinary state feedback [22], and Proposition 5 again.

(b) Since the CRPD output feedback (18) can be regarded as a special case of the CRPD state feedback (17), it follows from (a) that $R(E, A, B) = R(\bar{E}, \bar{A}, B)$. We have $N(\bar{E}, \bar{A}, B, C) = N(R_{-\theta}(\bar{E}, \bar{A}, B, C)) = N(\hat{g}_0(F) \circ R_{-\theta}(E, A, B, C)) = N(R_{-\theta}(E, A, B, C)) = N(E, A, B, C)$, where we have used successively Proposition 6, the Fundamental Lemma (Remark 11), the invariance of the unobservable subspace of a regular system under ordinary output feedback [22], and Proposition 6 again. ■

Part (a) of Proposition 8 appears in a somewhat different form in [17], [18].

V. CONTROL SYNTHESIS PROCEDURE

In this section, we give a systematic procedure for control synthesis using constant ratio proportional and derivative feedback. The key feature of this procedure is that a synthesis problem for a generalized system using CRPD feedback is solved by solving a synthesis problem for a corresponding

regular system using ordinary state (or output) feedback. Thus, all of the available techniques for state (or output) feedback synthesis of regular systems can be applied to the CRPD feedback synthesis of generalized systems. This procedure in no way uses the Weierstrass decomposition of the generalized system. The corresponding regular system is obtained from the given generalized system by a system rotation.

Corollary 7 to the Fundamental Lemma provides the basis for the following control synthesis procedure:

1. Rotate the system. Given a control synthesis problem for an admissible system (E, A, B) , choose θ such that $(E, A, B) \in \Sigma_\theta(n, m)$. Rotate by $-\theta$ to obtain the regular system $(E_0, A_0, B) \triangleq R_{-\theta}(E, A, B) \in \Sigma_0(n, m)$.
2. Rotate the performance specifications. Determine what properties a regular system $(\hat{E}_0, \hat{A}_0, B) \in \Sigma_0(n, m)$ must have in order for the system $(\hat{E}, \hat{A}, B) \triangleq R_\theta(\hat{E}_0, \hat{A}_0, B) \in \Sigma_\theta(n, m)$ to satisfy the given performance specifications of the synthesis problem.
3. Solve a state feedback synthesis problem for the regular system (E_0, A_0, B) . Choose a gain matrix F so that the closed loop regular system, call it $(\hat{E}_0, \hat{A}_0, B)$, which is obtained from (E_0, A_0, B) via the state feedback law $u = Fx + v$, satisfies the rotated performance specifications determined in Step 2. Note that $(\hat{E}_0, \hat{A}_0, B) = g_0(F)(E_0, A_0, B)$.
4. Implement the CRPD feedback law $u = F(\cos\theta x - \sin\theta \dot{x}) + v$ for the original generalized system (E, A, B) using the gain F determined in Step 4. Let (\hat{E}, \hat{A}, B) denote the resulting closed loop system $g_\theta(F)(E, A, B)$. By Corollary 7 of the Fundamental Lemma, $(\hat{E}, \hat{A}, B) = R_\theta \circ g_0(F) \circ R_{-\theta}(E, A, B) = R_\theta(\hat{E}_0, \hat{A}_0, B)$. From Steps 2 and 3, it

follows that the closed loop system (\hat{E}, \hat{A}, B) satisfies the given performance specifications of the control synthesis problem.

Remark 12: It is clear that the solution of a given CRPD feedback synthesis problem by the above procedure requires two results. It requires the solution of a corresponding state feedback synthesis problem for regular systems (for Step 3) together with knowledge of how the relevant system properties (e.g. pole location) are transformed by system rotation (for Step 2). The former results are generally known (see e.g [22]), while we have presented some of the latter results in Section III (Propositions 4,5,6).

Remark 13: Since a given admissible system (E, A, B) belongs to $\sum_{\theta}(n, m)$ for all but finitely many $\theta \in [0, \pi)$, it is possible to implement the above control synthesis procedure with θ regarded as an additional design parameter to be determined. This flexibility in the choice of θ is useful in certain synthesis problems such as disturbance decoupling. This will be illustrated below.

Remark 14: By virtue of Remark 11, there is an analogous procedure for control synthesis using CRPD output feedback in which the synthesis problem is solved by solving a corresponding ordinary output feedback synthesis problem for a corresponding regular system.

A. Disturbance Decoupling Problem

We will use the control synthesis procedure above to solve the disturbance decoupling problem for a generalized linear system using CRPD feedback. The disturbance decoupling problem refers to a control system

$$\begin{aligned} E \dot{x}(t) &= Ax(t) + Bu(t) + Dw(t) \\ y(t) &= Cx(t) \end{aligned} \quad (21)$$

where $w(t) \in \mathbb{R}^r$ is a disturbance input which is not directly measurable. The disturbance decoupling problem is to find feedback (of a specified type) such that the output of the closed loop system is independent of the disturbance input. This is equivalent to requiring that, for the closed loop system, the controllable subspace from the disturbance input is contained in $\ker C$.

In the special case where (21) is a regular system (i.e. E nonsingular) and the feedback is required to be state feedback, we have the following well known result of Basile-Marro [12] [13] and Wonham-Morse [14]. (See e.g. [22].)

Theorem 1 (Basile-Marro, Wonham-Morse): Let (E,A,B,C,D) be a regular system. Let V^* be the largest subspace V which is contained in $\ker C$ and satisfies $A(V) \subset E(V) + \text{Im } B$. The disturbance decoupling problem is solvable via state feedback, $u=Fx+v$, if and only if $\text{Im } D \subset E(V^*)$.

