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Dynamic, Transient and Stationary Behavior of the M/GI/1 Queue

by

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ABSTRACT

An exponential martingale is associated with the Markov chain of the number of customers in the $M/GI/1$ queue. This together with renewal theory are shown to provide a unified probabilistic framework for deriving several well-known generating functions for the $M/GI/1$ queue, including the Pollaczek-Khinchine formula, the transient generating function of the number of customers at departure epochs and the generating function of the number of customers served in a busy period.

INTRODUCTION

An exponential martingale is associated with the Markov chain of the number of customers at departure epochs in the $M/GI/1$ queue. Basic regularity properties of this martingale and standard arguments from Renewal Theory are shown to provide a unified probabilistic framework for deriving three well-known analytical formulae which respectively characterize the dynamic, transient and stationary behavior of this queue. The main theoretical interest of this new approach stems from a general equivalence relationship between the law of this embedded Markov chain and the law of the forward recurrence time of a discrete-time renewal process associated with this chain (Theorem 3). This equivalence produces several results which appear to be new, at least to the best of the authors' knowledge. For instance, a new probabilistic representation is established for the generating function of the Markov chain of the number of customers at the n^{th} departure epoch (Corollary 7). The usual representation of the transient generating function in terms of the double generating function of this quantity, both in time and space, is shown to be derivable from this new probabilistic representation.

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The paper is organized as follows: The exponential martingale is introduced in Section 1, and its definition is followed by a summary of its key regularity properties, already established by the authors in a previous paper [1]. Section 2 contains the derivation of the analytical results mentioned above, with some of the calculations collected in Section 3.

It should be emphasized that the authors' aim was not to give here a comprehensive list of the applications of this new approach, but rather to rather to illustrate the power of the approach. For instance, the results on the transient generating function are limited to the analysis of a simple particular case. Similarly, only the hitting time to the zero state (also known as busy period) was considered in the study of the dynamic properties of the chain. However, it is easy to see that most known formulae concerning the transient dynamic behavior can simply be obtained through similar arguments. These techniques are of independent interest and apply to other queueing systems, like for instance queues in random environments [2].

1. PRELIMINARIES

1.1. Notations

All the random variables (RV) and stochastic elements occurring in this paper are implicitly defined on a fixed probability triple $(\Omega, \mathcal{F}, \mathcal{P})$. Throughout, the characteristic function of any event A in $\mathcal{F}$ is denoted by $I[A]$. The collection of all non-negative integers is denoted by $\mathbb{N}$ and $\mathbb{R}$ denotes the set of real numbers.

Consider an $M/GI/1$ queue with Poissonian arrival pattern of intensity λ. The consecutive service times form a sequence of i.i.d. $\mathbb{R}$-valued RV's $\{S_n\}_1^\infty$ assumed independent of the arrival process. Throughout, the common probability distribution of the service times and its Laplace-Stieltjes transform are denoted by S and S^*, respectively. The initial queue size is given by an $\mathbb{N}$-valued RV Ξ.

1.2. The embedding

At time $t = 0$, a dummy customer is assumed to complete service and by leaving the system, it generates the 0^{th} departure. For $n = 0, 1, \ldots$, let X_n denote the number of customers in the system as seen by the n^{th} departing customer. For $n = 1, 2, \ldots$, let A_n denote the number of arrivals during the n^{th} service period. ¿From these definitions, the queue size sequence $\{X_n\}_0^\infty$ is readily seen to satisfy the recursion

$$X_{n+1} = X_n + A_{n+1} - I[X_n \neq 0], \qquad n = 0, 1, \ldots \tag{1.1}$$

with random initial condition $X_0 = \Xi$. Under the enforced assumptions, the RV's $\{A_{n+1}\}_0^\infty$ are i.i.d. RV's independent of the initial queue size Ξ, and the $\mathbb{N}$-valued process $\{X_n\}_0^\infty$ is an *irreducible* Markov chain with countable state space [3,5].

