ABSTRACT

Title of Dissertation: SPECIAL LAGRANGIANS IN MILNOR FIBERS AND ALMOST LAGRANGIAN MEAN CURVATURE FLOW

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Doctor of Philosophy, 2024

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The focus of this thesis is twofold:

(1) We solve the Shapere–Vafa Problem: We construct embedded special Lagrangian spheres in Milnor fibers. We give a necessary and sufficient condition for the existence of embedded special Lagrangian spheres in Milnor fibers.

(2) We solve the Thomas–Yau Problem for Milnor fibers: We prove the Thomas–Yau conjecture for the almost Lagrangian mean curvature flow (ALMCF) for Milnor fibers, under the assumption that the initial Lagrangian is an embedded positive Lagrangian sphere satisfying a natural stability condition proposed by Thomas–Yau but adapted to Milnor fibers by us.

In addition, we formulate a new approach to resolving the Thomas–Yau conjecture in arbitrary almost Calabi–Yau manifolds.

The Thomas–Yau conjecture proposes certain stability conditions on the initial Lagrangian under which the Lagrangian mean curvature flow (LMCF) exists for all time and converges to the

unique special Lagrangian in the Hamiltonian isotopy class, and therefore also homology class of the initial Lagranigan.

One of the reasons for studying LMCF in Calabi–Yau manifolds (or ALMCF in almost Calabi–Yau manifolds) is that the Lagrangian condition, as well as homotopy and homology classes, are preserved. Therefore, if the flow converges, it converges to a special Lagrangian.

We develop a method for finding special Lagrangian spheres in Milnor fibers. We provide examples which illustrate different situations which occur (the total number of special Lagrangian spheres is at least $\deg f - 1$ and at most $\frac{1}{2} \deg f(\deg f - 1)$, where f is the polynomial defining the Milnor fiber).

We show that the almost Lagrangian mean curvature flow of Lagrangian spheres in Milnor fibers can be reduced to a generalized mean curvature flow of paths in $\mathbb{C}$. This reduction is different from the one found by Thomas–Yau. We show that the limit of the flow is either a straight line segment or a polygonal line, corresponding to a special Lagrangian sphere or a chain of such spheres. We prove that under certain conditions (more general than the ones achieved by Thomas–Yau) the flow results in a special Lagrangian sphere.

Finally, we develop a method for associating a curve in $\mathbb{C}$ with a compact Lagrangian in a more general setting of an almost Calabi–Yau manifold. We show that when the Lagrangian flows by ALMCF that the corresponding curve remains convex and shortens its length. The limit is either a straight line segment corresponding to a special Lagrangian or a polygonal line resulting in a decomposition of the original Lagrangian.

SPECIAL LAGRANGIANS IN MILNOR FIBERS AND ALMOST LAGRANGIAN MEAN CURVATURE FLOW

by

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Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2024

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Acknowledgments

I wish to express my deep gratitude to my advisor, Yanir Rubinstein. He suggested to me a problem which I found rich and interesting. He provided many useful comments, questions and guidance.

A deep thank you to Jonathan Rosenberg.

I learned a lot from many wonderful professors: Jonathan Rosenberg, Matei Machedon, Konstantina Trivisa, Richard Wentworth, Karin Melnick, Tamás Darvas, Harry Tamvakis, Larry Washington, Leonid Koralov.

Thank you to Sigurd Angenent for very helpful email exchange.

Thank you to Trystan Denhard, Jemma Natanson, Ayala Nurieli, William Schildknecht, Julie Daberkow.

Thank you to Douglas Pinsky.

Finally, I am grateful for the support of the Anne G. Wylie dissertation fellowship, Hauptman summer fellowship, and the National Science Foundation through grants DMS-1515703, DMS-1906370, DMS-2204347.

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Chapter 1: Introduction

1.1 Overview

1.1.1 Introduction

The study of special Lagrangian submanifolds (known as BPS states in the physics literature) in Milnor fibers goes back to Shapere–Vafa in 1999 [1]. An important problem since has been to characterize the existence of special Lagrangians in a given homology class in a Milnor fiber. Despite the large degree of symmetry the problem has remained open since. In 2002, Thomas–Yau attempted to tackle this problem in some homology classes using a flow approach. Their work, however, contained three key limitations. First, it was limited to a subset of homology classes, phrased in terms of the existence of a certain representative for the class, a condition that cannot be easily verified even in explicit examples. Second, it relied on a heuristic argument of Shapere–Vafa for a key barrier argument in order to obtain the long-time behavior of the flow. Third, it contained a flaw in a key a priori estimate. A fix for the last point appeared in Dellatorre's thesis in 2019 [2], which required to further strengthen the condition required on the representative for the class.

In this thesis we resolve the Shapere–Vafa problem, i.e., we give a complete characterization of the existence of special Lagrangians in a given homology class in a Milnor fiber. Our

approach is constructive and ultimately rather elementary, using conical Riemann surfaces, complex analysis, and calculus of variations. We also demonstrate the method with many examples that give new explicit special Lagrangians.

In addition, we develop a different approach to studying the almost Lagrangian mean curvature flow of Lagrangian spheres in Milnor fibers. We reduce the flow of the Lagrangian spheres to a flow of curves as do Thomas–Yau [3] but the curves we use are not the ones used by Thomas–Yau; this is one of our novelties. We reduce the flow of Lagrangian spheres to a flow of paths, with fixed endpoints. We investigate this flow which we refer to as generalized curve shortening flow and we use it to prove Thomas–Yau conjecture for almost Calabi–Yau manifolds (Conjecture 1.2.5) in the setting of Milnor fibers. This solves the Thomas–Yau problem in this special setting.

Finally, we initiate a new approach for the Thomas–Yau conjecture in arbitrary almost Calabi–Yau manifolds.

1.1.2 History

The following sequence of results motivated this thesis.

1. In their 1982 seminal work *Calibrated Geometries* [4] Harvey and Lawson introduced the concept of calibration by closed differential forms. One of the principal examples of calibrated geometries is the special Lagrangian geometry.

   A calibration on a Riemannian manifold X is a closed p-form φ satisfying

$$\varphi|_\xi \leq dVol|_\xi \tag{1.1}$$

for all points $q \in X$ and all oriented tangent p-planes $\xi \subset TX_q$. A calibrated manifold is a Riemannian manifold X together with a calibration.

Submanifolds of dimension p for which equality holds in (1.1) are the special objects of the calibrated geometry. *A special object in a calibrated geometry has minimal volume in its homology class.*

Special Lagrangian geometry in $\mathbb{C}^n$ is given by the calibrating form $dz_1 \wedge \cdots \wedge dz_n$. In a Calabi–Yau manifold (X, J, ω, Ω) special Lagrangian geometry is given by the calibrating form Ω.

2. Mean curvature flow was first investigated for closed curves in the Euclidean plane and eventually on other surfaces by Gage–Hamilton [5], Grayson [6], [7] (*curve shortening flow*). When a closed curve evolves by mean curvature it shrinks to a round point in a finite amount of time, or, in case of surfaces possibly the flow exists for all time and converges to a closed geodesic.

   Significantly, Angenent investigated very general flows of closed curves which can be characterized by parabolic equations [8], [9].

   When generalizations to higher dimensions were made, first by Huisken [10], it was realized that there are other possible singularities of the flow. One of the applications of the mean curvature flow in higher dimensions is finding minimal submanifolds. *As the volume of the evolving object decreases under mean curvature flow, the flow, if it converges, can be used to find objects of minimal volume.*

3. Smoczyk proved that a unique short time solution exists for the mean curvature flow equa-

tion of a compact smooth Lagrangian in a Calabi–Yau manifold and that the solution remains Lagrangian [11, Theorem 1.9].

4. Finding special Lagrangian submanifolds in a Calabi–Yau manifold is of interest also in physics (string theory). In the physics literature related to our subject matter, compact special Lagrangian submanifolds are BPS (Bogomol'nyi, Prasad, Sommerfield) states [1, p. 8].The ideas behind calibrated geometries with their special objects and BPS bounds with BPS states are analogous. For each calibrated geometry there is a fixed differential form for which a certain inequality holds. Special objects are those for which the inequality is in fact equality. Similarly, BPS bounds are inequalities and BPS states are solutions of differential equations obtained from the BPS bounds when the bounds are "saturated", i.e., equality holds [12].

   Shapere–Vafa discuss what mathematicians refer to as Milnor fibers of dimension $n = 3$ (the dimension which is relevant in physics). They claim [1, p. 9 lines 7-10] that in order to find supersymmetric cycles one needs to find paths for which a certain complex integral has constant phase. The reference for that claim is a physics paper by Greene–Vafa–Warner [13] (reference 15 in Shapere–Vafa) but it appears to us that the reference should be [14, Equation (3.5)] (reference 13 in Shapere–Vafa). Furthermore, they claim that for any given phase, there exist three solutions emanating from a zero of the polynomial which is used in the complex integral [1, Section 5].

   The physics approach of Shapere–Vafa substantially agrees with our approach in dimension three. Some problems exist in the preprint which, to our knowledge, has not been published. They do not provide justification for how the integral which is minimized by

special Lagrangians is obtained. In the reference [14, Equation (3.5)] the derivation relies on the low dimension and does not imply the formula [1, Equation (4.2)]. This is expanded upon in Remark 3.9.1. Another issue is the lack of rigorous proof for the solution of the differential equation [1, Equation (5.1)] They proceed to show that if a path between two zeros of a polynomial such that the phase of the integral is constant exists, then the path is unique. They also look at two examples: (i) polynomial with three zeros, (ii) polynomial whose roots are the roots of unity [1, pp 16-18 and pp 18-20].

5. Thomas–Yau are inspired by Shapere–Vafa's work on Milnor fibers of complex dimension three [1, Sections 4 and 5]. They generalize it to Milnor fibers of complex dimension n. This is the setting in which they attempt to prove a version of their conjecture about the convergence of the almost mean curvature flow of Lagrangians (ALMCF) to a special Lagrangian.

   Thomas–Yau claim two main results:

   (i) Pick a connected graded Lagrangian L whose obstructions to the existence of its Floer cohomology vanish, and whose second Stieffel–Whitney class w_2 is the restriction of a class in $H^2(X;\mathbb{Z}/2)$ on the whole manifold (for instance if L is spin). Then there can be at most one smooth special Lagrangian in the Hamiltonian deformation class of L. In particular, SL (special Lagrangian) homology spheres are unique in their Hamiltonian deformation class in dimension 3 and above. [3, Theorem 4.3]

   (ii) Suppose that γ is a curve in $\mathbb{C}$, with endpoints at zeros of f, and otherwise missing the zeros of f, such that its pointwise phase $\theta := \arg(\gamma) + (\frac{n}{2} - 1)\arg(f(\gamma))$ satisfies

$$[\phi(L_1), \phi(L_2)] \not\subseteq (\inf_L \theta, \sup_L \theta), \tag{1.2}$$

for all Lagrangians $L_i = \gamma_i^n$, $i = 1, 2$, fibred over curves γ_i in the base, and also

$$S - I := \sup_\gamma \theta - \inf_\gamma \theta < \frac{2\pi}{3}. \tag{1.3}$$

Then the flow of γ by the vector

$$\frac{1}{1 + \frac{|f'(\gamma)|^2}{4|f(\gamma)|}} \big(\vec{\kappa} + (1 - \frac{n}{2})\vec{n}(\log|f(\gamma)|)\big), \tag{1.4}$$

where $\vec{\kappa}$ is the curvature vector and $\vec{n}$ is the unit normal vector of γ, exists for all time and converges in C^∞ to a smooth curve whose phase function is constant. [3, Theorem 7.6]

6. Dellatorre [2, Theorem 5.5.11] corrects the assumption (1.3) to

$$\sup_\gamma \theta - \inf_\gamma \theta < \frac{n\pi}{2(n-1)}, \ n > 1 \tag{1.5}$$

in order to correct a part of the proof of [3, Lemma 7.8] which claims that, during the flow (1.4), the curve over which the Lagrangian sphere is defined, does not start to spiral around a zero of f. Specifically, the assumption (1.5) is used by Dellatorre to show [2, Lemma 5.6.2]:

$$\limsup_{|s-s'|\to 0} |\arg(\gamma'(s,t)) - \arg(\gamma'(s',t))| < \pi,$$

for all time t for which the flow exists and precludes spiraling [2, p. 147, line 4]. This is

the inequality which Thomas–Yau attempt to prove [3, Lemma 7.8].

Dellatorre also notes [2, Remark 5.5.10] that Thomas–Yau consider Shapere-Vafa's physics approach (1.1.2) for $n = 3$ to be mathematically correct, and, moreover, valid for all n. To fix this Dellatorre assumes [2, Remark 5.5.10 equation (5.58)]:

There exist C^1 solutions, γ_c, to the one-dimensional initial value problem for the special Lagrangian equation:

$$\arg(\gamma_c') + (\frac{n}{2} - 1)\arg(f(\gamma_c)) = c, \quad \text{for } c \in \{\inf_{L_\gamma} \theta, \sup_{L_\gamma} \theta\}, \tag{1.6}$$

where $\gamma_c(0)$ is any of the two roots of f that are the endpoints of γ.

In sum, Dellatorre prove the following theorem

• Let $L_\gamma \subset X_f^n$ be an $O(n)$-invariant Lagrangian sphere. Suppose that the grading, θ, of L_γ satisfies the condition (1.2), that (1.6) holds, and that

$$\sup_\gamma \theta - \inf_\gamma \theta < \begin{cases} \pi, & \text{if } n = 1, \\ \frac{n\pi}{2(n-1)}, & \text{if } n > 1. \end{cases} \tag{1.7}$$

Then the almost Lagrangian mean curvature flow of L_γ exists for all time and converges in C^∞ to a smooth special Lagrangian [2, Theorem 5.5.11].

1.2 The Thomas–Yau conjecture

There are two different problems associated with the Thomas–Yau conjecture [3,]:

1. Find a special Lagrangian in a given homology class which contains Lagrangians. We will refer to this problem as the Shapere–Vafa problem.

2. If it exists, can a special Lagrangian be obtained from a stable Lagrangian in its homology class by means of Lagrangian mean curvature flow? We will refer to this problem as the Thomas–Yau problem.

The two problems are related since one of the reasons for studying the mean curvature flow of Lagrangians is to find special Lagrangians in the same homology class as a given Lagrangian.

• The key starting point for the investigation of ALMCF are two consequences of the flow [3, Equations (7.4) and (7.5)]:

$$\frac{d\theta}{dt} = -\Delta\theta + \frac{\langle d\theta, d|\Omega|\rangle}{|\Omega|} \tag{1.8}$$

$$\frac{d}{dt}(|\Omega\,|dV_{g_L}) = -|d\theta|^2(|\Omega\,|dV_{g_L}) \tag{1.9}$$

From (1.8) it follows that θ obeys the maximum principle for parabolic equations. As a consequence, if the Lagrangian angle of the initial Lagrangian satisfies a condition on its range, such as (1.3), then the condition holds during the flow.

From (1.9) it follows that ALMCF decreases the volume measured by the weighted volume form $|\Omega|dV_{g_L}$

Thomas–Yau use methods and results of Angenent [8], [9] for the proof of a technical estimate which deals with curvature blow-up of the curves defining Lagrangians in the flow [3, Lemma 7.9].

Uniqueness: Thomas–Yau prove that (under technical assumptions which hold for Lagrangian homology spheres) if L is a connected graded Lagrangian then there is at most one special Lagrangian in the Hamiltonian deformation class of L [3, Theorem 4.3].

Thomas–Yau explore the almost Lagrangian mean curvature flow of Lagrangian spheres in almost Calabi–Yau Milnor fibers. They attempt to prove a theorem which implies a version of their conjecture [3, Theorem 7.6] in that setting.

Our strategy is as follows:

• First we find the special Lagrangian spheres in Milnor fibers directly, in a constructive way, thus not utilizing ALMCF: Suppose that we would like to find a special Lagrangian sphere over a path connecting w_j and w_l, zeros of f. We define a map $Q : \mathbb{C} \to \mathbb{C}$, where $Q'(w) = (f(w))^{\frac{n-2}{2}}$. Then we find the images $Q(w_j)$ and $Q(w_l)$ and the straight line segment $\overline{Q(w_j)Q(w_l)}$. If the path we are looking for exists, it is contained in $Q^{-1}(\overline{Q(w_j)Q(w_l)})$. (This constructive method is presented in Chapter 3.)

• Second, we prove a version of the Thomas–Yau conjecture in the setting of Milnor fibers using the flow of paths in $Q(\mathbb{C})$:

Conjecture 1.2.1. *Let L be a positive Lagrangian submanifold in an almost Calabi–Yau manifold such that $\phi(L) = 0$. If L satisfies:*

$$[\phi(L_1), \phi(L_2)] \nsubseteq (\inf \theta_L, \sup \theta_L),$$

or

$$\int_{L_\gamma} |\Omega| dV_{g_{L_\gamma}} \leq \Big| \int_{L_{\gamma_1}} \Omega|_{L_{\gamma_1}} \Big| + \Big| \int_{L_{\gamma_2}} \Omega|_{L_{\gamma_2}} \Big|$$

for all graded connect sums $L_1\#L_2$ in the same Hamiltonian deformation class as L, then almost mean curvature flow for L exists for all time and converges to a special Lagrangian in its Hamiltonian deformation class. The special Lagrangian is unique in its Hamiltonian deformation class.

If γ is a path such that L_{γ_0} is an SL sphere then $Q(\gamma_0)$ is a straight line segment. Another path, γ having the same endpoints as γ_0 gives rise to L_γ which could potentially flow to L_{γ_0} under ALMCF. Now, $Q(\gamma)$ is a path which has the same endpoints as $Q(\gamma_0)$. ALMCF of L_γ induces a flow of γ [2, Equation (5.60)], which in turn induces a flow of $Q(\gamma)$ (Theorem 6.1.3).

The flow of $Q(\gamma)$ is simpler than the flow of γ (the flow used by Thomas–Yau).

We prove convergence of $Q(\gamma)$ assuming only that the variation of the phase θ is less than π. Thomas–Yau [3] assumed the variation of θ to be less than $\frac{2\pi}{3}$ and Dellatorre noticed that it needed to be changed to $\frac{n\pi}{2(n-1)}$ [2]. (Chapter 6 is dedicated to studying the flow and proving the convergence.)

1.2.1 The conjectures

The Thomas–Yau conjecture relates the convergence to a special Lagrangian of the Lagrangian mean curvature flow to a stability condition on the initial Lagrangian. Predating the Thomas–Yau Conjecture in [3] is Thomas's Conjecture in [15]. Prior to stating the conjectures we briefly explain some of the notation and notions involved:

• A connect sum of Lagrangians L_1 and L_2 intersecting at a point is a standard topological construction obtained by removing a small ball around the intersection point in each of L_1 and L_2, and gluing the two thus obtained boundaries. We will usually think of $L_1\#L_2$ as mostly

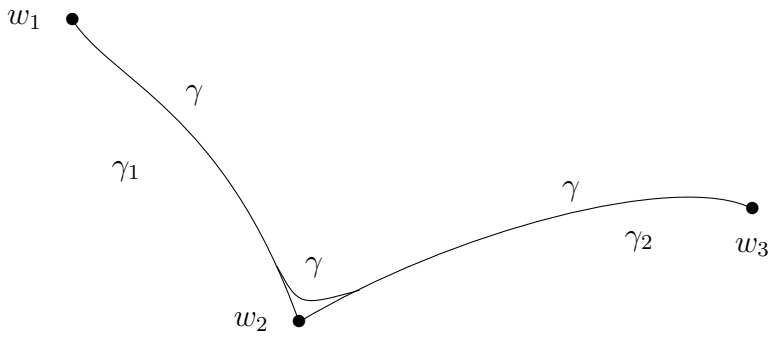


Figure 1.1: Connect sum (paths)

coinciding with $L_1 \cup L_2$ except for a small neighborhood of $L_1 \cap L_2$ which is transformed to create a smooth submanifold. Another distinction from a purely topological construction is that we will consider those connect sums for which $L_1 \# L_2$ is a graded Lagrangian. If θ_1 and θ_2 are gradings on L_1 and L_2 respectively, then, if it exists, $\theta_1 \# \theta_2$ denotes a grading on $L_1 \# L_2$ which agrees with θ_1 on $L_1 \# L_2 \cap L_1$ and with θ_2 on $L_1 \# L_2 \cap L_2$. respectively.

We illustrate Lagrangian connect sums in the context of Lagrangian spheres in Milnor fibers:

In Figure 1.1, w_1, w_2, and w_3 are zeros of the polynomial f. Let γ_1 be a smooth path between w_1 and w_2, and let γ_2 be a smooth path between w_2 and w_3, such that $\gamma_1 \cap \gamma_2 = \varnothing$. Then L_{γ_1} and L_{γ_2} are Lagrangian spheres which have one common point. L_γ is a connect sum of L_{γ_1} and L_{γ_2} and the Lagrangian angle θ_{L_γ} on L_γ is such that $(L_\gamma, \theta_{L_\gamma})$ is a graded connect sum.

• The average phase of a Lagrangian is

$$\phi(L) := \arg \int_L \Omega|_L. \tag{1.10}$$

If two Lagrangians are in the same homology class then the value $\left| \int_L \Omega|_L \right|$ is the same for each of them.

• Given a differentiable function H on a symplectic manifold, the *Hamiltonian vector field* for the function H is X_H defined by:

$$\omega(X_H, \cdot) = dH.$$

• Lagrangian L' is in the *Hamiltonian deformation class* of a Lagrangian L if there exists an isotopy from L to L' such that at any time t the flow vector field is the Hamiltonian vector field for some differentiable function on a neighborhood of L_t.

Thomas defines a notion of stability for Lagrangians in [15, Definition 5.1]:

Definition 1.2.2. *Take graded Lagrangians (L_1, θ_1) and (L_2, θ_2), Hamiltonian isotoped to intersect cleanly, and such that the graded Lagrangian connect sums $(L_1 \# L_2, \theta_1 \# \theta_2)$ exist. Then a Lagrangian L of Maslov class zero is said to be destabilised by the L_i if it is Hamiltonian isotopic to such an $L_1 \# L_2$, and the phases (real numbers, induced by the gradings) satisfy $\phi(L_1) \geq \phi(L_2)$. If L is not destabilised by any such L_i then it is called stable.*

In the same article, Thomas makes a conjecture [15, Conjecture 5.2] relating the existence of a special Lagrangian in a Hamiltonian deformation class of a given Lagrangian to that Lagrangian being stable in the sense of Definition 1.2.2:

Conjecture 1.2.3. *(Thomas) A Lagrangian of Maslov class zero has a special Lagrangian in its Hamiltonian deformation class if and only if it is stable, and this special Lagrangian representative is unique.*

The Thomas–Yau conjecture, although closely related to the Thomas conjecture, is quite different in that both the existence of a special Lagrangian and the convergence of the Lagrangian

mean curvature flow are expected [3, p. 1101, Conjecture 7.3], provided that the initial Lagrangian is stable in some sense:

Conjecture 1.2.4. (Thomas–Yau) *Let L be a Lagrangian submanifold in a Calabi–Yau manifold and $\phi(L) = 0$. If L satisfies:*

$$[\phi(L_1), \phi(L_2)] \nsubseteq (\inf \theta_L, \sup \theta_L), \tag{1.11}$$

or

$$VolL \leq \int_{L_1} \mathrm{e}^{-i\phi(L_1)}\Omega + \int_{L_2} \mathrm{e}^{-i\phi(L_2)}\Omega, \tag{1.12}$$

for all graded connect sums $L_1 \# L_2$ in the same Hamiltonian deformation class as L, then mean curvature flow for L exists for all time and converges to a special Lagrangian in its Hamiltonian deformation class, the unique special Lagrangian conjectured in Thomas [15].

1.2.2 Modifications of the Thomas–Yau conjecture

Thomas–Yau conjecture 1.2.4 is about Calabi–Yau manifolds. Lacking a proof of the conjecture in its generality, Thomas and Yau test the conjecture in almost Calabi–Yau Milnor fibers. The version of the conjecture which they attempt to prove is [3, Theorem 7.6] which is quoted above (1.1.2).

To reformulate the conjecture for almost Calabi–Yau manifolds the condition (1.12) needs to be modified while the condition (1.11) remains the same. Instead of (1.12), we have:

$$\int_{L_\gamma} |\Omega| dV_{g_{L_\gamma}} \leq \left| \int_{L_{\gamma_1}} \Omega|_{L_{\gamma_1}} \right| + \left| \int_{L_{\gamma_2}} \Omega|_{L_{\gamma_2}} \right| \tag{1.13}$$

Thomas–Yau and Dellatorre use only the phase condition, (1.11), for the Milnor fiber proof and thus the correct reformulation (1.13) of the condition (1.12) is not made either in [3] or in [2].

Thomas [15] and Thomas–Yau [3] conjectures inspired an active area of research. Notably, Joyce formulates his own conjecture based on Bridgeland stability [16, Conjecture 3.34]. Previously, Neves found counterexamples to Conjecture 1.2.4 in [17] but, as Joyce points out, with the additional assumption that the Lagrangians are positive, the Neves examples are not counterexamples to the Conjecture of Thomas–Yau [16, p. 3].

The following is the correct formulation of the Thomas–Yau conjecture for almost Calabi–Yau manifolds:

Conjecture 1.2.5. *Let L be a positive Lagrangian submanifold in an almost Calabi–Yau manifold such that $\phi(L) = 0$. If L satisfies:*

$$[\phi(L_1), \phi(L_2)] \nsubseteq (\inf \theta_L, \sup \theta_L),$$

or

$$\int_{L} |\Omega| dV_{g_L} \leq \left| \int_{L_1} \Omega|_{L_1} \right| + \left| \int_{L_2} \Omega|_{L_2} \right|$$

for all graded connect sums $L_1 \# L_2$ in the same Hamiltonian deformation class as L, then almost mean curvature flow for L exists for all time and converges to a special Lagrangian in its Hamiltonian deformation class. The special Lagrangian is unique in its Hamiltonian deformation

class.

1.3 Results

The first of our main results, set forth as Theorem 1.3.5 below, is in form similar to Thomas's Conjecture 1.2.3. Note that our definition of stability (Definition 1.3.3) is different from Thomas's (Definition 1.2.2) and our setting is a Milnor fiber.

Conditions (1.21) and (1.22) in Theorem 1.3.6 are in form similar to those present in the Thomas–Yau Conjecture (Conjecture 1.2.4). We prove the equivalence of (iii) and (i) of Theorem 1.3.6 by showing that condition (1.21) implies stability of the Lagrangian sphere. Similarly, we prove the equivalence of (i) and (iv) by showing that condition (1.22) implies stability of the Lagrangian sphere for which it holds.

The first step towards proving the main results is Theorem 1.3.1. In Theorem 1.3.1 we establish that for any given initial point, and any given phase, there exists a curve γ which gives rise to a special Lagrangian manifold. Those special Lagrangians are not necessarily compact. Compact special Lagrangians are obtained only for finitely many values of the phase. Depending on the positions of the $\deg f$ distinct zeros of the polynomial f which is used in the definition of the Milnor fiber, there exist at least $\deg f - 1$ and at most $\frac{1}{2}\deg f(\deg f - 1)$ such special Lagrangian spheres in the Milnor fiber X_f^n. We briefly explain here the reason for these extrema: The zeros of f can be viewed as vertices of a graph. Edges of the graph are paths connecting the vertices having the property that their corresponding Lagrangians are special. We will see in Section 2.2 that this graph must be connected. The values $\deg f - 1$ and $\frac{1}{2}k\deg f(\deg f - 1)$ are respectively the minimal and the maximal number of edges in a connected graph with $\deg f$

vertices - hence the extrema. For any $n \in \mathbb{N}$:

• If all the zeros of f are colinear then there are exactly $(\deg f - 1)$ edges in the described graph.

• If the zeros of f form a regular polygon then there are $\frac{1}{2} \deg f(\deg f - 1)$ edges in the graph.

Theorem 1.3.1. *For any zero w_j of the polynomial $f(w)$ and for any given $\theta \in [0, 2\pi)$, there exist n integral curves γ of the equation*

$$\arg\left(\gamma'(u) f(\gamma(u))^{\frac{n-2}{2}}\right) = \theta \tag{1.14}$$

with the initial condition

$$\gamma(0) = w_j.$$

Remark 1.3.2. Theorem 1.3.1 fills the gap which exists in the Thomas–Yau proof [3, Section 7]. They assume the validity of the theorem for all n based on the heuristic proof by Shapere-Vafa. (see 1.1.2)

Proof. For proof see the proof of Theorem 2.3.3 in Section 2.3 which is more general. □

There are finitely many values of θ such that the Lagrangian submanifold corresponding to the integral curve whose existence is claimed in Theorem 1.3.1 is non-compact. Since the initial point of a path γ (integral curve from Theorem 1.3.1) is a zero of the polynomial f over which the S^{n-1} fiber shrinks to a point, if the integral curve reaches another zero of f, the fiber degenerates above it as well, and the Lagrangian compactifies.

Therefore, in order to find a special Lagrangian, we find values of θ for which respective

integral curves of the initial value problem (1.14) extend to another zero of the polynomial f. All possible candidates for such values of θ can be found explicitly by Theorem 3.8.5.

To state Theorems 1.3.5 and 1.3.6, and Definition 1.3.3, we introduce some notation:

• For $f \in \mathcal{P}_{simple}$ (2.3) let

$$V(f) := \{w \in \mathbb{C} \,:\, f(w) = 0\}. \tag{1.15}$$

• For any $w_j, w_\ell \in V(f)$, $j \neq \ell$, let

$$\mathcal{A}_{j\ell} \tag{1.16}$$

be the set of simple C^1 paths connecting w_j and w_ℓ which contain no other zeros of $f(w)$.

• For any $w_j, w_\ell \in V(f)$, $j \neq \ell$, let $\mathcal{H}_{j\ell}$ denote the set of Lagrangian spheres defined over paths in $\mathcal{A}_{j\ell}$.

• For (X, J, ω, Ω) almost Calabi–Yau let the modulus of Ω be the following function $|\Omega| : X \to \mathbb{R}_{>0}$:

$$|\Omega|(p) := |\Omega(e_1, \ldots, e_n)|, \tag{1.17}$$

for any orthonormal (with respect to g_p) basis $e_1, \ldots, e_n$ of T_pX as a complex n-dimensional vector space.

It is shown in Lemma 2.1.7 that $|\Omega|$ is well-defined.

