ABSTRACT

Title of Dissertation: LANGLANDS-KOTTWITZ METHOD ON MODULI SPACES OF GLOBAL SHTUKAS

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We apply the approach of Scholze to compute the trace of Hecke operator twisted by some power of Frobenius on the cohomology of the moduli spaces of global shtukas in the case of bad reduction. We find a formula that involves orbital integrals and twisted orbital integrals which can be compared with the Arthur-Selberg trace formula. This extends the results of Ngo and Ngo Dac on counting points of moduli spaces of global shtukas over finite fields. The main problem lies in finding a suitable compactly supported locally constant function that will be plugged into the twisted orbital integrals. Following Scholze, we construct locally constant functions called the test functions by using deformation spaces of bounded local shtukas. Then we establish certain local-global compatibility to express the trace on the nearby cycle sheaves on the moduli space of global shtukas to the trace on the deformation spaces.

LANGLANDS-KOTTWITZ METHOD ON MODULI SPACES OF GLOBAL SHTUKAS

by

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Chapter 1: Introduction

The Langlands program is a web of conjectures that unites seemingly unrelated fields of mathematics such as number theory and harmonic analysis. One of the central goals of the program is to establish a correspondence between automorphic representations (an object in harmonic analysis) and Galois representations (an object in number theory). To be precise, let G be a connected reductive group over a global function field F of characteristic $p > 0$. The Langlands program predicts that there exists a canonical map

$$\left\{ \text{cuspidal automorphic representations of } G(\mathbb{A}) \right\} \to \left\{ L\text{-parameters for } G \text{ over } F \right\}$$

where $\mathbb{A}$ is the ring of adeles of F and L-parameters are certain conjugacy classes of representations of the Galois group $\mathrm{Gal}(\overline{F}/F)$.

One of the main strategies to build such a connection is to introduce algebraic geometry into the picture. It turns out that studying the action of $G(\mathbb{A})$ and the Galois group on the cohomologies of geometric objects proved to be useful in constructing reciprocity laws between harmonic analysis and number theory. In the worlds of number fields, systems (Sh_K) of varieties over a number field called Shimura varieties are proven to be useful in realizing various Langlands correspondences in their cohomology. Here the system is indexed by open compact subgroups K of $G(\mathbb{A})$ where G is a reductive group over $\mathbb{Q}$. One of the ways to understand the cohomology of

Shimura varieties is to obtain a formula called the unstabilized Langlands-Kottwitz formula for the trace of the Hecke operator composed with a power of Frobenius involving a sum of (twisted) orbital integrals. The Langlands-Kottwitz formula (after stabilization) resembles the geometric side of the Arthur-Selberg trace formula which appears in the automorphic representation theory. The comparison between the two trace formulas should give rise to the sought-for correspondence between automorphic representations and Galois representations. This approach to proving cases of Langlands reciprocity is called the Langlands-Kottwitz method.

Let p be a prime number. In [Kot92], Kottwitz computed the alternating sum of the traces of certain Hecke operators composed with powers of Frobenius for compact PEL type Shimura varieties Sh_K with good reduction at p. Here K is an open compact subgroup of $G(\mathbb{A})$. This amounts to saying that we require the p-component K_p of K to be hyperspecial. Then Scholze in [Sch12] generalized the Langlands-Kottwitz method to the cases of bad reduction at p for certain proper PEL type Shimura varieties.

The main goal of this thesis is to establish the approach of Scholze for the moduli spaces of global shtukas which are the analogue of Shimura varieties in the function field setting. To state the main theorem, let us introduce some notations. Let X be a smooth projective geometrically irreducible curve over $\mathbb{F}_q$. Let $F := \mathbb{F}_q(X)$ be the function field of X. Then the places of F are in bijection with the closed points of X. Fix an n-tuple of closed points $\underline{x} = (x_i)_{i=1}^n \in X^n$ and write F_{x_i} the completion of F at x_i. We denote by $\mathscr{O}_{x_i}$ the valuation ring of F_{x_i}. Let G be a connected reductive group over F and let $\mathscr{G}$ be a parahoric model of G over X. We would like to understand the cohomology of moduli spaces Sht_K of global shtukas with level K-structure where K is an open compact subgroup of $G(\mathbb{A})$ by computing the alternating sum of the traces of certain Hecke operators $f^{\underline{x}}$ away from $\underline{x}$ composed with a power σ^r of Frobenius σ at $\underline{x}$.

If K is a subgroup of $G(\mathbb{A})$ such that the x_i-component $K_{x_i} = \mathscr{G}(\mathscr{O}_{x_i})$ is hyperspecial maximal subgroup, Ngô and Ngô Dac [ND07] show that the elliptic regular part of the alternating sum of trace of $\sigma^r \circ f^{\underline{x}}$ on the special fiber is equal to sum of the form

$$\sum_{(\gamma_0;\gamma,\delta)} |\ker^1(F, G_{\gamma_0})| \mathrm{vol}(G_{\gamma_0}(F)\backslash G_{\gamma_0}(\mathbb{A})/\Xi) O_\gamma(f^{\underline{x}}) \prod_{x_i \in \underline{x}} TO_{\delta\sigma}(\phi_{r,i}^{\mu_i}) \tag{1.1}$$

where the sum is over certain Kottwitz triples which will be defined in Chapter 5, $\phi_{r,i}$ is the characteristic function of the double coset corresponding to the coweight μ_i which measures the modification of the shtuka morphism at x_i, $O_\gamma(f^{\underline{x}})$ is the orbital integral of $f^{\underline{x}}$, and $TO_{\delta\sigma}(\phi_{r,i})$ is the twisted orbital integral of $\phi_{r,i}$.

We generalize the result above to arbitrary deep compact open subgroups $K_{x_i} \subset \mathscr{G}(\mathscr{O}_{x_i})$. There are several obstructions to naively applying the technique of Scholze to moduli spaces of global shtukas. Firstly, the moduli stack $\mathrm{Sht}_{\mathscr{G},K}$ is a Deligne-Mumford stack of finite type over X^n for some n. The author is not aware of a well-behaved theory of nearby cycle sheaves for Deligne-Mumford stacks. Therefore, we will further impose boundedness conditions by a tuple of cocharacters $\underline{\mu}$ and truncations by Harder-Narashimhan parameters λ. On top of that, we will quotient out by a cocompact lattice Ξ so that our moduli spaces $\mathrm{Sht}^{\underline{\mu},\leq\lambda}_{\mathscr{G},\Xi K}$ of global shtukas are representable by a scheme of finite type over $\mathbb{F}_q$. Also to avoid the theory of nearby cycle sheaves over an arbitrary base, we will work with two legs with one leg fixed.

However, the moduli space $\mathrm{Sht}^{\underline{\mu},\leq\lambda}_{\mathscr{G},\Xi K}$ is not stable under the Hecke action. To be precise, the Hecke action does not preserve the Harder-Narashimhan filtration. However, when $\mathrm{Sht}^{\underline{\mu},\leq\lambda}_{\mathscr{G},\Xi K}$ is quasi-compact, we have $\mathrm{Sht}^{\underline{\mu},\leq\lambda}_{\mathscr{G},\Xi K} = \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\Xi K}$. Therefore, we will work with proper moduli spaces of global shtukas. This is a non-trivial assumption because moduli spaces of shtukas can be nonproper even for anisotropic groups.

Finally, we only consider the elliptic terms of the trace formula. A Kottwitz triple $(\gamma_0;\gamma,\delta)$ is a triple where $\gamma_0 \in G^*(F)$ is a stable conjugacy class where G^* is the quasi-split inner form of G. Contrary to the case of Shimura varieties, γ_0 is not automatically semisimple. Therefore, we only consider the fixed points that give rise to semisimple Kottwitz triples. As we are assuming that our shtuka spaces are compact, all semisimple elements are elliptic. Therefore, we are considering the semisimple part of the trace formula. To simplify the exposition, we will compute the elliptic regular terms of the trace formula.

Now we state our main theorem (Theorem 5.9.2).

Theorem. *Let x and ∞ be two distinct closed points in X. Let $\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi}$ be a proper moduli space of shtuka with two legs and with one leg fixed over $\mathrm{Spec}\,\mathscr{O}_{y,\infty'}$ (defined in §4.7). Denote by $F_{y,\infty'}$ the generic fiber of $\mathscr{O}_{y,\infty'}$. Let K be a compact open subgroup of $\mathscr{G}(\mathscr{O}_x)\times G(\mathbb{A}^{\infty,x})$ the form $K = K_xK^x\mathscr{G}(\mathscr{O}_\infty)$ where K_x is an open compact subgroup of $\mathscr{G}(\mathscr{O}_x)$ and K^x is a sufficiently small open compact subgroup of $G(\mathbb{A}^{\infty,x})$. Here $\mathbb{A}^{\infty,x}$ is the adeles of the global field $F=\mathbb{F}_q(X)$ away from ∞ and x.*

Let $f = f_x\otimes f^x$ where $f_x\in C_c^\infty(\mathscr{G}(\mathscr{O}_x))$ and $f^x\in C_c^\infty(G(\mathbb{A}^x))$ where the ∞-component of f^x is given by the characteristic function of $\mathscr{G}(\mathscr{O}_\infty)$. Then the elliptic regular part of the trace of $\sigma^r\times f$ is given by

$$\begin{aligned}&\mathrm{tr}(\sigma^r\times f \mid H^*(\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi,K}\otimes \overline{F}_{y,\infty'},\overline{\mathbb{Q}}_\ell))^{\mathrm{el,reg}}\\ &\qquad = \sum_{(\gamma_0;\gamma,\delta)} |\ker^1(F,G^*_{\gamma_0})|\mathrm{vol}(G^*_{\gamma_0}(F)\backslash G^*_{\gamma_0}(\mathbb{A})/\Xi)O_\gamma(f)TO_{\delta_t}(\phi^{\mu_x}_{r,f_x})TO_{\delta_\infty}(\phi^{\mu_\infty}_{r,f_\infty})\end{aligned}$$

where the sum runs over Kottwitz triples $(\gamma_0;\gamma,\delta)$ that is attached to a fixed point (see §5.5) with *vanishing invariant (see §5.8).*

In some sense, the only difference with the trace formula of (1.1) is the *test functions* $\phi^{\mu_t}_{r,f_t}$ for $t \in \{x, \infty\}$. In both cases, the above sum is a point counting formula obtained by the Lefschetz trace formula on the special fiber. Then the test function can be thought as a weight factor to this point counting formula.

We briefly explain the process. The cohomology of the generic fiber with nontrivial level K_x at x can be interpreted as the cohomology of the generic fiber with good reduction at x with some appropriate coefficients. Then this cohomology is isomorphic to the cohomology of the special fiber with nearby cycle sheaves because we assumed that our moduli space is proper. Then we apply the Lefschetz trace formula and get a summation of the form

$$\sum_{e \in \mathrm{Fix}} \mathrm{tr}(u)$$

where e runs over the fixed points corresponding to the Hecke-Frobenius action and $\mathrm{tr}(u)$ is the naive local factors. We show that $\mathrm{tr}(u)$ is $\phi^{\mu_x}_{r,f_x}(\delta_x)\phi^{\mu_\infty}_{r,f_\infty}(\delta_\infty)$. In chapter 3, we will explain how we define the test functions $\phi^{\mu_t}_{r,f_t}$. The test functions are local in the sense that it is defined as the trace of $\sigma \times f_t$ on the cohomology of the generic fiber of the deformation space of the associated local shtukas at t. The reason why we can use the local theory in our global application is because Berkovich proved that the nearby cycle sheaves only depend on the formal completion of the moduli space at the given point. Then we get a local-global compatibilty in the sense that the generic fiber of the deformation space is the tubular neighborhood defined by that given point.

Organization. In §2, we review the theory of torsors and introduce affine flag varieties, Beilinson-Drinfeld affine Grassmannians, and Schubert varieties which will set the foundation for defining moduli spaces of local and global shtukas. Next, in §3 we study the deformation theory of local shtukas for unramified groups and show that the generic fibers of deformation spaces with level structures form a finite étale tower. Next, in §4, we define the moduli spaces of iterated bounded global shtukas with level structures. Finally, in §5 we prove local-global compatibility between global shtukas and local shtukas. Then we compute the trace of the alternating sum as a formula involving orbital integrals and twisted orbital integrals using the Lefschetz fixed point theorem.

Chapter 2: Preliminaries

Almost all of the stacks that are used in this paper will be built in terms of torsors. Therefore, we will begin this chapter with a quick review of torsors over a curve, the geometric properties of their moduli spaces, and a filtration defined by Harder-Narashimhan parameters. Then we will define Hecke stacks, Beilinson-Drinfeld affine Grassmannians, affine Schubert varieties, and affine flag varieties for smooth affine group schemes. Next, we will introduce the notions of bounds developed by Arasteh-Rad-Bieker-Hartl-Viehmann-Xu ([HV11], [ARH14], [Bie24], [HX23]) to bound the relative positions between the torsors. Even though the only bounds we consider will be induced from BD Schubert varieties, we will work with bounds in full generality.

2.1 $\mathscr{G}$-torsors

Let $\mathbb{F}$ be a scheme. The category of $\mathbb{F}$-schemes is denoted by $(\mathbb{F}\text{-Sch})$. The Grothendieck topologies we consider are fpqc, fppf, and ét. When we say $*$-topology, we always mean $* \in \{\text{fpqc}, \text{fppf}, \text{ét}\}$. We denote by $\mathscr{G}$ the sheaf of groups for the $*$-topology that is either representable by a group scheme of a group ind-scheme over a $S \in (\mathbb{F}\text{-Sch})$.

Both the local and global shtukas will be defined as a tuple of torsors with additional structures. We begin by discussing the generalities of $\mathscr{G}$-torsors where $\mathscr{G}$ is either a group scheme or ind-group-scheme.

Definition 2.1.1. Let $\mathscr{E}$ be a presheaf over $\mathbb{F}$-scheme S. We say that $\mathscr{E}$ is a **pseudo-$\mathscr{G}$-torsor** over S if there exists an action $\mathscr{E} \times \mathscr{G} \to \mathscr{E}$ such that for all S-DM-stack T, either $\mathscr{E}(T)$ is empty, or $\mathscr{E}(T) \times \mathscr{G}(T) \to \mathscr{E}(T)$ is a simply transitive action. A morphism $\mathscr{E} \to \mathscr{E}'$ of pseudo-$\mathscr{G}$-torsors is a map of sheaves that is $\mathscr{G}$-equivariant, i.e. the diagram

$$\begin{array}{ccc} \mathscr{E}(T) \times \mathscr{G}(T) & \longrightarrow & \mathscr{E}(T) \\ \downarrow & & \downarrow \\ \mathscr{E}'(T) \times \mathscr{G}(T) & \longrightarrow & \mathscr{E}'(T) \end{array}$$

commutes for all S-schemes T.

Example 2.1.2. The action $\mathscr{G} \times \mathscr{G} \to \mathscr{G}$ defined by (right) multiplication $(g', g) \mapsto g'g$ makes $\mathscr{G}$ a pseudo-$\mathscr{G}$-torsor. We call $\mathscr{G}$ the **trivial $\mathscr{G}$-torsor**.

Definition 2.1.3. Let $\mathscr{E}$ be a pseudo-$\mathscr{G}$-torsor over $\mathbb{F}$-DM-stack S. We say that $\mathscr{E}$ is a **$\mathscr{G}$-torsor** over S for the $*$-topology if $\mathscr{E}$ is a sheaf for the $*$-topology such that $*$-locally $\mathscr{E}$ is isomorphic to $\mathscr{G}$ as pseudo-$\mathscr{G}$-torsors. In other words, there exists a $*$-cover $\{S_i \to S\}_{i \in I}$ of S such that $\mathscr{E}|_{S_i} \xrightarrow{\sim} \mathscr{G}|_{S_i}$ is an isomorphism of pseudo-$\mathscr{G}$-torsors for all $i \in I$.

Remark 2.1.4.

(i) Any morphism between torsors is an isomorphism.

(ii) Any $\mathscr{G}$-torsor $\mathscr{E}$ over an (ind-)scheme are relatively representable by a (ind-)scheme. Therefore, we may view $\mathscr{E}$ over an (ind-)scheme as an (ind-)scheme that trivializes over $*$-covers.

We introduce two operations on torsors that will be used throughout this paper. If $f : S' \to S$ is a morphism of $\mathbb{F}$-schemes and $\mathscr{E}$ is an H-torsor over S, then the **pullback** of $\mathscr{E}$ along f is denoted $f^*\mathscr{E}$, ${}^f\mathscr{E}$, $\mathscr{E} \times_S S'$, or $\mathscr{E}|_{S'}$. One can show that the pullback $f^*\mathscr{E}$ is an $\mathscr{G}$-torsor over S'. The next operation concerns with the change of (ind-)group schemes. Let $\varphi : \mathscr{G} \to \mathscr{G}'$ be a morphism of (ind-)group schemes and $\mathscr{E}$ be a $\mathscr{G}$-torsor over S. Then one defines the **associated $\mathscr{G}'$-torsor** $\mathscr{E} \times^{\mathscr{G}} \mathscr{G}'$ of $\mathscr{E}$ to be the $*$-sheaf associated to the presheaf

$$T \mapsto \{\mathscr{G}(T)\text{-orbits in } \mathscr{E}(T) \times \mathscr{G}'(T)\} \tag{2.1}$$

where the action is given by the anti-diagonal action. Here $\mathscr{G}(T)$ acts on $\mathscr{G}'(T)$ via the action $g' \cdot g = \varphi(g)^{-1}g'$ in $\mathscr{G}'(T)$ for $g \in \mathscr{G}(T)$ and $g' \in \mathscr{G}'(T)$. As the name suggests, $\mathscr{E} \times^{\mathscr{G}} \mathscr{G}'$ is an $\mathscr{G}'$-torsor over S where the action of $\mathscr{G}'(T)$ on $(\mathscr{E} \times^{\mathscr{G}} \mathscr{G}')(T)$ is defined by right multiplication on the second factor $\mathscr{G}'(T)$.

2.2 Torsors over a Curve and Harder-Narashimhan Truncation

Let X be a smooth projective geometrically irreducible curve over a finite field $\mathbb{F}$. We consider the case where $\mathscr{G}$ is a smooth affine group scheme over X. Denote by $\mathrm{Bun}_{\mathscr{G}}$ the category fibered in groupoids over $(\mathrm{Sch}/\mathbb{F})$ where

$$\mathrm{Bun}_{\mathscr{G}}(S) = \{\mathscr{E} \mid \mathscr{E} \text{ is a } \mathscr{G}\text{-torsor over } X_S\}$$

for all $S \in (\mathrm{Sch}/\mathbb{F})$.

Example 2.2.1. There is an isomorphism between $\mathrm{Bun}_{\mathrm{GL}_r}(S)$ to the stack Vect_X of vector bundles over X of rank r. For a GL_r-torsor $\mathscr{E}$, the corresponding vector bundle is defined by $\mathscr{V}(\mathscr{E}) = (\mathscr{E} \times_X \mathbb{V}(\mathscr{O}_X^r))/\mathrm{GL}_r$ where the action is given by the *anti-diagonal* action $(e, v)g = (eg, g^{-1}v)$. Here $\mathbb{V}(\mathscr{E})$ denote the vector bundle associated to a quasi-coherent $\mathscr{O}_X$-module, i.e. $\mathbb{V}(\mathscr{E}) = \underline{\mathrm{Spec}}_X(\mathscr{E})$. Conversely, given a vector bundle V, we associate $\mathscr{E}(V) := \underline{\mathrm{Isom}}_X(\mathscr{O}_X^r, V)$

which is a GL_r-torsor with the action of $\mathrm{GL}_r = \underline{\mathrm{Isom}}_X(\mathscr{O}_X^r, \mathscr{O}_X^r)$ given by composition. In the same manner, we can think of SL_r-torsors as vector bundles over X with trivial determinant.

Proposition 2.2.2 ([ARH21, Theorem 2.5])**.** *The category* $\mathrm{Bun}_{\mathscr{G}}$ *fibered in groupoids is a smooth Artin stack locally of finite type over* $\mathbb{F}$.

Proof. We only sketch the proof. By [ARH21, Proposition 2.2], there exists a faithful representation $\rho : \mathscr{G} \to \mathrm{GL}(\mathscr{V})$ for some vector bundle $\mathscr{V}$ of rank r over X such that the quotient $\mathrm{GL}(\mathscr{V})/\mathscr{G}$ is quasi-affine and satisfies the condition (2.1) in [ARH21]. Then ρ induces a map $\rho^* : \mathrm{Bun}_{\mathscr{G}} \to \mathrm{Bun}_{\mathrm{GL}(\mathscr{V})} \cong \mathrm{Vect}_X$ of stacks where Vect_X is an Artin stack that parametrizes vector bundles over X_S of rank r. Then [ARH21, Theorem 2.6] shows that ρ^* is representable by a morphism of schemes that is quasi-affine and of finite presentation. Now it follows that $\mathrm{Bun}_{\mathscr{G}}$ is locally of finite type over $\mathbb{F}$ because Vect_X is an Artin stack locally of finite type over $\mathbb{F}$ via [LMB00, Theoreme 4.6.2.1]. To show that $\mathrm{Bun}_{\mathscr{G}}$ is smooth, it suffices to show that $\mathrm{Bun}_{\mathscr{G}}$ is formally smooth. This is exactly [Beh03, Proposition 4.5.1] where he used the vanishing of the second cohomology of coherent sheaves on curves. □

The stack $\mathrm{Bun}_{\mathscr{G}}$ is only locally of finite type over $\mathbb{F}_q$ in general. This is undesirable for us because we eventually want to count points on a moduli space defined based on $\mathrm{Bun}_{\mathscr{G}}$ over the finite fields. Therefore, we will define certain truncations of $\mathrm{Bun}_{\mathscr{G}}$ defined by a generalization of Harder-Narashimhan filtration of vector bundles over algebraically closed fields.

Consider the case when $\mathscr{G} = \mathrm{SL}_r$. Let Λ^+ denote the monoid of dominant rational cocharacters of SL_r. For a $\lambda \in \Lambda^+$, denote by $\mathrm{Bun}_{\mathrm{SL}_r}^{\leq\lambda}$ the substack of $\mathrm{Bun}_{\mathrm{SL}_r}$ that parametrizes SL_r-bundles whose Harder-Narashimhan polygon is less or equal to λ. Then $\mathrm{Bun}_{\mathrm{SL}_r}^{\leq\lambda}$ is an open algebraic substack of $\mathrm{Bun}_{\mathrm{SL}_r}^{\leq\lambda}$ and is quasi-compact. (cf. [Sch14, Theorem 2.1]).

For a general smooth affine group $\mathscr{G}$ over X, let $\mathscr{G}^{\mathrm{ad}}$ be the image of $\mathscr{G}$ in G^{ad}. By [ARH21, Proposition 2.2], there exists a closed embedding $\iota : \mathscr{G}^{\mathrm{ad}} \to \mathrm{SL}_r$ such that the quotient $\mathrm{SL}_r/\mathscr{G}^{\mathrm{ad}}$ is quasi-affine and satisfies the condition (2.1) of [ARH21]. By [ARH21, Theorem 2.6], the induced morphism $\iota^* : \mathrm{Bun}_{\mathscr{G}^{\mathrm{ad}}} \to \mathrm{Bun}_{\mathrm{SL}_r}$ is representable, quasi-affine, and of finite presentation.

Definition 2.2.3. Let $\lambda \in \Lambda^+$. We define the category $\mathrm{Bun}_{\mathscr{G}}^{\leq\lambda}$ fibered in groupoids over $(\mathrm{Sch}/\mathbb{F})$ by the Cartesian diagram

$$\begin{array}{ccccc} \mathrm{Bun}_{\mathscr{G}}^{\leq\lambda} & \longrightarrow & \mathrm{Bun}_{\mathscr{G}^{\mathrm{ad}}}^{\leq\lambda} & \longrightarrow & \mathrm{Bun}_{\mathrm{SL}_r}^{\leq\lambda} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Bun}_{\mathscr{G}} & \longrightarrow & \mathrm{Bun}_{\mathscr{G}^{\mathrm{ad}}} & \xrightarrow{\iota^*} & \mathrm{Bun}_{\mathrm{SL}_r} \end{array} \tag{2.2}$$

To be precise, let $\iota' : \mathrm{Bun}_{\mathscr{G}} \to \mathrm{Bun}_{\mathrm{SL}_r}$ be the composition of the bottom morphisms in the diagram above. Then the S-points are given by

$$\mathrm{Bun}_{\mathscr{G}}^{\leq\lambda}(S) = \{\mathscr{E} \in \mathrm{Bun}_{\mathscr{G}}(S) \mid \text{ the associated } \mathrm{SL}_r\text{-torsor } \iota'(\mathscr{E}) \text{ is bounded by } \lambda\}.$$

Proposition 2.2.4. *The stack $\mathrm{Bun}_{\mathscr{G}^{\mathrm{ad}}}^{\leq\lambda}$ is an open substack of $\mathrm{Bun}_{\mathscr{G}^{\mathrm{ad}}}$ and is an Artin stack of finite type over $\mathbb{F}_q$. Also, the stack $\mathrm{Bun}_{\mathscr{G}}^{\leq\lambda}$ is an open substack of $\mathrm{Bun}_{\mathscr{G}}$.*

Proof. Since the diagram (2.2) is Cartesian, it follows from the fact that $\mathrm{Bun}_{\mathrm{SL}_r}^{\leq\lambda}$ is an open quasi-compact substack of $\mathrm{Bun}_{\mathrm{SL}_r}$ (see [Sch14, Theorem 2.1]), and ι^* is of finite presentation. □

Remark 2.2.5.

(i) In general, $\mathrm{Bun}_{\mathscr{G}}^{\leq\lambda}$ is not quasi-compact because it may have infinitely many connected components. We will later have to take the quotient of $\mathrm{Bun}_{\mathscr{G}}^{\leq\lambda}$ by a (co-)lattice inside the center to obtain stacks of finite type.

(ii) The reason why we used $\mathscr{G}^{\mathrm{ad}} \to \mathrm{SL}_r$ instead of $\mathscr{G} \to \mathrm{SL}_r$ is because we want the Harder-Narashimhan truncations to be stable under the action of the center of $\mathscr{G}$. This way, we can later take quotient modulo any lattice inside the center.

(iii) On the other hand, we may have used the embedding $\mathscr{G} \to \mathrm{GL}_r$ instead of $\mathscr{G} \to \mathrm{SL}_r$, but one of the advantages of using SL_r is that the elements in Λ^+ is bounded by a large enough mulitple of the sum $2\rho^\vee$ of positive coroots in SL_r.

2.3 Affine Flag Variety

Let $\mathscr{O}$ be a complete discrete valued ring with residual characteristic $p > 0$ and denote by F the fraction field of $\mathscr{O}$. Let G be a connected reductive group over F. Let us fix a parahoric group scheme $\mathscr{G}$ over $\operatorname{Spec} \mathscr{O}$ with generic fiber G.

Definition 2.3.1. The **(local) loop group** LG (resp. **(local) positive loop group** $L^+\mathscr{G}$) of G (resp. $\mathscr{G}$) is the fpqc sheaf over $(\mathrm{Sch}/\mathbb{F})$ associated to the presheaf

$$S \mapsto G(\Gamma(S, \mathscr{O}_S)(\!(\varpi)\!)) \quad \Big(\text{resp. } S \mapsto G(\Gamma(S, \mathscr{O}_S)[\![\varpi]\!])\Big).$$

The loop group LG is representable by an ind-scheme of ind-finite type over $\mathbb{F}$, and the positive loop group $L^+\mathscr{G}$ is an infinite-dimensional affine group scheme over $\mathbb{F}$ that is pro-smooth (cf. [HR20, Lemma 3.2(ii)]).

Definition 2.3.2. The **affine flag variety** $\mathscr{F}\!l_{\mathscr{G}}$ of $\mathscr{G}$ is the fpqc sheaf on $(\mathrm{Sch}/\mathbb{F})$ associated to the presheaf quotient $LG/L^+\mathscr{G}$.

Suppose there exists a smooth affine group scheme $\mathscr{G}$ over a smooth projective geometrically irreducible curve X with a closed point t such that $\mathscr{G} = \mathscr{G} \times_X \mathrm{Spec}\ \mathscr{O}_t$ where $\mathscr{O}_t$ is the complete local ring at t.

Proposition 2.3.3.

