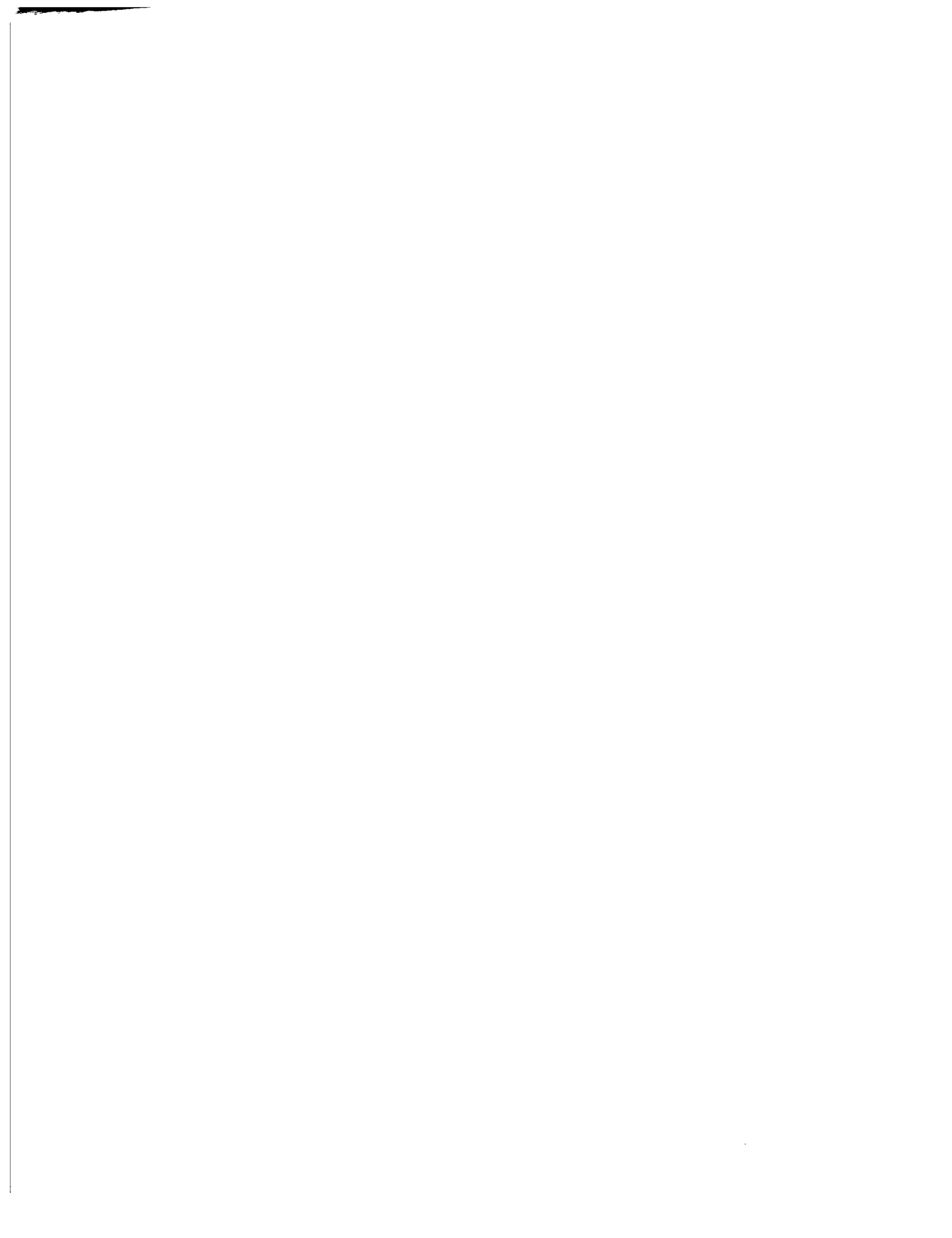


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**Small Degree Solutions for the  
Polynomial Bezout Equation**

**by**

**C.A. Berenstein and D.C. Struppa**



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0. The purpose of this paper is to briefly report on some new advances due to W.D. Brownawell [12] on the problem of explicit (concretely computable) solutions to the Bezout equations, which are based on some recent developments in complex analysis due to A. Yger and the authors [6], [9].