• For $f \in \mathcal{P}_{simple}$ (2.3) and $n \in \mathbb{N}$ let

$$q = (f(w))^{\frac{n-2}{2}}, \tag{1.18}$$

and let Q be a primitive function of q:

$$\frac{d}{dw}Q(w) = q(w), \quad Q(0) = 0 \tag{1.19}$$

Our notion of stability of Lagrangian spheres in Milnor fibers is:

Definition 1.3.3. *Let $\gamma \in \mathcal{A}_{j\ell}$. We call the Lagrangian L_γ stable if the path $Q(\gamma)$, together with the straight line segment $\overline{Q(w_j)Q(w_\ell)}$ encloses a region in the $\mathbb{C}$ whose closure does not contain the image under Q of any other zero of f. A curve γ such that L_γ is stable is called admissible.*

Remark 1.3.4. Our notion of stability is different from the one of Thomas–Yau. We will use Figures 1.2 and 1.3 to explain the difference.

The following facts are proven in Chapter 3 (expressed here for the path γ but valid for any path):

- $\text{length}(Q(\gamma)) = \int_{L_\gamma} |\Omega| dV_{g_{L_\gamma}}$
- $|Q(w_j)Q(w_l)| = \left| \int_{L_\gamma} \Omega|_{L_\gamma} \right|$
- $\tan\theta =$ slope of the tangent line to $Q(\gamma)$
- $\phi(L_\gamma) = \arg\left(Q(w_\ell) - Q(w_j)\right)$

In Figure 1.2, the length of the path $Q(\gamma)$ is larger than the length of $|Q(w_j)Q(w_p)|+|Q(w_p)Q(w_\ell)|$ which translates into:

$$\int_{L_\gamma} |\Omega| dV_{g_{L_\gamma}} > \left| \int_{L_{\gamma_1}} \Omega|_{L_{\gamma_1}} \right| + \left| \int_{L_{\gamma_2}} \Omega|_{L_{\gamma_2}} \right| \tag{1.20}$$

for any $\gamma_1 \in \mathcal{A}_{jp}$ and $\gamma_2 \in \mathcal{A}_{p\ell}$. This means that the rephrasing of the Thomas–Yau condition 1.12 in the setting of almost Calabi–Yau manifolds is not satisfied (1.20).

Similarly, in Figure 1.2, the angle condition of Thomas–Yau 1.11 fails to be satisfied since $\inf_{L_\gamma} \theta < \phi(L_{\gamma_1}), \phi(L_{\gamma_2}) < \sup_{L_\gamma} \theta$.

On the other hand, there exists a homotopy of paths in $\mathcal{A}_{j\ell}$ which maps to a homotopy of paths starting from $Q(\gamma)$ and ending with $\overline{Q(w_j)Q(w_\ell)}$ which does not pass through any other $Q(w_s)$, w_s a zero of f (Theorem 3.10.2). Such homotopy of paths induces a homotopy of corresponding Lagrangian spheres which ends at a special Lagrangian sphere. Therefore, our notion of stability guarantees the existence of a special Lagrangian sphere in a homotopy class when the Thomas–Yau theorem fails.

In Figure 1.3, $Q(\gamma)$ together with $\overline{Q(w_j)Q(w_\ell)}$ encompasses $Q(w_p)$ so γ is not admissible. Both Thomas–Yau conditions fail to be satisfied:

- Any path between $Q(w_j)$ and $Q(w_\ell)$ encompassing $Q(w_p)$ is longer than the polygonal line $Q(w_j)Q(w_p)Q(w_\ell)$.
- It is not possible for θ to be in $[\phi(L_{\gamma_1}), \phi(L_{\gamma_2})]$ for all points of γ and at the same time for $Q(\gamma)$ to go around $Q(w_p)$. Therefore, the following implication holds: if L_γ is not stable according to our definition then it is not stable according to Thomas–Yau.

In conclusion, based on our notion of stability the set of stable Lagrangian spheres is strictly larger.

The following theorem shows that stability of a Lagrangian sphere L_γ is the criterion for the existence of a special Lagrangian sphere in the homotopy class $[L_\gamma]$. It also gives a new proof of a result of Solomon–Yuval on the uniqueness [18, Corollary 1.3].

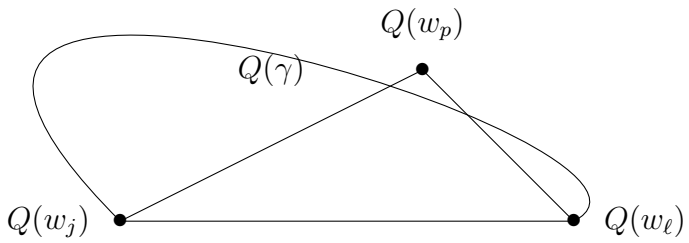


Figure 1.2: Admissible curve $Q(\gamma)$ from $Q(w_j)$ to $Q(w_\ell)$ with length greater than $|Q(w_j)Q(w_p)| + |Q(w_p)Q(w_\ell)|$

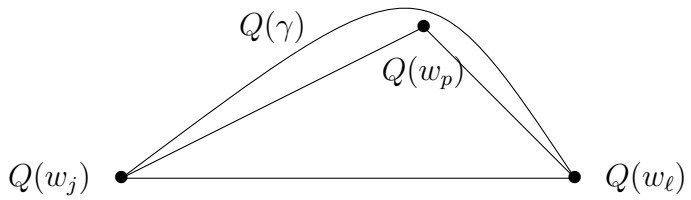


Figure 1.3: Non-admissible curve $Q(\gamma)$ from $Q(w_j)$ to $Q(w_\ell)$

Theorem 1.3.5. *A special Lagrangian sphere exists in a homology class of Lagrangian spheres in a Milnor fiber if the class contains a stable (Definition 1.3.3) Lagrangian sphere. If a special Lagrangian sphere exists in a homology class then it is unique.*

Proof. Sketch of Proof. The proof is in Section 3.10.1. In it we rephrase the statement of Theorem 1.3.5 in terms of paths in $\mathbb{C}$ so that Theorem 3.10.2 is equivalent to Theorem 1.3.5. The idea is that if there exists a special Lagrangian sphere over a path connecting two zeros of f then it is contained in $Q^{-1}\left(\overline{Q(w_j)Q(w_\ell)}\right)$. The stability condition makes it possible to find a homotopy of paths which does not contain another zero of f, and which starts with the path defining the stable Lagrangian sphere and ends with a path defining a special Lagrangian sphere. □

Theorem 1.3.6. *Let $w_j, w_\ell \in V(f)$, $j \neq \ell$. The following are equivalent:*

(i) There exists a special Lagrangian sphere in $\mathcal{H}_{j\ell}$.

(ii) There exists a path $\gamma \in \mathcal{A}_{j\ell}$ such that for the Lagrangian sphere, $L_\gamma \in \mathcal{H}_{j\ell}$:

$$\int_{L_\gamma} |\Omega_f^n| d_{L_\gamma} < \Big| \int_{L_{\gamma_1}} \Omega_f^n dV_{L_{\gamma_1}} \Big| + \Big| \int_{L_{\gamma_2}} \Omega_f^n dV_{L_{\gamma_2}} \Big|, \tag{1.21}$$

for all $w_p \in V(f)$, $p \notin \{j, \ell\}$, and for all paths $\gamma_1 \in \mathcal{H}_{jp}$ and $\gamma_2 \in \mathcal{H}_{p\ell}$.

(iii) There exists a path $\gamma \in \mathcal{A}_{j\ell}$ such that for the Lagrangian sphere, $L_\gamma \in \mathcal{H}_{j\ell}$:

$$\Big[\arg \int_{L_{\gamma_1}} \Omega_f^n dV_{g_{L_{\gamma_1}}}, \arg \int_{L_{\gamma_2}} \Omega_f^n dV_{g_{L_{\gamma_2}}} \Big] \not\subseteq (\inf_L \theta_L, \sup_L \theta_L), \tag{1.22}$$

for all $w_p \in V(f)$, $p \notin \{j, \ell\}$, and for all paths $\gamma_1 \in \mathcal{H}_{jp}$ and $\gamma_2 \in \mathcal{H}_{p\ell}$.

When a special Lagrangian sphere exists in H, it is unique.

Proof. Sketch of Proof The proof of Theorem 1.3.6 is in Section 3.11. The idea is to interpret the inequalities in (ii) and (iii) in $Q(\mathbb{C})$ where $\int_\gamma |\Omega|$ is the length of $Q(\gamma)$ and $\tan\theta$ is the slope of the tangent line to $Q(\gamma)$. □

It is important to note that the value $\int_{L_{\gamma_1}} \Omega_f^n$ does not depend on the choice of $\gamma_1 \in \mathcal{H}_{jp}$ and $\int_{L_{\gamma_2}} \Omega_f^n$ does not depend on the choice of $\gamma_2 \in \mathcal{H}_{p\ell}$. Therefore, if $L_\gamma \in H$ is given, there are only finitely many values of the expression on the right hand side of the inequality to which the value of the left hand side should be compared.

Theorem 1.3.6 gives a necessary and sufficient condition for the existence of a special Lagrangian sphere over a path connecting any selected pair of zeros of f. Special Lagrangian spheres minimize the value of $\int_L |\Omega_f^n| \, dV_{g_L}$ over a homology class because of the following:

- $\int_L |\Omega_f^n| \, dV_{g_L} \geq \left| \int_L (\Omega_f^n)|_L \right|$ for any Lag L,
- $\int_L (\Omega_f^n)|_L$ is constant over a homology class, and
- $\int_L |\Omega_f^n| \, dV_{g_L} = \left| \int_L (\Omega_f^n)|_L \right|$ when L is special.

When a path giving rise to a special Lagrangian does not exist between w_j and w_ℓ, the minimizing path is in the closure of the set of all paths between w_j and w_ℓ, i.e., it is a path from w_j to w_ℓ which contains one or more of the other zeros of f.

This situation is illustrated by an example in Figure 1.4. In Figure 1.4, w_1, w_2, and w_3 are zeros of $f(w)$. A special Lagrangian sphere exists over a path connecting w_1 and w_2, and also over a path connecting w_2 and w_3, but not over a path connecting w_1 and w_3. Lagrangian spheres are illustrated in a way which shows that they are fibrations over paths in $\mathbb{C}$. Each fiber, represented by a vertical line segment, is a copy of $S^{n-1} \subset \mathbb{C}^n$.

In general, each segment bounded by two zeros and containing no other zero of f gives rise to a special Lagrangian sphere and the path gives rise to a chain of spheres, consecutive ones having one point in common. Such a chain, consisting of two spheres, is shown in Figure 1.4. It is a singular Lagrangian since the point shared by the two spheres is singular. A connect sum of the Lagrangians in the chain does belong to $H_{j\ell}$. This provides a decomposition of a Lagrangian in $H_{j\ell}$ as described by Thomas and Yau [3, Definition 5.3].

We reduce the almost Lagrangian mean curvature flow to the generalized mean curvature flow in Theorem 6.1.3.

In Theorem 6.2.16 we show the existence of the generalized mean curvature flow.

The flow of a closed curve by its curvature vector was studied by Gage–Hamilton [5], Grayson [6], [7], and others. Huisken [19] developed a different approach to the problem. Also,

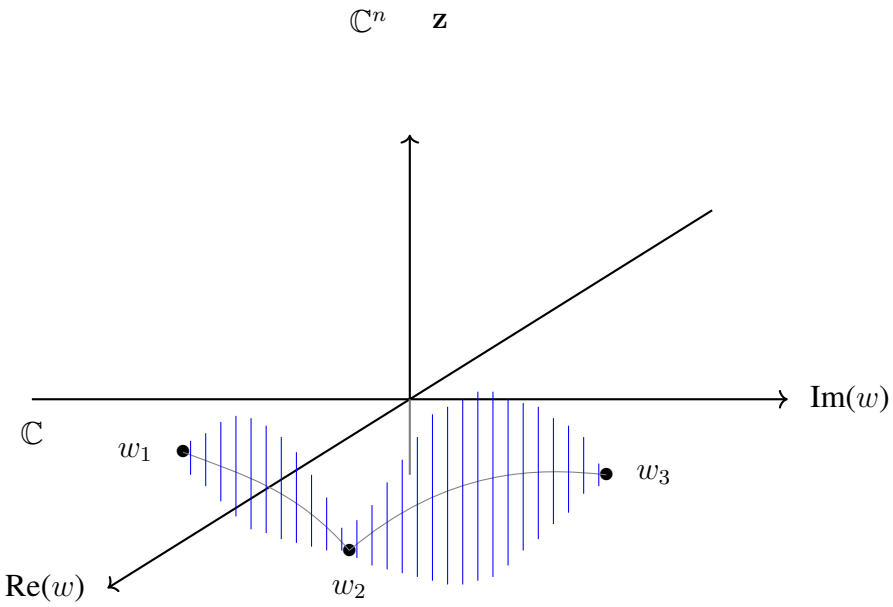


Figure 1.4: SL spheres (Path between w_1 and w_2 gives rise to an SL, so does path between w_2 and w_3. No such path exists between w_1 and w_3.)

Angenent [8], [9] studied a very general form of a curve shortening flow.

We use some of their methods to study the flow of paths which fixes endpoints and is in the direction of the curvature vector. Theorem 6.4.6 proves the Thomas–Yau conjecture for Milnor fibers.

1.4 Organization

Chapter 2 *Special Lagrangians in Milnor Fibers*

· Section 2.1 is background:

· In subsections 2.1.1 and 2.1.2 we provide background information on almost Calabi–Yau manifolds (Definition 2.1.2) and their Lagrangian submanifolds. We include the proof (proof of Lemma 2.1.7) of the facts that the existence of a nowhere vanishing holomorphic n-form Ω implies the existence of a nowhere vanishing global function $|\Omega|$, and that on a Lagrangian

submanifold, Ω induces a well-defined function, the phase of the Lagrangian (Definition 2.1.8).

· Subsection 2.1.3 is the background on Milnor fibers (Definition 2.1.13).

· In subsection 2.1.4 we describe the construction of Lagrangian submanifolds of Milnor fibers defined as S^{n-1} fibrations over paths in $\mathbb{C}$ which are central to this thesis. The construction yields Lagrangian submanifolds diffeomorphic to S^n (Definition 2.1.19). We demonstrate how the Lagrangian angle (phase) of such a submanifold is determined (Lemma 2.1.23).

· In Section 2.2 the existence of paths giving rise to special Lagrangian spheres is formulated in terms of a functional Φ (Definition 2.2.4) on the set of Lagrangian spheres. We show that finding special Lagrangians is equivalent to finding minimizers of the functional Φ.

· In Section 2.3 we consider the ODE (2.31) whose integral curves have the property that the Lagrangian spheres over them have constant phase. We prove the existence of solutions and we study the relationships between integral curves in subsection 2.3.2. The uniqueness of special Lagrangian sphere over a path between two fixed zeros of f follows.

Chapter 3 *The Underlining Riemann Surface, Pull-back Conical Metric*

· We introduce the primitive function $Q(w)$ for the function $q(w) = (f(w))^{\frac{n-2}{2}}$ (Definition 3.1). We construct an underlying Riemann surface associated with the Milnor fiber and use pull-back by the covering map Q to find curves which give rise to special Lagrangians.

In section 3.6 we introduce a singular metric on $\mathbb{C}$, pull-back by Q of the Euclidean metric, having a cone singularity of angle $n\pi$ at each zero of f (Theorem 3.6.4).

· In Section 3.7 we show that the curves γ such that the corresponding Lagrangians, L_γ are special, are geodesics in the pull-back singular metric. In subsection 3.10.1 the key result is

Theorem 3.10.2 which establishes a condition for existence of a special Lagrangian sphere in a homology class of Lagrangian spheres.

· In Section 3.11 we prove Theorem 1.3.6, which provides three necessary and sufficient conditions for the existence of a special Lagrangian sphere over a path between two fixed zeros of f.

Chapter 4 *Examples*

· One example presented is for n odd 4.1, and one for n even 4.2, but both use the same polynomial $f(w) = w^6 - 1$ for the definition of X_f^n. In both cases all possible (i.e.,$\frac{6(6-1)}{2} = 15$) paths giving rise to special Lagrangians exist. In general, for other choices of f, the number of special Lagrangian spheres is an integer between $\deg f - 1$ and $\frac{\deg f(\deg f-1)}{2}$ (inclusive).

· In example 4.3, $\deg f = 4$, $n = 4$, and the minimal number $(4 - 1 = 3)$ of paths giving rise to special Lagrangians exist.

· In section 4.4 we show that the limiting curve as $n \to \infty$ satisfies an equation. We illustrate with an example the limit curves.

Chapter 5 *Equivariant Lagrangians*

· We describe a family of $SO(n)$-invariant Lagrangians in $\mathbb{C}^n$ from Harvey–Lawson in [4] and we demonstrate a close relationship between them and Lagrangian spheres in Milnor fibers.

·

Chapter 6 *Almost Lagrangian Mean Curvature Flow*

· We shift the focus to the almost Lagrangian mean curvature flow in Milnor fibers. Theorems 6.1.3 and 6.2.16 are the main theorems in the chapter. Proofs of statements related to the generalized mean curvature flow are inspired by proofs of Gage–Hamilton [5], Grayson [6], [7], and Huisken [19] and follow their exposition, which is for the mean curvature flow, closely.

Chapter 7 *Thomas–Yau Conjecture*

· In this Chapter the setting is an almost Calabi–Yau manifold (not necessarily a Milnor fiber). We use a new approach to obtain partial results related to the Thomas–Yau conjecture. This chapter represents work in progress.

Chapter 2: Special Lagrangians in Milnor Fibers

2.1 Background

(X, J, ω, Ω) is an *almost Calabi–Yau manifold* if (X, J, ω) is a Kähler manifold of complex dimension n, with Kähler form ω and complex structure J, and Ω is a nowhere vanishing holomorphic $(n, 0)$-form on X.

Since Ω is a holomorphic $(n, 0)$-form, it follows (Lemma 2.1.3) that there exists a simple relationship between the forms

$$(-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \Omega \wedge \overline{\Omega}$$

and

$$\frac{\omega^n}{n!},$$

where $\frac{\omega^n}{n!}$ is the volume form on X coming from the Riemannian metric associated with the Kähler form ω, $g(\cdot, \cdot) := \omega(\cdot, J\cdot)$. That relationship is given by (Lemma 2.1.6):

$$(-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \Omega \wedge \overline{\Omega} = e^{2\nu} \frac{\omega^n}{n!}, \tag{2.1}$$

where $e^{2\nu}$ is some real, positive function on X.

If $e^{2\nu} \equiv 1$ we say that Ω is compatible with the metric and the manifold (X, J, ω, Ω) is

Calabi–Yau.

A real orientable submanifold $L \subset X$ of dimension $\dim_{\mathbb{R}} L = \dim_{\mathbb{C}} X = n$, with the property $\omega|_{TL} \equiv 0$ is called a *Lagrangian submanifold*. As L is a real submanifold of X, the restriction of the Riemannian metric $g(\cdot,\cdot) := \omega(\cdot, J\cdot)$ to L, g_L, is a Riemannian metric on L.

Let dV_{g_L} denote the Riemannian volume form induced by the Riemannian metric g_L. Assuming that L is orientable, the restriction to L of the nowhere vanishing $(n,0)$-form Ω is closely related to the volume form on L, as is seen in Lemma 2.1.6:

$$\Omega|_L = e^{\nu}\, e^{i\theta}\, dV_{g_L}, \tag{2.2}$$

where $\nu : X \to \mathbb{R}$, and $e^{i\theta} : L \to \{z \in \mathbb{C} \,:\, |z| = 1\} = S^1$.

$e^{i\theta}$ is called the *Lagrangian phase* (Definition 2.1.8).

If $e^{i\theta}$ lifts to a real-valued function θ then (L,θ) is a *graded Lagrangian*. In that case, $\theta : L \to \mathbb{R}$ is referred to as a *grading* or as a *Lagrangian angle* (Definition 2.1.10).

A priori it may not be possible to lift $E^{i\theta}$ to a real-valued θ. In this thesis we will be working only with graded Lagrangians. Note that any choice of a real-valued function θ satisfying (2.2) is called a grading, i.e., a graded Lagrangian has infinitely many possible gradings.

The Lagrangian angle was introduced in the seminal work of Harvey–Lawson [4, p. 96, Equation (2.18)]. *Special Lagrangian* submanifolds, introduced by Harvey–Lawson [4, pp. 49, 89], are those Lagrangian submanifolds for which the Lagrangian phase is constant.

The setting in the major part of the thesis is a *Milnor fiber* X_f^n. The two defining characteristics of X_f^n are:

- $n \in \mathbb{N}$ is the dimension of X_f^n, and

- f is a polynomial in the class $\mathcal{P}_{simple}$:

$$\mathcal{P}_{simple} := \{f \in \mathbb{C}[w] \,:\, f(w_0) = 0 \implies f'(w_0) \neq 0\}. \tag{2.3}$$

Given $n \in \mathbb{N}$ and $f \in \mathcal{P}_{simple}$, Milnor fiber $X_f^n \subset \mathbb{C}^{n+1}$ is given by:

$$X_f^n := \Big\{(w, z_1, \ldots, z_n) \in \mathbb{C}^n : z_1^2 + \cdots + z_n^2 = f(w)\Big\}. \tag{2.4}$$

X_f^n has the structure of an *almost Calabi–Yau manifold* (Proposition 2.1.17):

- It is a complex hypersurface in $\mathbb{C}^{n+1}$, hence Kähler, and
- A nowhere vanishing holomorphic $(n, 0)$-form, Ω_f^n, exists on X_f^n. It is given by (equation (2.20), Lemma 2.1.16):

$$\Omega_f^n = \frac{dz_1 \wedge \cdots \wedge dz_n}{f'(w)} = \frac{dz_1 \wedge \cdots \wedge dz_{i-1} \wedge dw \wedge dz_{i+1} \wedge \cdots \wedge dz_n}{2z_i}. \quad 1 \leq i \leq n. \tag{2.5}$$

(The two expressions agree wherever they are both defined. The choice of f as a polynomial in $\mathcal{P}_{simple}$ ensures that Ω_f^n is defined for all $p \in X_f^n$: If at some $p = (w^0, z_1^0, \ldots, z_n^0) \in X$, $f'(w^0) = 0$ then $f(w^0) \neq 0$, and so $\exists i$ such that $z_i^0 \neq 0$.)

There exist compact Lagrangian submanifolds of X_f^n diffeomorphic to a sphere S^n which we can describe explicitly:

Given any simple C^1 path γ connecting two distinct zeros of f in $\mathbb{C}$, such a Lagrangian is

an S^{n-1}-fibration over the given path :

$$L_\gamma := \left\{(w, r_1\sqrt{f(w)}, \ldots, r_n\sqrt{f(w)}) : w \in \mathrm{Im}\gamma,\ (r_1, \ldots, r_n) \in S^{n-1}(1) \subset \mathbb{R}^n\right\}. \tag{2.6}$$

(It is shown in Lemma 2.1.20 that L_γ is a Lagrangian submanifold of X_f^n.)

Due to the $O(n)$-invariance (Lemma 2.1.21 and [2, Subsection 5.4.4]) of these Lagrangian spheres, statements about thus obtained Lagrangian submanifolds are equivalent to statements about paths over which they are defined.

Consequently, lemmas and propositions, as well as their proofs will generally be about curves in the complex plane.

The S^1-valued Lagrangian phase is defined on these Lagrangian spheres (Definition 2.1.8) and is constant on the fiber over any point of the path γ. Let

$$\pi : L_\gamma \to \gamma,$$

$$\pi : (\gamma(u), r_1\sqrt{f(\gamma(u))}, \ldots, r_n\sqrt{f(\gamma(u))}) \mapsto \gamma(u).$$

be the projection map sending all the points in the fiber over $\gamma(u)$ to $\gamma(u)$. If the path γ is given by $\gamma : [0, 1] \to \mathbb{C}$ then

$$\theta \circ \pi^{-1} \circ \gamma : [0, 1] \to \mathbb{R}$$

is a well defined function.

The Lagrangian angle can be explicitly determined as a function of the variable u parametrizing the path γ. Thomas–Yau [3, Equation (6.4)] show that for a path γ parametrized by $u \in \mathbb{R}$,

the Lagrangian angle of the corresponding Lagrangian submanifold is (Lemma 2.1.23):

$$\big(\theta \circ \pi^{-1} \circ \gamma\big)(u) = \arg\left[\gamma'(u) f(\gamma(u))^{\frac{n-2}{2}}\right].$$

We will abuse the notation slightly and write θ for $\theta \circ \pi^{-1} \circ \gamma$ from now on. So,

$$\theta(u) = \arg\left[\gamma'(u) f(\gamma(u))^{\frac{n-2}{2}}\right]. \tag{2.7}$$

Remark 2.1.1. Lagrangians which are diffeomorphic to S^n are orientable. In addition, we will be assuming that the Lagrangian spheres are *positive*. Positive means that the range of the Lagrangian angle θ is contained in $(-\frac{\pi}{2}, \frac{\pi}{2})$, so that $\mathrm{Re}\Omega|_L$ is a volume form on L (Solomon [20]). The term *positive* is due to Solomon [20, p. 671]. Alternatively we could assume that Lagrangian spheres are *almost calibrated*. Almost calibrated means that the range of the Lagrangian angle θ is contained in $(\phi - \frac{\pi}{2}, \phi + \frac{\pi}{2})$, for some ϕ, i.e., the total variation of θ is less than π. This terminology is somewhat misleading, as almost calibrated Lagrangians are not close to being calibrated. There is no practical difference in assuming almost calibration instead of positivity. The two notions are essentially the same as replacing the form Ω by $e^{-i\phi}\Omega$, has the effect of changing a Lagrangian from almost calibrated by Ω to a positive Lagrangian with respect to $e^{-i\phi}\Omega$.

2.1.1 Almost Calabi–Yau manifolds

In Lemma 2.1.7 we look closely at the restriction of the holomorphic $(n, 0)$-form Ω to a Lagrangian submanifold of the almost Calabi–Yau manifold. The existence of a well-defined phase of a Lagrangian submanifold (Definition 2.1.8) follows from the proof of the theorem as

does the existence of a well-defined nowhere vanishing function $|\Omega| : X \to \mathbb{R}$. Recall the definition:

Definition 2.1.2. *An almost Calabi–Yau manifold, (X, J, ω, Ω) is a Kähler manifold X of dimension n with Kähler form ω and complex structure J, equipped with a nowhere vanishing holomorphic volume form Ω.*

If, in addition, the form Ω is compatible with the metric, i.e.

$$\frac{\omega^n}{n!} = \big(-1\big)^{\frac{n(n-1)}{2}} \Big(\frac{i}{2}\Big)^n \Omega \wedge \overline{\Omega}, \tag{2.8}$$

then (X, J, ω, Ω) is a Calabi–Yau manifold.

Lemma 2.1.3. *Let (X, J, ω, Ω) be an almost Calabi–Yau manifold. Then there exists a function $\nu : X \to \mathbb{R}$ such that*

$$e^{2\nu} \frac{\omega^n}{n!} = \big(-1\big)^{\frac{n(n-1)}{2}} \Big(\frac{i}{2}\Big)^n \Omega \wedge \overline{\Omega}. \tag{2.9}$$

Remark 2.1.4. Equality (2.9) replaces the compatibility with the metric equality holding in Calabi–Yau manifolds (2.8).

Proof. Let (X, J, ω, Ω) be an almost Calabi–Yau manifold. Since Ω is an $(n, 0)$-form it follows that using any local holomorphic coordinates, $z_1, \ldots, z_n$, in a neighborhood of a point $p \in X$, the form Ω can be expressed as

$$\Omega = \phi \, dz_1 \wedge \cdots \wedge dz_n, \tag{2.10}$$

where ϕ is a holomorphic function, $\phi \neq 0$. Furthermore, in the same neighborhood,

$$\begin{aligned}\Omega \wedge \overline{\Omega} &= |\phi|^2 \, dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n \\ &= |\phi|^2 \, (dx_1 + i dy_1) \wedge \cdots \wedge (dx_n + i dy_n) \wedge (dx_1 - i dy_1) \wedge \cdots \wedge (dx_n - i dy_n) \\ &= |\phi|^2 (-1)^{\frac{n(n+1)}{2}} (2i)^n \, dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n.\end{aligned}$$

Thus $\Omega \wedge \overline{\Omega}$ is a nowhere vanishing top level form on X and

$$(-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \Omega \wedge \overline{\Omega} = |\phi|^2 \, dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n$$

is $\mathbb{R}_{>0}$-valued, thus a volume form. Then there exists a simple relationship between $(-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \Omega \wedge \overline{\Omega}$ and $\frac{\omega^n}{n!}$, the volume form on X for the Riemannian metric associated with the Káhler form ω, $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$:

$$(-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \Omega \wedge \overline{\Omega} = e^{2\nu} \frac{\omega^n}{n!},$$

where $\nu : X \to \mathbb{R}$.

If $e^{2\nu} \equiv 1$ then Ω is compatible with the metric and the manifold (X, J, ω, Ω) is Calabi–Yau. □

Definition 2.1.5. *The modulus of the holomorphic $(n, 0)$-form Ω is $|\Omega| : X \to \mathbb{R}_{>0}$ given by:*

$$|\Omega|(p) := e^{\nu(p)}, \quad p \in X, \tag{2.11}$$

with ν as in Lemma 2.1.3.

2.1.2 Lagrangian submanifolds

In this subsection we show some of the properties of Lagrangian submanifolds which follow from the fact that there exists a holomorphic nowhere vanishing $(n,0)$-form Ω on our manifold (Lemma 2.1.7). Lemma 2.1.7 is an extension of a result of Harvey–Lawson [4, Theorem 1.10] to almost Calabi–Yau manifolds. Their result is for $\mathbb{C}^n$ but holds in the case of X being a Calabi–Yau manifold [4, Section V.3.].

However, in our setting of almost Calabi–Yau manifolds, the form Ω is not necessarily compatible with the metric in the sense of equation (2.8). We show that the equation (2.9) holds instead. A statement similar to the equation (2.9) is in Dellatorre [2, p. 134] where it is stated without proof.

Any Kähler manifold is a symplectic manifold. In a symplectic manifold of dimension $2n$, a Lagrangian submanifold is an isotropic submanifold of maximal dimension, i.e., a submanifold L of dimension n such that the symplectic form vanishes when restricted to TL. Lagrangian submanifolds considered in this thesis will always be assumed to be orientable.

If (X,ω,J,Ω) is almost Calabi–Yau with Riemannian metric $g=\omega(\cdot,J\cdot)$, and L,

$$\iota : L \to X,$$

is an embedded orientable Lagrangian submanifold, we will simplify the notation by identifying L with its image in X.

Lemma 2.1.6. *Let L be an orientable Lagrangian submanifold of an almost Calabi–Yau manifold X. Then there exists an S^1-valued function on L, $e^{i\theta}$ such that*

$$\Omega|_L = |\Omega|\, e^{i\theta}\, dV_{g_L}, \tag{2.12}$$

where $|\Omega|$ is as in Definition 2.1.5, and dV_{g_L} is the volume form on L induced by the restriction of the Riemannian metric on X to L.