We represent the system (21) by the matrix quintuple (E,A,B,C,D) . In place of $\hat{\Sigma}(n,m)$, $\Sigma(n,m)$, and $\Sigma_\theta(n,m)$, we define sets $\hat{\Delta}(n,m,p,r)$, $\Delta(n,m,p,r)$, and $\Delta_\theta(n,m,p,r)$ as follows: Let $\hat{\Delta}(n,m,p,r)$ denote the space of all matrix quintuples $(E,A,B,C,D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n} \times \mathbb{R}^{n \times r}$. Let $\Delta(n,m,p,r) \triangleq \{(E,A,B,C,D) \in \hat{\Delta}(n,m,p,r) : \det(sE-A) \neq 0\}$, and let $\Delta_\theta(n,m,p,r) \triangleq \{(E,A,B,C,D) \in \Delta(n,m,p,r) : \det(\cos\theta E - \sin\theta A) \neq 0\}$. In analogy to Proposition 1, $\{\Delta_\theta(n,m,p,r) : \theta \in [0,\pi)\}$ is a cover of $\Delta(n,m,p,r)$ by open and dense subsets. Abusing notation slightly, we use the symbol R_ϕ to denote the mapping of $\hat{\Delta}(n,m,p,r)$ given by $R_\phi(E,A,B,C,D) \triangleq (\cos\phi E + \sin\phi A, -\sin\phi E + \cos\phi A, B, C, D)$, and the symbol $g_\theta(F)$ to denote the mapping of $\Delta_\theta(n,m,p,r)$ given by $g_\theta(F)(E,A,B,C,D) \triangleq (E + \sin\theta BF, A + \cos\theta BF, B, C, D)$. With these redefinitions of R_ϕ and $g_\theta(F)$, we again have $R_{-\theta} \circ g_\theta(F) =$

$g_0(F) \circ R_{-\theta}$, as in Corollary 7. Consequently, we have the obvious analogue of the control synthesis procedure.

We now derive a generalization of Theorem 1 by following step-by-step the synthesis procedure:

1. Given $(E, A, B, C, D) \in \Delta(n, m, p, r)$, choose θ such that $(E, A, B, C, D) \in \Delta_{\theta}(n, m, p, r)$. Let $(E_0, A_0, B, C, D) \stackrel{\Delta}{=} R_{-\theta}(E, A, B, C, D)$.
2. Let $(\hat{E}_0, \hat{A}_0, B, C, D) \in \Delta_0(n, m, p, r)$ and let $(\hat{E}, \hat{A}, B, C, D) \stackrel{\Delta}{=} R_{\theta}(\hat{E}_0, \hat{A}_0, B, C, D)$. $(\hat{E}, \hat{A}, B, C, D)$ is disturbance decoupled if and only if $R(\hat{E}, \hat{A}, D) \subset \ker C$. Applying Proposition 5, this is equivalent to the condition $R(\hat{E}_0, \hat{A}_0, D) \subset \ker C$ on the regular system $(\hat{E}_0, \hat{A}_0, B, C, D)$.
3. We need to choose a state feedback gain matrix F so that the closed loop regular system $(\hat{E}_0, \hat{A}_0, B, C, D) \stackrel{\Delta}{=} g_0(F)(E_0, A_0, B, C, D)$ satisfies the rotated performance specifications-i.e. such that $R(\hat{E}_0, \hat{A}_0, D) \subset \ker C$.
4. If the gain matrix F required in Step 3 exists, then the CRPD feedback law $u = F(\cos\theta x - \sin\theta \dot{x}) + v$ solves the disturbance decoupling problem for the system (E, A, B, C, D) .

From the above procedure, it follows that the disturbance decoupling problem for $(E, A, B, C, D) \in \Delta_{\theta}(n, m, p, r)$ is solvable using the CRPD state feedback (17) if and only if the disturbance decoupling problem for the regular system (E_0, A_0, B, C, D) is solvable using ordinary state feedback. Applying Theorem 1, we immediately obtain the following result.

Theorem 2: Let $(E, A, B, C, D) \in \Delta_{\theta}(n, m, p, r)$, and let V_{θ}^* be the largest subspace V which is contained in $\ker C$ and satisfies

$$(\sin\theta E + \cos\theta A)(V) \subset (\cos\theta E - \sin\theta A)(V) + \text{Im } B.$$

The disturbance decoupling problem is solvable using CRPD state feedback $u = F(\cos\theta x - \sin\theta \dot{x}) + v$ if and only if

$$\text{Im } D \subset (\cos\theta E - \sin\theta A)(V_\theta^*).$$

Remark 15: In the special case where $\theta=0$, $\Delta_\theta(n,m,p,r)$ consists of the regular systems, and the CRPD feedback $u = F(\cos\theta x - \sin\theta \dot{x}) + v$ coincides with ordinary state feedback. Thus, Theorem 1 is recovered as a special case of Theorem 2. The thesis of Zhou [17] and paper of Zhou-Shayman-Tarn [18] contain a result which can be shown to be equivalent to Theorem 2 provided $\theta \neq 0$. However, since this result does not apply in the case $\theta=0$, it does not offer the insight which Theorem 2 provides—namely that the well known result of Basile-Marro and Wonham-Morse (Theorem 1) is a special case of a broader result (Theorem 2) which applies to all admissible systems, regardless of whether or not they are regular.

Theorem 2 is of interest for regular systems as well as for singular systems. If (E,A,B,C,D) is a regular system, then by definition we have $(E,A,B,C,D) \in \Delta_0(n,m,p,r)$. However, it is also true that $(E,A,B,C,D) \in \Delta_\theta(n,m,p,r)$ for all but finitely many values of the parameter θ . Consequently, it is permissible to apply the CRPD feedback (17) to the system (E,A,B,C,D) choosing a nonzero value of θ . The following example shows that it is possible to use CRPD feedback to disturbance decouple regular systems which cannot be decoupled using ordinary state feedback—i.e. for which the condition $\text{Im } D \subset E(V^*)$ of Theorem 1 is not satisfied.