(i) *The affine flag variety $\mathscr{F}l_{\mathscr{G}}$ is represented by an ind-projective ind-scheme over $\mathbb{F}$ and the obvious morphism $LG \to \mathscr{F}l_{\mathscr{G}}$ admits sections locally for the étale topology. Furthermore, $\mathscr{F}l_{\mathscr{G}}$ is ind-projective if and only if $\mathscr{G}$ is a parahoric group scheme over $\mathscr{O}_F$ for $\mathscr{G}$ a smooth affine model of G with geometrically connected fibers.*

(ii) *There exists a canonical isomorphism of ind-schemes over* $\mathrm{Spf}\ \mathscr{O}_t$

$$\mathrm{Gr}_{\mathscr{G},\{1\}} \times_X \mathrm{Spf}\ \mathscr{O}_t \xrightarrow{\sim} \mathscr{F}l_{\mathscr{G}} \widehat{\times}_{\mathbb{F}_t} \mathrm{Spf}\ \mathscr{O}_t$$

Proof. For (i), see [Ric15], and for (ii) see [HX23, Corollary 2.5.9]. □

2.4 Beilinson-Drinfeld Affine Grassmannian

The Schubert varieties inside the BD Grassmannian will be used to bound the relative positions of the modifications between $\mathscr{G}$-torsors so that the moduli spaces we consider has some finiteness properties. Let X be either a smooth projective geometrically irreducible curve over a finite field $\mathbb{F}$ or $X = \mathrm{Spec}\ \mathscr{O}$ where $\mathscr{O} = \mathbb{F}[\![\varpi]\!]$.

Definition 2.4.1. The **Beilinson-Drinfeld affine Grassmannian** (or **BD affine Grassmannian**) $\mathrm{Gr}_{\mathscr{G},\boldsymbol{I}}$ is the fpqc sheaf of sets on $(\mathbb{F}\text{-Sch})$ such that the S-valued points are given by a pair $(\underline{s}, (\mathscr{E}_j)_{j=0}^k, (\tau_j)_{j=1}^k, \varepsilon)$ where

- $\underline{s} \in X^I(S)$ are S-points of X^I and called **legs** or **characteristic places**,

- $\mathscr{E}_j$ are $\mathscr{G}$-bundles over X_S, i.e. in $\mathrm{Bun}_{\mathscr{G}}(S)$, for $0 \leq j \leq k$,

- the isomorphisms $\tau_j : \mathscr{E}_{j-1}|_{X_S - \bigcup_{i\in I_j}\Gamma_{s_i}} \xrightarrow{\sim} \mathscr{E}_j|_{X_S - \bigcup_{i\in I_j}\Gamma_{s_i}}$ away from $\bigcup_{i\in I_j}\Gamma_{s_i}$ are called **modifications**,

- $\varepsilon : \mathscr{E}_k \xrightarrow{\sim} \mathscr{G} \times_X X_S$ is a trivialization of $\mathscr{E}_k$.

To ease the notation, we make the following shorthand notations. For an I-tuple $\underline{s}$ and a subset $I_j \subset I$, denote by $\Gamma_{\underline{s}_j} := \bigcup_{i\in I_j}\Gamma_{s_i}$. Also, instead of $\tau_j : \mathscr{E}_{j-1}|_{X_S - \bigcup_{i\in I_j}\Gamma_{s_i}} \xrightarrow{\sim} \mathscr{E}_j|_{X_S - \bigcup_{i\in I_j}\Gamma_{s_i}}$, we will simply write

$$\tau_j : \mathscr{E}_{j-1} \overset{X_S - \Gamma_{\underline{s}_j}}{\dashrightarrow} \mathscr{E}_j.$$

Remark 2.4.2.

(i) For $I = \{1\}$, we have

$$\mathrm{Gr}_{\mathscr{G},\{1\}}(S) = \{(s, \mathscr{E}_0, \mathscr{E}_1, \tau_1 : \mathscr{E}_0 \overset{X_S - \Gamma_s}{\dashrightarrow} \mathscr{E}_1, \varepsilon : \mathscr{E}_k \xrightarrow{\sim} \mathscr{G} \times_X X_S)\}.$$

Combining τ_1 with ε, we have the alternative description

$$\mathrm{Gr}_{\mathscr{G},\{1\}}(S) = \{(s, \mathscr{E}, \varepsilon) \mid \varepsilon : \mathscr{E} \overset{X_S - \Gamma_s}{\dashrightarrow} \mathscr{G} \times_X X_S\}.$$

(ii) Let $X = \operatorname{Spec} \mathscr{O}$ and assume that there exists a smooth curve X_0 and a closed point x_0 of X_0 such that $\mathscr{O} \cong \widehat{\mathscr{O}}_{X_0,x_0}$. Let $\mathscr{G}$ be a smooth affine group scheme over $\mathbb{F}[\![\varpi]\!]$ with reductive generic fiber. Let R be an $\mathbb{F}[\![\varpi]\!]$-algebra. By [Ric19, §0.3], the R-valued points in the BD affine Grassmannian $\mathrm{Gr}_{\mathscr{G},\{1\}}$ can be given by tuple $(\mathscr{E},\varepsilon)$ where

- $\mathscr{E}$ is a $\mathscr{G}$-torsor on $\mathrm{Spec}(R[\![\varpi-\zeta]\!])$,
- $\varepsilon : \mathscr{E}|_{\mathrm{Spec}\ R(\!(\varpi-\zeta)\!)} \xrightarrow{\sim} \mathscr{G}|_{\mathrm{Spec}\ R(\!(\varpi-\zeta)\!)}$ is a trivialization.

where ζ is an additional formal variable.

By Proposition 4.2.2 which will be proved in the later section, we have the following properties of BD affine Grassmannian.

Proposition 2.4.3. *The BD affine Grassmannian $\mathrm{Gr}_{\mathscr{G},\underline{I}}$ is representable by an ind-quasi-projective ind-scheme over X^I. The ind-scheme is ind-projective if and only if $\mathscr{G}$ is a parahoric group scheme over X.*

Before we define the BD-Schubert varieties, we need an alternative description of BD affine Grassmannian. Let S be an $\mathbb{F}$-scheme. For a relative effective divisor $D \subset X_S$, denote by $\hat{D} = (D, \hat{\mathscr{O}}_{X_S,D})$ the completion of D in X_S. Let $\mathscr{I} \subset \hat{\mathscr{O}}_D$ be the ideal defining D. Then we denote by $D^{(r)} := \mathrm{Spec}(\hat{\mathscr{O}}_D/\mathscr{I})$ the r-th formal neighborhood of D. The divisor D is a closed subscheme of $\hat{D}$, and we write $\hat{D}^\circ = \hat{D}\backslash D$.

Proposition 2.4.4. *Let X be a smooth projective geometrically irreducible curve over a field $\mathbb{F}$.*

(i) *Let S be an $\mathbb{F}$-scheme and D be an relative effective Cartier divisor in X_S. Let $\mathscr{F}$ be a functor from $\mathrm{Bun}_{\mathscr{G}}$ to the category of triples of the form $(\mathscr{E}_1,\mathscr{E}_2,\alpha)$ where $\mathscr{E}_1$ is a $\mathscr{G}$-bundle on $X_S\backslash D$, $\mathscr{E}_2$ is a $\mathscr{G}$-bundle on $\hat{D}$, and $\alpha : \mathscr{E} \xrightarrow{\hat{D}\backslash D} \mathscr{E}_2$ is an isomorphism of $\mathscr{G}$ bundles over $\hat{D}\backslash D$ defined by*

$$\mathscr{E} \mapsto (\mathscr{E}|_{X_S\backslash D}, \mathscr{E}|_{\hat{D}}, \mathrm{id} : \mathscr{E} \xrightarrow{\hat{D}\backslash D} \mathscr{E})$$

Then $\mathscr{F}$ is an equivalence of categories.

(ii) *The S-points in $\mathrm{Gr}_{\mathscr{G},\underline{I}}$ can be alternatively described as a tuple*

$$(\underline{s}, (\mathscr{E}_j)_{j=0}^{k}, (\tau_j)_{j=1}^{m}, \varepsilon)$$

where $\mathscr{E}_j$ is a $\mathscr{G}$-torsors over $\hat{\Gamma}_{\underline{s}}$, $\tau_j : \mathscr{E}_{j-1} \xrightarrow{\hat{\Gamma}_{\underline{s}} - \hat{\Gamma}_{\underline{s}_j}} \mathscr{E}_j$, and ε is a trivialization of $\mathscr{E}_k$ over $\hat{\Gamma}_{\underline{s}}$.

Proof. (i) follows from Tannakian formalism. Namely, we may view $\mathscr{G}$-torsors over X_S as an exact faithful tensor functor $\mathrm{Rep}_{\mathscr{O}_X}(\mathscr{G}) \to \mathrm{Vect}_{X_S}$ where Vect_{X_S} is the category of vector bundles on X_S. Hence it suffices to prove in the case of vector bundles which was done in [BD, Theorem 2.12.1]. Then (ii) follows from (i). □

Using Beauville-Laszlo, one can prove that the iterated BD affine Grassmannian is the product of BD affine Grassmannian for $I = \{1\}$ away from the diagonals.

Proposition 2.4.5. *Let $U = \{(x_i)_{i\in I} \in X^I \mid x_i \neq x_j$ whenever $i \neq j$ in $I\}$ denote the complement of all diagonals in X^I. Then the map $\mathrm{Gr}_{\mathscr{G},\boldsymbol{I}}|_U \to \left(\prod_{i\in I} \mathrm{Gr}_{\mathscr{G},\{1\}}\right)|_U$ defined by*

$$(\underline{s}, (\mathscr{E}_j), (\tau_j), \varepsilon) \mapsto (s_i, \mathscr{E}_0|_{\hat{\Gamma}_{s_i}}, \mathscr{E}_k|_{\hat{\Gamma}_{s_i}}, \tau_k \circ \cdots \circ \tau_1|_{\hat{\Gamma}_{s_i}\setminus\Gamma_{s_i}}, \varepsilon)_{i\in I}$$

is an isomorphism.

2.5 Global Loop Group and BD Schubert Varieties

Definition 2.5.1. The **global loop group** $\mathscr{L}_{X^I}\mathscr{G}$ is the functor on the category of $\mathbb{F}$-schemes defined by $R \mapsto \{(\underline{s}, g) \mid \underline{s} \in X^I(S), g \in \mathscr{G}(\hat{\Gamma}^{\circ}_{\underline{s}})\}$. The **positive loop group** $\mathscr{L}^+_{X^I}\mathscr{G}$ is the functor on the category of $\mathbb{F}$-schemes defined by $R \mapsto \{(\underline{s}, g) \mid \underline{s} \in X^I(S), g \in \mathscr{G}(\hat{\Gamma}_{\underline{s}})\}$. Finally, $\mathscr{L}^{(r)}_{X^I}\mathscr{G}$ is defined by $S \mapsto \{(\underline{s}, g) \mid \underline{s} \in X^I(S), g \in \mathscr{G}(\hat{\Gamma}^{(r)}_{\underline{s}})\}$.

The global loop group $\mathscr{L}_{X^I}\mathscr{G}$ is representable by an ind-affine group ind-scheme over X^I and the positive global loop group $\mathscr{L}^+_{X^I}\mathscr{G}$ is representable by a closed affine subgroup scheme of $\mathscr{L}_{X^I}\mathscr{G}$ over X^I by [Hei09, Proposition 2]. Moreover, $\mathscr{L}^+_{X^I}\mathscr{G}$ is pro-smooth in the sense that there is a canonical isomrphism $\mathscr{L}^+_{X^I}\mathscr{G} \cong \varprojlim_r \mathscr{L}^{(r)}_{X^I}\mathscr{G}$ where $\mathscr{L}^{(r)}_{X^I}\mathscr{G}$ is smooth for all r. By [Ric15, Lemma 2.11], all $\mathscr{L}^{(r)}_{X^I}\mathscr{G} \to X^I$ are representable by flat affine group schemes of finite type and the transition maps for varying r are affine. There is an action of $\mathscr{L}^+_{X^I}\mathscr{G}$-action on $\mathrm{Gr}_{\mathscr{G},\boldsymbol{I}}$ by changing the trivialization ε.

Before we define the iterated version of BD Schubert varieties, let us consider the case when $\boldsymbol{I} = I = \{1\}$ is a singleton. The following treatment is from [Bie24]. Let G be the generic fiber of $\mathscr{G}$ and we assume that G is a reductive group scheme over the function field F of X. Fix a separable closure $\overline{F}$, and let $\boldsymbol{B}$ be a Borel subgroup such that $\boldsymbol{T}_{\overline{F}} \subset \boldsymbol{B} \subset G_{\overline{F}}$. Denote by $X^+_*(\boldsymbol{T})$ be the set of dominant cocharacters of $\boldsymbol{T}_{\overline{F}}$ with respect to $\boldsymbol{B}$. This can be identified as the set of conjugacy classes of cocharacters of $G_{\overline{F}}$.

Let $\{\mu\}$ be a conjugacy class of geometric cocharacter of G and μ be a dominant representative in $\{\mu\}$. Let F' be a finite separable extension of F such that G splits. Then the field extension F'/F corresponds to a finite, flat, surjective morphism $\widetilde{X}_{F'} \to X$ of curves where $\widetilde{X}_{F'}$ is the normalization of X in $\operatorname{Spec} F'$. By [Ric19, Section 0.2] and [Ric15, Corollary 2.14], the generic fiber of the BD affine Grassmannian $\mathrm{Gr}_{\mathscr{G},\{1\}}$ can be identified as the classical affine Grassmannian Gr_G associated to the group $G \times_F \mathrm{Spec}\, F[\![\varpi_\eta]\!]$, i.e. the fpqc sheafs on F-algebras associated to the presheaf $R \mapsto G(R(\!(\varpi_\eta)\!))/G(R[\![\varpi_\eta]\!])$ where ϖ_η is the local coordinate at the generic point $\mathrm{Spec}\, F$. We may and will view μ as an element in $X_*(\boldsymbol{T}_{F'})^+$ where $\boldsymbol{T}_{F'}$ is a maximal torus in $G_{F'}$ so that $\mu(\varpi_\eta) \in \mathrm{Gr}_{G_{F'}}(F')$. Then the **Schubert variety** $\mathrm{Gr}^\mu_{G_{F'}}$ of μ is defined as the schematic-image of the orbit map $L^+G_{F'} \to \mathrm{Gr}_{G_{F'}}$ given by $g \mapsto g\mu(\varpi_\eta)$ of $\mu(\varpi_\eta)$ in the affine Grassmannian. Equivalently,

$$\mathrm{Gr}^\mu_{G_{F'}} := \overline{L^+G_{F'}\mu(\varpi)L^+G_{F'}/L^+G_{F'}} \subset \mathrm{Gr}_{G_{F'}}$$

where the closure is given the reduced structure. The field of definition of μ will be called the **reflex field** of μ and will be denoted by E_μ. Then one can show that the Schubert variety $\mathrm{Gr}^\mu_{G_{F'}}$ associated to μ descends to E_μ.

The iterated BD-Schubert variety will then be defined as a schematic-closure of a product of the Schubert variety in the BD affine Grassmannian. Let X_μ be the normalization of X in E_μ and call it the **reflex scheme** of μ.

Definition 2.5.2. The **BD-Schubert variety** associated to μ is

$$\mathrm{Gr}^\mu_{\mathscr{G},\{1\}} := \mathrm{im}\left(\mathrm{Gr}^\mu_G \hookrightarrow \mathrm{Gr}_{\mathscr{G},\{1\}} \times_X X_\mu\right).$$

For an I-tuple of cocharacters $\underline{\mu} = (\mu_i)_{i\in I} \in X_*^+(\boldsymbol{T})^I$, set $X^I_{\underline{\mu}} := \prod_{i\in I} X_{\mu_i}$ and call it the **reflex scheme** of the tuple $\underline{\mu}$. Then the **BD-Schubert variety** associated to $\underline{\mu}$ is defined by the schematic image

$$\mathrm{Gr}^{\underline{\mu}}_{\mathscr{G},\underline{I}} := \mathrm{im}\left(\prod_{i\in I}\mathrm{Gr}^{\mu_i}_{\mathscr{G},\{1\}}\Big|_{U_{F_{\underline{\mu}}}} \hookrightarrow \mathrm{Gr}_{\mathscr{G},\boldsymbol{I}} \times_{X^I} X^I_{\underline{\mu}}\right)$$

where the composition is given by

$$\prod_{i\in I}\mathrm{Gr}^{\mu_i}_{\mathscr{G},\{1\}}\Big|_{U_{F_{\underline{\mu}}}} \hookrightarrow \prod_{i\in I}\mathrm{Gr}_{\mathscr{G},\{1\}}\Big|_{U_{F_{\underline{\mu}}}} \cong \mathrm{Gr}_{\mathscr{G},\underline{I}}\Big|_{U_{F_{\underline{\mu}}}} \hookrightarrow \mathrm{Gr}_{\mathscr{G},\underline{I}} \times_{X^I} X^I_{\underline{\mu}}$$

where the isomorphism is from Proposition 2.4.5. Here $U \subset X^I$ is the complement of all diagonals, and $U_{F_{\underline{\mu}}} := U \times_{X^I} X^I_{\underline{\mu}}$.

2.6 Bounds

Axiomatically defined bounds are certain closed subschemes of BD affine Grassmannians that will bound the relative positions between two torsors. Imposing such boundedness conditions will shrink the moduli of global shtukas which is an ind-Deligne-Mumford stack to a Deligne-Mumford stack locally of finite type. The main references are [ARH14], [ARH21], [HV21], and [HX23].

Let X be either a smooth projective geometrically irreducible curve over $\mathbb{F}_q$ or the spectrum of a complete DVR $\mathscr{O}$. Denote by F the function field of X or the fraction field of $\mathscr{O}$. In both cases, write $\mathbb{F}$ for the residue field. For a finite separable extension E of F, denote by $\widetilde{X}_E$ is the normalization of X in E. Then the induced map $\widetilde{X}_E \to X$ is finite and faithfully flat. If we are given an I-tuple $\underline{E} = (E_i)_{i\in I}$ of finite separable extensions of F, we denote by $\widetilde{X}^I_{\underline{E}} = \prod_{i\in I}\widetilde{X}_{E_i}$ where the product is taken over $\mathbb{F}$.

Definition 2.6.1.

(i) Let $\underline{E} = (E_i)$ and $\underline{E}' = (E_i')$ be two I-tuples of finite separable extensions of F. Let Z_1 and Z_2 be quasi-compact closed subschemes of $\mathrm{Gr}_{\mathscr{G},\underline{I}} \times_{X^I} \widetilde{X}_{\underline{E}}$ and $\mathrm{Gr}_{\mathscr{G},\underline{I}} \times_{X^I} \widetilde{X}_{\underline{E}'}$ respectively. We say that Z_1 and Z_2 are **equivalent** if there exists an I-tuple $\underline{E}'' = (E_i'')$ with E_i'' a common finite separable extension of E_i and E_i' such that $Z_1 \times_{\widetilde{X}^I_{\underline{E}}} \widetilde{X}^I_{\underline{E}''} = Z_2 \times_{\widetilde{X}^I_{\underline{E}'}} \widetilde{Y}^I_{\underline{E}''}$.

(ii) Let $\mathscr{Z}$ be an equivalence class of some quasi-compact closed subschemes $Z_{\underline{E}} \subset \mathrm{Gr}_{\mathscr{G},\underline{I}} \times_{X^I} \widetilde{X}_{\underline{E}}$. We choose an I-tuple $\underline{E} = (E_i)_{\in I}$ of field extensions of F such that E_i/F is a finite Galois extension. Set $\mathrm{Aut}_{\mathscr{Z}}(\underline{E}) = \{(g_i)_{i\in I} \in \mathrm{Gal}(E_i/F) \mid (g_i)^*(\mathscr{Z}) = \mathscr{Z}\}$. Then the **reflex scheme** $X^I_{\mathscr{Z}}$ is defined to be the quotient

$$X^I_{\mathscr{Z}} = \widetilde{X}^I_{\underline{E}}/\mathrm{Aut}_{\mathscr{Z}}(\underline{E})$$

(iii) A **bound** is an equivalence class $\mathscr{Z}$ of some quasi-compact closed subscheme $Z_{\underline{E}}$ of $\mathrm{Gr}_{\mathscr{G},\underline{I}} \times_{X^I} \widetilde{X}^I_{\underline{E}}$ such that all its representatives $Z_{\underline{E}}$ are stable under the left $\mathscr{L}^+_{X^I}\mathscr{G} \times_{X^I} \widetilde{X}^I_{\underline{E}}$-action on $\mathrm{Gr}_{\mathscr{G},\underline{I}} \times_{X^I} \widetilde{X}^I_{\underline{E}}$. We call $\mathscr{Z}$ a **global bound** if X is a curve and a **local bound** if $X = \mathrm{Spec}\,\mathscr{O}$.

Remark 2.6.2.

(i) The reflex scheme is independent of the choice of the representative with Galois extensions (cf. [Bie24, Lemma 2.2.3]). In general, the reflex scheme does not have to be a product.

(ii) It is not clear there exists a representative over the reflex scheme in general.

(iii) In [HV11] and [ARH14], *local bound* is defined in terms of completed affine flag varieties $\widehat{\mathscr{Fl}}_{\mathscr{G}}$ and finite ring extensions R of $\mathbb{F}[\![\zeta]\!]$ where ζ is the additonal formal variable in

2.4.2(ii). Then the local bound is an equivalence classes of closed ind-subschemes $\widehat{\mathscr{Z}}_R$ of $\widehat{\mathscr{Fl}}_{\mathscr{G},R} := \widehat{\mathscr{Fl}}_{\mathscr{G}} \widehat{\times}_{\mathbb{F}((\zeta))} \mathrm{Spf}\ R$ that are stable under the left $L^+\mathscr{G}_t$-action and the special fibers are quasi-compact subschemes of $\mathscr{Fl}_{\mathscr{G}} \hat{\times}_{\mathbb{F}} \mathrm{Spec}\ \kappa_R$ where κ_R is the residue field of R. By using the isomorphism in Proposition 2.3.3 (ii), we obtain the local bound from the way we defined.

Example 2.6.3. The BD Schubert variety $\mathrm{Gr}^{\underline{\mu}}_{\mathscr{G},\underline{I}}$ corresponding to an I-tuple $\underline{\mu}$ of conjugacy classes of cocharacters of $G_{\overline{F}}$ is a bound. By construction, the reflex scheme of BD-Schubert variety is $X_{\underline{\mu}} := \prod_{i \in I} X_{\mu_i}$. Furthermore, if $\mu_i : \mathbb{G}_m \to G_{\overline{F}}$ and t is a closed point of X, one can view μ_i as a geometric cocharacter $\mathbb{G}_m \to G_{\overline{F}_t}$ of G_{F_t}. Then the local bound $\hat{Z}_{\preceq \mu, R}$ induced by the global Schubert variety in [HV21, Example 2.7] agrees with $\widehat{\mathrm{Gr}}^{\mu_i}_{\mathscr{G}_t} := \mathrm{Gr}^{\mu_i}_{G,\{1\}} \times_X \mathrm{Spf}\ \mathscr{O}_t$. In particular, the local bound $\widehat{\mathrm{Gr}}^{\mu_i}_{\mathscr{G}_t}$ satisfies properties (i)-(v) in [HV21, Definition 2.2(b)].

Chapter 3: Local Shtukas and the Test Function

Our focus will be to define *test functions* that contains the information on the ramification in the bad reduction of the moduli space of global shtukas. As a result, we will obtain a *weighted* point counting formula contrary to unramified case as in [ND07] and [ND13]. Such test functions will be constructed using the deformation spaces of bounded local shtukas. As local shtukas describe the local behaviors of global shtukas, they serve as the function field analogue of p-divisible groups of abelian varieties (with extra structures). We remark that we need bounded local shtukas to establish the correct analogy with p-divisible groups. More precisely, we will consider deformation spaces of local shtukas whose relative position is bounded by a conjugacy class of cocharacters. This is motivated from the fact that F-isocrystals and Dieudonné modules also have bounded Hodge slopes. This way, we obtain the representability result of the deformation spaces as formal schemes formally locally of finite type.

In this chapter, we review the definition and the deformation theory of bounded local shtukas for unramified groups. The main referenes are [HV11], [ARH14], and [HV21]. We first prove that the deformation functors of a trivializable bounded local shtuka are pro-representable by a complete noetherian local ring. Then we will introduce the notion of dual Tate modules to define deformation spaces with level structures. Varying the level structures, we obtain a finite étale tower of deformation spaces with level structrures which, in turn, will be used to define the desired test functions ϕ as the trace of the cohomology of deformation spaces with level structures with coefficients in $\overline{\mathbb{Q}}_\ell$. We will conclude the chapter by showing that the test functions are locally constant and the definition is independent of ℓ not equal to the residue characteristic.

3.1 Bounded Local Shtukas

Let $\mathscr{O}$ be a equal-characteristic complete discrete valuation ring with residue field $\mathbb{F}$ of characteristic $p > 0$ and write F for its fraction field. Then choosing an uniformizer ϖ of $\mathscr{O}$, we obtain non-canonical isomorphisms $\mathscr{O} \cong \mathbb{F}[\![\varpi]\!]$ and $F \cong \mathbb{F}(\!(\varpi)\!)$. The main examples are the ring of integers of a local function field, i.e. finite extensions of the formal Laurent series $\mathbb{F}_q(\!(t)\!)$ over a finite field $\mathbb{F}_q$ of order q.

Let G be a connected reductive group over F. We assume that there exists a reductive group scheme $\mathscr{G}$ over $\mathbb{D} := \operatorname{Spec} \mathscr{O}$ whose generic fiber is G. In particular, we assume that G is an unramified group over F. For an $\mathbb{F}$-DM-stack S, we say LG-torsor (respectively, $L^+\mathscr{G}$-torsor) over S to mean LG-torsor (respectively, $L^+\mathscr{G}$-torsor) for the fppf topology. Since $\mathscr{G}$ is smooth over $\mathbb{D}$, we automatically get that LG-torsors are torsors for the étale topology by [ARH14, Proposition 2.4], i.e. torsors are étale-locally trivial. Denote by $H^1(S, \mathscr{G})$ to mean the groupoid of $\mathscr{G}$-torsors over S. The proof uses the fact that there is a natural isomorphism between the groupoid of $L^+\mathscr{G}$-torsors over $\operatorname{Spec} \mathbb{F}$ and the groupoid of *formal* $\mathscr{G}$-torsors over $\widehat{\mathbb{D}} := \operatorname{Spf} \mathscr{O}$. Given a $L^+\mathscr{G}$-torsor $\mathscr{L}$, the LG-torsor $L\mathscr{L} := \mathscr{L} \times^{L^+\mathscr{G}} LG$ defined as in (2.1) will be called the **associated LG-torsor** of $\mathscr{L}$.

Definition 3.1.1.

(i) A **local $\mathscr{G}$-shtuka** over a $\mathbb{F}$-DM-stack S is a pair $\underline{\mathscr{L}} = (\mathscr{L}, \hat{\tau})$ where $\underline{\mathscr{L}}$ is a $L^+\mathscr{G}$-torsor over S and $\hat{\tau} : {}^{\sigma}L\mathscr{L} \xrightarrow{\sim} L\mathscr{L}$ is an isomorphism of LG-torsors over S. Here $L\mathscr{L}$ is an associated LG-torsor of $\mathscr{L}$ and ${}^{\sigma}LG$ is the pullback of LG under the Frobenius σ. If $\mathscr{L}$ was instead an LG-torsor, the pair is called **local G-isoshtuka** over S.

(ii) Let $\underline{\mathscr{L}}$ be a local $\mathscr{G}$-shtuka over S. If $\hat{\tau}$ comes from an isomorphism $\hat{\tau}_+ : {}^{\sigma}\mathscr{L} \xrightarrow{\sim} \mathscr{L}$ of $L^+\mathscr{G}$-torsors, we say $\underline{\mathscr{L}}$ is **étale**.

(iii) A **morphism** $\underline{\mathscr{L}} = (\mathscr{L}, \hat{\tau}) \to \underline{\mathscr{L}}' = (\mathscr{L}', \hat{\tau}')$ of local $\mathscr{G}$-shtukas over S is an isomorphism $f : \mathscr{L} \to \mathscr{L}'$ of $L^+\mathscr{G}$-torsors over S such that $Lf \circ \hat{\tau} = \hat{\tau}' \circ {}^{\sigma}Lf$.

Example 3.1.2. Let $\mathscr{G} = \mathrm{GL}_r$, then giving an étale local GL_r-shtuka over S is the same as giving an étale local shtuka over S which is a pair $(M, \hat{\tau})$ consisting of a locally free sheaf M of $\mathscr{O}_S[\![z]\!]$-modules of finite rank on S and an isomorphism $\hat{\tau} : \sigma^* M \otimes_{\mathscr{O}_S[\![\varpi]\!]} \mathscr{O}_S(\!(\varpi)\!) \to M \otimes_{\mathscr{O}_S[\![\varpi]\!]} \mathscr{O}_S(\!(\varpi)\!)$. It is called **étale** if $\hat{\tau}$ comes from an isomorphism $\hat{\sigma^*} M \to M$ of $\mathscr{O}_S[\![\varpi]\!]$-modules.