Proof. As L is a real submanifold of X, the restriction of the Riemannian metric $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$ to L, g_L, is a Riemannian metric on L. The volume form associated with the metric g_L is dV_{g_L}.

Now, as Ω is an n-form and $\dim L = n$, $\Omega|_L$ is a top level form on L. We will show that it is nowhere vanishing.

Suppose that $e_1, \ldots, e_n \in T_pL$ is an orthonormal basis of T_pL (with respect to the metric g_L). Then, as L is Lagrangian,

$$e_1, \ldots, e_n, Je_1, \ldots Je_n$$

is an orthonormal basis of T_pX as a real $2n$-dimensional vector space and

$$e_1, \ldots, e_n$$

is a basis of the tangent space T_pX, viewed as an n-dimensional complex vector space. Now, as Ω is nowhere vanishing on X, there exist $v_1, \ldots v_n \in T_pX$ such that

$$\Omega|_p(v_1, \ldots, v_n) \neq 0.$$

Since $e_1, \ldots, e_n$ is a basis of T_pX, there exists a complex linear transformation $A : (e_1, \ldots, e_n) \mapsto$

$(v_1, \ldots, v_n)$. Evaluating Ω on $v_1, \ldots v_n$, we get

$$0 \neq \Omega(v_1, \ldots, v_n) = \Omega(A(e_1, \ldots, e_n)) = \det(A)\, \Omega(e_1, \ldots, e_n),$$

as Ω is an alternating, multilinear map. Consequently, $\Omega(e_1, \ldots e_n) \neq 0$. Now, the point $p \in L$ was arbitrary and therefore $\Omega|_L$ is a nowhere vanishing n-form on L. As the space of top level forms is one-dimensional, there exists a function $\phi : L \to \mathbb{C}$ such that

$$\Omega|_L = \phi dV_{g_L} = |\phi| \frac{\phi}{|\phi|} dV_{g_L}. \tag{2.13}$$

□

Lemma 2.1.7. *Suppose that (X, J, ω, Ω) is almost Calabi–Yau and L is an orientable Lagrangian submanifold of X. Then there exists a well-defined function*

$$|\Omega| : X \to \mathbb{R}_{>0} \tag{2.14}$$

and there exists a well-defined function

$$\alpha_L : L \to S^1 = \{z \in \mathbb{C} \,:\, |z| = 1\} \tag{2.15}$$

such that

$$\Omega_L = |\Omega| \alpha_L dV_{g_L}.$$

Proof. Let p be a point in $L \subset X$, and let $z_1, \ldots, z_n$ be local coordinates centered at p which give

rise to orthonormal dual bases for the tangent space T_pX and the cotangent space T_p^*X at p. In these local coordinates, in a neighborhood U of p, the nowhere vanishing $(n,0)$-form Ω can be written as

$$\Omega|_U = F(z_1, \ldots, z_n) dz_1 \wedge \cdots \wedge dz_n,$$

where $F \neq 0$ is some holomorphic function. Let

$$e_1, \ldots, e_n$$

be the orthonormal basis of the tangent space T_pX dual to

$$dz_1|_p, \ldots, dz_n|_p.$$

Let

$$r_1, \ldots, r_n$$

be an orthonormal basis of T_pL. T_pL is a Lagrangian plane in T_pX viewed as a real $2n$-dimensional vector space. L being a Lagrangian submanifold in a Kähler manifold implies that $r_1, \ldots, r_n, Jr_1, \ldots, Jr_n$ is an orthonormal basis of T_pX over $\mathbb{R}$. $e_1, \ldots, e_n, Je_1, \ldots, Je_n$ is also an orthonormal basis for T_pX over $\mathbb{R}$, and $r_1, \ldots, r_n$ is an orthonormal basis for T_pX over $\mathbb{C}$.

There exists a unitary matrix A such that

$$A : (e_1, \ldots, e_n) \mapsto (r_1, \ldots, r_n), \tag{2.16}$$

i.e.

$$A(e_j) = r_j.$$

Now evaluate Ω on $r_1, \ldots, r_n$:

$$\begin{aligned}
\Omega|_p(r_1, \ldots, r_n) &= F(p) dz_1 \wedge \cdots \wedge dz_n(Ae_1, \ldots, Ae_n) \\
&= F(p) \det(A) dz_1 \wedge \cdots \wedge dz_n(e_1, \ldots, e_n) \\
&= F(p) \det(A) dV_L(r_1, \ldots, r_n).
\end{aligned}$$

Define $\phi : L \to \mathbb{C}$ as

$$\phi(p) := F(p) \det(A).$$

We saw that $|\phi|$ is not affected if another orthonormal base of X is picked. So we can define

$$|\Omega(p)| := |\phi(p)|. \tag{2.17}$$

We need to show that ϕ is well defined on L. First we will show that it is independent of the choice of an orthonormal basis of T_pL, and second, we will show that it is independent of the choice of coordinates at p.

Let

$$r'_1, \ldots, r'_n$$

be another orthonormal basis of T_pL having the same orientation as $r_1, \ldots, r_n$.

Then there exists an orthogonal matrix B with $\det(B) = 1$ such that

$$r'_j = B(r_j).$$

It follows that

$$\begin{aligned}
\Omega|_p(r'_1, \ldots, r'_n) &= F(p) dz_1 \wedge \cdots \wedge dz_n(BAe_1, \ldots, BAe_n) \\
&= F(p) \det(BA) \\
&= F(p) \det(B) \det(A) dV_L(r'_1, \ldots, r'_n) \\
&= F(p) \det(A) dV_L(r'_1, \ldots, r'_n).
\end{aligned}$$

$\det(A)$ is a complex number of modulus 1. Therefore, the function ϕ satisfies:

$$|\phi| = |F| \qquad \text{and} \qquad \arg(\phi) = \arg(F) + \arg(\det(A)) \mod 2\pi.$$

We have shown that ϕ is independent of the choice of a basis of T_pL. Let

$$\alpha := e^{i \arg(\phi)}$$

We can write

$$\Omega_L(p) = |\Omega(p)| \alpha(p) (dV_{g_L})_p.$$

It remains to check that the function ϕ is independent of the choice of a coordinate chart

around p in X.

Suppose that $w_1, \ldots, w_n$ are some other coordinates around p. Let

$$P := \left(\frac{\partial w_j}{\partial z_l} \right)_{j\ell}$$

be the Jacobian matrix for the change of coordinates between $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$.

In w coordinates the expression for Ω at p is

$$\Omega_p = G(p) dw_1 \wedge \cdots \wedge dw_n,$$

where $G \neq 0$ is some holomorphic function.

$$\begin{aligned}
\Omega|_p(r_1, \ldots, r_n) &= G(p) dw_1 \wedge \cdots \wedge dw_n(r_1, \ldots, r_n) \\
&= G(p) \det(P) dz_1 \wedge \cdots \wedge dz_n(r_1, \ldots, r_n) \\
&= G(p) \det(P) \det(A) \\
&= G(p) \det(AP) \\
&= G(p) \det(AP) dV_L(r_1, \ldots, r_n).
\end{aligned}$$

On the other hand,

$$\Omega|_p(r_1, \ldots, r_n) = F(p) \det(A),$$

so it follows that

$$G(p) \det(P) = F(p).$$

We have

$$G(p)\det(A') = F(p)\det(A),$$

where $A' = AP$ is the change of basis matrix mapping the basis of T_pX dual to $dw_1, \dots dw_n$. It follows that

$$\phi = G(p)\det(A') = F(p)\det(A)$$

is a well-defined function on L.

□

Definition 2.1.8. *Let ϕ be as in Lemma 2.1.7. The function $\alpha : L \to S^1$,*

$$\alpha(p) := \mathrm{e}^{i \arg \phi(p)} \tag{2.18}$$

is called the phase of the Lagrangian submanifold L.

Definition 2.1.9. *A Lagrangian submanifold $L \subset X$ is called a special Lagrangian submanifold if the phase function is constant on L.*

Definition 2.1.10. *If there exists a continuous function $\theta : L \to \mathbb{R}$, $p \mapsto \theta(p)$, such that*

$$\alpha(p) = \mathrm{e}^{i\theta(p)}, \quad p \in L,$$

then the Lagrangian L is called a graded Lagrangian. The real-valued function θ is called the Lagrangian angle or grading of L.

Remark 2.1.11. Lagrangian angle / grading is not unique. Clearly, two gradings differ by a constant.

Remark 2.1.12 (Notation)**.** We will abbreviate 'Lagrangian (submanifold)' by Lag and 'special Lagrangian (submanifold)' by SL.

2.1.3 Milnor fibers

For the exposition in this subsection we follow Dellatorre [2] and Thomas–Yau [3]. We define Milnor fibers X_f^n (Definition 2.1.13), as in Dellatorre [2, Chapter 5] and Thomas–Yau [3, Section 6]. Then, in Lemma 2.1.16, we demonstrate that a Milnor fiber has the structure of an almost Calabi–Yau manifold by producing a holomorphic $(n,0)$-form Ω_f^n (Definition 2.1.15), and showing that it does not vanish anywhere on X_f^n.

Definition 2.1.13. *Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial in $\mathcal{P}_{simple}$ (2.3). Milnor fiber of dimension n defined by the polynomial f is the set $X_f^n \subset \mathbb{C}^{n+1}$ given by:*

$$X_f^n := \{(w, z_1, \ldots, z_n) : z_1^2 + \cdots + z_n^2 = f(w)\}. \tag{2.19}$$

Proposition 2.1.14. *X_f^n is a complex submanifold of $\mathbb{C}^{n+1}$ of dimension n.*

Proof. Let

$$F(w, z_1, \ldots, z_n) := z_1^2 + \cdots + z_n^2 - f(w) : \mathbb{C}^{n+1} \to \mathbb{C}.$$

Then $X_f^n = F^{-1}(0)$. The differential map is given by the matrix

$$\begin{bmatrix} -f'(w) & 2z_1 & \ldots & 2z_n \end{bmatrix}.$$

The rank of the matrix representing the differential map is never 0 and therefore always maximal because of the assumption that any zero of f has order one. By the holomorphic implicit function

theorem [21, p. 19], X_f^n is a complex submanifold of $\mathbb{C}^{n+1}$ of complex dimension n. □

It remains to show that there exists a nowhere vanishing holomorphic $(n,0)$-form on X_f^n. First we will give an ad hoc definition of an $(n,0)$-form. Immediately after we will proceed to show that the chosen form indeed has the needed properties (and is well-defined!).

Definition 2.1.15. *Let Ω_f^n be the holomorphic $(n,0)$-form on X_f^n given by*

$$\Omega_f^n := \frac{dz_1 \wedge \cdots \wedge dz_n}{f'(w)} = \frac{dz_1 \wedge \cdots \wedge dz_{j-1} \wedge dw \wedge dz_{j+1} \wedge \cdots \wedge dz_n}{2z_j}, \tag{2.20}$$

for $j = 1, \ldots n$.

Lemma 2.1.16. *Assume that $f \in \mathcal{P}_{simple}$ (2.3), $n \in \mathbb{N}$, and that X_f^n is a Milnor fiber (Definition 2.1.13). Then Ω_f^n, given in Definition 2.1.15, is*

(i) well-defined.

(ii) a nowhere vanishing holomorphic $(n,0)$-form on X_f^n.

Proof. (i) As $f \in \mathcal{P}_{simple}$, $V(f)$ and $V(f')$ are finite sets such that

$$V(f) \cap V(f') = \varnothing.$$

The first expression for Ω_f^n (2.20),

$$\frac{dz_1 \wedge \cdots \wedge dz_n}{f'(w)},$$

is defined on

$$X_f^n \setminus \{(w, z_1, \ldots, z_n) \,:\, w \in V(f')\}.$$

If at some point $(w^0, z_1^0, \dots z_n^0) \in X_f^n$, $f'(w^0) = 0$, then $f(w^0) \neq 0$ since otherwise w_0 would be a zero of order higher than one. Since

$$0 \neq f(w^0) = (z_1^0)^2 + \cdots + (z_n^0)^2,$$

at least one $z_j^0 \neq 0$. Then there is an open set $U \subset X_f^n$, $(w^0, z_1^0, \dots z_n^0) \in U$, such that $z_j \neq 0$ for all $(w, z_1, \dots, z_n) \in U$. On U, we use the second expression for Ω_f^n. The expressions are interchangeable on any set where $z_j f'(w) \neq 0$, which can be seen as follows:

$$z_1^2 + \cdots + z_n^2 = f(w),$$

$$2z_1 dz_1 + \cdots + 2z_n dz_n = f'(w)dw.$$

Wedge the equation with $dz_1 \wedge \cdots \wedge dz_{j-1}$ on the left and with $dz_{j+1} \wedge \cdots \wedge dz_n$ on the right to obtain:

$$2z_j dz_1 \wedge \cdots \wedge dz_n = f'(w)dz_1 \wedge \cdots \wedge dz_{j-1} \wedge dw \wedge dz_{j+1} \wedge \cdots \wedge dz_n,$$

and therefore,

$$\frac{dz_1 \wedge \cdots \wedge dz_n}{f'(w)} = \frac{dz_1 \wedge \cdots \wedge dz_{j-1} \wedge dw \wedge dz_{j+1} \wedge \cdots \wedge dz_n}{2z_j}, \tag{2.21}$$

at any point $(w, z_1, \dots, z_n) \in X_f^n$ where $z_j f'(w) \neq 0$.

We also need to see that at any point $(w, z_1, \dots, z_n) \in X_f^n$ where $z_j z_\ell \neq 0$,

$$\frac{dz_1 \wedge \cdots \wedge dz_{j-1} \wedge dw \wedge dz_{j+1} \wedge \cdots \wedge dz_n}{2z_j}$$

and

$$\frac{dz_1 \wedge \cdots \wedge dz_{\ell-1} \wedge dw \wedge dz_{\ell+1} \wedge \cdots \wedge dz_n}{2z_\ell}$$

are equal.

By (2.21), at those points where $z_j z_\ell f'(w) \neq 0$ both expressions equal to

$$\frac{dz_1 \wedge \cdots \wedge dz_n}{f'(w)},$$

and therefore they are equal to each other.

Suppose therefore that $\tilde{w} \in V(f')$ and suppose that the point p is

$$(\tilde{w}, \tilde{z_1}, \ldots \tilde{z_n}), \quad \text{with} \quad \tilde{z_j}\tilde{z_\ell} \neq 0.$$

Now, let $H \subset X_f^n$ be the hypersurface defined by the equation:

$$w = \tilde{w}.$$

Then $p \in H$. Furthermore, as $V(f')$ is discrete, there exists a sequence of points

$$p_\nu := (w_\nu, (z_1)_\nu, \ldots, (z_n)_\nu),$$

such that $f'(w_\nu) \neq 0$ and $(z_i)_\nu (z_j)_\nu \neq 0$, and such that

$$\lim_{\nu \to \infty} p_\nu = p,$$

and $p_\nu \in X_f^n \setminus H$, for all ν. Since $f'(w_\nu) \neq 0$ we have the equality of the two forms on the sequence $\{p_\nu\}$. As both forms are smooth sections away from the hypersurfaces $z_j = 0$ and $z_\ell = 0$ they agree on the limit point p as well.

(ii) From its defining expressions (2.20) it is immediately evident that Ω_f^n is a holomorphic $(n,0)$-form and that it does not vanish anywhere. □

Proposition 2.1.17. $(X_f^n, J, \omega, \Omega_f^n)$ *is an almost Calabi–Yau manifold.*

Proof. X inherits the Kähler structure from $\mathbb{C}^{n+1}$. A nowhere vanishing holomorphic $(n,0)$-form is Ω_f^n from Definition 2.1.15 which is well-defined as shown in Lemma 2.1.16. □

Lemma 2.1.18. *There is an action of the orthogonal group* $O(n)$ *on* X_f^n.

Proof. Let A be an $n \times n$ matrix in $O(n)$. The action is defined by

$$(w, \mathbf{z}) \mapsto (w, \mathbf{z}A),$$

where $\mathbf{z} = (z_1, \ldots, z_n)$. By definition of X_f^n,

$$\mathbf{z}\mathbf{z}^T = f(w).$$

Then

$$(\mathbf{z}A)(\mathbf{z}A)^T = \mathbf{z}AA^T(\mathbf{z})^T = \mathbf{z}\mathbf{z}^T = f(w).$$

Thus $(w, \mathbf{z}A) \in X_f^n$. □

2.1.4 Lagrangian spheres in Milnor fibers

Compact Lag submanifolds of X_f^n diffeomorphic to S^n (Definition 2.1.19) exist due to the existence of a discrete set of simple zeros of the function f.

The idea is, roughly, that such submanifolds are defined using paths between zeros of $f(w)$ in the copy of $\mathbb{C}$ corresponding to the variable w, with an S^{n-1} fiber above any point on the path. The S^{n-1} fiber is defined by restricting the equation $z_1^2 + \cdots + z_n^2 = f(w)$ to a real subspace of $\mathbb{C}^n$. The size of the S^{n-1} fiber decreases when an endpoint is approached and at the endpoint the fiber degenerates into a single point.

The presentation of the material follows the exposition in Dellatorre [2, Section 5.4.4]. The proof of Lemma 2.1.20 is new. In it we show that the Lag spheres in Milnor Fibers are Lag submanifolds diffeomorphic to spheres, and we explicitly calculate the phase (Definition 2.1.8).

Let $\gamma : [0,1] \to \mathbb{C}$ be a simple C^1 path, $\gamma \in \mathcal{A}_{j\ell}$. (We will abuse notation slightly and use γ both for the function and for the image, $\gamma[0,1]$.)

Definition 2.1.19. *$L_\gamma \subset X_f^n$ is given by:*

$$L_\gamma := \left\{(w, r_1\sqrt{f(w)}, \ldots, r_n\sqrt{f(w)}) : w \in \gamma,\ (r_1, \ldots, r_n) \in S^{n-1}(1) \subset \mathbb{R}^n\right\}. \tag{2.22}$$

Lemma 2.1.20. *L_γ is a Lag submanifold of X_f^n.*

Proof. To show that L_γ is Lag we need to check that the Kähler form ω vanishes on L_γ. In order to do that we find a basis for the tangent space of L_γ.

Let $w \in \gamma$ be fixed. The cross section of L_γ at w is a copy of S^{n-1}.

Fix $(r_1, \ldots, r_n) \in S^{n-1}$. Let $\sqrt{f(w)}$ denote a continuous choice of the square root along γ. The following vectors form a basis of the tangent space of L_γ at the point $\left(w, r_1\sqrt{f(w)}, \ldots, r_n\sqrt{f(w)}\right)$:

- a basis of the tangent space of the fiber S^{n-1} at $\left(w, r_1\sqrt{f(w)}, \ldots, r_n\sqrt{f(w)}\right)$:

$$z_2\partial_{z_1} - z_1\partial_{z_2}, z_3\partial_{z_2} - z_2\partial_{z_3}, \ldots, z_n\partial_{z_{n-1}} - z_{n-1}\partial_{z_n} \tag{2.23}$$

  and

- vector

$$\gamma'\left(\partial_w + \frac{f'(\gamma)}{2f(\gamma)}\sum z_i\partial_{z_i}\right) \tag{2.24}$$

  tangent to the curve $\beta_r : [0,1] \to L_\gamma$ given by:

$$\beta_r : u \mapsto \left(\gamma(u), r_1\sqrt{f(\gamma(u))}, \ldots, r_n\sqrt{f(\gamma(u))}\right), \tag{2.25}$$

  for $r = (r_1, \ldots, r_n) \in S^{n-1}$ fixed.

Now, $\omega(V_1, V_2) = 0$ for any 2 vectors in the tangent space of the cross sectional copy of S^{n-1} because it is contained in $\sqrt{f(\gamma(u))}\mathbb{R}^n$ which is isotropic in $\mathbb{C}^{n+1}$, and also

$$\omega\left(\gamma'\left(\partial_w + \frac{f'(\gamma(u))}{2f(\gamma(u))}\sum z_i\partial_{z_i}\right), z_j\partial_{z_i} - z_i\partial_{z_j}\right) = 0.$$

It remains to see that the symplectic form ω vanishes when restricted to the tangent space to L_γ at an endpoint of γ.

First we find the limit of the unit tangent vector to the curve β_r as w approaches an endpoint

of the path γ (where $f(w)=0$). The unit tangent vector to β is:

$$\frac{\gamma'\left(\partial_w+\frac{f'(\gamma(u))}{2f(\gamma(u))}\sum z_i\partial_{z_i}\right)}{|\gamma'\left(\partial_w+\frac{f'(\gamma(u))}{2f(\gamma(u))}\sum z_i\partial_{z_i}\right)|}. \tag{2.26}$$

To find the limit, we modify the expression ()2.26, using $z_j=r_j\sqrt{f}$, $j=1,\dots,n$:

$$\frac{\gamma'\left(\partial_w+\frac{f'(\gamma(u))}{2f(\gamma(u))}\sum z_i\partial_{z_i}\right)}{|\gamma'\left(\partial_w+\frac{f'(\gamma(u))}{2f(\gamma(u))}\sum z_i\partial_{z_i}\right)|}=\frac{\gamma'}{|\gamma'|}\frac{1}{\sqrt{1+\frac{|f'|^2}{4|f|}}}\left(1,\frac{f'}{2\sqrt{f}}r_1,\dots,\frac{f'}{2\sqrt{f}}r_n\right)$$

$$=\frac{\gamma'}{|\gamma'|}\left(\frac{1}{\sqrt{1+\frac{|f'|^2}{4|f|}}},\frac{f'}{\sqrt{4|f|+|f'|^2}}\frac{\sqrt{|f|}}{\sqrt{f}}r_1,\dots,\frac{f'}{\sqrt{4|f|+|f'|^2}}\frac{\sqrt{|f|}}{\sqrt{f}}r_n\right)$$
$$=\frac{\gamma'}{|\gamma'|}\left(\frac{2\sqrt{|f|}}{\sqrt{4|f|+|f'|^2}},\frac{f'}{\sqrt{4|f|+|f'|^2}}\frac{\sqrt{\bar{f}}}{\sqrt{|f|}}r_1,\dots,\frac{f'}{\sqrt{4|f|+|f'|^2}}\frac{\sqrt{\bar{f}}}{\sqrt{|f|}}r_n\right).$$

As $\lim\frac{\gamma'}{|\gamma'|}=C_1$ and $\lim\frac{f'}{\sqrt{4|f|+|f'|^2}}\frac{\sqrt{\bar{f}}}{\sqrt{|f|}}=\frac{f'}{|f'|}\sqrt{\lim\frac{\bar{f}}{|f|}}=C_2$ exist and are complex numbers of modulus one, the limit of the unit vector to β_r is

$$C_1\left(0,C_2r_1,\dots,C_2r_n\right)=C_1C_2\left(0,r_1,\dots,r_n\right). \tag{2.27}$$

C_1 and C_2 depend on the curve γ but are independent of $r_1,\dots,r_n$. As $r=(r_1,\dots,r_n)\in S^{n-1}$ varies we get different unit vectors in $C_1C_2\mathbb{R}^n$. Thus $C_1C_2\mathbb{R}^n$ is the tangent space to the Lag sphere L_γ at an endpoint of γ where the fiber, S^{n-1}, degenerates into one point. The symplectic form ω vanishes on $C_1C_2\mathbb{R}^n$ since it is an isotropic plane in $\mathbb{C}^{n+1}$ equipped with the standard symplectic form. □

Lemma 2.1.21. *Each fiber of L_γ is $O(n)$ invariant. L_γ is $O(n)$ invariant with fixed points of the*

action $(w_j, 0, \ldots, 0)$, $w_j \in V(f)$.

Proof. Let $(w, \mathbf{z}) \in L_\gamma$. Then

$$\mathbf{z} = \sqrt{f(w)}\mathbf{r},$$

for some $\mathbf{r} \in S^{n-1}$. Let $A \in O(n)$.

$$\mathbf{z}A = \sqrt{f(w)}\mathbf{r}A = \sqrt{f(w)}\mathbf{r}',$$

where $\mathbf{r}' = \mathbf{r}A \in S^{n-1}$, as S^{n-1} is $O(n)$-invariant. Therefore $(w, \mathbf{z}A)$ is in the same fiber as $(w, \mathbf{z})$. □

Solomon–Yuval [18, Proposition 3.7] prove the following fact which for us follows from the proof of Lemma 2.1.20:

Corollary 2.1.22. *Let γ be smooth. Then L_γ is diffeomorphic to a sphere.*

Proof. We explicitly evaluate the tangent vectors at points where the fiber degenerates (2.27).

L_γ is a trivial fibration over a simple path with fiber S^{n-1}. A diffeomorphism between L_γ and S^n is established by a map between the path γ and a diameter of S^n. Let $\gamma : [-1, 1] \subset \mathbb{R} \subset \mathbb{R}^{n+1} \to \mathbb{C}$. Then there is a map

$$\Gamma : S^{n+1} \to L_\gamma$$

$$\Gamma : (u, r_1, \ldots r_n) \to \left(\gamma(u), r_1\sqrt{f(\gamma(u))}, \ldots, r_n\sqrt{f(\gamma(u))}\right).$$

It is straightforward to see that Γ is a bijection. Smoothness is clear for $u \in (-1, 1)$, and is not a priori clear only for $u \in \{-1, 1\}$. In the proof of Lemma 2.1.20 we show that the tangent space to L_γ at $(w_j, 0, \ldots, 0)$ (where w_j is a zero of f and w_j an endpoint of the path γ) is well defined,

i.e., L_γ has no singularity at $(w_j, 0, \dots, 0)$. It follows from the proof of Lemma 2.1.20 that the differential map extends continuously to $T_{(-1,0,\dots,0)}S^{n+1} \to T_{(w_j,0,\dots,0)}L_\gamma$. □

Lemma 2.1.23. *Let L_γ be a Lag sphere in a Milnor fiber X_f^n. Then the Lagrangian angle θ_γ (Definition 2.1.10), defined on L_γ is given by*

$$\theta_\gamma(u) = \arg\left[(\gamma'(u))(f(\gamma(u)))^{\frac{n-2}{2}}\right], \tag{2.28}$$

Proof. Fix a point on the curve γ and evaluate the form Ω_f^n on a basis of tangent vectors at a point of L_γ which is in the fiber above the fixed point $\gamma(u)$. The coordinates of that point are:

$$\left(\gamma(u), \sqrt{f(\gamma(u))}r_1, \dots \sqrt{f(\gamma(u))}r_n\right),$$

for some $(r_1, \dots, r_n) \in S^{n-1}$. A basis of the tangent space to L_γ at that point is:

$$\gamma'(u)\left(1, \frac{f'(\gamma(u))}{2f(\gamma(u))}z_1, \frac{f'(\gamma(u))}{2f(\gamma(u))}z_2, \dots, \frac{f'(\gamma(u))}{2f(\gamma(u))}z_n\right),$$

$$(0, -z_2, z_1, 0 \dots, 0), \dots, (0, -z_n, 0, \dots 0, z_1).$$

Ω_f^n evaluated on these vectors is:

$$\frac{1}{f'(\gamma(u))}\gamma'\frac{f'(\gamma(u))}{2f(\gamma(u))}\left(z_1^n + z_1^{n-2}z_2^2 + \dots + z_1^{n-2}z_n^2\right) = \frac{1}{2}\gamma'\big(f(\gamma(u))\big)^{\frac{n-2}{2}}r_1^{n-2}.$$

□

2.2 Minimization Approach

In section 3.7 we will give an explicit way of finding paths γ between zeros of f such that L_γ is special. That procedure does not always yield a solution simply because sometimes the shortest path between two cone points in a surface with conical singular points with cone angles larger than 2π is not along a geodesic connecting them.

To establish that a shortest path does exist, we use a functional on the set of paths. We will later see that the value of the functional is the length of the path in the conical pull-back metric (subsection 3.6).

In Lemma 2.2.1 we show that $\int_{L_\gamma} \Omega_f^n$ and $\int_\gamma q(\gamma)$ are directly proportional and consequently we can work with line integrals instead of integrals over n-dimensional Lags. We introduce a functional $\Phi_{j\ell}$ (Definition 2.2.4) on the set of paths connecting two zeros of f, $\mathcal{A}_{j\ell}$. We show that the functional $\Phi_{j\ell}$ is minimized by paths which give rise to SLs/chains of SLs (Theorem 2.2.5).

2.2.1 Line integrals

In this section we show that evaluating $\int_{L_\gamma} \Omega_f^n$ can be reduced to evaluating a line integral over γ (Lemma 2.2.1). Shapere–Vafa [1] use the integral $n = 3$. Their reference for it is [13]

A homotopy of paths in $\mathbb{C}$ gives rise to a homotopy of Lag spheres corresponding to them.

Let $\gamma \in \mathcal{A}_{j\ell}$ (1.3) for some $j \neq \ell$ and recall the definition of a Lag sphere over γ:

$$L_\gamma := \left\{ \left(w, \sqrt{f(w)}r_1, \ldots, \sqrt{f(w)}r_n\right) \in X_f^n \,:\, w \in \gamma,\, r_1^2 + \cdots + r_n^2 = 1,\, r_1, \ldots, r_n \in \mathbb{R} \right\}$$

Remark: Regardless of the choice of branch of the square root we take in the defining equation, L_γ is the same set. In Lemma 2.1.20 we established that L_γ is an S^{n-1} fibration over γ, diffeomorphic to S^n and is a Lag submanifold of the manifold X_f^n.

Lemma 2.2.1. *Let $\gamma \in \mathcal{A}_{j\ell}$ (1.3). Then*

$$\int_{L_\gamma} \Omega_f^n = \frac{n}{2} \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2}+1)} \int_\gamma \left(\sqrt{f(\gamma(u))}\right)^{n-2} \gamma'(u) du, \tag{2.29}$$

where the direction of integration along γ and the value of $\sqrt{f(\gamma(u))}$ are chosen in such a way that $\exp\left(i \arg[(\sqrt{f(\gamma(u))})^{n-2}\gamma'(u)]\right)$ is the phase of L_γ at $(\gamma(u), z_1, \ldots, z_n) \in L_\gamma$ (Lemma 2.1.23).

Proof. L_γ is a fibration over γ. The fiber over a point $w \in \gamma$ is the $(n-1)$-sphere:

$$\left\{ \left(\sqrt{f(w)}r_1, \ldots, \sqrt{f(w)}r_n\right) : (r_1, \ldots, r_n) \in S^{n-1} \right\}.$$

(For n odd, the orientation of the basis of $\sqrt{f(w)}\mathbb{R}^n$, $(\sqrt{f(w)}r_1, \ldots, \sqrt{f(w)}r_n)$ changes if the choice of the square root branch is changed. For n even, there is no change.)