Example 1: Consider the regular system (E,A,B,C,D) , where

$$E \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A \triangleq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B \triangleq \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D \triangleq \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad C = [1 \quad -1].$$

Since $\text{Im } D$ is not contained in $E(\ker C)$, it follows immediately from Theorem 1 that this system cannot be disturbance decoupled using ordinary state feedback.

Regarding θ as a parameter to be determined, we note that it follows from Theorem 2 that a necessary condition for the solvability of the decoupling problem using the CRPD feedback (17) is

$$\text{Im } D \subset (\cos\theta E - \sin\theta A)(\ker C) \quad (22)$$

It is easily verified that (22) is satisfied if and only if $\tan\theta = 1/2$. Using this choice for θ and following the control synthesis procedure, the CRPD feedback gain F is obtained by solving a decoupling problem for the regular system $R_{-\theta}(E, A, B, C, D)$. One obtains $F = [\delta \sqrt{5} - \delta]$ with δ arbitrary. Using any such F , the CRPD feedback law (17) decouples the disturbance. For any choice of δ except $2\sqrt{5}$, the resulting closed loop system is a regular system.

B. Pole Placement

We will use the control synthesis procedure above to extend to generalized systems with CRPD state feedback the following well known result concerning pole placement for regular systems with ordinary state feedback. Theorem 3 ([15]): Let (E, A, B) be a regular system. (E, A, B) is controllable if and only if for every self-conjugate set Ω of n complex numbers, there exists state feedback $u = Fx + v$ such that Ω is the set of system eigenvalues of the resulting closed loop system $(E, A + BF, B)$.

We have the following generalization of Theorem 3.

Theorem 4: Let $(E, A, B) \in \sum_{\theta}(n, m)$. (E, A, B) is controllable if and only if for every self-conjugate set Ω of n numbers from $\mathbb{C} \cup \{\infty\} - \{ctn\theta\}$, there exists CRPD state feedback $u = F(\cos\theta x - \sin\theta \dot{x}) + v$ such that Ω is the set of system eigenvalues of the resulting closed loop system $g_{\theta}(F)(E, A, B)$.

Proof: We prove this result by following the control synthesis procedure given above. (1) Rotate to obtain the regular system $(E_0, A_0, B) \triangleq R_{-\theta}(E, A, B)$. (2) Let $(\hat{E}_0, \hat{A}_0, B) \in \Sigma_0(n, m)$ and let $(\hat{E}, \hat{A}, B) \triangleq R_{\theta}(\hat{E}_0, \hat{A}_0, B)$. It follows from Proposition 4 that (\hat{E}, \hat{A}, B) has system eigenvalues Ω if and only if the system eigenvalues of $(\hat{E}_0, \hat{A}_0, B)$ are

$$\Omega_0 \triangleq \left\{ \begin{array}{l} (\cos\theta)\alpha + \sin\theta \\ -(\sin\theta)\alpha + \cos\theta \end{array} : \alpha \in \Omega \right\}.$$

(3) Consider the problem of choosing a state feedback gain matrix F so that the closed loop regular system $(\hat{E}_0, \hat{A}_0, B) \triangleq g_0(F)(E_0, A_0, B)$ satisfies the rotated performance specifications-i.e. such that $(\hat{E}_0, \hat{A}_0, B)$ has system eigenvalues Ω_0 . Since Ω is an arbitrary self-conjugate set of n numbers from $\mathbb{C} \cup \{\infty\} - \{\text{ctn } \theta\}$, Ω_0 is an arbitrary self-conjugate set of n numbers from \mathbb{C} . By Theorem 3, this problem is solvable for all such Ω_0 if and only if the regular system (E_0, A_0, B) is controllable. By Proposition 5, the controllability of (E_0, A_0, B) is equivalent to the controllability of (E, A, B) . (4) Since the solvability of the problem in Step (3) is equivalent to the solvability of the original problem of choosing F so that the set of system eigenvalues of $g_{\theta}(F)(E, A, B)$ is Ω , it follows that this problem is solvable for arbitrary Ω if and only if (E, A, B) is controllable. ■

Remark 16: Theorem 4 shows that if (E, A, B) is a controllable system in $\Sigma_{\theta}(n, m)$, then the CRPD state feedback (17) can be used to assign the system eigenvalues arbitrarily (subject to self-conjugacy) in the Riemann sphere (i.e. the extended complex plane) with the single point $\{\text{ctn } \theta\}$ deleted. Since $\Sigma_{\theta}(n, m)$ is invariant under the CRPD feedback (17) (Proposition 7), and $\Sigma_{\theta}(n, m)$ consists of those systems which do not have an eigenvalue at $\text{ctn } \theta$, the CRPD feedback (17) can never place an eigenvalue at $\text{ctn } \theta$. In the special case where $\theta=0$, this means that "the eigenvalues of a controllable regular system can be assigned arbitrarily by ordinary state feedback in the

Riemann sphere with the point at infinity deleted-i.e. in the complex plane. Thus, the pole placement result for $\sum_{\theta}(n,m)$ using CRPD feedback with a given nonzero value of θ is completely analogous to the well known result (Theorem 3) for regular systems using ordinary state feedback. The only difference is that the point $\text{ctn}\theta$ which is deleted from the Riemann sphere no longer happens to be the point at infinity.

Remark 17: Let θ be fixed, and let $H_{\theta}(s)$ denote the mapping of the extended complex plane into itself defined by the linear fractional transformation

$$H_{\theta}(s) = \frac{(\cos\theta)s + \sin\theta}{(-\sin\theta)s + \cos\theta}.$$

Let (E,A,B) , (E_0, A_0, B) , Ω , Ω_0 be as in the proof of Theorem 4. It follows from the proof of Theorem 4 that the problem of finding CRPD feedback to shift the eigenvalues of (E,A,B) to Ω is equivalent to the problem of finding ordinary state feedback to shift the eigenvalues of the regular system (E_0, A_0, B) to $\Omega_0 = H_{\theta}(\Omega)$.