1.3. The martingale

For all $n = 0, 1, \ldots$, the RV's $\{\Xi, A_k, 0 < k \leq n\}$ generate the σ-field $\mathcal{F}_n$; in view of (1.1) the RV's $\{X_k\}_0^n$ are all $\mathcal{F}_n$-measurable. Consider an arbitrary $\mathcal{F}_n$-stopping time σ and define the $\mathbb{N}$-valued RV $\nu(\sigma)$ by

$$\nu(\sigma) = \begin{cases} \inf\{n \geq 1 : X_{\sigma+n} = 0\} & \text{if } \sigma < \infty \text{ and this set is non empty ;} \\ \infty & \text{otherwise .} \end{cases} \tag{1.2}$$

Let a denote the generating function of the i.i.d. sequence $\{A_{n+1}\}_0^\infty$, namely

$$a(z) = E[z^{A_n}] = S^*(\lambda(1-z)), \quad 0 \le z \le 1. \qquad n = 0,1,\ldots (1.3)$$

For $0 < z \le 1$, the $\mathcal{F}_n$-measurable $\mathbb{R}_+$-valued RV's $\{M_n(z)\}_0^\infty$ are defined by

$$M_n(z) = \begin{cases} z^{X_0} & \text{if } n = 0 \; ; \\ z^{X_n} \dfrac{z^{\sum_{k=0}^{n-1} I[X_k \neq 0]}}{a(z)^n} & \text{for } n = 1,2,\ldots \end{cases} \qquad (1.4)$$

The following result was established in [1, Thm. 2 pp. 181,186].

Theorem 1. *For all $0 < z < 1$, the RV's $\{M_n(z)\}_0^\infty$ form a positive and integrable $\mathcal{F}_n$-martingale. If $\rho \le 1$, the stopping time $\nu(\sigma)$ is regular for this martingale and the relation*

$$E[I[\sigma < \infty, \nu(\sigma) < \infty][\frac{z}{a(z)}]^{\nu(\sigma)} \mid \mathcal{F}_\sigma] = I[\sigma < \infty] z^{I[X_\sigma = 0]} z^{X_\sigma} \quad \text{a.s.} \qquad (1.5)$$

holds for all $0 < z \le 1$.

The relation (1.5) is a simple consequence of the regularity of the stopping time σ and of Doob's Optional Sampling Theorem [6 Cor. IV-2-6, p. 67]. Moreover, $z < a(z)$ for $0 < z < 1$ under the condition $\rho \le 1$ (Takàcs' Lemma [7, p. 46], and letting $z \uparrow 1$ in (1.5) yields

$$P[\sigma < \infty, \nu(\sigma) < \infty \mid \mathcal{F}_\sigma] = I[\sigma < \infty] \quad \text{a.s.} \qquad (1.6)$$

as an immediate consequence of the Bounded Convergence Theorem.

2. TRANSFORMS

Let $\{\tau_n\}_0^\infty$ be the sequence of $\mathcal{F}_n$-stoppin[illegible]ned by the recursion

$$\tau_{n+1} = \tau_n + \nu(\tau_n) \qquad n = 0,1,..(2.1)$$

with $\tau_0 = 0$. With $\sigma = 0$, (1.6) specializes to

$$P[\tau_1 < \infty \mid \mathcal{F}_0] = 1 \quad \text{a.s.} \qquad (2.2)$$

Lemma 2. *If $\rho \le 1$, the RV's $\{\tau_n\}_0^\infty$ form a possibly delayed recurrent renewal process.*

Proof. It is plain from (1.5) that the RV $\nu(\tau_n)$ and the σ-field $\mathcal{F}_{\nu(\tau_n)}$ are independent for all $n = 0,1,\ldots$ Consequently the RV's $\{\tau_n\}_0^\infty$ form a renewal process if $\Xi = 0$ a.s. and a delayed renewal process otherwise. For $\rho \le 1$, the recurrence property is immediately obtained from (1.6). □

The forward recurrence times $\{\mu(n)\}_0^\infty$ of the recurrent renewal process $\{\tau_n\}_0^\infty$ are defined by

$$\mu(n) = \begin{cases} \inf\{m \ge 0 : X_{n+m} = 0\} & \text{if this set is non empty ;} \\ \infty & \text{otherwise .} \end{cases} \qquad n = 0,1,...(2.3)$$

It turns out that the generating function of the number X_n of customers at service completion is very simply related to the generating function of the forward recurrence time $\mu(n)$. The key relationship is provided in the next theorem. For every $0 < z \leq 1$, it is convenient to introduce the quantity $\xi(z)$ as the ratio

$$\xi(z) = \frac{z}{a(z)}. \tag{2.4}$$

Theorem 3. *Assume $\rho < 1$. For all $0 < z \leq 1$, the relation*

$$E[z^{X_n}] = E[\xi(z)^{\mu(n)}] \qquad n = 0, 1, ... (2.5)$$

holds true.

Proof. Note from (1.2) and (2.3) the easy facts

$$[X_n = 0] = [\mu(n) = 0] \qquad n = 0, 1, ... (2.6)$$

and

$$\nu(n) = \mu(n) \quad \text{on} \quad [X_n \neq 0]. \qquad n = 0, 1, ... (2.7)$$

It is then plain that for each $y > 0$, the relations

$$y^{\mu(n)} I[\mu(n) \neq 0] = y^{\mu(n)} I[X_n \neq 0] = y^{\nu(n)} I[X_n \neq 0] \qquad n = 0, 1, ... (2.8)$$

hold true.

