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National Science Foundation
Engineering Research Center
Program (NSFD CD 8803012),
Industry and the University

Interpolating Varieties for Spaces of Meromorphic Functions

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TR 92-39

Interpolating Varieties for Spaces of Meromorphic Functions*

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Abstract

Various interesting results on interpolation theory of entire functions with given growth conditions have been obtained by imposing conditions on multiplicity varieties and weights. All the results discussed in the literature are limited to the space of entire functions. In this paper, we shall extend and generalize the interpolation problem of entire functions to meromorphic functions. The analytic conditions sufficient and necessary for a given multiplicity variety to be interpolating for meromorphic functions with given growth conditions will be obtained. Moreover, purely geometric characterization of interpolating varieties will be given for slowly decreasing radial weights which enable us to determine whether or not a given multiplicity variety is an interpolating variety by direct calculation. When weights grow so rapidly as to allow infinite order functions in the considered space, the geometric conditions would become more delicate. For such weights $p(z)$, we also find purely geometric sufficient as well as necessary conditions provided that $\log p(e^r)$ is convex . As corollaries of our results, one obtains the corresponding results for the interpolation of entire functions .

* This research is supported in part by NSF grants DMS-90-00616 and CDR 88-03012.

Interpolating Varieties for Spaces of Meromorphic Functions*

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1. Introduction

Suppose that $f(z)$ is a meromorphic function in the complex plane $\mathbf{C} = \{z : |z| < \infty\}$ and $\{z_k\}_{k=1}^{\infty}$ is a sequence of complex numbers. By $n(r, z_k, \frac{1}{f})$ and $n(r, z_k, f)$ we denote the numbers of zeros and poles (counted with multiplicity) of f in the disc $B(z_k, r) = \{z : |z - z_k| \leq r\}$, respectively. Then we have the following Laurent expansion near each z_k:

$$f(z) = \sum_{l=-n_k}^{\infty} f_{k,l}(z - z_k)^l,$$

where

$$f_{k,l} = \frac{1}{2\pi i}\int_{\gamma_k} \frac{f(z)}{(z - z_k)^{l+1}}dz,$$

$$\gamma_k = \{z : |z - z_k| = \rho_k, \rho_k > 0 \text{ some small number}\},$$

and

$$n_k = n(0, z_k, f) - n(0, z_k, \frac{1}{f}).$$

In particular, if $f(z)$ is regular at some point z_k, then $f_{k,l} = \frac{f^{(l)}(z_k)}{l!}$.

A subharmonic function $p(z)$:$\mathbf{C}$—$[0, \infty)$ is called a weight function if it satisfies the following conditions :

(i) $\log(1 + |z|^2) = O(p(z))$

(ii) there exist constants C and D such that $|\zeta - z| \leq 1$ implies

$$p(\zeta) \leq Cp(z) + D.$$

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* This research is supported in part by NSF grants DMS-90-00616 and CDR 88-03012.

We shall always assume that $p(z)$ is a weight function throughout this paper. Now let us recall the following definitions (see e.g.[4]).

Definition 1.1. $\mathbf{A}_p = \{f \text{ entire} : |f(z)| \leq A\exp(Bp(z)) \text{ for some } A, B > 0\}$.

Remark 1.2. It is easy to check that all polynomials belong to $\mathbf{A}_p$ and $\mathbf{A}_p$ is closed under differentiation.

Remark 1.3. The two basic examples of such weight functions are $p(z) = |z|^\rho (\rho > 0)$ and $p(z) = |Imz| + \log(1+|z|^2)$ corresponding to the space $\mathbf{A}_p$ of all entire functions of order$\leq \rho$ and finite type and the space $\hat{\mathcal{E}}'$ of Fourier transforms of distributions with compact support in the real line (see e.g.[8]).

Definition 1.4. Let $V = \{(z_k, m_k)\}_{k=1}^{\infty}$ be a multiplicity variety , that is , a sequence of points $\{z_k\}_{k=1}^{\infty} \subset \mathbf{C}$ with $|z_k| \uparrow \infty$, and a sequence of positive integers $\{m_k\}_{k=1}^{\infty}$ corresponding to the multiplicities of the points z_k. Then

$$\mathbf{A}_p(V) := \{a = \{a_{k,l}\}_{\substack{k\in\mathbf{N} \\ 0\leq l<m_k}} : \sum_{l=0}^{m_k-1} |a_{k,l}| \leq A\exp(Bp(z_k))$$

for some constants $A, B > 0$ independent of k but depending on $a\}$, where $\mathbf{N}$:= the set of all positive integers.

With the above notations, the interpolation problems usually considered are typified by the following model(see e.g.[4]) : Under what conditions is it true that for any doubly indexed sequence $\{a_{k,l}\} \in \mathbf{A}_p(V)$ there exists an entire function $f \in \mathbf{A}_p$ such that $f_{k,l} := \frac{f^{(l)}(z_k)}{l!} = a_{k,l}$ for any $k \in \mathbf{N}$ and $0 \leq l \leq m_k - 1$, that is, $f(z)$ has given Taylor coefficients?

Definition 1.5. If the above problem has a solution for every $\{a_{k,l}\} \in \mathbf{A}_p(V)$, we will say that V is an interpolating variety for the space $\mathbf{A}_p$ of entire functions.

Various interesting results are obtained by imposing conditions on $z_k, m_k, a_{k,l}$ or weights $p(z)$ (see e.g [4],[5],[13],[16] and [17]). The interpolation results with growth conditions discussed in the literature are limited to the space of entire functions which is a proper subspace of the space of meromorphic functions. For the latter, to extend and generalize the classic Mittag-Leffler theorem and the interpolation theory of entire functions , it's natural to try to find a meromorphic function with given Laurent coefficients (see Part 3 in the present paper). This provides the model for the general interpolation problems in the spaces of meromorphic functions which follows. Since any meromorphic function f is the quotient of two entire functions f_1, f_2, i.e., $f = \frac{f_1}{f_2}$ we give the following

Definition 1.6. $\mathbf{M}_p = \{f = \frac{f_1}{f_2} \text{ meromorphic} : f_i \in \mathbf{A}_p, i = 1,2\}$.

Remark 1.7. If $p(z) = \log(1+|z|^2)$, then $\mathbf{M}_p$ is the space of all rational functions. If $p(z) = |z|^\rho (\rho > 0)$, then $\mathbf{M}_p$ is the field of quotient of the rings of all entire functions of order $\leq \rho$ and finite type, or the field of all meromorphic functions of order $\leq \rho$ and finite type (see [18]).

Definition 1.8. Let $V = \{(z_k, m_k, n_k)\}$ be a multiplicity variety (for meromorphic functions) with $|z_k| \uparrow \infty$, $m_k \geq 1$ and $n_k \geq 0$. Then

$$\mathbf{M}_p(V) := \{a = \{a_{k,l}\}_{\substack{k \in \mathbf{N} \\ -n_k \leq l < m_k}} : \sum_{l=-n_k}^{m_k - 1} |a_{k,l}| \leq A \exp(Bp(z_k))$$

for some positive constants A, B independent of k but depending on $a\}$.

We pose the following interpolation problem for meromorphic functions : under what conditions is it true that for any sequence $a = \{a_{k,l}\} \in \mathbf{M}_p(V)$, there exists a meromorphic function $f(z) \in \mathbf{M}_p$ such that for any $k \in \mathbf{N}$,

$$f_{k,l} = \begin{cases} 0, & \text{if } l < -n_k \\ a_{k,l}, & \text{if } -n_k \leq l \leq m_k - 1 \end{cases} \quad ?$$

That is, when does there exist a meromorphic function $f \in \mathbf{M}_p$ such that $f(z)$ has given Laurent coefficients, in particular, $f(z)$ has given singular parts at poles z_k and given Taylor coefficients at regular points z_k ?

To find good choices of f, we also ask that the first nonvanishing Taylor coefficient of the denominator of f at z_k is not 'too small' in absolute value , i.e. not less than $\varepsilon \exp(-Lp(z_k))$ for some positive numbers ε, L independent of k but depending on the sequence $a = \{a_{k,l}\}$.

Definition 1.9. If the above problem has a solution for every sequence $\{a_{k,l}\} \in \mathbf{M}_p(V)$, we will call that $V = \{(z_k, m_k, n_k)\}$ is an interpolating variety for the space $\mathbf{M}_p$ of meromorphic functions.

The present paper is divided into five sections. As background , the first two sections contain the basic notions and notations. In the third section, we obtain the analytic conditions necessary and sufficient for V to be an interpolating variety for M_p, which give a complete solution to the above problem. The fourth and fifth sections are concerned with the geometry of interpolating varieties for finite and infinite order functions. Purely geometric characterization of interpolating varieties will be given in Section 4 for radial weights $p(z)$ with $p(\alpha z) = O(p(z))$ for some $\alpha > 1$. And purely geometric sufficient as well as necessary conditions will be given in Section 5 for radial weights $p(z)$ with $\log p(e^r)$ being convex. As corollaries, one obtains the corresponding results for entire functions studied by Berenstein, Taylor, Squires and others in [4],[16] and [17] , etc.

2. Preliminaries

In the following, we need to introduce some more notations . Let's start with some concepts in the standard Nevanlinna theory (see e.g.[9],[18]). If $f(z)$ is a meromorphic function in $\mathbf{C}$,then

$$Z(f) = \{z : f(z) = 0\}$$

(counted with multiplicity),

$$M(r,f) = \sup_{|z|=r}\{|f(z)|\},$$

$$N(r,f) = \int_0^r \frac{n(t,f)-n(0,f)}{t}dt + n(0,f)\log r,$$

where $n(t,f) = n(t,0,f)$ was defined in §1,

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|d\theta,$$

and

$$T(r,f) = N(r,f) + m(r,f)$$

is the Nevanlinna characteristic. Similarly, we have the functions

$$N(r,z_0,f), \quad m(r,z_0,f) \quad \text{and} \quad T(r,z_0,f)$$

defined with respect to the disk $\{|z-z_0| \leq r\}$.

If $Z = \{z_k\}$ is a sequence of not necessary distinct complex numbers such that $z_n \to \infty$ as $n \to \infty$, then

$$n(r,Z) = \sum_{|z_k|\leq r} 1$$

is the number of points of Z in $|z| \leq r$,

$$N(r,Z) = \int_0^r \frac{n(t,f)-n(0,Z)}{t}dt + n(0,Z)\log r,$$

$$S(r,n,Z) = \frac{1}{n}\sum_{|z_k|\leq r, z_k\neq 0} (\frac{1}{z_k})^n, n \in \mathbf{N},$$

and

$$S(r_1,r_2,n,Z) = S(r_2,n,Z) - S(r_1,n,Z), 0 < r_1 \leq r_2.$$

Similarly we define $n(r, z_0, Z)$ and $N(r, z_0, Z)$ with respect to the disk$\{|z - z_0| \le r\}$.

Suppose that $\lambda(r)$ is a non–negative function for $r \ge 0$. We shall say that $f(z)$ is of finite λ-type if $T(r, f) \le A\lambda(Br)$ for some constants $A, B > 0$ and all $r \ge 0$.

We shall say that Z has finite λ-density if $N(r, Z) \le A\lambda(r)(r \ge 0)$ for some $A > 0$; Z is λ–balanced if

$$S(r_1, r_2, n, Z) \le \frac{A\lambda(r_1)}{r_1^n} + \frac{A\lambda(r_2)}{r_2^n}(0 < r_1 \le r_2, n \in \mathrm{N})$$

for some $A > 0$; Z is λ –admissible if Z has finite λ–density and is λ –balanced.

Throughout the paper, we shall use A and B to denote positive constants the actual values of which may vary from one occurrence to the next.

3. Interpolating varieties for M_p, I

As is pointed out in Section 1, the original model for the interpolation problem in M_p is the following result:

If $V = \{(z_k, m_k, n_k)\}$ is a multiplicity variety, then for any sequence $\{a_{k,l}\}$ $(-n_k \le l < m_k)$ of complex numbers, there exists a meromorphic function such that

$$f_{k,l} = \begin{cases} 0, & \text{if } l < -n_k \\ a_{k,l}, & \text{if } -n_k \le l < m_k \end{cases} \qquad k \in \mathrm{N}. \tag{3.1}$$

That is, $f(z)$ has given Laurent coefficients. In particular, $f(z)$ has given singular parts at poles z_k and given Taylor coefficients at regular points z_k.

This result can be proved by using Theorem 3 in [4] or by the following direct argument: using the Weierstrass theorem, there exists an entire function $g(z)$ such that near each z_k,

$$g(z) = g_{k,m_k}(z - z_k)^{m_k} + g_{k,m_k+1}(z - z_k)^{m_k+1} + \cdots, \qquad \text{where } g_{k,m_k} \ne 0.$$

Set

$$R_k(z) = \sum_{i=1}^{n_k+m_k} c_{k,i}(z - z_k)^{-i},$$

where the $c_{k,i}$ will be chosen in what follows. We then have, about z_k,

$$g(z)R_k(z) = \frac{1}{(z - z_k)^{n_k}}(g_{k,m_k} + g_{k,m_k+1}(z - z_k) + \cdots)$$

$$\times \left(c_{k,n_k+m_k} + \cdots + c_{k,1}(z - z_k)^{n_k+m_k-1}\right).$$

Since $g_{k,m_k} \neq 0$, we can choose $c_{k,i}(1 \leq i \leq m_k + n_k)$ such that

$$g(z)R_k(z) = \frac{1}{(z - z_k)^{n_k}}(a_{k,-n_k} + \cdots + a_{k,-1}(z - z_k)^{n_k-1} +$$

$$+ a_{k,0}(z - z_k)^{n_k} + \cdots + a_{k,m_k-1}(z - z_k)^{n_k+m_k-1}).$$

(Just compare the coefficients of $1, (z - z_k), \cdots, (z - z_k)^{n_k+m_k-1}$ on both sides).

Now by the Mittag-Leffler theorem, there is a meromorphic function $h(z)$ such that the singular part of $h(z)$ at z_k is $R_k(z)$ for all k. Then we can readily check that $f(z) = g(z)h(z)$ satisfies (3.1). The result follows.