For closed-loop stability, Ω will be a subset of the open left half-plane. However, the left half-plane is not invariant under the mapping H_{θ} . Consequently, Ω_0 will generally not be a subset of the left half-plane. This is an unusual feature of the state feedback synthesis problem to be solved for (E_0, A_0, B) . Ω_0 will be a subset of the image of the open left half-plane under H_{θ} . There are four cases to consider. If $\theta = 0$, the region is trivially the left half-plane, while if $\theta = \pi$, the region is the right half-plane. If $0 < \theta < \pi/2$, the region is the interior of the circle of radius $1/|\sin 2\theta|$ and center at $-\text{ctn} 2\theta$ on the real axis, while if $\pi/2 < \theta < \pi$, the region is the exterior of this circle.

C. Choosing the Ratio Parameter θ

A given system (E,A,B) belongs to $\sum_{\theta}(n,m)$ for all but finitely-many values of the ratio parameter θ . Consequently, if CRPD feedback $u = F(\cos\theta \dot{x} - \sin\theta \ddot{x}) + v$

(or the analogous output feedback) is applied to the system, almost any value of θ may be chosen. We briefly discuss some of the issues involved in the choice of θ .

For an application such as disturbance decoupling, the additional flexibility offered by the choice of θ may be required to make the problem solvable. This is apparent from Theorem 2, and is illustrated by Example 1. where the disturbance decoupling problem is solvable if and only if $\tan\theta = 1/2$. For such applications, problem-solvability dictates the choice of θ .

The role of the parameter θ in pole-placement is quite different from its role in disturbance decoupling. From a theoretical viewpoint, θ has a minimal effect on the solvability of the pole-assignment problem. If (E,A,B) is controllable, Theorem 4 implies that CRPD feedback of the form $u = F(\cos\theta x - \sin\theta \dot{x}) + v$ can place the system eigenvalues anywhere in the extended complex plane except at the point $\cot\theta$. Thus, changing the value of θ has no effect on pole-assignability except to change the position of the lone forbidden eigenvalue location.

However, from a practical viewpoint, the choice of θ is more complex. Basic to the control synthesis procedure is the rotation of $(E,A,B) \in \Sigma_\theta(n,m)$ to the regular system $(E_0, A_0, B) \stackrel{\Delta}{=} R_{-\theta}(E,A,B) \in \Sigma_0(n,m)$. If θ is such that (E,A,B) is near the boundary of $\Sigma_\theta(n,m)$, then (E_0, A_0, B) will be near the boundary of $\Sigma_0(n,m)$. In other words, (E_0, A_0, B) will be a nearly-singular regular system. This is undesirable from a computational viewpoint since Step 3 of the synthesis procedure requires the solution of a feedback synthesis problem for (E_0, A_0, B) .

Example 2: Consider the system (E,A,B) where

$$E \triangleq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad A \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B \triangleq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$(E, A, B) \in \Sigma_\theta(n, m)$ for any nonzero θ . Then

$$E_0 = \begin{bmatrix} -\sin\theta & \cos\theta \\ 0 & -\sin\theta \end{bmatrix} \quad A_0 = \begin{bmatrix} \cos\theta & \sin\theta \\ 0 & \cos\theta \end{bmatrix}$$

The ordinary state-space pair associated with the regular system (E_0, A_0, B)

is $(E_0^{-1}A_0, E_0^{-1}B)$ with

$$E_0^{-1}A_0 = \begin{bmatrix} -\operatorname{ctn}\theta & -\operatorname{csc}^2\theta \\ 0 & -\operatorname{ctn}\theta \end{bmatrix} \quad E_0^{-1}B = \begin{bmatrix} -\operatorname{ctn}\theta \operatorname{csc}\theta \\ -\operatorname{csc}\theta \end{bmatrix}$$

If θ is nearly 0, then (E_0, A_0, B) is nearly-singular, and the entries of $(E_0^{-1}A_0, E_0^{-1}B)$ approach infinity.

To avoid the computational difficulties associated with the solution of synthesis problems for nearly-singular regular systems, it is desirable to choose θ such that (E, A, B) is well within the interior of $\Sigma_\theta(n, m)$. In other words, the open-loop system eigenvalues should not be too close to the point $\operatorname{ctn}\theta$.

Different complications may arise if a desired closed-loop eigenvalue is near the point $\operatorname{ctn}\theta$. If this is the case, the state feedback gain F determined in Step 3 of the synthesis procedure must be such that the closed-loop regular system $g_0(F)(E_0, A_0, B)$ has an eigenvalue near infinity. Consequently, F must be high-gain.

Example 3: Let (E, A, B) be as in Example 2. Suppose that the desired closed-loop eigenvalues are both at the point -1 . In the resulting CRPD feedback law $u = F(\cos\theta \, x - \sin\theta \, \dot{x}) + v$, the entries of F are found to be

$$f_1 = \frac{-\sin\theta}{1+\sin 2\theta} \quad f_2 = \frac{-2\sin\theta - \cos\theta}{1+\sin 2\theta}$$

As $\theta \rightarrow 3\pi/4$ (i.e. as $\cot\theta \rightarrow -1$), the entries of F go to infinity.

The use of high-gain feedback in the presence of unmodeled high-frequency dynamics can result in instability of the closed-loop system. This potential problem can be avoided by choosing θ such that $\cot\theta$ is not too close to any of the desired closed-loop eigenvalues.