Specialize (1.5) to $\sigma = n$ and multiply both sides of the resulting equation by $I[X_n \neq 0]$. For all $0 < z \leq 1$, the relation

$$E[I[\nu(n) < \infty, X_n \neq 0] \xi(z)^{\nu(n)} \mid \mathcal{F}_n] = I[X_n \neq 0] z^{X_n} \quad \text{a.s.} \tag{2.9}$$

readily follows since the RV X_n is $\mathcal{F}_n$-measurable, and taking the mathematical expectation of both sides of (2.9) implies

$$E[I[\nu(n) < \infty, X_n \neq 0] \xi(z)^{\nu(n)}] = E[I[X_n \neq 0] z^{X_n}]. \tag{2.10}$$

Starting with a simple decomposition argument, it is now plain that for all $n = 0, 1, ...$, the relations

$$\begin{aligned} E[z^{X_n}] &= P[X_n = 0] + E\left[z^{X_n} I[X_n \neq 0]\right] \\ &= P[\mu(n) = 0] + E\left[I[\nu(n) < \infty, X_n \neq 0] \xi(z)^{\nu(n)}\right] \\ &= P[\mu(n) = 0] + E\left[I[\nu(n) < \infty, \mu(n) \neq 0] \xi(z)^{\mu(n)}\right] \end{aligned} \tag{2.11}$$

hold true. In this chain of equalities, the second one is a consequence of (2.6) and (2.10), while the last equality follows from (2.8). Under the foregoing assumptions, $P[\nu(n) < \infty \mid \mathcal{F}_n] = 1$ a.s. by virtue of (1.6), whence $\mu(n) < \infty$ a.s. since $0 \leq \mu(n) \leq \nu(n)$ as a consequence of (2.6) and (2.7). This remark readily validates the passage from (2.11) to (2.5). □

The aim of the remainder of the section is to recover various known transforms from (1.5) and (2.5), thus avoiding the usual analytical calculations.

2.1. Generating function of the number of customers served in a busy period

Denote by $\{f(n)\}_0^\infty$ the point mass distribution function of the number of customers served in a busy period of the $M/GI/1$ queue. On the event $[X_0 = 0]$, the number of customers served in the first busy period coincides with $\nu(0)$, so that the generating function of $\{f(n)\}_0^\infty$, denoted by F^*, is given by the relation

$$F^*(y) = E[y^{\nu(0)} \mid X_0 = 0], \quad 0 \leq y \leq 1. \tag{2.12}$$

In other words, F^* is also the generating function of the interevent times of the discrete-time renewal process $\{\tau_n\}_0^\infty$. It is now shown that F^* can be obtained as an immediate consequence of (1.5).

Lemma 4. *Assume $\rho \leq 1$. For each $0 < \xi \leq 1$, the equation in the unknown variable z*

$$z = \xi a(z) \tag{2.13}$$

has a unique root in the interval $[0, 1]$, denoted by $Z(\xi)$. The generating function F^* of the number of customers served in a busy period, or equivalently of the interevent times of the discrete-time renewal process $\{\tau_n\}_0^\infty$, is given by

$$F^*(y) = Z(y), \quad 0 \leq y \leq 1, \tag{2.14}$$

and the mean value m of this distribution function is given by

$$m = E[\nu(0) \mid X_0 = 0] = \begin{cases} \frac{1}{1-\rho} & \text{if } \rho < 1 \ ; \\ \infty & \text{if } \rho = 1 \ . \end{cases} \tag{2.15}$$

Proof. The first statement concerning the roots of (2.13) follows from classical convexity arguments [7, Lemma 1, p. 47] and its proof is therefore omitted for the sake of brevity. Upon specializing again (1.5) to $\sigma = 0$, it is plain that for all $0 < z \leq 1$,

$$E[\xi(z)^{\nu(0)} \mid X_0 = 0] = z \quad \text{a.s.} \tag{2.16}$$

Let ξ be a real number such that $0 \leq \xi \leq 1$ and let $Z(\xi)$ denote the root corresponding to (2.13). It is now immediate from (2.16) and from the definition of $Z(\xi)$ that

$$F^*(\xi) = E[\xi^{\nu(0)} \mid X_0 = 0] = Z(\xi) \quad \text{a.s.}, \tag{2.17}$$

and (2.14) is obtained. Equation (2.15) is now obtained by differentiating (2.16) with respect to z in a left neighborhood of 1 and by letting z go to 1 in the resulting expression. The differentiation step is validated by well-known properties of uniform convergence for generating functions. □

2.2 Generating function of the number of customers at steady state

The next step consists in establishing the Pollaczek-Khinchine transform.