Suppose $\underline{\mathscr{L}} = (\mathscr{L}, \hat{\tau})$ is **trivializable** local $\mathscr{G}$-shtuka over S, i.e. $\mathscr{L}$ is isomorphic to $(L^+\mathscr{G})_S$ as an $L^+\mathscr{G}$-torsor over S. Then $\hat{\tau}$ can be identified as an element b in $\mathrm{Aut}(LG) = LG(S)$. Choosing a different trivialization is amounts to choosing $b' \in LG(S)$ that is $L^+\mathscr{G}(S)$-σ-conjugate of b by an element in $L^+\mathscr{G}(S)$. This means that there exists $g \in L^+\mathscr{G}(S)$ such that $b' = g^{-1}b\sigma(g)$. Hence $\underline{\mathscr{L}}$ defines a $L^+\mathscr{G}(S)$-σ-conjugacy class $[b]_+ := \{g^{-1}b\sigma(g) \mid g \in L^+\mathscr{G}(S)\}$. When S is a spectrum of a finite field or a separably closed field, we have that local shtukas are trivializable.

Proposition 3.1.3. *Let $\underline{\mathscr{L}}$ be a local $\mathscr{G}$-shtuka over* $\mathrm{Spec}\ \kappa$ *where κ is either a finite field or an separably closed field. Then $\underline{\mathscr{L}}$ can be trivialized, i.e. there exists an isomorphism $\underline{\mathscr{L}} \cong ((L^+\mathscr{G})_\kappa, b\sigma)$ for some $b \in LG(\kappa)$.*

Proof. Since $\underline{\mathscr{L}}$ is étale-locally trivial, local $\mathscr{G}$-shtuka over an separably closed field is trivial. Now suppose κ is a finite field. Let $f_n : \mathrm{Spec}\ \kappa[\![t]\!]/(t^n) \to \mathrm{Spec}\ \kappa$ be the morphism of schemes induced by the injection $\kappa \hookrightarrow \kappa[\![t]\!]/(t^n)$. Then we show that there are natural isomorphisms of groupoids

$$H^1(\kappa, \mathscr{G}) \xrightarrow{\sim} \varprojlim_n H^1(\kappa[\![t]\!]/(t^n), \mathscr{G}_n) \xrightarrow{\sim} \varprojlim_n H^1(\kappa, f_{n,*}\mathscr{G}_n)$$

To give the first map, let $\mathscr{L} \in H^1(\kappa, \mathscr{G})$. Choose an fpqc covering $\mathrm{Spec}\ \kappa' \to \mathrm{Spec}\ \kappa$. The trivialization of $\mathscr{L}$ over κ' gives a 1-cocycle $g \in L^+\mathscr{G}(\kappa'')$ where $\kappa'' = \kappa' \otimes_\kappa \kappa'$. Then g modulo t^n can be identified with a descent datum on $\mathscr{G} \otimes_{\mathscr{O}} \kappa[\![t]\!]/(t^n)$. Since $\mathscr{G}$ is affine and $\mathrm{Spec}\ \kappa'[\![t]\!]/(t^n) \to \mathrm{Spec}\ \kappa[\![t]\!]/(t^n)$ is a fpqc cover, the descent datum is effective. Therefore we have an finitely presented affine scheme $\mathscr{L}_n$ that is a $\mathscr{G}_n := \mathscr{G} \otimes_{\mathscr{O}} \mathscr{O}/(t^n)$-torsor over $\kappa[\![t]\!]/(t^n)$. By [ARH14, Proposition 2.4], the map $\mathscr{L} \mapsto (\mathscr{L}_n)$ is a natural isomorphism of groupoids.

Next, we show that $H^1(\kappa[\![t]\!]/(t^n), \mathscr{G}_n) \to H^1(\kappa, f_{n,*}\mathscr{G}_n)$ defined by $\mathscr{L} \mapsto f_{n,*}\mathscr{L}$ is an isomorphism. This is essentially Shapiro's lemma. Note that if $\mathscr{L}$ is a $f_{n,*}\mathscr{G}_n$-torsor, then $f_n^*\mathscr{L}$ is an $f_n^* f_{n,*}\mathscr{G}_n$-torsor. Pushing forward along the counit $f_n^* f_{n,*}\mathscr{G}_n \to \mathscr{G}_n$ induces a $\mathscr{G}_n$-torsor $i(\mathscr{L})$. The map $\mathscr{L} \to i(\mathscr{L})$ is a quasi-inverse to $\mathscr{L} \mapsto f_{n,*}\mathscr{L}$.

To conclude, we have that $H^1(\kappa, f_{n,*}\mathscr{G}_n)$ is trivial for all n by Lang's theorem. This shows that $H^1(\kappa, \mathscr{G})$ is trivial. ☐

Denote by $\mathscr{N}ilp_{\mathscr{O}}$ the category of $\mathscr{O}$-schemes S where the image ζ of ϖ in S is locally nilpotent. Equivalently, $\mathscr{N}ilp_{\mathscr{O}}$ is the category of $\mathscr{O}$-schemes which factor through $\mathrm{Spf}\ \mathscr{O}$. As in the case of p-divisible groups, we have a notion of quasi-isogeny for local shtukas over $S \in \mathscr{N}ilp_{\mathscr{O}}$.

Definition 3.1.4. A **quasi-isogeny** $\underline{\mathscr{L}} = (\mathscr{L}, \hat{\tau}) \to \underline{\mathscr{L}}' = (\mathscr{L}', \hat{\tau}')$ of local $\mathscr{G}$-shtukas over $S \in \mathscr{N}ilp_{\mathscr{O}}$ is an isomorphism of associated LG-torsors $L\mathscr{L} \to L\mathscr{L}'$ such that $f \circ \hat{\tau} = \hat{\tau}' \circ {}^\sigma f$.

Denote by $\mathrm{QIsog}_S(\underline{\mathscr{L}}, \underline{\mathscr{L}}')$ for the set of quasi-isogenies from $\underline{\mathscr{L}}$ to $\underline{\mathscr{L}}'$.

Then we have the following rigidity statement analogous to p-divisible groups.

Proposition 3.1.5. *Let $S \in \mathscr{N}ilp_{\mathscr{O}}$ and let $j : \overline{S} \to S$ be a closed immersion defined by a locally nilpotent sheaf of ideals $\mathscr{I}$. Then*

$$\mathrm{QIsog}_S(\underline{\mathscr{L}}, \underline{\mathscr{L}}') \to \mathrm{QIsog}_{\overline{S}}(j^*\underline{\mathscr{L}}, j^*\underline{\mathscr{L}}')$$

defined by $f \mapsto j^*f$ is a bijection of sets.

Proof. The proposition is proved in [HV11, Proposition 3.9] when $\mathscr{G} = G_0 \times_{\mathbb{F}} \operatorname{Spec} \mathscr{O}$ where G_0 is a split reductive group over $\mathbb{F}$. However, the argument generalizes to any smooth affine group scheme $\mathscr{G}$ without any change. □

As mentioned in the introduction, we need to consider local shtukas whose relative position $\hat{\tau}$ is bounded.

Definition 3.1.6. Let $\mathscr{Z}$ be a local bound as in 2.6.1(iii). Let E be the reflex field of $\mathscr{Z}$ and $\mathscr{O}_E$ be its valuation ring. Let $S' \to S$ be an étale cover such that $\underline{\mathscr{L}}_{S'}$ trivializes. Fix a trivialization $\varepsilon : \underline{\mathscr{L}}_{S'} \xrightarrow{\sim} ((L^+\mathscr{G})_{S'}, b'\sigma)$. Then $(\underline{\mathscr{L}}_{S'}, \varepsilon)$ defines an element $g \in (\mathrm{Gr}_{\mathscr{G},\{1\}} \widehat{\times}_{\operatorname{Spec} \mathscr{O}} \operatorname{Spf} \mathscr{O}_E)(S')$. Here we used the equivalence between the category of $L^+\mathscr{G}$ torsors over S with a *formal* $\hat{\mathscr{G}}$-torsor over $\operatorname{Spf} \mathscr{O} \widehat{\times} S$ where $\widehat{\mathscr{G}}$ is the completion of $\mathscr{G}$ along $V(\varpi)$. We say that $\underline{\mathscr{L}}$ is **bounded** by $\mathscr{Z}$ if for some (or equivalently for all) representative Z of $\mathscr{Z}$ over a finite separable extension E'/E, the map

$$g \times \mathrm{id} : S' \times_{\operatorname{Spf}(\mathscr{O})} \operatorname{Spf}(\mathscr{O}_{E'}) \to (\mathrm{Gr}_{\mathscr{G},\{1\}} \times_{\operatorname{Spec}(\mathscr{O})} \operatorname{Spf}(\mathscr{O}_{E'}))$$

factors through the $Z_{E'} \times_{\operatorname{Spec}(\mathscr{O}_{E'})} \operatorname{Spf}(\mathscr{O}_{E'}) \subset \mathrm{Gr}_{\mathscr{G},\{1\}} \times_{\operatorname{Spec}(\mathscr{O})} \operatorname{Spf}\,(\mathscr{O}_{E'})$.

3.2 Deformation Space of Bounded Local Shtukas

Let $\mu : \mathbb{G}_m \to G_{\overline{F}}$ be a conjugacy class of geometric cochracter of G. Let $\widehat{Z}_\mu$ be the local bound associated to the global Schubert variety $\widehat{\mathrm{Gr}}^{\mu}_{\mathscr{G}}$ as in Example 2.6.3. Write E for the field of definition of μ and $\mathscr{O}_E$ for the ring of integers of E.

We would like to form a deformation problem of a local $\mathscr{G}$-shtuka $\overline{\underline{\mathscr{L}}}$ over $\operatorname{Spec} \kappa$ where κ is an $\mathscr{O}_E$-algebra that is also a field. We are mainly interested in the case where $\overline{\underline{\mathscr{L}}}$ is trivializable, e.g. κ is a finite field or an algebraically closed field by Proposition 3.1.3.

Let $(\mathrm{Art}_{\mathscr{O}_E,\kappa})$ be the category of Artinian local $\mathscr{O}_E$-algebras R with residue field κ. Then we define $\mathrm{Def}^{\mu}_{\overline{\underline{\mathscr{L}}}}$ to be the functor $(\mathrm{Art}_{\mathscr{O}_E,\kappa}) \to (\mathrm{Set})$ that parametrizes the isomorphism classes of pairs $(\underline{\mathscr{L}}, \alpha)$ where

- $\underline{\mathscr{L}}$ is a local $\mathscr{G}$-shtuka over $\operatorname{Spec} R$ bounded by μ,
- $\alpha : \overline{\underline{\mathscr{L}}} \xrightarrow{\sim} \underline{\mathscr{L}} \otimes_R \kappa$ is an isomorphism of local $\mathscr{G}$-shtukas over κ

Proposition 3.2.1. *The functor $\mathrm{Def}^{\mu}_{\overline{\underline{\mathscr{L}}}}$ is pro-representable by a complete Notherian local $\mathscr{O}_E$-algebra $R^{\mu}_{\overline{\underline{\mathscr{L}}}}$ with residue field κ.*

Proof. This is proved in [HV11, Theorem 5.6] in the split constant case $\mathscr{G} := G_0 \times_{\mathbb{F}_q} X$ where G_0 is a connected reductive group over $\mathbb{F}_q$. For general reductive groups over $\mathscr{O}$, the proposition is proved in [VW18, Proposition 2.6]. The ring $R^{\mu}_{\overline{\underline{\mathscr{L}}}}$ will be defined as a completed local ring of some BD Schubert variety. Since we assumed that $\overline{\underline{\mathscr{L}}}$ is trivializable, we have an isomorphism $\overline{\underline{\mathscr{L}}} \underset{\sim}{\xrightarrow{\varepsilon}} (L^+\mathscr{G}_\kappa, b\sigma)$ where $b \in LG(\kappa)$. Then using the equivalence between the category of $L^+\mathscr{G}$-torsor over $\operatorname{Spec} \kappa$ with the category of formal $\hat{\mathscr{G}}$-torsor over $\operatorname{Spf} \kappa[\![\varpi]\!]$, the pair $(\mathscr{L}, \varepsilon)$ defines an κ-valued point x^{-1} in $\mathrm{Gr}_{\mathscr{G}} \times_{\mathscr{O}} \operatorname{Spf} \kappa[\![\varpi]\!]$ which lies in $\widehat{\mathrm{Gr}}^{\mu}_{\mathscr{G}}$ as in Example 2.6.3. Then following the proof of the mentioned references above, the completed local ring $R^{\mu}_{\underline{\mathscr{L}}}$ of $\widehat{\mathrm{Gr}}^{\mu}_{\mathscr{G}}$ at x pro-represents the deformation functor $\mathrm{Def}^{\mu}_{\underline{\mathscr{L}}}$. □

Denote by k the complete unramified extension of E with residue field κ and write $\mathscr{O}_k$ for the valuation ring of k. Then denote by $X^{\mu}_{\underline{\mathscr{L}}}$ be the generic fiber of the formal scheme $\mathscr{X}^{\mu}_{\underline{\mathscr{L}}} := \mathrm{Spf}\ R^{\mu}_{\underline{\mathscr{L}}}$ as a k-analytic space. By [Ber96, §1], $X^{\mu}_{\underline{\mathscr{L}}}$ can be written as an increasing sequence of affinoid domains $X_1 \subset X_2 \subset \cdots$ that exhausts $X^{\mu}_{\underline{\mathscr{L}}}$, i.e. $X^{\mu}_{\underline{\mathscr{L}}} = \bigcup_{n\geq 1} X_n$. If $X_n = \mathscr{M}(R_n)$ where $\mathscr{M}$ denotes the Berkovich spectrum and R_n is some affinoid algebra, then the canonical homomorphisms $R^{n}_{\underline{\mathscr{L}}} \to R_n$ are continuous, and the image of $R^{\mu}_{\underline{\mathscr{L}}} \otimes_{\mathscr{O}_k} k$ in each R_n is everywhere dense. We can further find one such that X_n is a Weierstrass domain in X_{n+1}. A k-analytic space with such an exhaustion by affinoid domains is called **Stein space**. Given an open compact subgroup K of $\mathscr{G}(\mathscr{O})$, we would like to define the deformation space $X^{\mu}_{\underline{\mathscr{L}},K}$ of $\overline{\underline{\mathscr{L}}}$ with level K-structures. Analogous to the deformation space of p-divisible groups, we will define level structures as certain orbits of trivializations of Tate module of the universal local $\mathscr{G}$-shtukas over $X^{\mu}_{\underline{\mathscr{L}}}$. We first define (dual) Tate modules of any étale local $\mathscr{G}$-shtukas. We will introduce Tate module of an étale local shtuka as a tensor functor from the certain category of representations. We first introduce the necessary notations.

Let $\underline{\mathscr{L}} = (\mathscr{L}, \hat{\tau})$ be an étale local shtuka over an connected k-analytic space X. Denote by $\mathrm{Rep}_{\mathscr{O}}\mathscr{G}$ the category of representations $\rho : \mathscr{G} \to \mathrm{GL}(V)$ of $\mathscr{G}$ in finite free $\mathscr{O}$-modules V as a morphism of algebraic groups over $\mathscr{O}$. Let $\overline{x}$ be a geometric point in X. Denote by $\mathrm{FMod}_{\mathscr{O}[\pi_1^{\text{ét}}(X,\overline{x})]}$ the category of finite free $\mathscr{O}$-modules equipped with a continuous action of $\pi_1^{\text{ét}}(X, \overline{x})$. Let $\rho \in \mathrm{Rep}_{\mathscr{O}}\mathscr{G}$. Let $\rho_*\underline{\mathscr{L}} = \underline{M} = (M, \hat{\tau}_M)$ be the associated étale local shtuka as in Example 3.1.2.

Here $\rho_*\mathscr{L}$ is the sheaf of $\mathscr{O}_S[\![\varpi]\!]$-modules associated to the presheaf

$$T \mapsto (\mathscr{L}(T) \times (V \otimes_{\mathscr{O}} \mathscr{O}_S[\![\varpi]\!](T)))/L^+\mathscr{G}(T).$$

Definition 3.2.2. The **Tate module** of an étale local $\mathscr{G}$-shtuka $\underline{\mathscr{L}}$ over X associated to $\rho \in \mathrm{Rep}_{\mathscr{O}}\mathscr{G}$ is defined to be

$$\breve{\mathscr{T}}_{\underline{\mathscr{L}},\overline{x}} := \{m \in \underline{M}_{\overline{x}} \mid \hat{\tau}_M(\sigma^* m) = m\}$$

where $\sigma^* m = m \otimes 1 \in \sigma^* M_{\overline{x}}$. Or, equivalently, we can write $\breve{\mathscr{T}}_{\underline{\mathscr{L}},\overline{x}}(\rho) = (\rho_*\underline{\mathscr{L}} \otimes_{\mathscr{O}_X[\![\varpi]\!]} \kappa(\overline{x})[\![\varpi]\!])^{\hat{\tau}}$. Then the **Tate functor** of $\underline{\mathscr{L}}$ is defined as the tensor functor $\breve{\mathscr{T}}_{\underline{\mathscr{L}},\overline{x}} : \mathrm{Rep}_{\mathscr{O}}\mathscr{G} \to \mathrm{FMod}_{\mathscr{O}[\pi_1^{\text{ét}}(X,\overline{x})]}$ defined by $\rho \mapsto \breve{\mathscr{T}}_{\underline{\mathscr{L}},\overline{x}}(\rho)$.

Suppose $\underline{\mathscr{L}}_0 = ((L^+\mathscr{G})_X, \sigma)$ is a trivial local $\mathscr{G}$-shtuka over X. Then the composition $\mathrm{Rep}_{\mathscr{O}} \xrightarrow{\breve{\mathscr{T}}_{\underline{\mathscr{L}}_0}} \mathrm{FMod}_{\mathscr{O}[\pi_1^{\text{ét}}(X,\overline{x})]} \xrightarrow{\mathscr{F}} \mathrm{FMod}_{\mathscr{O}}$ with the forgetful functor $\mathscr{F}$ is naturally isomorphic to the fiber functor $\omega^\circ : \mathrm{Rep}_{\mathscr{O}}\mathscr{G} \to \mathrm{FMod}_{\mathscr{O}}$ which forgets the $\mathscr{G}$-action.

Now the following definition is from [ARH14, Definition 3.5].

Remark 3.2.3. There are few alternative ways to define the Tate module of an étale local shtuka $\underline{\mathscr{L}} = (\mathscr{L}, \hat{\tau})$ over an $\mathbb{F}_q$-scheme S. The **Tate module** $\mathscr{T}_{\underline{\mathscr{L}}}$ can be defined as a moduli problem of isomorphisms of local $\mathscr{G}$-shtukas

$$\mathscr{T}_{\underline{\mathscr{L}}} := \underline{\mathrm{Isom}}((L^+\mathscr{G}_S, \sigma), \underline{\mathscr{L}}).$$

We will use this definition to build global shtukas with infinite levels in the next chapter.

Remark 3.2.4. For tensor functors $\mathscr{A}$ and $\mathscr{B}$ from $\mathrm{Rep}_{\mathscr{O}}$ to $\mathrm{FMod}_{\mathscr{O}}$. Denote by $\mathrm{Aut}^{\otimes}(\mathscr{A})$ to be the tensor automorphisms of $\mathscr{A}$ and $\mathrm{Isom}^{\otimes}(\mathscr{A}, \mathscr{B})$ to be the tensor isomorphisms from $\mathscr{A}$ to $\mathscr{B}$ which has a natural $\mathrm{Aut}^{\otimes}(\mathscr{A})$-action via composition. In the proof of [ARH14, Proposition 3.6], one shows that there is a natural isomorphisms between $\mathscr{T}_{\underline{\mathscr{L}}}$ and $\mathrm{Isom}^{\otimes}(\omega^{\circ}, \mathscr{F} \circ \breve{\mathscr{T}}_{\underline{\mathscr{L}}})$ where $\mathscr{F}$ was the forgetful functor $\mathrm{FMod}_{\mathscr{O}[\pi_1^{\text{ét}}(X,\overline{x})]} \to \mathrm{FMod}_{\mathscr{O}}$ that forgets the action of the étale fundamental group $\pi_1^{\text{ét}}(S, \overline{s})$.

Proposition 3.2.5. *The set* $\mathrm{Isom}^{\otimes}(\omega^{\circ}, \mathscr{F} \circ \breve{\mathscr{T}}_{\underline{\mathscr{L}}})$ *is a* $\mathscr{G}$*-torsor over* S *for the pro-étale topology.*

Proof. By [ARH21, Lemma 6.2], the set $\mathrm{Isom}^{\otimes}(\omega^{\circ}, \breve{\mathscr{T}}_{\underline{\mathscr{L}}})$ is non-empty and is a $\mathscr{G}$-torsor. □

By following the arguments in [HV21, Remark 6.7] for bounded Rapoport-Zink spaces, one can obtain an universal étale local $\mathscr{G}$-shtuka $\underline{\mathscr{L}}^{\mu}_{\text{univ}} = (\mathscr{L}^{\mu}_{\text{univ}}, \hat{\tau}^{\mu}_{\text{univ}})$ over $X^{\mu}_{\underline{\mathscr{L}}} = \left(\mathrm{Spf}\ R^{\mu}_{\underline{\mathscr{L}}}\right)^{\text{an}}$.

Definition 3.2.6. Let $K \subset \mathscr{G}(\mathscr{O})$ be an open subgroup. Fix a geometric base point $\overline{x}$ of connected k-analytic space X. Then an **integral level K-structure** is a $\pi_1(X, \overline{x})$-invariant K-orbit of $\mathrm{Isom}^{\otimes}(\omega^{\circ}, \mathscr{F} \circ \breve{\mathscr{T}}_{\underline{\mathscr{L}}^{\mu}_{\text{univ}}})(\mathscr{O})$. Define $X^{\mu}_{\underline{\mathscr{L}},K}$ to be the functor on the category of k-analytic spaces over $X^{\mu}_{\underline{\mathscr{L}}}$ that parametrizes integral K-level structure on the étale G-shtuka $\underline{\mathscr{L}}^{\mu}_{\text{univ}}$ over X.

Proposition 3.2.7. *Let* $K \subset \mathscr{G}(\mathscr{O})$ *be a normal open subgroup. The functor* $X^{\mu}_{\underline{\mathscr{L}},K}$ *is representable by the finite étale covering space of* $X^{\mu}_{\underline{\mathscr{L}}}$ *that is Galois with Galois group* $\mathscr{G}(\mathscr{O})/K$.

Proof. See [HV21, Proposition 7.5]. □

As in Scholze [Sch12], we introduce the notion of controlled cohomology.

Definition 3.2.8.

(i) Let X be a paracompact k-analytic spaces such that X is exhausted by a sequence of affinoid spaces $X_0 \subset X_1 \subset \cdots \subset X$ such that $X = \bigcup_n X_n$. Then for $\ell \neq p$, we say that

X has a **controlled ℓ-cohomology** if the map

$$H_c^i(X_n \otimes_k \hat{\bar{k}}, \mathbb{Z}/\ell\mathbb{Z}) \to H_c^i(X \otimes \hat{\bar{k}}, \mathbb{Z}/\ell\mathbb{Z})$$

is an isomorphism for all i for big enough n.

(ii) Let $\underline{\mathscr{L}}$ be a local $\mathscr{G}$-shtuka over a finite extension κ of κ_E bounded by μ. We say that $\underline{\mathscr{L}}$ has a **controlled cohomology** if $X^{\mu}_{\underline{\mathscr{L}},K}$ has controlled cohomology for all open normal pro-p subgroups $K \subset \mathscr{G}(\mathscr{O})$, and $\ell \neq p$.

Lemma 3.2.9. *Let $\underline{\mathscr{L}}$ be a local $\mathscr{G}$-shtuka over κ which is a finite extension of κ_E bounded by μ. For all compact open subgroups $K \subset \mathscr{G}(\mathscr{O})$, the cohomology groups $H^i(X^{\mu}_{\underline{\mathscr{L}},K} \otimes_k \hat{\bar{k}}, \overline{\mathbb{Q}}_\ell)$ vanish for big enough $i > \dim X^{\mu}_{\underline{\mathscr{L}},K}$.*

Proof. We adapt the proof of [Han19, Corollary 3.4]. Let d be the dimension of $X^{\mu}_{\underline{\mathscr{L}}}$. We have that $X^{\mu}_{\underline{\mathscr{L}}}$ is a k-analytic space over the analytification of the Schubert variety $\mathrm{Gr}^{\mu,\mathrm{an}}_{\mathscr{G}}$ in the generic fiber which is a projective scheme over E_μ. Since $X^{\mu}_{\underline{\mathscr{L}}}$ is the generic fiber of a formal completion of a projective scheme at a closed point, $X^{\mu}_{\underline{\mathscr{L}}}$ is a Stein space in the sense of $X^{\mu}_{\underline{\mathscr{L}}} = \bigcup_{m=1}^{\infty} X_m$ where X_m is a Weierstrass domain in X_{m+1}. Since $X^{\mu}_{\underline{\mathscr{L}},K}$ is a finite étale covering of $X^{\mu}_{\underline{\mathscr{L}}}$, we have that $X^{\mu}_{\underline{\mathscr{L}},K} = \bigcup_{m=1}^{\infty} Y_m$ is a Stein space with Y_m is a Weierstrass domain in Y_{m+1}. Then by [Hen16, Proposition 5.23], the embedding of an affinoid domain translates to open immersions of open affinoids in the sense of adic spaces. Denote by Y_m^{ad} the associated adic space of Y_m. Then for each Y_m, we have $H^i(Y_m^{\mathrm{ad}} \otimes_k \hat{\bar{k}}, \mathbb{Z}/\ell^n\mathbb{Z}) = 0$ for $i > d$ by [Han19, Theorem 1.4]. Denote by

$Y_{m,\hat{\bar{k}}} = Y_m \otimes_k \hat{\bar{k}}$. Then by [Hub96, Lemma 3.9.2], we have a short exact sequence

$$0 \to \varprojlim_m{}^{(1)} H^{i-1}(Y^{\mathrm{ad}}_{m,\hat{\bar{k}}}, \mathbb{Z}/\ell^n\mathbb{Z}) \to H^i(X^{\mu,\mathrm{ad}}_{\underline{\mathscr{L}},K,\hat{\bar{k}}}, \mathbb{Z}/\ell^n\mathbb{Z}) \to \varprojlim_m H^i(Y^{\mathrm{ad}}_{m,\hat{\bar{k}}}, \mathbb{Z}/\ell^n\mathbb{Z}) \to 0$$

As both all $H^i(Y^{\mathrm{ad}}_{m,\hat{\bar{k}}}, \mathbb{Z}/\ell^n\mathbb{Z}) = 0$ for $i > d$, and $H^i(Y_{m,\hat{\bar{k}}}, \mathbb{Z}/\ell^n\mathbb{Z})$ is finite as in the proof of (i), so $\varprojlim_m^{(1)}$ vanishes. Therefore, we have the desired result. □

Remark 3.2.10. For all compact open subgroups $K \subset \mathscr{G}(\mathscr{O})$, we need that the cohomology groups $H^i(X^{\mu}_{\underline{\mathscr{L}},K} \otimes_k \hat{\bar{k}}, \overline{\mathbb{Q}}_\ell)$ are finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces for all $i \geq 0$ so that the trace of the space is a number. The cohomology groups $H^i(X^{\mu}_{\underline{\mathscr{L}},K} \otimes_k \hat{\bar{k}}, \mathbb{Z}/\ell^n\mathbb{Z})$ with torsion coefficients can be proved to be finite $\mathrm{Gal}(\overline{F}/F)$-module. As $\pi : X^{\mu}_{\underline{\mathscr{L}},K} \to X^{\mu}_{\underline{\mathscr{L}}}$ is a finite étale map, it suffices to prove that $H^*(X^{\mu}_{\underline{\mathscr{L}}} \otimes_k \hat{\bar{k}}, \pi_*\mathbb{Z}/\ell^n\mathbb{Z})$ is finite. By [Ber15, Corollary 3.1.2], we need to show that $\pi_*\mathbb{Z}/\ell^n\mathbb{Z}$ is a $\mathrm{Spf}\ R^{\mu}_{\underline{\mathscr{L}}}$-constructible sheaf on $X^{\mu}_{\underline{\mathscr{L}}}$. As $\pi_*\mathbb{Z}/\ell^n\mathbb{Z}$ is finite locally constant for the étale topology on $X^{\mu}_{\underline{\mathscr{L}}}$ by [Ber93, Proposition 4.4.6] and have a discrete action of $\mathrm{Gal}(\overline{F}/F)$, we have that $H^i(X^{\mu}_{\underline{\mathscr{L}}}, \pi_*\mathbb{Z}/\ell^n\mathbb{Z}) = H^i(X^{\mu}_{\underline{\mathscr{L}},K}, \mathbb{Z}/\ell^n\mathbb{Z})$ is a finite discrete $\mathrm{Gal}(\overline{F}/F)$-module.