Other issues which are relevant to the choice of θ are sensor noise, plant parameter variations, and load disturbances which cannot be completely decoupled. To minimize the effect of measurement noise, it is natural to choose θ to be very small. In this case, CRPD feedback can be viewed as ordinary proportional feedback perturbed by the addition of a very small derivative term. However, if (E, A, B) is singular, then there is an open-loop eigenvalue at infinity, which will be close to $\cot\theta$ if θ is chosen to be small. Consequently, there must be a trade-off between the objective of choosing θ to minimize the effect of sensor noise and choosing θ so that (E_0, A_0, B) is not nearly-singular. The implications of plant parameter variations and load disturbances for the choice of θ in CRPD feedback remain to be investigated. It is known [34] that derivative feedback can be useful for the reduction of sensitivity in regular systems.

VI. FEEDBACK CLASSIFICATION

In this section, we generalize Brunovsky's well known classification of controllable regular systems under the action of the state feedback group [16] [23] [24] [22]. We begin by reviewing the definition of the state feedback group.

Consider the ordinary state-space model

$$\dot{x}(t) = A x(t) + B u(t) \quad (23)$$

where $(A,B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$. We consider three types of elementary transformations on the system (23). They are (1) change of basis in the state-space, $x=Pz$ with P a nonsingular $n \times n$ matrix; (2) change of basis in the input space, $u=Qv$ with Q a nonsingular $m \times m$ matrix; (3) state feedback, $u=Fx+v$. These operations transform the matrix pair (A,B) as follows:

$$(A,B) \rightarrow (P^{-1}AP, P^{-1}B) \quad (24-1)$$

$$(A,B) \rightarrow (A, BQ) \quad (24-2)$$

$$(A,B) \rightarrow (A+BF, B). \quad (24-3)$$

The transformation group generated by (24-1), (24-2), (24-3) can be conveniently represented in the following way. Recall that a right group action of a group G on a set X is a mapping $\eta: X \times G \rightarrow X$ satisfying the conditions

$$\eta(x, e) = x$$

$$\eta(x, g_1 g_2) = \eta(\eta(x, g_1), g_2)$$

where e denotes the identity element of G . If $x \in X$, the orbit of x , denoted xG , consists of the subset $\{\eta(x, g) : g \in G\}$ of X .

Let $C(n,m)$ denote the space of all matrix pairs $(A,B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$ which are controllable. Let $H(n,m)$ denote the group consisting of all nonsingular $(n+m) \times (n+m)$ matrices of the form

$$\begin{bmatrix} P & 0 \\ F & Q \end{bmatrix}$$

with P $n \times n$, F $m \times n$, Q $m \times m$. We refer to $C(n,m)$ as the space of controllable pairs and to $H(n,m)$ as the state feedback group. Define a right group action of $H(n,m)$ on $C(n,m)$ by

$$\eta((A,B), \begin{bmatrix} P & 0 \\ F & Q \end{bmatrix}) \triangleq (P^{-1}AP + P^{-1}BF, P^{-1}BQ). \quad (25)$$

The transformations (24-1), (24-2), (24-3) correspond to the special cases of (25) where $F=0$ and $Q=I$, $P=I$ and $F=0$, $P=I$ and $Q=I$ respectively.

It is of interest to know when two systems (A_1, B_1) and (A_2, B_2) are related by a transformation in the state feedback group. In other words, it is useful to have a classification of the orbits of the group action (25). This is provided by Brunovsky's Theorem [16].

Theorem 5 (Brunovsky): The distinct orbits of $H(n, m)$ acting on $C(n, m)$ are in one-to-one correspondence with the partitions of n into m integer parts, $n = n_1 + n_2 + \dots + n_m$, $n_1 \geq n_2 \geq \dots \geq n_m \geq 0$. On each orbit, there is exactly one pair (A_c, B_c) of the form

$$A_c = \begin{bmatrix} J_{n_1} & 0 & 0 & \dots & 0 \\ 0 & J_{n_2} & 0 & \dots & 0 \\ 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & J_{n_r} \end{bmatrix}$$

$$B_c = \left[\begin{array}{cccccc|ccc} e_{n_1} & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & e_{n_2} & 0 & \dots & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & e_{n_r} & 0 & \dots & 0 \end{array} \right]$$

where $r \triangleq \text{rank } B$, J_k is a $k \times k$ matrix of the form

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ 0 & 0 & \cdot & \cdot & 0 \end{bmatrix}$$

and e_k is a k -dimensional column vector in which the only nonzero component is the last, which is 1.

We will refer to the pair (A_c, B_c) in Theorem 5 as the Brunovsky canonical form associated with the partition (n_1, \dots, n_m) of n . Note that n_1, \dots, n_r are nonzero while n_{r+1}, \dots, n_m are zero.

In order to generalize the preceding result, we begin by considering four types of elementary transformations applied to the generalized linear system (1). They are (1) change of basis in the state-space, $x=Pz$ with P a nonsingular $n \times n$ matrix; (2) change of basis in the input space, $u=Qv$ with Q a nonsingular $m \times m$ matrix; (3) proportional and derivative feedback, $u = F_1x - F_2\dot{x} + v$; (4) left multiplication by a nonsingular $n \times n$ matrix R^{-1} . These operations transform the matrix triple (E,A,B) as follows:

$$(E,A,B) \rightarrow (P^{-1}EP, P^{-1}AP, P^{-1}B) \quad (26-1)$$

$$(E,A,B) \rightarrow (E, A, BQ) \quad (26-2)$$

$$(E,A,B) \rightarrow (E+BF_2, A+BF_1, B) \quad (26-3)$$

$$(E,A,B) \rightarrow (R^{-1}E, R^{-1}A, R^{-1}B). \quad (26-4)$$

The transformation group generated by (26-1), (26-2), (26-3), (26-4) can be conveniently represented in the following way. Let $G(n,m)$ denote the group consisting of all nonsingular $(3n+m) \times (3n+m)$ matrices of the form

$$\begin{bmatrix} R & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & F_2 & F_1 & Q \end{bmatrix}$$

with R and P $n \times n$, Q $m \times m$, and F_1, F_2 $m \times n$. We refer to $G(n,m)$ as the proportional and derivative state feedback group. Define a right group

action of $G(n,m)$ on $\hat{\Sigma}(n,m)$ by

$$\lambda((E,A,B), \begin{bmatrix} R & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & F_2 & F_1 & Q \end{bmatrix}) \triangleq (R^{-1}EP+R^{-1}BF_2, R^{-1}AP+R^{-1}BF_1, R^{-1}BQ). \quad (27)$$