Let  $P_1, \dots, P_m \in \mathbb{C}[z_1, \dots, z_n] = \mathbb{C}[\underline{z}]$  be polynomials with degrees  $\deg(P_j)$ ; if the  $P_j$ 's have no common zeros in  $\mathbb{C}^n$ , then the well known Hilbert's Nullstellensatz shows that the so called Bezout equation

$$(0.1) \quad P_1 Q_1 + \dots + P_m Q_m = 1$$

has a solution  $\underline{Q} = (Q_1, \dots, Q_m)$  with  $Q_j \in \mathbb{C}[\underline{z}]$ .

If  $n = 1$  (i.e., only one complex variable is involved), the solution  $\underline{Q}$  can be explicitly obtained with the use of the euclidean algorithm; even for  $n > 1$ , the existence of  $\underline{Q}$  does not require the full use of the Nullstellensatz, as it can be derived (not explicitly though) by elimination theory [13]; this approach, also, enables one to deduce an upper bound on the degrees of the  $Q_j$ 's in terms of the  $P_j$ 's which, however, is quite high for all "practical purposes."

The reason for mentioning "practical purposes" need not be explained in detail in this volume, and we will be satisfied with two rather well known examples. The first one arises in the study of the problem of stabilizability of a strictly causal MIMO (multiple input-multiple output) system, in which case, as it is shown, e.g., in [15], one is directly lead to the study of a matrix version of (0.1) (in the case of single input and output,

the scalar version (0.1) suffices). More generally, many problems connected with the stabilization of MIMO weakly causal systems lead to similar considerations; we refer the reader to [11] for more details on this and related subjects. We would only like to recall here that the matrix valued Bezout equation can be reduced to solving a single equation of the type (0.1) (cf. [6]).

A different applied area in which the Bezout equation (0.1) arises quite naturally is connected with the many problems from the (increasingly important) field of "image reconstruction" (i.r. in the sequel) techniques. As this was the origin of our first interest in the subject, let us briefly summarize how i.r. can be linked to Bezout equations. A naïve approach to the i.r. problem could consist in using a single sensing device (a lens which diffracts the signal to determine or anything else which transforms in an "explicit" way the image we wish to determine), which is usually mathematically modeled as a convolutor related to a compactly supported distribution (this modeling, of course, depends on some specific physical assumptions we ask about the sensing device, like its time-invariancy, its causality. etc.). In other words, if  $f$  is the unknown signal (e.g., represented by a distribution), and if  $g$  is the distribution describing the transformed signal (the one we receive), it is (up to noise)

$$(0.2) \quad g = \mu * f$$

with  $\mu$  some compactly supported distribution, i.e., if  $f, g$  and  $\mu$  are  $C^\infty$  functions,  $\mu$  with compact support

$$(0.3) \quad g(x) = \int_{\mathbb{R}^n} f(x-t)\mu(t)dt,$$

$\mathbb{R}^n$  being the euclidean space (of suitable dimension) in which our physical problem takes place.

The question, however, of recovering the unknown  $f$  from  $g$  is easily seen to be (generally speaking) an ill-posed problem (see, e.g., sec. 1 of [4]), so that the i.r. problem naturally leads to a multiple-sensor approach. From a physical point of view this simply means that we try to reconstruct the signal from the action on it of several (suitably related, in a sense which is precise and which we will explain in a while) sensors; from a mathematical point of view, on the other hand, we can provide the following model: we will assume the unknown signal to be represented by  $f \in \mathcal{E}(\mathbb{R}^n)$  ( $\mathcal{E}$  will denote the space of  $C^\infty$  functions, with the usual topology of uniform convergence on compact subsets: topology plays a quite relevant role in this problem!), and we will represent our sensing devices with compactly supported distributions  $\mu_1, \dots, \mu_m \in \mathcal{E}'(\mathbb{R}^n)$ , which produce the received signals  $g_1 = \mu_1 * f, \dots, g_m = \mu_m * f$ . Clearly, the i.r. problem will be well posed (and solvable) only when the map

$$(0.4) \quad f \mapsto (g_1, \dots, g_m)$$

from  $\mathcal{E}$  to  $\mathcal{E}^m$  has a continuous inverse (which can be explicitly produced). The link between the i.r. problem and the Bezout equation is now given by the following well known result [17], [18].