Assumption. In this thesis, we make an assumption that we assume that the $\overline{\mathbb{Q}}_\ell$-vector space $H^i(X^{\mu}_{\underline{\mathscr{L}},K}, \otimes_k \hat{\bar{k}}, \overline{\mathbb{Q}}_\ell)$ is a finite dimensional $\overline{\mathbb{Q}}_\ell$ vector space for local $\mathscr{G}$-shtukas with controlled cohomology. This should be done by proving that the comparing the generators of $H^i(X^{\mu}_{\underline{\mathscr{L}},K}, \otimes_k \hat{\bar{k}}, \mathbb{Z}/\ell^n\mathbb{Z})$ and $X^{\mu}_{\underline{\mathscr{L}},K}, \otimes_k \hat{\bar{k}}, \mathbb{Z}/\ell^{n+1}\mathbb{Z})$, but we will be explored in a future draft of this article.

3.3 The test function

Recall that we assumed that G is unramified. Therefore, E/F is an unramified extension. Denote by $\sigma_E \in \mathrm{Gal}(\overline{E}/E)$ the geometric Frobenius element in W_E. Fix an integer $j \geq 0$ and set $r = j[\kappa_E : \mathbb{F}]$ where κ_E is the residue field of E of our chosen conjugacy class μ of geometric cocharacter. Denote by F_r the degree-r unramified extension of F and let $\mathscr{O}_r$ be the ring of integers of F_r. Then we have the following definition of the test function.

Definition 3.3.1. Let $\delta \in G(F_r)$ and $f \in C_c^\infty(G(\mathscr{O}))$. Let $\overline{\underline{\mathscr{L}}}_\delta = ((L^+\mathscr{G})_{\kappa_r}, \delta\sigma)$. Then we define the test function $\phi_{r,f} : G(F_r) \to \overline{\mathbb{Q}}_\ell$ by

$$\phi^\mu_{r,f}(\delta) = \mathrm{tr}(\sigma^r \times f \mid H^*(X^\mu_{\overline{\underline{\mathscr{L}}}_\delta, K} \otimes_k \hat{\bar{k}}, \overline{\mathbb{Q}}_\ell)$$

if $\overline{\underline{\mathscr{L}}}_\delta$ is bounded by μ and has controlled cohomology. Here we chose $K \subset \mathscr{G}(\mathscr{O})$ such that f is bi-K-invariant. For other δ, we simply put $\phi^\mu_{r,f}(\delta) = 0$. We may and will assume that f is the characteristic function of KgK for some $g \in \mathscr{G}(\mathscr{O})$ for some open compact normal subgroup K of $\mathscr{G}(\mathscr{O})$.

Proposition 3.3.2. *The function $\phi^\mu_{j,f}$ is a locally constant and compactly supported function $G(F_r) \to \overline{\mathbb{Q}}_\ell$.*

Proof. Let $\mathscr{O}_r$ be the ring of integers of F_r. Let $\delta \in G(F_r)$. View δ as an element in $G(\breve{F}) = LG(\overline{\mathbb{F}})$. Then we have $\mathrm{Supp}(\phi^\mu_{j,f}) \subset \mathscr{G}(\mathscr{O}_r)\mu(\varpi)\mathscr{G}(\mathscr{O}_r)$. As the double coset $A := \mathscr{G}(\mathscr{O}_r)\mu(\varpi)\mathscr{G}(\mathscr{O}_r)$ is closed, the complement $G(F_r)\backslash A$ is an open neighborhood of $\delta \in G(F_r)\backslash A$ where $\phi^\mu_{r,f}(\delta') = 0$ for all δ' in the complement. Hence we need to show that there exists an open neighborhood of

$\delta \in \mathscr{G}(\mathscr{O}_r)\mu(\varpi)\mathscr{G}(\mathscr{O}_r)$ such that $\phi^{\mu}_{r,f}(\delta') = \phi^{\mu}_{r,f}(\delta)$ for all $\delta' \in \mathscr{G}(\mathscr{O}_r)\mu(\varpi)\mathscr{G}(\mathscr{O}_r)$. In order to show this, we need the following lemma.

Lemma 3.3.3. *Let $K \subset \mathscr{G}(\mathscr{O})$ be a open compact subgroup. Then there exists $n \geq 1$ such that if g is an automorphism of $\underline{\overline{\mathscr{L}}}$ that acts trivially on $\underline{\overline{\mathscr{L}}}[\varpi^n] := \underline{\overline{\mathscr{L}}} \times^{\mathscr{G}} \mathscr{G}/\mathscr{G}_n$ where $\mathscr{G}_n := \mathscr{G} \otimes_{\mathscr{O}} \mathscr{O}/(\varpi^{n+1})$, then g acts trivially on $H^i(X^{\mu}_{\underline{\overline{\mathscr{L}}},K} \otimes_k \hat{\bar{k}}, \overline{\mathbb{Q}}_\ell)$.*

The proof then follows from [Sch12, Proposition 3.15] because we assumed that the local $\mathscr{G}$-shtuka $\underline{\overline{\mathscr{L}}}_\delta$ has controlled cohomology. Choose m from the lemma, and let V be the open neighborhood of δ, viewed as an element in $G(\breve{F})$, defined as the image of $\ker(G(\breve{\mathscr{O}}) \to G(\breve{\mathscr{O}}/(\varpi^m))$ under the map $g \mapsto g^{-1}\delta\sigma(g)$. To see that V is open, we need to show that the map $g \mapsto g^{-1}\delta\sigma(g)$ is an open map. By the inverse function theorem in [Ser92, Theorem 2, page 83], it suffices to look at the map $\breve{F}^n \to \breve{F}^n$ defined by $-x + (\mathrm{Ad}\ \delta)x^{\sigma}$ is open. The following [Sch12, Lemma 4.5], we see that the map is open.

Let $U = V \cap G(F_r)$ and $\delta' \in U$. Since the twisting by an element in $\ker(G(\breve{\mathscr{O}}) \to G(\breve{\mathscr{O}}/(\varpi^m))$ does not change $\underline{\overline{\mathscr{L}}}_\delta[\varpi^m]$, the actions of σ^r on

$$X_{\underline{\overline{\mathscr{L}}}_\delta,K} \otimes_k \hat{\bar{k}} \cong X_{\underline{\overline{\mathscr{L}}}_{\delta'},K} \otimes_k \hat{\bar{k}}$$

differs by some action of g in the automorphism group of $\underline{\overline{\mathscr{L}}}_\delta$ over $\overline{\mathbb{F}}$ that is trivial on ϖ^m-torsion points. Hence the induced action of σ^r on the cohomology is trivial by Lemma 3.3.3. Therefore, the two trace of $\sigma^r \times f$ on the cohomology groups are the same, and this shows that $\phi^{\mu}_{r,f}$ is locally constant. □

Chapter 4: Moduli of Global Shtukas

The goal of this chapter is to review the properties of moduli spaces of global shtukas with two legs and one leg fixed at ∞. We first recall some facts about general iterated global shutkas and their moduli spaces. We then introduce the adelic level structures using the variant of Beauville-Laszlo glueing introduced in [HK24]. We conclude the chapter by showing that the the moduli stack is representable by a quasi-projective scheme over the curve when the level structures is *small enough* after bounding the relative positions by BD-Schubert varities and truncating the moduli space by Harder-Narashimhan filtrations. The reason why we fix one leg is to work with one-dimensional bases for the moduli space so that we can use the theory of nearby cycle sheaves avoiding the theory over arbitrary bases.

4.1 Notations

Let X be a smooth projective geometrically irreducible curve over a finite field $\mathbb{F}_q$ of order q. Denote by $F := \mathbb{F}_q(X)$ its function field. For a fixed separable closure $\overline{F}$ of F, denote by $\Gamma = \mathrm{Gal}(\overline{F}/F)$ the absolute Galois group of F. We write $|X|$ for the set of closed points in X. For each $t \in |X|$, denote by $\mathscr{O}_t := \hat{\mathscr{O}}_{X,t}$ the completion of the local ring $\mathscr{O}_{X,t}$ at t, and write F_t for its fraction field. Write $\mathbb{A} := \prod'_{t\in|X|}(F_t, \mathscr{O}_t)$ for the ring of adeles of F, and write $\mathbb{O} := \prod_{t\in|X|} \mathscr{O}_t$ for the ring of integral adeles.

In this chapter, we only consider a connected reductive group G over F. Let $\mathscr{G}$ be a parahoric integral model of G over X. In other words, $\mathscr{G}$ is a smooth affine group scheme over X such that it has geometrically connected fibers, its generic fiber is G, and for all $t \in |X|$, $\mathscr{G}_t := \mathscr{G} \times_X \operatorname{Spec} \mathscr{O}_t$ is a parahoric group scheme over $\mathscr{O}_t$ in the sense of [BT84, Definition 5.2.6]. We can always construct such a model as follows. There exists an open dense subset $U \subset X$ and a reductive group scheme $\mathscr{G}_U$ over U whose generic fiber is G. For each $t \in X \backslash U$, choose a parahoric model $\mathscr{G}_t$ over $\mathscr{O}_t$ whose generic fiber is G_{F_t}. As $U \amalg \coprod_{t \in X\backslash U} \operatorname{Spec} \mathscr{O}_t \to X$ is an fpqc-cover, we obtain parahoric integral model $\mathscr{G} \to X$ via glueing $\mathscr{G}_U$ with all $\mathscr{G}_t$ using fpqc-descent. We fix once and for all distinct closed point x and ∞ in $|X|$ where $\infty \in X(\mathbb{F}_q)$. The closed point x will serve as the place of bad reduction in our setting. We will further assume that G is unramified at x, i.e. $\mathscr{G}_x$ is a connected reductive group over $\operatorname{Spec} \mathscr{O}_x$.

For an $\mathbb{F}_q$-scheme S, we denote by $\sigma_S : S \to S$ the absolute q-Frobenius endomorphism that acts as the identity on the underlying topological space of S and as the q-power map $s \mapsto s^q$ on the structure sheaf. For Deligne-Mumford stacks Y and Z over $\mathbb{F}_q$, we simply write $Y \times_{\mathbb{F}_q} Z$ instead of $Y \times_{\operatorname{Spec} \mathbb{F}_q} Z$.

4.2 Iterated Bounded Global Shtukas

We now recall definitions and properties of moduli spaces of global shtukas. Let I be a finite set. The underline $\underline{I} = I_1 \sqcup \cdots \sqcup I_k$ will always denotes a partition of I.

Definition 4.2.1. The **Hecke stack** $\operatorname{Hk}_{\mathscr{G},\underline{I}}$ is the category fibered in groupoids on $(\mathrm{Sch}/\mathbb{F})$ defined by

$$S \mapsto \{(\underline{s}, (\mathscr{E}_j)_{j=0}^k, (\tau_j)_{j=1}^k)\}$$

where

- $\underline{s} \in X^I(S)$ are S-points of X^I,
- $\mathscr{E}_j$ are $\mathscr{G}$-bundles over X_S,
- the isomorphisms $\tau_j : \mathscr{E}_{j-1}|_{X_S - \bigcup_{i\in I_j}\Gamma_{s_i}} \xrightarrow{\sim} \mathscr{E}_j|_{X_S - \bigcup_{i\in I_j}\Gamma_{s_i}}$ away from $\bigcup_{i\in I_j}\Gamma_{\underline{s}_i}$.

The Hecke stack $\mathrm{Hk}_{\mathscr{G},\underline{I}}$ parametrizes the same data as the BD affine Grassmannian $\mathrm{Gr}_{\mathscr{G},\underline{I}}$ except the trivialization of the last $\mathscr{G}$-torsor.

The morphisms between two S-points with the same legs in $\mathrm{Hk}_{\mathscr{G},\underline{I}}(S)$ is a tuple of morphisms in $\mathrm{Bun}_{\mathscr{G}}(S)$ that are compatible with the modifications away from the graph.

Proposition 4.2.2. *The stack* $\mathrm{Hk}_{\mathscr{G},\underline{I}}$ *is an ind-Artin stack locally of ind-finite type over* X. *The morphism* $\mathrm{Hk}_{\mathscr{G},\underline{I}} \to X^I \times_{\mathbb{F}_q} \mathrm{Bun}_{\mathscr{G}}$ *defined by* $(\underline{s}, (\mathscr{E}_j), (\tau_j)) \mapsto (\underline{s}, \mathscr{E}_k)$ *is relatively representable by a morphism of schemes which is of ind-finite type and quasi-projective. It is even projective if and only if* $\mathscr{G}$ *is a parahoric group scheme.*

Proof. See [ARH21, Proposition 3.9 and Proposition 3.12]. □

Definition 4.2.3. Let $\mathscr{G}$ be a smooth affine group scheme over X. The **moduli space of global $\mathscr{G}$-shtukas** is the stack fibered in groupoids over $(\mathrm{Sch}/\mathbb{F}_q)$ defined by the Cartesian diagram

$$\begin{array}{ccc} \mathrm{Sht}_{\mathscr{G},\underline{I}} & \longrightarrow & \mathrm{Bun}_{\mathscr{G}} \\ \downarrow & & \downarrow \\ \mathrm{Hk}_{\mathscr{G},\underline{I}} & \longrightarrow & \mathrm{Bun}_{\mathscr{G}} \times \mathrm{Bun}_{\mathscr{G}} \\ (\underline{s}, (\mathscr{E}_j)_{j=0}^k, (\tau_j)_{j=1}^k) & \longmapsto & ((\mathscr{E}_k, \psi_k), (\mathscr{E}_0, \psi_0)) \end{array} \qquad \begin{array}{c} (\mathscr{E}_k, \psi_k) \\ \downarrow \\ ((\mathscr{E}_k, \psi_k), {}^{\sigma}(\mathscr{E}_k, \psi_k)) \end{array}$$

Then the S-valued points of $S \in (\mathrm{Sch}/\mathbb{F}_q)$ is given by tuples $\underline{\mathscr{E}} := (\underline{s}, (\mathscr{E}_j)_{j=0}^k, (\tau_j)_{j=0}^k)$ where

- $(\underline{s}, (\mathscr{E}_j)_{j=0}^k, (\tau_j)_{j=1}^k) \in \mathrm{Hk}_{\mathscr{G},\underline{I}}(S)$,

- $\tau_0 : {}^\sigma\mathscr{E}_k \xrightarrow{\sim} \mathscr{E}_0$ is an isomorphism of $\mathscr{G}$-torsors. Here $\sigma := \mathrm{id}_X \times \sigma_S$ and ${}^\sigma\mathscr{E}$ denotes the pullback of $\mathscr{E}$ under σ.

Note that we get a structure morphism $\mathrm{Sht}_{\mathscr{G},\underline{I}} \to X^I$ by projection to the legs which can be explictly written as $\underline{\mathscr{E}} = (\underline{s}, (\mathscr{E}_j), (\tau_j)) \mapsto \underline{s}$. Unfortunately, the moduli $\mathrm{Sht}_{\mathscr{G},\underline{I}}$ is not suitable for point counting as it is an ind-Deligne-Mumford stack that is ind-separated and locally of ind-finite type over X^I by [ARH21, Proposition 3.15]. Therefore, we introduce *bounded* global shtukas.

Let $S = \mathrm{Spec}\ R$ be an affine scheme over $\mathbb{F}_q$ and $\mathscr{Z}$ be a bound as in 2.6.1 (iii). Let $\underline{\mathscr{E}} \in (\mathrm{Sht}_{\mathscr{G},\underline{I}} \times_{X^I} X^I_{\mathscr{Z}})(S)$. For an étale cover $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$[1] such that $\hat{\Gamma}_{\underline{s}_{R'}}$ trivializes $\mathscr{E}_k$, i.e. there exists a trivialization $\varepsilon : \mathscr{E}_k|_{\hat{\Gamma}_{\underline{s}_{R'}}} \xrightarrow{\sim} \mathscr{G}|_{\hat{\Gamma}_{\underline{s}_{R'}}}$, we have

$$e = (\underline{s}, (\mathscr{E}_j|_{\hat{\Gamma}_{\underline{s}_{R'}}}), (\tau_j|_{\hat{\Gamma}_{\underline{s}_{R'}}}), \varepsilon) \in (\mathrm{Gr}_{\mathscr{G},\underline{I}} \times_{X^I} X^I_{\mathscr{Z}})(R') \tag{4.1}$$

Definition 4.2.4.

(i) Let $\mathscr{Z}$ be a bound. We say that $\underline{\mathscr{E}} \in (\mathrm{Sht}_{\mathscr{G},\underline{I}} \times_{X^I} X^I_{\mathscr{Z}})(S)$ is **bounded by** $\mathscr{Z}_{F'}$ for an affine scheme $S = \mathrm{Spec}\ R$ if for some representative Z of $\mathscr{Z}_{F'}$ over a finite extension $F'/F_{\mathscr{Z}}$, the morphism $e \times \mathrm{id}_{X_{F'}} : \mathrm{Spec}\ R' \times_{X^I_{\mathscr{Z}}} X^I_{F'} \to (\mathrm{Gr}_{\mathscr{G},\underline{I}} \times_{X^I} X^I_{F'})$ factors through $Z_{F'}$. Here $R \to R'$ is an étale cover that trivializes $\underline{E}$ and e is the element in $(\mathrm{Gr}_{\mathscr{G},\underline{I}} \times_{X^I} X^I_{\mathscr{Z}})(R')$ in (4.1). The definition is independent of the choice of the étale cover and the trivialization as $\mathscr{Z}$ is invariant under the left $\mathscr{L}^+_{X^I}\mathscr{G}$-action by definition.

(ii) We denote by $\mathrm{Sht}^{\mathscr{Z}}_{\mathscr{G},\underline{I}}$ the substack of $\mathrm{Sht}_{\mathscr{G},\underline{I}} \times_{X^I} X^I_{\mathscr{Z}}$ that parametrizes global $\mathscr{G}$-shtukas bounded by $\mathscr{Z}$. When $\mathscr{Z} = \mathrm{Gr}^{\mu}_{\mathscr{G},\underline{I}}$ is the BD-Schubert variety corresponding to an I-tuple

[1] such étale cover exists by [HR20, Lemma 3.4]

of cocharacters $\underline{\mu}$, we simply write $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I}}$.

Proposition 4.2.5. *Let $\mathscr{G}$ be a smooth affine group scheme over the curve X and $K = K_T \times \prod_{t \notin T} \mathscr{G}(\mathscr{O}_x)$ be an compact open subgroup of $G(\mathbb{A})$. Then the stack $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I}}$ is a Deligne-Mumford stack locally of finite type and separated over $(X \backslash T)^I$.*

Proof. This is proved in [ARH21, Theorem 3.15]. □

Definition 4.2.6. Let $\underline{\mathscr{E}}$ and $\underline{\mathscr{E}}'$ be two global $\mathscr{G}$-shtukas over S with the same legs. A **quasi-isogeny** between two global $\mathscr{G}$-shtukas $\underline{\mathscr{E}} \dashrightarrow \mathscr{G}'$ is a tuple of isomorphisms $f_j : \mathscr{E}_j \xrightarrow{X_S \backslash D_S} \mathscr{E}'_j$ for all $0 \leq j \leq k$ away from some effective Cartier divisor D of X such that $f_j \circ \tau_j = \tau'_j \circ f_{j-1}$ for all $1 \leq j \leq k$ and $\tau'_0 \circ {}^{\sigma} f_k = f_0 \circ \tau_0$.

4.3 Associated Local Shtukas

We fix two distinct closed points x and ∞ of X. Let $\mathbb{A}^{\infty,x} := \prod'_{t \in |X| \backslash \{\infty,x\}} (F_t, \mathscr{O}_t)$ be the ring of adeles away from x and ∞. The aim of this section is to define level structures defined by an open compact subgroup of the form $K^x \subset G(\mathbb{A}^{\infty,x})$ and an open compact subgroup $K = K_x K^x$ of $G(\mathbb{A}^\infty)$ where K_x is an open normal subgroup of $\mathscr{G}(\mathscr{O}_x)$. With abuse of notation, we may still denote by K the subgroup $K_x K^x \mathscr{G}(\mathscr{O}_\infty)$ of $G(\mathbb{A})$ when there is no confusion. The main references are [HK24, Section 3.4] and [Neu16, Section 3.2].

Recall that $\underline{I} = I_1 \sqcup \cdots \sqcup I_k$ is a partition of I. To define level structures, we need to first attach $(k+1)$-many local shtukas to a global $\mathscr{G}$-shtuka for each closed point of X. Let $\underline{\mathscr{E}} = ((s_i)_{i \in I}, (\mathscr{E}_j)_{j=0}^k, (\tau_j)_{j=0}^k)$ be a global $\mathscr{G}$-shtukas over $S \in (\mathrm{Sch}/\mathbb{F}_q)$. For any closed point

$t \in |X|$ and $0 \le j \le k$, define

$$\underline{\mathscr{E}}[t^\infty] := ((\mathscr{E}_j[t^\infty])_{j=0}^k, (\tau_{j,t})_{j=0}^k) \tag{4.2}$$

and

$$\underline{\mathscr{E}_j}[t^\infty] := (\mathscr{E}_j[t^\infty], \hat{\tau}'_{j,t}) \tag{4.3}$$

as follows. To begin with, $\mathscr{E}_j[t^\infty]$ is a $\mathscr{G}_t$-torsor obtained by the base change of $\mathscr{E}_j$ to $\mathrm{Spf}\ \mathscr{O}_t \widehat{\times}_{\mathbb{F}_q} S$ under the composition $\mathrm{Spf}\ \mathscr{O}_t \widehat{\times}_{\mathbb{F}_q} S \to \mathrm{Spec}\ \mathscr{O}_t \times_{\mathbb{F}_q} S \to X_S$. The fiber product $\widehat{\times}$ is taken in the category of ind-schemes. By [Neu16, Lemma 3.2.1 and Lemma 3.2.4], there exists equivalences of categories

$$\left\{ \begin{array}{c} \mathscr{G}_t\text{-torsors over} \\ \mathrm{Spf}\ \mathscr{O}_t \widehat{\times}_{\mathbb{F}_q} S \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} L^+\mathscr{G}_t\text{-torsors over} \\ \{t\} \times_{\mathbb{F}_q} S \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} L^+\mathrm{Res}_{\mathscr{O}_t/\mathbb{F}_q[\![\varpi_t]\!]}\mathscr{G}_t\text{-torsors} \\ \text{over } S \end{array} \right\}.$$

For now, we view $\mathscr{E}_j[t^\infty]$ as a $L^+\mathscr{G}_t$-torsor over $\{t\} \times S$.

Note that $\tau_j : \mathscr{E}_{j-1} \dashrightarrow \mathscr{E}_j$ might not descend to a morphism of $\mathscr{E}_{j-1}[t^\infty] \to \mathscr{E}_j[t^\infty]$ as the morphism is an isomorphism away from the graph $\Gamma_{\underline{s}_j} := \bigcup_{i \in I_j} \Gamma_{s_i}$. In particular, τ_j won't descend if t is in the image of s_i for some $i \in I_j$. However, τ_j induces an isomorphism of $L\mathscr{G}_t$-torsors $L\mathscr{E}_{j-1}[t^\infty] \to L\mathscr{E}_j[t^\infty]$ over $\{t\} \times_{\mathbb{F}_q} S$ which we call $\tau_{j,t}$. We now denote by

$$\hat{\tau}'_{j,t} := \hat{\tau}_{j,t} \circ \cdots \circ \hat{\tau}_0 \circ {}^\sigma\hat{\tau}_{k,t} \circ \cdots \circ {}^\sigma\hat{\tau}_{j+1,t} : {}^{\sigma_t}L\mathscr{E}_j[t^\infty] \xrightarrow{\sim} L\mathscr{E}_j[t^\infty].$$

the composition over $\{t\} \times S$. Here $\sigma_t := \mathrm{id}_{\mathbb{F}_t} \times_{\mathbb{F}_q} \sigma_S$. By [Neu16, Lemma 3.2.1], there exists an equivalences of categories

$$\left\{\begin{array}{c} L\mathscr{G}_t\text{-torsors over} \\ \\ \{t\}\times_{\mathbb{F}_q} S \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} L\mathrm{Res}_{\mathscr{O}_t/\mathbb{F}_q[\![\varpi_t]\!]}\mathscr{G}_t\text{-torsors} \\ \\ \text{over } S \end{array}\right\}.$$

Therefore $\underline{\mathscr{E}_j}[t^\infty]$ from (4.3) is a local $\mathrm{Res}_{\mathscr{O}_t/\mathbb{F}_q[\![\varpi_t]\!]}\mathscr{G}_t$-shtukas over S. Note that each $\tau_j : \mathscr{E}_{j-1} \dashrightarrow \mathscr{E}_j$ induces an quasi-isogeny $\hat{\tau}_{j,t} : \underline{\mathscr{E}_{j-1}}[t^\infty] \to \underline{\mathscr{E}_j}[t^\infty]$ of local $\mathrm{Res}_{\mathscr{O}_t/\mathbb{F}_q[\![\varpi_t]\!]}\mathscr{G}_t$-shtukas.

Suppose $t \notin \bigcup_{i\in I_j} \mathrm{im}(s_i)$ for some j. Then since $\tau_j : \mathscr{E}_{j-1} \dashrightarrow \mathscr{E}_j$ is an isomorphism away from $\Gamma_{\underline{s}_j} := \bigcup_{i\in I_j} \Gamma_{s_i}$, the corresponding quasi-isogeny $\hat{\tau}_{j,t} : \underline{\mathscr{E}_{j-1}}[t^\infty] \to \underline{\mathscr{E}_j}[t^\infty]$ is an isomorphism. In the special case where the $t \notin \bigcup_{i\in I} \mathrm{im}(s_i)$, the map τ_j' exists over $\mathrm{Spf}\ \mathscr{O}_t \widehat{\times}_{\mathbb{F}_q} S$. Therefore, the map $\hat{\tau}_j'$ in (4.3) comes from ${}^\sigma\mathscr{E}_j[t^\infty] \to \mathscr{E}_j[t^\infty]$. In other words, $\underline{\mathscr{E}_j}[t^\infty]$ is an étale $\mathrm{Res}_{\mathscr{O}_t/\mathbb{F}_q[\![\varpi_t]\!]}\mathscr{G}_t$-local shtuka. Furthermore, each quasi-isogeny $\hat{\tau}_j$ is an isomorphism, so all $\underline{\mathscr{E}_j}[t^\infty]$ can be constructed from $\underline{\mathscr{E}_0}[t^\infty]$. In this situation, we call $\underline{\mathscr{E}_0}[t^\infty]$ to be the **local $\mathrm{Res}_{\mathscr{O}_t/\mathbb{F}_q[\![\varpi_t]\!]}\mathscr{G}_t$-shtukas associated to $\underline{\mathscr{E}}$ at t.**

4.4 Adelic Level Structures

Following [Neu16, Section 3.4] and [HK24, Section 3.6], we define adelic level structures using Tate modules of local shtukas. Recall that if $\underline{\mathscr{L}} := (\mathscr{L}, \hat{\tau})$ is an étale local shtuka over S, we defined the t-adic Tate module of $\mathscr{L}$ to be $\mathscr{T}_{\underline{\mathscr{L}}} := \underline{\mathrm{Isom}}((L^+\mathscr{G}_S, \sigma), \underline{\mathscr{L}})$. We denote by $\mathscr{T}_{\underline{\mathscr{E}}^{\mathrm{univ}}[t^\infty]}$ to be the t-adic Tate module of the associated local shtuka of the universal global shtuka $\underline{\mathscr{E}}^{\mathrm{univ}}(t)$ over $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I}}|_{X^I_{\underline{\mu}}\setminus\pi^{-1}((t)_{i\in I})}$.

Definition 4.4.1. Let $T = \{t_1, \ldots, t_n\}$ be a finite subset of $|X|$. We define the **moduli space of**

$\mathscr{G}$**-shtukas with infinite level structures at** T **bounded by** $\underline{\mu}$ to be

$$\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},T} := \mathscr{T}_{\underline{\mathscr{E}^{\mathrm{univ}}}[t_1^\infty]} \times_{\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I}}} \cdots \times_{\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I}}} \mathscr{T}_{\underline{\mathscr{E}^{\mathrm{univ}}}[t_n^\infty]}$$

Then the S-valued points are given by $(\underline{\mathscr{E}}, (\psi_t)_{t\in T})$ where $\underline{\mathscr{E}} = (\underline{s}, (\mathscr{E}_j), (\tau_j))$ is a global $\mathscr{G}$-shtuka in $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I}}(S)$ such that $\underline{s}$ is in $(X^I_{\underline{\mu}} \backslash \pi^{-1}(T^I))(S)$ and $\psi_t : (L^+\mathrm{Res}_{\mathscr{O}_t/\mathbb{F}_q[\![\varpi_t]\!]}\mathscr{G}_t, \sigma) \xrightarrow{\sim} \underline{\mathscr{E}}[t^\infty]$ is an isomorphism of local shtukas.