For the general interpolation problem in M_p, the first observation is that a result similar to the one as above is not true any more. Along the lines of the above result, the only interesting case occurs when $V = \{(z_k, m_k, n_k)\}$ satisfies

$$V^* = \{(z_k, \max\{m_k, n_k\})\} \subset Z(f)$$

for some $f \in A_p$ (see Theorem 1 below) . Consider, for example, the case when $p(z) = |z|^\rho (\rho > 0)$. It's easy to construct a multiplicity variety $V = \{(z_k, m_k, n_k)\}$ such that $V^* = \{(z_k, \max\{m_k, n_k\})\}$ is not of finite p -density (e.g. for $Z = \{z_k\} = \{\log k\}(k > 1), N(r, Z) \geq \int_{\frac{r}{2}}^{r} \frac{n(t,Z)}{t} dt \geq n(\frac{r}{2}, Z)\log 2 = [e^{\frac{r}{2}}]\log 2$ and thus Z is not of finite p-density). Then such multiplicity variety can not be an interpolation variety for

A_p, since otherwise there exists a $f \in A_p$ such that $V^* \subset Z(f)$ (see Theorem 1) and thus by the Nevanlinna first fundamental theorem (see e.g. [9])

$$N(r, V^*) \leq N(r, \frac{1}{f}) \leq T(r, \frac{1}{f}) = T(r, f) + O(1) \leq \log M(r, f) + O(1) \leq Ap(r)$$

for some $A > 0$, i.e. V^* is of finite p- density, a contradiction.

The following both sufficient and necessary conditions give a complete solution to the interpolation problem for the space M_p.

Theorem 3.1. A multiplicity variety $V = \{(z_k, m_k, n_k)\}$ is an interpolating variety for M_p if and only if there exists an entire function $f \in A_p$ such that

$$V^* := \{(z_k, t_k)\} \subset Z(f),$$

where $t_k := \max\{m_k, n_k\}$ and

$$|f_{k,t_k}| \geq \varepsilon \exp\left(-cp(z_k)\right), k \in \mathbf{N}, \tag{3.2}$$

for some constants $\varepsilon, c > 0$.

To prove this theorem, we shall need the following lemmas.

Lemma 3.2. Let $g(z)$ be analytic in $D = \{z : |z| < 1\}$ and satisfy $|g(z)| \leq M$ for some constant $M > 0$. If $g(a) = 0$ for some $a \in D$. Then $|a| \geq \frac{|g(0)|}{2M}$.

Proof. Set

$$F(z) = \frac{g(z) - g(0)}{2M}.$$

Then $F(0) = 0$ and $|F(z)| \leq 1$ for $z \in D$. Hence by the Schwarz Lemma [7],

$$\frac{|g(0)|}{2M} = |F(a)| \leq |a|.$$

The proof is finished. ∎

Lemma 3.3. If $V = \{(z_k, m_k, n_k)\}$ is an interpolating variety for $\mathbf{M_p}$ and $M > 0$, then there exist four constants $A, B, L_1, \varepsilon_1 > 0$, two sequences $\{q_k\}$ and $\{v_k\}$ of integers with $q_k \geq n_k, v_k \geq 0$, and two sequences $\{g_k\}$ and $\{h_k\}$ of functions in the space A_p such that

$$(g_k)_{i,l} = 0, i \in \mathbf{N}, 0 \leq l \leq q_i + m_i - 1, \text{except that} |(g_k)_{k,q_k+m_k-1}| \geq \varepsilon_1 \exp(-L_1 p(z_k)),$$

$$(h_k)_{i,l} = 0, i \in \mathbf{N}, 0 \leq l \leq v_i + m_i - 1, \text{except that} \quad |(h_k)_{k,v_k}| \geq \varepsilon_1 \exp(-L_1 p(z_k)).$$

Moreover

$$|g_k(z)| \leq A \exp(Bp(z))/\exp(Mp(z_k)),$$

and

$$|h_k(z)| \leq A \exp(Bp(z))/\exp(Mp(z_k)).$$

Proof. Set

$$S = \{a = \{a_{k,l}\}_{k \in \mathbf{N}, -\infty < l < m_k} : a_{k,l} = 0, l < -n_k, \quad \text{and}$$
$$(\sum_{l=-n_k}^{m_{k-1}} |a_{k,l}|) \exp(-Mp(z_k)) \leq 1, k \in \mathbf{N}\}.$$

Then the space S is complete under the metric induced by the norm

$$||a|| := sup\{(\sum_{l=-n_k}^{m_k-1} |a_{k,l}|) \exp(-Mp(z_k)) : k \in \mathbf{N}\}.$$

Let

$$A_{p,N} = \{f \in \mathbf{A}_p : |f_{k,\tau_k}| \geq \frac{1}{N} \exp(-Np(z_k)), \quad \text{where} \quad \tau_k = n(0, z_k, 1/f)\},$$

and

$$S_{m,n,N} = \{\left((\frac{\alpha}{\beta})_{k,l}\right)_{k \in \mathbf{N}, -\infty < l < m_k} \in S : (\alpha, \beta) \in A_p \times A_{p,N},$$
$$|\alpha(z)| \leq m \exp(mp(z)), |\beta(z)| \leq n \exp(np(z))\},$$

where $m, n, N \in \mathbf{N}$. It's easy to see that

$$S = \cup_{m,n,N=1}^{\infty} S_{m,n,N}$$

by the hypotheses of the lemma.

The proof follows the lines of the open mapping theorem (c.f. [10,p.294],[4,p.125] or [16]) to show that at least one of $S_{m,n,N}$ has non-empty interior which gives the functions required in the lemma. To this end, we first prove that each $S_{m,n,N}$ is a closed subset of S. In fact, if

$$a_i = \left((\frac{\alpha_i}{\beta_i})_{k,l} \right) \in S_{m,n,N}$$

and $a_i \to a$, as $i \to \infty$, then

$$(\alpha_i, \beta_i) \in A_p \times A_{p,N},$$

$$|\alpha_i(z)| \leq m \exp(mp(z)),$$

and

$$|\beta_i(z)| \leq n \exp(np(z)).$$

It follows that $\{\alpha_i\}$ and $\{\beta_i\}$ are locally bounded (by the property (ii) of $p(z)$) and hence normal families. By passing to subsequences, we can assume , without loss of generality, that $(\alpha_i, \beta_i) \to (\alpha, \beta)$ locally uniformly. Clearly

$$|\alpha(z)| \leq m \exp(mp(z)), |\beta(z)| \leq n \exp(np(z)). \tag{3.3}$$

We claim that $\beta(z) \not\equiv 0$. Otherwise, $\beta_i(z)$ converges to 0 uniformly in $D_1 = \{z : |z - z_1| \leq 1\}$. Thus there exists a i such that

$$|\beta_i(z)| \leq \frac{\varepsilon_1}{2}$$

for any $z \in D_1$, where $\varepsilon_1 = \frac{1}{N} \exp(-Np(z_1))$. But for all l,

$$|(\beta_i)_{1,l}| = \left| \frac{\beta_i^{(l)}(z_1)}{l!} \right| = \left| \frac{1}{2\pi i} \int_{\partial D_1} \frac{\beta_i(z)}{(z - z_1)^{l+1}} dz \right| \leq \frac{1}{2\pi} \cdot 2\pi \cdot \frac{\varepsilon_1}{2} = \frac{\varepsilon_1}{2},$$

which is a contradiction , since $\beta_i \in A_{p,N}$ and so

$$|(\beta_i)_{1,r_1}| \geq \frac{1}{N} \exp(-Np(z_1)) = \varepsilon_1,$$

where $\tau_1 = n(0, z_1, \frac{1}{\beta_i})$.

Now we can apply the Hurwitz theorem [2,7] to $\beta(z)$ and conclude that $\frac{\alpha_i}{\beta_i} \to \frac{\alpha}{\beta}$ normally . Therefore $\frac{\alpha}{\beta}$ is meromorphic and for each k there exists a $\rho > 0$ such that

$$(\frac{\alpha_i}{\beta_i})_{k,l} = \frac{1}{2\pi i}\int_{|z-z_k|=\rho} \frac{\alpha_i(z)/\beta_i(z)}{(z-z_k)^{l+1}}dz$$

$$\to \frac{1}{2\pi}\int_{|z-z_k|=\rho} \frac{\alpha(z)/\beta(z)}{(z-z_k)^{l+1}}dz = (\frac{\alpha}{\beta})_{k,l} \tag{3.4}$$

as $i \to \infty$. Hence

$$a = \left((\frac{\alpha}{\beta})_{k,l} \right)_{k\in N, -n_k \le l < m_k} \in S. \tag{3.5}$$

Next we prove that $\beta \in A_{p,N}$ and thus that $(\alpha, \beta) \in A_p \times A_{p,N}$. To this end, write, about z_k,

$$\beta(z) = \beta_{k,\tau_k}(z-z_k)^{\tau_k} + \beta_{k,\tau_{k+1}}(z-z_k)^{\tau_k+1} + \cdots,$$

where $\beta_{k,\tau_k} \neq 0$, $\tau_k = n(0, z_k, \frac{1}{\beta})$, and

$$\beta_i(z) = (\beta_i)_{k,0} + \cdots + (\beta_i)_{k,\tau_k}(z-z_k)^{\tau_k} + \cdots.$$

Then

$$(\beta_i)_{k,j} \to 0, \quad \text{for} \quad 0 \le j < \tau_k$$

and

$$(\beta_i)_{k,\tau_k} \to \beta_{k,\tau_k}, \qquad \text{as } i \to \infty. \tag{3.6}$$

Let $\varepsilon_k = \frac{1}{N}\exp(-Np(z_k))$. Then there exists a $I_k > 0$ such that when $i \ge I_k$,

$$|(\beta_i)_{k,j}| < \frac{\varepsilon_k}{2}$$

for $0 \le j < \tau_k$ and

$$(\beta_i)_{k,\tau_k} \neq 0. \tag{3.7}$$

We claim that $(\beta_i)_{k,0} = 0$ for all $i \ge I_k$. Otherwise suppose that $(\beta_{i_0})_{k,0} \neq 0$ for some $i_0 \ge I_k$. Then

$$\frac{\varepsilon_k}{2} > |(\beta_{i_0})_{k,0}| \ge \varepsilon_k,$$

since $\beta_{i_0} \in A_{p,N}$, a contradiction. By the same argument, we deduce that

$$(\beta_i)_{k,0} = (\beta_i)_{k,1} = \cdots = (\beta_i)_{k,\tau_k - 1} = 0.$$

In view of (3.7), we have that $\tau_k = n(0, z_k, \frac{1}{\beta_i})$ for $i \geq I_k$ and so that $(\beta_i)_{k,\tau_k} \geq \varepsilon_k$ again since $\beta_i \in A_{p,N}$.

Using (3.6), we obtain that

$$|\beta_{k,\tau_k}| \geq \varepsilon_k = \frac{1}{N} \exp\left(-Np(z_k)\right),$$

which shows that $\beta(z) \in A_{p,N}$.

Combining this result with (3.3), (3.4) and (3.5), we get that

$$a_i \rightarrow a = \left((\frac{\alpha}{\beta})_{k,l} \right) \in S_{m,n,N},$$

i.e., $S_{m,n,N}$ is closed.

Now it follows from the fact that $S = \cup_{m,n,N} S_{m,n,N}$ and the well-known Baire -category theorem (see e.g.[12]) that some $S_{m,n,N}$ has non-empty interior. We can assume, without loss of generality, that $S_{m,n,N} \supset \{a : \| a \| < \varepsilon\}$, from which we readily obtain two sequences $\{\alpha_k\} \subset A_p$ and $\{\beta_k\} \subset A_{p,N}$ such that

$$(\frac{\alpha_k}{\beta_k})_{i,l} = 0, i \in \mathbf{N}, -\infty < l \leq m_i - 1, \quad \text{except that } (\frac{\alpha_k}{\beta_k})_{k,m_k - 1} = \exp\left(Mp(z_k)\right), \tag{3.8}$$

and

$$|\alpha_k(z)| \leq A \exp\left(Bp(z)\right), |\beta_k(z)| \leq A \exp\left(Bp(z)\right)$$

for some constants $A, B > 0, k \in \mathbf{N}$. Let $\alpha_k{}' = \alpha_k / \exp\left(Mp(z_k)\right)$. Then

$$|\alpha'_k| \leq A \exp\left(Bp(z_k)\right) / \exp\left(Mp(z_k)\right)$$

and

$$(\frac{\alpha_k{}'}{\beta_k})_{i,l} = 0, i \in \mathbf{N}, -\infty < l \leq m_i - 1, \quad \text{except that } (\frac{\alpha_k{}'}{\beta_k})_{k,m_k - 1} = 1.$$

On the other hand, since $V = \{(z_k, m_k, n_k)\}$ is an interpolating variety for M_p, it is not difficult to deduce that there exist

$$r(z) \in A_p, \quad \text{and} \quad \eta(z) \in A_{p,N_0}$$

for some $n_0 \in \mathbf{N}$ such that

$$\left(\frac{r(z)}{\eta(z)}\right)_{k,-n_k} \neq 0$$

and

$$(\eta(z))_{k,\tau_k} \geq \varepsilon \exp\left(-Lp(z_k)\right)$$

for some constants $\varepsilon, L > 0$, where $\tau_k = n(0, z_k, \frac{1}{\eta}) \geq n_k$. Set

$$\begin{aligned} g_k(z) &= \frac{{\alpha_k}'}{\beta_k} \cdot \beta_k \cdot \eta(z) \\ &= {\alpha_k}' \cdot \eta(z). \end{aligned} \tag{3.9}$$

Then

$$|g_k(z)| \leq A \exp\left(Bp(z)\right) / \exp\left(Mp(z_k)\right), k \in \mathbf{N},$$

for some $A, B > 0$.