We proceed with calculations needed to establish the validity of the claim,

$$z_j = r_j\sqrt{f(w)}, \quad dz_j = dr_j\sqrt{f(w)} + r_j \frac{f'(w)}{2\sqrt{f(w)}} dw$$

Recall that

$$\Omega_f^n = \frac{dz_1 \wedge \cdots \wedge dz_n}{f'(w)}.$$

Note that $dr_1 \wedge \cdots \wedge dr_n = 0$ since $r_1^2 + \cdots + r_n^2 = 1$.

$$\begin{aligned}
\Omega_f^n|_{L_\gamma} &= \frac{dz_1 \wedge \cdots \wedge dz_n}{f'(w)} \\
&= \frac{\left(dr_1\sqrt{f(w)} + r_1\frac{f'(w)}{2\sqrt{f(w)}}dw\right) \wedge \cdots \wedge \left(dr_n\sqrt{f(w)} + r_n\frac{f'(w)}{2\sqrt{f(w)}}dw\right)}{f'(w)} \\
&= \frac{1}{2}\left(\sqrt{f(w)}\right)^{n-2}\left(dr_1 \wedge \cdots \wedge dr_{n-1} \wedge r_n dw + \cdots + r_1 dw \wedge dr_2 \wedge \cdots \wedge dr_n\right) \\
&= \frac{1}{2}\left(\sqrt{f(w)}\right)^{n-2} dw \wedge \left(r_1 dr_2 \wedge \cdots \wedge dr_n - \cdots + (-1)^{n-1} r_n dr_1 \wedge \cdots \wedge dr_{n-1}\right).
\end{aligned}$$

Now, evaluate $\int_{L_\gamma} \Omega_f^n$,

$$\begin{aligned}
\int_{L_\gamma} \Omega_f^n &= \int_{L_\gamma} \frac{dz_1 \wedge \cdots \wedge dz_n}{f'(w)} \\
&= \int_{L_\gamma} \frac{1}{2}\left(\sqrt{f(w)}\right)^{n-2} dw \wedge \sum_{j=1}^{n} (-1)^{j-1} r_j dr_1 \wedge \cdots \wedge \widehat{dr_j} \wedge \cdots \wedge dr_n \\
&= \frac{1}{2}\int_{\gamma} \left(\sqrt{f(w)}\right)^{n-2} dw \int_{S^{n-1}} \left(\sum_{j=1}^{n} (-1)^{j-1} r_j dr_1 \wedge \cdots \wedge \widehat{dr_j} \wedge \cdots \wedge dr_n\right)
\end{aligned}$$

By Stokes' Theorem

$$\begin{aligned}
&\int_{r_1^2+\cdots+r_n^2=1} \left(r_1 dr_2 \wedge \cdots \wedge dr_n + \cdots + (-1)^{n-1} dr_1 \wedge \cdots \wedge dr_{n-1}\right) = \\
&\qquad = \int_{r_1^2+\cdots+r_n^2\leq 1} d\Big(r_1 dr_2 \wedge \cdots \wedge dr_n + \cdots + (-1)^{n-1} dr_1 \wedge \cdots \wedge dr_{n-1}\Big) \\
&\qquad = \int_{r_1^2+\cdots+r_n^2\leq 1} n dr_1 \wedge \cdots \wedge dr_n = \frac{n\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2}+1)}.
\end{aligned}$$

Therefore,

$$\int_{L_\gamma} \Omega_f^n = \frac{n}{2} \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2}+1)} \int_\gamma \left(\sqrt{f(w)}\right)^{n-2} dw = \frac{n}{2} \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2}+1)} \int_\gamma \left(\sqrt{f(\gamma(u))}\right)^{n-2} \gamma'(u) du.$$

□

Lemma 2.2.2. *Let L_{γ_1} and L_{γ_2} be the Lag submanifolds arising from the paths γ_1 and γ_2. If $\gamma_1, \gamma_2 \in \mathcal{A}_{j\ell}$ (1.3), for some $j, l \in \{1, \dots, k\}$, and the closed curve formed by γ_1 and γ_2 does not contain any zeros of $f(w)$ in its interior, then L_{γ_1} and L_{γ_2} are in the same homology class in $\mathcal{H}_{j\ell}$ (Definition 1.3).*

Proof. Suppose that the region of $\mathbb{C}$ enclosed by $\gamma_1 - \gamma_2$, Γ, contains no zeros of $f(w)$. Γ lifts to:

$$Y = \left\{(w, z_1, \dots z_n) \in \mathbb{C}^{n+1} : w \in \Gamma, (z_1, \dots, z_n) = (r_1, \dots, r_n)\sqrt{f(w)}, r_1^2 + \cdots + r_n^2 = 1\right\}$$

a submanifold of X_f^n of real dimension $n+1$ with boundary $L_{\gamma_1} - L_{\gamma_2}$. Then the homology class $[L_{\gamma_1} - L_{\gamma_2}] = 0$ so $[L_{\gamma_1}] = [L_{\gamma_2}]$. □

Remark 2.2.3. If $\gamma_1, \gamma_2 \in \mathcal{A}_{j\ell}$ are two paths such that the closed curve $\gamma_1 - \gamma_2$ does not contain any zeros of $f(w)$ in its interior then

$$\int_{\gamma_1} \left(\sqrt{f(\gamma_1(u))}\right)^{n-2} \gamma_1'(u) du = \int_{\gamma_2} \left(\sqrt{f(\gamma_2(u))}\right)^{n-2} \gamma_2'(u) du,$$

and

$$\int_{L_{\gamma_1}} \Omega_f^n = \int_{L_{\gamma_2}} \Omega_f^n.$$

By Lemma 2.2.2, L_{γ_1} and L_{γ_2} are in the same homology class and $L_{\gamma_1} - L_{\gamma_2}$ is the boundary

of a real submanifold Y of X_P of dimension $n+1$. Using Stokes' theorem:

$$\int_{L_{\gamma_1}} \Omega_f^n - \int_{L_{\gamma_2}} \Omega_f^n = \int_{\partial Y} \Omega_f^n = \int_Y d\Omega_f^n = 0$$

since Ω_f^n is closed. Therefore

$$\int_{L_{\gamma_1}} \Omega_f^n = \int_{L_{\gamma_2}} \Omega_f^n,$$

which by Lemma 2.2.1 implies

$$\int_{\gamma_1} (f(w))^{\frac{n-2}{2}}\, dw = \int_{\gamma_2} (f(w))^{\frac{n-2}{2}}\, dw.$$

2.2.2 Functional Φ

We define the functional $\Phi_{j\ell}$ (Definition 2.2.4). In Theorem 2.2.5 we show that the minimizer for $\Phi_{j\ell}$ always exists.

Definition 2.2.4. *Let $\Phi_{j\ell}(\gamma)$ be a real functional on the closure of the set $\mathcal{A}_{j\ell}$ given by:*

$$\Phi_{j\ell}(\gamma) := \int_\gamma \left| (f(w))^{\frac{n-2}{2}}\, dw \right| = \int_0^1 \left| \left(\sqrt{f(\gamma(u))}\right)^{n-2} \gamma'(u) \right| du. \tag{2.30}$$

Theorem 2.2.5. *Let X_f^n be as in Definition 2.1.13, let $\mathcal{A}_{j\ell}$ be as in (1.3) and L_γ as in Definition 2.1.19. Let $w_j, w_\ell \in V(f)$ and $j \neq \ell$. Then there exists a path $\gamma \in \overline{\mathcal{A}_{j\ell}}$ for which the functional $\Phi_{j\ell}$ (Definition 2.2.4) has a minimum.*

Proof. Using the notation $q = (f(w))^{\frac{n-2}{2}}$, let

$$e^{i\theta_\gamma(u)} = \frac{q(\gamma(u))}{|q(\gamma(u))|}.$$

We can write

$$\int_\gamma q = \int_\gamma q(\gamma(u))\gamma'(u)du = \int_\gamma |q(\gamma)\gamma'(u)|e^{i\theta_\gamma(u)}du = \int_\gamma e^{i\theta_\gamma(u)}|q|.$$

Triangle inequality,

$$\Big|\int_\gamma q\Big| \leq \int_\gamma |q|,$$

gives a lower bound for $\Phi_{j\ell}(\gamma) = \int_\gamma |q|$ for γ in a fixed homology class. There are finitely many homology classes of paths in $\mathcal{A}_{j\ell}$ since $|V(f)| < \infty$ (Lemma 2.2.1, Remark 2.2.3).

$$\inf_{\gamma\in A}\int_\gamma |q|$$

is achieved on the closure of $\mathcal{A}_{j\ell}$. □

Remark 2.2.6. If for some $\gamma \in \mathcal{A}_{j\ell}$, $\theta_\gamma = \text{const.}$, then the lower bound of $\Big|\int_\gamma q\Big|$, is achieved for this $\gamma \in A$ since

$$\Big|\int_\gamma q\Big| = \Big|\int_\gamma |q|e^{i\theta_\gamma}\Big| = \Big|e^{i\theta_\gamma}\int_\gamma |q|\Big| = \int_\gamma |q|.$$

It is possible that the lower bound is achieved on an element of the closure of a class $A \in \mathcal{A}_{j\ell}$ which is a concatenation of paths, $\gamma_1 \dots \gamma_m$, connecting zeros of f, such that $\theta_{\gamma_1} = \text{const.}$, $\dots$ $\theta_{\gamma_m} = \text{const.}$

In Theorem 3.10.2 we gave a necessary and sufficient condition for the existence of a path $\gamma \in A_{j\ell}$, such that L_γ has constant phase.

Remark 2.2.7. $\Phi_{j\ell}(\gamma)$ is the length of the path γ in the pullback metric $|q|^2|dw|^2$ (Lemma 3.6.4) and a minimizer of the functional $\Phi_{j\ell}$ is the pullback of a length minimizer, i.e.,the pullback of a segment of a geodesic line, or is a concatenation of paths where each constituent is the pullback of a segment of a geodesic line.

Corollary 2.2.8. *Under the assumptions of Theorem 2.2.5, the Lag sphere corresponding to the minimizer, L_γ, is the unique compact SL submanifold of X_f^n in its homology class. If $\gamma \in \overline{A}_{j\ell} \backslash A_{j\ell}$ and $\gamma = \gamma_1 \cup \cdots \cup \gamma_m \in \overline{A}_{j\ell}$, then there is no SL $L_\gamma \in A_{j\ell}$ but each L_{γ_p}, $1 \leq p \leq m$ is a compact SL submanifold of X_f^n, unique in its homology class.*

2.3 ODE Approach

In this section we show that the special Lagrangian ODE (2.31) can be solved for any value of θ and any initial point. This extends and also proves the claim of Shapere–Vafa [1, Section 5] (for $n = 3$ and initial point in $V(f)$).

$$\arg\left[\left(\sqrt{f(\gamma(t))}\right)^{n-2}\gamma'(t)\right] \equiv \theta. \tag{2.31}$$

The main results of the section is Theorem 2.3.3 in which we show that for a generic initial point there is a unique solution and that for the initial point in the set $V(f)$, there are n solutions of the equation (2.31).

In subsection 2.3.1 we solve the initial value problem, in 2.3.2 we analyse properties of integral curves, and in 2.3.3 we solve the boundary value problem, specifically when the boundary

condition requires the endpoints of the integral path to be distinct points in $V(f)$.

2.3.1 Initial value problem and integral curves

Lemma 2.3.1. *Let $\theta \in [0, 2\pi)$ be given. Let $w_j \in V(f)$. Then there are n directions for the tangent vector at w_j for integral curves of the equation*

$$\arg\left[\frac{d}{du}Q(w)\right] = \arg\left[\Big(f(w)\Big)^{\frac{n-2}{2}}\frac{dw}{du}\right] = \theta. \tag{2.32}$$

Consecutive directions form an angle of $\frac{2\pi}{n}$.

Proof. The curve passing through $Q(w_j)$ whose tangent vector has constant phase is the straight line through $Q(w_j)$ with the slope $\tan\theta$. Near w_j,

$$q(w) = (w - w_j)^{\frac{n-2}{2}}\left(a_0 + a_1(w - w_j) + a_2(w - w_j)^2 + \dots\right),$$

and

$$Q(w) = (w - w_j)^{\frac{n}{2}}\left(\frac{2}{n}a_0 + \frac{2}{n+2}a_1(w - w_j) + \frac{2}{n+4}a_2(w - w_j)^2 + \dots\right) + Q(w_j).$$

We can parametrize the straight line through $Q(w_j)$ by a parameter t so that the points on the line are given by $Q(w_j) + te^{i\theta}$, $t \in \mathbb{R}$, $\theta = \text{const}$.

Therefore,

$$\begin{aligned} t\mathrm{e}^{i\theta} &= (w-w_j)^{\frac{n}{2}}\left[\frac{2}{n}a_0 + (w-wj)\left(\frac{2}{n+2}a_1 + \frac{2}{n+4}a_2(w-w_j)+\dots\right)\right] \\ &= (w-w_j)^{\frac{n}{2}}\frac{2}{n}a_0 + O\big((w-w_j)^{\frac{n}{2}+1}\big). \end{aligned}$$

From here,

$$\theta = \arg\left[(w-w_j)^{\frac{n}{2}}\frac{2}{n}a_0 + O\big((w-w_j)^{\frac{n}{2}+1}\big)\right].$$

Taking the limit $w \to w_j$, along γ, and using the mean value theorem, we get

$$\theta = \lim_{w\to w_j} \arg\left[(\gamma')^{\frac{n}{2}}a_0\right].$$

Therefore, the direction of the tangent line to γ at w_j is given by one of

$$\mathrm{e}^{i\left[\frac{2(\theta-\arg(a_0))}{n}+m\frac{2\pi}{n}\right]}, \quad m = 0,1,\dots,n-1.$$

□

Lemma 2.3.2. *Given, $\theta_1, \theta_2 \in [0, 2\pi)$, $\theta_1 \neq \theta_2$. Suppose that γ_1 is an integral curve of the equation*

$$\arg\left[\Big(f(w)\Big)^{\frac{n-2}{2}}\frac{dw}{dt}\right] = \theta_1 \tag{2.33}$$

and γ_2 is an integral curve of the equation

$$\arg\left[\Big(f(w)\Big)^{\frac{n-2}{2}}\frac{dw}{dt}\right] = \theta_2, \tag{2.34}$$

and $\gamma_1 \cap \gamma_2 = \{p\}$. *Then*

(i) If $p \notin V(f)$ *then* γ_1 *and* γ_2 *form an angle measuring* $|\theta_1 - \theta_2|$.

(ii) If $p \in V(f)$ *then the angle between* γ_1 *and* γ_2 *is* $\frac{2}{n}|\theta_1 - \theta_2| \pm m\frac{2\pi}{n}$, *for some* $m = 0, 1, 2, \ldots n-1$.

Proof. (i) $\gamma_1 \cap \gamma_2 = \{p\}$, and $f(p) \neq 0$.

$$\arg\left[\left(\sqrt{f(p)}\right)^{n-2}\gamma_1'(p)\right] = \theta_1 \quad \rightarrow \quad \frac{\left(\sqrt{f(p)}\right)^{n-2}\gamma_1'(p)}{\left|\left(\sqrt{f(p)}\right)^{n-2}\gamma_1'(p)\right|} = \mathrm{e}^{i\theta_1}, \tag{2.35}$$

and similarly,

$$\arg\left[\left(\sqrt{f(p)}\right)^{n-2}\gamma_2'(p)\right] = \theta_2 \quad \rightarrow \quad \frac{\left(\sqrt{f(p)}\right)^{n-2}\gamma_2'(p)}{\left|\left(\sqrt{f(p)}\right)^{n-2}\gamma_2'(p)\right|} = \mathrm{e}^{i\theta_2}. \tag{2.36}$$

From (2.35) and (2.36) we get:

$$\frac{|\gamma_1'|}{\gamma_1'}\mathrm{e}^{i\theta_1} = \frac{\left(\sqrt{f(p)}\right)^{n-2}}{\left|\left(\sqrt{f(p)}\right)^{n-2}\right|} = \frac{|\gamma_2'|}{\gamma_2'}\mathrm{e}^{i\theta_2},$$

and finally,

$$\frac{\frac{\gamma_1'}{|\gamma_1'|}}{\frac{\gamma_2'}{|\gamma_2'|}} = \mathrm{e}^{i(\theta_1-\theta_2)},$$

at the point p, as claimed.

(ii) $\gamma_1 \cap \gamma_2 = \{p\}$ and $f(p) = 0$. By Lemma 2.3.1 there are n integral curves through p corresponding to θ_1, and n integral curves corresponding to θ_2 and the integral curves have tangents in the directions of

$$\mathrm{e}^{i[\frac{2(\theta_1-\arg(a_0))}{n}+m\frac{2\pi}{n}]}, \quad m=0,1,\ldots,n-1, \tag{2.37}$$

and

$$\mathrm{e}^{i[\frac{2(\theta_2-\arg(a_0))}{n}+m\frac{2\pi}{n}]}, \quad m=0,1,\ldots,n-1. \tag{2.38}$$

Choosing one direction from (2.37) and one from (2.38) results in the following angles between them:

$$\frac{2}{n}(\theta_1-\theta_2)\pm m\frac{2\pi}{n}, \quad m=0,1,\ldots,n-1.$$

□

Theorem 2.3.3. *Given any $\theta\in[0,2\pi)$, through any point $w_0\in\mathbb{C}$ there exists a solution $\gamma=\gamma(t)$ of the initial value problem*

$$\begin{cases} \arg\left[\left(\sqrt{f(\gamma(t))}\right)^{n-2}\gamma'(t)\right]\equiv\theta, \\ \gamma(0)=w_0. \end{cases} \tag{2.39}$$

If $w_0\notin V(f)$ then the curve is unique. If $w_0\in V(f)$ then there are exactly n curves, consecutive ones at angles of $\frac{2\pi}{n}$ with respect to each other.

Proof. First we show that at any point w_0 such that $f(w_0)\neq 0$ the direction of the tangent vector is uniquely determined. Then we show that if $f(w_0)=0$ there are n possibilities for the direction of the tangent vector. Finally, we show the existence.

Assume that there exists a curve γ satisfying

$$\arg\left[\left(\sqrt{f(\gamma(t))}\right)^{n-2}\gamma'(t)\right] \equiv \theta.$$

and $w_0 \in \gamma([0,1])$. Then

$$\left(\sqrt{f(\gamma)}\right)^{n-2}\gamma' = \mathrm{e}^{i\theta}\left|\left(\sqrt{f(\gamma)}\right)^{n-2}\gamma'\right|$$

From this equation we can express the unit tangent vector to the curve:

$$\frac{\gamma'}{|\gamma|} = \mathrm{e}^{i\theta}\frac{|\sqrt{f(\gamma)}|^{n-2}}{\left(\sqrt{f(\gamma)}\right)^{n-2}}. \tag{2.40}$$

If w_0 is not a zero of $f(w)$ then (2.40) is uniquely determined as

$$\frac{\gamma'}{|\gamma|}\Big|_{w=w_0} = \mathrm{e}^{i\theta}\frac{\left|\sqrt{f(w_0)}\right|^{n-2}}{\left(\sqrt{f(w_0)}\right)^{n-2}}. \tag{2.41}$$

Claim: If $w_0 \in V(f)$ then

$$\lim_{w\to w_0}\frac{\left|\sqrt{f(w_0)}\right|^{n-2}}{\left(\sqrt{f(w_0)}\right)^{n-2}}$$

exists.

Proof of Claim: Let $\gamma = \gamma(t), t \in [0,1]$ be a path, $w_0 = \gamma(0)$. Recall that $f(w) = (w-w_1)(w-w_2)\ldots(w-w_k)$ and without loss of generality assume that $w_0 = w_1$. When w

approaches w_1 over any differentiable path the following limit exists and

$$\lim_{w \to w_1} (w - w_2) \dots (w - w_k) = (w_1 - w_2) \dots (w_1 - w_k) \neq 0.$$

Let

$$\beta_1 := \mathrm{e}^{i \arg\left[\left(\sqrt{(w_1 - w_2) \dots (w_1 - w_k)}\right)^{n-2}\right]}.$$

Then

$$\lim_{t \to 0} \frac{\left|\sqrt{f(w_0)}\right|^{n-2}}{\left(\sqrt{f(w_0)}\right)^{n-2}} = \frac{1}{\beta_1} \lim_{t \to 0} \mathrm{e}^{-i \frac{n-2}{2} \arg(\gamma(t) - \gamma(0))} = \frac{\mathrm{e}^{-i \frac{n-2}{2} \arg(\gamma'(0))}}{\beta_1}.$$

This ends the proof of the claim.

Let

$$v_1 := \mathrm{e}^{\frac{i}{2} \arg(\gamma'(0))}.$$

If γ in addition to containing x_1 satisfies (2.41), then substitute v_1 in (2.41) to get

$$v_1^2 = \frac{\mathrm{e}^{i\theta}}{\beta_1 v_1^{n-2}},$$

so

$$v_1^n = \frac{\mathrm{e}^{i\theta}}{\beta_1}.$$

$v_1, \beta_1, \mathrm{e}^{i\theta}$ are all unit complex numbers. $\mathrm{e}^{i\theta}$ is given and β_1 doesn't depend on γ but only on the polynomial f so it is also fixed. There are therefore n different solutions for v_1 and they define the n different possible directions (v_1^2) for the tangent vector to γ at w_1.

When w_0 is not a zero of $f(w)$ then the first order ODE:

$$\begin{cases} \gamma'(t) = \mathrm{e}^{i\theta} \dfrac{\left|\sqrt{f(w_0)}\right|^{n-2}}{\left(\sqrt{f(w_0)}\right)^{n-2}}, \quad t \in (0,1] \\ \gamma(0) = w_0 \end{cases} \tag{2.42}$$

has a unique solution in a neighborhood of w_0 by Picard-Lindelöf.

If $w_0 \in V(f)$, say $w_0 = w_1$ choose one of the n possible directions for the tangent vector at w_1,

$$\left(\frac{\mathrm{e}^{i\theta}}{\beta_1}\right)^{\frac{2}{n}} \mathrm{e}^{m\frac{2\pi}{n}i}, \quad m = 0, 1, \ldots, n-1.$$

Choose a small angle η and consider a wedge W with central angle 2η of a small circle of radius δ with center at w_1 so that the direction $\left(\frac{\mathrm{e}^{i\theta}}{\beta_1}\right)^{\frac{2}{n}}$ bisects W and W is away from all of $w_2, \ldots w_k$.

Then the initial value problem:

$$\begin{cases} \gamma'(t) = \mathrm{e}^{i\theta} \dfrac{\left|\sqrt{f(w_0)}\right|^{n-2}}{\left(\sqrt{f(w_0)}\right)^{n-2}}, \quad t \in (0,1], \quad \gamma'(0) = \left(\frac{\mathrm{e}^{i\theta}}{\beta_1}\right)^{\frac{2}{n}} \\ \gamma(0) = w_1 \end{cases} \tag{2.43}$$

has a solution in W for $t \in [0, t_0)$, for some $t_0 \in (0,1)$, by Peano's existence theorem. It remains to show that there cannot be two integral curves having the same tangent vector at w_1.

Suppose that there are two such solutions, γ_1 and γ_2. Refer to Figure 2.1 for the following. Take a point on γ_1 close to w_j. Through that point there exists an integral curve for the equation (2.31) with $\mathrm{e}^{i\theta}$ replaced by $\mathrm{e}^{i(\theta+\frac{\pi}{2})}$, γ_3. γ_3 intersects the curves γ_1 and γ_2 at right angles (by Lemma 2.3.2). We can claim that it intersects both γ_1 and γ_2 because they have the same tangent at their common point and the function q being close to a constant where γ_3 is constructed means

that the tangent direction to γ_3 is close to a constant. The point on γ_1 can be chosen in such a way that the curvilinear triangle formed by γ_1, γ_2, and γ_3 does not contain any other zero of the polynomial f (w_j is one of its vertices). Then choose another point on γ_1, closer to w_j than the first one, and construct γ_4, the unique solution of (2.31) through that point with $\mathrm{e}^{i\theta}$ replaced by $\mathrm{e}^{i(\theta+\frac{\pi}{2})}$. γ_1, γ_2, γ_3, and γ_4 form a curvilinear quadrilateral containing no zeros of f. q is holomorphic in an open set containing the quadrilateral $\gamma_2\gamma_3\gamma_1\gamma_4$.

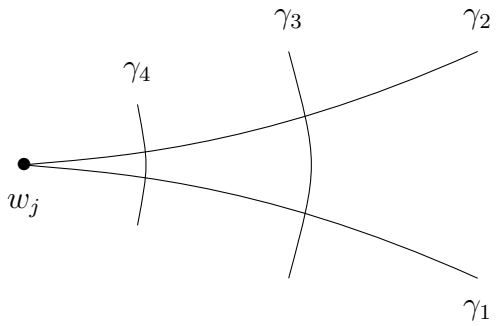


Figure 2.1: γ_1 and γ_2, integral curves at w_j for phase θ, γ_3 and γ_4, integral curves for phase $\theta+\frac{\pi}{2}$.

Therefore

$$\oint_{\gamma_2\cdot\gamma_3\cdot\gamma_1\cdot\gamma_4} q = 0.$$

Along each side of the quadrilateral the integrand has constant phase. Therefore

$$\oint_{\gamma_1\cdot\gamma_2\cdot\gamma_3\cdot\gamma_4} q = \mathrm{e}^{i\theta}\int_{\gamma_2}|q| - \mathrm{e}^{i(\theta+\frac{\pi}{2})}\int_{\gamma_3}|q| - \mathrm{e}^{i\theta}\int_{\gamma_1}|q| + \mathrm{e}^{i(\theta+\frac{\pi}{2})}\int_{\gamma_4}|q|.$$

The integrals over γ_3 and γ_4 don't cancel since $|q|$ along γ_4 is less than $|q|$ along γ_3 and the length of the segment of γ_4 is less than that of γ_3. (It is possible to choose the points on γ_1 in such a way that this is satisfied because γ_1 and γ_2 intersect at w_0 at angle equal to 0 and $|q(w)| \to 0$ when $w \to w_j$. Thus the expression on the right hand side is not equal to 0. Contradiction.

Therefore there is a unique solution of the initial value problem (2.43).

The same reasoning for the other possible values of $\gamma'(0)$ provides the existence and uniqueness for other solutions emanating from w_1.

□

2.3.2 Some properties of integral curves

In Proposition 2.3.4 we show that the intersection of one integral curve of equation (2.31) for a value θ_1, and one integral curve for a value θ_2, with $\theta_1 \neq \theta_2$, contains at most one point.

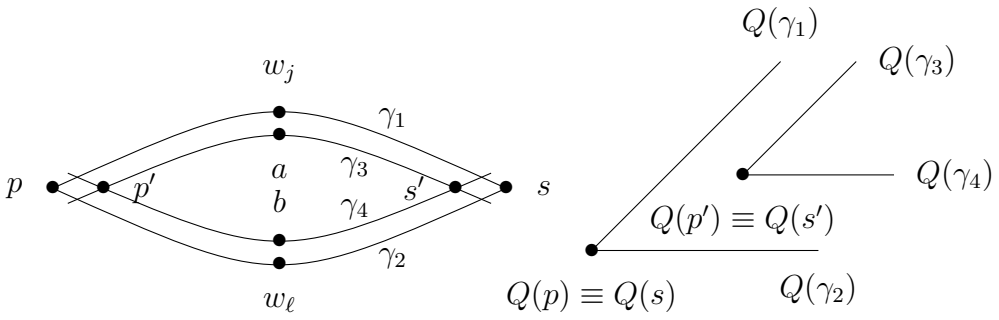
In Proposition 2.3.5 we show that the intersection of two integral curves of (2.31) for the same value of θ contains at most one point, and if the intersection is non-empty, the point of intersection is a zero of f.

These properties will be used in subsection 2.3.3.

Proposition 2.3.4. *Suppose that γ_1 and γ_2 are integral curves of the equation (2.31) for θ_1 and θ_2 respectively, and $\theta_1 \neq \theta_2$. Then $|\gamma_1 \cap \gamma_2| \leq 1$.*

Proof. Refer to Figure 2.2. $Q(\gamma_1)$ and $Q(\gamma_2)$ are non-parallel straight lines. They have a unique common point. If there exists a point $p \in \gamma_1 \cap \gamma_2$ then $\{Q(p)\} = Q(\gamma_1) \cap Q(\gamma_2)$. Suppose that there is another point of intersection, s. Then $Q(s) = Q(p)$ since that is the only point of intersection of two distinct straight lines. If there are more than two points in $\gamma_1 \cap \gamma_2$ choose s so that the arcs $\overset{\frown}{ps} \subset \gamma_1$ and $\overset{\frown}{ps} \subset \gamma_2$ only have the endpoints in common. Now, when a point, w, traverses the arc $\overset{\frown}{ps} \subset \gamma_1$ from p to s, its image moves from $Q(p)$ to $Q(s) = Q(p)$ along the straight line containing $Q(\gamma_1)$. At some point w of the arc $\overset{\frown}{ps} \subset \gamma_1$, the distance $d(Q(w), Q(p))$ is maximal. At that point $q(\gamma_1(t))\gamma_1'(t) = 0$ since the tangent to the path $Q(\gamma_1(t))$ switches from pointing to one direction of the straight line containing $Q(\gamma_1)$ to pointing in the opposite direction

as the point then moves back closer to $Q(s) = Q(p)$. Therefore, the arc $\widehat{ps} \subset \gamma_1$ has to contain a zero of f (w_j in Figure 2.2) and so does the arc $\widehat{ps} \subset \gamma_2$ (w_ℓ in Figure 2.2). Choose points a and b in the interior of the closed curve consisting of the two arcs so that a is closer than b to γ_1 and b is closer than a to γ_2. Through a there exists an integral curve for θ_1 and through b for θ_2. These two integral curves intersect in two points. This is because two integral curves for the same value of θ do not have common points so γ_1 and the integral curve through a don't intersect, and neither do γ_2 and the integral curve through b. Now the two new curves form the same picture as the two initial curves, γ_1 and γ_2, and therefore each of them contains a zero of f. This process can be repeated and we get an infinite number of zeros of f. Contradiction. □



Integral curves for θ_1 and θ_2 Images under Q

Figure 2.2: for Proposition 2.3.4

Proposition 2.3.5. *If γ_1 and γ_2 are two integral curves of the equation (2.31), for the same value of θ, then*

$$w \in \gamma_1 \cap \gamma_2 \Longrightarrow f(w) = 0,$$

and

$$\gamma_1 \cap \gamma_2 \neq \varnothing \Longrightarrow |\gamma_1 \cap \gamma_2| = 1.$$

Proof. By Theorem 2.3.3, through a point w_0 for which $f(w_0) \neq 0$ there exists a unique integral

curve for a given θ. Therefore γ_1 and γ_2 could only intersect at a point $w_j \in V(f)$.