Now we consider open compact subgroup K of $\mathscr{G}(\mathbb{O})$ of the form

$$K = K_T \times \prod_{t\notin T} \mathscr{G}(\mathscr{O}_t) \tag{4.4}$$

where $K_T \subset \prod_{t\in T} \mathscr{G}(\mathscr{O}_t)$. As $\mathrm{Aut}(L^+\mathrm{Res}_{\mathscr{O}_t/\mathbb{F}_q[\![\varpi_t]\!]}\mathscr{G}_t) = \mathscr{G}(\mathscr{O}_t)$, the element $(g_t)_{t\in T}$ in $\prod_{t\in T} \mathscr{G}(\mathscr{O}_t)$ acts on $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},T}$ via post-composition, i.e. $(\underline{E}, (\psi_t)_{t\in T}) \cdot (g_t) = (\underline{E}, (\psi_t \cdot g_t)_{t\in T})$. We define the **moduli space of** $\mathscr{G}$**-shtukas with level** K**-structure bounded by** $\underline{\mu}$ to be the quotient stack $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},K} := [\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},T}/K_T]$.

Proposition 4.4.2. *Let $S = \mathrm{Spec}\,\Omega$ be a spectrum of a algebraically closed field. The S-points of $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},K}$ is given by a pair $(\underline{\mathscr{E}}, \psi K)$ where $\underline{\mathscr{E}} \in \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},T}(S)$ and $\psi K \in \prod_{t\in T} \mathscr{T}_{\underline{\mathscr{E}}[t^\infty]}/K$.*

Proof. Let $\underline{K}$ be the constant group scheme associated to K. The S-valued points are $(\mathscr{B}, \mathscr{B} \xrightarrow{\varphi} \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},K})$ where $\mathscr{B}$ is a $\underline{K}$-torsor over S and the latter map is $\underline{K}$-equivariant map. Since Ω is algebraically closed field, the global section $\mathscr{B}(S)$ is not empty. Then there exists an isomorphism $\mathscr{B}(S) \xrightarrow{\sim} K$ as S is connected. Therefore giving $\underline{K}$-equivariant map φ is the same as giving $\{g \in K \mid (\underline{\mathscr{E}}, \psi g)\}$, i.e. $(\underline{\mathscr{E}}, \psi K)$ with $\psi K \in \prod_{t\in T} \mathscr{T}_{\underline{\mathscr{E}}[t^\infty]}/K$. $\square$

Remark 4.4.3.

(i) Let $K' \subset K \subset \mathscr{G}(\mathbb{O})$ where K is of the form (4.4) and $K' = K'_{T'} \times \prod_{t \notin T} \mathscr{G}(\mathscr{O}_t)$. Then we can replace both T and T' by its union $T'' = T \cup T'$ such that both are of the form $K = K_{T''} \times \prod_{t \notin T''} \mathscr{G}(\mathscr{O}_t)$ and $K' = K'_{T''} \times \prod_{t \notin T''} \mathscr{G}(\mathscr{O}_t)$.

(ii) Another common level structures on global $\mathscr{G}$-shtukas is given by finite closed subschemes $D \subset X$. A level D-structure on a $\mathscr{G}$-torsor $\mathscr{E}$ over X_S is a trivialization

$$\psi : \mathscr{E} \times_{X_S} D_S \xrightarrow{\sim} \mathscr{G} \times_X D_S$$

Then denote by $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},D}$ the stack fibered in groupoids over $(\mathrm{Sch}/\mathbb{F}_q)$ whose S-valued points are given by $(\underline{\mathscr{E}}, (\psi_j))$ where $\underline{\mathscr{E}} \in \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I}}(S)$ such that the sections are in $(X^I_{\underline{\mu}} \backslash \pi^{-1}(D^I))(S)$ and ψ_js are level D-structures on $\mathscr{E}_j$. For more detail, see [Var04], [Laf18], or [ARH21]. Let $K_D := \ker(\mathscr{G}(\mathbb{O}) \to \mathscr{G}(\mathscr{O}_D))$. Then using the same arguments in [ARH21, Theorem 6.5], one can show that $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},D}$ is isomorphic to $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},K_D}$.

Proposition 4.4.4.

(i) *The stack* $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},T}$ *is a* $\prod_{t \in T} \mathscr{G}(\mathscr{O}_t)$*-torsor on* $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I}} \times_{X^I_{\underline{\mu}}} (X^I_{\underline{\mu}} \backslash \pi^{-1}(T^I))$.

(ii) *The stack* $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},K} \to \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I}} \times_{X^I_{\underline{\mu}}} (X^I_{\underline{\mu}} \backslash \pi^{-1}(T^I))$ *is relatively representable by a finite étale morphism with Galois group* $\mathscr{G}(\mathbb{O})/K$. *In particular, let* $K' \subset K$ *be two open compact subgroups of* $\mathscr{G}(\mathbb{O})$ *of the form as in (4.4) such that* K' *is a normal subgroup of* K. *Then*

$$\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},K'} \to \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},K}$$

defined by natural quotient map $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},K'} \to [\mathrm{Sht}_{\mathscr{G},\underline{I},K'}/(K/K')] = \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},K}$ *is finite étale* *morphism with Galois group* K/K'.

Proof.

(i) For each $t \in T$, the Tate module $\mathscr{T}_{\underline{\mathscr{E}}^{\mathrm{univ}}[t^\infty]}$ is a $\mathscr{G}(\mathscr{O}_t)$-torsor for the pro-étale topology by Proposition 3.2.5. Therefore, $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},T}$ is a $\prod_{t\in T}\mathscr{G}(\mathscr{O}_t)$-torsor for the pro-étale topology.

(ii) In the case $K = K_D$ for some finite closed subscheme of X, the proposition is proved in [Var04, Proposition 2.16 b)] for constant group scheme $\mathscr{G} := \mathscr{G}_0 \times_{\mathbb{F}_q} X$ and in [ARH21, Theorem 3.15] for any flat affine group scheme $\mathscr{G}$ of finite type over X. Since we assumed that $K_T \subset \prod_{t\in T}\mathscr{G}(\mathscr{O}_t)$ is a compact open subgroup, there exists a finite closed subscheme $D \subset X$ such that $K_D \subset K$. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},K_D} & \xrightarrow{\quad f \quad} & \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},K} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I}} \times_{X^I_{\underline{\mu}}} (X^I_{\underline{\mu}} \backslash \pi^{-1}(T^I)) & \end{array}$$

where f is the natural quotient map. To check surjectivity of f, it suffices to look at $\operatorname{Spec}\Omega$-points Ω is an algebraically closed field extension of $\mathbb{F}_q$. But this is clear because the map is simply given by the forgetful morphism $(\underline{\mathscr{E}}, \psi K') \mapsto (\underline{\mathscr{E}}, \psi K)$ over $\operatorname{Spec}\Omega$ by Proposition 4.4.2. Furthermore, the map is étale. To see this, let $S \to \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},K}$ be a map where S is an $\mathbb{F}_q$-scheme. This is given as a (K/K')-torsor $\mathscr{B}$ with an (K/K')-equivariant map $\mathscr{B} \to \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},K'}$. Then the base change of f to S is exactly given by (K/K')-torsor $\mathscr{B} \to S$ which is étale as (K/K') is étale. Then by [Sta18, 02K6], q is étale.

□

4.5 Variant of Beauville-Laszlo and Rational Level Structures

We introduce a variant of Beauville-Laszlo glueing defined using local shtukas instead of torsors over the completed effective Cartier divisor $\hat{D}$ of D following [HK24, Section 4.3]. This way, we can generalize the definition of infinite level at a finite number of place to infinite numbers of places and obtain a finite étale cover from an arbitrary deep level at x (which we fixed from the beginning) over the generic fiber.

To begin with, let X be a curve over $\mathbb{F}_q$ and S be an $\mathbb{F}_q$-scheme. We consider D a relative effective Cartier divisor of X_S. We will define the punctured version of $\hat{D}$. Let $\mathscr{U}$ be the set of all affine opens $U = \operatorname{Spec} R \subset X_S$ such that $D \cap X_S = V(f)$ for some regular element f in R. Write $\mathscr{U}_D := \{U \cap D \mid U \in \mathscr{U}\}$. Then $\mathscr{U}_D$ is a form a basis for the topology on D.

Definition 4.5.1. Denote by $\hat{\mathscr{O}}^{\circ}_{X_S,D}$ to be a sheaf on $\mathscr{U}_D$ (hence a sheaf on D) defined by

$$\widehat{\mathscr{O}}^{\circ}_{X_S,D}(V) := \widehat{\mathscr{O}}_{X_S,D}(V) = \widehat{R}[f^{-1}]$$

where $V = (\operatorname{Spec} R) \cap D \in \mathscr{U}_D$ and $\widehat{R}$ is the f-adic completion of R. Then the **punctured formal neighborhood** of D is defined to be the topologically ringed space $\widehat{D}^{\circ} := (D, \widehat{\mathscr{O}}^{\circ}_{X_S,D})$.

To define the glueing data of the Beauville-Laszlo descent, we will construct two $\widehat{\mathscr{O}}^{\circ}_{X_S,D}$-sheaves where one is from a vector bundle $\mathscr{V}$ over $X_S \backslash D$ and the other is from a vector bundle over $\widehat{D}$.

Definition 4.5.2.

(i) Let $\mathscr{V}$ be a vector bundle on $X_S \backslash D$. The **analytification** $\mathscr{V}^{\text{an}}$ is sheaf on $\widehat{D}^{\circ}$ given by extending the sheaf on $\mathscr{U}_D$ defined by

$$V \mapsto \mathscr{V}(\widehat{\mathscr{O}}^{\circ}_{X_S,D}(V))$$

for $V \in \mathscr{U}_D$.

(ii) For a vector bundle $\mathscr{V}$ over $\widehat{D}$, define $\mathscr{V}[D^{-1}] := \mathscr{V} \otimes_{\widehat{\mathscr{O}}_{X_S,D}} \widehat{\mathscr{O}}^{\circ}_{X_S,D}$.

Then we can now formulate the reinterpretation of Beauville-Laszlo descent for vector bundles following [HK24, Proposition 4.6].

Proposition 4.5.3. *The functor $\mathscr{V} \mapsto (\mathscr{V}|_{X_S \backslash D}, \mathscr{V}|_{\widehat{D}}, \mathrm{can}_{\mathscr{V}})$ is an equivalence of categories between the category of vector bundles on X and category of triples $(\mathscr{V}, \mathscr{V}', \varphi)$ where $\mathscr{V}$ is an vector bundle over $X_S \backslash D$, $\mathscr{V}'$ is a vector bundle over $\widehat{D}$, and $\varphi : \mathscr{V}^{\mathrm{an}} \to \mathscr{V}[D^{-1}]$.*

Now we turn our attention to the case where $D = \{t\} \times S$ where $t \in |X|$. One can define the **analytification** of $\mathscr{E}^{\mathrm{an}}$ of a $\mathscr{G}$-torsor $\mathscr{E}$ on $X_S \backslash D$ to be a sheaf defined by $\mathrm{Spec}\ R \mapsto \mathscr{E}(\widehat{\mathscr{O}}_{X_S,D}(\mathrm{Spec}\ R)) = \mathscr{E}(R(\!(\varpi_t)\!))$ where ϖ_t is the local parameter corresponding to x in R. Then $\mathscr{E}^{\mathrm{an}}$ is a $\mathscr{G}$-torsor over $\widehat{D}^{\circ}$. Then similar to the case of vector bundles, one has an equivalence

$$\mathscr{E} \mapsto (\mathscr{E}|_{X_S \backslash D}, \mathscr{E}|_{\widehat{D}}, \varphi_{\mathscr{E}} : (\mathscr{E}|_{X_S \backslash D})^{\mathrm{an}} \xrightarrow{\sim} (\mathscr{G}^{\mathrm{an}} \times_{\mathscr{G}|_{\widehat{D}}} \mathscr{E}|_{\widehat{D}})) \tag{4.5}$$

of categories between the category of $\mathscr{G}$-torsors over X_S and the category of triples $(\mathscr{E}, \mathscr{E}', \varphi)$ where $\mathscr{E}$ is a $\mathscr{G}$-torsor over $X_S \backslash D$, $\mathscr{E}'$ is a $\mathscr{G}$-torsor over $\widehat{D} := \mathrm{Spf}\ \mathscr{O}_t \times S$, and $\varphi : \mathscr{E}^{\mathrm{an}} \to (\mathscr{G}^{\mathrm{an}} \times_{\mathscr{G}|_{\widehat{D}}} \mathscr{E}')$ is an isomorphism.

To show that (4.5) is an equivalence, first observe that there exists a vector bundle $\mathscr{V}$ and a line bundle $\mathscr{L} \subset \mathscr{V}^{\otimes}$ such that $\mathscr{G} = \mathrm{Aut}(\mathscr{V}, \mathscr{L})$ by [Bro13, Theorem 1.1]. Hamacher and Kim then viewed $\mathscr{G}$-torsors over X_S as a $(\mathscr{V},\mathscr{L})$-twists [Bro13, Corollary 1.4] which is a pair $(\mathscr{V}',\mathscr{L}')$ where $\mathscr{V}'$ is a vector bundle with a line bundle $\mathscr{L}' \subset \mathscr{V}'^{\otimes}$ that is étale-locally isomorphic to $(\mathscr{V},\mathscr{L})$. Then one can deduce the Beauville-Lazslo descent lemma for $\mathscr{G}$-torsors from the results for vector bundles. Under the identification

$$\left\{ \mathscr{G}\text{-torsors over } \widehat{D} \right\} \leftrightarrow \left\{ L^+\mathrm{Res}_{\mathscr{O}_t/\mathbb{F}_q[\![\varpi_t]\!]}\mathscr{G}_t\text{-torsors over } S \right\}$$
$$\left\{ \mathscr{G}^{\mathrm{an}}\text{-torsors over } \widehat{D}^\circ \right\} \leftrightarrow \left\{ L\mathrm{Res}_{\mathscr{O}_t/\mathbb{F}_q[\![\varpi_t]\!]}\mathscr{G}_t\text{-torsors over } S \right\}$$

We can replace the category of triples in (4.5) with the category of triples $(\mathscr{E},\mathscr{E}',\varphi)$ where $\mathscr{E}'$ is a $L^+\mathrm{Res}_{\mathscr{O}_t/\mathbb{F}_q[\![\varpi_t]\!]}\mathscr{G}_t$-torsors over S and φ is an isomorphism of $L\mathrm{Res}_{\mathscr{O}_t/\mathbb{F}_q[\![\varpi_t]\!]}\mathscr{G}_t$-torsors over S.

Finally, we define the adelic loop groups. Let T be a subset of $|X|$ such that $(X\backslash T, \mathscr{O}_X|_{X\backslash T})$ is a scheme. Define adeles at T and integral adeles at T to be

$$\mathbb{A}_T := \prod_{t\in T}{}^{'}(F_t,\mathscr{O}_t) \quad \text{and} \quad \mathbb{O}_T := \prod_{t\in T}\mathscr{O}_t$$

Then for any $\mathbb{F}_q$-algeba R, we define $\mathbb{A}_T(R) := (R\widehat{\otimes}_{\mathbb{F}_q}\mathbb{O}_T)\otimes_{\mathbb{O}_T}\mathbb{A}_T$ and view $\mathbb{A}_T$ as an fpqc-sheaf on $\mathbb{F}_q$-algebras. Then the adelic loop group at T is given by

$$L_{\mathbb{A}_T}\mathscr{G}(R) := \prod_{t\in T}{}^{'}(L\mathrm{Res}_{\mathscr{O}_t/\mathbb{F}_q[\![\varpi_t]\!]}\mathscr{G}_t(R), L^+\mathrm{Res}_{\mathscr{O}_t/\mathbb{F}_q[\![\varpi_t]\!]}\mathscr{G}_t(R))$$

Now we would like to define two $L_{\mathbb{A}_T}G$-torsors over S from $\mathscr{G}$-torsor $\mathscr{E}$ over $(X\backslash T)_S$ and T-tuples of $L^+\mathrm{Res}_{\mathscr{O}_t/\mathbb{F}_q[\![\varpi_t]\!]}\mathscr{G}_t$-torsors so that we have a Beauville-Laszlo glueing data in terms of adelic loop groups. For a $\mathscr{G}$-torsor $\mathscr{E}$ over $(X\backslash T)_S$, define $\mathscr{E}^{\mathrm{an}}_{\mathbb{A}_T}$ by $\mathrm{Spec}(R) \mapsto \mathscr{E}(\mathbb{A}_T(R))$. For

$\mathscr{E}_t$ be $L^+\mathrm{Res}_{\mathscr{O}_t/\mathbb{F}_q[\![\varpi_t]\!]}\mathscr{G}_t$-torsors over S, we attach a $L_{\mathbb{A}_T}\mathscr{G}$-torsor

$$\mathscr{L}_{\mathbb{A}_T}((\mathscr{E}_t)_{t\in T}) := L_{\mathbb{A}_T}\mathscr{G} \times^{\prod_{t\in T} L^+\mathrm{Res}_{\mathscr{O}_t/\mathbb{F}_q[\![\varpi_t]\!]}\mathscr{G}_t} \prod_{t\in T}\mathscr{E}_t$$

The definition of $(-)^{\mathrm{an}}_{\mathbb{A}_T}$ and $\mathscr{L}_{\mathbb{A}_T}(-)$ can be extended to global $\mathscr{G}$-shtukas over $(X\backslash T)$ and T-tuples of local $\mathrm{Res}_{\mathscr{O}_t/\mathbb{F}_q[\![\varpi_t]\!]}\mathscr{G}_t$-shtukas over S.

Let $T \subset |X|$ be a subset such that the locally ringed space $(X\backslash T, \mathscr{O}_X|_{X\backslash T})$ is a scheme. For instance, this happens when T is finite or cofinite. Define the moduli of global $\mathscr{G}$-shtukas bounded by $\underline{\mu}$ with infinite level at T to be

$$\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},T} := \prod_{t\in T}\mathscr{T}_{\underline{\mathscr{E}^{\mathrm{univ}}}[t^\infty]}$$

where the product is over $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I}}$.

Proposition 4.5.4 ([HK24, Proposition 4.11, Proposition 4.13])**.**

(i) *There exists a fully faithful functor from* $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I}}$ *defined by*

$$\underline{\mathscr{E}} \mapsto (\underline{\mathscr{E}}|_{(X\backslash T)_S}, (\underline{\mathscr{E}}[t^\infty])_{t\in T}, \varphi)$$

where $\varphi : \underline{\mathscr{E}}^{\mathrm{an}}_{\mathbb{A}_T} \xrightarrow{\sim} \mathscr{L}_{\mathbb{A}_T}((\underline{\mathscr{E}}[t^\infty])_{t\in T})$ *where its essential image contains those triples such that the underlying* $L\mathrm{Res}_{\mathscr{O}_t/\mathbb{F}_q[\![\varpi_t]\!]}\mathscr{G}_t$*-torsors of* $\underline{\mathscr{E}}[t^\infty]$ *are trivial. When* T *is finite, then the functor is an equivalence of categories.*

(ii) *The* S*-valued points in* $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},T}$ *can be identified as a tuple* $(\underline{s}, (\mathscr{E}_j)_{j=0}^k, (\tau_j)_{j=0}^k, \psi)$ *where* $(\underline{s}, (\mathscr{E}_j)_{j=0}^k, (\tau_j)_{j=0}^k) \in \mathrm{Hk}^{\underline{\mu}}_{\mathscr{G}|_{X\backslash T},\underline{I}}(S)$ *and* $\psi : (L_{\mathbb{A}_T}G, \sigma) \xrightarrow{\sim} (\mathscr{E}^{\mathrm{an}}_{0,\mathbb{A}_T}, (\tau'_{0,t}))$ *is an isomorphism* *of $L_{\mathbb{A}_T}G$-isoshutkas.*

Now we are ready to define finite étale cover over the generic fiber. Let $K \subset \mathscr{G}(\mathbb{A})$ be an open compact subgroup such that $K = K_x K^x \mathscr{G}(\mathscr{O}_\infty)$ where K_x is an open compact normal subgroup of $\mathscr{G}(\mathscr{O}_x)$ and K^x is an open compact subgroup of $G(\mathbb{A}^{\infty,x})$ such that $[K : K_D]$ is finite for some finite closed subscheme $D \subset X$. Let $D = nx + D'$ as an effective divisor of X.

Proposition 4.5.5. *The projection map $\pi_{KK^x} : \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},K} \to \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},\mathscr{G}(\mathscr{O}_x)K^x\mathscr{G}(\mathscr{O}_\infty)}$ is a finite étale cover with Galois group $\mathscr{G}(\mathscr{O}_x)/K_x$.*

Proof. Suppose $T = |X|$. Then we have a commutative diagram over the generic fiber $\eta^I_{\underline{\mu}} := \prod_{i \in I} \eta_{\mu_i}$.

$$\begin{array}{ccc} \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},K_D} & \longrightarrow & \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},K} \\ \downarrow & & \downarrow \\ \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},K_{D'}} & \longrightarrow & \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},\mathscr{G}(\mathscr{O}_x)K^x\mathscr{G}(\mathscr{O}_\infty)} \end{array}$$

Here the horizontal maps are finite étale cover (hence surjective) with Galois groups K/K_D and $K^x/K_{D'}$ respectively. Then again by [Sta18, 02K6], $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},K} \to \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},\mathscr{G}(\mathscr{O}_x)K^x\mathscr{G}(\mathscr{O}_\infty)}$ is a finite étale cover over the generic fiber. □

4.6 Representability

Let $\lambda \in \Lambda^+$ where Λ^+ is the monoid of dominant rational cocharacters of SL_r. Fix a closed embedding of $\mathscr{G}^{\mathrm{ad}} \to \mathrm{SL}_r$. Define $\mathrm{Sht}^{\underline{\mu},\leq\lambda}_{\mathscr{G},\underline{I}}$ to be the open substack of $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I}}$ parametrizing global $\mathscr{G}$-shtukas such that $\mathscr{E}_0 \in \mathrm{Bun}^{\leq\lambda}_{\mathscr{G}}$. Then the collection $(\mathrm{Sht}^{\underline{\mu},\leq\lambda}_{\mathscr{G},\underline{I}})_{\lambda\in\Lambda^+}$ form an open cover of $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I}}$. In fact, $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I}}$ is an increasing union of $\mathrm{Sht}^{\underline{\mu},\leq\lambda}_{\mathscr{G},\underline{I}}$.

Proposition 4.6.1. *For every $\lambda \in \Lambda^+$, the stack $\mathrm{Sht}^{\underline{\mu},\leq\lambda}_{\mathscr{G},\underline{I},K}$ is a Deligne-Mumford stack locally of finite type. If K is small enough with respect to λ, then $\mathrm{Sht}^{\underline{\mu},\leq\lambda}_{\mathscr{G},\underline{I},K}$ is representable by a scheme locally of finite type.*

Proof. The composition $\mathscr{G} \to \mathscr{G}^{\mathrm{ad}} \to \mathrm{SL}_r$ yields a natural map

$$\mathrm{Sht}^{\underline{\mu},\leq\lambda}_{\mathscr{G},\underline{I},K} \to \mathrm{Sht}^{\underline{\mu}',\leq\lambda}_{\mathrm{SL}_r,\underline{I},K'} \times_{\mathrm{Sht}^{\underline{\mu}',\leq\lambda}_{\mathrm{SL}_r,\underline{I}}} \mathrm{Sht}^{\underline{\mu},\leq\lambda}_{\mathscr{G},\underline{I}}$$

which is representable by a finite morphism of schemes where K' and $\underline{\mu}'$ are the images of K and $\underline{\mu}$ in SL_r respectively. The stack $\mathrm{Sht}^{\underline{\mu}',\leq\lambda}_{\mathrm{SL}_r,\underline{I},K'}$ is representable by a scheme locally of finite type for sufficiently small K'. In fact, the connected components of $\mathrm{Sht}^{\underline{\mu},\leq\lambda}_{\mathscr{G},\underline{I},K}$ are quasi-projective schemes for small enough K' by [Var04, Proposition 2.16]. $\square$

In general, $\mathrm{Sht}^{\underline{\mu},\leq\lambda}_{\mathscr{G},\underline{I},K}$ may have infinitely many connected components. Therefore, we need to control the center in the following sense. Let Z be the center of G and $\Xi \subset Z(F)\backslash Z(\mathbb{A})$ be a cocompact lattice. This means that Ξ is a torsion-free subgroup such that $Z(F)\backslash Z(\mathbb{A})/\Xi$ is compact. Since we defined the Harder-Narashimhan truncation $\mathrm{Sht}^{\underline{\mu},\leq\lambda}_{\mathscr{G},\underline{I}}$ via the closed embedding $\iota : \mathscr{G}^{\mathrm{ad}} \to \mathrm{SL}_r$, the image of Z in SL_r is trivial. Therefore, the natural action of $Z(F)\backslash Z(\mathbb{A})$ (and hence Ξ) stabilizes $\mathrm{Sht}^{\underline{\mu},\leq\lambda}_{\mathscr{G},\underline{I},K}$. Therefore, it makes sense to take the quotient $\mathrm{Sht}^{\underline{\mu},\leq\lambda}_{\mathscr{G},\underline{I},\Xi K} := [\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},\Xi}/\Xi]$.

Proposition 4.6.2. *Let $\lambda \in \Lambda^+$. The stack $\mathrm{Sht}^{\underline{\mu},\leq\lambda}_{\mathscr{G},\underline{I},\Xi K}$ is a Deligne-Mumford stack of finite type. If K is small enough, $\mathrm{Sht}^{\underline{\mu},\leq\lambda}_{\mathscr{G},\underline{I},\Xi K}$ is a scheme of finite type.*

Proof. Let K^{ad} be the image of $K \subset \mathscr{G}(\mathscr{O}_x) \times G(\mathbb{A}^x)$ in $\mathscr{G}^{\mathrm{ad}}(\mathscr{O}_x) \times G^{\mathrm{ad}}(\mathbb{A}^x)$. Let $\underline{\mu}^{\mathrm{ad}}$ be the tuple (μ_i^{ad}) where $\mu_i^{\mathrm{ad}} : \mathbb{G}_m \to G_{\overline{F}} \to G^{\mathrm{ad}}_{\overline{F}}$ defined by composition. Recall from Proposition

2.2.4 that $\mathrm{Bun}_{\mathscr{G}^{\mathrm{ad}}}^{\leq\lambda}$ is an Artin stack of finite type over $\mathbb{F}_q$. This shows that the moduli $\mathrm{Sht}_{\mathscr{G}^{\mathrm{ad}},\underline{I},K^{\mathrm{ad}}}^{\underline{\mu}^{\mathrm{ad}}}$ is Deligne-Mumford stack of finite type. As $\mathrm{Sht}_{\mathscr{G},\underline{I},\Xi K}^{\underline{\mu}}$ is a finite étale cover of $\mathrm{Sht}_{\mathscr{G}^{\mathrm{ad}},\underline{I},K^{\mathrm{ad}}}^{\underline{\mu}^{\mathrm{ad}}}$ with Galois group $Z(F)\backslash Z(\mathbb{A})/(\Xi(Z(\mathbb{A})\cap K))$. It follows that $\mathrm{Sht}_{\mathscr{G},\underline{I},\Xi K}^{\underline{\mu},\leq\lambda}$. See [Laf18, Lemme 12.19] and [HK24, Corollary 4.22] for more details. □

Recall that the S-point in $\mathrm{Sht}_{\mathscr{G},\underline{I},\Xi K}^{\underline{\mu},\leq\lambda}$ for $\mathbb{F}_q$-scheme S is of the form $(\underline{\mathscr{E}},\psi\Xi K)$. The morphism $\mathrm{Sht}_{\mathscr{G},\underline{I},\Xi K}^{\underline{\mu}} \to \mathrm{Sht}_{\mathscr{G},\underline{I},\mathscr{G}(\mathscr{O}_x)K^x\mathscr{G}(\mathscr{O}_\infty)}^{\underline{\mu}}$ descends to a morphism

$$\mathrm{Sht}_{\mathscr{G},\underline{I},\Xi K}^{\underline{\mu},\leq\lambda} \to \mathrm{Sht}_{\mathscr{G},\underline{I},\Xi\mathscr{G}(\mathscr{O}_x)K^x\mathscr{G}(\mathscr{O}_\infty)}^{\underline{\mu},\leq\lambda}$$

as $\underline{\mathscr{E}}$ is unchanged. Therefore the same argument shows that this is an finite étale cover.

4.7 Moduli of Shtukas with two legs and with one leg fixed

We fix two closed distinct points $x,\infty \in |X|$. The place x will be playing the role of the place at bad reduction and ∞ will be the place where we fix one of the legs. The reason why we do this is to avoid having to use the general theory of nearby cycles sheaves over higher dimensional base scheme. With two legs and one leg fixed at ∞, our moduli of global shtukas will be defined over a one-dimensional base.

Let $\underline{\mu} = (\mu_x,\mu_\infty)$ be a pair of conjugacy classes of cocharacters of $G_{\overline{F}}$. We defined $X_{\underline{\mu}}^I$ with $I=\{1,2\}$ to be the product $X_{\mu_x}\times_{\mathbb{F}_q} X_{\mu_\infty}$. To fix a leg at ∞, we need to first choose a point ∞' over X_{μ_∞} over ∞. Then we will base change the moduli space of global shtukas along the map $X_{\mu_x}\times_{\mathbb{F}_q}\infty' \to X_{\mu_x}\times_{\mathbb{F}_q} X_{\mu_\infty}$. Let us simply write $X_{\underline{\mu}}$ (without the I) for $X_{\mu_x}\times_{\mathbb{F}_q}\infty'$.