Theorem 0.1. The map (0.4) has a continuous inverse iff the Fourier transform  $\hat{\mu}_j$  of the distributions  $\mu_j$  (the  $\hat{\mu}_j$  are entire functions on  $\mathbb{C}^n$  of exponential type and of polynomial growth on  $\mathbb{R}^n \subset \mathbb{C}^n$ ) satisfy the following condition:  $\exists A > 0$

such that

$$(0.5) \quad |\hat{\mu}_1(\underline{z})| + \dots + |\hat{\mu}_m(\underline{z})| \geq A(1+|\underline{z}|)^A \exp(A|\operatorname{Im} \underline{z}|), \quad \forall \underline{z} \in \mathbb{C}^m;$$

condition (0.5), in turn, is equivalent to the existence of distributions  $v_1, \dots, v_m \in \mathcal{E}'(\mathbb{R}^n)$  such that

$$(0.6) \quad \mu_1 * v_1 + \dots + \mu_m * v_m = \delta$$

( $\delta$  the Dirac delta) or (equivalently) to the existence of

$Q_1, \dots, Q_m \in (\mathcal{E}'(\mathbb{R}^n))^\wedge$  such that

$$(0.7) \quad \hat{\mu}_1 \cdot Q_1 + \dots + \hat{\mu}_m \cdot Q_m = 1.$$

In this case the i.r. problem is solved with the construction of the inverse of (0.4), i.e., by

$$(0.8) \quad (g_1, \dots, g_m) \rightarrow f = v_1 * g_1 + \dots + v_m * g_m.$$

Thus, at least in the case in which the sensors have punctual supports, the solution to the Bezout equation immediately provides the solution to a particular i.r. problem.

Notice that condition (0.5) can be, quite often, translated into physical conditions on the sensing devices; consider, for example, the case  $n = m = 2$ , e.g., two sensing devices in the plane, which were taken to be the diffraction in two circular lenses of radiuses  $R_1$  and  $R_2$  (the details are discussed in [8]); in this case it can be shown that, in order for (0.5) to hold, it is sufficient the existence of a positive constant  $C$  such that

$$\left| \frac{R_1}{R_2} - \frac{p}{q} \right| \geq \frac{C}{|q|^2},$$

where  $p, q (\neq 0)$  describe the set of zeros of the first Bessel

function  $J_1$  (which in this problem arises as a part of the  $\hat{\mu}_j$ 's).

These motivations are probably sufficient to justify the great interest which, in these last few years, has developed around the construction of explicit solutions to (0.1). Note however, that in this paper we are concerned with a very simple Bezout equation, in which only polynomials are concerned, so that, in this case, Theorem 0.1 is essentially superfluous, as (0.5) reduces to the condition that the  $P_j$ 's have no common zeros, and so it does not yield any more information than the Nullstellensatz.

Still, Theorem 0.1 is worth looking at, since it can be used towards the goal of explicitly constructing the  $Q_j$ 's. Indeed, Theorem 0.1 (an important breakthrough in complex analysis, at the time, due to the power of the new tool of the  $L^2$ -estimates for the  $\bar{\partial}$ -operator) is only an existence theorem which provides no clues with regard to the construction of the  $v_j$ 's (or, equivalently, to the construction of the  $Q_j$ 's), as it is based on the purely existential techniques of  $L^2$ -estimates; in Hörmander's arguments, the  $Q_j$ 's are "constructed" in a quite natural way: one first constructs a  $C^\infty$ -solution to (0.7), which is then "corrected" into a holomorphic one, with growth control, via the existence of solutions to the inhomogeneous Cauchy-Riemann equation  $\bar{\partial}u = f$  (for  $f$  a  $\bar{\partial}$ -closed  $(0,1)$ -form). In view of this procedure, the work of M. Andersson and B. Berndtsson [1] on explicit solutions to the inhomogeneous Cauchy-Riemann equations becomes immediately of crucial interest, even though their formulas do not satisfy the necessary stability requirements, at least



in the most general case (we should mention that a different approach to the representation formulas of [1], which avoids the use of the  $\bar{\partial}$ -techniques, has been recently established by M. Andersson and M. Passare [2]). Still, these formulas work fairly well in the case of (0.1), i.e., in the case of the polynomial Bezout equation (of which (0.7) is the entire holomorphic version), and in a series of papers by B.A. Taylor, A. Yger and the authors [6], [8], [9], special versions and modifications of it have been applied towards an explicit solution of (0.1) and (0.7): in section 1 we will briefly outline these results.

As a consequence of these (long) considerations, it has probably become clear the necessity of providing good bounds for the degrees of the  $Q_j$ 's in (0.1); this, of course, is necessary to even consider the possibility of implementing a symbolic calculation which would effectively produce the  $Q_j$ 's.