We can readily see that $g_k \in A_p$ and

$$(g_k)_{i,l} = 0, i \in \mathbf{N}, 0 \leq l \leq q_i + m_i - 1 \text{ except that } (g_k)_{k,q_k+m_k-1} \geq \varepsilon_1 \exp\left(-L_1 p(z_k)\right)$$

for some constants $\varepsilon_1, L_1 > 0$, where $q_k \geq n_k$.

Using the same reasoning as above , changing (3.8) into

$$(\frac{\alpha_k}{\beta_k})_{i,l} = 0, i \in \mathbf{N}, -\infty < l \leq m_i - 1, \quad \text{except that} \quad (\frac{\alpha_k}{\beta_k})_{k,0} = \exp(Mp(z_k))$$

and changing (3.9) into

$$h_k(z) = \frac{{\alpha_k}'}{\beta_k} \cdot \beta_k,$$

we can get a sequence $\{h_k\} \in A_p$ and a sequence $\{v_k\}$ of non-negative integers such that

$$|h_k(z)| \leq A \exp(Bp(z)) / \exp(Mp(z_k))$$

for some constants $A, B > 0$ and

$$(h_k)_{i,l} = 0, i \in \mathbf{N}, 0 \leq l \leq v_i + m_i - 1, \quad \text{except that } |(h_k)_{k,v_k}| \geq \varepsilon_1 \exp(-L_1 p(z_k))$$

for some constants $\varepsilon_1, L_1 > 0$. The proof is thus complete. ∎

Lemma 3.4. If $V = \{(z_k, m_k, n_k)\}$ is an interpolating variety for M_p. Then

$$d_k := \inf_{j \neq k} \{|z_j - z_k|\} \geq A \exp(-Bp(z_k)), k \in \mathbf{N},$$

for some constants $A, B > 0$.

Proof. We apply Lemma 3.3 with $M = 1$ to get entire functions $g_k \in A_p$ and integers $q_k \geq n_k$ such that

$$(g_k)_{i,l} = 0, i \in \mathbf{N}, 0 \leq l \leq q_i + m_i - 1, \text{except that } (g_k)_{k,q_k+m_k-1} \geq \varepsilon_1 \exp(-L_1 p(z_k))$$

for some constants $\varepsilon_1, L_1 > 0$.

Set

$$G_k(z) = \frac{g_k(z)}{(z - z_k)^{q_k+m_k-1}}.$$

Then on $|z - z_k| = 1$, and so in $|z - z_k| \leq 1$, by the Maximum Modulus Theorem,

$$|G_k(z)| \leq A \exp(Bp(z_k)),$$

for some constants $A, B > 0$. Applying Lemma 1 to $G_k(z)$ and noting that the $z_j (j \neq k)$ are all zeros of $G_k(z)$, we obtain that

$$d_k \geq \frac{|G_k(z_k)|}{2A \exp(Bp(z_k))} \geq A \exp(-Bp(z_k))$$

for $k \in \mathrm{N}$. ∎

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. $\Longleftarrow$:

Let $\{(\xi_k, s_k)\} = Z(f^2) - \{(z_k, 2t_k)\}$. Clearly there exist positive numbers $\rho_k \leq \frac{1}{2}$ such that

$$B(\xi_k, \rho_k) \cap B(\xi_j, \rho_j) = \phi$$

for $k \neq j$. Take a sequence $\{\eta_k\}_{k=1}^{\infty}$ such that $\eta_k \geq 1$ and $\sum_{k=1}^{\infty} \frac{s_k}{\eta_k} = L < +\infty$. Set

$$\varepsilon_k = \frac{\rho_k}{\eta_k} \leq \frac{1}{2}$$

and

$$g(z) = f^2(z)\Pi_{k=1}^{\infty}(1 + \frac{\varepsilon_k}{z - \xi_k})^{s_k}.$$

Then g is an entire function and it is obvious that

$$\{(z_k, 2t_k)\} = Z(f^2, g) := \{z \in C | f^2(z) = g(z) = 0\}. \tag{3.10}$$

We assert that

$$g(z) \in A_p. \tag{3.11}$$

In fact, if $z \notin \cup_{k=1}^{\infty} B(\xi_k, \rho_k)$, then $|z - \xi_k| \geq \rho_k$ for all k. Thus

$$|g(z)| \leq |f(z)|^2 \Pi_{k=1}^{\infty}(1 + \frac{\varepsilon_k}{\rho_k})^{s_k}$$
$$\leq |f(z)|^2 \cdot e^L \leq Ae^{Bp(z)}.$$

If $z \in \cup_{k=1}^{\infty} B(\xi_k, \rho_k)$, then there is a k_0 such that $z \in B(\xi_{k_0}, \rho_{k_0})$. Therefore

$$|g(z)| \leq \max_{w \in \partial B(\xi_{k_0}, \rho_{k_0})} \{|g(w)|\} = g(w_k)$$

for some $w_k \in \partial B(\xi_{k_0}, \rho_{k_0})$. By the above estimate of $|g(z)|$ on $\partial B(\xi_{k_0}, \rho_{k_0})$ and in view of the property (ii) of $p(z)$, we have that $|g(z)| \leq Ae^{Bp(w_k)} \leq Ae^{Bp(z)}$ for some constants $A, B > 0$. This shows that $g \in A_p$.

Now by (3.2), (3.10), (3.11), and using Theorem 4 in [4,p.124], we obtain that $V_1 := \{(z_k, 2t_k)\}$ is an interpolating variety for the space A_p. In view of the fact that $n_k < n_k + m_k \leq 2t_k$, we can get an entire function $h(z) \in A_p$ such that

$$h(z) = h_{k,n_k}(z - z_k)^{n_k} + h_{k,n_k+1}(z - z_k)^{n_k+1} + \cdots,$$

where

$$h_{k,n_k} = 1. \tag{3.12}$$

Applying the Cauchy formula, for $j \geq n_k$,we have

$$\begin{aligned} |h_{k,j}| &= |\frac{1}{2\pi i} \int_{|z-z_k|=1} \frac{h(z)}{(z-z_k)^{j+1}} dz| \\ &\leq \max_{|z-z_k|=1} |h(z)| \leq \max_{|z-z_k|=1} |Ae^{Bp(z)}| \leq Ae^{Bp(z_k)} \end{aligned} \tag{3.13}$$

by the property (ii) of $p(z)$.

For any $\{a_{k,l}\}_{k\in\mathbf{N},-n_k\leq l<m_k} \in M_p(V)$, we have that, about z_k,

$$h(z) \sum_{l=-n_k}^{m_k-1} a_{k,l}(z-z_k)^l = (h_{k,n_k} + h_{k,n_k+1}(z-z_k) + \cdots) \times$$

$$\left(a_{k,-n_k} + a_{k,-n_k-1}(z-z_k) + \cdots + a_{k,m_k-1}(z-z_k)^{m_k+n_k-1}\right)$$

$$= c_{k,0} + c_{k,1}(z-z_k) + \cdots + c_{k,m_k+n_k-1}(z-z_k)^{n_k+m_k-1} + \cdots, \quad \text{say.}$$

Comparing the coefficients of $1, z - z_k, \cdots, (z - z_k)^{m_k+n_k-1}$ on both sides, we deduce that

$$h_{k,n_k} a_{k,-n_k} = c_{k,0},$$

$$h_{k,n_k+1} a_{k,-n_k} + h_{k,n_k} a_{k,-n_k+1} = c_{k,1},$$

$$\cdots\cdots$$

$$h_{k,n_k+m_k+n_k-1} a_{k,-n_k} + h_{k,n_k+m_k+n_k-2} a_{k,-n_k+1} + \cdots + h_{k,n_k} a_{k,m_k-1} = c_{k,m_k+n_k-1}.$$

By (3.13), we have

$$|c_{k,j}| \leq Ae^{Bp(z_k)} \sum_{l=-n_k}^{m_k-1} |a_{k,l}| \leq Ae^{Bp(z_k)}$$

and thus that

$$\sum_{l=0}^{m_k+n_k-1} |c_{k,l}| \leq (m_k+n_k)Ae^{Bp(z_k)} \leq 2t_kAe^{Bp(z_k)}. \tag{3.14}$$

Also by (3.2),

$$\varepsilon \exp(-cp(z_k)) \leq |f_{k,t_k}| = |\frac{1}{2\pi i}\int_{|z-z_k|=e} \frac{f(z)}{(z-z_k)^{t_k+1}}dz|$$

$$\leq A\exp(Bp(z_k)) / \exp(t_k)$$

and so

$$t_k \leq e^{t_k} \leq A\exp(Bp(z_k)).$$

It follows from (3.14) that

$$\sum_{l=0}^{m_k+n_k-1} |c_{k,l}| \leq A\exp(Bp(z_k)),$$

for some $A, B > 0$. Clearly $m_k + n_k - 1 \leq 2t_k - 1$. We construct a sequence

$$\{b_{k,l}\}_{k\in\mathbf{N},0\leq l<2t_k} \in A_p(V_1)$$

such that

$$b_{k,l} = c_{k,l} \quad \text{if} \quad 0 \leq l \leq m_k + n_k - 1$$

and

$$b_{k,l} = 0 \quad \text{if} \quad m_k + n_k - 1 < l \leq 2t_k - 1.$$

Recalling that V_1 is an interpolating variety for A_p, we obtain an entire function $q(z) \in A_p$ such that $q_{k,l} = b_{k,l}$ for $k \in \mathbf{N}$ and $0 \leq l \leq m_k + n_k - 1$.

Set

$$\lambda(z) = \frac{q(z)}{h(z)}.$$

Then $\lambda(z) \in M_p$ and it's not difficult to check that

$$\lambda_{k,l} = \begin{cases} 0, & \text{if } l < -n_k \\ a_{k,l}, & \text{if } -n_k \leq l \leq m_k - 1\,, \end{cases} \qquad k \in \mathbf{N}$$

In view of (3.12), it follows that V is an interpolating variety for M_p.

$\Longrightarrow$:

Suppose that V is an interpolating variety for M_p. By Lemma 3.3, for any $M > 0$. there exist functions $g_k(z)$ and $h_k(z)$ having the properties stated in that lemma.

Set

$$f(z) = \sum_{k=1}^{\infty} (z - z_k)^{t_k - q_k - v_k - m_k + 1} g_k(z) h_k(z) := \sum_{k=1}^{\infty} f_k^*(z),$$

where $t_k = \max\{m_k, n_k\}, q_k \geq n_k$ and $v_k \geq 0$ are the same as in Lemma 3.3. This series converges to an entire function $f \in A_p$. Assuming this for the moment, then it is readily seen that

$$|f^*{}_{k,t_k}| \geq \varepsilon \exp\left(-cp(z_k)\right), k \in \mathrm{N}$$

for some constants $\varepsilon, c > 0$ and $\{z_k, t_k\} \subset Z(f)$.

Therefore, the necessity will hold once we show that $f \in A_p$. To this end, we first consider the upper bound of f_k^* for each fixed k.

If $|z - z_k| \geq 1$, then in view of the property (i) of $p(z)$, there exist two positive constants A_1 and B_1 independent of M such that

$$\begin{aligned} |f_k^*(z)| &= |(z - z_k) g_k(z) h_k(z) / (z - z_k)^{(q_k + v_k + m_k) - t_k}| \\ &\leq |z - z_k| |g_k(z)| |h_k(z)| \\ &\leq \left(A_1 e^{B_1 p(z)} + A_1 e^{B_1 p(z_k)}\right) A e^{Bp(z)} / \exp\left(2Mp(z_k)\right) \\ &\leq A e^{Bp(z)} e^{(B_1 - 2M) p(z_k)} \end{aligned} \tag{3.15}$$

for some constants $A, B > 0$.

If $|z - z_k| = 1$, then by the property (ii) of $p(z)$,there exists a $B_2 > 0$ independent of M such that

$$|f_k^*(z)| = |g_k(z) h_k(z)| \leq A e^{Bp(z)} e^{-2Mp(z_k)} \leq A e^{(B_2 - 2M) p(z_k)}.$$

By the maximum Modulus theorem, on $|z - z_k| \le 1$,

$$|f_k^*(z)| \le Ae^{(B_2-2M)p(z_k)}.$$

This and (3.15) yield that for any $z \in \mathbf{C}$,

$$|f_k^*(z)| \le Ae^{Bp(z)}e^{(B_3-M)p(z_k)}$$

for some constants $A, B, B_3 > 0$ (B_3 is independent of M).

Let

$$d_k = \inf_{j \ne k}\{|z_j - z_k|\}, \varepsilon_k = \frac{1}{2}\min\{1, d_k\},$$

and

$$D_k = \{z : |z - z_k| < \varepsilon_k\}.$$

Then by lemma 3.4 and the property (ii) of $p(z)$, we can get two positive constants B_4 and B_5 independent of M such that

$$\begin{aligned}\sum_{k=1}^{\infty} e^{(B_3-M)p(z_k)} &\le \sum_{k=1}^{\infty}\int_{D_k} Ae^{(B_4-M)p(z_k)}dxdy \\ &\le A\sum_{k=1}^{\infty}\int_{D_k} e^{(B_5-M)p(z)}dxdy \\ &\le A_2\int_{\mathbf{C}} e^{(B_5-M)p(z)}dxdy < \infty,\end{aligned}$$

provided that M is large (in view of the property (i) of $p(z)$).

This shows that the series $\sum_{k=1}^{\infty} f_k^*(z)$ is uniformly convergent in compact sets and

$$|f(z)| \le Ae^{Bp(z)}\sum_{k=1}^{\infty} e^{(B_3-M)p(z_k)} \le Ae^{Bp(z)}$$

for some constants $A, B > 0$ so that $f(z) \in A_p$.

The proof of the theorem is thus complete. ∎

From the proof of Theorem 3.1 , if $n_k = 0$ for all k, we have the following result for the space A_p of entire functions, which may also be showed by using the results in [4] and [16].

Corollary 3.5. A multiplicity variety $V = \{(z_k, m_k)\}$ is an interpolating variety for A_p if and only if there exists an entire function $f(z) \in A_p$ such that $V \subset Z(f(z))$ and

$$|f_{k,m_k}| \geq \varepsilon \exp\big(-cp(z_k)\big), k \in \mathbf{N},$$

for some constants $\varepsilon, c > 0$.