Corollary 5. *Assume $\rho < 1$. For all $0 < z \leq 1$, $\lim_n E[z^{X_n}]$ exists (when n goes to ∞) and is given by*

$$\lim\nolimits_n E[z^{X_n}] = (1 - \rho)\frac{(1 - z)a(z)}{a(z) - z}. \tag{2.18}$$

can be used in (3.1) to readily yield the relation

$$g(y,n) = \frac{y}{2i\pi}\int_C \frac{F^*(u)}{u-y}\frac{du}{u^{n+1}} \qquad n = 1,2,\ldots(3.3)$$

after standard algebraic manipulations. It is now plain from (2.22), (2.28) and (3.3) that

$$\begin{aligned} G^*(y,t) &==1 + \frac{y}{2i\pi}\int_C \frac{F^*(u)}{u-y}\sum_{n=1}^{\infty}\frac{t}{u}^n\frac{du}{u} \\ &=1 + \frac{yt}{2i\pi}\int_C \frac{F^*(u)}{(u-y)u(u-t)}du \end{aligned} \tag{3.4}$$

where the passage from the first to the second equality follows from standard facts on geometric series. Finally, substitute the decomposition

$$\frac{1}{(u-y)u(u-t)} = \frac{1}{(u-y)}\frac{1}{y(y-t)} + \frac{1}{(u-t)}\frac{1}{t(t-y)} + \frac{1}{uty} \tag{3.5}$$

in the last expression of (3.4), and note by a straightforward application of Cauchy's Formula that

$$G^*(y,t) = 1 + \frac{tF^*(y) - yF^*(t)}{y-t} + F^*(0). \tag{3.6}$$

The identity (2.14) and the fact that $Z(0) = 0$ now imply that

$$G^*(y,t) = 1 + \frac{tZ(y) - yZ(t)}{y-t} \tag{3.7}$$

and (2.31) is now obtained by some elementary algebra after using this last expression in (2.32). □

Observe that $N^*(z,t)$ also admits an integral representation in the form

$$N^*(z,t) = \frac{1}{1-Z(t)}\left[1 + \frac{\xi(z)t}{2i\pi}\int_C \frac{Z(u)}{(u-\xi(z))u(u-t)}du\right], \tag{3.8}$$

which follows from (2.32) and (3.4).

REFERENCES

[1] F. Baccelli and A.M. Makowski, "Martingale arguments for stability: The $M/GI/1$ case," *Systems & Control Letters* **6** (1986), pp. 181-186.

[2] F. Baccelli and A.M. Makowski, "Exponential martingales for queues in random environment," in preparation, 1988.

[3] E. Çinlar, *Introduction to Stochastic Processes*, Prentice-Hall, Englewood Cliffs, NJ (1975).

[4] J.W. Cohen, *The Single Server Queue*, North-Holland, Amsterdam (The Netherlands) (1969).

[5] L. Kleinrock, *Queueing Systems I: Theory*, J. Wiley & Sons, New York (1976).

[6] J. Neveu, *Discrete-time Martingales*. English Translation , North Holland, Amsterdam (The Netherlands) (1975).

[7] L. Takács, *Introduction to the Theory of Queues*, Oxford University Press, New York (1962).

[8] M. Woodroofe, *Nonlinear Renewal Theory in Sequential Analysis*, CBMS-NSF Regional Conference Series in Applied Mathematics, S.I.A.M. Philadelphia, PA (1982).

and the representation (2.23) follows immediately from (2.5) and (2.25). □

Classically [7, Eqn. (59), p. 70], the transient behavior of the queue is given in terms of the double transform N^* given by

$$N^*(z,t) = \sum_{n=0}^{\infty} t^n E[z^{X_n}], \quad 0 \le z \le 1,\ 0 \le t < 1. \tag{2.26}$$

In order to recover the classical xpression for this function from (2.23), it is convenient to introduce the generating functions

$$R^*(t) = \sum_{n=0}^{\infty} R(n)t^n \tag{2.27}$$

and

$$G^*(y,t) = \sum_{n=0}^{\infty} g(y,n)t^n \tag{2.28}$$

These generating functions are expressed in term of the root function $\xi \to Z(\xi)$ of Lemma 4 through the relations

$$R^*(t) = \frac{1}{1-Z(t)} \tag{2.29}$$

and

$$G^*(y,t) = 1 + \frac{tZ(y) - yZ(t)}{y-t} \tag{2.30}$$

defined for $0 \le t < 1$. The derivation of (2.29) is immediate from (2.21), while the derivation of (2.30) from (2.22) is established in Section 3. The well-known formula for $N^*(z,t)$ given in [7, Eqn. (59), p. 70] is now directly recovered from these relations.