Some algebraic approaches to this question, when (0.1) is replaced by the more general

$$(0.9) \quad P_1 Q_1 + \dots + P_m Q_m = C, \quad C \in \mathbb{C}[\underline{z}],$$

but with strong hypotheses on the  $P_j$ 's have been given in [3], [14], while, on the other hand, if the  $P_j$ 's have no common zeros at infinity either (think of the  $P_j$ 's as homogeneous polynomials in  $\mathbb{C}^n$ ), basic results of elimination theory [19] show that the  $Q_j$ 's can be chosen with

$$\deg(Q_j) \leq n(D-1) + 1.$$

$$D = \max_i (\deg(p_i)).$$

Until the recent results of Brownawell, the best one could do

in the general case was to employ the classical methods of Hermann [16], and, in particular, D.W. Masser and G. Wüstholz [20] proved that one can solve (0.1) with

$$\deg(Q_j) \leq 2(2D)^{2^{n-1}}$$

which, of course, is a terrible bound, being a double exponential.

Recently, however, Brownawell exploited the explicit solutions to (0.1) described before, to drastically reduce this bound, as the following theorem [12] shows:

Theorem 0.2. Let  $P_1, \dots, P_m \in \mathbb{C}[z]$  have no common zeros, and let  $D = \max_i(\deg(P_i))$ . Then (0.1) can be solved with  $Q_j$ 's such that

$$\deg(P_j Q_j) \leq 3\mu n D^\mu,$$

for  $\mu = \min(m, n)$ .

The purpose of section 2 of this paper will be to outline this result.

Acknowledgements. The authors are indebted to W.D. Brownawell for discussing with them his paper [12].

1. In this section we describe our explicit integral formula (formula (2.11) in [6]) which constitutes the analytic part in Brownawell's argument; actually, this formula is just an explicit translation of Andersson-Berndtsson's formulas, in the case of polynomials, in which case the original kernels become quite manageable. In order to give the flavor of its construction, let us briefly sketch what happens in the simple case of one variable, but for entire functions with growth restrictions: take, e.g.,  $F_1, F_2 \in (\mathcal{E}'(\mathbb{R}))^\wedge$ , which satisfy (0.5), and let us try to construct  $G_1, G_2$  in  $(\mathcal{E}'(\mathbb{R}))^\wedge$  such that

$$(1.1) \quad F_1 \cdot G_1 + F_2 \cdot G_2 = 1.$$

It is well known [7], that, by (0.5), one can interpolate the values of  $1/F_1$  on  $\{z \in \mathbb{C} : F_2(z) = 0\}$  with a function  $H_1 \in \hat{\mathcal{E}}'$  and the values of  $1/F_2$  on  $\{z \in \mathbb{C} : F_1(z) = 0\}$  with  $H_2 \in \hat{\mathcal{E}}'$ . Then the pair  $(H_1, H_2)$  might well be a candidate for a solution of (1.1), and one could "reasonably" think to express  $H_1, H_2$  via a "basis" of functions of the like

$$\zeta \mapsto F_2(\zeta) / (\zeta - \alpha) F_2'(\alpha),$$

where  $\alpha$  is a simple zero of  $F_2$ .

This idea, however, does meet (generally speaking) with some difficulties, which make necessary the introduction (see [9], section 2) of some extra conditions on  $(F_1, F_2)$  which, however, are always satisfied in the polynomial case. Without dwelling in these details we simply state the following result [9]:

Theorem 1.1. Let  $F_1, F_2$  satisfy (0.5) and suppose that their zeros are simple and lie in the region

$$\{z : |\operatorname{Im} z| \leq C \log(2 + |\operatorname{Re} z|)\}.$$

Then there exists  $q \geq 1$  such that the series

$$G_1(z) = \sum_{F_2(\beta)=0} \frac{1}{\beta^q F_1(\beta) F_2'(\beta)} \cdot \frac{F_2(z)}{(z-\beta)}, \quad z \in \mathbb{C}$$

$$G_2(z) = \sum_{F_1(\alpha)=0} \frac{1}{\alpha^q F_2(\alpha) F_1'(\alpha)} \cdot \frac{F_1(z)}{(z-\alpha)}, \quad z \in \mathbb{C}$$

are normally convergent in  $\hat{\mathcal{E}}'(\mathbb{R})$ , then, on  $\mathbb{C}$ ,

$$1 = z^q F_1(z) G_1(z) + z^q F_2(z) G_2(z) + F_1(z) F_2(z) P(z)$$

with

$$P(z) = \operatorname{Res}_{\zeta=0} \left[ \frac{z^{q-1} + \zeta z^{q-2} + \dots + \zeta^{q-1}}{\zeta^q F_1(\zeta) F_2(\zeta)} \right].$$