The transformations (26-1), (26-2), (26-3), (26-4) correspond to the special cases of (27) where $R=P$, $F_1=F_2=0$, and $Q=I$; $R=P=I$ and $F_1=F_2=0$; $R=P=I$ and $Q=I$; $P=I$, $F_1=F_2=0$, and $Q=I$ respectively.

Since we are not interested in proportional and derivative feedback in its full generality, we are interested in those subgroups of $G(n,m)$ obtained by requiring the proportional and derivative feedback $u=F_1x-F_2\dot{x}+v$ to be the CRPD state feedback (17). For each θ , let $G_\theta(n,m)$ denote the subgroup of $G(n,m)$ consisting of all nonsingular $(3n+m) \times (3n+m)$ matrices of the form

$$\begin{bmatrix} R & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & \sin\theta F & \cos\theta F & Q \end{bmatrix}. \quad (28)$$

We refer to the family of subgroups $(G_\theta(n,m): \theta \in [0, \pi])$ as the CRPD state feedback groups.

Remark 18: Let $Z(\phi)$ denote the $(3n+m) \times (3n+m)$ matrix

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & \cos\phi I & \sin\phi I & 0 \\ 0 & -\sin\phi I & \cos\phi I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

It is easily verified that

$$G_{\theta+\phi}(n,m) = Z(\phi) G_\theta(n,m) Z(\phi)^{-1}.$$

Hence, the CRPD state feedback groups $(G_\theta(n,m): \theta \in [0, \pi])$ are a one-parameter family of conjugate subgroups of the general linear group $GL(3n+m, \mathbb{R})$, each of which is contained in $G(n,m)$.

Let $g_\theta(R,P,Q,F)$ denote the transformation on $\hat{\Sigma}(n,m)$ induced by the matrix (28) in $G_\theta(n,m)$. In other words,

$$g_\theta(R,P,Q,F)(E,A,B) \triangleq (R^{-1}EP + \sin\theta R^{-1}BF, R^{-1}AP + \cos\theta R^{-1}BF, R^{-1}BQ). \quad (29)$$

Proposition 9: $\hat{\Sigma}_\theta(n,m)$ is invariant under the action of $G_\theta(n,m)$.

Proof: It follows from (29) that

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$$g_\theta(R, P, Q, F) = g_\theta(I, I, Q, 0) g_\theta(I, I, I, F) g_\theta(I, P, I, 0) g_\theta(R, I, I, 0). \quad (30)$$

By Proposition 7, $\Sigma_\theta(n, m)$ is invariant under $g_\theta(I, I, I, F)$, and it is trivial to check that $\Sigma_\theta(n, m)$ is invariant under $g_\theta(I, I, Q, 0)$, $g_\theta(I, P, I, 0)$, and $g_\theta(R, I, I, 0)$. Hence, it follows from (30) that $\Sigma_\theta(n, m)$ is invariant under $g_\theta(R, P, Q, F)$. ■

Proposition 10: The following is a commutative diagram:

$$\begin{array}{ccc} \Sigma_\theta(n, m) & \xrightarrow{g_\theta(R, P, Q, F)} & \Sigma_\theta(n, m) \\ R_\phi \downarrow & & \downarrow R_\phi \\ \Sigma_{\theta+\phi}(n, m) & \xrightarrow{g_{\theta+\phi}(R, P, Q, F)} & \Sigma_{\theta+\phi}(n, m) \end{array}$$

I.e. $R_\phi \circ g_\theta(R, P, Q, F) = g_{\theta+\phi}(R, P, Q, F) \circ R_\phi$.

Proof: By direct verification. ■

Remark 19: Since $g_\theta(I, I, I, F) = g_\theta(F)$, the Fundamental Lemma of CRPD Feedback is a special case of Proposition 10.

Let $C_\theta(n, m)$ denote the set of those systems $(E, A, B) \in \Sigma_\theta(n, m)$ which are controllable.

Proposition 11: $C_\theta(n, m)$ is invariant under the action of $G_\theta(n, m)$.

Proof: Let $(E, A, B) \in C_\theta(n, m)$ and let $(\hat{E}, \hat{A}, \hat{B}) \triangleq g_\theta(R, P, Q, F)(E, A, B)$. By Proposition 9, $(\hat{E}, \hat{A}, \hat{B}) \in \Sigma_\theta(n, m)$, so it suffices to show that $(\hat{E}, \hat{A}, \hat{B})$ is controllable. Let $(E_0, A_0, B_0) \triangleq R_{-\theta}(E, A, B)$ and let $(\hat{E}_0, \hat{A}_0, \hat{B}_0) \triangleq R_{-\theta}(\hat{E}, \hat{A}, \hat{B})$. Since controllability is invariant under system rotation (Proposition 5), (E_0, A_0, B_0) is controllable, and it suffices to show that $(\hat{E}_0, \hat{A}_0, \hat{B}_0)$ is controllable.