Let $K\subset G(\mathbb{A})$ of the form $K_T\times\prod_{t\notin T}\mathscr{G}(\mathscr{O}_t)\times\mathscr{G}(\mathscr{O}_\infty)$ with $K_T\subset G(\mathbb{A}_T)$ where T is some finite subset of $|X|$ and $\mathbb{A}_T$ is the ring of adeles at T. At this point, T may contain our fixed closed point x. Denote by $T^I_{\underline{\mu}}$ and $T_{\underline{\mu}}$ to be the inverse image of T^I in $X^I_{\underline{\mu}}$ and the inverse image of T in $X_{\underline{\mu}}$ respectively. Finally, Ξ is a cocompact lattice in $Z(F)\backslash Z(\mathbb{A})$.

Definition 4.7.1. We denote by

$$\begin{aligned} \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\Xi K} &:= \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\{1\}\sqcup\{2\},\Xi K} \times_{(X^I_{\underline{\mu}}\backslash T^I_{\underline{\mu}})} (X_{\underline{\mu}}\backslash T_{\underline{\mu}}) \\ {}'\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\Xi K} &:= \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\{1,2\},\Xi K} \times_{(X^I_{\underline{\mu}}\backslash T^I_{\underline{\mu}})} (X_{\underline{\mu}}\backslash T_{\underline{\mu}}) \end{aligned}$$

Therefore the projection morphism defines both stacks as a Deligne-Mumford stack of finite type over $(X_{\underline{\mu}}\backslash T_{\underline{\mu}})$ and even a scheme of finite type if K is small enough by Proposition 4.6.2. Explicitly, the S-points of $\mathrm{Sht}^{\underline{\mu},\leq\lambda}_{\mathscr{G},\Xi K}$ is given by

$$\underline{\mathscr{E}} := (s, \mathscr{E}, \mathscr{E}', \tau_0, \tau_1, \psi)$$

where $s : S \to (X_{\underline{\mu}}\backslash T_{\underline{\mu}})$ is the **leg** of $\underline{\mathscr{E}}$, $(s, \mathscr{E}, \mathscr{E}', \tau_0, \tau_1) \in \mathrm{Hecke}^{\underline{\mu}}_{\mathscr{G},\{1\}\sqcup\{2\}}(S)$, and ψ is a level ΞK-structure. The set of S-points of ${}'\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\Xi K}$ is given similarly.

To use nearby cycle sheaves on the moduli space of shtukas, we need to view the moduli of global shtuka as a scheme over a henselian trait. Let y be a point of X_{μ_x} above x. Then $(y, \infty') \in X_{\mu_x} \times \infty'$. Write $\mathscr{O}_y := \widehat{\mathscr{O}}_{\mathscr{O}_{\mu_x},y}$ to be the completed local ring of X_{μ_x} at y. Also denote by $\mathbb{F}_{\infty'}$ the residue field of ∞'. Fix a uniformizer ϖ_y of $\mathscr{O}_y$ and fix an isomorphism $\mathscr{O}_y \cong \mathbb{F}_y[\![\varpi_y]\!]$. Since $\mathrm{Spec}\, \mathscr{O}_y \times_{\mathbb{F}_q} \{\infty'\} = \mathrm{Spec}(\mathbb{F}_y[\![\varpi_y]\!] \otimes_{\mathbb{F}_q} \mathbb{F}_{\infty'})$ is not a spectrum of a DVR, we consider the composition

$$\mathrm{Spec}\, \mathbb{F}_{y,\infty'}[\![\varpi_y]\!] \to \mathrm{Spec}\, \mathscr{O}_y \times_{\mathbb{F}_q} \{\infty'\} \to X_{\mu_x} \times_{\mathbb{F}_q} \{\infty'\}$$

where $\mathbb{F}_{y,\infty'}$ is the compositum of $\mathbb{F}_y$ and the $\mathbb{F}_{\infty'}$. The map $\mathbb{F}_y[\![\varpi_y]\!] \otimes_{\mathbb{F}_q} \mathbb{F}_{\infty'} \to \mathbb{F}_{y,\infty'}[\![\varpi_y]\!]$ is defined using the universal property of tensor products. Namely, we use the obvious inclusions $\mathbb{F}_y[\![\varpi_y]\!] \to \mathbb{F}_{y,\infty'}[\![\varpi_y]\!]$ and $\mathbb{F}_{\infty'} \to \mathbb{F}_{y,\infty'}[\![\varpi_y]\!]$.

Definition 4.7.2. (i) Write $\mathscr{O}_{y,\infty'} := \mathbb{F}_{y,\infty'}[\![\varpi_y]\!]$ and call its maximal ideal $\mathfrak{m}_{y,\infty'}$. Let $E_{y,\infty'} := \mathbb{F}_{y,\infty'}(\!(\varpi_y)\!)$ be the fraction field of $\mathscr{O}_{y,\infty'}$. If we denote by $S_{y,\infty'} := \mathrm{Spec}(\mathscr{O}_{y,\infty'})$, $s_{y,\infty'} := \mathrm{Spec}(\mathscr{O}_{y,\infty'}/\mathfrak{m}_{y,\infty'})$, and $\eta_{y,\infty'} := \mathrm{Spec}(E_{y,\infty'})$, then the triple $(S_{y,\infty'}, s_{y,\infty'}, \eta_{y,\infty'})$ is an henselian trait.

(ii) We denote by

$$\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi K,\mathscr{O}_{y,\infty'}} := \mathrm{Sht}^{\mu}_{\mathscr{G},\Xi K} \times_{X_{\underline{\mu}} \backslash T_{\underline{\mu}}} \mathrm{Spec}\ \mathscr{O}_{y,\infty'}$$

Let S be a scheme over $\mathrm{Spec}\ \mathscr{O}_{y,\infty'}$. In the case $\underline{\mathscr{E}} \in \mathrm{Sht}^{\mu}_{\mathscr{G},\Xi K,\mathscr{O}_{y,\infty'}}(S)$ is such that the section $s : S \to X$ factors through $\mathrm{Spf}\ \mathscr{O}_{y,\infty'}$, we would like to attach a local shtuka at the legs in a slightly different setting compared to Section 4.3. Let $t \in \{x, \infty\}$ and $\deg(t) = [\mathbb{F}_t : \mathbb{F}_q]$ be the inertial degree of t. We have the decomposition

$$\mathrm{Spf}\ \mathscr{O}_{y,\infty'} \widehat{\times}_{\mathbb{F}_q} S \cong \coprod_{j \in \mathbb{Z}/(\deg(t))} V(\mathfrak{a}_{t,j})$$

where $\mathfrak{a}_{t,j} := \langle a \otimes 1 - 1 \otimes s^*(a)^{q^j} \mid a \in \mathbb{F}_t \rangle$ is the ideal, and $V(\mathfrak{a}_{t,j})$ is the closed subset defined by $\mathfrak{a}_{t,j}$ in $\mathrm{Spf}\ \mathscr{O}_{y,\infty'} \widehat{\times}_{\mathbb{F}_q} S$. Then by above, we have decomoposition of the group as

$$\mathscr{G} \widehat{\times}_{X_S} (\mathrm{Spf}\ \mathscr{O}_{y,\infty'} \widehat{\times}_{\mathbb{F}_q} S) \cong \coprod_{j \in \mathbb{Z}/(\deg\ t)} \mathscr{G} \widehat{\times}_{X_S} V(\mathfrak{a}_{t,j}).$$

Then the pair $(\mathscr{G} \widehat{\times}_{X_S} V(\mathfrak{a}_{t,0}), \tau_0^{\deg(t)})$ can be viewed as a local $\mathscr{G}_t$-shtuka over S where $\mathscr{G}_t :=$

$\mathscr{G} \times_X \operatorname{Spec} \mathscr{O}_t$. See [ARH14, §5.2] for more details.

Definition 4.7.3. Let $\underline{\mathscr{E}} \in \mathrm{Sht}^{\mu}_{\mathscr{G},\Xi K,\mathscr{O}_{y,\infty'}}(S)$ such that the section $s : S \to X$ factors through $\operatorname{Spf} \mathscr{O}_{y,\infty'}$. Let $t \in \{x, \infty\}$. We write

$$\underline{\mathscr{L}}_t(\underline{\mathscr{E}}) := (\mathscr{E} \widehat{\times}_{X_S} V(\mathfrak{a}_{t,0}), \tau_0^{\deg t})$$

for the **local $\mathscr{G}_t$-shtuka associated to $\underline{\mathscr{E}}$ at** t.

Remark 4.7.4. There exists a fully faithful functor from the category of local $\mathscr{G}_t$-shtukas to the category of local $\operatorname{Res}_{\mathscr{O}_t/\mathbb{F}_q[\![\varpi_t]\!]}\mathscr{G}_t$-shtukas by [ARH14]. Therefore, the local $\mathscr{G}_t$-shtuka $\underline{\mathscr{L}}_t(\underline{\mathscr{E}})$ may be viewed as a local $\operatorname{Res}_{\mathscr{O}_t/\mathbb{F}_q[\![\varpi_t]\!]}\mathscr{G}_t$. However, this setup is more useful when we form Serre-Tate theory and local-global compatibility in Chapter 5.

Chapter 5: Counting Points

5.1 Setup

We establish the notations and assumptions that will be used throughout this chapter. Fix closed points x and ∞ of X. Let $\underline{\mu} = (\mu_t)_{t\in\{x,\infty\}}$ be a pair of conjugacy classes of cocharacters of $G_{\overline{F}}$ with reflex field $X^I_{\underline{\mu}}$. Choose a point ∞' in X_{μ_∞} lying over ∞ and denote by $X_{\underline{\mu}} := X_{\mu_x} \times_{\mathbb{F}_q} \infty'$ which comes with a generically étale cover $\pi : X_{\underline{\mu}} \to X$. We denote by $\eta_{\underline{\mu}} := \mathrm{Spec}\ F_{\underline{\mu}}$ the generic fiber of $X_{\underline{\mu}}$. Let G be a connected reductive group over F. We fix a parahoric group scheme over X with generic fiber G. Let Ξ be a cocompact lattice in $Z(F)\backslash Z(\mathbb{A})$. Fix a point y in $X_{\underline{\mu}}$ over x.

We will only consider the level structures of the form

$$K = K_x K^x \mathscr{G}(\mathscr{O}_\infty) \tag{5.1}$$

where $K_x \subset \mathscr{G}(\mathscr{O}_x)$ and $K^x \subset G(\mathbb{A}^{\infty,x})$ with $K^x = K_T \times \prod_{t\notin T} \mathscr{G}(\mathscr{O}_t)$ for some finite subset $T \subset |X|\backslash\{\infty, x\}$. We would like to study the trace of finite Hecke correspondence on $\mathrm{Sht}^{\mu,\le\lambda}_{\mathscr{G},\Xi K}$ for $\lambda \in \Lambda^+$.

Remark 5.1.1. Let $g \in \mathscr{G}(\mathscr{O}_x) \times G(\mathbb{A}^x)$ and choose a open compact subgroup $K \subset \mathscr{G}(\mathscr{O}_x) \times G(\mathbb{A}^x)$. Let $\mathbf{1}_{KgK}$ be the characteristic function of KgK. Unfortunately, the Hecke correspondence

$[KgK]$ that we will define in the later part of the chapter does not stabilize $\mathrm{Sht}^{\mu,\leq\lambda}_{\mathscr{G},\Xi K}$. Instead, there exists $\lambda' \in \Lambda^+$ such that the correspondence yields a morphism

$$R\Gamma([KgK]) : R\Gamma_c(\mathrm{Sht}^{\mu,\leq\lambda}_{\mathscr{G},\Xi K}) \to R\Gamma_c(\mathrm{Sht}^{\mu,\leq\lambda+\lambda'}_{\mathscr{G},\Xi K})$$

This is why we have to impose the following assumption throughout the chapter.

Assumption.

(i) We assume that G^{der} is simply connected.

(ii) We assume that the local group G_x is unramified, i.e. $\mathscr{G}_x$ is a connected reductive group scheme over $\mathrm{Spec}\, \mathscr{O}_x$. We also assume that $\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi}$ is quasi-compact (and hence $\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi K}$ is quasi-compact for all level structures K). Here we view Ξ as a subgroup of $G(\mathbb{A})$. Let

$$\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi K^x,\mathscr{O}_{y,\infty'}} := \mathrm{Sht}^{\mu}_{\mathscr{G},\Xi\mathscr{G}(\mathscr{O}_x)K^x\mathscr{G}(\mathscr{O}_\infty)} \times_{(X_{\underline{\mu}}\backslash T_{\underline{\mu}})} \mathrm{Spec}\, \mathscr{O}_{y,\infty'}$$

be the base change to $\mathrm{Spec}\, \mathscr{O}_{y,\infty'}$. We further assume that the structure morphism $\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi K^x,\mathscr{O}_{y,\infty'}} \to \mathrm{Spec}\, \mathscr{O}_{y,\infty'}$ is proper.

Remark 5.1.2.

(i) Assumption (i) should be a condition that can relieved. However, assumption (ii) is necessary because in general, the Hecke correspondence does not stabilize the Harder-Narashimhan filtration. Also, properness was necessary in order to work with nearby cycle sheaves.

(ii) We assumed that the base change $\mathscr{G}_x$ to x is connected reductive group scheme over $\mathrm{Spec}\, \mathscr{O}_x$ because the theory of infinitesimal deformation theory for such groups is well established.

(iii) Recall that $(\mathrm{Sht}^{\mu,\leq\lambda}_{\mathscr{G},\Xi})_{\lambda\in\Lambda^+}$ is an open covering of $\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi}$ by quasi-compact substacks given by Harder-Narashimhan parameters $\lambda \in \Lambda^+$. If $\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi}$ is already quasi-compact, then one can find a convex enough λ such that $\mathrm{Sht}^{\mu,\leq\lambda}_{\mathscr{G},\Xi} = \mathrm{Sht}^{\mu}_{\mathscr{G},\Xi}$. However, in the general case where $\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi}$ is not quasi-compact, one of the strategies is to find a suitable compactification of the moduli space of shtukas so that one can systematically work-around this obstacle. This was crucially used in the work of Laurent Lafforgue in his proof of the Langlands correspondence for GL_n (cf. [Laf02]).

Example 5.1.3. We give an example of a moduli stack of shtukas that satisfies the assumption. Let D is a central division algebra over F of dimension d^2. Let $\mathscr{D}$ be a locally free $\mathscr{O}_X$-algebra with generic fiber D which locally is a maximal order. Let $\mathscr{G} := \mathscr{D}^\times$. Choose an idele a of nonzero degree, then set $\Xi = a^{\mathbb{Z}}$. Let $\underline{\mu} = (\mu_i)_{i\in I}$ be an I-tuple of geometric cocharacters of G. Let $\underline{I} = \{1\} \sqcup \cdots \sqcup \{d\}$. Then each μ_i can be identified as $\mu_i = (\mu_i^1, \ldots, \mu_i^d) \in (\mathbb{Z}^d)_+$ where $(\mathbb{Z}^d)^+$ is the set of non-increasing sequences of integers. Suppose the sum of all μ_i^j vanishes. Then the stack $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\underline{I},\Xi}$ is of finite type by [Lau07, Proposition 1.11]. Let $X' \subset X$ be the locus where $\mathscr{G}$ is reductive. If we further assume that $\underline{\mu}$ satisfies the criterion that

$$\sum_{t\in X}[m\,\mathrm{inv}_t(D)] > \sum_{i=1}^{2}\sum_{j=1}^{d-m}\mu_i^j$$

for every integer m such that $0 < m < d$, the stack $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\Xi} \to X'$ is proper by [Lau07, Theorem A]

5.2 Finite Hecke Correspondences

We wish to compute the trace of Hecke actions twisted by a power of Frobenius on the alternating sum of the cohomology

$$H_c^*(\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi} \times_{\eta_{y,\infty'}} \overline{\eta}_{y,\infty'}, \overline{\mathbb{Q}}_\ell) = \sum (-1)^i H_c^i(\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi} \times_{\eta_{y,\infty'}} \overline{\eta}_{y,\infty'}, \overline{\mathbb{Q}}_\ell).$$

over the geometric generic fiber $\overline{\eta}_{y,\infty'} := \mathrm{Spec}\ \overline{E}_{y,\infty'}$. Here we viewed $\Xi \subset Z(F)\backslash Z(\mathbb{A})$ as a subgroup $\Xi \subset Z(\mathbb{A}) \subset G(\mathbb{A})$ so that we view $\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi}$ as a stack over the generic fiber $\eta_{y,\infty'}$. Denote by $\Gamma_{y,\infty'} := \mathrm{Gal}(\overline{E}_{y,\infty'}/E_{y,\infty'})$ the absolute Galois group of $E_{y,\infty'}$ with the Frobenius element $\sigma_{y,\infty'}$. Let us show that there exists an action of $\Gamma_{y,\infty'} \times C_c^\infty(\mathscr{G}(\mathscr{O}_x) \times G(\mathbb{A}^x))/\Xi)$ on the cohomology $H_c^*(\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi} \times_{\eta_{y,\infty'}} \overline{\eta}_{y,\infty'}, \overline{\mathbb{Q}}_\ell)$.

For the Galois action , let $\tau \in \Gamma_{y,\infty'}$. Then τ acts on $\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi} \times_{\eta_{y,\infty'}} \overline{\eta}_{y,\infty'}$ via $\mathrm{id} \times \tau$, hence it defines the action on the cohomology. For the Hecke action, recall that the smooth action of $(\mathscr{G}(\mathscr{O}_x) \times G(\mathbb{A}^x))/\Xi$ on the cohomology H^i induces an action of the Hecke algebra

$$\mathscr{H} := C_c^\infty(\mathscr{G}(\mathscr{O}_x) \times G(\mathbb{A}^x))/\Xi, \overline{\mathbb{Q}}_\ell)$$

via convolution. To elaborate, consider a compact open subgroup K of $\mathscr{G}(\mathscr{O}_x) \times G(\mathbb{A}^x)$ and let $f = \mathbf{1}_{KgK}$ be the characteristic function of the double coset KgK for some $g \in \mathscr{G}(\mathscr{O}_x) \times G(\mathbb{A}^x)$. Let $H := H_c^i(\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi} \times_{\eta_y} \overline{\eta}_y, \overline{\mathbb{Q}}_\ell)$ and $\rho : \mathscr{G}(\mathscr{O}_x) \times G(\mathbb{A}^x) \to \mathrm{GL}(H)$ be the group homomorphism asscociated to H as a representation. Then we define

$$\mathrm{tr}(f | H_c^*(\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi} \times_{\eta_{y,\infty'}} \overline{\eta}_{y,\infty'}, \overline{\mathbb{Q}}_\ell)) := \mathrm{tr}(\rho(f) | H_c^*(\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi} \times_{\eta_{y,\infty'}} \overline{\eta}_{y,\infty'}, \overline{\mathbb{Q}}_\ell)^K)$$

where $\rho(f)v = \int_{\mathrm{G}(\mathscr{O}_x)\times G(\mathbb{A}^x)} f(g)\rho(g)v\, dg$. Then the definition extends to all $f \in \mathscr{H}$ by linearity. As $f \in \mathscr{H}$ is all of the form

$$f = \sum_{i=1}^{n} \alpha_i \mathbf{1}_{K_i g_i K_i} \in \mathscr{H}$$

where K_i is an open compact subgroup of $\mathscr{G}(\mathscr{O}_x) \times G(\mathbb{A}^x)$ and by linearity, it suffices to compute the trace of f of the form $\mathbf{1}_{KgK}$.

The pullback along the natural projection $\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi K} \to \mathrm{Sht}^{\mu}_{\mathscr{G},\Xi}$ induces an embedding of cohomology groups $H^i_c(\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi K} \times_{\eta_{y,\infty'}} \overline{\eta}_{y,\infty'}, \overline{\mathbb{Q}}_\ell) \hookrightarrow H^i_c(\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi} \times_{\eta_{y,\infty'}} \overline{\eta}_{y,\infty'}, \overline{\mathbb{Q}}_\ell)$. One can identify $H^i_c(\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi} \times_{\eta_{y,\infty'}} \overline{\eta}_{y,\infty'})^K$ with $H^i_c(\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi K} \times_{\eta_{y,\infty'}} \overline{\eta}_{y,\infty'})$ via this embedding. In particular, we only need to compute the traces of the form

$$\mathrm{tr}(\sigma^j_{y,\infty'} \circ \mathbf{1}_{KgK} | H^*_c(\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi K} \times_{\eta_{y,\infty'}} \overline{\eta_{y,\infty'}}, \overline{\mathbb{Q}}_\ell)$$

for a positive integer $j \geq 0$.

Next, we introduce finite Hecke correspondence $[KgK]^{(r)}$ where KgK is a double coset in $\mathscr{G}(\mathscr{O}_x) \times G(\mathbb{A}^x)$ such that $KgK = K_x g_x K_x \times K^x g^x K^x$ with $K_x \subset \mathscr{G}(\mathscr{O}_x)$, $g_x \in \mathscr{G}(\mathscr{O}_x)$, $K^x \subset G(\mathbb{A}^x)$, and $g^x \in G(\mathbb{A}^x)$. We will show that the trace of $[KgK]^{(r)}$ on the cohomology coincides with the a trace of the characteristic function $\mathbf{1}_{KgK}$.

Set $r = j[\mathbb{F}_{y,\infty'} : \mathbb{F}_q]$. Then the finite étale correspondence $[KgK]^{(r)}$ is defined by

$$\begin{array}{ccccc} \mathrm{Sht}^{\mu}_{\mathscr{G},\Xi K} & \xleftarrow{\sigma^j_y \circ p_1} & \mathrm{Sht}^{\mu}_{\mathscr{G},\Xi K_g} & \xrightarrow{p_2} & \mathrm{Sht}^{\mu}_{\mathscr{G},\Xi K} \\ (\sigma^j_y)^*(\mathscr{E},\tau,\psi\Xi K) & \leftarrow\!\shortmid & (\mathscr{E},\tau,\psi\Xi K_g) & \mapsto & (\mathscr{E},\tau,\psi g\Xi K) \end{array} \tag{5.2}$$

Here $K_g := K \cap gKg^{-1}$. In fact, one can rewrite

$$(\sigma_y^j)^*(\mathscr{E}, \tau, \psi\Xi K) \cong (\sigma^r)^*(\mathscr{E}, \tau, \psi\Xi K)$$

where $\sigma := \sigma_q$ is the absolute q-Frobenius. Denote by $p_1^{(r)} := \sigma_y^j \circ p_1$. The Hecke correspondence naturally extends to a cohomological correspondence $u : p_{2!}(p_1^{(r)})^*\overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_\ell$. Then we get an induced endomorphism of $R\Gamma_c(\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi K}, \overline{\mathbb{Q}}_\ell)$ via the composition

$$\begin{aligned} R\Gamma_c(\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi K}, \overline{\mathbb{Q}}_\ell) &\to R\Gamma_c(\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi K_g}, (p_1^{(r)})^*\overline{\mathbb{Q}}_\ell) \to R\Gamma_c(\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi K_g}, p_2^!\overline{\mathbb{Q}}_\ell) \\ &= R\Gamma_c(\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi K}, p_{2!}p_2^!\overline{\mathbb{Q}}_\ell) \to R\Gamma_c(\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi K}, \overline{\mathbb{Q}}_\ell) \quad (5.3) \end{aligned}$$

Therefore, the trace $\mathrm{tr}([KgK]^{(r)}|H_c^i(\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi K}, \overline{\mathbb{Q}}_\ell))$ can be defined.

We can now establish the relationship between the trace of $\mathbf{1}_{KgK}$ on $R\Gamma_c(\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi}, \overline{\mathbb{Q}}_\ell)$ and the trace of $R\Gamma_c([KgK]^{(r)})$. Let $p_K : H_c^i(\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi}, \overline{\mathbb{Q}}_\ell) \to H_c^i(\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi K}, \overline{\mathbb{Q}}_\ell)$ be defined by $v \mapsto \int_K g \cdot v \, dg$. Then a straightforward calculation shows that

$$(\rho(\mathbf{1}_{KgK}))v = ([KgK] \circ p_K)(v).$$

for $v \in H_c^i(\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi}, \overline{\mathbb{Q}}_\ell)$. Therefore,

$$\begin{aligned} \mathrm{tr}(\mathbf{1}_{KgK} \mid R\Gamma_c(\mathrm{Sht}^{\mu}_{\mathscr{G},\Xi}, \overline{\mathbb{Q}_\ell})) &= \mathrm{vol}_{dg}(K)\mathrm{tr}(R\Gamma_c[KgK]) \\ &= \mathrm{vol}_{dg_x}(K_x)\mathrm{vol}_{dg^x}(K^x)\mathrm{tr}(R\Gamma_c[KgK]). \end{aligned}$$

where dg_x denotes the Haar measure on $G(F_x)$ normalized such that $G(\mathscr{O}_x)$ has volume 1 and

dg^x denotes the Haar measure on $G(\mathbb{A}^x)$ normalized such that K^x has volume 1. To conclude,

$$\mathrm{tr}(f|R\Gamma_c(\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\Xi}, \overline{\mathbb{Q}_\ell})) = \mathrm{tr}([KgK]|R\Gamma_c(\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\Xi}, \overline{\mathbb{Q}_\ell}))$$

for $f = \frac{1}{\mathrm{vol}_{dg_x}(K_x)}\mathbf{1}_{K_x g_x K_x} \otimes \frac{1}{\mathrm{vol}_{dg^x}(K^x)}\mathbf{1}_{K^x g^x K^x}$. We will only focus on the finite étale correspondence $[KgK]$ and its twist by a power of Frobenius from this point onwards.

5.3 Lefschetz Trace Formula

The main goal of this paper is to obtain a formula for the trace

$$\mathrm{tr}([KgK]^{(r)}|H^*_c) := \mathrm{tr}([KgK]^{(r)} \mid H^*_c(\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\Xi K} \times_{\eta_{y,\infty'}} \overline{\eta}_{y,\infty'}, \overline{\mathbb{Q}}_\ell))$$

involving orbital integrals away from x and twisted orbital integrals at x.

Following the ideas of Kottwitz [Kot92] and Scholze [Sch12], we wish to utilize the Lefschetz trace formula to obtain the required formula for $\mathrm{tr}([KgK]^{(r)}|H^*_c)$. Therefore, we need to express the trace of the cohomology on the generic fiber as a trace of the cohomology of the special fiber with coefficients in the nearby cycle sheaves. Using the base change of the finite étale cover $\pi_{KK^x} : \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\Xi K} \to \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\Xi\mathscr{G}(\mathscr{O}_x)K^x\mathscr{G}(\mathscr{O}_\infty)} \cong \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\Xi K^x} \times_{X_{\underline{\mu}}\backslash T_{\underline{\mu}}} \eta$ to $\eta_{y,\infty'}$, we obtain the following isomorphism

$$H^i_c(\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\Xi K} \times_{\eta_{y,\infty'}} \overline{\eta}_{y,\infty'}, \overline{\mathbb{Q}}_\ell) \cong H^i_c(\mathrm{Sht}_{\mathscr{G},\Xi K^x,\mathscr{O}_{y,\infty'}} \times \overline{\eta}_{y,\infty'}, \pi_{KK^x*}\overline{\mathbb{Q}}_\ell). \tag{5.4}$$

Then we obtain the following correspondence away from x.

$$\begin{array}{ccccc} \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\Xi K^x} & \xleftarrow{\sigma_y^j\circ\tilde{p}_1} & \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\Xi K^x_{g^x}} & \xrightarrow{\tilde{p}_2} & \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\Xi K^x} \\ (\sigma^r)^*(\mathscr{E},\tau,\psi\Xi K^x) & \leftarrow\!\shortmid & (\mathscr{E},\tau,\psi\Xi K^x_{g^x}) & \mapsto & (\mathscr{E},\tau,\psi g\Xi K^x) \end{array} \tag{5.5}$$

Here $K^x_{g^x} := K^x \cap g^x K^x (g^x)^{-1}$. Then we can naturally extend to obtain a cohomological correspondence $\tilde{u} : \tilde{p}_{2!}(\tilde{p}_1^{(r)})^*\pi_{KK^x*}\overline{\mathbb{Q}}_\ell \to \pi_{KK^x*}\overline{\mathbb{Q}}_\ell$ and an endomorphism of $R\Gamma_c(\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\Xi K^x}, \pi_{KK^x*}\overline{\mathbb{Q}}_\ell)$ as we did in (5.3).