Sketch of the proof. The proof of this result is essentially based on a suitable application of the Cauchy formula to the function  $w(\zeta) = \zeta^q F_1(\zeta) F_2(\zeta)$ , where  $q$  has to be chosen large enough (in a sense that will become clear in the sequel). Indeed, one needs to find a sequence of real numbers  $r_n \nearrow +\infty$ , and a sequence of Jordan curves  $\Gamma_n \subset \Gamma_{n+1}$ , together with positive constants  $\lambda, M > 0$ , such that

$$(1.2) \quad \begin{cases} |\zeta| \sim r_n & \text{on } \Gamma_n, \quad \text{length}(\Gamma_n) = o(r_n), \\ |\zeta^M F_1(\zeta)| \geq \lambda(1+|\zeta|) & \text{on } \Gamma_n. \end{cases}$$

(Notice that, for polynomials, condition (1.2) can be easily satisfied). If then  $D_n$  is the bounded open set whose boundary is  $\Gamma_n$ , the Cauchy formula gives, for  $z \in D_n$  and  $q \geq M$ .

$$(1.3) \quad 1 = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{w(z) - w(\zeta)}{w(\zeta)(z - \zeta)} d\zeta - \frac{w(z)}{2\pi i} \int_{\Gamma_n} \frac{\partial \zeta}{w(\zeta)(z - \zeta)},$$

from which one then gets the above result.

If we now want to extend Theorem 1.1 to the case of  $n > 1$ , (for  $n = 1$  and  $m > 2$ , no real difficulties arise, see [9]), we have to substitute, in (1.3), the Cauchy integral formula with the so called Koppelman formula [10]; this result states that, for a bounded domain  $D \subset \mathbb{C}^n$ , with  $C^1$  boundary, there are kernels  $K$  and  $P$  such that for any  $u \in C^1(\bar{D})$ , the following representation holds for  $z \in D$ ,

$$(4.10) \quad u(z) = \frac{-1}{n!(2\pi i)^n} \left[ \int_{\partial D} u(\zeta) K(z, \zeta) - \int_D \bar{\partial} u(\zeta) \wedge K(z, \zeta) - \int_D u(\zeta) P(z, \zeta) \right],$$

where  $K$  and  $P$  are differential forms in  $\zeta$  of type, respectively,  $(n, n-1)$  and  $(n, n)$ , and their concrete construction is given in [10]. Instead of describing the general case, which is rather complicated, as well as beyond our immediate interests, we will confine ourselves to the explicit description of the situation for the case of the polynomial Bezout equation.

Our notations will be those from complex differential calculus, which can be found in [24]. From this point of view, a function  $\varphi$  on  $\mathbb{C}^n$  will be regarded as a function in the  $2n$  variables  $\zeta_1, \dots, \zeta_n, \bar{\zeta}_1, \dots, \bar{\zeta}_n$ , whose complex differentials are defined by

$$\partial\varphi = \sum_{j=1}^n \frac{\partial\varphi}{\partial\zeta_j} d\zeta_j, \quad \bar{\partial}\varphi = \sum_{j=1}^n \frac{\partial\varphi}{\partial\bar{\zeta}_j} d\bar{\zeta}_j,$$

for  $\frac{\partial}{\partial\zeta_j}$ ,  $\frac{\partial}{\partial\bar{\zeta}_j}$  the usual operators. If  $w$  is a  $(1,0)$ -type dif-

ferential form, i.e.,  $w = \sum_{j=1}^n w_j d\zeta_j$ , one defines

$$\bar{\partial}w = \sum_{j=1}^n \bar{\partial}w_j \wedge d\zeta_j$$

and

$$w^1 = w \quad \dots \quad w \quad (1 \text{ times}).$$

If now  $P_1, \dots, P_m$  are polynomials in  $\mathbb{C}[z]$  with no common zeros, it is well known that there exist  $\varepsilon, C > 0$ , such that, on  $\mathbb{C}^n$

$$(1.5) \quad |P_1(z)| + \dots + |P_m(z)| \geq \varepsilon(1 + |z|)^{-L};$$

the value of  $L$ , which classically [23] is known to be estimated a priori in terms of the degrees of the  $P_j$ 's is crucial to what follows, and we will return to it later on.