Refer to Figure 2.3 for the following. Suppose that $w_j, w_\ell \in \gamma_1 \cap \gamma_2$, with $j \neq \ell$. Without loss of generality assume that there are no other integral curves of the equation (2.31) connecting w_j with w_ℓ and contained in the compact region of $\mathbb{C}$ with boundary $\gamma_1 \cup \gamma_2$ (otherwise replace γ_2 with a curve with this property). Now, through any point in the interior there exists an integral curve of the equation (2.31). Choose a point in the interior of the region bounded by $\gamma_1 \cup \gamma_2$. An integral curve through such a point is closed, a loop. Wlog, the closed integral curve, γ, can be chosen in such a way that there exists an open set containing the compact region bounded by γ which does not contain any zeros of f. In that open set the function q is holomorphic and therefore

$$\oint_\gamma q = 0.$$

But,

$$\oint_\gamma q = \int_0^1 q(\gamma(t))\gamma'(t)dt = \mathrm{e}^{i\theta} \int_0^1 |q(\gamma(t))\gamma'(t)|dt \neq 0,$$

since the integrand is strictly positive. Contradiction. □

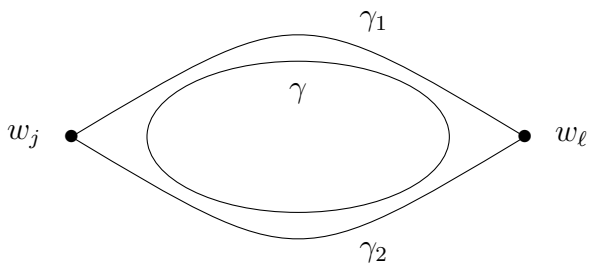


Figure 2.3: for Proposition 2.3.5 (Closed integral curve is not possible)

2.3.3 Boundary value problem

In this subsection we prove results related to the existence of a path connecting $w_j \in V(f)$ and $w_\ell \in V(f)$ and satisfying the equation (2.31) for some value of θ, i.e.,giving rise to an SL.

In Proposition 2.3.6, we show that if the zeros of f are colinear then each straight line segment connecting two consecutive zeros of f (consecutive on the line that contains them all) is a solution of the boundary value problem 2.44 for the corresponding boundary value conditions. There is a total of $k-1$ paths between zeros of f which give rise to SLs.

Proposition 2.3.7 represents the other extreme: we show that if the zeros of f are vertices of a regular k-gon, then for each of $\frac{1}{2}k(k-1)$ pairs of vertices there is a path which connects them and gives rise to an SL.

Shapere–Vafa [1, Section 5.1 Examples] note that both the minimum, $k-1$, and maximum $\frac{1}{2}k(k-1)$ can be achieved, for $n=3$.

In Theorem 2.3.8 we prove that for three zeros of f if two of the three possible paths exists then if the third one exists then the angle between the first two is less than $\frac{2\pi}{n}$. In Theorem 2.3.9 a partial converse to Theorem 2.3.8 is given, i.e., if the angle between the first two paths is less than $\frac{2\pi}{n}$ then the union of the two paths can't be a minimizer of the functional Φ (Definition 2.2.4) for the two endpoints.

There are two steps in the solution of the boundary value problem:

$$\begin{cases} \arg\left[\gamma'(t)\left(\sqrt{f(w_0)}\right)^{n-2}\right] = \theta \\ \gamma(0) = w_j \\ \gamma(1) = w_\ell \end{cases} \tag{2.44}$$

The first is to find a value $\theta \in [0, 2\pi)$ for which an integral curve of the equation (2.31) may contain a path connecting two chosen zeros of f, and the second is to establish the existence or non-existence of such a path.

Suppose that we are looking at two distinct zeros of f, w_j and w_ℓ. We established in Theorem 3.8.5 that $\tan\theta$, where $e^{i\theta}$ is the phase of the potential SL sphere generated by a path between w_j and w_ℓ, is calculated from the values $Q(w_j)$ and $Q(w_\ell)$. Also, an equation defining a curve in $\mathbb{C}$ which contains the path sought, if it exists, is given in the same Theorem. It is the inverse image under Q of the straight line defined by $Q(w_j)$ and $Q(w_\ell)$.

The existence is dependent on the positions of the zeros of f and on n, the dimension of the almost Calabi–Yau manifold X_f^n.

Proposition 2.3.6. *If the zeros of f are colinear then the line containing the zeros is an integral curve of the equation (2.31).*

Proof. Suppose that the zeros of f, $w_1, \ldots, w_k$ are on a line l and let w be a point on l. Let γ be the segment $w_j w_{j+1}$. Suppose that w moves from w_j to w_{j+1} along the line segment $w_j w_{j+1}$. Then, for every $a = 1, \ldots, k$, $w - w_a$ is a vector contained in l and has one of two opposite fixed directions. Of course, γ' has constant argument as γ is a straight line segment. It follows

that $\left(\sqrt{(f(w)}\right)^{n-2}\gamma'(w)$ has constant argument and therefore satisfies the equation (2.31) with a suitable choice of θ. □

Proposition 2.3.7. *If the zeros of f are vertices of a regular polygon then for any two vertices there exists a path connecting them which generates an SL sphere.*

Proof. The proof is not presented here but in the Examples section the statement of the proposition is shown for $k = 6$ (hexagon) and values of $n = 3$ and $n = 4$. Proof in the general case is similar. The key reason why this proposition holds is that the vertices of the regular polygon (zeros of f) are mapped by the mapping Q to vertices of a regular polygon. □

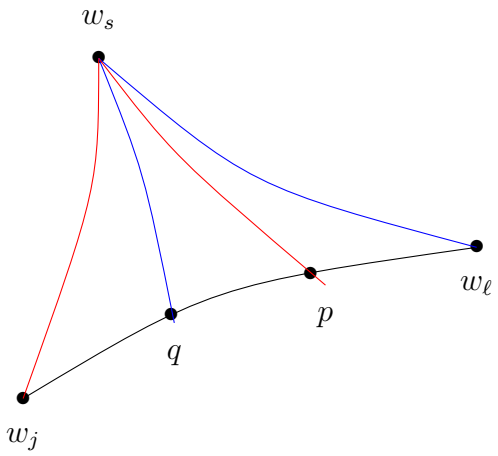


Figure 2.4: Angle between $\widehat{w_s w_j}$ and $\widehat{w_s w_\ell}$ larger than $\frac{2\pi}{n}$

The following is a partial result about existence/non-existence of paths which solve the boundary value problem. It is similar to the observation of Shapere–Vafa in the case $n = 3$ [1, pp. 14-15 Jumping Phenomena].

Theorem 2.3.8. *Let $w_j, w_\ell, w_s \in V(f)$ and $|\{j, \ell, s\}| = 3$. Assume that two of the three paths of constant phase (not containing any other zeros of f) exist between these three points. If the third one exists then the angle between the first two is less than $\frac{2\pi}{n}$. Consequently, if a cycle of length*

three exists in the graph with vertex set $V(f)$ and edges the paths of constant phase between them, then the measures of all three angles in such a curvilinear triangle are less than $\frac{2\pi}{n}$.

Proof. Let the path $\widehat{w_s w_j}$ be part of an integral curve of the equation (2.31) with θ_1 and let the path $\widehat{w_s w_\ell}$ be part of an integral curve of the equation (2.31) with θ_2. Suppose that there exists also the third path, $\widehat{w_j w_\ell}$ and suppose that the angle between the first two at w_s is more than $\frac{2\pi}{n}$. By Lemma 2.3.1 there is another integral curve for θ_1, starting at w_s and having the angle $\frac{2\pi}{n}$ with the path $\widehat{w_s w_j}$, and another integral curve for θ_2, starting at w_s and having the angle $\frac{2\pi}{n}$ with the path $\widehat{w_s w_\ell}$, so that these two curves are inside the angle formed by the paths $\widehat{w_s w_j}$ and $\widehat{w_s w_\ell}$, as in Figure 2.4.

As two integral curves starting at the same zero of f are contained in the preimages of the same straight line, it follows that $Q(p) = Q(w_j)$, as the images under Q of $\widehat{w_j w_\ell}$ and $\widehat{w_s w_j}$ intersect at a single point, $Q(w_j)$. Similarly, $Q(q) = Q(w_\ell)$. But then, when a point transverses the path from w_j to w_ℓ, its image under Q transverses a segment of the straight line $Q(w_j)Q(w_\ell)$, starting from $Q(w_j)$, reaching $Q(q) = Q(w_\ell)$ and going back to $Q(p) = Q(w_j)$ and then finally reaching $Q(w_\ell)$ again. This would require the tangent vector to switch directions discontinuously which can happen only at points $Q(w)$ for which $Q'(w) = q(w) = 0$, i.e.,p and q themselves would have to be zeros of f. Contradiction. □

The converse does not quite hold. Instead we have the following:

Theorem 2.3.9. *Let w_j, w_s, and w_ℓ be three zeros of f. If paths of constant phase $\widehat{w_s w_j}$ and $\widehat{w_s w_\ell}$ exist, don't contain any other zeros of f, and the angle between them is less than $\frac{2\pi}{n}$ then the path $\widehat{w_j w_s} \cup \widehat{w_s w_\ell}$ is not a minimizer for the functional $\Phi_{j\ell}$ (see Definition 2.2.4).*

Proof. Refer to Figure 2.5 for the following. Suppose that that paths $w_s w_j$ and $w_s w_\ell$ of constant

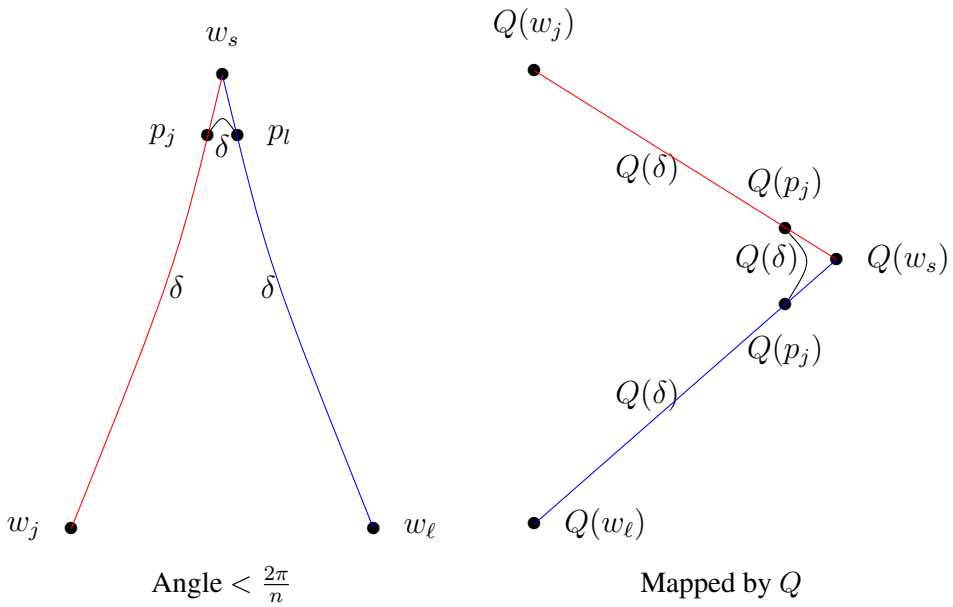


Figure 2.5: Angle between paths of constant phase, $\widehat{w_s w_j}$ and $\widehat{w_s w_\ell}$, less than $\frac{2\pi}{n}$

phases θ_1 and θ_2 exist and that the angle between them is less than $\frac{2\pi}{n}$. Then, the angle between the straight line segments $Q(w_s)Q(w_j)$ and $Q(w_s)Q(w_\ell)$ is less than π by Lemma 2.3.2. Pick points p_j and p_l, on paths $w_s w_j$ and $w_s w_\ell$ respectively, very close to w_s where $q(w) \approx C(w - w_s)^{\frac{n-2}{2}}$ and $Q(w) \approx D(w - w_s)^{\frac{n}{2}}$, for appropriate constants $C, D \in \mathbb{C}$. We look at a smooth curve, δ, which coincides with paths $w_s w_j$ and $w_s w_\ell$ except for an arc between p_j and p_l which replaces $\widehat{p_j w_s} \cup \widehat{w_s p_l}$. The goal is to show that for this variation of the path $\widehat{w_j w_s} \cup \widehat{w_s w_\ell}$, the point $Q(w_s)$ is in the exterior of the region bounded by the straight line segment $Q(w_j)Q(w_\ell)$ together with $Q(\delta)$. When a point w traverses δ from p_j to p_l, $\arg(w - w_s)$ changes by $\frac{2}{n}|\theta_2 - \theta_1| < \frac{2\pi}{n}$ and $\arg(Q(w) - Q(w_s))$ changes by $|\theta_2 - \theta_1| < \pi$. Therefore, $Q(w)$ does not go around $Q(w_s)$ because that would require a change in $\arg(Q(w) - Q(w_s))$ of more than π.

Points p_j and p_l, as well as the arc between can be chosen in such a way that there is no zero of f in the interior or the boundary of the "triangle" $Q(p_j)Q(w_s)Q(p_l)$, and the length of the arc $\widehat{Q(p_j)Q(p_l)}$ is strictly less than the sum of the lengths of $Q(p_j)Q(w_s)$ and $Q(w_s)Q(p_l)$.

It follows that $\Phi_{j\ell}(\delta) < \Phi_{j\ell}(\widehat{w_j w_s} \cup \widehat{w_s w_\ell})$. If there are no images under Q of other zeros of f in the triangle $\triangle Q(w_s)Q(w_\ell)Q(w_j)$ then by Theorem 3.10.2 the preimage of the straight line segment $Q(w_j)Q(w_\ell)$ contains a path between w_j and w_ℓ of constant phase which is then the minimizer of $\Phi_{j\ell}$.

□

Remark 2.3.10. The configuration described at the end of the proof of Theorem 2.3.8 could occur if f and n are chosen in such a way that for four zeros of f, w_j, w_ℓ, w_a, w_b, $Q(w_j) = Q(w_a) \neq Q(w_\ell) = Q(w_b)$. But then the path of constant phase between w_j and w_ℓ gives rise to a chain of three SL spheres, the neighboring ones touching at $(w_a, 0, \ldots, 0)$ and $(w_b, 0, \ldots, 0)$ respectively.

Chapter 3: The underlying Riemann surface, pull-back conical metric

3.1 Introduction

In this chapter we apply the concepts of covering maps between Riemann surfaces to Milnor fibers.

In sections 3.2 and 3.3 we include definitions and standard results on covering maps and specifically ramified covering maps between Riemann surfaces. The references used for the background material are [22] and [23].

In section 3.4 n is assumed to be even. We define a map $Q : \mathbb{C} \to \mathbb{C}$, and show that it is a ramified covering map of degree $\frac{1}{2}k(n-2)+1$. The case of n being odd is discussed in 3.5. It is less straightforward because q is now a multi-valued function. We define a map Q which is now multi-valued.

In section 3.6 the n-even and n-odd cases are unified. The main result of the subsection is Theorem 3.6.4 in which we show that the pull-back of the Euclidean metric by the map Q is a metric with conical singularities given by the expression $|q|^2|dw|^2$.

3.2 Covering maps

All the topological spaces will be assumed to be locally compact Hausdorff (LCH).

Definition 3.2.1 (Proper map)**.** *A map $F : X \to Y$ of topological spaces is proper if, for $K \subset Y$,*

$$K \text{ compact} \quad \Rightarrow \quad F^{-1}(K) \text{ compact.}$$

Definition 3.2.2 (Local homeomorphism)**.** *A map $F : X \to Y$ of topological spaces is a local homeomorphism if for every $p \in X$, there exists an open neighborhood U_p, such that*

$$F|_{U_p} : U_p \to F(U_p)$$

is a homeomorphism.

Definition 3.2.3 (Covering map)**.** *A map $F : X \to Y$ of topological spaces is a covering map if for every $q \in Y$ there exists an open neighborhood V_q such that*

$$F^{-1}(V_q) = \bigsqcup_{\alpha \in A} U_\alpha \quad \textit{(disjoint union)}$$

for some open sets $U_\alpha \subset X$, $\alpha \in A$. and

$$F|_{U_\alpha} : U_\alpha \to V_q$$

is a homeomorphism for every $\alpha \in A$.

Proposition 3.2.4. *If a map $F : X \to Y$ of topological spaces is a proper local homeomorphism then it is a covering map. Moreover, it is a finite covering map, i.e., $|F^{-1}(y)| < \infty$ for any $y \in Y$, and $|F^{-1}(y)|$ is locally constant.*

Proof. Let $y \in F(X)$. The singleton $\{y\}$ is compact and therefore, by the properness, of F, the fiber $F^{-1}(y)$ is compact. As F is a local homeomorphism, for every $x \in F^{-1}(y)$ there exists an open neighborhood U_x such that U_x and $F(U_x)$ are homeomorphic. $\{U_x\}_{x \in F^{-1}(y)}$ is an open cover of the compact set $F^{-1}(y)$. It follows that there exists a finite subcover, $\{U_{x_1}, \ldots, U_{x_k}\}$. As each U_x contains exactly one element of $F^{-1}(y)$ since otherwise $F|_{U_x}$ would not be one-to-one, and the set $F^{-1}(y)$ is covered by finitely many of these open sets, the fiber $F^{-1}(y)$ is a finite set. Now, if the sets U_{x_j} are not all pairwise disjoint, there exist open subsets of these sets which are pairwise disjoint since X is Hausdorff. To avoid new notation we may then assume that $U_{x_1}, \ldots, U_{x_k}$ are pairwise disjoint. Let

$$V_y := \bigcap_{j=1}^{k} F(U_{x_k}).$$

Then V_y is an open set as a finite intersection of open sets.

$$F^{-1}(V_y) = \bigsqcup_{j=1}^{k} \big(F^{-1}(V_y) \cap U_{x_j}\big).$$

The union is disjoint as the sets $U_{x_1}, \ldots, U_{x_k}$ are pairwise disjoint. $F|_{U_{x_j}}$ is a homeomorphism, thus its restriction to a subset is a homeomorphism:

$$F|_{F^{-1}(V_y) \cap U_{x_j}} : F^{-1}(V_y) \cap U_{x_j} \to F(F^{-1}(V_y) \cap U_{x_j}) = V_y.$$

Now, to see that $F^{-1}(y)$ is locally constant take any $z \in V_y$. Then z has one and only one preimage in each of the sets $F^{-1}(V_y) \cap U_{x_j}$ and has no other preimages. Therefore $|F^{-1}(z)| = |F^{-1}(y)|$. □

Corollary 3.2.5. *If a map $F : X \to Y$ of topological spaces is a proper local homeomorphism and Y is connected, then $|F^{-1}(y)|$ is constant.*

Definition 3.2.6. *The degree of a covering $F : X \to Y$, is $|F^{-1}(y)|$ for any $y \in Y$.*

3.3 Maps between Riemann surfaces

A Riemann surface is a complex manifold X of $\dim_{\mathbb{C}} X = 1$. The coordinate charts are subsets of $\mathbb{C}$ and the transition functions are holomorphic.

Lemma 3.3.1. *Let $F : X \to Y$ be a non-constant holomorphic map between Riemann surfaces. Let $p \in X$ and let $(U_p, \tilde{U}_p, \psi_p)$ and $(U_{F(p)}, \tilde{U}_{F(p)}, \psi_{F(p)})$ be holomorphic charts centered at p and $F(p)$ respectively, $\tilde{U}_p, \tilde{U}_{F(p)} \subset \mathbb{C}$. Let $f : \tilde{U}_p \to \tilde{U}_{F(p)}$ represent the map F, i.e. $f = \psi_{F(p)} \circ F \circ \psi_p^{-1}$, and assume that $f(\tilde{U}_p) \subset \tilde{U}_{F(p)}$. Let $z \in \tilde{U}_p$. Let*

$$f(z) = \sum_{n=1}^{\infty} a_n z^n$$

be the power series expansion of f is a neighborhood of 0. (f is holomorphic and $f(0) = 0$.) Then

(i)

$$e_p := \min_{a_n \neq 0}(n)$$

is independent of the choice of coordinate charts U_p and $U_{F(p)}$, and

(ii) There exists a neighborhood of p and a coordinate chart centered at p in which F is repre-

sented by:

$$w^{e_p}.$$

Proof. (i) Consider another coordinate chart, with variable w, centered at p, $(V_p, \tilde{V}_p, \phi_p)$. Restricting to a smaller neighborhood of p contained in $U_p \cap V_p$, the holomorphic transition function between a subset of $\tilde{V}_p$ and a subset of $\tilde{U}_p$ can be represented by a power series:

$$z(w) = \sum_{m=1}^{\infty} b_m w^m,$$

with $b_1 \neq 0$, since $\frac{dz}{dw}|_{w=0} \neq 0$.

$$\begin{aligned} f(z(w)) &= \sum_{n=e_p}^{\infty} a_n (z(w))^n \\ &= \sum_{n=e_p}^{\infty} a_n (\sum_{m=1}^{\infty} b_m w^m)^n. \end{aligned}$$

In the last expression, the term of lowest degree (with a non-zero coefficient) is $a_{e_p} b_1^{e_p} w^{e_p}$, as claimed.

(ii) Start from

$$\sum_{n=e_p}^{\infty} a_n z^n = z^{e_p} \sum_{n=0}^{\infty} a_{e_p+n} z^n = z^{e_p} g(z),$$

where $g(z)$ is a holomorphic function such that $g(0) \neq 0$. Then $g(z) \neq 0$ in a neighborhood of zero and there exits a holomorphic function $h(z)$ such that $g(z) = (h(z))^{e_p}$. Let

$$\begin{aligned} H : \tilde{U}_p &\to \mathbb{C} \\ z &\mapsto w := zh(z). \end{aligned}$$

Then

$$\frac{dw}{dz}|_{z=0} = (h(z) + zh'(z))|_{z=0} = h(0) \neq 0,$$

so that there exists a possibly smaller neighborhood of 0, $\tilde{U}$ in which $\frac{dw}{dz} \neq 0$ so that H is invertible and $H \circ \psi_p$ is a homeomorphism. Let $U := \psi_p^{-1}(\tilde{U})$. Then

$$(U, H(\tilde{U}), H \circ \psi_p)$$

is a chart such that F is represented by w^{e_p} since

$$w^{e_p} = (zh(z))^{e_p} = z^{e_p}(h(z))^{e_p} = z^{e_p}g(z) = f(z).$$

□

Definition 3.3.2. *Let $F : X \to Y$ be a non-constant holomorphic map between Riemann surfaces. Let $p \in X$ and let z be a local coordinate centered at p. Let f represent the map F. f is a holomorphic function and in neighborhood of 0 it can be expressed as a convergent power series. Let*

$$f(z) = f(0) + \sum_{n=1}^{\infty} a_n z^n$$

be the power series expansion of f at p. Then the index of p for the map F is

$$e_p := \min_{n \in \mathbb{N}, a_n \neq 0} n.$$

Proposition 3.3.3. *Let $F : X \to Y$ be a non-constant holomorphic map between connected Riemann surfaces. Then:*

(i) $R := \{p \in X \,:\, e_p > 1\}$ *is discrete.*

(iii) If F *is proper* $B := F(\{p \in X \,:\, e_p > 1\})$ *is discrete.*

(iii) If F *is proper then* $F|_{X \setminus F^{-1}(F(R))}$ *is a finite covering map.*

Proof. Let F be represented by f in a chart centered at p.

(i) If $e_p > 1$ then $f'(0) = 0$. As f is a holomorphic function, so is f'. The claim now follows from the fact that the set of zeros of a holomorphic function is discrete.

(ii) This is true without the assumption that the map is between Riemann surfaces. The image of a discrete set under a proper map is discrete. If the image were not discrete then it would have an accumulation point. The accumulation point would have a compact neighborhood containing infinitely many images of elements of the discrete set. The inverse image of that compact neighborhood would be compact and therefore infinitely many elements of a discrete would be contained in a compact set. Contradiction.

(iii) By Lemma 3.3.1, for any $p \notin F^{-1}(F(R))$, since $e_p = 1$, there exists a coordinate chart in which the map F is represented by the identity map. The identity map is clearly a homeomorphism onto its image. Therefore $F|_{X \setminus F^{-1}(F(R))}$ is a local homeomorphism. By Proposition 3.2.4, a proper local homeomorphism is a finite covering map.

□

Let

$$d(q) := \sum_{p \in F^{-1}(q)} e_p.$$

Proposition 3.3.4. *Let* $F : X \to Y$ *be a proper non-constant holomorphic map of connected Riemann surfaces. Then* $d : Y \to \mathbb{N}$ *is a constant map.*

Proof. By Corollary 3.2.5, d is constant on $Y \setminus F(R)$. Let $q \in F(R)$. Let $p \in F^{-1}(q)$. By Lemma 3.3.1 there is a chart centered at p such that F is represented by w^{e_p} in it. The map w^{e_p} maps $\mathbb{C} \to \mathbb{C}$ in such a way that any $0 \neq c \in \mathbb{C}$ has exactly e_p preimages. From the properness of F it follows that q has finitely many preimages, p_m, $1 \leq m \leq n_q$. For each of these preimages, p_m, there is a neighborhood U_m, and a local representation, $w^{e_{pm}}$ of the function F. Let

$$V_q := \bigcap_{1 \leq m \leq n_q} F(U_m).$$

Then $d(q) = d(y)$ for any $y \in V_q$ so d is locally constant and therefore constant since Y is connected.

□

Definition 3.3.5. *Let $F : X \to Y$ be a non-constant, proper, holomorphic map between connected Riemann surfaces. Then*

- $R := \{p \in X : e_p > 1\} \subset X$ *is the set of ramification points of the map F.*
- $B := F(\{p \in X : e_p > 1\})$ *is the set of branch points of the map F.*
- $\deg(F) := d(y)$, *for any $y \in Y$ is the degree of the map F.*

3.4 *n* even

We can now proceed to use the results of subsections 3.2 and 3.3 to understand the underlining structure of Milnor fibers. First we discuss the more straightforward case of n being even (and thus $\frac{1}{2}(n-2)$ being an integer). The results of this section are not new; they represent an application of known results to the specific situation of Milnor fibers.

We introduce the following notation:

- Let $f \in \mathcal{P}_{simple}$ (2.3) and let $n \in \mathbb{N}$. Let

$$q(w) := (f(w))^{\frac{n-2}{2}}. \tag{3.1}$$

- Fix one primitive function of q, Q:

$$q(w) = Q'(w). \tag{3.2}$$

Maps defined by polynomials are proper, holomorphic maps $\mathbb{C} \to \mathbb{C}$.

Lemma 3.4.1. *Let $P \in \mathbb{C}[w]$. Then:*

(i) $P : \mathbb{C} \to \mathbb{C}$ is a ramified covering map of degree equal to the degree of the polynomial $P(w)$.

(ii) The set of ramification points of the map P is $V(P')$. The set of branch points is $P(V(P'))$.

(iii) The ramification index of $w_j \in V(P')$ is equal to the order of w_j as a zero of $P(w) - P(w_j)$.

(iv) P extends to $\widehat{\mathbb{C}} = \mathbb{C} \cup \{p_\infty\}$, the one-point compactification of $\mathbb{C}$ by $P(p_\infty) = p_\infty$. The index of p_∞ is $e_\infty = \deg(P)$.

Proof. (i) The degree of a ramified covering is the size of the fiber over a generic point. Let $C \in \mathbb{C}$ be generic, i.e., $C \in \mathbb{C} \setminus P(V(P'))$. Then the equation

$$P(w) - C = 0$$

has $\deg(P)$ different solutions as

$$(P(w) - C)'|_{w=c} = P'(c) \neq 0.$$

Therefore $|P^{-1}(C)| = \deg(P)$ for all $C \in \mathbb{C} \setminus P(V(P'))$.

Now, to show that

$$P : \mathbb{C} \setminus P^{-1}(P(V(P'))) \to \mathbb{C} \setminus P(V(P'))$$

is a covering map, let $c \in P^{-1}(C)$, for some $C \in \mathbb{C} \setminus P(V(P'))$. Then, as $P'(c) \neq 0$, for every $c \in P^{-1}(C)$ there exists an open neighborhood U_c of c such that P is invertible on U_c. It can be assumed, without loss of generality, that if $c' \in P^{-1}(C)$ and $c' \neq c$ then $U_c \cap U_{c'} = \varnothing$.

Let

$$V := \bigcap_{c \in P^{-1}(C)} P(U_c).$$

Then $P^{-1}(V)$ is a union of disjoint open sets, each homeomorphic to V:

$$P^{-1}(V) = \bigcup_{c \in P^{-1}(C)} (P^{-1}(V) \cap U_c).$$

(ii) Let $w_j \in V(P')$. Then $P'(w_j) = 0$ so there is no neighborhood of w_j in which P is invertible. Therefore, $V(P')$ is the set of ramification points of the map P and $P(V(P'))$ is the set of branch points.

(iii) This is immediate from the definition of the index (Definition 3.3.2).

(iv) Given a covering map $F : E \to X$ of compact spaces the Riemann–Hurwitz formula

relates the genera of the two spaces, $g(E)$ and $g(X)$, the degree of the covering $\deg(F)$, and indices of points in E ($e_p, p \in E$):

$$2 - 2g(E) = \deg(F)(2 - 2g(X)) - \sum_{p \in E}(e_p - 1).$$

In our case both E and X are $\widehat{\mathbb{C}}$, and so $g(E) = g(X) = 0$.

$$2 - 0 = \deg(P)(2 - 0) - \sum_{p \in V(P')} (e_p - 1) - (e_\infty - 1).$$

Now, $e_p - 1$ is the order of p as a zero of P' and therefore

$$\sum_{p \in V(P')} (e_p - 1) = \deg(P') = \deg(P) - 1.$$

Thus,

$$2 = 2\deg(P) - \deg(P) + 1 - e_\infty + 1,$$

and consequently, $e_\infty = \deg(P)$.

□

Corollary 3.4.2. *Let n be even. Let q and Q be as in (3.1) and (3.2). Then:*

(i) Both $q(w)$ and $Q(w)$ are polynomials.

(ii) $Q : \mathbb{C} \to \mathbb{C}$ is a ramified covering map and $\deg(Q) = \deg Q(w)$ (degree of the map is equal to the degree of the polynomial).

(iii) The branch points of the map Q are: $Q(w_j), w_j \in V(f)$.

(iv) Each ramification point $w_j \in V(q) = V(f)$ has index $\frac{n}{2}$.

Proof. (i) Since $n-2$ is even and f is a polynomial,

$$q = f^{\frac{n-2}{2}} = (w-w_1)^{\frac{n-2}{2}} \dots (w-w_k)^{\frac{n-2}{2}}$$

and hence also Q are polynomials.

(ii) and (iii) are immediate.

(iv) Let $w_j \in V(q)$. Then w_j is a zero of multiplicity $\frac{n}{2}$ of the polynomial $Q(w) - Q(w_j)$, since

$$\Big(Q(w) - Q(w_j)\Big)(w_j) = 0 \tag{3.3}$$

and

$$\Big(Q(w) - Q(w_j)\Big)' = q(w) = (w-w_1)^{\frac{n-2}{2}} \dots (w-w_j)^{\frac{n-2}{2}} \dots (w-w_k)^{\frac{n-2}{2}},$$

so that the order of w_j as a solution of the equation $Q(w) = Q(w_j)$ is $\frac{1}{2}(n-2)+1 = \frac{n}{2}$. Therefore $e_{w_j} = \frac{n}{2}$.