4. Interpolating varieties for M_p, II

This section is concerned with purely geometric conditions for V to be interpolating which only depend on the distribution of the points of the given multiplicity variety and thus enable us to determine whether or not V is interpolating for M_p by direct calculation. This will be done (see Theorem 4.1) for radial weights $p(z)$ with $p(\alpha z) = O(p(z))$ for some $\alpha > 1$, i.e., $p(|z|) = p(z)$ and

$$p(\alpha z) \leq Mp(z) + C \qquad (z \in \mathbf{C}) \tag{4.1}$$

for some $M, C \geq 0$ (e.g. $p(z) = |z|^\rho$ for $\rho > 0$). We shall assume these conditions throughout this section.

We note here that some interesting results on the geometry of interpolating varieties for A_p have been obtained (see e.g. [5],[17]). But only for the weight $p(z) = |z|$, purely geometric characterization has been completely known (see (6) and (7) in [5,p.3]). Now as a direct consequence of Theorem 4.1, a more general result will be given in Corollary 4.8.

Theorem 4.1. A multiplicity variety $V = \{(z_k, m_k, n_k)\}$ is an interpolating variety for M_p if and only if for some $A, B > 0$,

$$N(|z|, V^*) \leq A + Bp(z) \tag{4.2}$$

and

$$N(|z_k|, z_k, V^*) \leq A + Bp(z_k), \tag{4.3}$$

where $V^* := \{(z_k, \max(m_k, n_k)\}$.

To prove our theorem, some preparations are required. We first set

$$t_k = \max\{m_k, n_k\};$$

$$\tau(r) =^{\#} \{k : |z_k| < r\};$$

$$s(r) = \sum_{k=1}^{\tau(r)} t_k;$$

$$\omega = \exp(\frac{2\pi i}{n_0 + 1}), \quad n_0 > \frac{\log M}{\log \alpha},$$

an integer, where M, α are the constants in (4.1) ;

$$\tilde{V} = \cup_{j=0}^{n_0} \omega^j V^* = \{(z_k, t_k)\} \cup (\cup_{j=1}^{n_0} \{(\omega^j z_k, t_k)\});$$

and

$$\tilde{\tilde{V}} = V^* \cup (\cup_{j=1}^{n_0} V_j) = \{(z_k, t_k)\} \cup (\cup_{j=1}^{n_0} \{(z_{j,k}, t_k)\}),$$

where $z_{j,k} := \omega^j(z_k + a_{j,k})$ is close to $\omega^j z_k$ with

(a) $|a_{j,k}| \leq \frac{1}{4}|z_k| \qquad 1 \leq j \leq n_0, k \in \mathbf{N}$;

(b) $n(r, \tilde{\tilde{V}}) = O(n(2r, V^*))$;

(c) $|S(r, n, \tilde{\tilde{V}})| r^k = O(n(2r, V^*)) \quad (1 \leq n \leq n_0 - 1)$;

(d) for any integer n and $z \in \mathbf{C}$ with $2^{n-1} \leq |z| \leq 2^n$, we have if

$$\prod_{k=1}^{\tau(2^{n+4})} \prod_{j=1}^{n_0} |z - z_{j,k}|^{t_k} \leq (c2^n)^{n_0 s(2^{n+4})}$$

then

$$\prod_{k=1}^{\tau(2^{n+4})} |z - \omega^j z_k|^{t_k} > (c2^n)^{s(2^{n+4})}$$

for all $0 \le j \le n_0$, where $0 < c < 1$ is some constant.

Such $z_{j,k}$'s do exist by Lemma 2.5 in [6] and its proof.

$\tilde{\tilde{V}}$ is called an adjacent variety to $\tilde{V}$.

Lemma 4.2. Let $\lambda(r)$ be a non-negative function satisfying

$$\lambda(\alpha r) \le M\lambda(r) + C \tag{4.4}$$

for some $\alpha > 1$ and $M, C \ge 0$. Then for any sequence $Z = \{z_k\}$ of finite $\lambda + 1$ -density, the sequence

$$\tilde{Z} := \cup_{j=0}^{n_0} \omega^j Z$$

is $\lambda + 1$- admissible,where ω and n_0 are defined as above.

Proof. The lemma is essentially due to [18]. Since Z has finite $\lambda+1$-density, there is a constant $A > 0$ such that

$$N(r, Z) \le A\,(\lambda(r) + 1) \tag{4.5}$$

for all $r \ge 0$. From this inequality and (4.4) we easily deduce that for $n \in \mathbf{N}$,

$$\begin{aligned}\int_r^\infty \frac{N(\alpha t, Z)}{t^{n+1}}dt &= \sum_{j=0}^{\infty} \int_{r\alpha^j}^{r\alpha^{j+1}} \frac{N(\alpha t, Z)}{t^{n+1}}dt \\ &\le \sum_{j=0}^{\infty} \frac{N(r\alpha^{j+2}, Z)}{n(r\alpha^j)^n} \\ &\le \sum_{j=0}^{\infty} \frac{A\left(\lambda(\alpha^{j+2}r) + 1\right)}{nr^n(\alpha^n)^j} \\ &\le \frac{A\,(\lambda(r)+1)}{r^n}\end{aligned} \tag{4.6}$$

for some $A > 0$ provided that $n \ge \frac{\log M}{\log \alpha}$. We can assume, without loss of generality, that $0 \notin Z$. Then

$$n(r, Z) = \frac{1}{\log \alpha} \int_r^{\alpha r} \frac{n(r, Z)}{t} dt \le \frac{1}{\log \alpha} N(\alpha r, Z).$$

Therefore for $r_1, r_2 > 0$ with $r_1 \le r_2$ and $n > n_0$ we have that , in view of (4.4),(4.5) and (4.6),

$$\begin{aligned}
S(r_1, r_2, n, \tilde{Z}) &= \frac{1}{n}\left| \sum_{\substack{r_1 \le |\xi_k| \le r_2 \\ \xi_k \in \tilde{Z}}} \left(\frac{1}{\xi_k}\right)^n \right| \\
&\le \frac{1}{n}\int_{r_1}^{r_2} \frac{1}{t^n} dn(t, \tilde{Z}) \\
&\le \frac{(n_0+1)n(r_2, Z)}{r_2^n} + (n_0+1)\int_{r_1}^{r_2} \frac{n(t,Z)}{t^{n+1}} dt \\
&\le \frac{(n_0+1)n(r_2, Z)}{r_2^n} + \frac{(n_0+1)}{\log \alpha}\int_{r_1}^{r_2} \frac{N(\alpha t,Z)}{t^{n+1}} dt \\
&\le \frac{A\left(\lambda(r_2)+1\right)}{r_2^n} + \frac{A\left(\lambda(r_1)+1\right)}{r_1^n}
\end{aligned}$$

for some $A > 0$. For $n \in \{1, 2, \cdots, n_0\}$, we have that

$$S(r_1, r_2, n, \tilde{Z}) = \left(\sum_{j=0}^{n_0} (\omega^{-n})^j \right) S(r_1, r_2, n, Z) = 0$$

since $\sum_{j=0}^{n_0} (\omega^{-n})^j = 0$. Clearly,

$$N(t, \tilde{Z}) \le (n_0+1)N(t, Z) \le (n_0+1)A\left(\lambda(t)+1\right).$$

Combining the above results, $\tilde{Z}$ is $\lambda + 1$ admissible. ∎

Lemma 4.3. If there is a $F \in A_p$ such that $\tilde{V} = Z(F) := \{z : F(z) = 0\}$, then there exists a $G \in A_p$ with

(α) $Z(G) = \tilde{\tilde{V}}$

(β) there are $\varepsilon, C > 0$ such that all components of $\Delta(G, \varepsilon, C) := \{z \in \mathbf{C} : |G(z)| < \varepsilon \exp(-Cp(z))\}$ which contain a $z_{j,k}$ are disjoint from $\Delta(F, \varepsilon, C)$.

(γ) the ideal $I := \{f \in A_p : f$ vanishes at z_k with multiplicity $\ge t_k\}$ is algebraically generated by F and G.

Proof. (α) and (β) follow from Proposition 2.6 in [6]. (γ) follows from Theorem 2.7 and its proof in [6]. ∎

Lemma 4.4. [18,Th 5.2]. Let $\lambda(r)$ be a non-negative function for $r \geq 0$ and $Z = \{z_k\}$ (repeated according to multiplicity) a sequence of complex numbers. If Z is λ-admissible, then there exists an entire function $f(z)$ of finite λ -type such that $Z = Z(f)$.

Lemma 4.5. Suppose that $g(z)$ is analytic in $|z| \leq (4e+1)R$ and satisfies $|g(z)| \leq M$. Let $a, |a| < R$, be such that $g(a) \neq 0$. Then there exists a $r, \frac{1}{2} < r < R$, such that

$$\min_{|z|=r}\{\log|g(z)|\} > -8\log M + 9\log|g(a)|$$

Proof. Let $G(z) = \frac{g(z)}{g(a)}$. Then $G(a) = 1$. Applying the minimum modulus theorem (see [14,p.21]) to $G(z)$ in $|z-a| \leq 4eR$, we then have that inside the circle $|z-a| \leq 2R$, but outside of a family of excluded circles the sum of whose radii is less than $\frac{1}{2}R$,

$$\log|G(z)| > -8\log(M/|g(a)|)$$

or

$$\log|g(z)| > -8\log M + 9\log|g(a)|.$$

Hence there must be a $r, \frac{1}{2} < r < R$, such that

$$\min_{|z|=r}\{\log|g(z)|\} > -8\log M + 9\log|g(a)|.$$

The proof is complete. ∎

Proof of the sufficiency of Theorem 4.1:

Since $N(|z|, V^*) \leq A + Bp(z) \leq A_1(p(z)+1)$ for some $A_1 > 0$, V^* is of finite $p+1$-density. By Lemma 4.2 with $\lambda(r) = p(r)$ and Lemma 4.4, there is an entire function $F(z)$ of finite $p+1$-type such that $\tilde{V} := \cup_{j=0}^{n_0}\omega^j V^* = Z(F)$, where $\omega = \exp(\frac{2\pi i}{n_0+1})$ for some integer $n_0 > \frac{\log M}{\log \alpha}$. Therefore

$$T(r, F) \leq A(p(Br)+1)$$

for some $A, B > 0$ and $r \geq 0$. We thus have

$$\log M(r, F) \leq \frac{\alpha r + r}{\alpha r - r} T(\alpha r, F) \leq \frac{\alpha + 1}{\alpha - 1} A\big(p(\alpha B r) + 1\big). \tag{4.7}$$

Applying the maximum principle for subharmonic functions to $p(r)$ (see e.g [7]), we deduce that $p(r)$ is non-decreasing for $r > 0$. Hence it follows from (4.7) that

$$\log M(r, F) \leq Ap(r) + B$$

for some $A, B > 0$, i.e., $F \in A_p$.

Now using Lemma 4.3, there exists a $G \in A_p$ such that $Z(G) = \tilde{\tilde{V}}$, where $\tilde{\tilde{V}} \supset V^*$ is an adjacent variety of $\tilde{V}$ defined in the beginning of this section. It is no loss of generality to assume that $G(0) = 1$. Applying the minimum modulus theorem [14] to $G(z)$ in $|z| \leq 4e|z_k|$, there exists a $r_k, |z_k| \leq r_k \leq 2|z_k|$, such that for $|z| = r_k$,

$$\log |G(z)| > -A_1 \log M(4e|z_k|, G) \geq -Ap(z_k) - B$$

for some $A_1, A, B > 0$.

Next we apply Lemma 4.5 to the function $G(z)$ in $|z - z_k| \leq (4e + 1)|z_k|$. Notice that

$$|G(z)| \leq M((4e + 2)|z_k|, G) \leq Ae^{Bp(z_k)}$$

for z satisfying $|z - z_k| \leq (4e + 1)|z_k|$, and $|\omega_k - z_k| \leq |z_k|$, where $\omega_k := r_k e^{\iota arg z_k}$.

We thus obtain a $\beta_k (0 < \beta_k < |z_k|)$ such that

$$\min_{|z - z_k| = \beta_k} \{\log |G(z)|\} > -8 \log(Ae^{Bp(z_k)}) + 9 \log |G(\omega_k)| \geq -Ap(z_k) - B,$$

for some $A, B > 0$.