We now associate to each  $P_j$  a differential form  $g^{(j)}$  in the variables  $\zeta_1, \dots, \zeta_n$  and parameters  $z_1, \dots, z_n$  by

$$g^{(j)}(\zeta, z) = \sum_{k=1}^n g_k^{(j)}(\zeta, z) d\zeta_k,$$

where

$$(1.6) \quad g_k^{(j)}(\zeta, z) = \int_0^1 \frac{\partial P_j}{\partial \zeta_k}(\zeta_1 + t(z_1 - \zeta_1), \dots, \zeta_n + t(z_n - \zeta_n)) dt.$$

Thus, the  $g^{(j)}$  are differential forms whose coefficients are polynomials in  $2n$  variables which (1.6) provides in an explicit fashion. (We remark that since the  $P_j$  are polynomials, these

integrals can be computed explicitly. We just leave them in this form to simplify the notation.) As the  $P_j$ 's have no common zeros, we can define one more differential form

$$Q = Q(\zeta, \bar{\zeta}, z) = \left[ \sum_{j=1}^n \overline{P_j(\zeta)} g^{(j)}(\zeta, z) \right] / \|P(\zeta)\|^2,$$

where, as customary,  $\|P(\zeta)\|^2 = \sum_{j=1}^n |P_j(\zeta)|^2$ . We are now going to

write an integral formula in which a sufficiently high exponent  $N$  must be chosen, in order to ensure the convergence of the integrals (as in Theorem 1.1); as it will appear in a second, the value of  $N$  depends explicitly on the value of the constant  $L$  which appears in (1.5), and can thus be estimated in terms of the degrees of the  $P_j$ 's: a relevant part of Brownawell's work consisted in improving as much as possible this estimate.

$$\text{Let then } \overline{P(\zeta)} \cdot P(z) = \sum_{j=1}^m \overline{P_j(\zeta)} \cdot P_j(z), \quad \bar{\zeta} \cdot z = \sum_{j=1}^n \bar{\zeta}_j \cdot z_j \quad \text{and,}$$

for  $s = \min(m, n+1)$ , set

$$C_k = \frac{(n-1)!}{k!(n-k)!} \cdot \frac{s!}{(s-k)!} \cdot \frac{N!}{(N-n+k)!}, \quad k = 0, \dots, s-1.$$

With all these notations set, we can finally write the Bezout equation (0.1);

$$(1.7) \quad 1 = \frac{1}{m!(2\pi i)^n} \left[ \int_{\mathbb{C}^n} \sum_{k=0}^{s-1} C_k \left( \frac{1+\bar{\zeta}z}{1+\|\zeta\|^2} \right)^{N-n+k} \cdot \left( \frac{\overline{P(\zeta)} \cdot P(z)}{\|P(\zeta)\|^2} \right)^{s-k} \cdot (\bar{\partial} \partial \log(1+\|\zeta\|^2))^{n-k} \wedge (\bar{\partial} Q)^k, \right]$$

with integration with respect to the variables  $\zeta$  and  $\bar{\zeta}$ .

Even though (1.7) might not resemble (0.1) at first sight, it takes only a moment to realize that each  $Q_j$  in (0.1) can be found in (1.7) by simply collecting the terms in which  $P_j$  appears; this provides, henceforth, an explicit solution to (0.1) which, from a concrete point of view, is now reduced to the computation of a finite number of definite integrals over  $\mathbb{C}^n$ . A few comments are in order on the practicality of this approach and on its stability: first one realizes that the computation of the integrals arising in (1.7) can be executed rather efficiently since the choice of  $N$  itself assures that the integrands decay rather quickly as functions on  $|\zeta|$ , and, more importantly, we have explicit estimates on this decay, which can enable us to control the errors; as for a more detailed consideration on the stability and the applicability of this kind of algorithm, we refer the reader to [4] and [5], where several concrete examples are discussed.



2. This last section is devoted to the brilliant result of Brownawell who, via the formula (1.7), and with a careful exploitation of elimination techniques in the theory of transcendental numbers (mainly based on the work of Yu.V. Nesterenko, [22], and its references) succeeded in proving Theorem 0.2.

This theorem, actually, turns out to be a corollary of the proof of a more refined statement for which we will concentrate in the sequel.