There are

$$\deg(Q) - \frac{n}{2} = \deg(f)\frac{n-2}{2} + 1 - \frac{n}{2} = \frac{(\deg(f)-1)(n-2)}{2}$$

points in the fiber of $P(w_j)$ different from w_j.

□

A covering map Q as in Corollary 3.4.2 is illustrated schematically in Figure 3.1. In the

chosen example, $\deg(f) = 3$ and $n = 6$ so that the degree of the covering is

$$\deg(Q) = \frac{1}{2}\deg(f)(n-2) + 1 = \frac{1}{2}3(6-2) + 1 = 7.$$

Zeros of f have index $\frac{1}{2}n = 3$ and the size of the fiber above $Q(w_j)$ is

$$\deg(Q) - \frac{1}{2}n + 1 = 7 - 3 + 1 = 5.$$

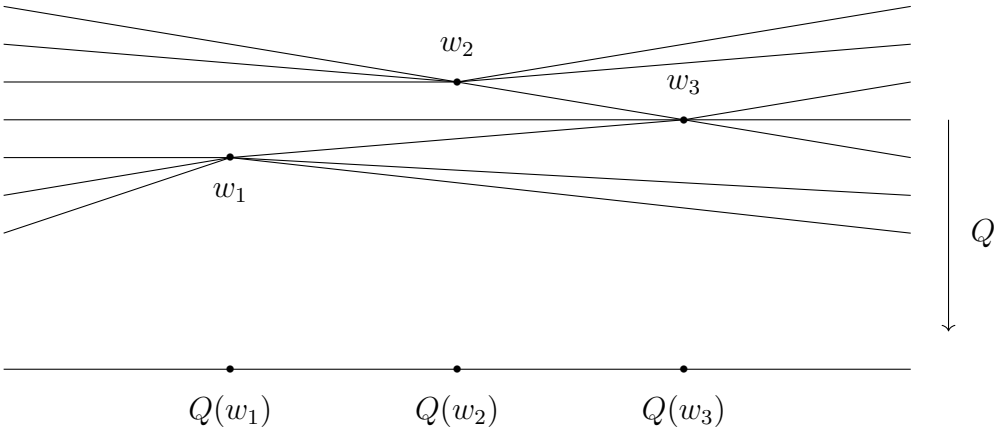


Figure 3.1: Example of the sheets of the covering map Q with $k = 3, n = 6$.

3.5 *n* odd

As n is odd, $\frac{n-2}{2}$ is not an integer and q is not a polynomial. But, as in the case n even, we would like to have a primitive function for q, Q.

Instead of making a choice of a branch of q (3.1), we will consider it as a two-valued function (single-valued at finitely many points): for any w, $w \notin V(f)$, there are two choices for the value $f(w)^{\frac{n-2}{2}}$. Changing the domain of q to a Riemann surface results in q being single-valued. We proceed to show that the Riemann surface needed is X_f^1.

From Definition 2.1.13 of Milnor Fibers with $n = 1$ we have (use z in place of z_1):

$$X_f^1 = \{(w, z) \,:\, w \in \mathbb{C}\,,\; z^2 = f(w)\}.$$

We know that X_f^1 is a smooth manifold (Lemma 2.1.14). X_f^1 is a one-dimensional complex manifold, a Riemann surface. Since it is defined by the polynomial equation $z^2 = f(w)$, where $f \in \mathcal{P}_{simple}$ the result is a *hyperelliptic curve*. The following Lemma states well-known properties of hyperelliptic curves.

Lemma 3.5.1. *(i)* $\pi : X_f^1 \to \mathbb{C}$, $(w, z) \mapsto w$ *is a ramified covering map of degree 2.*

(ii) $I : X_f^1 \to X_f^1$, $(w, z) \mapsto (w, -z)$ *is an automorphism of* X_f^1. I *is an involution with fixed points* $V(f) \times \{0\}$.

(iii) X_f^1 *can be compactified by the addition of one or two points at infinity:* $\widehat{X}_f^1 = X_f^1 \cup \{p_\infty\}$, *if* $\deg(f)$ *is odd,* $\widehat{X}_f^1 = X_f^1 \cup \{p_+, p_-\}$, *if* $\deg(f)$ *is even.*

(iv) π *can be extended to* $\pi : \widehat{X}_f^1 \to \widehat{\mathbb{C}}$ *by* $\pi(p_\infty) = \infty$, *if* $\deg(f)$ *is odd, and* $\pi(p_+) = \pi(p_-) = \infty$, *if* $\deg(f)$ *is even.*

(v) $\widehat{X}_f^1$ *is an orientable compact surface of genus* $\left\lfloor \frac{\deg f - 1}{2} \right\rfloor$.

(vi) I *extends to* $\widehat{X}_f^1$. $I(p_\infty) = I(p_\infty)$, $I(p_+) = p_-$.

Proof. (i) $\pi|_{X_f^1 \setminus V(f)}$ is a covering map: for any (w_0, z_0) with $z_0 \neq 0$ there exists a neighborhood U such that

$$U \cap U' = \varnothing,$$

where $U' := \{(w, -z) \, : \, (w, z) \in U$. Then

$$\pi^{-1}(\pi(U)) = U \cup U'.$$

If $w \notin V(f)$ then $z^2 = f(w) \neq 0$ has two different solutions for z so any $w \in \mathbb{C} \setminus V(f)$ is the image of exactly two different points of X_f^1.

If $w \in V(f)$ then $\pi^{-1}(\{w\}) = \{(w, 0)\}$. Then $(w, 0)$ is a ramification point of index two.

(iii) To form the Riemann Surface X_f^1, Form a simple path passing through the zeros of f: $w_1 w_2 \ldots w_k$. Cut slits along the path, the slits connecting w_{2j-1} and w_{2j}, $j = 1, 2, \ldots \lfloor \frac{k}{2} \rfloor$. If there is an odd number of zeros the last slit should be along a ray with vertex w_k, and not intersecting with the path $w_1 w_2 \ldots w_k$. The two sheets are connected in such a way that the points w_j are identified and the side of a slit in one of the sheets is glued to the opposite side of the slit in the other sheet. After the gluing, a small neighborhood of each w_j looks like the Riemann surface for the square root function.

If $\deg(f)$ is even (see Figure 3.2), after the gluing, outside of a compact set no gluing has occurred and the two sheets are disjoint. They can each be compactified by the addition of one point. We denote these two points by p_+ and p_-.

If $\deg(f)$ is odd (see Figure 3.3), one of the slits along which the two sheets are glued together is a ray. Therefore the two sheets remain connected and the addition of one point, p_∞, compactifies the Riemann surface.

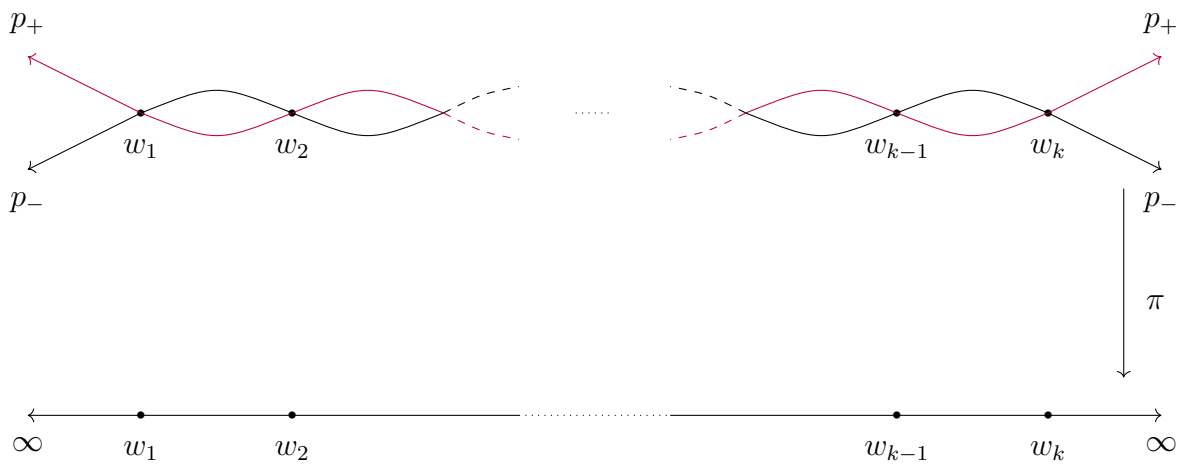


Figure 3.2: $\deg(f)$ even

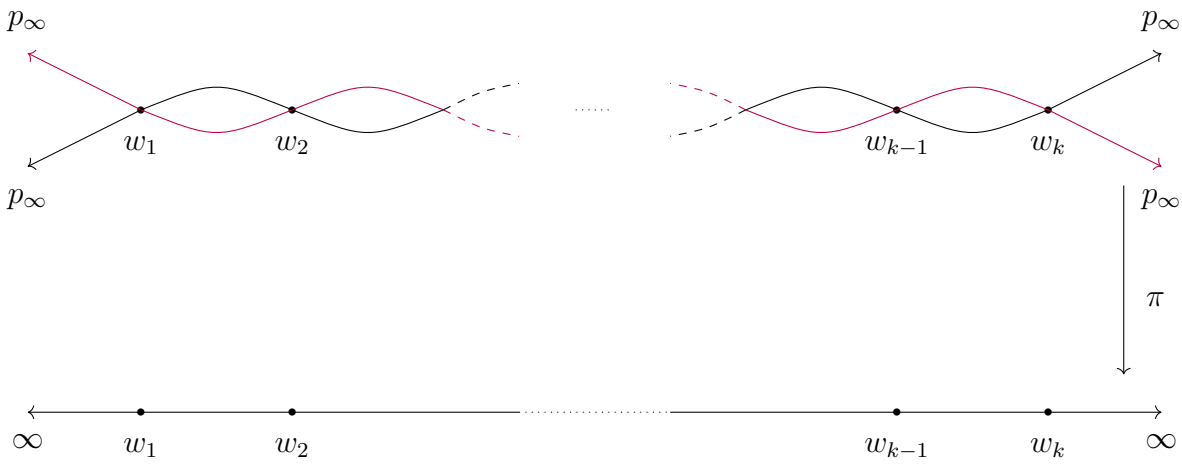


Figure 3.3: $\deg(f)$ odd

(iv) Riemann–Hurwitz formula holds:

$$2 - 2g(\widehat{X}_f^1) = 2(2 - 2g(\widehat{\mathbb{C}})) - \sum_{e_p \in \widehat{X}_f^1} (e_p - 1),$$

so

$$2g(\widehat{X}_f^1) = \sum_{e_p \in \widehat{X}_f^1} (e_p - 1) - 2 = \#\text{ of ramification points } - 2.$$

It follows that the number of ramification points has to be even. If $\deg(f)$ is odd then the ramification set of the extension of the map π is $\pi^{-1}(V(f)) \cup \{p_\infty\}$. If $\deg(f)$ is even, the set of the ramification points is $\pi^{-1}(V(f))$.

Finally, it follows from Riemann–Hurwitz formula that the genus of the compact Riemann surface $\widehat{X}_f^1$ is

$$g(\widehat{X}_f^1) = \Big\lfloor \frac{\deg f - 1}{2} \Big\rfloor.$$

□

If $\deg(f) = 3$ or 4 then $g(\widehat{X}_f^1) = 1$ and $\widehat{X}_f^1$ is an elliptic curve. If $\deg(f) > 4$ then $g(\widehat{X}_f^1) > 1$ and $\widehat{X}_f^1$ is a hyperelliptic curve. [24, Examples 1.7 and 1.8 Hyperelliptic Curves].

In the n even case, $q = f^{\frac{n-2}{2}}$ is a polynomial. In the n odd case, changing the domain of q from $\mathbb{C}$ to X_f^1 results in a (single-valued) function. We will abuse the notation slightly by continuing to use the same name, q:

$$q(w, z) := z^{n-2}.$$

Lemma 3.5.2. *qdw is a holomorphic 1-form on X_f^1.*

Proof. We have shown in Lemma 2.1.16 that

$$\Omega_f^1 = \frac{dw}{2z} = \frac{dz}{f'(w)}$$

is a nowhere vanishing holomorphic 1-form on X_f^1. By Lemma 2.1.16 and

$$qdw = z^{n-2}dw = 2z^{n-1}\frac{dw}{2z} = 2(f(w))^{\frac{n-1}{2}}\Omega_f^1,$$

the claim follows. □

As in the n even case, we want to choose a primitive function for q, Q, such that $dQ = qdw$. Q can be obtained by integrating the 1-form $z^{n-2}dw$ along paths starting from some fixed point in X_f^1. Since $\widehat{X}_f^1$ has non-zero genus, the value of the integral $\int_\gamma qdw$ depends on the homology class of the path γ.

To take this dependence into account, we have the following:

Definition 3.5.3. *Let $w_j, w_l \in V(f)$. Let γ be the straight path between w_j and w_l in $\mathbb{C}$. Let $Y \subset \mathbb{C}$ be a maximal open set such that*

(i) Y contains the interior of γ,

(ii) $V(f) \cap Y = \varnothing$,

(iii) Y is simply connected.

Lemma 3.5.4. *There exist open sets Y_1 and Y_2, $Y_1, Y_2 \subset X_f^1$ such that*

(i) $\pi^{-1}(Y) = Y_1 \cup Y_2$,

(ii) $Y_1 \cap Y_2 = \varnothing$,

(iii) $I(Y_1) = Y_2$,

(iv) Y_1 *and* Y_2 *are simply connected.*

Proof. Y does not contain a loop around any of the zeros of f. Then it is possible to make a continuous choice for $\sqrt{f}$ on all of Y. Therefore there exist two holomorphic functions ϕ_1 and ϕ_2 on Y, such that $(\phi_1(w))^2 = (\phi_2(w))^2 = f(w)$ and $\phi_1(w) = -\phi_2(w)$ for all $w \in \mathbb{C}$.

Let $Y_1 := \{(w, \phi_1(w)) : w \in Y\}$ and $Y_2 := \{(w, \phi_2(w)) : w \in Y\}$.

(i) By definition of Y_1 and Y_2, every $w \in Y$ has two distinct preimages in $Y_1 \cup Y_2$. As π is a ramified covering map od degree two there are no other preimages.

(ii) As $V(f) \cap Y = \varnothing$ it follows that for every $w \in Y$, $\phi_1(w) \neq \phi_2(w)$. Therefore $Y_1 \cap Y_2 = \varnothing$.

(iii) Let $(w, \phi_1(w)) \in Y_1$. Then $I((w, \phi_1(w))) = (w, -\phi_1(w)) = (w, \phi_2(w)) \in Y_2$ and $I((w, \phi_2(w))) = (w, -\phi_2(w)) = (w, \phi_1(w))$. Thus $I(Y_1) = Y_2$ and $I(Y_2) = Y_1$.

(iv) $\pi|_{Y_1}$ and $\pi|_{Y_2}$ are homeomorphisms and $\pi(Y_1) = \pi(Y_2) = Y$ As Y is simply connected so are Y_1 and Y_2.

□

Definition 3.5.5. *Let* $Q_1 : Y \to \mathbb{C}$ *and* $Q_2 : Y \to \mathbb{C}$ *be given by*

$$Q_1(w) := \int_\alpha q dw, \quad Q_2(w) := \int_\beta q dw, \tag{3.4}$$

where $\alpha \subset Y_1$ *is any path from* $(w_j, 0)$ *to* (w, z) *in* Y_1 *and* $\beta \subset Y_2$ *is any path from* $(w_j, 0)$ *to* $(w, -z)$ *in* Y_2.

Lemma 3.5.6.

$$Q_1(w) = -Q_2(w)$$

for all $w \in Y$.

Proof. Let α be a path in Y_1 such that $Q_1(w) = \int_\alpha q dw$. Let $\beta := I(\alpha)$. For any point $p \in \alpha$,

$$(qdw)|_p = -(qdw)|_{I(p)}.$$

Therefore

$$Q_1(w) = \int_\alpha qdw = -\int_\beta qdw = -Q_2(w).$$

□

Lemma 3.5.7. *(i)* Q_1 *and* Q_2 *can be extended to* $\mathrm{cl}(Y)$.

(ii) $Q_1(w_m) = -Q_2(w_m), w_m \in V(f)$.

(iii) $I(\widehat{X}_f^1 \setminus (Y_1 \cup Y_2)) = \widehat{X}_f^1 \setminus (Y_1 \cup Y_2)$.

Proof. (i) The integral defining Q_1 does not depend on the path as long as the path is contained in Y_1. $Q_1(p) = \lim_{q \to p, q \in Y_1} Q_1(q)$.

(ii) By Lemma 3.5.6, $Q_1(w) = -Q_2(w)$, for all $w \in Y$. This property extends to $\mathrm{cl}(Y)$ and $V(f) \subset \mathrm{cl}(Y)$.

(iii) Since $I(Y_1) = Y_2$ and $I(Y_2) = Y_1$ it follows that $I(Y_1 \cup Y_2) = Y_1 \cup Y_2$. I is a bijection therefore also $I((Y_1 \cup Y_2)^c = (Y_1 \cup Y_2)^c$.

□

We can use the fact that $\left(\frac{f}{w-w_j}\right)^{\frac{n-2}{2}}$ is holomorphic in a neighborhood of w_j in X_f^1, and thus has a convergent power series expansion, to find expressions for Q_1 and Q_2 near w_j.

$$\pm\left(\frac{f(w)}{w-w_j}\right)^{\frac{n-2}{2}} = \pm\left(\sqrt{(w-w_1)\dots\widehat{(w-w_j)}\dots(w-w_k)}\right)^{n-2}$$

$$\pm\Big(\frac{f(w)}{w-w_1}\Big)^{\frac{n-2}{2}} = \pm\sum_{l=0}^{\infty} a_l(w-w_1)^l\,, \quad w\in U.$$

Next we express q,

$$\pm q(w) = \pm\Big(f(w)\Big)^{\frac{n-2}{2}} = \pm\Big(\sqrt{w-w_j}\Big)^{n-2}\Big(a_0+a_1(w-w_j)+a_2(w-w_j)^2+\dots\Big).$$

We can integrate term by term to find a primitive function:

$$\pm\int_{w_j}^{w} q\,dw = \pm\left(\frac{2}{n}a_0(w-w_j)^{\frac{n}{2}} + \frac{2}{n+2}a_1(w-w_j)^{\frac{n+2}{2}} + \frac{2}{n+4}a_2(w-w_j)^{\frac{n+4}{2}}+\dots\right). \tag{3.5}$$

The last expression represents $Q_1(w,z)$ and $Q_2(w,-z)$.

Definition 3.5.8. *Let $Q: Y_1\cup Y_2 \to Q(Y_1\cup Y_2)\subset\mathbb{C}$ be given by:*

$$Q(w,z) = \begin{cases} Q_1(w), & (w,z)\in Y_1, \\ \\ Q_2(w), & (w,z)\in Y_2. \end{cases} \tag{3.6}$$

Lemma 3.5.9. *Q is a covering map of degree* $\deg(Q) = (n-2)\deg(f)+2$.

Proof. First we need to add points at infinity to Y_1 and Y_2. Recall that completing X_f^1 to $\widehat{X}_f^1$ re-

quires two points when $\deg f$ is even and one point when $\deg f$ is odd (Lemma 3.5.1). Nevertheless, since $Y_1 \cap Y_2 = \varnothing$, we add one point at infinity to each whether $\deg f$ is even or odd. Let the two added points be p_1^∞ and p_2^∞. Now take any compact subset $K \subset (Y_1 \cup \{p_1^\infty\}) \cup (Y_2 \cup \{p_2^\infty\})$, $p_1^\infty, p_2^\infty \in K$. We can use the Riemann–Hurwitz formula for the map $Q|_K$, with genus 0 for both the domain and the codomain.

$$-2 = N(-2) + (e_1^\infty - 1) + (e_2^\infty - 1)$$

In Lemma 3.12.4 we find that the index of the points at infinity is $(n-2)\deg f + 2$. Then it follows that the degree of the covering is $N = (n-2)\deg f + 2$. As $(Y_1 \cup \{p_1^\infty\}) \cup (Y_2 \cup \{p_2^\infty\})$ is exhausted by such compact sets K, we conclude that $\deg Q = (n-2)\deg f + 2$. □

3.6 Pull-back metric with conical singularities

We begin with a discussion on Riemann surfaces with conical singularities. Then we show, in Theorem 3.6.4, that the pull-back under the map Q of the Euclidean metric is a singular metric on $\mathbb{C}$ which has conical singularities.

3.6.1 Conical singularity

The following definition follows Troyanov [25, p. 796]:

Definition 3.6.1. *(Conical singularity - first definition) Let Σ be a Riemann surface equipped with a metric which has a discrete set of singularities. A point $p \in \Sigma$ is a conical singularity of cone angle $2\pi\beta$ (or, of order $\beta - 1$) for some $\beta \in (0,1) \cup (1,\infty)$, if in a neighborhood of p the*

metric can be expressed as

$$\rho(w)|w|^{2(\beta-1)}|dw|^2 \tag{3.7}$$

where w is a holomorphic coordinate function such that the point p is defined by $w = 0$, and $\rho(w)$ is a positive continuous function.

An equivalent definition following Mazzeo, Rubinstein, Sesum [26, Section 2.2]:

Definition 3.6.2. *(Conical singularity - second definition) Let Σ be a Riemann surface equipped with a metric which has a discrete set of singularities. A point $p \in \Sigma$ is a conical singularity of cone angle $2\pi\beta$ (or of order $\beta - 1$) for some $\beta \in (0,1) \cup (1,\infty)$, if in a neighborhood of p the metric can be expressed as*

$$\rho(r,\theta)(dr^2 + r^2\beta^2 d\theta^2), \tag{3.8}$$

where $\rho(r,\theta)$ is a continuous positive function, $d\theta^2$ is a metric on S^1, and $r \geq 0$ is a radial variable, ($r = 0$ defines p).

As discussed in [26, Section 2.2], it is not difficult to verify the equivalence of the two definitions. Given the first definition, introduce polar coordinates:

$$w = Re^{i\theta}.$$

Then,

$$dw = (dR + Rid\theta)e^{i\theta},$$

and

$$|dw|^2 = dR^2 + R^2 d\theta^2$$

The expression for metric becomes

$$\rho(w)|w|^{2(\beta-1)}|dw|^2 = \rho(w(R,\theta))R^{2(\beta-1)}(dR^2 + R^2 d\theta^2).$$

Now, let

$$r = \frac{1}{\beta}R^{\beta}.$$

Then

$$dr = R^{\beta-1}dR,$$

and

$$R^{2(\beta-1)}(dR^2 + R^2 d\theta^2) = dr^2 + \beta^2 r^2 dr^2.$$

After substituting we obtain the expression for metric from the second definition (3.8):

$$\rho(r,\theta)(dr^2 + \beta^2 r^2 d\theta^2).$$

Reversing the procedure, we get the equivalence of the two definitions.

3.6.2 Singular metric

Lemma 3.6.3. *Let $f(w) \in \mathcal{P}_{simple}$, let $n \in \mathbb{N}$, $n > 2$, and let $q(w)^2 = (f(w))^{n-2}$. Let $|dw|^2$ be the Euclidean metric on $\mathbb{C}$. Then $|q(w)|^2|dw|^2$ is a singular metric on $\mathbb{C}$ whose singular set is $V(f)$. At each singular point, the metric $|q(w)|^2|dw|^2$ has a conical singularity, with cone angle of $n\pi$. If $n = 2$ the metric $|q(w)|^2|dw|^2$ is not singular.*

Proof. On the set $\mathbb{C} \setminus V(f)$, the function $|q(w)|^2$ is positive. Thus $|q(w)|^2|dw|^2$ is a metric on

$\mathbb{C} \setminus V(f)$, conformal with the Euclidean metric. Since $|q(w)|^2$ vanishes on $V(f)$, $V(f)$ is the singular set for the metric.

Near $w_j \in V(f)$,

$$|q(w)|^2 = \prod_{m=1}^{k} |w - w_m|^{n-2} = D_j(w)|w - w_j|^{n-2},$$

where

$$D_j(w) = \prod_{m \neq j} |w - w_m|^{n-2},$$

is a continuous positive function and

$$\lim_{w \to w_j} D_j(w) = D_j(w_j) > 0.$$

Therefore,

$$|q(w)|^2 |dw|^2 = D_j(w)|w - w_j|^{n-2}|dw|^2 = D_j(w)|w - w_j|^{2(\frac{n}{2}-1)}|dw|^2.$$

Comparing the last expression with the equation (3.7) in Definition 3.6.1, we can see that there is a conical singularity at w_j with cone angle $2\pi\frac{n}{2} = n\pi$.

If $n = 2$ then $\frac{n-2}{2} = 0$ so $q = f^{\frac{n-2}{2}} = 1$. The metric $|q(w)|^2|dw|^2$ is the Euclidean metric $|dw|^2$. Note that the cone angle at $w_j \in V(f)$ is 2π in this case and there is no singularity. □

Theorem 3.6.4. *The singular metric of Lemma 3.6.3 is the pull-back by Q (3.1) of the Euclidean metric, g_{Euc}, on $\mathbb{C}$, if n is even. If n is odd, it is the pull-back by Q_1 or Q_2 (3.5.5), for some choice of $Y \subset \mathbb{C} \setminus V(f)$ (3.5.3). (When $n = 2$, there is no singularity and Q can be chosen to be the*

identity map.)

Proof. For any point $w \notin V(f)$, there exists a neighborhood U on which Q is one-to-one. Consequently, the pull back of the Euclidean metric on $Q(U)$ to U is a metric. From

$$Q(w) = z$$

we have

$$dz = q(w)dw.$$

Since $g_{\text{Euc}} = |dz|^2$, the pull-back metric in a neighborhood of w with $w \notin V(f)$, is

$$Q^*(g|_{Q(U)}) = |q(w)|^2|dw|^2,$$

which is a metric on $\mathbb{C} \setminus V(f)$ which is conformal to the Euclidean metric.

Similarly, for n odd,

$$z = Q_1(w) \quad \text{or} \quad z = Q_2(w)$$

$$dz = q(w)dw \quad \text{or} \quad dz = -q(w)dw.$$

And again

$$|dz|^2 = |q(w)|^2|dw|^2.$$

The domain of Q_1 and Q_2, Y, is a proper subset of $\mathbb{C} \setminus V(f)$ so if only one choice of Y is considered the pull-back metric is defined on Y only. But, for any point p in $\mathbb{C} \setminus V(f)$ there exists a set Y, maximal simply connected subset of $\mathbb{C} \setminus V(f)$, such that $p \in Y$.

□

3.7 Existence of geodesics between cone points

Recall that L_γ (Definition 2.1.19) is special if and only if

$$\theta_\gamma(u) = \text{const.}$$

The main results of this section are Theorem 3.8.3, in which we prove that a path γ gives rise to an SL if and only if γ is contained in a geodesic of the pull-back metric, $|q(w)|^2 dw^2$, and Theorem 3.8.5 where we give an explicit equation for such a curve.

The starting point is the result of Lemma 2.1.23, due to Thomas–Yau [3], where we established that the Lagrangian angle of a Lag sphere L_γ, as a function of the parameter u of the path γ, is

$$\theta_\gamma(u) = \arg\left[\gamma'(u)\big(f(\gamma(u))\big)^{\frac{n-2}{2}}\right].$$

In the following lemma we prove the chain rule for the composition of a complex valued function of a real variable and a holomorphic function. We include it because it is not usually covered in complex analysis texts.

Lemma 3.7.1. *Let $Q : \mathbb{C} \to \mathbb{C}$ be a holomorphic function, and let $q = Q'$. Let $\gamma : (a, b) \subset \mathbb{R} \to \mathbb{C}$, $u \mapsto \gamma(u)$, be a C^1 curve in the complex plane. Then*

$$\frac{d}{du} Q(\gamma(u)) = \gamma'(u) q(\gamma(u)).$$

Proof. Let $Q = Q(w)$ and $w = x + iy$. Along the curve γ, x and y are functions of the parameter u, $x = x(u)$ and $y = y(u)$. Let

$$\dot{x} := \frac{dx}{du} \quad \text{and} \quad \dot{y} := \frac{dy}{du}.$$

Then, using Cauchy–Riemann equations for the holomorphic function Q,

$$\begin{aligned} q = \frac{\partial Q}{\partial w} &= \frac{1}{2}\left(\frac{\partial Q}{\partial x} + \frac{1}{i}\frac{\partial Q}{\partial y}\right) \\ &= \frac{1}{2}\left(\frac{\partial \mathrm{Re}(Q)}{\partial x} + i\frac{\partial \mathrm{Im}(Q)}{\partial x} + \frac{1}{i}\left(\frac{\partial \mathrm{Re}(Q)}{\partial y} + i\frac{\partial \mathrm{Im}(Q)}{\partial y}\right)\right) \\ &= \frac{1}{2}\left(\frac{\partial \mathrm{Re}(Q)}{\partial x} + i\frac{\partial \mathrm{Im}(Q)}{\partial x} + \frac{1}{i}\left(-\frac{\partial \mathrm{Im}(Q)}{\partial x} + i\frac{\partial \mathrm{Re}(Q)}{\partial x}\right)\right) \\ &= \frac{\partial \mathrm{Re}(Q)}{\partial x} + i\frac{\partial \mathrm{Im}(Q)}{\partial x}. \end{aligned}$$

The following calculation proves the claim:

$$\begin{aligned} q\gamma' &= \left(\frac{\partial \mathrm{Re}(Q)}{\partial x} + i\frac{\partial \mathrm{Im}(Q)}{\partial x}\right)(\dot{x} + i\dot{y}) \\ &= \left(\frac{\partial \mathrm{Re}(Q)}{\partial x}\dot{x} - \frac{\partial \mathrm{Im}(Q)}{\partial x}\dot{y}\right) + i\left(\frac{\partial \mathrm{Im}(Q)}{\partial x}\dot{x} + \frac{\partial \mathrm{Re}(Q)}{\partial x}\dot{y}\right) \\ &= \left(\frac{\partial \mathrm{Re}(Q)}{\partial x}\dot{x} + \frac{\partial \mathrm{Re}(Q)}{\partial y}\dot{y}\right) + i\left(\frac{\partial \mathrm{Im}(Q)}{\partial x}\dot{x} + \frac{\partial \mathrm{Im}(Q)}{\partial y}\dot{y}\right) \\ &= \frac{d}{du}\mathrm{Re}(Q) + i\frac{d}{du}\mathrm{Im}(Q) = \frac{d}{du}Q. \end{aligned}$$

□

3.8 Paths giving rise to special Lagrangians

We now arrive to the characterization of paths γ between zeros of the polynomial f with the property that the corresponding Lag, L_γ is an SL.