By the Nevanlinna first fundamental theorem, we have

$$\log |G_{k,t_k}| + T(\beta_k, z_k, \frac{1}{G}) = T(\beta_k, z_k, G).$$

Thus,

$$\begin{aligned}\log|G_{k,t_k}| &= m(\beta_k, z_k, G) - m(\beta_k, z_k, \frac{1}{G}) - N(\beta_k, z_k, \tilde{\tilde{V}}) \\ &\geq \frac{1}{2\pi}\int_0^{2\pi} \log|G(z_k + \beta_k e^{i\theta})d\theta - N(|z_k|, z_k, \tilde{\tilde{V}}) \\ &\geq -Ap(z_k) - B - N(|z_k|, z_k, \tilde{\tilde{V}}). \end{aligned} \tag{4.8}$$

Recall that $\tilde{\tilde{V}} = V^* \cup V_*$, where

$$V_* := \cup_{j=1}^{n_0} V_j = \cup_{j=1}^{n_0}\{(z_{j,k}, t_k)\}.$$

Thus

$$N(|z_k|, z_k, \tilde{\tilde{V}}) \leq N(|z_k|, z_k, V^*) + N(|z_k|, z_k, V_*). \tag{4.9}$$

By Lemma 4.3, $z_{j,k} \notin \Delta(F, \varepsilon, C)$. Hence $F(z_{j,k}) \neq 0$. This implies that $z_k \notin V_*$ since $F(z_k) = 0$. We conclude that

$$\begin{aligned} N(|z_k|, z_k, V_*) &= \int_0^{|z_k|} \frac{n(t, z_k, V_*) - n(0, z_k, V_*)}{t} dt + n(0, z_k, V_*)\log|z_k| \\ &= \sum_{\substack{i \\ |z_{j,i} - z_k| \leq |z_k|}} \sum_{j=1}^{n_0} \log\frac{|z_k|^{t_i}}{|z_{j,i} - z_k|^{t_i}}. \end{aligned} \tag{4.10}$$

Assume that k is such that $2^{n-1} \leq |z_k| \leq 2^n$ for some integer n. Then when $|z_{j,i} - z_k| \leq |z_k|$, we have $|z_{j,i}| \leq 2|z_k| \leq 2^{n+1}$. But

$$z_{j,i} = \omega^j(z_i + a_{j,i})$$

for some $a_{j,i}$ with $|a_{j,i}| \leq \frac{1}{4}|z_i|$ by (a). It follows that

$$|z_i| \leq |z_i + a_{j,i}| + |a_{j,i}| \leq |z_{j,i}| + |a_{j,i}| \leq 2^{n+1} + \frac{1}{4}|z_i|$$

and so that $|z_i| \leq \frac{4}{3}2^{n+1} \leq 2^{n+3}$. From this we deduce that

$$i \leq^{\#} \{q : |z_q| \leq |z_i|\} \leq^{\#} \{q : |z_q| \leq 2^{n+3}\} \leq^{\#} \{q : |z_q| < 2^{n+4}\} = \tau(2^{n+4}).$$

Now by (4.10),

$$
\begin{aligned}
N(|z_k|, z_k, V_*) &\le \sum_{i=1}^{\tau(2^{n+4})} \sum_{j=1}^{n_0} \log \frac{|z_k|^{t_i}}{|z_{j,i} - z_k|^{t_i}} \\
&= \log \prod_{i=1}^{\tau(2^{n+4})} \prod_{j=1}^{n_0} \frac{|z_k|^{t_i}}{|z_{j,i} - z_k|^{t_i}} \\
&\le \log \frac{(2^n)^{n_0 s(2^{n+4})}}{\prod_{i=1}^{\tau(2^{n+4})} \prod_{j=1}^{n_0} |z_{j,i} - z_k|^{t_i}}.
\end{aligned}
\tag{4.11}
$$

We claim that

$$
\prod_{i=1}^{\tau(2^{n+4})} \prod_{j=1}^{n_0} |z_{j,i} - z_k|^{t_i} > (c2^n)^{n_0 s(2^{n+4})},
$$

where c is the constant in (d). Otherwise, by (d) we have

$$
\prod_{i=1}^{\tau(2^{n+4})} |z_k - \omega^j z_i|^{t_i} > (c2^n)^{s(2^{n+4})}
$$

for $0 \le j \le n_0$. Especially letting $j = 0$, we obtain that $|z_k - z_i| \ne 0$ for $1 \le i \le \tau(2^{n+4})$. Notice that

$$
k \le^{\#} \{q : |z_q| \le |z_k|\} \le^{\#} \{q : |z_q| \le 2^n\} \le \tau(2^{n+4}).
$$

Thus $|z_k - z_k| \ne 0$, a contradiction.

Hence by (4.11), we have

$$
\begin{aligned}
N(|z_k|, z_k, V_*) &\le \log \frac{(2^n)^{n_0 s(2^{n+4})}}{(c2^n)^{n_0 s(2^{n+4})}} \\
&= s(2^{n+4}) n_0 \log \frac{1}{c} \\
&\le n(2^{n+4}, V^*) n_0 \log \frac{1}{c} \\
&\le n(2^5 |z_k|, V^*) n_0 \log \frac{1}{c}.
\end{aligned}
$$

But

$$
\begin{aligned}
n(r, V^*) &\le \frac{1}{\log \alpha} \int_r^{\alpha r} \frac{n(t, V^*)}{t} dt \\
&\le \frac{1}{\log \alpha} N(\alpha r, V^*) \\
&\le \frac{1}{\log \alpha} (A + Bp(\alpha r)) \le A_1 + B_1 p(r)
\end{aligned}
$$

for some $A_1, B_1 > 0$. This shows that

$$N(|z_k|, z_k, V_*) \leq A_2 + B_2 p(2^5|z_k|) \leq A_2 + B_3 p(z_k)$$

and so that , by (4.3),(4.9) and (4.8) ,

$$\log |G_{k,t_k}| \geq -Ap(z_k) - B$$

for some constants $A, B > 0$. The sufficiency now follows from Theorem 3.1. ∎

Next we are going to prove the necessity of Theorem 4.1 . It is obviously contained in the following statement.

Proposition 4.6. A multiplicity variety $V = \{(z_k, m_k, n_k)\}$ is an interpolating variety for M_p if and only if for some $A, B > 0$

$$N(|z|, V^*) \leq A + Bp(z) \tag{4.12}$$

and

$$\min\{N(|z_k|, z_k, \tilde{V}), N(|z_k|, z_k, \tilde{\tilde{V}})\} \leq A + Bp(z_k), \tag{4.13}$$

where

$$\tilde{V} := \cup_{j=0}^{n_0} \omega^j V^*, \omega = \exp(\frac{2\pi i}{n_0 + 1})$$

for some $n_0 > \frac{\log M}{\log \alpha}$ and $\tilde{\tilde{V}}$ is an adjacent variety of $\tilde{V}$.

Since $V^* \subset \tilde{V}, V^* \subset \tilde{\tilde{V}}$, we see that

$$N(|z_k|, z_k, V^*) \leq \min\{N(|z_k|, z_k, \tilde{V}), N(|z_k|, z_k, \tilde{\tilde{V}})\}.$$

Therefore (4.12) and (4.13) imply (4.2) and (4.3) and the sufficiency of Proposition 4.6 follows from the one of Theorem 4.1. We only need to prove the necessity of Proposition 4.6 which is stronger than the one of Theorem 4.1 . We note here that in Proposition 4.6 the condition (4.13) can not be replaced by the simpler condition:

$$N(|z_k|, z_k, \tilde{V}) \leq A + Bp(z_k) \tag{4.14}$$

although this condition together with (4.12) is also sufficient . A counter example will be given later.

Proof of the necessity of Proposition 4.6.

If V is an interpolating variety for M_p, then by Theorem (3.1) there exists a $f \in A_p$ such that $V^* \subset Z(f)$ and

$$|f_{k,t_k}| \geq \varepsilon \exp(-cp(z_k)), k \in \mathbf{N} \tag{4.15}$$

for some constants $\varepsilon, c > 0$. Thus

$$\begin{aligned} N(r, V^*) &\leq N(r, \frac{1}{f}) \\ &= T(r, f) + A_1 \\ &\leq \log M(r, f) + A_1 \leq A + Bp(r) \end{aligned}$$

for some constants $A_1, A, B > 0$, i.e., (4.12) holds. This also shows that V^* is of finite $p+1$-density. Exactly like in the proof of the sufficiency of Theorem 4.1 , there exists a function $F \in A_p$ such that $\tilde{V} = Z(F)$. Again applying Lemma 4.3, there is a function $G \in A_p$ such that $\tilde{V} = Z(G)$ and the ideal

$$I := \{g \in A_p : g \quad \text{vanishes at } z_k \quad \text{with multiplicity} \geq t_k\}$$

is algebraically generated by the two elements F and G. Obviously $f \in I$. Thus there exist two functions $\phi(z), \gamma(z) \in A_p$ such that

$$f = \phi F + \gamma G$$

and thus

$$f_{k,t_k} = (\phi F)_{k,t_k} + (\gamma G)_{k,t_k}.$$

By (4.15), we deduce that either

$$|(\phi F)_{k,t_k}| \geq \frac{\varepsilon}{2} \exp(-cp(z_k)) \tag{4.16}$$

or

$$|(\gamma G)_{k,t_k}| \geq \frac{\varepsilon}{2}\exp(-cp(z_k)). \tag{4.17}$$

If (4.16) holds, then noting that $\tilde{V} = Z(F) \subset Z(\phi F)$ and the Nevanlinna first fundamental theorem

$$\begin{aligned} N(|z_k|, z_k, \tilde{V}) &\leq N(|z_k|, z_k, \frac{1}{\phi F}) \\ &\leq T(|z_k|, z_k, \frac{1}{\phi F}) \\ &= T(|z_k|, z_k, \phi F) + \log \frac{1}{|(\phi F)_{k,t_k}|} \\ &\leq \log M(|z_k|, z_k, \phi F) + A + Bp(z_k) \\ &\leq A + Bp(z_k) \end{aligned}$$

for some $A, B > 0$, since

$$M(|z_k|, z_k, \phi F) \leq \max_{|z|=2|z_k|} \{|\phi F|\} \leq Ae^{Bp(z_k)}.$$

If (4.17) holds, then from the fact that $\tilde{\tilde{V}} = Z(G) \subset Z(\gamma G)$, we deduce, using exactly the same reasoning , that

$$N(|z_k|, z_k, \tilde{\tilde{V}}) \leq A + Bp(z_k)$$

for some $A, B > 0$.

Thus (4.13) holds in every case. This completes the proof. ∎

Example 4.7. We have proved that (4.12) and (4.14) are sufficient for V to be interpolating, and (4.12) is also necessary. This example will show that (4.14) is not necessary. Consider the weight $p(z) = |z|$. Then $p(2z) = 2p(z)$ and so (4.1) is satisfied with $M = 2$. For any $n_0 > \frac{\log M}{\log 2} = 1$, we set $\omega = \exp(\frac{2\pi i}{n_0+1})$. Take a j $(0 < j \leq n_0)$ such that $arg(\omega^{-j}) \in [-\frac{3\pi}{2}, -\frac{\pi}{2}]$, i.e., ω^{-j} lies in the second or third quadrant. Let

$$V_1 = \{\theta_k\} := \{k\}, \quad V_2 = \{\omega_k\} := \{\omega^{-j}(k + e^{-k^2})\}$$

for $k \in \mathbf{N}$ and $V = V_1 \cup V_2$ with multiplicity $m_k = 1$ and $n_k = 0$ for any k. Then

$$\begin{aligned} N(r,V) &= \int_0^r \frac{n(t,V)-n(0,V)}{t}dt + n(0,V)\log r \\ &= \int_1^r \frac{n(t,V)}{t}dt \\ &\leq \int_1^r \frac{2t}{t}dt \leq 2\int_1^r dt \leq 2r = 2p(r). \end{aligned}$$

For any $\theta_k = k$, the disk $A_k := \{z : |z-\theta_k| \leq |\theta_k|\}$ does not contain any ω_k. Thus

$$\begin{aligned} N(|\theta_k|,\theta_k,V) &= \int_0^{|\theta_k|} \frac{n(t,\theta_k,V_1)-n(0,\theta_k,V_1)}{t}dt + n(0,\theta_k,V_1)\log|\theta_k| \\ &\leq \int_1^{|\theta_k|} \frac{2t}{t}dt + \log|\theta_k| \\ &\leq 2|\theta_k| + \log|\theta_k| \leq 3|\theta_k| = 3p(\theta_k). \end{aligned}$$

For any $\omega_k = \omega^{-j}(k+e^{-k^2})$, the disk $\theta_k := \{z : |z-\omega_k| \leq |\omega_k|\}$ does not contain any points $\theta_k \in V_1$. Hence

$$\begin{aligned} N(|\omega_k|,\omega_k,V) &= \int_0^{|\omega_k|} \frac{n(t,\omega_k,V_2)-n(0,\omega_k,V_2)}{t}dt + n(0,\omega_k,V_2)\log|\omega_k| \\ &\leq \int_{\delta_k}^{|\omega_k|} \frac{2t+1}{t}dt + \log|\omega_k|, \end{aligned}$$

where

$$\delta_k = \min\{|\omega_{k+1}-\omega_k|, |\omega_k-\omega_{k-1}|\}.$$

For large k , we have that $\frac{1}{2} \leq \delta_k < |\omega_k|$ and thus that

$$N(|\omega_k|,\omega_k,V) \leq 2(|\omega_k|-\delta_k) + 2|\omega_k| + \log 2 \leq 4p(\omega_k) + \log 2.$$

The above shows that the multiplicity V satisfies all conditions in Theorem 4.1. i.e. (4.2) and (4.3). V is thus an interpolating variety for $p(z) = |z|$. But

$$\begin{aligned} \tilde{V} &:= \cup_{i=0}^{n_0}\omega^i V \supset V \cup \omega^j V \\ &\supset V_1 \cup \omega^j V_2 = \{k\} \cup \{k+e^{-k^2}\} := V_0. \end{aligned}$$

Thus

$$\begin{aligned} N(|\theta_k|,\theta_k,\tilde{V}) &\geq N(|\theta_k|,\theta_k,V_1\cup\omega^jV_2) \\ &= \int_0^{|\theta_k|}\frac{n(t,\theta_k,V_0)-n(0,\theta_k,V_0)}{t}dt+n(0,\theta_k,V_0)\log|\theta_k| \\ &\geq \int_{e^{-k^2}}^{|\theta_k|}\frac{dt}{t}+\log|\theta_k| \\ &\geq \log\frac{k}{c^{-k^2}}\geq k^2=|\theta_k|^2=p(|\theta_k|)^2. \end{aligned}$$

This shows that (4.14) can not hold for any constants $A,B>0$.

Specializing Theorem 4.1 to entire functions , we have

Corollary 4.8. A multiplicity variety $V=\{(z_k,m_k)\}$ is an interpolating variety for A_p if and only if for some constants $A,B>0$,

$$N(|z|,V)\leq A+Bp(z)$$

and

$$N(|z_k|,z_k,V)\leq A+Bp(z_k).$$

Remark 4.9. Compared with Corollary 4.8, the following conditions by Squires [17] , namely,

$$\int_0^{|z_k|}\frac{n(t,z_k,V)}{t}dt\leq A+Bp(z_k)$$

and

$$m_k\leq\frac{A+Bp(z_k)}{log|z_k|}$$

are necessary for $V=\{(z_k,m_k)\}$ to be interpolating for A_p. But they are not sufficient conditions except for some special multiplicity varieties(see [17,Th.2]).