Let us recall a definition from commutative algebra, [21]: we say that  $P_1, \dots, P_m \in \mathbb{C}[\underline{z}]$  form a regular sequence if  $P_1 \neq 0$  and, for  $i = 2, \dots, m$ ,  $P_i$  is not a zero divisor in  $\mathbb{C}[\underline{z}]/(P_1, \dots, P_{i-1})$ , where  $(P_1, \dots, P_{i-1})$  denotes the ideal generated by  $P_1, \dots, P_{i-1}$  in  $\mathbb{C}[\underline{z}]$ . The theorem we mentioned before is:

Theorem 2.1. Let  $P_1, \dots, P_m \in \mathbb{C}[\underline{z}]$  form a regular sequence and let  $D_i = \deg(P_i) > 0$ . If the  $P_j$ 's have no common zeros then there exist  $Q_1, \dots, Q_m$  in  $\mathbb{C}[\underline{z}]$  such that

$$P_1 Q_1 + \dots + P_m Q_m = 1 \text{ on } \mathbb{C}^n$$

and

$$\deg(P_i Q_i) \leq 2\mu n D_i \cdot \dots \cdot D_\mu + 3mD$$

with  $\mu = \min(m, n)$ ,  $D = \max_i(D_i)$ .

The key step in the proof of Theorem 2.1 is an interesting result on a lower bound on the maximum modulus of a regular sequence with no common zeros, namely:

Theorem 2.2. If  $P_1, \dots, P_m \in \mathbb{C}[\underline{z}]$  is a regular sequence with no common zeros and  $\deg(P_i) = D_i > 0$ ,  $i = 1, \dots, m$ , then there exists a constant  $C > 0$ , depending only on  $P_1, \dots, P_m$  such that

for all  $\zeta \in \mathbb{C}^n - \{0\}$ , with  $|\zeta| = \max_i (|\zeta_i|) \geq 2$ , it is

$$\max_i |P_1(\zeta)| \geq C |\zeta|^{1 - (\mu-1)D_1 \cdots D_\mu},$$

for  $\mu = \min(m, n)$ .

We do not wish to spend any time on the complicated proof of Theorem 2.2 (which, in Brownawell's paper, relies on Nesterenko's use of the Chow form of homogeneous ideals in  $\mathbb{C}[z]$ ), for which we refer the reader to [12]; on the other hand, we wish to show how the algebraic result given in Theorem 2.2 can be used to prove Theorem 2.1 and how this theorem, in turn, can be used to obtain the bounds of Theorem 0.2. Henceforth, throughout the sequel, we will assume Theorem 2.2.

Proof of Theorem 2.1. In this proof, we have to explicitly refer to the constructions of (1.6) and (1.7). Indeed, by construction

$$\deg_{\zeta} (g_k^{(j)}) \leq D_j - 1,$$

so that, if we write

$$\bar{\partial}Q = \sum_{i,j=1}^n a_{ij}(\zeta, \bar{\zeta}, z) d\zeta_i \wedge d\bar{\zeta}_j,$$

we have, for  $z \in \mathbb{C}^n$  fixed, and as  $\|\zeta\| \rightarrow +\infty$ ,

$$\max_{i,j} |a_{ij}(\zeta, \bar{\zeta}, z)| = O(\|\zeta\|^{2B+2(D-1)}),$$

where, again,  $D = \max_i (D_i)$ , and  $B = 1 - (\mu-1)D_1 \cdots D_\mu$ . In a

similar way, if we write

$$\bar{\partial} \partial \log(1 + \|\zeta\|^2) = \sum_{i,j=1}^n b_{ij}(\zeta, \bar{\zeta}) d\zeta_i \wedge d\zeta_j,$$

we have, for  $\|\zeta\| \rightarrow +\infty$ ,

$$\max |b_{ij}(\zeta, \bar{\zeta})| = O(\|\zeta\|^{-2}).$$

Hence, Theorem 2.2 shows that the convergence in the integrals of (1.7) is guaranteed if, for each  $k = 0, \dots, m-1$  (notice that the regularity hypothesis on  $P_1, \dots, P_m$  implies  $m \leq n+1$ , hence  $s = \min(m, n+1) = m$ ), it is

$$N - n + k > (m-k)B - 2(n-k) + 2k(B+D-1) + 2n,$$

i.e.,

$$(2.1) \quad N > (m+k)B + 2kD + n - k.$$

The worst case of (2.1) occurs for  $k = m-1$ , i.e., the convergence of the integrals in (1.7) is implied by

$$N = (2m-1)B + 2(m-1)D + n - m + 2.$$

Now, the coefficients of  $\bar{\partial}Q$  have degree, in  $z$ , less than or equal to  $D - 1$ , so that, from Theorem 2.2, we immediately get that the degree of  $P_i Q_i$  is bounded, for every  $i$ , by

$$(2.2) \quad N - n + mD = (2m-1)(n-1)D_1 \cdot \dots \cdot D_\mu + (3m-2)D - 3m + 3.$$

Theorem 2.1 now follows immediately: indeed, if  $\mu = m$  this is obvious, while if  $\mu = n$ ,  $m = n + 1$ , so that  $(2m-1)(n-1) = 2\mu n - n - 1$  which concludes the proof.  $\square$

We now sketch how Theorem 0.2 is actually a consequence of the proof of Theorem 2.2 and, more precisely, of (2.2).