We now define the geometric special fiber of the moduli of global shtukas. We fix a geometric point $\overline{x}$ over y and hence over x. Similarly, we fix a geometric point $\overline{\infty}$ over a place ∞' in X_{μ_∞} and hence over ∞. Let us denote $\overline{(x,\infty)}$ to be the fiber product

$$\overline{(x,\infty)} := (\overline{x} \times_{\mathbb{F}_q} \overline{\infty}) \times_{X_{\underline{\mu}}} \mathrm{Spec}\, \mathscr{O}_{y,\infty'}.$$

We denote by

$$\mathcal{S}^{\underline{\mu}}_{K^x,\overline{x}} := \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\Xi K^x,\mathscr{O}_{y,\infty'}} \times_{\mathscr{O}_{y,\infty'}} \overline{(x,\infty)}$$

the **geometric special fiber** at $\overline{x}$. To rewrite (5.4) as a cohomology on the geometric special fiber, we need to recall the definition of nearby cycle sheaves.

Definition 5.3.1. Let X be a scheme of finite type over the henselian trait (S,s,η). Let $\overline{\eta}$ be the separable closure of η, and $\overline{S}$ the normalization of S in $\overline{\eta}$ with residue field $\overline{s}$.

Denote by $D^b_c(X,\overline{\mathbb{Q}}_\ell)$ be the "derived category" of $\overline{\mathbb{Q}}_\ell$-sheaves in the sense of [KW01, page 94] and $D^b_c(X\times_s\eta,\overline{\mathbb{Q}}_\ell)$ to be the category of sheaves in $D^b_c(X_{\overline{s}},\overline{\mathbb{Q}}_\ell)$ together with a continuous action of $\mathrm{Gal}(\overline{\eta}/\eta)$ compatible with the action on $X_{\overline{s}}$. Then for $\mathscr{F} \in D^b_c(X_\eta,\overline{\mathbb{Q}}_\ell)$, the **nearby**

cycle sheaf is defined by

$$R\Psi^X(\mathscr{F}) := \overline{i}^* R\overline{j}_*(\mathscr{F}_{\overline{\eta}}) \in D_c^b(X \times_s \eta, \overline{\mathbb{Q}}_\ell)$$

where $X_{\overline{s}} \overset{\overline{i}}{\hookrightarrow} X_{\overline{S}} \overset{\overline{j}}{\hookleftarrow} X_{\overline{\eta}}$ are the closed (resp. open) immersions of geometric special (resp. generic) fiber of X over S and $\mathscr{F}_{\overline{\eta}}$ is the pullback of $\mathscr{F}$ along the canonical projection $X_{\overline{\eta}} \to X_\eta$. When there is no confusion, we omit X from the notation and simply write $R\Psi\mathscr{F}$.

Since we assumed that $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\Xi K^x,\mathscr{O}_{y,\infty'}} \to \mathrm{Spec}\,\mathscr{O}_{y,\infty'}$ is proper, we obtain a $\mathrm{Gal}(\overline{\eta}_{y,\infty'}/\eta_{y,\infty'})$-equivariant isormophism

$$H^i(\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\Xi K^x,\mathscr{O}_{y,\infty'}} \times_{\mathrm{Spec}\,\mathscr{O}_{y,\infty'}} \overline{\eta}_{y,\infty'}, \pi_{KK^x*}\overline{\mathbb{Q}}_\ell) \cong H_c^i(\mathscr{S}^{\underline{\mu}}_{K^x,\overline{x}}, R\Psi\pi_{KK^x*}\overline{\mathbb{Q}}_\ell)$$

by the proper base change theorem.

Recall that the action of $[KgK]$ was determined by $g = (g_x, g^x) \in \mathscr{G}(\mathscr{O}_x) \times G(\mathbb{A}^x)$. We describe the algebraic correspondence on the special fiber

$$\mathscr{S}^{\underline{\mu}}_{K^x,\overline{x}} \xleftarrow{\overline{p}_1^{(r)}} \mathscr{S}^{\underline{\mu}}_{K^x_{g^x},\overline{x}} \xrightarrow{\overline{p}_2} \mathscr{S}^{\underline{\mu}}_{K^x,\overline{x}}$$

Here $K^x_{g^x} = g^x K^x (g^x)^{-1}$. We will separate the action of g on $R\Psi\pi_{KK^x*}\overline{\mathbb{Q}}_\ell$ into an action of g_x and g^x. To distinguish between different constant sheaves, we write $(\overline{\mathbb{Q}}_\ell)_K$ and $(\overline{\mathbb{Q}}_\ell)_{K^x}$ for constant $\overline{\mathbb{Q}}_\ell$-adic sheaf on $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\Xi K}$ and $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\Xi K^x}$ respectively. Then we claim that

$$R\Psi\pi_{KK^x*}(\overline{\mathbb{Q}}_\ell)_K \cong (\overline{\mathbb{Q}}_\ell)_{K^x} \otimes R\Psi\pi_{KK^x*}(\overline{\mathbb{Q}}_\ell)_K$$

such that g^x acts on $(\overline{\mathbb{Q}}_\ell)_{K^x}$ and g_x acts on $R\Psi\pi_{KK^x*}(\overline{\mathbb{Q}}_\ell)_K$. Then for big enough $r \gg 0$, we obtain the scheme of fixed points by the Cartesian product

$$\begin{array}{ccc} \mathrm{Fix}[K^x g^x K^x]^{\underline{\mu},(r)} & \longrightarrow & \mathscr{S}^{\underline{\mu}}_{K^x_{g^x},\overline{x}} \\ \downarrow & & \downarrow {\scriptstyle (\overline{p}_1^{(r)},\overline{p}_2)} \\ \mathscr{S}^{\underline{\mu}}_{K^x,\overline{x}} & \xrightarrow{\Delta} & \mathscr{S}^{\underline{\mu}}_{K^x,\overline{x}} \times \mathscr{S}^{\underline{\mu}}_{K^x,\overline{x}} \end{array}$$

where Δ is the diagonal morphism. We now apply the Lefschetz trace formula ([Var07, Theorem 2.3.2]) to obtain

$$\mathrm{tr}([KgK]^{\underline{\mu},(r)} \mid H_c^*) = \sum_{e \in \mathrm{Fix}[K^x g^x K^x]^{(r)}} \mathrm{tr}(u_e) \tag{5.6}$$

where $u_e : ((\overline{\mathbb{Q}}_\ell)_{K^x} \otimes R\Psi\pi_{KK^x*}(\overline{\mathbb{Q}}_\ell)_K)_{\overline{p}_1^{(r)}(e)} \to ((\overline{\mathbb{Q}}_\ell)_{K^x} \otimes R\Psi\pi_{KK^x*}(\overline{\mathbb{Q}}_\ell)_K)_{\overline{p}_2(e)}$. This can be interpreted as counting the fixed points with certain weights given by the trace.

5.4 Local-Global Compatibility

The Serre-Tate theorem says that the deformations of global $\mathscr{G}$-shtukas is given by the deformations of their associated local shtukas at the legs. One of the main ingredients are the rigidities of both local and global shtukas. The Serre-Tate theorem for shtukas has been proved in [ARH14, Theorem 5.10] for unbounded shtukas with distinct legs, in [Bie24, Proposition 3.3.5] for bounded shtukas, and [HX23, Proposition 2.9.30] for *colliding legs*.

Let $\overline{\underline{\mathscr{E}}} \in \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\Xi K^x,\mathscr{O}_{y,\infty'}}(\kappa)$ where κ is an $\mathscr{O}_{y,\infty'}$-algebra that is also a finite field or a separably closed field. Denote by $\mathrm{Def}^{\underline{\mu}}_{\mathscr{O}_{y,\infty'}}(\overline{\underline{\mathscr{E}}})$ be the deformation problems given by $R \mapsto (\underline{\mathscr{E}}, \alpha)$ where $\underline{\mathscr{E}}$ is a global $\mathscr{G}$-shtuka over κ and $\alpha : \overline{\underline{\mathscr{E}}} \xrightarrow{\sim} \underline{\mathscr{E}} \otimes_R \kappa$ is an isomorphism. Then we have the following theorem.

Theorem 5.4.1 (Serre-Tate)**.** *Let* $\underline{\overline{\mathscr{E}}} = (\overline{\mathscr{E}}, \overline{\tau}) \in \mathrm{Sht}^{\mu}_{\mathscr{G},\Xi K^x}(\kappa)$. *The functor*

$$\mathrm{Def}^{\mu}_{\mathscr{O}_{y,\infty'}}(\underline{\overline{\mathscr{E}}}) \to \prod_{t\in\{x,\infty\}} \mathrm{Def}^{\mu_t}_{\mathscr{O}_{y,\infty'}}(\underline{\mathscr{L}}_t(\underline{\overline{\mathscr{E}}}))$$

defined by $(\underline{\mathscr{E}}, \alpha) \mapsto (\underline{\mathscr{L}}_t(\underline{\mathscr{E}}), \underline{\mathscr{L}}_t(\alpha))_{t\in\{x,\infty'\}}$ *induces an natural isomorphism.*

Proof. We only give a sketch of [ARH14, Theorem 5.10]. The inverse functor is constructed as follows. Let$(\underline{\mathscr{L}}_x, \hat{\alpha}_x)$ and $(\underline{\mathscr{L}}_\infty, \hat{\alpha}_\infty)$ be a pair of deformations over over an artinian local $\mathscr{O}_{y,\infty'}$-algebra R with residue field κ. Let $\mathfrak{m}_R$ be its maximal ideal, and write $j : \mathrm{Spec}\ \kappa \hookrightarrow \mathrm{Spec}(R)$ for the closed embedding. It suffices to assume that $\mathfrak{m}_R^q = 0$. Then the Frobenius $\sigma_R : \mathrm{Spec}(R) \xrightarrow{\sigma'} \mathrm{Spec}\ \kappa \xrightarrow{j} \mathrm{Spec}(R)$. Pick any global $\mathscr{G}$-shtuka $\underline{\widetilde{\mathscr{E}}}$ over R. Then $\overline{\tau}$ defines a quasi-isogeny between $j^*\underline{\widetilde{\mathscr{E}}} \to \underline{\overline{\mathscr{E}}}$ which is an isomorphism away from the legs.

Let $(\underline{\overline{\mathscr{L}}}_t, \hat{\overline{\tau}}_t)_{t\in\{x,\infty\}}$ be the tuple of local shtukas associated to $\underline{\overline{\mathscr{E}}}$. Then the composition $j^*\underline{\mathscr{L}}_t \xrightarrow{\hat{\alpha}_t} \underline{\overline{\mathscr{L}}}_t \xrightarrow{\hat{\overline{\tau}}_t} j^*\underline{\widetilde{\mathscr{L}}}_t$ is a quasi-isogeny of local $\mathscr{G}_t$-shtukas. By Proposition 3.1.5, the quasi-isogenty lifts to $\gamma_t : \underline{\mathscr{L}}_t \to \underline{\widetilde{\mathscr{L}}}_t$. Then by 4.5.4(i), one obtains a global $\mathscr{G}$-shtuka $\underline{\mathscr{E}}$ over R with a quasi-isogeny $\gamma : \underline{\mathscr{E}} \to \tilde{\mathscr{E}}$ which is an isomorphism away from the legs and the local $\mathscr{G}_t$-shtuka $\underline{\mathscr{E}}[t^\infty]$ can be identified as $\underline{\mathscr{L}}_t$ for $t \in \{x, \infty\}$ under the equivalence in Remark 4.7.4. Then the inverse functor is given by

$$(\underline{\mathscr{L}}_t, \hat{\alpha}_t)_{t\in\{x,\infty\}} \mapsto (\underline{\mathscr{E}}, \hat{\tau} \circ j^*\gamma).$$

□

Next, we establish the following global-local compatibility that compares the global theory with the local theory.

Proposition 5.4.2. *Let* κ *be a finite or separably closed field of characteristic* p *over* $\mathscr{O}_{y,\infty'}$ *and* *let* $e := \underline{\overline{\mathscr{E}}} \in \mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\Xi K^x,\mathscr{O}_{y,\infty'}}(\kappa)$. *Let* $\underline{\overline{\mathscr{L}}}_i$ *be the local* $\mathscr{G}_t$*-shtuka associated to* $\underline{\overline{\mathscr{E}}}$ *over* κ *where* $t \in \{x,\infty\}$. *Let* k *be the complete unramified extension of* $E_{y,\infty'}$ *with residue field* κ. *Then the complete local ring* R_e *of* $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\Xi K^x,\mathscr{O}_{y,\infty'}}$ *at* e *is isomorphic to the completed tensor product of deformation rings*

$$R^{\mu_x}_{\underline{\overline{\mathscr{L}}}_x} \widehat{\otimes}_{\mathscr{O}_{y,\infty'}} R^{\mu_\infty}_{\underline{\overline{\mathscr{L}}}_\infty}$$

Let $\mathfrak{X}_e := \mathrm{Spf}\ R_e$. *There exists a natural specialization map* $\mathrm{sp} : |(\mathfrak{X}_e)_\eta| \to |\mathfrak{X}_e|$. *Then inverse image* $\mathrm{sp}^{-1}(\{e\}) = \prod_{t\in\{x,\infty\}} X^{\mu_t}_{\underline{\overline{\mathscr{L}}}_t}$ *is identified as a tubular neighborhood in* $\mathrm{Sht}^{\underline{\mu}}_{\mathscr{G},\Xi K^x}$. *Here the product* $\prod_{t\in\{x,\infty\}} X^{\mu_t}_{\underline{\overline{\mathscr{L}}}_t}$ *is taken in the category of formal schemes. For any open subgroups* $K_x \subset \mathscr{G}(\mathscr{O}_x)$*, we have the Cartesian diagram*

$$\begin{array}{ccc} \prod_{t\in\{x,\infty\}} X^{\mu_t}_{\underline{\overline{\mathscr{L}}}_t, K_t} & \longrightarrow & \prod_{t\in\{x,\infty\}} X^{\mu_i}_{\underline{\overline{\mathscr{L}}}_i} \\ \downarrow & & \downarrow \\ \mathrm{Sht}^{\underline{\mu},\mathrm{an}}_{\mathscr{G},\Xi K,\mathscr{O}_{y,\infty'}} & \longrightarrow & \mathrm{Sht}^{\underline{\mu},\mathrm{an}}_{\mathscr{G},\Xi K^x,\mathscr{O}_{y,\infty'}} \end{array}$$

Here we set $K_\infty = \mathscr{G}(\mathscr{O}_\infty)$.

Proof. By a standard argument, R_e pro-represents the deformation functor $\mathrm{Def}^{\underline{\mu}}_{\mathscr{O}_{y,\infty'}}(\underline{\overline{\mathscr{E}}})$. Then by Serre-Tate theory, we have the first statement. To show that the diagram is Cartesian, we consider the fiber product

$$\mathrm{Sht}^{\underline{\mu},\mathrm{an}}_{\mathscr{G},\Xi K,\mathscr{O}_{y,\infty'}} \widehat{\times}_{\mathrm{Sht}^{\underline{\mu},\mathrm{an}}_{\mathscr{G},\Xi K^x,\mathscr{O}_{y,\infty'}}} \prod_{t\in\{x,\infty\}} X^{\mu_i}_{\underline{\overline{\mathscr{L}}}_i}$$

Let Y be a connected affinoid strictly k-analytic space with geometric point $\overline{y}$ and $\mathfrak{Y}$ be its formal admissible model. Then the Y-valued point in the fiber product is given by pair $(\underline{\mathscr{E}}, (\psi_x, \psi_\infty, \psi^{x,\infty}))$ and $(\underline{\mathscr{L}}_x, \underline{\mathscr{L}}_\infty)$ over $\mathfrak{Y}$ such that its image in $\mathrm{Sht}^{\underline{\mu},\mathrm{an}}_{\mathscr{G},\Xi K^x,\mathscr{O}_{y,\infty'}}$ is the pullback $j^*\underline{\overline{\mathscr{E}}}$ of $\underline{\overline{\mathscr{E}}}$ along $j : \mathrm{Spec}\ \kappa \to \mathfrak{Y}$. In particular, the only data that is not determined by $\underline{\overline{\mathscr{E}}}$ is the K_x-level structure ψ_x and the level K_∞-level structure ψ_∞. The pair (ψ_x, ψ_∞) is exactly given by $\prod_{t\in\{x,\infty\}} X^{\mu_t}_{\underline{\overline{\mathscr{L}}}_t}$. $\square$

Remark 5.4.3. We would like to say that $\underline{\overline{\mathscr{L}}}_t$ has controlled cohomology for $t \in \{t, \infty\}$ as in [Sch12, Proposition 5.4]. The proof in [Sch12] uses the fact that the deformation space is smooth which is in general not true unless the cocharacters μ_x and μ_∞ are minuscule cocharacters. We expect that the deformations spaces $X^{\mu_t}_{\underline{\overline{\mathscr{L}}}_t}$ has controlled cohomology, and we assume that this is true. Then we have a $\mathrm{Gal}(\overline{k}/k)$-equivariant isomorphism

$$(R^i\Psi\pi_{KK^x*}\overline{\mathbb{Q}}_\ell)_{\overline{e}} \cong \bigotimes_{t\in\{x,\infty\}} H^i(X^{\mu_t}_{\underline{\overline{\mathscr{L}}}_t,K_t} \otimes_k \hat{\overline{k}}, \overline{\mathbb{Q}}_\ell)$$

where $\overline{e} = \overline{p}_1^{(r)}(e) = \overline{p}_2(e)$ and an isomorphism

$$H^i(X^{\mu_t}_{\underline{\overline{\mathscr{L}}}_t} \otimes_k \hat{\overline{k}}, \overline{\mathbb{Q}}_\ell) \cong H^i(X^{\mu_t}_{\underline{\overline{\mathscr{L}}}_t,K_t} \otimes_k \hat{\overline{k}}, \overline{\mathbb{Q}}_\ell)^{K_t}.$$

5.5 Moduli Description of the Fixed Points

Let us recall the moduli description of the $\overline{\mathbb{F}}_q$-points of the geometric special fiber $\mathscr{S}^{\underline{\mu}}_{K^x_{g^x},\overline{x}}$. We denote by $\overline{X} := X_{\underline{\mu}} \otimes_{\mathbb{F}_q} \overline{x} \cong X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$. Then

$$\mathscr{S}^{\underline{\mu}}_{K^x_{g^x},\overline{x}}(\overline{\mathbb{F}}_q) = \left\{ \underline{\mathscr{E}} = (s, \mathscr{E}, \tau, \psi\Xi K^x_{g^x}) \;\middle|\; \begin{array}{l} \bullet\ s : \overline{x} \to (X_{\underline{\mu}} \backslash T_{\underline{\mu}}) \text{ that factors through } y \\ \bullet\ \mathscr{E} \text{ is a } \mathscr{G}\text{-bundle over } \overline{X}. \\ \bullet\ \tau : {}^\sigma\mathscr{E}|_{X_{\underline{\mu}}-\Gamma_s-\Gamma_\infty} \xrightarrow{\sim} \mathscr{E}|_{X_{\underline{\mu}}-\Gamma_s-\Gamma_\infty} \text{ bounded by } \underline{\mu} \\ \bullet\ \psi\Xi K^x_{g^x} \text{ is a } \Xi K^x_{g^x}\text{-level structure on } \mathscr{E} \\ \quad \Xi K^x_{g^x}\text{-orbit of } \psi : (L_{\mathbb{A}^{\infty,x}}G, \sigma) \xrightarrow{\sim} (\mathscr{E}^{\mathrm{an}}_{\mathbb{A}^{\infty,x}}, \tau^{\mathrm{an}}_{\mathbb{A}^{\infty,x}}) \end{array} \right\}$$

Note that we do not need to specify the section s for $\overline{\mathbb{F}}_q$-points $\mathscr{S}^{\underline{\mu}}_{K^x_{g^x},\overline{x}}(\overline{\mathbb{F}}_q)$ as s must be that composition $s : \overline{x} \to y \to (X_{\underline{\mu}} \backslash T_{\underline{\mu}})$. Therefore, we write the tuple $(s, \mathscr{E}, \tau, \psi \Xi K^x_{g^x})$ in $\mathcal{S}^{\underline{\mu}}_{K^x_{g^x}, \overline{x}}(\overline{\mathbb{F}}_q)$ simply as a triple $(\mathscr{E}, \tau, \psi \Xi K^x_{g^x})$.

We would like to parametrize the fixed points $\mathrm{Fix}[K^x g^x K^x]^{\underline{\mu},(r)}$ as G-isoshtukas which can be thought as the generic fibers of global $\mathscr{G}$-shtukas. Then G-isoshtukas will be associated with some group-theoretic data. By definition, elements in $\mathrm{Fix}[K^x g^x K^x]^{\underline{\mu},(r)}$ are of the form $(\mathscr{E}, \tau, \psi \Xi K^x_{g^x})$ that induces an isomorphism

$$(\sigma^r)^*(\mathscr{E}, \tau, \psi \Xi K^x) \cong (\mathscr{E}, \tau, \psi g^x \Xi K^x)$$

in $\mathcal{S}^{\underline{\mu}}_{K^x, \overline{x}}(\overline{x})$. The above isomorphism then induces an isomorphism $\tau' : (\sigma^r)^* \mathscr{E} \to \mathscr{E}$ that commutes with τ, i.e. the following diagram

$$\begin{array}{ccc} {}^{\sigma^{r+1}}\mathscr{E} & \xrightarrow{{}^{\sigma}\tau'} & {}^{\sigma}\mathscr{E} \\ {\scriptstyle {}^{\sigma^r}\tau}\downarrow & & \downarrow{\scriptstyle \tau} \\ {}^{\sigma^r}\mathscr{E} & \xrightarrow{\tau'} & \mathscr{E} \end{array} \tag{5.7}$$

commutes and τ' sends $(\sigma^r)^*(\psi \Xi K^x)$ to $\psi g^x \Xi K^x$. To summarize,

Proposition 5.5.1. *The $\overline{\mathbb{F}}_q$-points of $\mathrm{Fix}[K^x g^x K^x]^{\underline{\mu},(r)}$ are parametrized by $(\mathscr{E}, \tau, \tau', \psi)$ where*

- *$\mathscr{E}$ is a $\mathscr{G}$-torsor over $\overline{X}$,*
- *$\tau : {}^{\sigma}\mathscr{E}|_{\overline{X} - \Gamma_x - \Gamma_\infty} \xrightarrow{\sim} \mathscr{E}|_{\overline{X} - \Gamma_x - \Gamma_\infty}$ is bounded by $\underline{\mu}$,*
- *$\tau' : {}^{\sigma^r}\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ is an isomorphism of $\mathscr{G}$-torsors over $\overline{X}$*
- *ψ^x is a ΞK^x-level structure on $\mathscr{E}$*

such that $(s, \mathscr{E}, \tau) \in \mathcal{S}^{\underline{\mu}}_{K^x_{g^x}, \overline{x}}(\overline{\mathbb{F}}_q)$, τ and τ' commute in the sense of (5.7) and $(\sigma^r)^*(\psi \Xi K^x) \cong$

$\psi g^x \Xi K^x$.

Remark 5.5.2. The above-fixed scheme is an analogue of the category $\mathscr{C}^I_{\lambda_{\overline{T}},\beta_{T'}}(T, T', s)$ of fixed points in [ND07] where I is the level structure given by a finite closed subscheme of X, $\lambda_{\overline{T}} = (\lambda_t)_{t\in T}$ is a T-tuple of (conjugacy classes of) cocharacters, and $\beta_{T'}$ is a T'-tuple of double cosets $K_{I,t}\beta_t K_{I,t}$ that induce the Hecke correspondence. The counting points formula in [ND07] uses Lefschetz trace formula for algebraic stacks instead of truncating the Harder-Narashimhan filtration to work with schemes.

Let $\eta = \operatorname{Spec} F$ be the generic fiber of the curve X. Denote by $\breve{F} := F \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ and $\breve{\eta} := \eta \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} \overline{\mathbb{F}}_q$. We would like to classify all points in an isogeny class of a fixed point. This leads to the definition of G-isoshtukas.

Definition 5.5.3.

(i) A G**-isoshtuka** over $\overline{\mathbb{F}}_q$ is a pair (E, c) where E is a G-torsor over $\breve{F} := F \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ and $c : {}^{\sigma}E \to E$ is an isomorphism of G-torsors. A morphism $f : (E, c) \to (E_0, c_0)$ of G-isoshtuka is an isomorphisms $f : E \to E_0$ of G-torsors over $\breve{F}$ that is compatible with c and c_0, i.e. the diagram

$$\begin{array}{ccc} {}^{\sigma}E & \xrightarrow{\ {}^{\sigma}f\ } & {}^{\sigma}E' \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c'} \\ E & \xrightarrow{\ f\ } & E' \end{array} \tag{5.8}$$

commutes.

(ii) A G**-isoshtuka with** r**-structure** is a triple (E, c, c') where (E, c) is a G-isoshtuka and $c' : {}^{\sigma^r}E \to E$ is an isomorphism of G-torsors such that

(a) c and c' commute, i.e.

$$\begin{array}{ccc} {}^{\sigma^{r+1}}E & \xrightarrow{{}^{\sigma}c'} & {}^{\sigma}E' \\ {}^{\sigma^r}c\downarrow & & \downarrow c \\ {}^{\sigma^r}E & \xrightarrow{c'} & E \end{array} \tag{5.9}$$

(b) $(E_t, c_t) := (E, c) \otimes_{\breve{F}} (\breve{F}\widehat{\otimes}_F F_t)$ is isomorphic to $(G_{\breve{F}\widehat{\otimes}_F F_t}, \sigma)$.

(c) $(E_t, c'_t) := (E, c') \otimes_{\breve{F}} (\breve{F}\widehat{\otimes}_F F_t)$ is isomorphic to $(G_{\breve{F}\widehat{\otimes}_F F_t}, \sigma^r)$ for $t \in \{x, \infty\}$.

We denote by $\mathrm{IsoSht}_G^{(r)}$ the set of G-isoshtukas over $\breve{F}$ with r-structures.

Remark 5.5.4. There is a bijection between the set of isogeny classes of global $\mathscr{G}$-shtuka over $\overline{\mathbb{F}}_q$ and the set of G-isoshtukas over $\breve{F}$ by restriction to the generic point, i.e. $\underline{\mathscr{E}} \mapsto \underline{\mathscr{E}} \times_{\overline{X}} \breve{\eta}$. Since $\breve{F}$ has cohomological dimension 1 by Tsen's theorem, G-torsor E over $\breve{F}$ is trivial. Hence $(E, c) \cong (G_{\breve{F}}, b\sigma)$ for some $b \in G(\breve{F})$. Then choosing a different trivialization is the same as replacing b by $g^{-1}b\sigma(g)$, so we have a bijection between the set of G-isoshtuka over $\overline{\mathbb{F}}_q$ and the set $BG(F)$ of σ-conjugacy classes in $G(\breve{F})$.

Let $(\mathscr{E}, \tau, \tau', \psi) \in \mathrm{Fix}[K^x g^x K^x]^{\underline{\mu},(r)}$ be a fixed point. Then the restriction $(E, c, c') = (\mathscr{E}, \tau, \tau') \times_{\overline{X}} \breve{\eta}$ defines a G-isoshtuka (E, c) over $\overline{\mathbb{F}}_q$ with an isomorphism $c' : {}^{\sigma^r}E \to E$. Since τ and τ' commute in the sense of (5.7), it follows that c and c' commute in the sense of (5.9).

For $t \notin \{x, \infty\}$, the t-component of the level structure $\psi : (L_{\mathbb{A}^{\infty,x}}G, \sigma) \xrightarrow{\sim} (\mathscr{E}^{\mathrm{an}}_{\mathbb{A}^{\infty,x}}, \tau^{\mathrm{an}}_{\mathbb{A}^{\infty,x}})$ is an isomorphism $\psi_t : ((L\mathrm{Res}_{\mathscr{O}_t/\mathbb{F}_q[\![\varpi_t]\!]}\mathscr{G}_t)_{\overline{\mathbb{F}}_q}, \sigma) \cong (\mathscr{E}^{\mathrm{an}}_t, \tau^{\mathrm{an}}_t)$. Then by the isomorphism

$$(\mathrm{Res}_{F_t/\mathbb{F}_q(\!(\varpi_t)\!)}G)(\breve{F}_t) \cong G(\breve{F}\widehat{\otimes}_F F_t)$$

we have an isomorphism $(E_t, c_t) \cong (G_{\breve{F}\widehat{\otimes}_F F_t}, \sigma)$.