Next we consider the interesting sequences $\{z_k\}$ satisfying

$$|z_{k+1}|\geq L|z_k| \qquad (k\in\mathbf{N}) \tag{4.18}$$

for some constant $L > 1$, which are frequently used in the theory of Picard set and the theory of factorization of meromorphic functions (see e.g.[1],[15]). For such sequences, we have

Theorem 4.10. A multiplicity variety $V = \{(z_k, m_k, n_k)\}$ satisfying (4.18) is an interpolating variety for M_p if and only if for some constants $A, B > 0$,

$$N(|z|, V^*) \leq A + Bp(z) \tag{4.19}$$

and

$$\max\{m_k, n_k\} \log |z_k| \leq A + Bp(z_k), \tag{4.20}$$

where $V^* = \{(z_k, \max\{m_k, n_k\})\}$.

Proof. $\Longrightarrow$:

Set $t_k = \max\{m_k, n_k\}$. If V is an interpolating variety, then by Theorem 4.1, (4.19) holds and for some $A, B > 0$

$$N(|z_k|, z_k, V^*) \leq A + Bp(z_k).$$

Thus

$$t_k \log |z_k| \leq \int_0^{|z_k|} \frac{n(t, z_k, V^*) - n(0, z_k, V^*)}{t} dt + n(0, z_k, V^*) \log |z_k|$$

$$= N(|z_k|, z_k, V^*) \leq A + Bp(|z_k|).$$

That is, (4.20) holds.

$\Longleftarrow$:

Using exactly the same argument as in the proof of the sufficiency of Theorem 4.1, we can find a $f \in \mathbf{A}_p$ such that $Z(f) = \tilde{V}$, where $\tilde{V} = \cup_{j=0}^{n_0} \omega^j V^*, \omega = \exp(\frac{2\pi i}{n_0+1})$ and $n_0 \geq 1$ is some integer. Let

$$\eta_k = \min\{|z_{k+1}| - |z_k|, |z_k| - |z_{k-1}|\}.$$

Then $\eta_k > 0$ by the hypotheses. Without loss of generality, we assume that $\eta_k = |z_k| - |z_{k-1}|$ (otherwise, we consider $\eta_k = |z_{k+1}| - |z_k|$ in the same way).Again like the proof of Theorem 4.1, applying the minimum modulus theorem to $f(z)$ in

$$|z| \leq 2eL|z_{k-1}|,$$

we obtain

$$\log|f(z)| \geq -Ap(z_k) - B \qquad (|z| = r_k)$$

for some $A, B > 0$, where $|z_{k-1}| \leq r_k \leq L|z_{k-1}| \leq |z_k|$.

Then applying Lemma 4.5 to f in

$$|z - z_k| \leq (4e+1)(|z_k| - |z_{k-1}|)$$

and noting that for $\omega_k = r_k e^{iargz_k}$,

$$|\omega_k - z_k| \leq |z_k| - |z_{k-1}|,$$

we get a $\beta_k, 0 < \beta_k < \eta_k$, such that

$$\min_{|z-z_k|=\beta_k} \{\log|f(z)|\} \geq -Ap(z_k) - B$$

for some constants $A, B > 0$.

It's obvious that

$$D_k = \{|z - z_k| < \beta_k\} \subset \{|z - z_k| < \eta_k\} \subset A_k := \{|z_{k-1}| < |z| < |z_{k+1}|\}$$

and

$$A_k \cap \{z_k\}_{k=1}^{\infty} = \{z_k\}, \qquad \text{a single point.}$$

Thus

$$n(\beta_k, z_k, \tilde{V}) \leq n_0 n(\beta_k, z_k, V^*) \leq n_0 t_k. \tag{4.21}$$

Denote

$$\delta = \min_{0<j\leq n_0} \{|\omega^j - 1|\}.$$

If

$$\omega^j z_k (0 < j \leq n_0) \notin D_k,$$

then

$$N(\beta_k, z_k, \tilde{V}) = n(0, z_k, \tilde{V}) \log \beta_k \leq n_0 t_k \log |z_k|.$$

Otherwise by (4.21) and (4.20),

$$\begin{aligned} N(\beta_k, z_k, \tilde{V}) &\leq \int_{\delta|z_k|}^{\beta_k} \frac{n(t, z_k, \tilde{V})}{t} dt + n(0, z_k, \tilde{V}) \log \beta_k \\ &\leq n_0 t_k \log \frac{\beta_k}{\delta |z_k|} + n(0, z_k, \tilde{V}) \log \beta_k \\ &\leq n_0 t_k \log |z_k| + C_1 \leq A + Bp(z_k) \end{aligned}$$

for some $A, B > 0$. Using the Nevanlinna first fundamental theorem , we obtain that

$$\begin{aligned} \log |f_{k,t_k}| &= \frac{1}{2\pi} \int_0^{2\pi} \log |f(z_k + \beta_k e^{i\theta})| d\theta - N(\beta_k, z_k, \tilde{V}) \\ &\geq -Ap(z_k) - B \end{aligned}$$

for some constants $A, B > 0$.

It then follows that V is an interpolating variety from Theorem 3.1. ∎

Corollary 4.11. A multiplicity variety $\{(z_k, m_k)\}$ satisfying (4.18) is an interpolating variety for $\mathbf{A}_p$ if and only if for some constants $A, B > 0$,

$$N(|z|, V) \leq A + Bp(z),$$

and

$$m_k \log |z_k| \leq A + Bp(z_k).$$

In particular, if the multiplicity $m_k = 1$ for all $k \in \mathbf{N}$, then we have, in view of the property (i) of $p(z)$:

Corollary 4.12. A sequence $V = \{z_k\}$ satisfying (4.18) is an interpolating variety for $\mathbf{A}_p$ if and only if for some constants $A, B > 0$,

$$N(|z|, V) \leq A + Bp(z).$$

5. Interpolating varieties for M_p, III

In last section,we gave the geometric characterization of interpolating varieties for weights $p(z)$ satisfying (4.1). In the case when weights $p(z)$ grow so rapidly as to allow infinite order functions in M_p, the geometric conditions would become more delicate . They are still unknown even for the interpolation by entire functions with the fundamental weight $p(z) = \exp(|z|)$ (c.f.[3,p.3]).

We shall discuss the above weight in this section. More generally, from now on, we shall assume that radial weights $p(z)$ only satisfy the weaker condition: $\log p(e^r)$ is convex in r (e.g. $p(z) = \exp(|z|)$). Purely geometric sufficient conditions as well as necessary conditions for such weights will be given. Meanwhile counter examples will show that the conditions given here are best possible in some sense. To state our results, some preparations are required.

We shall always assume that $0 \leq \phi(r)$ is a non-increasing logarithmically convex function of $r \geq 0$ satisfying

$$\int_a^\infty \frac{\phi(r)}{r} dr < \infty \quad (a \geq 0) \tag{5.1}$$

such that

$$p(r(1 + q(r)) = O\{(p(r)\} \tag{5.2}$$

for $r \geq 0$, where $q(r) = \phi(p(r))$.

Remark 5.1. Such $\phi(r)$ exists for any weight $p(r)$. We remark here that, by a well-known theorem due to Borel (see [9] or [11]), (5.2) holds for any non-decreasing $p(r)$ except possibly a union of some intervals with finite logarithmic measure provided that $\phi(r)$ is a non-negative and non-increasing function satisfying (5.1) , and thus (5.2) holds for all $r \geq 0$ after some modification of values of $\phi(r)$ (Note that $p(r)$ is non-decreasing for $r \geq 0$ by the maximum principle for subharmonic functions).

Proportion 5.2. If $\phi(r)$ is a non-negative and non-increasing function satisfying (5.1), then

$$q(r)\log p(r) \to 0 \quad \text{as} \quad r \to \infty \tag{5.3}$$

Proof. By (5.1), we know that for any $\varepsilon > 0$,

$$\int_{r^{1/2}}^{r} \frac{\phi(r)}{r} dr < \varepsilon$$

for large r. But

$$\int_{r^{1/2}}^{r} \frac{\phi(r)}{r} dr \geq \phi(r) \int_{r^{1/2}}^{r} \frac{1}{r} dr = \frac{1}{2}\phi(r)\log r.$$

Therefore $\phi(r)\log r \to 0$ as $r \to 0$. This implies that $q(r)\log p(r) \to 0$ as $r \to \infty$. ∎

Remark 5.3. If $p(r) = e^r$. Then for any non-negative $\phi(r)$ satisfying (5.1) (e.g. $\phi(r) = (\log r)^{-(1+\varepsilon)}(\varepsilon > 0)$), (5.2) holds for all $r \geq 0$. In fact

$$p(r + rq(r)) = e^{r+rq(r)} = p(r)e^{q(r)\log p(r)} \leq Ap(r)$$

for some $A > 0$ by (5.3).

Now we are going to state our results.

Theorem 5.4. Let $V = \{(z_k, m_k, n_k)\}$ be a multiplicity variety. If

$$n(r, V^*) \leq Aq^3(r)p(r) \tag{5.4}$$

and

$$N(q(z_k), z_k, V^*) \leq A + Bp(z_k) \tag{5.5}$$

for some $A, B > 0$, where $V^* := \{(z_k, \max(m_k, n_k)\}$. Then V is an interpolating variety for M_p.

Corollary 5.5. Let $V = \{(z_k, m_k, n_k)\}$ be a multiplicity variety. If

$$n(r, V^*) \leq A\frac{1}{r^{3+3\varepsilon}}e^r \tag{5.6}$$

and

$$N(\frac{1}{|z_k|^{1+\varepsilon}}, z_k, V^*) \leq A + Be^{|z_k|} \tag{5.7}$$

for some $A, B, \varepsilon > 0$. Then V is an interpolating variety for $M_{\exp(|z|)}$.

Specializing these results to the interpolation problem for entire functions, we have

Corollary 5.6. Let $V = \{(z_k, m_k)\}$ be a multiplicity variety. If

$$n(r, V) \leq Aq^3(r)p(r)$$

and

$$N(q(z_k), z_k, V) \leq A + Bp(z_k)$$

for some $A, B > 0$. Then V is an interpolating variety for A_p.

Corollary 5.7. Let $V = \{(z_k, m_k)\}$ be a multiplicity variety. If

$$n(r, V) \leq A\frac{1}{r^{3+3\varepsilon}}e^r$$

and

$$N(\frac{1}{|z_k|^{1+\varepsilon}}, z_k, V) \leq A + Be^{|z_k|}$$

for some $A, B, \varepsilon > 0$. Then V is an interpolating variety for $A_{\exp(|z|)}$.

To prove our results, we need several lemmas.

Lemma 5.8. If $\log p(e^r)$ is convex, then

$$p(r) \geq br^{\alpha} + c \tag{5.8}$$

$(r \geq 0)$ for some $b, \alpha > 0$ and c.

Proof. For any function $\lambda(r)$, we define the operator D_+, the right-hand derivative, as follows:

$$D_+\lambda(r) = lim_{h\to 0^+}\frac{\lambda(r+h)-\lambda(r)}{h}.$$

Then $D_+(\log p(e^r))$ is non-decreasing and non-negative. Thus there exists a $r_0 \geq 0$ such that $r \geq r_0$,

$$D_+(\log p(e^r)) \geq D_+(\log p(e^{r_0})) := \alpha > 0.$$

That is

$$D_+(\log p(e^r) - \alpha r) \geq 0.$$

This shows that for $r \geq r_0$

$$\log p(e^r) - \alpha r \geq \log p(e^{r_0}) - \alpha r_0 := \log b,$$

and so that $p(e^r) \geq be^{\alpha r}$. Hence we can find a c such that

$$p(r) \geq br^{\alpha} + c$$

for $r \geq 0$. ∎

Lemma 5.9. [18,Prop 1.14]. Suppose that $\lambda(r)$ is a positive continuous function and $Z = \{z_k\}$ (repeated according to multiplicity) is a λ-balanced sequence, i.e., for some $A > 0$,

$$S(r_1, r_2, k, Z) \leq \frac{A\lambda(r_1)}{r_1^k} + \frac{A\lambda(r_2)}{r_2^k}$$

for all $r_1, r_2 > 0$ with $r_1 \le r_2$ and $k \in \mathbf{N}$. Then there exists a sequence $\{\alpha_k\}$ such that for some $A_1 > 0$,

$$|\alpha_k + S(r, k, Z)| \le \frac{A_1 \lambda(r)}{r^k}$$

for all $r > 0$ and $k \in \mathbf{N}$.

Lemma 5.10. Under the hypothesis (5.4) in Theorem 5.4, we can find a sequence $\{\alpha_k\}$ and $A, \beta > 0$ such that

$$|\alpha_k + S(r, k, \tilde{V})| \le \frac{Aq^2(r)p(r)}{r^k} \tag{5.9}$$

for all $r > 0$ and $k \in \mathbf{N}$, where $\tilde{V} := V^* \backslash \cup_{|z_k| \le \beta} \{(z_k, t_k)\}, t_k := \max\{m_k, n_k\}$.

Proof. Set

$$\rho(r) = \log q^2(e^r)p(e^r) = 2 \log q(e^r) + \log p(e^r)$$

Then $\rho(r)$ is convex since $\log p(e^r)$ and $\log q(e^r) = \log \phi(p(e^r))$ are all convex (Note that $p(r)$ is convex since $p(r)$ is subharmonic). Thus

$$\begin{aligned} \frac{q^2(r)p(r)}{r^k} &= \exp\{\log q^2(r)p(r) - k \log r\} \\ &= \exp\{\rho(0) + \int_0^x (D_+\rho(t) - k)dt\} \end{aligned}$$

by letting $r = e^x$. The right-hand derivative of ρ , $D_+\rho(t)$, is non-decreasing. Hence there exists a sequence $\{R_k\}_{k=1}^{\infty}$ which is unbounded and non-decreasing such that $q^2(r)p(r)/r^k$ is non-increasing for $r \le R_k$ and non-decreasing for $r \ge R_k, k \in \mathbf{N}$.

Let

$$\beta = |\frac{a-c}{b}|^{\frac{1}{\alpha}},$$

where a, b, c and α are defined in (5.1) and (5.8), and

$$m = \min\{k : R_k \ge \beta\}.$$

Then $R_m \ge \beta$ and $R_{m-1} \le \beta (R_0 := 0)$.