Proof of Theorem 0.2. The idea of the proof [12] is, reasonably enough, only algebraic, as it essentially tends to show that, starting with the  $P_j$ 's, one can (if the theorem does not hold

immediately) produce a regular sequence  $(Q_1, \dots, Q_m)$ , with no common zeros, where each  $Q_j$  is a  $\mathbb{C}$ -linear combination of the  $P_j$ 's; (2.2) then gives the thesis. To be more precise, let  $i = 1, \dots, \nu = \min(m, n+1)$ ; the induction hypothesis is that either Theorem 0.2 holds true, or else there are polynomials  $Q_1, \dots, Q_i$  which are linear combinations (over  $\mathbb{C}$ ) of  $P_1, \dots, P_m$  and which form a regular sequence. For  $i = 1$ , just take  $Q_1$  to be any non-zero  $P_j$ . Suppose we have now constructed such a sequence  $Q_1, \dots, Q_i$  for  $i < \nu$ . If the  $Q_j$ 's have no common zeros, the induction step follows from (2.2) in an obvious way. If the  $Q_j$ 's, on the other hand, have common zeros, it is a consequence of Lemma 1 of [20] the existence of  $Q_{i+1}$ , again a linear combination of  $P_1, \dots, P_m$ , such that  $Q_1, \dots, Q_{i+1}$  is still regular. The conclusion is now a simple matter of applying (2.2).  $\square$

We now wish to conclude with a remark on the concrete possibilities that formula (2.2) has to be applied, in view also of Theorem 0.2. Indeed, at least in the case of polynomial Bezout equations, our methods have an important "opponent" in the algebraic method which is due essentially to B. Buchberger (1965), and which relies on the so called Gröbner bases. It would take us too far afield to describe this method (for which we can refer the reader to the excellent survey given by Buchberger himself in [13]), but a couple of words may help to understand the different nature of this elegant method. Let  $F$  be a finite set of polynomials in  $\mathbb{C}^n$ : the so called "simplification problem modulo the ideal generated by  $F^n$ ", i.e., the problem of finding unique representatives in the residue classes modulo the ideal generated by

$F$ , was first posed explicitly in [16], and the main objective in the method of Gröbner bases is exactly to solve this problem. The basic idea of the method is to transform  $F$ , in an explicit way, which is simple enough to be taught to a computer (and experiments in this area have been going on in the last twenty years, with rather good results), into a standard form, called "Gröbner basis for  $F$ ." Once the Gröbner basis  $G$  for  $F$  is constructed, one can easily solve a large number of problems concerning the ideal generated by  $F$  (or, which is the same, by  $G$ ). In particular, one is able to provide an explicit construction for the  $Q_j$ 's in (0.1): a detailed description of the algorithm is given (under the name of "method G.13") in [13], to which we refer the reader once more. Thus our method (which, however, holds also for the entire holomorphic case) and Buchberger's one, provide two radically different approaches to the same problem; it would be therefore of great interest to be able to comparatively discuss the complexity built into each of these methods; as pointed out in [13], much is now known about the complexity of the algorithm centered on Gröbner bases: in particular, the degrees of the polynomials in  $G$  can be (almost always, in the sense of probability) bounded by  $D_1 + \dots + D_m - n + 1$  where  $D_1, \dots, D_m$  are the degrees of the polynomials in  $F$ ; therefore, one is now lead to the study of a new equation (0.1), in which the degrees of the  $P_j$  may be significantly increased: on the other hand, this bound may not be sufficient for some exceptional cases. In fact, since the method of Gröbner bases is capable of deciding the harder problem of when is  $P$  in the ideal generated by  $P_1, \dots, P_m$  (even when

they have common zeros) one expects a double exponential bound to appear. The other annoying fact is that the polynomials that cause difficulties are often of integral coefficients! (After all, they have only measure zero!). As far as our method is concerned, a thorough complexity analysis has not yet been attempted, even though some computer implementation of these techniques is discussed in [4]; now, in view of Brownawell's result, we know that, for each  $Q_j$ , we have to determine a number of coefficients which is of the order of magnitude in  $D^{n^2}$ , which seems to be rather high: still, some symmetries in the kernels which appear in (1.7) seem to suggest that the actual number of computations may be drastically reduced.

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