By Proposition 10, $(\hat{E}_0, \hat{A}_0, \hat{B}_0) = g_0(R, P, Q, F)(E_0, A_0, B_0)$. This implies that

$$(\hat{E}_0^{-1} \hat{A}_0, \hat{E}_0^{-1} \hat{B}_0) = \eta((E_0^{-1} A_0, E_0^{-1} B_0), \begin{bmatrix} P & 0 \\ F & Q \end{bmatrix}). \quad (31)$$

Since the regular system (E_0, A_0, B_0) is controllable, $(E_0^{-1}A_0, E_0^{-1}B_0)$ is a controllable pair-i.e. $(E_0^{-1}A_0, E_0^{-1}B_0) \in C(n,m)$. Since $C(n,m)$ is invariant under the action of the state feedback group $H(n,m)$, (31) implies that $(\hat{E}_0^{-1}\hat{A}_0, \hat{E}_0^{-1}\hat{B}_0) \in C(n,m)$. Hence, the regular system $(\hat{E}_0, \hat{A}_0, \hat{B}_0)$ is controllable. ■

By virtue of Proposition 11, we can restrict the action of $G_\theta(n,m)$ on $\hat{\Sigma}(n,m)$ to the invariant subset $C_\theta(n,m)$. For the remainder of this section, we study the action of the CRPD state feedback group $G_\theta(n,m)$ on the subset $C_\theta(n,m)$ consisting of the controllable systems in $\hat{\Sigma}_\theta(n,m)$.

Remark 20: In the special case where $\theta=0$, $C_\theta(n,m)$ consists of the controllable regular systems, and $G_\theta(n,m)$ can be regarded as the state feedback group $H(n,m)$ augmented by left-multiplication.

We now classify the systems in $C_\theta(n,m)$ relative to the action of the CRPD state feedback group $G_\theta(n,m)$.

Theorem 6: The distinct orbits of $G_\theta(n,m)$ acting on $C_\theta(n,m)$ are in one-to-one correspondence with the partitions of n into m integer parts, $n = n_1 + n_2 + \dots + n_m$, $n_1 \geq n_2 \geq \dots \geq n_m \geq 0$. On each orbit, there is exactly one triple $(\tilde{E}, \tilde{A}, \tilde{B})$ of the form

$$\begin{aligned}\tilde{E} &= \cos\theta I + \sin\theta A_c \\ \tilde{A} &= -\sin\theta I + \cos\theta A_c \\ \tilde{B} &= B_c\end{aligned}$$

where (A_c, B_c) is the Brunovsky canonical form associated with the partition (n_1, \dots, n_m) .

Proof: Let $(E, A, B) \in C_\theta(n,m)$, and let $(E_0, A_0, B_0) \stackrel{\Delta}{=} R_{-\theta}(E, A, B)$. Let (A_c, B_c) denote the Brunovsky canonical form of the pair $(E_0^{-1}A_0, E_0^{-1}B_0) \in C(n,m)$. By Theorem 5, there exist P, Q, F such that

$$(A_c, B_c) = (P^{-1}E_0^{-1}A_0P + P^{-1}E_0^{-1}B_0F, P^{-1}E_0^{-1}B_0Q).$$

Letting $R \stackrel{\Delta}{=} E_0P$, it follows that

$$g_0(R,P,Q,F)(E_0,A_0,B_0) = (I,A_c,B_c).$$

Applying R_θ to both sides of this equation and invoking Proposition 10 gives

$$g_\theta(R,P,Q,F)(E,A,B) = (\cos\theta I + \sin\theta A_c, -\sin\theta I + \cos\theta A_c, B_c)$$

as required.

It remains to show that the form $(\tilde{E},\tilde{A},\tilde{B})$ is unique. Let (A_c,B_c) be a pair in Brunovsky canonical form, and suppose there exist R,P,Q,F such that

$$g_\theta(R,P,Q,F)(E,A,B) = R_\theta(I,A_c,B_c)$$

It follows from Proposition 10 that

$$g_0(R,P,Q,F)(E_0,A_0,B_0) = (I,A_c,B_c).$$

This implies that $R=E_0P$ and

$$(A_c,B_c) = \eta((E_0^{-1}A_0, E_0^{-1}B_0), \begin{bmatrix} P & 0 \\ F & Q \end{bmatrix}).$$

From the uniqueness part of Theorem 5, we conclude that (A_c,B_c) is unique.

Since $(\tilde{E},\tilde{A},\tilde{B}) = R_\theta(I,A_c,B_c)$, it follows that $(\tilde{E},\tilde{A},\tilde{B})$ is unique. ■

Remark 21: The canonical form for the action of $G_\theta(n,m)$ on $C_\theta(n,m)$ given in Theorem 6 is a rotated version of Brunovsky canonical form. In the special case where $\theta=0$, the canonical form is (I,A_c,B_c) with (A_c,B_c) in Brunovsky canonical form. Thus, Brunovsky's result (Theorem 5) can be regarded as a special case of Theorem 6, although there is a slight difference in form which is due to the fact that $C_0(n,m)$ consists of matrix triples whereas $C(n,m)$ consists of matrix pairs, and $G_0(n,m)$ consists of $H(n,m)$ augmented by left-multiplication.