Define

$$\tilde{V} = V^* \setminus \cup_{|z_k| \le \beta} \{(z_k, t_k)\}.$$

We assert that for any $k \in \mathrm{N}, r_1, r_2 > 0$ with $r_1 \le r_2$,

$$S(r_1, r_2, k, \tilde{V}) \le \frac{Aq^2(r_1)p(r_1)}{r_1^k} + \frac{Aq^2(r_2)p(r_2)}{r_2^k} \tag{5.10}$$

for some $A > 0$.

First, by the definition, we have

$$\begin{aligned} S(r_1, r_2, k, \tilde{V}) &= \frac{1}{k} \Big| \sum_{r_1 \le |\zeta_n| \le r_2, \zeta_n \in \tilde{V}} (\frac{1}{\zeta_n})^k \Big| \\ &\le \frac{1}{k} \int_{r_1}^{r_2} \frac{1}{t^k} dn(t, \tilde{V}) \\ &\le \frac{n(r_2, \tilde{V})}{r_2^k} + \int_{r_1}^{r_2} \frac{n(t, \tilde{V})}{t^{k+1}} dt \\ &\le \frac{Aq^3(r_2)p(r_2)}{r_2^k} + A \int_{r_1}^{r_2} \frac{q^3(t)p(t)}{t^{k+1}} dt. \end{aligned} \tag{5.11}$$

Using Lemma 5.8 , we have , for any r, that

$$\begin{aligned} \int_\beta^\infty \frac{\phi(p(t))}{t} dt &\le \int_\beta^\infty \frac{\phi(bt^\alpha + c)}{t} dt \\ &= \frac{1}{\alpha} \int_{b\beta^\alpha + c}^\infty \frac{\phi(r)}{r - c} dr \\ &= \frac{1}{\alpha} \int_{b\beta^\alpha + c}^\infty \frac{\phi(r)}{r} (1 + \frac{c}{r - c}) dr \\ &\le \frac{1}{\alpha} (1 + \frac{|c|}{b\beta^\alpha}) \int_a^\infty \frac{\phi(r)}{r} dr := A_1. \end{aligned} \tag{5.12}$$

If $r_2 \le \beta$, it follows clearly that $S(r_1, r_2, k, \tilde{V}) = 0$. Thus we assume $r_2 > \beta$ in the following. Denote $r_1' = \max\{r_1, \beta\}$. Then,by (5.11),

$$\begin{aligned} S(r_1, r_2, k, \tilde{V}) &= S(r_1', r_2, k, \tilde{V}) \\ &\le \frac{Aq^3(r_2)p(r_2)}{r_2^k} + A \int_{r_1'}^{r_2} \frac{q^3(t)p(t)}{t^{k+1}} dt. \end{aligned} \tag{5.13}$$

We discuss two cases: $k < m$ and $k \ge m$.

If $k < m$. Then $R_k \le R_{m-1} \le \beta \le r_1'$. Hence

$$\begin{aligned}\int_{r_1'}^{r_2} \frac{q^3(t)p(t)}{t^{k+1}}dt &\le \frac{q^2(r_2)p(r_2)}{r_2^k}\int_{r_1'}^{r_2}\frac{q(t)}{t}dt\\ &\le \frac{q^2(r_2)p(r_2)}{r_2^k}\int_{\beta}^{\infty}\frac{\phi(p(t))}{t}dt\\ &\le A_1\frac{q^2(r_2)p(r_2)}{r_2^k} \qquad (5.14)\end{aligned}$$

(by(5.12)) . Therefore (5.10) holds by (5.13) and (5.14).

If $k \ge m$. Then by the fact that $r_1' \ge \beta$ and $R_k \ge R_m \ge \beta$, we have that

$$\begin{aligned}\int_{r_1'}^{r_2} \frac{q^3(t)p(t)}{t^{k+1}}dt &= \int_{r_1'}^{R_k}\frac{q^3(t)p(t)}{t^{k+1}} + \int_{R_k}^{r_2}\frac{q^3(t)p(t)}{t^{k+1}}dt\\ &\le \frac{q^2(r_1)p(r_1)}{r_1^k}\left|\int_{r_1'}^{R_k}\frac{q(t)}{t}dt\right| + \frac{q^2(r_2)p(r_2)}{r_2^k}\left|\int_{R_k}^{r_2}\frac{q(t)}{t}dt\right|\\ &\le \frac{q^2(r_1)p(r_1)}{r_1^k}\int_{\beta}^{\infty}\frac{\phi(p(t))}{t}dt + \frac{q^2(r_2)p(r_2)}{r_2^k}\int_{\beta}^{\infty}\frac{\phi(p(t))}{t}dt\\ &\le A_1\frac{q^2(r_1)p(r_1)}{r_1^k} + A_1\frac{q^2(r_2)p(r_2)}{r_2^k}\end{aligned}$$

(by(5.12)), independent of whether R_k is in $[r_1', r_2]$ or not. This together with (5.13) yields (5.10).

The conclusion of the lemma now follows from Lemma 5.9. ∎

Similar to Lemma 4.4, the following lemma relates $\tilde{V}$ to the set of zeros of an entire function f in A_{qp}.

Lemma 5.11. With the same hypothesis as in Lemma 5.10, there exists an entire function $f(z)$ such that $|f(z)| \le \exp(Aq(z)p(z))$ for some $A > 0$ and $Z(f) = \tilde{V}$, which was defined in Lemma 5.10.

Proof. By Lemma 5.10, we can find a sequence $\{\alpha_k\}$ satisfying (5.9). We define, in $|z| < r, g_r(z)$ and $f_r(z)$ as follows:

$$g_r(z) = \sum_{k\ge1}\{\alpha_k + S(r,k,\tilde{V})\}z^k = \sum_{k\ge1}\{\alpha_k + \frac{1}{k}\sum_{|z_n|\le r, z_n\in\tilde{V}}(\frac{1}{z_n})^k\}z^k,$$

and

$$f_r(z) = \exp(g_r(z)) \prod_{|z_n| \le r, z_n \in \tilde{V}} (1 - \frac{z}{z_n}). \tag{5.15}$$

By (5.9), $g_r(z)$ and so $f_r(z)$ are analytic in $|z| < r$. It turns out that for any fixed r, $f_r(z)$ defines a single entire function by means of (5.15). In fact, for any r_1, r_2 with $r_1 < r_2$, if $|z| \le r_1$, then

$$\begin{aligned} f_{r_2}(z) &= \exp\{\sum_{k \ge 1}(\alpha_k + \frac{1}{k}\sum_{|z_n| \le r_1}(\frac{1}{z_n})z^k + \frac{1}{k}\sum_{r_1 < |z_n| \le r_2}(\frac{1}{z_n})^k)z^k\} \cdot \\ &\quad \prod_{|z_n| \le r_1}(1 - \frac{z}{z_n}) \prod_{r_1 < |z_n| \le r_2}(1 - \frac{z}{z_n}) \\ &= f_{r_1}(z)\exp(\sum_{k \ge 1}\frac{1}{k}\sum_{r_1 < |z_n| \le r_2}(\frac{z}{z_n})^k)\exp(\sum_{r_1 < |z_n| \le r_2}\log(1 - \frac{z}{z_n})) \\ &= f_{r_1}(z)\exp\{\sum_{r_1 < |z_n| \le r_2}(\sum_{k \ge 1}\frac{1}{k}(\frac{z}{z_n})^k + \log(1 - \frac{z}{z_n})\} = f_{r_1}(z). \end{aligned}$$

Therefore, we obtain an entire function $f(z)$ defined as above which obviously satisfies $Z(f) = \tilde{V}$. For $|z| \le r$, we have that, in view of (5.9) and (5.2),

$$\begin{aligned} |g_{r+rq(r)}(z)| &\le \sum_{k \ge 1}\frac{Aq^2(r + rq(r))p(r + rq(r))}{(r + rq)^k}r^k \\ &\le Aq^2(r)p(r)\sum_{k \ge 1}(\frac{1}{1+q})^k \\ &= Aq^2(r)p(r)\frac{1}{q(r)} = Aq(r)p(r) \end{aligned}$$

and so that

$$\begin{aligned} \log|f(z)| &\le |g_{r+rq(r)}(z)| + \sum_{|z_n| \le r + rq(r), z_n \in \tilde{V}}\log(1 + \frac{r}{|z_n|}) \\ &\le Aq(r)p(r) + \int_0^{r+rq(r)}\log(\frac{3r}{t})d(n(t,\tilde{V}) - n(0,\tilde{V})) \\ &\le Aq(r)p(r) + n(r + rq(r),\tilde{V})\log 3 + \int_0^{r+rq(r)}\frac{n(t,\tilde{V}) - n(0,\tilde{V})}{t}dt \\ &\le Aq(r)p(r) + (\log 3)n(r + rq(r),\tilde{V}) + n(r + rq(r),\tilde{V})\log\frac{r + rq(r)}{\beta}. \end{aligned}$$

Noting (5.4),(5.2),(5.8) and (5.3),we obtain that

$$|f(z)| \le \exp(Aq(z)p(r))$$

for some $A > 0$. ∎

Lemma 5.12. (Cartan's Theorem [14]). Given any number $h > 0$ and complex number $a_1, a_2, \cdots, a_n$, we have that

$$\prod_{j=1}^{n} |z - a_j| > (\frac{h}{e})^n$$

for any $z \in \mathbf{C}$ outside a union of circles with the sum of the radii $\leq 2h$.

Lemma 5.13. Suppose that $f(z)$ is an entire function satisfying

$$f(0) \neq 0, \quad |f(z)| \leq \exp(Aq(z)p(z)), \tag{5.16}$$

and

$$n(r, \frac{1}{f}) \leq Aq(r)p(r) \tag{5.17}$$

for some $A > 0$. Then

$$\log |f(z)| > -A_1 p(r) \qquad (|z| \leq r + \frac{1}{2} rq(r))$$

for some $A_1 > 0$ outside a union of excluded circles with the sum of radii $\leq \frac{1}{4} q(r)$.

Proof. It is no loss of generality to assume that $f(0) = 1$. We will use the classic argument (c.f.[14,p.21]) to set $q = q(r)$ and

$$R(z) = \frac{(-(r + rq)^n}{a_1 a_2 \cdots a_n} \prod_{k=1}^{n} \frac{(r + rq)(z - a_k)}{(r + rq)^2 - \bar{a}_k z},$$

where $a_1, a_2, \cdots, a_n$ are the zeros of $f(z)$ in $|z| < r + rq(r)$, and

$$S(z) = f(z)/R(z).$$

Recall that if a function $G(z)$ is regular in $|z| \leq R$ with no zeros and $G(0) = 1$ then

$$\log |G(z)| \geq -\frac{2r}{R - r} \log M(R, f) \quad (|z| \leq r < R)$$

(see [14,p.19]). Applying this result to $S(z)$ in $|z| \leq r + rq$. Then in $|z| \leq r + \frac{1}{2}rq$, in view of (5.16) and (5.2), we have

$$\log|S(z)| \geq \frac{-2(r+\frac{1}{2}rq)}{r+rq-(r+\frac{1}{2}rq)}\{\log M(r+rq,f) - \log\frac{(r+rq)^n}{|a_1a_2\cdot a_n|}\}$$
$$\geq -\frac{A}{q}qp(z) = -Ap(z). \tag{5.18}$$

Using Lemma 5.12 with $h = \frac{1}{8}q$, we have

$$\prod_{k=1}^{n}|z-a_k| > (\frac{\frac{1}{8}q}{e})^n$$

for any z outside a union of circles with the sum of radii $\leq \frac{1}{4}q$. Therefore for such z in $\{|z| \leq r + \frac{1}{2}rq\}$, we have

$$|R(z)| \geq \frac{(r+rq)^n}{(r+rq)^n}\frac{(r+rq)^n}{2^n(r+rq)^{2n}}(\frac{q}{8e})^n = (\frac{q}{16e(r+rq)})^n$$

and consequently , in view of (5.17),(5.8) and (5.3),

$$\log|R(z)| \geq n\log\frac{q(r)}{16e(r+rq(r))}$$
$$\geq Ap(r)q(r)\log\frac{q(r)}{16er(1+q(r))}$$
$$\geq -Ap(r)q(r)(\log p(r) + A_1) - p(r)q(r)\log\frac{1}{q(r)}$$
$$\geq -Ap(r). \tag{5.19}$$

By (5.18) and (5.19), we obtain that

$$\log|f(z)| > -A_1p(r) \qquad (|z| \leq r + \frac{1}{2}rq(r))$$

foe some $A_1 > 0$ outside a union of circles with the sum of radii $\leq \frac{1}{4}q(r)$. ∎

Now we are ready to prove Theorem 5.4 , 5.6 and corollaries.

Proof of Theorem 5.4. First by Lemma 5.11, there exists an entire function $f(z)$ such that $Z(f) = \tilde{V}$ and (5.16) holds. Clearly by (5.4) we have that

$$n(r,\frac{1}{f}) = n(r,\tilde{V}) \leq n(r,V^*) \leq Aq(r)p(r).$$

By Lemma 5.13, we know

$$\log|f(z)| > -Ap(r) \qquad (|z| \le |z_k| + \frac{1}{2}|z_k|q(z_k))$$

outside a family of excluded circles with the sum of radii $\le \frac{1}{4}q(z_k)$. Hence, for large k, one can get a $\rho_k (0 < \rho_k < q(z_k))$ such that on $|z - z_k| = \rho_k$,

$$\log|f(z)| > -Ap(z_k).$$

By the Nevanlinna first fundamental theorem, we have, for $z_k \in \tilde{V}$, that

$$\log|f_{k,t_k}| + T(\rho_k, z_k, \frac{1}{f}) = T(\rho_k, z_k, f),$$

where $t_k := \max\{m_k, n_k\}$, and thus that

$$\begin{aligned}\log|f_{k,t_k}| &= m(\rho_k, z_k, f) - m(\rho_k, z_k, \frac{1}{f}) - N(\rho_k, z_k, \frac{1}{f}) \\ &\ge \frac{1}{2\pi}\int_0^{2\pi} \log|f(z_k + \rho_k e^{i\phi})d\phi - N(q(z_k), z_k, \tilde{V}) \\ &\ge -Ap(z_k) - B\end{aligned}$$

for some constants $A, B > 0$.