Recall that Q is a primitive function of $q = f^{\frac{n-2}{2}}$ (3.1), and $f \in \mathcal{P}_{simple}$ (2.3). To simplify the notation, in this section we use Q for both even and odd values of n, i.e., we use Q in place of Q_1 or Q_2, for n odd. When n is odd there is a practical problem of establishing a suitable choice of the domain Y (3.5.3). Everything else regarding the procedure is the same whether n is odd or even. Having to make a choice for Y in the case of n odd really means that we may have to make several different choices for Y in order to find all the paths γ for which L_γ is special. In our Example 4.1 we are able to use only one such set Y for all the paths.

In Lemma 3.8.1 we show that zeros of f are the only possible singular points of $Q^{-1}(\gamma)$. In Lemma 3.8.2 we prove a formula relating the rate of change along γ of θ_γ and the curvature of $Q(\gamma)$. In Theorem 3.8.3 we conclude that L_γ is special if and only if γ is geodesic in the pull-back metric. Finally, in Theorem 3.8.5 we give an explicit equation which defines a curve giving rise to a Lag of constant phase. The explicit equation enables us to create figures for examples in section 4.

Lemma 3.8.1. *Let $\Gamma \subset \mathbb{C}$ be a simple C^1 curve. Then*

$$Q^{-1}(\Gamma)$$

is a C^1 curve except possibly at one or more zeros of f.

Proof. If $\gamma = \gamma(u)$ is a preimage of Γ, as (Lemma 3.7.1)

$$\frac{d}{du}Q(\gamma(u)) = q(\gamma(u))\gamma'(u),$$

and as q is smooth, the tangent vector to $\Gamma(u)$ on the left, and the tangent vector to $\gamma(u)$, $\gamma'(u)$, are of the same class (at least C^1) as long as $q(\gamma(u)) \neq 0$. Since the zeros of q are the same as the zeros of f, the statement of the Lemma follows. □

As observed before, the function θ_γ is constant on the fiber of L_γ above any point of the path γ. Therefore we think of the function θ_γ as a function of one variable only, the variable parametrizing the path γ, i.e., $\theta_\gamma = \theta_\gamma(u)$. In the next Lemma we find the relationship between the rate of change, $\frac{d}{du}\theta_\gamma(u)$ (Definition 2.1.10) and the curvature of $Q(\gamma(u))$.

Lemma 3.8.2. *The rate of change along γ of the phase θ_γ of L_γ can be expressed as:*

$$\frac{d}{du}\theta_\gamma(u) = \kappa(Q(\gamma(u)))|q(\gamma(u))||\gamma'(u)|.$$

Proof. $Q(\gamma)$ can be parametrized by the same variable, u, as γ. It can also be parametrized by its arclength, s. The curvature of the curve $Q(\gamma)$, parametrized by its arclength, s, is:

$$\kappa(Q(\gamma)) = \frac{d}{ds}\arg\left[\frac{d}{ds}Q(\gamma)\right].$$

From Lemma 2.1.23, we have

$$\theta_\gamma(u) = \arg\left[\gamma'(u)q(\gamma(u))\right].$$

Then

$$\frac{d}{du}\theta_\gamma = \frac{d}{du}\arg\big[\gamma' q\big] = \frac{d}{ds}\arg\big[\gamma' q\big]\frac{ds}{du} = \frac{d}{ds}\arg\Big[\frac{d}{du}Q\Big]\frac{ds}{du}. \tag{3.9}$$

Using the fact that $\frac{ds}{du}$ is real and positive, we have,

$$\frac{d}{ds}\arg\Big[\frac{d}{du}Q\Big]\frac{ds}{du} = \frac{d}{ds}\arg\left[(\frac{d}{ds}(Q))(\frac{ds}{du})\right]\frac{ds}{du} = \frac{d}{ds}\arg\left[\frac{d}{ds}Q\right]\frac{ds}{du} = \kappa(Q)\frac{ds}{du}. \tag{3.10}$$

Furthermore, note that the arc length of $Q(\gamma)$ is given by:

$$s = \int_0^u \sqrt{\left(\frac{d}{du}\Big(\mathrm{Re}\big(Q(\gamma)\big)\Big)\right)^2 + \left(\frac{d}{du}\Big(\mathrm{Im}\big(Q(\gamma)\big)\Big)\right)^2}\,du$$

i.e.

$$\frac{ds}{du} = \sqrt{\left(\frac{d}{du}\Big(\mathrm{Re}\big(Q(\gamma)\big)\Big)\right)^2 + \left(\frac{d}{du}\Big(\mathrm{Im}\big(Q(\gamma)\big)\Big)\right)^2},$$

and therefore,

$$(\frac{ds}{du})^2 = \left(\frac{d}{du}\Big(\mathrm{Re}\big(Q(\gamma)\big)\Big)\right)^2 + \left(\frac{d}{du}\Big(\mathrm{Im}\big(Q(\gamma)\big)\Big)\right)^2 \tag{3.11}$$

For $Q(\gamma)$ we have:

$$\begin{aligned}
Q(\gamma(u)) &= \mathrm{Re}(Q(\gamma(u))) + i\mathrm{Im}(Q(\gamma(u))) \\
\frac{d}{du}Q(\gamma(u)) &= \frac{d}{du}\mathrm{Re}(Q(\gamma(u))) + i\frac{d}{du}\mathrm{Im}(Q(\gamma(u))) \\
Q'(\gamma(u))\gamma' &= \frac{d}{du}\mathrm{Re}(Q(\gamma(u))) + i\frac{d}{du}\mathrm{Im}(Q(\gamma(u))),
\end{aligned}$$

Combining the last equality with (3.11) we obtain,

$$(\frac{ds}{du})^2 = (Q'(\gamma(u))\gamma')\overline{(Q'(\gamma(u))\gamma')} = |q(\gamma)|^2|\gamma'|^2. \tag{3.12}$$

From (3.9), (3.10), and (3.12),

$$\frac{d\theta_\gamma}{du} = \kappa(Q)\frac{ds}{du} = \kappa(Q(\gamma(u)))|q(\gamma(u))||\gamma'(u)|,$$

as claimed.

□

Theorem 3.8.3. *Let $f \in \mathcal{P}_{simple}$, let $n \in \mathbb{N}$, $n > 2$, and let $|dw|^2$ be the Euclidean metric on $\mathbb{C}$. Let γ be a geodesic path in $\mathbb{C}$ with the pull-back metric $|q(w)|^2|dw|^2$, and $\overline{\gamma}$ its closure. Then $L_{\overline{\gamma}}$ is an SL.*

Proof. By definition, L_γ is special iff $\frac{d}{du}\theta_\gamma(u) \equiv 0$. We showed in Lemma 3.8.2 that

$$\frac{d}{du}\theta_\gamma(u) = \kappa(Q(\gamma(u))|q(\gamma(u))||\gamma'(u)|.$$

Condition $\kappa(Q(\gamma(u)) = 0$ means that $Q(\gamma)$ is a straight line segment in the complex plane, i.e., $Q(\gamma)$ is contained in a geodesic line of the Euclidean metric and therefore γ is contained in a geodesic of the pull-back metric. □

In other words, if γ is a simple C^1-path in $\mathbb{C}$ connecting two distinct zeros of f then the corresponding Lag L_γ is special if and only if γ is contained in a geodesic of the pull-back metric of Lemma 3.6.4.

Corollary 3.8.4. *If L_γ is an SL, i.e., $\theta_\gamma = \text{const.}$, then the slope of the straight line containing $Q(\gamma)$ is*

$$\tan\theta_\gamma.$$

Proof. By Lemma 3.7.1, the tangent vector to the curve $Q(\gamma(u))$ is

$$\frac{d}{du}(Q(\gamma(u)) = q(\gamma(u))\gamma'(u),$$

and by Lemma 2.1.23

$$\theta_\gamma = \arg[q(\gamma(u))\gamma'(u)].$$

When $Q(\gamma)$ is contained in a straight line, the slope of that straight line is the tangent of the argument of the tangent vector to $Q(\gamma)$, which of course has the same direction as the line.

□

The following theorem gives a constructive way of finding an SL corresponding to a path between two given zeros of f.

Theorem 3.8.5. *Let $w_j, w_\ell \in V(f)$ and $j \neq \ell$. Assume $Q(w_j) \neq Q(w_\ell)$. Then*

$$Q^{-1}\Big(\{(1-u)Q(w_j) + uQ(w_l) \ : \ u \in \mathbb{R}\}\Big). \tag{3.13}$$

is a curve passing through the points w_j and w_ℓ. If there exists a branch of the curve passing through both w_j and w_ℓ then the path from w_j to w_ℓ

$$\gamma \subseteq \bigcup_{u\in[0,1]} Q^{-1}\Big((1-u)Q(w_j) + uQ(w_l)\Big).$$

defines an SL sphere L_γ.

Proof. (3.13) is the preimage of the straight line through the points $Q(w_j)$ and $Q(w_\ell)$. If there exists a path γ between w_j and w_ℓ such that L_γ is an SL then graph of γ is contained in the graph of the curve defined by (3.13). Conversely, if a branch of the curve connects w_j and w_ℓ then by Theorem 3.8.3 the path defines an SL sphere.

□

In Theorem 3.8.5 we assumed that $Q(w_j) \neq Q(w_\ell)$ but that situation can occur. We will now consider it.

Theorem 3.8.6. *Let $w_j, w_\ell \in V(f)$ and $j \neq \ell$. Assume that $Q(w_j) = Q(w_\ell)$. Then there does not exist a path from w_j to w_ℓ giving rise to an SL sphere.*

Proof. $Q(w_j) \equiv Q(w_\ell)$. Let ℓ be any straight line through that point. Then $Q^{-1}(\ell)$ is a curve consisting of several branches and $w_j, w_\ell \in Q^{-1}(\ell)$. Suppose that β is a branch of $Q^{-1}(\ell)$ such that $w_j, w_\ell \in \beta$.

Claim: There is no path γ, contained in β connecting w_j and w_ℓ, such that L_γ is an SL sphere.

Proof of Claim: Suppose that there is such a path, γ, connecting w_j and w_ℓ, $\gamma \subset \beta$. Then its image under Q is a path along the straight line ℓ from $Q(w_j)$ to a point $p \in \ell$ and back (to $Q(w_\ell)$). Let γ be parametrized by arclength, u. By Lemma 3.7.1, a tangent to the line ℓ is

$$\frac{d}{du}Q(\gamma(u)) = q(\gamma(u))\gamma'(u).$$

Since ℓ is a straight line, the direction of the tangent vector is constant, i.e.,

$$\arg\left(\frac{d}{du}Q(\gamma(u))\right) = \text{const.}$$

from $Q(w_j)$ to p. At the point p there is a discontinuity of the tangent vector to $Q(\gamma)$ since coming back from p to $Q(w_\ell)$ the direction is the opposite of the direction of the tangent vector when moving along $Q(\gamma)$ from $Q(w_j)$ to p. By Lemma 3.8.1, if a point on $Q^{-1}(\ell)$ is singular then it is a zero of $f(w)$ so $p = Q(w_s)$ for some $w_s \neq w_j, w_\ell$, $f(w_s) = 0$ and $w_s \in \gamma$. Then the path γ from w_j to w_ℓ contains a third zero of $f(w)$, w_s, in its interior. Therefore, L_γ, the Lag generated by γ, is a union of two or more SL spheres.

□

3.9 Connection to Physics

The practical approach to constructing paths which give rise to SLs in this chapter turns out to be substantially the same in dimension $n = 3$ as the construction in the physics article of Shapere–Vafa [1, Sections 4,5, 6].

The origin of the method of Shapere–Vafa is [14]. The derivation is specific for the physics notions and corresponding dimension. The polynomial which is central in the physics construction is the characteristic polynomial of the Higgs field.

Shapere–Vafa provide a proof of uniqueness of paths [1, pp 11-13]. They do not provide a proof for solving the equation (3.27). The formulation is of the problem is also informal and perhaps misleading as x is a complex variable. The solution which they give as $x = (\frac{3}{2}\alpha t)^{\frac{2}{3}}$ does not explain integral curves shown in [1, Figure 1].

3.9.1 Summary of Shapere-Vafa

The purpose of the article is to provide an answer to the question: What is the number of BPS states for slight deformations of conformal theories in dimensions greater than two away from the conformal point.

• In section 2. they describe the kind of singular Calabi–Yau manifold with an isolated singularity which is of interest in physics. It is a hypersurface of $\mathbb{C}^{d+1}$ defined by $W(x_i) = 0$, where W is a quasihomogeneous function of x_i, meaning [1, Equation (2.1)]:

$$W(\lambda^{q_i} x_i) = \lambda W(x_i). \tag{3.14}$$

By $W(x_i)$ they seem to mean $W(x_1, \ldots, x_{d+1})$ and by (3.14) they mean

$$W(\lambda^{q_1} x_1, \ldots, \lambda^{q_{d+1}} x_{d+1}) = \lambda W(x_1, \ldots, x_{d+1}).$$

• They define the isolated singularity by:

$$dW = 0 \quad \text{iff} \quad x_i = 0.$$

We understand this as saying that the single isolated singularity is assumed to be the point $(0, 0, \ldots, 0)$. Elsewhere, $dW \neq 0$.

• They say that the function W can be viewed as an LG (Landau-Ginzburg) superpotential of an $\mathcal{N} = 2$ theory in two dimensions. This theory would flow to a superconformal theory with

central charge:

$$\hat{c} = \sum_{i=1}^{d+1}(1 - 2q_i) = d + 1 - 2\sum_{i=1}^{d+1} q_i. \tag{3.15}$$

It is not explained here what the flow is but a reference is given [27].

• They claim that, according to [28], for the singularity to be of physical interest [1, Equation (2.3)]:

$$\hat{c} < d - 1. \tag{3.16}$$

Clearly from (3.15) and (3.16) it follows that

$$\sum_{i=1}^{d+1} q_i > 1 \quad \text{iff} \quad \hat{c} < d - 1.$$

• The singularity ring of W is defined near the top of page 3 [1, p. 3] as

$$\mathcal{R} := \mathbb{C}[x_i]/dW. \tag{3.17}$$

The meaning of the definition is that $\mathcal{R}$ is generated by monomials $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and that the ring of polynomials with complex coefficients in $n + 1$ variables is taken modulo the ideal generated by $\frac{\partial W}{\partial x_1}, \dots, \frac{\partial W}{\partial x_{d+1}}$.

• The charge of x^α is defined as

$$Q_\alpha := \sum_{i=1}^{d+1} q_i \alpha_i. \tag{3.18}$$

Now they go back and restrict the set of generators of the ring $\mathcal{R}$ to those monomials x^α for which

$$0 \leq Q_\alpha \leq \hat{c}.$$

- Two equations follow for which the reference [29] is cited:

1. The degeneracy of the ring elements is given by the function:

$$\sum_{x^\alpha \in \mathcal{R}} t^{Q_\alpha} = \prod_{i=1}^{d+1} \frac{1 - t^{1-q_i}}{1 - t^{q_i}}.$$

2. The dimension of the ring $\mathcal{R}$ is

$$N = \dim \mathcal{R} = \prod_{i=1}^{d+1} \frac{1 - q_i}{q_i}.$$

- Near the top of page 4 [1, p. 4] a holomorphic d-form is defined:

$$\Omega := \frac{\prod_{i=1} d + 1}{dW}.$$

This is taken to mean that for a choice of i, $\frac{dx_i}{dW} := \frac{1}{\frac{\partial W}{\partial x_i}}$. Say the chosen variable is x_{d+1} for the convenience of notation, then

$$\Omega = \frac{\prod_{i=1}^{d} dx_i}{\frac{\partial W}{\partial x_{d+1}}}. \tag{3.19}$$

- Expressed thus, Ω is very similar to the form Ω_f^n (Definition 2.1.15). Recall that in our

setting, the manifold X_f^n is the hypersurface in $\mathbb{C}^{n+1}$ defined by

$$f(w) - z_1^2 - \cdots - z_n^2 = 0.$$

Or, in the notation od Shapere–Vafa ($n = d$, $w = x_{d+1}$, $z_j = x_j$),

$$W = f(x_{d+1}) - x_1^2 - \ldots x_d^2 = 0.$$

Clearly, $\frac{\partial W}{\partial x_{d+1}} = f'(x_{d+1})$. So, in Shapere–Vafa notation, Ω_f^d is

$$\frac{dx_1 \wedge \cdots \wedge dx_n}{f'(x_{d+1})} = \frac{dx_1 \wedge \cdots \wedge dx_n}{\frac{\partial W}{\partial x_{d+1}}}. \tag{3.20}$$

(3.20) and (3.19) differ only in the wedge product vs product. It may be that Shapere-Vafa intend the product to mean wedge product as they refer to Ω as a holomorphic d-form.

• On the same page [1, p. 4]) they say that a Dd-brane wrapped around a d-dimensional supersymmetric cycle C gives a particle with BPS mass

$$M := \int_C \Omega. \tag{3.21}$$

• Now, we already understand Shapere–Vafa's form Ω to be analogous to our holomorphic form Ω_f^d. The d-dimensional supersymmetric cycle C is analogous to a Lagrangian sphere which is the central topic of this thesis (Definition 2.1.19). This is not obvious at this point in the article but it becomes clear in section 4.

• We have shown that there is a simple relationship between $\int_{L_\gamma} \Omega_f^n | L_\gamma$ and $\int_\gamma q d\gamma$ (Lemma

2.2.1). In the case of dimension three, it is

$$\int_{L_\gamma} \Omega_f^3|_{L_\gamma} = 2\pi \int_0^1 \sqrt{f(\gamma(t))}\gamma'(t)dt, \tag{3.22}$$

where $\gamma : [0,1] \to \mathbb{C}$. Here, in the physics setting, the left hand side is mass (3.21).

• In subsection 2.2, examples of W in dimension 3 which may be of interest are given in the form [1, Equation (2.6)]:

$$W = F(x,y) + z^2 + w^2.$$

According to [14], the function F can be thought of as defining the Seiberg-Witten Riemann surface, $F(x,y) = 0$. They say that Since the Calabi–Yau is singular, so is the Riemann surface.

$$F(x,y) = x^n + y^2$$

(bottom of page 5, [1]) is a special case of this which corresponds to Argyres-Douglas points.

• Comment: Shapere–Vafa are referring to the manifolds which we call almost Calabi–Yau as Calabi–Yau.

• In section 4. Solitons in dimension three, they study Calabi–Yau three-fold singularities corresponding to Argyres–Douglas points. These points correspond to local singularities given by [1, Equation (4.1)]:

$$W = P(x) + y^2 + z^2 + w^2 = 0, \tag{3.23}$$

where P(x) is a polynomial of degree n in x.

• Superconformal point occurs when

$$P(x) = x^n.$$

When $P(x) \neq x^n$, it is considered as a small perturbation near the superconformal point. There is a physical meaning attached to the coefficients of the polynomial P, half of them correspond to the expectation values of fields and half to dual mass parameters:

$$P(x) = x^n + g_{n-2}x^{n-2} + g_{n-3}x^{n-3} + \cdots + g_0.$$

• In [14] it is explained that $P(x)$ is the characteristic polynomial of a Higgs field.

• BPS states correspond to D3âbranes wrapped around supersymmetric 3-cycles. To find these supersymmetric 3-cycles they follow the procedure developed in [14]. Such a 3-cycle is a copy of S^3 realized as an S^2 fibration over a path in $\mathbb{C}$, the copy belonging to the variable x in $P(x)$. The fiber S^2 above $x \in \mathbb{C}$ is defined by (bottom of page 8, [1]:

$$y^2 + z^2 + w^2 = -P(x)$$

by choosing a real subspace.

• Comment: This is completely analogous to the Lagrangian sphere L_γ (Definition 2.1.19).

• As a consequence of the polynomial P being the characteristic polynomial of a Higgs field and the definition of a rank 2 Higgs bundle it is shown in [14] that the condition of having a supersymmetric cycle becomes:

There exists a path with endpoints at zeros of P such that the phase of:

$$\int_{S^2} \Omega = \sqrt{P(x)}dx \tag{3.24}$$

is constant [1, Equation (4.2)].

• Comment: What they mean by integration over S^2 in this equation is that they integrate the form Ω over the fiber above the point $x \in \mathbb{C}$, the fiber being a copy of S^2. The right hand side is a 1-form because Ω is a 3-form and integration is over a two-dimensional sphere. Clearly the formula is not correct as it is written. We derive the precise formula (3.22) in Lemma 2.2.1. In the notation of this section, our formula states:

$$\int_{L_\gamma} \Omega|_{L_\gamma} = 2\pi \int_0^1 \sqrt{P(\gamma(t))}\gamma'(t)dt, \tag{3.25}$$

$\gamma : [0,1] \to \mathbb{C}$.

Remark 3.9.1. It is not evident that (3.24) is justified by [14]. In that reference they claim that, in dimension 2, $\int_y \Omega = d\lambda$ for a one-form λ [14, p. 10, Equation (2.15)]. This seems to be the idea behind the claim of Shapere–Vafa expressed by the formula (3.24). Still, there is no justification for it in the reference [14]. It seems that (3.24) is an ad-hoc formula, which happens to be correct up to a constant factor. In this thesis we provide the framework and the proof of the correct formula 3.25.

• This condition guarantees saturation in the BPS inequality which is: [1, p. 9]

$$\int_{S^3} \big|\Omega\big| \geq \Big| \int_{S^3} \Omega \Big|.$$

Comment: This is not explained but is simple to prove.

• Another way of looking at the condition yields that the path which is sought is $x(t)$ defined by:

$$W(x(t)) := \int_{x(0)}^{x(t)} \sqrt{P(x)}dx = \alpha t, \tag{3.26}$$

where α is the constant phase from 3.24, and t is a real parameter.

• The meaning is, in their words, that the image of the path under the Jacobian map with respect to the reduced 1-form is a straight line in the flat W-plane.

• Now, this W is not the same as the previously used W (3.23). On the other hand, this is analogous to what we do with the map Q (3.2, 3.5.5)), i.e. we prove that the image under the map Q of the path, over which the Lagrangian is special, is contained in a straight line (Theorem 3.8.3).

• Another problem with (3.26) is that the right hand side, αt defines a line through the origin, not an arbitrary line. If they fix a point through which they want to find solution curve and map it to 0, then the map itself depends on the choice of a point which is mapped to 0.

• Continuing in [1], from here, to find the paths for a fixed value of α means to find integral curves for the differential equation:

$$\frac{dx}{dt} = \frac{\alpha}{\sqrt{P(x)}}, \tag{3.27}$$

where $x = x(t)$ is a complex function and α is a complex number such that $|\alpha| = 1$.

• Removing the zeros of P from the complex plane, creates a space with non-trivial homology. The set of paths between two distinct zeros of P is thus divided into classes. Tho paths are in the same class if the closed curve which they form contains no zero of P in its interior.

• Shapere–Vafa note that since $\int_\gamma \sqrt{P(x)}dx$ does not depend on γ within a homology class

(up to a sign), it is possible, again up to a sign, to determine α which would correspond to a path, in the given homology class of paths between a chosen pair of zeros of P, which would give rise to a BPS state [Equation (5.2) and below] [1].

• Comment: They do not discuss that $\sqrt{P}$ is not a single valued function and, in addition, when a branch of the square root is chosen in order to have a single-valued function, it is not an entire function and that some choices need to be made for the domain of definition. We do this in Section 3.5.

• Next they show that there is at most one such path for a chosen pair of zeros of P. They use the integral for the winding of the phase and work through the possibilities. This is done in a detailed way and it constitutes a proof [1, pp 10-13].

• They establish that locally, around any zero of P, the equation is approximately:

$$\frac{dx}{dt} = \frac{\alpha}{\sqrt{x}}.$$

Thus there are three integral curves emanating from a zero of P, because the solutions of the approximate equation are given by [1, p. 10]:

$$x(t) = \left(\frac{3}{2}\alpha t\right)^{\frac{2}{3}}.$$

• The conclusion that there are three integral curves at $\frac{2\pi}{3}$ angles is correct. Using the approximate equation one can anticipate this result. We prove it (for arbitrary dimension) in Theorem 2.3.3.

• They also show that there cannot be more than one integral curve through a point which

is not a root of P. Examples are discussed at length. They are based on a generic polynomial of degree three and $x^6 - 1$ [1, pp 16-20].

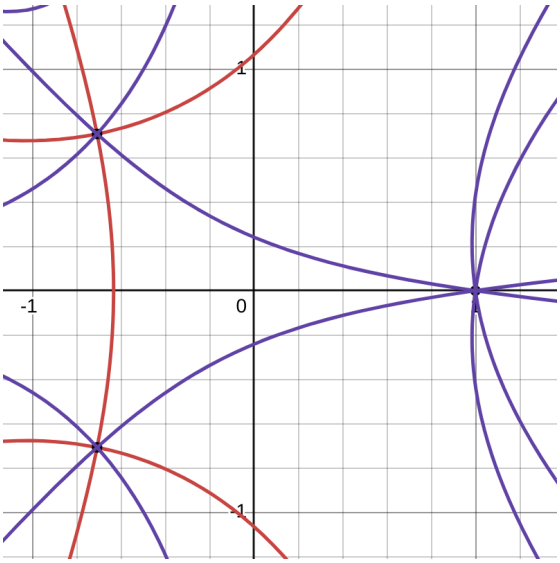
• Shapere–Vafa are interested in *jumping phenomena*. We will explain what they mean by that term: Recall that they think of the roots of P as close to 0, the polynomial P being a perturbation of x^n. Think of the roots of P as n distinct points in the complex plane and suppose that the paths which generate BPS states are known and also in this picture. Keep all the points except one fixed and the one chosen point is free to move. When the point moves, the paths also move and we see that there are positions when the paths seem to switch in the sense that some points may cease to be connected while others, previously unconnected, become connected. Looking at the picture as a graph, it is always a connected graph. Of course, more than one point can be moving at the same time.

• We illustrate this with an example (Figure 3.4). Let

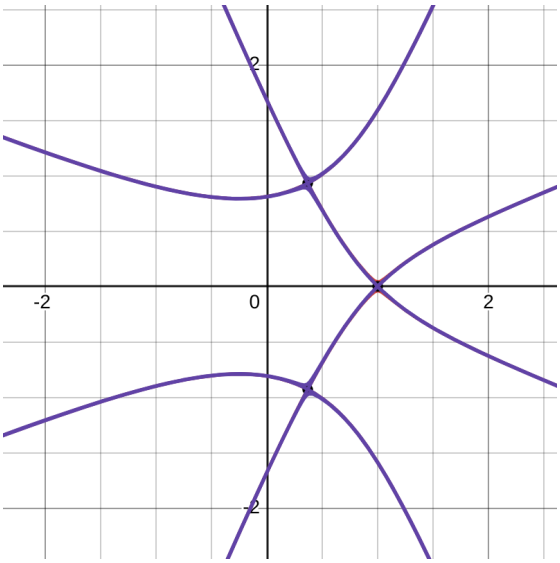
$$P(x) = (x-1)(x-e^{ia})(x-e^{-ia}),$$

where $a \in \mathbb{R}$ is the parameter which we change to cause the two conjugate complex roots to move. (For the graphs we chose the dimension to be four for ease of calculation and for the greater precision of graphs.) The jumping phenomenon occurs when the images under Q of the three points become colinear.

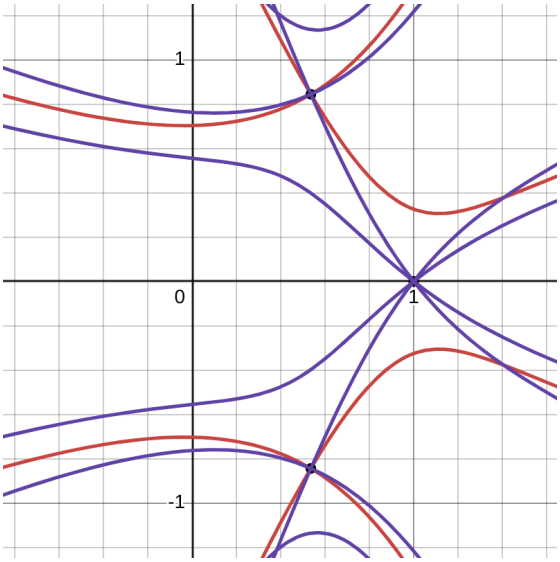
• Shapere-Vafa [1, Figure 5, p. 14] correctly note the following: Suppose that the configuration of three points p_1, p_2, p_3 is such that paths p_1p_2 and p_2p_3 exist. Then if the angle at p_2 between these two paths is more than $\frac{2\pi}{3}$ then path p_1p_3 does not exist, if the angle is less than $\frac{2\pi}{3}$ the path p_1p_3 exists and if the angle is exactly $\frac{2\pi}{3}$ the path p_1p_3 passes through p_2 but the phase is



All three paths exist



Position at which the path between conjugate complex roots ceases to exist



Two paths exist

Figure 3.4: Jumping phenomenon

constant on the whole path. These three situations are ilustrated in Figure 3.4.

They use the winding number again to argue this. The underlining reason for this phenomenon is understood best by looking at the Riemann surface picture we develop in Section 3.5. For arbitrary dimension d, since at each zero of the polynomial the conical metric has cone angle $d\pi$, each section of angle $\frac{2\pi}{d}$ corresponds to a sheet of the covering Riemann surface.

3.10 Admissible paths

The main result of the section is Theorem 3.10.2 in which we show that an admissible path $\gamma \in \mathcal{A}_{j\ell}$ (Definition 3.10.1) exists, if and only if the inverse image of the straight line segment $\overline{Q(w_j)Q(w_\ell)}$ contains a path between w_j and w_ℓ. The map Q is, as before, a primitive function for $q = f^{\frac{n-2}{2}}$ (3.1). In the case of n being odd, Q stands for Q_1 or Q_2 (Definition 3.5.5). In subsection 3.12 we show that the asymptotic behavior of geodesics of the pull-back metric, outside of a compact set containing $V(f)$ depends on $\dim X_f^n = n$ and $\deg f$ only.

Let $\mathcal{A}_{j\ell}$ be the set of paths between w_j and w_l for some $w_j, w_l \in V(f)$.

Definition 3.10.1. *We call a path $\gamma \in \mathcal{A}_{j\ell}$ admissible if the closure of the region in $\mathbb{C}$ enclosed by $Q(\gamma)$ and $\overline{Q(w_j)Q(w_\ell)}$ contains no image under Q of a zero of f.*

Theorem 3.10.2. *If there exists an admissible path $\gamma \in \mathcal{A}_{j\ell}$, then the inverse image of the straight line segment $\overline{Q(w_j)Q(w_\ell)}$, contains a path in $\mathcal{A}_{j\ell}$.*

Proof. Suppose that there exists an admissible path $\gamma : [0,1] \to \mathbb{C}$, $\gamma \in \mathcal{A}_{j\ell}$.