Example 4: Let $n=5$, $m=3$, and $(n_1,n_2,n_3) = (3,2,0)$. The canonical form for the orbit associated with the partition (n_1,n_2,n_3) (of the action of $G_\theta(n,m)$ on $C_\theta(n,m)$) is

$$\tilde{E} = \left[\begin{array}{ccc|cc} \cos\theta & \sin\theta & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & 0 & 0 \\ \hline 0 & 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & 0 & 0 & \cos\theta \end{array} \right]$$

$$\tilde{A} = \left[\begin{array}{ccc|cc} -\sin\theta & \cos\theta & 0 & 0 & 0 \\ 0 & -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & -\sin\theta & 0 & 0 \\ \hline 0 & 0 & 0 & -\sin\theta & \cos\theta \\ 0 & 0 & 0 & 0 & -\sin\theta \end{array} \right]$$

$$\tilde{B} = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right]$$

If $\eta: X \times G \rightarrow X$ is a group action, the stabilizer of an element $x \in X$ is the subgroup of all $g \in G$ such that $\eta(x, g) = x$. For the action (25) of the state feedback group $H(n, m)$ on the space of controllable pairs $C(n, m)$, Brockett [24] has described the structure of the stabilizer subgroups. In particular, he has shown that if (A_c, B_c) is the Brunovsky canonical form corresponding to the partition (n_1, \dots, n_m) of n , and if r denotes the rank of B_c , then the dimension of the stabilizer of (A_c, B_c) is equal to

$$(n+m)(m-r) + \sum_{\{i,j: n_i \geq n_j\}} (n_i + 1 - n_j). \quad (32)$$

We now describe additional properties of the action of the CRPD state feedback group $G_\theta(n, m)$ on the space $C_\theta(n, m)$ of controllable systems in $\Sigma_\theta(n, m)$. Let $\text{Orb}_\theta(n_1, \dots, n_m)$ denote the orbit of this action corresponding to the partition (n_1, \dots, n_m) of n . In other words, if (A_c, B_c) denotes the Brunovsky canonical form corresponding to (n_1, \dots, n_m) , then $\text{Orb}_\theta(n_1, \dots, n_m)$ is the orbit of the triple $R_\theta(I, A_c, B_c) \in C_\theta(n, m)$.

Theorem 7:

- (a) R_ϕ maps $\text{Orb}_\theta(n_1, \dots, n_m)$ isomorphically onto $\text{Orb}_{\theta+\phi}(n_1, \dots, n_m)$.
- (b) The dimension of $\text{Orb}_\theta(n_1, \dots, n_m)$ is $2n^2 + (n+m)r - \sum_{(i,j: n_i \geq n_j)} (n_i + 1 - n_j)$ (where r is the rank of B_c).

Proof: Let $(E, A, B) \in C_\theta(n, m)$, and let (A_c, B_c) denote the Brunovsky canonical form corresponding to (n_1, \dots, n_m) . It follows from Proposition 10 that $g_0(R, P, Q, F) \circ R_{-\theta}(E, A, B) = (I, A_c, B_c)$ if and only if $g_\theta(R, P, Q, F)(E, A, B) = R_\theta(I, A_c, B_c)$. Consequently, $R_{-\theta}(E, A, B) \in \text{Orb}_0(n_1, \dots, n_m)$ if and only if $(E, A, B) \in \text{Orb}_\theta(n_1, \dots, n_m)$. Thus, $R_\theta(\text{Orb}_0(n_1, \dots, n_m)) = \text{Orb}_\theta(n_1, \dots, n_m)$, which easily implies that

$$R_\phi(\text{Orb}_\theta(n_1, \dots, n_m)) = \text{Orb}_{\theta+\phi}(n_1, \dots, n_m).$$

Since the map R_ϕ is an isomorphism, this proves (a).

By virtue of (a), it suffices to prove the dimension formula in the special case where $\theta=0$. It is easy to check that $g_0(R, P, Q, F)(I, A_c, B_c) = (I, A_c, B_c)$ if and only if $R=P$ and $\begin{bmatrix} P & 0 \\ F & Q \end{bmatrix}$ belongs to the stabilizer of (A_c, B_c) under the group action (25). Consequently, the dimension of the stabilizer of (I, A_c, B_c) under the action of $G_0(n, m)$ is given by (32). Since the dimension of $G_0(n, m)$ is $2n^2 + m^2 + mn$, and the dimension of $\text{Orb}_0(n_1, \dots, n_m)$ is the difference between the dimension of $G_0(n, m)$ and the dimension of the stabilizer, (b) follows immediately. ■

Remark 22: Hayton [25] has studied the action of the state feedback group $H(n, m)$ on the space of generalized systems. Our approach differs considerably since we do not permit ordinary state feedback transformations (i.e. no derivative contribution) to be applied to the singular systems.

VII. CONCLUSION

We have presented a unified theory of control synthesis which applies to all admissible generalized linear systems, both regular and singular. Our approach differs considerably from those in the literature. We do not

attempt to apply ordinary static state feedback and output feedback to singular systems. Instead, we cover the space of all admissible systems by a family of open and dense subsets indexed by the unit circle. One of these subsets coincides with the set of all regular systems. There is a group of symmetry transformations of this cover, which can be identified with the group of rotations of the plane. On each of the open and dense subsets, we define an admissible class of feedback transformations by fixing the ratio of state to state-derivative (or output to output-derivative) in the feedback law. Thus, the class of allowable feedback transformations is specifically adapted to each subset. On the subset consisting of the regular systems, the admissible feedback coincides with ordinary static state (or output) feedback. Thus, the theory of static state feedback (and static output feedback) for regular systems is included as a special case of our theory.

Using this approach, we obtain a general procedure of control synthesis of constant-ratio proportional and derivative feedback for generalized linear systems which systematically reduces each synthesis problem to an ordinary static state feedback (or output feedback) synthesis problem for a corresponding regular system. The regular system is obtained by system rotation, not by the Weierstrass decomposition. In particular, this means that the determination of the associated regular system is trivial from a computational point of view. We apply the control synthesis procedure to derive generalizations of three fundamental results in the theory of state feedback for regular systems. These include the Disturbance Decoupling Theorem, the Pole Assignment Theorem, and the Brunovsky Classification Theorem. In the case of disturbance decoupling, our result is interesting even when specialized to the regular systems. We are able to show that constant-ratio proportional and derivative feedback can be used to disturbance-decouple regular systems which cannot be decoupled using ordinary state feedback.

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