Corollary 8. *Assume* $\rho < 1$ *and* $\Xi = 0$. *For all* $0 < z \le 1$, *the relation*

$$N^*(z,t) = \frac{z(1-Z(t)) - (1-z)ta(z)}{(1-Z(t))(z-ta(z))} \tag{2.31}$$

holds with Z *as defined in Lemma 4.*

Proof. Taking the generating function in n of both sides of Equation (2.23) readily yields the relation

$$N^*(z,t) = R^*(t)G^*(\xi(z),t) \qquad n = 0,1,\ldots(2.32)$$

and (2.31) is a now a direct consequence of (2.29) and (2.30). □

3. Derivation of (2.30)

It is plain from (2.22) that the relation

$$g(y,n) = E[y^{\tau_1 - n} I[\tau_1 > n]] = \sum_{k=n+1}^{\infty} f(k)y^{k-n} \tag{3.1}$$

holds for all $n = 1,2,\ldots$ and $0 < y \le 1$. With C denoting the unit circle in the complex plane, the integral representation

$$f(k) = \frac{1}{2i\pi}\int_C \frac{F^*(u)}{u^{k+1}}du \qquad k = 0,1,\ldots(3.2)$$

can be used in (3.1) to readily yield the relation

$$g(y,n) = \frac{y}{2i\pi} \int_C \frac{F^*(u)}{u-y} \frac{du}{u^{n+1}} \qquad n = 1,2,\ldots (3.3)$$

after standard algebraic manipulations. It is now plain from (2.22), (2.28) and (3.3) that

$$\begin{aligned} G^*(y,t) &= = 1 + \frac{y}{2i\pi} \int_C \frac{F^*(u)}{u-y} \sum_{n=1}^{\infty} \frac{t}{u}^n \frac{du}{u} \\ &= 1 + \frac{yt}{2i\pi} \int_C \frac{F^*(u)}{(u-y)u(u-t)} du \end{aligned} \tag{3.4}$$

where the passage from the first to the second equality follows from standard facts on geometric series. Finally, substitute the decomposition

$$\frac{1}{(u-y)u(u-t)} = \frac{1}{(u-y)} \frac{1}{y(y-t)} + \frac{1}{(u-t)} \frac{1}{t(t-y)} + \frac{1}{uty} \tag{3.5}$$

in the last expression of (3.4), and note by a straightforward application of Cauchy's Formula that

$$G^*(y,t) = 1 + \frac{tF^*(y) - yF^*(t)}{y-t} + F^*(0). \tag{3.6}$$

The identity (2.14) and the fact that $Z(0) = 0$ now imply that

$$G^*(y,t) = 1 + \frac{tZ(y) - yZ(t)}{y-t} \tag{3.7}$$

and (2.31) is now obtained by some elementary algebra after using this last expression in (2.32). □

Observe that $N^*(z,t)$ also admits an integral representation in the form

$$N^*(z,t) = \frac{1}{1-Z(t)} \left[1 + \frac{\xi(z)t}{2i\pi} \int_C \frac{Z(u)}{(u-\xi(z))u(u-t)} du \right], \tag{3.8}$$

which follows from (2.32) and (3.4).

REFERENCES

[1] F. Baccelli and A.M. Makowski, "Martingale arguments for stability: The $M/GI/1$ case," *Systems & Control Letters* **6** (1986), pp. 181-186.

[2] F. Baccelli and A.M. Makowski, "Exponential martingales for queues in random environment," in preparation, 1988.

[3] E. Çinlar, *Introduction to Stochastic Processes*, Prentice-Hall, Englewood Cliffs, NJ (1975).

[4] J.W. Cohen, *The Single Server Queue*, North-Holland, Amsterdam (The Netherlands) (1969).

[5] L. Kleinrock, *Queueing Systems I: Theory*, J. Wiley & Sons, New York (1976).

[6] J. Neveu, *Discrete-time Martingales.* English Translation , North Holland, Amsterdam (The Netherlands) (1975).

[7] L. Takács, *Introduction to the Theory of Queues*, Oxford University Press, New York (1962).

[8] M. Woodroofe, *Nonlinear Renewal Theory in Sequential Analysis*, CBMS-NSF Regional Conference Series in Applied Mathematics, S.I.A.M. Philadelphia, PA (1982).