Set

$$F(z) = f(z)\prod_{i=1}^{I}(z - z_i)^{t_i},$$

where $I := \max\{k : |z_k| \le \beta\}$, β was defined in Lemma 5.10. Then by the definition of $\tilde{V}$, we will have $Z(F) = V^*$. It is immediate to see that for any $z_k \in V^*$, we still have

$$\log|F_{k,t_k}| \ge -Ap(z_k) - B$$

for some $A, B > 0$, and

$$\log|F(z)| \le A_1 \log|z| + \log|f(z)| \le Ap(z) + B$$

by (5.8) and (5.16), i.e., $F \in A_p$. Now using Theorem 3.1, we conclude that V is an interpolating variety for M_p.

The proof is thus complete. ∎

Proof of Corollary 5.5. Taking $\phi(r) = \frac{1}{(\log r)^{1+\varepsilon}} (\varepsilon > 0)$, we see that

$$\int_a^\infty \frac{\phi(r)}{r} dr < \infty$$

where $a = 1 + \varepsilon_0$ with $\varepsilon_0 > 0$. By Remark 5.3,

$$p(r(1 + q(r)) = O(p(r))$$

for all $r \geq 0$, where $q(r) = \phi(p(r)) = \frac{1}{r^{1+\varepsilon}}$.

The corollary follows directly from Theorem 5.4. ∎

Next, we consider the converse direction to Theorem 5.4 and its corollaries. We will prove that (5.5) is also necessary. Moreover, we have

Theorem 5.14. Suppose that $V = \{(z_k, m_k, n_k)\}$ is an interpolating variety for M_p. Then

$$n(r, V^*) \leq A \frac{1}{q(r)} p(r) \tag{5.20}$$

for some $A > 0$ and (5.5) holds.

Corollary 5.15. Suppose that $V = \{(z_k, m_k, n_k)\}$ is an interpolating variety for $M_p, p(z) = \exp(|z|)$. Then

$$n(r, V^*) \leq A r^{1+\varepsilon} e^r$$

for some $\varepsilon, A > 0$ and (5.7) holds.

Given any double multiplicity variety $V = \{(z_k, m_k)\}$, by changing V^* into V in Theorem 5.14 and Corollary 5.15, one can get necessary conditions for V to be interpolation for A_p.

Proof of Theorem 5.14. By Theorem 3.1, there is a function $f \in A_p$ such that

$$V^* \subset Z(f) \quad \text{and} \quad |f_{k,t_k}| \geq \varepsilon \exp(-cp(z_k)), k \in \mathbf{N}, \tag{5.21}$$

for some $\varepsilon, c > 0$, where $t_k = \max\{m_k, n_k\}$.

Thus for large r,

$$\begin{aligned} n(r, V^*) &\leq n(r, \frac{1}{f}) \\ &\leq \frac{1}{\log(1+q(r))} \int_r^{r+rq(r)} \frac{n(t, \frac{1}{f})}{t} dt \\ &\leq \frac{A}{q(r)} N(r + rq(r), \frac{1}{f}) \\ &\leq \frac{A}{q(r)} T(r + rq(r), \frac{1}{f}) \\ &\leq \frac{A}{q(r)} \{\log M(r + rq(r), f) + C\} \\ &\leq \frac{A}{q(r)} p(r) \end{aligned} \tag{5.22}$$

for some C and $A > 0$. By an appropriate modification of value of A, (5.22) holds for all $r \geq 0$. Also,

$$\begin{aligned} N(q(z_k), z_k, V^*) &\leq N(q(z_k), z_k, \frac{1}{f}) \\ &\leq T(q(z_k), z_k, \frac{1}{f}) \\ &= T(q(z_k), z_k, f) + \log \frac{1}{|f_{k,t_k}|} \\ &\leq \log M(q(z_k), z_k, f) + \log \frac{1}{|f_{k,t_k}|} \\ &\leq Ap(z_k) + B \end{aligned}$$

for some $A, B > 0$ by (5.2) and (5.21). The proof is thus finished. ∎

Corollary 5.15 is a consequence of Theorem 5.14.

Combining Theorem 5.4 with 5.14, we have

Theorem 5.16. The multiplicity variety $V = \{(z_k, m_k, n_k,)\}$ satisfying (5.4) is an interpolating variety for M_p if and only if

$$N(q(z_k), z_k, V^*) \leq Ap(z_k) + B$$

for some $A, B > 0$.

For the weight $p(z) = \exp(|z|)$, we obtain

Corollary 5.17. The multiplicity variety $V = \{(z_k, m_k, n_k)\}$ satisfying (5.6) is an interpolating variety for $M_{\exp|z|}$ if and only if

$$N(\frac{1}{|z_k|^{1+\varepsilon}}, z_k, V^*) \leq A\exp(|z_k|) + B$$

for some $A, B, \varepsilon > 0$.

Similarly, we have the corresponding results for interpolation by entire functions in A_p.

Obviously we can not drop the hypothesis (5.4) in Theorem 5.16 (resp.(5.6) in Corollary 5.17). But observing Theorem 5.4 and Theorem 5.14, it is natural to question whether or not one could find a real number α such that

$$n(r, V^*) \leq Aq^{\alpha}(r)p(r) \tag{5.23}$$

and (5.5) are both sufficient and necessary geometric conditions for V to be interpolating for M_p. A negative answer will be provided by the following counterexamples, which also show that for the sufficient conditions in Theorem 5.4, (5.4) can not be weakened into

$$n(r, V^*) \leq Ap(r) \tag{5.24}$$

and for the necessary condition in Theorem 5.14, (5.20) also can not be strengthened into (5.24).

Example 5.18. This example will show that (5.23) and (5.5) are not sufficient for V to be interpolating if $\alpha \leq 0$. Let's consider the weight $p(z) = \exp(|z|)$. Denote

$$\phi(r) = \frac{1}{(\log r)^{3+2\varepsilon} e^r}$$

and

$$q(r) = \phi(e^r) = \frac{1}{r^{3+2\varepsilon} e^{e^r}},$$

where ε is a small positive number. Let

$$z_k = \log k \quad \text{and} \quad \omega_k = z_k e^{i\theta_k}$$

with $\theta_k > 0$ satisfying

$$|\omega_k - z_k| = \frac{1}{|z_k|^{3+2\varepsilon} \exp(\exp|z_k|)};$$

and

$$V = \{z_k\}_{k=1}^{\infty} \cup \{\omega_k\}_{k=1}^{\infty}$$

with $m_k = 1, n_k = 0$ for $k \in \mathbf{N}$. Then obviously

$$n(r, V) = 2[e^r] \leq 2p(r) \leq 2q^{\alpha}(r)p(r)$$

for any $\alpha \leq 0$, i.e., (5.23) holds for $\alpha \leq 0$.

It is easy to see that for any $\zeta_k \in V$,

$$\begin{aligned} N(q(\zeta_k), \zeta_k, V) &= \int_0^{q(\zeta_k)} \frac{n(t, \zeta_k, V) - n(0, \zeta_k, V)}{t} dt + n(0, \zeta_k, V) \log q(\zeta_k) \\ &= \log q(\zeta_k) \leq A \end{aligned}$$

for some constant $A > 0$. That is , (5.5) holds.

But V is not an interpolating variety for $p(z) = \exp(|z|)$. Since otherwise, by Corollary 5.15 we should have

$$N(\frac{1}{|z_k|^{1+\varepsilon}}, z_k, V) \leq Ae^{|z_k|} + B \tag{5.25}$$

for some constants $A, B > 0$. However,

$$N(\frac{1}{|z_k|^{1+\varepsilon}}, z_k, V) = \int_0^{|z_k|^{-1-\varepsilon}} \frac{n(t, z_k, V) - n(0, z_k, V)}{t} dt + n(0, z_k, V) \log |z_k|^{-1-\varepsilon}$$

$$= \int_{|z_k|^{-3-2\varepsilon} e^{-e^{|z_k|}}}^{|z_k|^{-1-\varepsilon}} \frac{1}{t} dt + \log |z_k|^{-1-\varepsilon}$$

$$= \log |z_k|^{2+\varepsilon} + e^{|z_k|} + \log |z_k|^{-1-\varepsilon}$$

$$= \log |z_k| + e^{|z_k|}.$$

This shows that (5.25) can not hold, a contradiction.

Example 5.19. We will show, by this example, that (5.23) is not necessary if $\alpha \geq 0$. Again we consider the weight $p(z) = \exp(|z|)$. Let $f(z) = e^{ie^z}$. Then for $z = x + iy = re^{i\theta}$,

$$|f(z)| = e^{Re(ie^z)} = e^{-(siny)e^x} \leq e^{e^x} \leq e^{e^r} = e^{p(r)},$$

which shows that $f \in A_p$ and thus $F(z) := f(z) - 1 \in A_p$.

Set

$$V = Z(F(z)) := \{(z_k, m_k)\}. \tag{5.26}$$

We see that, at each z_k,

$$F'(z_k) = f'(z_k) = e^{ie^{z_k}} ie^{z_k} = ie^{z_k} \neq 0. \tag{5.27}$$

Therefore $m_k = 1$. Moreover, by (5.27), for $z_k = x_k + iy_k$,

$$|F'(z_k)| = |ie^{z_k}| = e^{x_k} \geq e^{-|z_k|} \geq e^{-e^{|z_k|}} = e^{-p(z_k)}. \tag{5.28}$$

Now (5.26) together with (5.28) shows that V is an interpolating variety for $p(z) = \exp(|z|)$ by Corollary 3.5.

However, (5.23) is not true for any $\alpha \geq 0$. In fact, for any $k \in \mathbf{N}$ and $n \in \mathbf{N}$, we have

$$f(\log 2k\pi + 2n\pi i) = e^{i \exp(\log 2k\pi + 2n\pi i)} = e^{2k\pi i} = 1.$$

Thus

$$V_1 := \{\log 2k\pi + 2n\pi i\}_{k,n} \subset V.$$

It is easy to check that

$$\frac{n(r, V_1)}{e^r} \to \infty \quad \text{as} \quad r \to \infty.$$

But $n(r, V) \geq n(r, V_1)$. Hence, for any $\alpha \geq 0$ and $\phi(r)$ satisfying (5.1) we can not find a $A > 0$ such that (5.23) holds.

Remark 5.20. Let us point out that all the above results still remain true if we replace the condition (ii) of $p(z)$ in §1 by the following Hormander's condition (see e.g. [5,p.4]): there exist four positive constants $c_1, \cdots, c_4$ such that $|\xi - z| \leq \exp(-c_1 p(z) - c_2)$ implies that $p(\xi) \leq c_3 p(z) + c_4$. We omit the details of modification here.

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for some constants $A, B > 0$. However,

$$\begin{aligned} N(\frac{1}{|z_k|^{1+\varepsilon}}, z_k, V) &= \int_0^{|z_k|^{-1-\varepsilon}} \frac{n(t, z_k, V) - n(0, z_k, V)}{t} dt + n(0, z_k, V) \log |z_k|^{-1-\varepsilon} \\ &= \int_{|z_k|^{-3-2\varepsilon} e^{-e^{|z_k|}}}^{|z_k|^{-1-\varepsilon}} \frac{1}{t} dt + \log |z_k|^{-1-\varepsilon} \\ &= \log |z_k|^{2+\varepsilon} + e^{|z_k|} + \log |z_k|^{-1-\varepsilon} \\ &= \log |z_k| + e^{|z_k|}. \end{aligned}$$

This shows that (5.25) can not hold, a contradiction.

Example 5.19. We will show, by this example, that (5.23) is not necessary if $\alpha \geq 0$. Again we consider the weight $p(z) = \exp(|z|)$. Let $f(z) = e^{ie^z}$. Then for $z = x + iy = re^{i\theta}$,

$$|f(z)| = e^{Re(ie^z)} = e^{-(\sin y)e^x} \leq e^{e^x} \leq e^{e^r} = e^{p(r)},$$

which shows that $f \in A_p$ and thus $F(z) := f(z) - 1 \in A_p$.

Set

$$V = Z(F(z)) := \{(z_k, m_k)\}. \tag{5.26}$$

We see that, at each z_k,

$$F'(z_k) = f'(z_k) = e^{ie^{z_k}} ie^{z_k} = ie^{z_k} \neq 0. \tag{5.27}$$

Therefore $m_k = 1$. Moreover, by (5.27), for $z_k = x_k + iy_k$,

$$|F'(z_k)| = |ie^{z_k}| = e^{x_k} \geq e^{-|z_k|} \geq e^{-e^{|z_k|}} = e^{-p(z_k)}. \tag{5.28}$$

Now (5.26) together with (5.28) shows that V is an interpolating variety for $p(z) = \exp(|z|)$ by Corollary 3.5.

However, (5.23) is not true for any $\alpha \geq 0$. In fact, for any $k \in \mathbb{N}$ and $n \in \mathbb{N}$, we have

$$f(\log 2k\pi + 2n\pi i) = e^{i \exp(\log 2k\pi + 2n\pi i)} = e^{2k\pi i} = 1.$$

Thus

$$V_1 := \{\log 2k\pi + 2n\pi i\}_{k,n} \subset V.$$

It is easy to check that

$$\frac{n(r, V_1)}{e^r} \to \infty \quad \text{as} \quad r \to \infty.$$

But $n(r, V) \geq n(r, V_1)$. Hence, for any $\alpha \geq 0$ and $\phi(r)$ satisfying (5.1) we can not find a $A > 0$ such that (5.23) holds.

Remark 5.20. Let us point out that all the above results still remain true if we replace the condition (ii) of $p(z)$ in §1 by the following Hormander's condition (see e.g. [5,p.4]): there exist four positive constants $c_1, \cdots, c_4$ such that $|\xi - z| \leq \exp(-c_1 p(z) - c_2)$ implies that $p(\xi) \leq c_3 p(z) + c_4$. We omit the details of modification here.

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