For $t \in \{x, \infty\}$, denote by $\tau'_t := \tau' \times_{\overline{X}} (\mathrm{Spf}\ \mathscr{O}_t \widehat{\times}_{\mathbb{F}_q} \mathrm{Spec}\ \overline{\mathbb{F}}_q)$ be the isomorphism ${}^{\sigma^r}\mathscr{E}_t \xrightarrow{\sim}$

$\mathscr{E}_t$. Since $\mathscr{E}_t$ is a $\mathscr{G}_t$-torsor over $(\mathrm{Spf}\ \mathscr{O}_t \widehat{\times}_{\mathbb{F}_q} \mathrm{Spec}\ \overline{\mathbb{F}}_q)$, it follows that $\mathscr{E}_t$ is isomorphic to the trivial $\mathscr{G}_t$-torsor. And we have $(\mathscr{E}_t, \tau_t') \cong (\mathscr{G}_t, b'\sigma)$ where $b' \in \mathscr{G}(\mathscr{O}_t \widehat{\otimes}_{\mathbb{F}_q} \overline{\mathbb{F}}_q)$. Then as every element in $\mathscr{G}(\mathscr{O}_t \widehat{\otimes}_{\mathbb{F}_q} \overline{\mathbb{F}}_q)$ is σ-conjugate to the identity, we have $(\mathscr{E}_t, \tau_t') \cong (\mathscr{G}_t, \sigma)$. As (E_t, c_t') is the generic fiber of $(\mathscr{E}_t, \tau_t')$, it follows that $(E_t, c_t) \cong (G_{\breve{F}\widehat{\otimes}_F F_t}, \sigma^r)$.

$$p : \mathrm{Fix}[K^x g^x K^x]^{\underline{\mu},(r)} \to \mathrm{IsoSht}_G^{(r)} \qquad \underline{\mathscr{E}} \mapsto \underline{\mathscr{E}} \times_{\overline{X}} \breve{\eta}.$$

5.6 Elliptic Kottwitz Triples

In [Kot92], Kottwitz associates to a virtual abelian variety (A, λ, i) with additional structure a triple $(\gamma_0; \gamma, \delta)$ called the Kottwitz triple. They are certain conjugacy classes in $G(\mathbb{Q}), G(\mathbb{A}_f^p)$, and $G(L_r)$ respectively where $\mathbb{A}_f^p$ is the finite adeles away from p, and L_r is some unramified extension of a local field.

We wish to similarly attach a Kottwitz triple to a G-isoshtuka with r-structures so that we can reindex the counting formula in (5.6) in terms of certain Kottwitz triples. We first give its definition.

Definition 5.6.1. Let G^* be a quasi-split inner form of G over F. A **degree-r-Kottwitz triple** $(\gamma_0; \gamma, \delta)$ consists of

(i) a stable conjugacy class of a semisimple element $\gamma_0 \in G^*(F)$,

(ii) a tuple of conjugacy class $\gamma = (\gamma_t) \in \prod_{t \neq \infty, x} G(F_t)$ that is stably conjugate to γ_0,

(iii) a σ-conjugacy class $\delta \in G(F_{x^r})$ such that $N\delta$ is stably conjugate to γ_0 with $[\delta] \in B(G_x, \mu_x)$ where $[\delta]$ is induced from $G(F_{x^r}) \subset G(\breve{F}_x) \twoheadrightarrow BG_x(F_x)$.

Unlike in its number field counterpart, it is not clear that G-isoshtukas gives rise to a semisimple global conjugacy class. To explain what we mean by a global conjugacy class, let $(E, c, c') \in \mathrm{IsoSht}_G^{(r)}$. We choose trivializations $(E, c) \cong (G_{\breve{F}}, b\sigma)$ and $(E, c') \cong (G_{\breve{F}}, b'\sigma^r)$. We can view both maps $b\sigma$ and $b'\sigma^r$ as an element in the semi-direct product $G(\breve{F}) \rtimes \langle\sigma\rangle$. Namely, the product is given by $(b\sigma^r) \cdot (b'\sigma^s) = b\sigma^r(b')\sigma^{r+s}$. Then we define $\gamma := c^r(c')^{-1}$. Observe that $(b\sigma)^r = N_r(b)\sigma^r$ where $N_r(b) = b\sigma(b)\cdots\sigma^{r-1}(b)$ is the norm element of b. Indeed, $(b\sigma)(N_{r-1}(b)\sigma^{r-1}) = (b\sigma(N_{r-1}(b))\sigma^r = N_r(b)\sigma^r$, so we get the desired equality by induction. In particular, we have $\gamma = N_r(b)b'^{-1}$. Then we claim that

$$\sigma(\gamma) = b^{-1}\gamma b.$$

The condition that $cc' = c'c$ can be rewritten as $b\sigma(b') = b'\sigma^r(b)$. Therefore $\sigma^r(b) = b'^{-1}b\sigma(b')$. Also, $\sigma(\gamma) = \sigma(N_r(b)b'^{-1}) = \sigma(N_r(b))\sigma(b')^{-1} = b^{-1}N_r(b)\sigma^r(b)\sigma(b')^{-1}$.

$$\sigma(\gamma) = b^{-1}N(b)(b'^{-1}b\sigma(b'))\sigma(b'^{-1}) = b^{-1}N(b)b'^{-1}b = b^{-1}\gamma b$$

Therefore, $\gamma \in J_b(F)$ where J_b is a reductive group over F defined by the functor $J_b(R) := \{g \in G(R \otimes_F \breve{F}) \mid g = b\sigma(g)b^{-1}\}$. Then J_b is an $\breve{F}$-inner form of M_b^* which is the centralizer of $\nu_G(b)$ in $G^*_{\breve{F}}$ by [HK21, Proposition 6.2]. We are in the situation of [HK24, Lemma 8.1], so the $G^*(\overline{F})$-conjugacy class of γ contains an element $\gamma_0 \in G^*(F)$.

Let us make the following auxiliary definition.

Definition 5.6.2.

(i) Let G be a reductive group over a field F. Then a torus T in G containing $Z(G)^\circ$ is said to be **elliptic** if the torus $T/Z(G)^\circ$ is F-anisotropic.

(ii) A semisimple element $g \in G(F)$ is **elliptic** if g is contained in $T(F)$ for some elliptic maximal torus $\boldsymbol{T}$ of G.

Definition 5.6.3. We say that $(E, c, c') \in \mathrm{IsoSht}_G^{(r)}$ is **semisimple** (resp. **elliptic**, **elliptic regular**) if $\gamma_0 \in G^*(F)$ is semisimple (resp. elliptic, elliptic regular). We say that $(\mathscr{E}, \tau, \tau', \psi)$ is **semisimple** (resp. **elliptic**, **elliptic regular**) if its associated G-isoshtuka with r-structure is semisimple (resp. elliptic, ellptic regular).

Let us denote by $\mathrm{IsoSht}_G^{(r),\mathrm{er}} \subset \mathrm{IsoSht}_G^{(r),\mathrm{el}} \subset \mathrm{IsoSht}_G^{(r),\mathrm{ss}}$ to denote the set of elliptic regular G-isoshtukas, elliptic G-isoshtukas, and semisimple G-isoshtukas over $\breve{F}$. And define $\mathrm{Fix}[K^x g^x K^x]^{(r),\mathrm{er}} \subset \mathrm{Fix}[K^x g^x K^x]^{(r),\mathrm{el}} \subset \mathrm{Fix}[K^x g^x K^x]^{(r),\mathrm{ss}}$ similarly.

By the existence of a level $K^{x,\infty}$-structure, we know that $(E_t, c_t) \cong ((L\mathrm{Res}_{\mathscr{O}_t/\mathbb{F}_q[\![t]\!]}\mathscr{G}_t)_{\overline{\mathbb{F}}_q}, \sigma)$ for all $t \neq \infty, x$. Therefore, we have that $(E_t, c_t') \cong ((L\mathrm{Res}_{\mathscr{O}_t/\mathbb{F}_q[\![t]\!]}\mathscr{G}_t)_{\overline{\mathbb{F}}_q}, \gamma_t^{-1}\sigma^r)$ where $\gamma_t \in L\mathrm{Res}_{\mathscr{O}_t/\mathbb{F}_q[\![t]\!]}\mathscr{G}_t(\overline{\mathbb{F}}_q) = \mathscr{G}(F_t \widehat{\otimes}_{\mathbb{F}_q} \overline{\mathbb{F}}_q) = G(\breve{F} \widehat{\otimes}_F F_t)$. Then we have the conjugacy class of $\gamma = (\gamma_t)_{t \notin \{x,\infty\}} \in G(\mathbb{A}^{\infty,x})$. Finally, for $t \in \{x, \infty\}$, denote by $\underline{\mathscr{L}}_t := \underline{\mathscr{L}}_t((s, \mathscr{E}, \tau)) = (\mathscr{L}_t, \hat{\sigma}_t)$ the associated local $\mathscr{G}_t$-shtuka over $\overline{\mathbb{F}}_q$. Note that $\tau' : {}^\sigma\mathscr{E} \to \mathscr{E}$ descends to an isomorphism ${}^{\sigma^r}\mathscr{L}_t \to \mathscr{L}_t$. Fix a trivialization $(\mathscr{L}_t, \hat{\tau}) \cong ((L^+\mathscr{G}_t)_{\overline{\mathbb{F}}_q}, \overline{\delta}_t\sigma)$ where $\overline{\delta}_t \in L\mathscr{G}_t(\overline{\mathbb{F}}_q) = \mathscr{G}_t(F_t^{\mathrm{un}})$. Then following the argument of [Sch12, Proposition 6.2], $\overline{\delta}_t$ descends to $\delta_t \in \mathscr{G}_t(F_{t^r})$.

5.7 Counting Points in an isogeny class

Let us fix an elliptic regular G-isoshtuka $(E_0, c_0, c_0') \in \mathrm{IsoSht}_G^{(r),\mathrm{er}}$ associated to a fixed point $(\underline{\mathscr{E}}_0, \tau_0, \tau_0', \psi_0) \in \mathrm{Fix}[K^x g^x K^x]^{(r),\mathrm{er}}$. Recall that we wanted to compute

$$\mathrm{tr}([KgK]^{\underline{\mu},(r)} \mid H_c^*) = \sum_{e \in \mathrm{Fix}[K^x g^x K^x]^{(r)}} \mathrm{tr}(u_e)$$

We can rewrite the formula as

$$\sum_{e \in \mathrm{Fix}[K^x g^x K^x]^{(r)}} \mathrm{tr}(u_e) = \sum_{(E_0, c_0, c_0')} \sum_{e \to (E_0, c_0, c_0')} \mathrm{tr}(u_e)$$

where the first summation runs over G-isoshtukas with r-structures coming from a fixed point. The second summation runs over fixed points whose generic fibers are isogenous to (E_0, c_0, c_0'). We want to calculate the contribution of a single isogeny class associated to (E_0, c_0, c_0'), i.e.

$$\sum_{e \to (E_0, c_0, c_0')} \mathrm{tr}(u_e).$$

Therefore, this section is devoted to proving the following proposition.

Proposition 5.7.1. *Let $(\gamma_0; \gamma, \delta)$ be the Kottwitz triple associated to a fixed G-isoshtuka (E_0, c_0, c_0') with r-structures. Let us assume that $(\gamma_0; \gamma, \delta)$ is elliptic regular. Then we have the equality*

$$\sum_{e \to (E_0, c_0, c_0')} \mathrm{tr}(u_e) = \mathrm{vol}(G^*_{\gamma_0}(F) \backslash G^*_{\gamma_0}(\mathbb{A})/\Xi) O_\gamma(f^x) TO_\delta(\phi^{\mu_x}_{r, f_x}) TO(\phi^{\mu_\infty}_{r, f_\infty})$$

Here

$$\begin{aligned} O_\gamma(f^x) &= \int_{G^*_{\gamma_0}(\mathbb{A}^{\infty,x}) \backslash G(\mathbb{A}^{\infty,x})} f^x(y^{-1}\gamma y)\, d\overline{y} \\ TO_{\delta_t}(\phi^{\mu_t}_{r, f_t}) &= \int_{G^*_{\gamma_0}(F_t) \backslash G(F_{t^r})} \phi^{\mu_t}_{r, f_t}(w^{-1}\delta_t \sigma(w))\, d\overline{w} \end{aligned}$$

are orbital integral and twisted integral.

Let us start by giving a description of the automorphism group of our fixed G-isoshtuka (E_0, c_0, c_0').

Lemma 5.7.2. *Let $(E_0, c_0, c_0') \in \mathrm{IsoSht}_G^{(r),\mathrm{er}}$ and $(\gamma_0; \gamma, \delta)$ be the associated Kottwitz triple. The automorphism group $\mathrm{Aut}(E_0, c_0, c_0')$ of (E_0, c_0, c_0') is the set of F-points of the centralizer $G^*_{\gamma_0}$ of γ_0 in G^*. Furthermore, $G^*_{\gamma_0}(F_t)$ is the centralizer of γ_0 in $G(F_t)$ for $t \notin \{x, \infty\}$ and $G^*_{\gamma_0}(F_t) = G_{\delta_t \sigma}(F_t)$ is the twisted centralizer of δ_t in $G(F_t \widehat{\otimes}_{\mathbb{F}_q} \mathbb{F}_{q^r})$.*

Proof. Since E is trivializable, we may assume that $E = G^*_{\breve{F}}$ where G^* is a quasi-split inner form of G. Recall that $\gamma_0 \in G^*(F)$ is defined as $c_0^r (c_0')^{-1} = \gamma_0$ over $\breve{F}$. Since c_0 and c_0' commute, both g and c_0 commute with γ_0. Therefore, both g and c_0 are elements in $G^*_{\gamma_0}(\breve{F})$. As γ_0 is regular by assumption, G_{γ_0} is a torus. Therefore c_0 and g commute.

$$
\begin{aligned}
\mathrm{Aut}(E_0, c_0, c_0') &= \{g \in G^*(\breve{F}) \mid gc_0 = c_0\sigma(g) \text{ and } g\gamma_0 = \gamma_0 g\} \\
&= \{g \in G^*_{\gamma_0}(\breve{F}) \mid c_0 g = c_0 \sigma(g)\} \\
&= \{g \in G^*_{\gamma_0}(\breve{F}) \mid g = \sigma(g)\} \\
&= G^*_{\gamma_0}(F)
\end{aligned}
$$

□

Remark 5.7.3. After trivializing E, c_0 is the same as $b\sigma$ where $b \in G^*(\breve{F})$. If γ_0 is semisimple but not necessarily regular, then the automorphism group is given by

$$
\{g \in G^*_{\gamma_0}(\breve{F}) \mid gb = b\sigma(g)\}
$$

which is the set of F-points of an inner form J_b of $G^*_{\gamma_0}$.

Let us follow the arguments from [Kot92, §16]. The set of fixed points for which its associated G-isoshtuka (E, c, c') is isogenous to (E_0, c_0, c_0') is equal to the set $G^*_{\gamma_0}(F)\backslash Y$ where Y is the set of pairs $(\underline{\mathscr{E}}, \varphi)$ where $\underline{\mathscr{E}}$ is a fixed point in $\mathrm{Fix}[K^x g^x K^x]^{(r),\mathrm{er}}$ whose associated G-isoshtuka (E, c, c') is isogenous to (E_0, c_0, c_0'), and $\varphi : (E, c, c') \to (E_0, c_0, c_0')$ is a morphism of G-isoshtukas with r-structures.

Therefore, we have that there is a bijection between $G^*_{\gamma_0}(F)\backslash Y$ with the set $G^*_{\gamma_0}(F)\backslash(Y_x \times Y_\infty \times Y^{\infty,x})$ where

$$\begin{aligned} Y^{\infty,x} &= \{y \in G(\mathbb{A}^{\infty,x})/K^x_{g^x} \mid y^{-1}\gamma y \in K^x(g^x)^{-1}\} \\ Y_t &= \{w_t \in G(F_{t^r})/\mathscr{G}(\mathscr{O}_{t^r}) \mid w_t^{-1}\delta\sigma(w_t) \in \mathscr{G}(\mathscr{O}_{t^r})\sigma(\mu(\varpi_t^{-1}))\mathscr{G}(\mathscr{O}_{t^r})\} \end{aligned}$$

Note that by Remark 5.5.4, we have

$$\begin{aligned} \sum_{e \to (E_0, c_0, c_0')} \mathrm{tr}(u_e) &= \sum_{e \to (E_0, c_0, c_0')} \mathrm{tr}(\sigma^r \circ g^{\infty,x} \mid (\overline{\mathbb{Q}}_\ell)_{\overline{e}}) \mathrm{tr}(\sigma^r \circ (g_x, 1) \mid (R\Psi\pi_{KK^x*}\overline{\mathbb{Q}}_\ell)_{\overline{e}}) \\ &= \sum_{e \to (E_0, c_0, c_0')} \prod_{t \in \{x, \infty\}} \mathrm{tr}(\tau \circ g_t \mid H^i(X^{\mu_t}_{\underline{\mathscr{L}}_t, K_t} \otimes_k \hat{\overline{k}}, \overline{\mathbb{Q}}_\ell)) \\ &= \sum_{e \to (E_0, c_0, c_0')} \phi^{\mu_x}_{r, f_x}(\delta_x)\phi^{\mu_\infty}_{r, f_\infty}(\delta_\infty) \end{aligned}$$

where $g_\infty = 1$. Then the summation becomes the integral

$$\int_{G^*_{\gamma_0}(F)\backslash(G(\mathbb{A}^{\infty,x}) \times G(F_{x^r}) \times G(F_{\infty^r}))} \tilde{f}^x(y^{-1}\gamma y)\phi^{\mu_x}_{r,f_x}(w_x^{-1}\delta\sigma(w_x))\phi^{\mu_\infty}_{r,f_\infty}(w_\infty^{-1}\delta_\infty\sigma(w_\infty)), \qquad (5.10)$$

where $\widetilde{f}^x$ is the characteristic function of $K^x(g^x)^{-1}$ and $\phi^{\mu_t}_{r,f_t}$ is the test function defined as before.

We use the Haar measure on $G^*_{\gamma_0}(F)$ giving points measure 1, the Haar measure on $G(\mathbb{A}^{\infty,x})$ giving $K^x_{g^x}$ measure 1, and the Haar measure of $G(F_{t^r})$ giving $\mathscr{G}(\mathscr{O}_{t^r})$ giving measure 1. As $O_\gamma(\widetilde{f^x}) = O_\gamma(f^x)$ where f^x was the characteristic function $K^x(g^x)^{-1}K^x$. Therefore, the integral (5.10) is given by

$$\mathrm{vol}(G^*_{\gamma_0}(F)\backslash G^*_{\gamma_0}(\mathbb{A})/\Xi)O_\gamma(f^x)TO_{\delta_x}(\phi^{\mu_x}_{r,f_x})TO_{\delta_\infty}(\phi^{\mu_\infty}_{r,f_\infty})$$

which completes the proof of Proposition 5.7.1.

5.8 Kottwitz invariants

Let $\Gamma = \mathrm{Gal}(\overline{F}/F)$ be the Galois group where $\overline{F}$ is a separable closure of F and $\widehat{G}$ be the dual group, i.e. the complex connected reductive group whose root datum is dual to that of G. Given a Kottwitz triple $(\gamma_0;\gamma,\delta)$, we want to determine when the triple comes from a fixed point $\mathrm{Fix}[K^x g^x K^x]^{\underline{\mu},(r)}$. To answer this questions, we will associate a character $\mathrm{inv}(\gamma_0;\gamma,\delta)$ called the **invariant** of $Z(\widehat{G}_{\gamma_0})^\Gamma$ to the Kottwitz triple $(\gamma_0;\gamma,\delta)$.

Let $(\gamma_0;\gamma,\delta)$ be a semisimple Kottwitz triple. Then $\gamma_0 \in G^*(F)$ is a semisimple element, and the centralizer $G^*_{\gamma_0}$ is a reductive group over F. We will define invariants locally and define the invariant as the sum of all those invariants.

Let $t \notin \{x,\infty\}$. Since γ_t is stably conjugate to γ_0, we can find an element $g \in G(\overline{F}_t)$ such that $\gamma_x = g\gamma_0 g^{-1}$. Since $H^1(\overline{F}_t/F^{\mathrm{un}}_t, G^*_{\gamma_0}(\overline{F}_t)) = 1$ by Steinberg's theorem, we can choose $g \in G(F_t\hat{\otimes}_{\mathbb{F}_q}\overline{\mathbb{F}}_q)$. As $\gamma_t = \sigma(\gamma_t)$, this translates to $g\gamma_0 g^{-1} = \sigma(g)\sigma(\gamma_0)\sigma(g)^{-1} = \sigma(g)b^{-1}\gamma_0 b\sigma(g)^{-1}$. Therefore, $g^{-1}\sigma(g)b^{-1}$ is a centralizer of γ_0 in $G(\overline{F}_t)$ and defines an element $B(G^*_{\gamma_0,F_t})$. Then we denote by $\mathrm{inv}_t(\gamma_0;\gamma,\delta)$ to be the restriction to $Z(\widehat{G}^*_{\gamma_0})^\Gamma$ of the image of $g^{-1}\sigma(g)b^{-1}$ under the

Kottwitz map $\kappa : B(G^*_{\gamma_0,F_t}) \to X^*(Z(\widehat{G}^*_{\gamma_0})^{\Gamma_t})$ for the group $G^*_{\gamma_0,F_t}$.

Let $t \in \{x, \infty\}$. Since $N(\delta_t)$ is stably conjugate to γ_0, there exists $g \in G(\overline{F}_t)$ such that $N(\delta_t) = g\gamma_0 g^{-1}$. As above, we can choose $g \in G(F_t \widehat{\otimes}_{\mathbb{F}_q} \overline{\mathbb{F}}_q)$. Then one has

$$\begin{aligned} \sigma(N(\delta_t)) &= \sigma(g)b^{-1}\gamma_0 b\sigma(g)^{-1} \\ \sigma(N(\delta_t)) &= \delta_t^{-1}N(\delta_t)\sigma^r(\delta_t) \\ &= \delta_t^{-1}N(\delta_t)\delta_t \\ &= \delta_t^{-1}g\gamma_0 g^{-1}\delta_t \end{aligned}$$

Therefore, we have $g^{-1}\delta_t\sigma(g)b^{-1}$ is the centralizer of γ_0 in $G(\overline{F}_t)$. Let $\mathrm{inv}_t(\gamma_0; \gamma, \delta) \in X^*(Z(\widehat{G}^*_{\gamma_0})^{\Gamma})$ be the restriction of the image of $g^{-1}\delta_t\sigma(g)b^{-1}$ under the Kottwitz map as above. Then we can finally define the **invariant** of $(\gamma_0; \gamma, \delta)$ to be

$$\mathrm{inv}(\gamma_0; \gamma, \delta) = \prod_{t \in |X|} \mathrm{inv}_t(\gamma_0; \gamma, \delta).$$

Here $\mathrm{inv}_t(\gamma_0; \gamma, \delta) = 1$ for almost all $t \in |X|$, so the product is a well-defined element in $X^*(Z(\widehat{G}^*_{\gamma_0})^{\Gamma})$.

We introduce an alternative definition of Kottwitz invariant for a fixed point $(\mathscr{E}, \tau, \tau', \psi) \in \mathrm{Fix}[K^x g^x K^x]^{\underline{\mu},(r)}$. In fact, we will define Kottwitz invariant for an element $b \in G^*_{\gamma_0}(\breve{F})$. Let (E, c, c') be the generic fiber of $(\mathscr{E}, \tau, \tau', \psi)$. Set $\gamma = c^r(c')^{-1}$ and let γ_0 be the element in $G^*(\overline{F})$-conjugacy class of γ as in the discussion above Definition 5.6.2. After choosing a trivialization $E \cong G_{\breve{F}}$, the isomorphism $c : {}^\sigma E \to E$ induces an isomorphism $c_0 : {}^\sigma G_{\breve{F}} \to G_{\breve{F}}$. Then c_0 is of the form $b\sigma$ where $b \in G(\breve{F})$. Since c and c' commute, c commutes with γ_0, so $b \in G^*_{\gamma_0}(\breve{F})$. For a place $t \in |X|$, let b_t be its t-component in $G^*_{\gamma_0}(F_t \widehat{\otimes}_{\mathbb{F}_q} \overline{\mathbb{F}}_q)$ and hence defines an element

$B(G^*_{\gamma_0}(F_t))$ which we again write b_t. We write $\mathrm{inv}_t(b)$ be the restriction of the image of b_t under the Kottwitz map to $Z(\widehat{G}^*_{\gamma_0})^\Gamma$. The (global) **invariant** of $b \in G^*_{\gamma_0}(\breve{F})$ is the product

$$\mathrm{inv}(\mathscr{E}, \tau, \tau', \psi) := \mathrm{inv}(b) := \prod_{t \in |X|} \mathrm{inv}_t(b).$$

Since $b \in B(G^*_{\gamma_0})$ is a global element, $b_t \in \mathscr{G}(\mathscr{O}_t \widehat{\otimes}_{\mathbb{F}_q} \overline{\mathbb{F}}_q)$ for almost all $t \in |X|$. This means that b_t is σ-conjugate to 1, so the local invariant $\mathrm{inv}_t(b)$ is the trivial character. Therefore, The product is well-defined.

Proposition 5.8.1. *If* $(\gamma_0; \gamma, \delta)$ *is the Kottwitz triple associated to the fixed point* $(\mathscr{E}, \tau, \tau', \psi) \in \mathrm{Fix}[K^x g^x K^x]^{\underline{\mu}, (r), \mathrm{ss}}$*, we have*

$$\mathrm{inv}(\gamma_0; \gamma, \delta) = \mathrm{inv}(\mathscr{E}, \tau, \tau', \psi) = 1.$$

Proof. We first prove that $\mathrm{inv}(\gamma_0; \gamma, \delta) = \mathrm{inv}(\mathscr{E}, \tau, \tau', \psi)$. In fact, we show that $\mathrm{inv}_t(\gamma_0; \gamma, \delta) = \mathrm{inv}_t(\mathscr{E}, \tau, \tau', \psi)$ for every $t \in |X|$. Since we assumed that $(E_t, c_t) \cong ($ $g \in G(F_t \widehat{\otimes}_{\mathbb{F}_q} \overline{\mathbb{F}}_q)$ such that $b = g^{-1}\sigma(g)$. Also, recall that $cc' = c'c$ translates to $b\sigma(b') = b'\sigma^r(b)$. Therefore, we have

$$b'\sigma^r(g^{-1})\sigma^{r+1}(g) = g^{-1}\sigma(g)\sigma(b')$$

Then one has $gb'\sigma^r(g^{-1}) = \sigma(gb'\sigma^r(g^{-1}))$, so we have $gb'\sigma^r(g^{-1}) \in G(F_t)$. Recall that γ_t^{-1} was defined via the isomorphism $(E, b'\sigma^r) \cong (G_{F_t \widehat{\otimes}_{\mathbb{F}_q} \overline{\mathbb{F}}_q}, \gamma_t^{-1}\sigma^r)$. Therefore, $gb'\sigma^r(g^{-1})$ is exactly γ_t^{-1}. As $\gamma_t = g\gamma_0 g^{-1}$, we see that the local invariants $\mathrm{inv}_t(\gamma_0; \gamma, \delta)$ and $\mathrm{inv}_t(\mathscr{E}, \tau, \tau', \psi)$ at $t \notin \{x, \infty\}$ are the same. Similar proof shows that the local invariant agrees at $t \in \{x, \infty\}$.

For any connected reductive group H over F and $b \in H(\breve{F})$, [ND07, Proposition 7.2] proves that $\mathrm{inv}(b) = 0$. In particular, we can conclude that $\mathrm{inv}(\mathscr{E}, \tau, \tau', \psi) = \mathrm{inv}(b) = 1$ as $G^*_{\gamma_0}$ is a connected reductive group over F for semisimple $\gamma_0 \in G^*_{\gamma_0}(F)$. □

5.9 Langlands-Kottwitz Formula

Lemma 5.9.1. *Let $(\gamma_0; \gamma, \delta)$ be a regular elliptic Kottwitz triple associated to a G-isoshtuka (E, c, c') with r-structures. Then the number of such isogeny classes is given by the order of the finite set*

$$\ker^1(F, G^*_{\gamma_0}) = \ker\left(H^1(F, G^*_{\gamma_0}) \to \prod_{t \in |X|} H^1(F_t, G^*_{\gamma_0})\right)$$

Proof. The same argument in [ND07, Proposition 11.1] should work in our case. □

Combining the Proposition 5.7.1 and Lemma 5.9.1, we have our main theorem.

Theorem 5.9.2. *Let $f^x \in C_c^\infty(G(\mathbb{A}^{x,\infty}))$, $h_t \in c_c^\infty(G(\mathscr{O}_t))$ and j be a positive integer and write $r := j[\mathbb{F}_{y,\infty'} : \mathbb{F}_q]$. Then*

$$\mathrm{tr}(\sigma^r \times f \mid H^*)^{\text{el,reg}}$$
$$= \sum_{(\gamma;\gamma,\delta)} |\ker^1(F, G^*_{\gamma_0})| \mathrm{vol}(G^*_{\gamma_0}(F) \backslash G^*_{\gamma_0}(\mathbb{A})/\Xi) O_\gamma(f) TO_{\delta_t}(\phi^{\mu_x}_{r,f_x}) TO_{\delta_\infty}(\phi^{\mu_\infty}_{r,f_\infty})$$

where the summation runs over Kottwitz triples $(\gamma_0; \gamma, \delta)$ that is associated with the fixed points such that the invariant is zero.

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