As γ is admissible, the closure of the region in $\mathbb{C}$ enclosed by $Q(\gamma)$ and $\overline{Q(w_j)Q(w_\ell)}$ contains no image under Q of a zero of f. Therefore, there exists a homotopy Γ between the path

$Q(\gamma)$ and the path $\overline{Q(w_j)Q(w_\ell)}$:

$$\Gamma(t,u) : [0,1] \times [0,1] \to \mathbb{C}, \quad \Gamma(0 \times [0,1]) = Q(\gamma), \quad \Gamma(1 \times [0,1]) = \overline{Q(w_j)Q(w_\ell)}.$$

such that

$$\forall t \in [0,1], \forall u \in (0,1), \Gamma(t,u) \neq Q(w_m), \text{ for any } w_m \in V(f),$$

and

$$\forall t \in [0,1], \quad \Gamma(t,0) = w_j, \quad \Gamma(t,1) = w_\ell.$$

Let us use the following notation:

$$\Gamma_t := \{\Gamma(t,u) \, : \, u \in [0,1]\}.$$

Claim : $\forall t \in [0,1]$, there exists $\gamma_t \subset Q^{-1}(\Gamma_t)$, which is a path between w_j and w_ℓ.

Proof of Claim: The idea of the proof is to show that the set of times for which $Q^{-1}(\Gamma_t)$ contains a path in $\mathcal{A}_{j\ell}$ is open in $[0,1]$ and that the same is true for the set of times for which $Q^{-1}(\Gamma_t)$ does not contain a path in $\mathcal{A}_{j\ell}$. The union of the two sets of times is $[0,1]$. But then the connected set $[0,1]$ is the union of two open sets and therefore one of them has to be empty. The set of times for which a path in $\mathcal{A}_{j\ell}$ exists as a subset of $Q^{-1}(\Gamma_t)$ is non-empty as it contains 0 ($\Gamma_0 \equiv Q(\gamma)$, and $\gamma \subset Q^{-1}(Q(\gamma))$), therefore it follows that it equals all of $[0,1]$, which proves the claim.

Now we proceed with the details of the proof. Q is locally invertible around any point which is not a zero of f since $Q' = q = f^{\frac{n-2}{2}}$. Let $\varepsilon > 0$ be small. Then for any $u \in [\varepsilon, 1-\varepsilon]$,

there exists a neighborhood of $\gamma(u)$, $U_0(u)$, on which Q is invertible. Therefore we have an open cover of $\gamma([\varepsilon, 1-\varepsilon])$:

$$\gamma([\varepsilon, 1-\varepsilon]) \subset \bigcup_{u\in[\varepsilon,1-\varepsilon]} U(u),$$

and an open cover of $Q(\gamma([\varepsilon, 1-\varepsilon]))$:

$$Q(\gamma([\varepsilon, 1-\varepsilon])) \subset \bigcup_{u\in[\varepsilon,1-\varepsilon]} Q(U(u)),$$

since Q is holomorphic and non-constant and therefore open by the Open mapping theorem [30, Theorem 6.2]. $\gamma([\varepsilon, 1-\varepsilon])$ and $Q(\gamma([\varepsilon, 1-\varepsilon]))$ are compact as continuous images of a compact set. Then there exist open sets $U_1, \ldots U_a$ such that

$$\{U_1, \ldots, U_a\}$$

is a finite cover of $\gamma([\varepsilon, 1-\varepsilon])$, and such that

$$\{Q(U_1), \ldots, Q(U_a)\}$$

is an open cover of $Q(\gamma([\varepsilon, 1-\varepsilon]))$.

Since there are finitely many sets in the cover, there exists $\varepsilon_0 > 0$ such that

$$\{z \in \mathbb{C} \,:\, d(z, Q(\gamma([\varepsilon, 1-\varepsilon])) \leq \varepsilon_0\} \subset \bigcup_{p=1}^{a} Q(U_p). \tag{3.28}$$

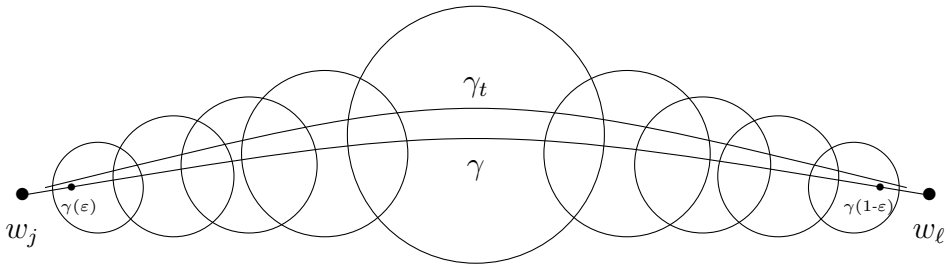


Figure 3.5: γ_t, the unique preimage of $\Gamma_t|_{[\varepsilon,1-\varepsilon]}$ contained in $\bigcup_{p=1}^{a} U_p$

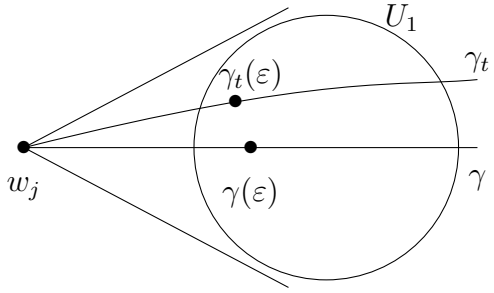


Figure 3.6: Neighborhood of w_j. γ_t extends continuously to w_j.

Γ is a continuous function of two variables, uniformly continuous on the compact set $[0,1]\times[0,1]$. Therefore, for $\varepsilon_0 > 0$, $\exists \delta_0(\varepsilon_0) > 0$ such that

$$d((t_1,u_1),(t_2,u_2)) < \delta_0 \;\rightarrow\; |\Gamma(t_1,u_1) - \Gamma(t_2,u_2)| < \varepsilon_0. \tag{3.29}$$

Combining (3.29) with (3.28) we have that as long as $t < \delta_0$, $\Gamma(t,u) \in \bigcup_{p=1}^{a} Q(U_p)$ for $\varepsilon \le u \le 1-\varepsilon$.

As Q is one-to-one on each of the sets U_p, it follows that $\Gamma_t|_{[\varepsilon,1-\varepsilon]}$, $t < \delta_0$, has a unique and continuous preimage under Q, γ_t (Figure 3.5), such that

$$\{\gamma_t(u) \,:\, u \in [\varepsilon, 1-\varepsilon]\} \subset \bigcup_{p=1}^{a} U_p.$$

It remains to see that γ_t extends to w_j and w_ℓ continuously.

As $\varepsilon > 0$ can be arbitrarily small, points on the path γ for $u \to \varepsilon$ are in a small neighborhood of w_j where $q \approx (w - w_j)^{\frac{n-2}{2}}$. There, Q is one-to-one on segments with angle $\frac{2\pi}{n}$ at w_j (Theorem 3.6.4). Since Q is one-to-one on U_1, U_1 has to be contained in such a segment (Figure 3.6). Then, because Q is one-to-one inside the segment, and Γ_t extends to $Q(w_j)$, it follows that there is a unique preimage of Γ_t, γ_t, that extends from $\gamma_t(\varepsilon)$ to $\gamma_t(0) = w_j$ and, similarly, to w_ℓ. Therefore, for $t < \delta_0$, $Q^{-1}(\Gamma_t)$ contains a continuous path γ_t connecting w_j and w_ℓ.

In the same manner, if we know that a continuous path $\gamma_{t_0} \subset Q^{-1}(\Gamma_{t_0})$ exists for some time t_0, we can find a $\delta > 0$ such that γ_t exists for $|t - t_0| < \delta$.

Therefore,

$$\{t \in [0,1] \ : \ \text{path } \gamma_t \text{ exists } \}$$

is open in $[0, 1]$.

Now, if for some $t_0 \in (0, 1)$ there is no path between w_j and w_ℓ contained in $Q^{-1}(\Gamma_{t_0})$ then any preimage through w_j is disjoint from any preimage through w_ℓ. Such preimages are shown in Figure 3.7.

A preimage of Γ_{t_0} through w_j and a preimage through w_ℓ are compact and are contained in disjoint open sets. Because of uniform continuity of Γ there exists $\delta > 0$ such that if $|t - t_0| < \delta$ then

Therefore,

$$\{t \in [0,1] \ : \ \text{path } \gamma_t \text{ does not exist } \}$$

is open in $[0, 1]$, as well.

We have both $\{t \in [0,1] \ : \ \text{path } \gamma_t \text{ exists}\}$ and $\{t \in [0,1] \ : \ \text{path } \gamma_t \text{ does not exist } \}$ are

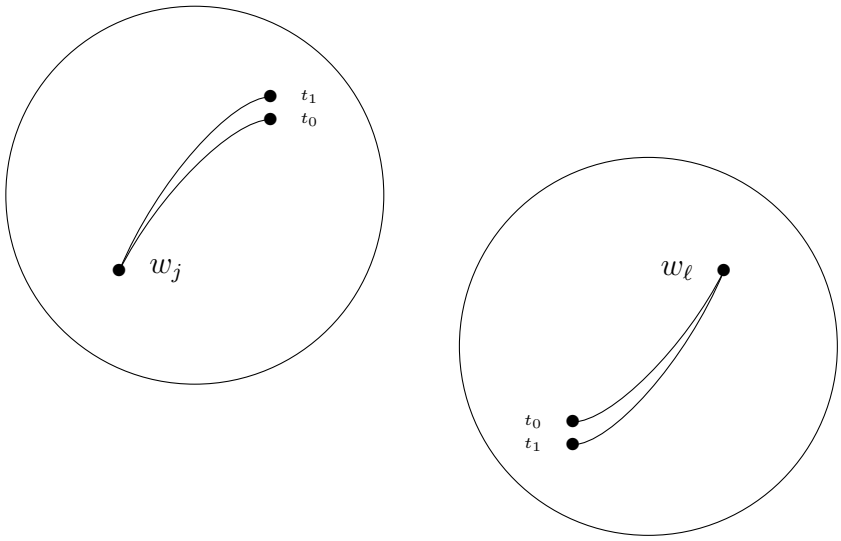


Figure 3.7: No path connecting w_j and w_ℓ in $Q^{-1}(\Gamma_{t_0})$

open and

$$[0,1] = \{t \in [0,1] \ : \ \text{path } \gamma_t \text{ exists}\} \cup \{t \in [0,1] \ : \ \text{path } \gamma_t \text{ does not exist }\}.$$

As $[0,1]$ is connected and

$$0 \in \{t \in [0,1] \ : \ \text{path } \gamma_t \text{ exists}\} \neq \varnothing,$$

it follows that

$$\{t \in [0,1] \ : \ \text{path } \gamma_t \text{ does not exist }\} = \varnothing.$$

This ends the proof of the Claim.

To finish the proof of the theorem, it remains to observe that by the Claim, $Q^{-1}(\Gamma_1)$ contains a path between w_j and w_ℓ, γ_1, and that L_{γ_1} is special by Theorem 3.8.3.

□

3.11 Version of Thomas–Yau in Milnor fibers

In Theorem 3.10.2 we gave a qualitative condition for the existence of an SL sphere in a given class. The reasoning there was roughly:

Suppose that we are interested in determining whether there exists a path γ between w_j and w_ℓ such that L_γ is SL. Then the procedure is as follows, first choose a primitive function Q for $q = f^{\frac{n-2}{2}}$, then construct the straight line segment connecting $Q(w_j)$ and $Q(w_\ell)$. If there exists a path γ_0 between w_j and w_ℓ such that there exists a homotopy, which fixes the endpoints, between $Q(\gamma_0)$ and $Q(w_j)Q(w_\ell)$, and which does not pass through any other $Q(w_m)$ for $w_m \in V(f)$, then there exists a subset of the preimage of $Q(w_j)Q(w_\ell)$ which is a path γ between w_j and w_ℓ, such that L_γ is SL.

The quantitative condition of Theorem 3.11.5 ensures that the qualitative one is satisfied. The same is true for the quantitative condition of Theorem 3.11.6. These quantitative conditions are almost the same as the ones in the Thomas–Yau conjecture. They give further support to the validity of the conjecture in some form. There are counterexamples to the conjecture in its original form and there have been attempts to reformulate it in order to prove its validity in some form, especially since there is evidence that it should hold.

Lastly, in Theorem 3.11.7 we show that either there is a path beween w_j and w_ℓ which gives rise to an SL sphere or the path which minimizes $\Phi_{j\ell}$ (Definition 2.2.4) contains other zeros of f but each of the segments between two zeros of f gives rise to an SL sphere so that we have a chain of SL spheres.

Let $V(f) = \{w_1, \ldots, w_k\}$ be the set of zeros of f. Recall that for a path $\gamma \in \mathcal{A}_{j\ell}$ between

two zeros of f there exists a Lag submanifold L_γ of X_f^n diffeomorphic to a sphere:

$$L_\gamma := \left\{(w, r_1\sqrt{f(w)}, \ldots, r_n\sqrt{f(w)}) : w \in Im\gamma,\ (r_1, \ldots, r_n) \in S^{n-1}(1) \subset \mathbb{R}^n\right\}.$$

Lemma 3.11.1. *Let $w', w'' \in \mathbb{C}$ be arbitrary. Let $\gamma : [0,1] \to \mathbb{C}$ be a piecewise C^1 path from w' to w''. Then*

$$\int_0^1 q(\gamma(u))\gamma'(u)du = Q(w'') - Q(w').$$

Furthermore,

$$\int_0^1 \left|q(\gamma(u))\gamma'(u)\right|du = \text{ the length of the path } Q(\gamma).$$

Proof. q is a holomorphic function so its integral is independent of the path. Q is a primitive function for q so the first statement follows. In the case of n being odd it is important that the choice of Q is such that the path γ is contained in the domain of holomorphicity of Q.

To show that the second integral is the length of the path $Q(\gamma)$ recall that we have shown in Lemma 3.7.1 that the tangent vector to the curve $Q(\gamma)$ is given by

$$\frac{d}{du}Q(\gamma(u)) = q(\gamma(u))\gamma'(u).$$

Then. the length of $Q(\gamma)$ is

$$\int_{u'}^{u''} \left|q(\gamma(u))\gamma'(u)\right|du = \int_\gamma |q|\, d\gamma,$$

where $\gamma(u') = w'$ and $\gamma(u'') = w''$.

□

Remark 3.11.2. If n is even q and Q are uniquelly determined and holomorphic. If n is odd, for q and Q to be holomorphic, a choice has to be made of rays to remove from the complex plane. Effectively, that is also a choice of homology classes of paths. In practical terms, in order to check all different homology classes of paths, we would have to repeat the procedure with a different choice of cuts in the complex plane. But according to Lemma 3.12.4 the result is not dependent on the choice of a primitive function, except for the phase θ_H which may differ by π.

Lemma 3.11.3. *For a Lag sphere $L_\gamma \in H$, $\gamma : [0,1] \to \mathbb{C}$, we have the inequality:*

$$|\int_0^1 q(\gamma(u))\gamma'(u)du| \leq \int_0^1 \left|q(\gamma(u))\gamma'(u)\right|du.$$

$L_\gamma \in H$ has constant phase if and only if:

$$|\int_0^1 q(\gamma(u))\gamma'(u)du| = \int_0^1 \left|q(\gamma(u))\gamma'(u)\right|du,$$

Proof. ($\to$) If the phase of L_γ, $\theta =$ const., then

$$\theta = \arg\left[q(\gamma(u))\gamma'(u)\right] = \text{const.},$$

and thus

$$q(\gamma(u))\gamma'(u) = e^{i\theta}\left|q(\gamma(u))\gamma'(u)\right|.$$

Consequently,

$$\mathrm{e}^{i\theta}\Big|\int_0^1 q(\gamma(u)\gamma'(u)du\Big| = \int_0^1 q(\gamma(u))\gamma'(u)du = \mathrm{e}^{i\theta}\int_0^1 \Big|q(\gamma(u)\gamma'(u)\Big|du,$$

so

$$\Big|\int_0^1 q(\gamma(u)\gamma'(u)du\Big| = \int_\gamma |q|$$

($\leftarrow$) Suppose that L_γ is not constant. It follows then that the curve $Q(\gamma)$ is not a straight line segment: the tangent vector of $Q(\gamma)$ has the direction of the phase of L_γ; if the tangent vector of $Q(\gamma)$ does not have constant argument then $Q(\gamma)$ is not contained in a straight line. Then, as the length of the straight line segment between the endpoints of the path $Q(\gamma)$ is $\Big|\int_0^1 q(\gamma(u)\gamma'(u)du\Big|$ by Lemma 3.11.1, it is strictly less than the length of the curve $Q(\gamma)$.

□

Recall that $\mathcal{H}_{j\ell}$ (1.3) is the set of Lag spheres arising from paths in $\mathcal{A}_{j\ell}$ (1.3). We know (Remark 2.2.3) that for each homology class of such Lag spheres, H,

$$\int_\gamma q(\gamma(u))du = \text{const.}, \quad L_\gamma \in H.$$

We introduce the following notation:

Definition 3.11.4. *Let*

$$\theta_H := \arg\int_\gamma q(\gamma(u))\gamma'(u)du, \quad L_\gamma \in H, \tag{3.30}$$

and let

$$a_H := \Big| \int_\gamma q(\gamma(u))\gamma'(u)du \Big|, \quad L_\gamma \in H, \tag{3.31}$$

i.e.,

$$\int_\gamma q = a_H \mathrm{e}^{i\theta_H}, \quad L_\gamma \in H. \tag{3.32}$$

Theorem 3.11.5. *Let $w_j, w_\ell \in V(f)$ and $j \neq \ell$. An SL sphere in $\mathcal{H}_{j\ell}$ exists if and only if there exists a Lag sphere, $L_\gamma \in \mathcal{H}_{j\ell}$ such that*

$$\int_\gamma |q| \leq a_{\mathcal{H}_{jp}} + a_{\mathcal{H}_{pl}}, \quad \forall p \neq j, l. \tag{3.33}$$

Proof. ($\rightarrow$) If there exists an SL sphere $L_\gamma \in \mathcal{H}_{j\ell}$ then $Q(\gamma)$ is $\overline{Q(w_j)Q(w_\ell)}$ and $\int_\gamma |q| = \big|\overline{Q(w_j)Q(w_\ell)}\big|$. The smallest possible value of $a_{\mathcal{H}_{jp}}$ is the length of $\overline{Q(w_j)Q(w_p)}$, and, similarly, the smallest possible value of $a_{\mathcal{H}_{p\ell}}$ is the length of $\overline{Q(w_p)Q(w_\ell)}$. These values are achieved if there exist SLs in $\mathcal{H}_{jp}$ and $\mathcal{H}_{j\ell}$. Therefore, the condition (3.33) is satisfied for SL L_γ due to triangle inequality (see Figure 3.8), provided that the points $Q(w_j), Q(w_\ell)$, and $Q(w_p)$ form a triangle. Now, if those three points are colinear, there are two possible situations:

(1) $Q(w_p)$ is between $Q(w_j)$ and $Q(w_\ell)$, or

(2) $Q(w_p)$ is not between $Q(w_j)$ and $Q(w_\ell)$.

Next we check what happens in each of the two cases:

(1) As L_γ is an SL sphere, γ does not pass through w_p but it does pass through another point

in $Q^{-1}(Q(w_p))$. Clearly,

$$|\overline{Q(w_j)Q(w_\ell)}| = |\overline{Q(w_j)Q(w_p)}| + |\overline{Q(w_p)Q(w_\ell)}|.$$

Since $a_{\mathcal{H}_{jp}} \geq |\overline{Q(w_j)Q(w_p)}|$ and $a_{\mathcal{H}_{p\ell}} \geq |\overline{Q(w_p)Q(w_\ell)}|$, the strict inequality is satisfied in (3.33) unsless SLs exist in both $\mathcal{H}_{jp}$ and $\mathcal{H}_{j\ell}$ in which case equality holds.

(2) For this configuration,

$$|\overline{Q(w_j)Q(w_\ell)}| < |\overline{Q(w_j)Q(w_p)}| + |\overline{Q(w_p)Q(w_\ell)}| \leq a_{\mathcal{H}_{jp}} + a_{\mathcal{H}_{p\ell}}.$$

Therefore the inequality (3.33) is strict.

($\leftarrow$) Suppose that there exists a Lag sphere $L_\gamma \in \mathcal{H}_{j\ell}$ satisfying (3.33). Then:

Claim: $Q(\gamma)$ is admissible (Definition 3.10.1).

Proof of Claim: (Proof by contradiction.) Suppose that the path $Q(\gamma)$ together with the straight line segment $Q(w_j)Q(w_\ell)$ forms a closed curve which includes another zero of f in its interior or the boundary (see Figure 3.8), i.e.,the path $Q(\gamma)$ does not satisfy the assumption of Theorem 3.10.2). Then $Q(\gamma)$ has length which is greater than the sum of the lengths of two sides of the triangle

$$\triangle Q(w_j)Q(w_p)Q(w_\ell), \tag{3.34}$$

which is

$$mQ(w_j)Q(w_p) + mQ(w_p)Q(w_\ell) = a_{\mathcal{H}_{jp}} + a_{\mathcal{H}_{pl}}, \tag{3.35}$$

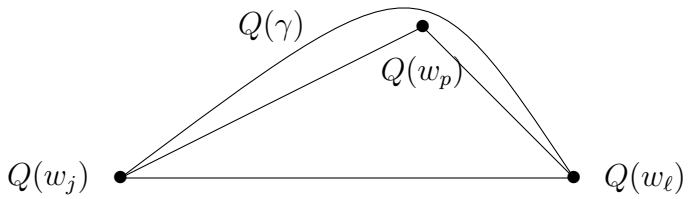


Figure 3.8: Curve $Q(\gamma)$ from $Q(w_j)$ to $Q(w_\ell)$ with length greater than $a_{H_{jp}} + a_{H_{pl}}$

and consequently the inequality (3.33) fails to be satisfied. This is a contradiction which ends the proof of the claim.

Now, since $Q(\gamma)$ is admissible, by Theorem 3.10.2, $Q^{-1}\left(\overline{Q(w_j)Q(w_\ell)}\right)$ contains a path in $A_{j\ell}$. The Lag sphere over that path is the SL in $\mathcal{H}_{j\ell}$.

□

Theorem 3.11.6. *Let $w_j, w_\ell \in V(f)$, $j \neq \ell$. An SL sphere in $\mathcal{H}_{j\ell}$ exists if and only if there exists a Lag sphere, $L = L_\gamma \in \mathcal{H}_{j\ell}$ such that*

$$[\theta_{\mathcal{H}_{jp}}, \theta_{\mathcal{H}_{pl}}] \nsubseteq (\inf_L \theta_L, \sup_L \theta_L), \quad \forall p \neq j, l. \tag{3.36}$$

Proof. ($\rightarrow$) If there exists an SL sphere $L_\gamma \in \mathcal{H}_{j\ell}$ then the condition (3.36) is satisfied for that SL since $\inf_L \theta_L = \sup_L \theta_L$ for an SL and consequently $(\inf_L \theta_L, \sup_L \theta_L) = \varnothing$, while $[\theta_{\mathcal{H}_{jp}}, \theta_{\mathcal{H}_{pl}}] \neq \varnothing$ even if $\theta_{\mathcal{H}_{jp}} = \theta_{\mathcal{H}_{pl}}$.

($\leftarrow$) Suppose that there exists a Lag L_γ for which the condition (3.36) holds.

Claim: $Q(\gamma)$ is admissible (Definition 3.10.1).

Proof of Claim: Suppose that γ is not admissible, as in Figure 3.8. Then, in order for the path $Q(\gamma)$ to go through or above the point $Q(w_p)$ the slope of the tangent line to γ has to

reach the value of the slope of the straight line $Q(w_j)Q(w_p)$. Similarly, it has to reach the value of the slope of $Q(w_p)Q(w_\ell)$. But then $(\inf_L \theta_L, \sup_L \theta_L)$ contains $[\theta_{H_{jp}}, \theta_{H_{pl}}]$, and consequently the condition (3.36) fails to be satisfied. Contradiction. End of proof of the claim.

As $Q(\gamma)$ is admissible by the Claim, by Theorem 3.10.2, $Q^{-1}\big(\overline{Q(w_j)Q(w_\ell)}\big)$ contains a path in $A_{j\ell}$. The Lag sphere over that path is the SL in $\mathcal{H}_{j\ell}$. □

If for some $w_j, w_\ell \in V(f)$ there exists no SL sphere in $H_{j\ell}$, there still exists a path $\gamma \in \mathrm{cl}\big(A_{j\ell}\big)$ such that

$$\int_\gamma |q(\gamma)|\, d\gamma$$

is minimized (Theorem 2.2.5). The image under Q of such path is a polygonal path with endpoints $Q(w_j)$ and $Q(w_\ell)$ and the rest of the vertices are of the form $Q(w_p)$, where $w_p \neq w_j, w_\ell$, and $w_p \in V(f)$. Each side of the polygonal path gives rise to an SL sphere. The resulting Lag L_γ is a singular submanifold: a chain of spheres where the consecutive ones have a single point in common.

Theorem 3.11.7. *Let $w_j, w_\ell \in V(f)$, $j \neq \ell$. Then there exists a union of SL spheres corresponding to a path from w_j to w_ℓ which may pass through other zeros of f.*

Proof. Consider the set

$$P := \Big\{ \int_\gamma |q| : \gamma \in \mathrm{cl}\big(\mathcal{A}_{j\ell}\big) \Big\} \subset \big(a_{\mathcal{H}_{j\ell}}, \infty\big)\,. \tag{3.37}$$

Then $\inf(P)$ exists and $\inf(P) \geq a_{\mathcal{H}_{j\ell}}$. By Lemma 3.11.1,

$$P := \Big\{ \text{length}(Q(\gamma)) : \gamma \in \text{cl}\big(\mathcal{A}_{j\ell}\big) \Big\}.$$

If $\inf(P) = a_{\mathcal{H}_{j\ell}}$ then the straight line segment $\overline{Q(w_j)Q(w_\ell)} = Q(\gamma)$ for a path $\gamma \in \text{cl}\big(\mathcal{A}_{j\ell}\big)$. If γ does not contain any other zeros of f then the corresponding Lag sphere L_γ is special. If γ contains other zeros of f then it gives rise to a chain of SL spheres since the fiber above each zero of f degenerates into a point.

If $\inf(P) > a_{\mathcal{H}_{j\ell}}$ then no path $\gamma \in \text{cl}\big(\mathcal{A}_{j\ell}\big)$ is admissible. $\inf(P)$ is achieved when $Q(\gamma)$ is a polygonal line with endpoints $Q(w_j)$ and $Q(w_\ell)$ and with other vertices $Q(w_p)$ for some (or all) of the other zeros of f.

□

Remark 3.11.8. Lotay–Gonçalo [31] develop an approach which is very similar to the one developed here. Ours was developed in 2019–2020 and is for Milnor fibers. Theirs is in the setting of hyperKähler manifolds of complex dimension two (which are Calabi–Yau) with a circle action.

Note that $n = 2 \Rightarrow Q = q$, and in this case the straight line segment $\gamma_{j\ell} = \overline{w_j w_\ell}$ has the property that $L_{\gamma_{j\ell}}$ is special, provided that $\overline{w_j w_\ell} \cap V(f) = \{w_j, w_\ell\}$.

3.12 Behavior of geodesics in the pull-back metric outside of a compact set

In addition to finding segments of geodesics in the metric $|q(w)|^2|dw|^2$ connecting zeros of f, which are of primary interest as they give rise to SL spheres, it is interesting to understand the asymptotic behavior of geodesics. It turns out that the behavior is determined by the degree

of f and the dimension, n, but is independent of the distribution of the k zeros of f. Shapere and Vafa [1, Example pg. 19] note this asymptotic behavior for an example in dimension three. Proposition 3.12.2 covers the case of n being even, and Proposition 3.12.4 the case of n being odd. In each case we show that preimages by Q^{-1} of a straight line, are asymptotic to n equiangulary distributed rays starting from 0.

3.12.1 n even

First we examine the asymptotic behavior of geodesics for n even. We find that the preimages under Q of a ray are asymptotic to rays, and we determine their directions.

Definition 3.12.1. *Let $w \in \mathbb{C}$ and let $\theta \in \mathbb{R}$.*

$$r_{w,\theta} := \{w + re^{i\theta} \ : \ r \in \mathbb{R}_+\}. \tag{3.38}$$

Proposition 3.12.2. *Let $\theta \in [0, 2\pi)$ be fixed, let $w_0 \in \mathbb{C}$ be arbitrary. Then each preimage under Q of the ray $r_{Q(w_0),\theta}$ (Definition 3.12.1) is asymptotic to a ray which has one of the following directions:*

$$\exp\left(i\frac{\theta + 2j\pi}{\frac{1}{2}k(n-2)+1}\right), \quad 0 \le j \le k\frac{n-2}{2}. \tag{3.39}$$

Proof. Let $\gamma(u), 0 \le u < \infty$ be one preimage of $r_{Q(w_0),\theta}$. As $Q(w) \to \infty$ if and only if $|w| \to \infty$, we know that $\lim_{u\to\infty} |\gamma(u)| = \infty$.

Now

$$f(\gamma(u)) = (\gamma(u) - w_1)\dots(\gamma(u) - w_k) = (\gamma(u))^k + a_{k-1}(\gamma(u))^{k-1} + \dots + a_0, \tag{3.40}$$

so that

$$\lim_{u\to\infty}\frac{f(\gamma(u))}{(\gamma(u))^k}=\lim_{u\to\infty}\left(1+\frac{a_{k-1}}{\gamma(u)}+\frac{a_{k-2}}{(\gamma(u))^2}+\cdots+\frac{a_0}{(\gamma(u))^k}\right)=1$$

Similarly,

$$\lim_{u\to\infty}\frac{(f(\gamma(u)))^{\frac{n-2}{2}}}{(\gamma(u))^{\frac{k(n-2)}{2}}}=1$$

i.e., $q(\gamma(u))$ is asymptotic to $(\gamma(u))^{\frac{k(n-2)}{2}}$. Consequently,

$$\lim_{u\to\infty}\Big(\arg(q(\gamma(u))-\arg(\gamma(u)^{\frac{1}{2}k(n-2)})\Big)=0. \tag{3.41}$$

Now, since by assumption,

$$Q(\gamma(u))=Q(w_0)+r\mathrm{e}^{i\theta},$$

we have

$$\frac{d}{du}Q(\gamma(u))=\frac{d}{du}\Big[Q(w_0)+r\mathrm{e}^{i\theta}\Big]$$
$$q(\gamma(u))\gamma'(u)=\frac{dr}{du}\mathrm{e}^{i\theta}.$$

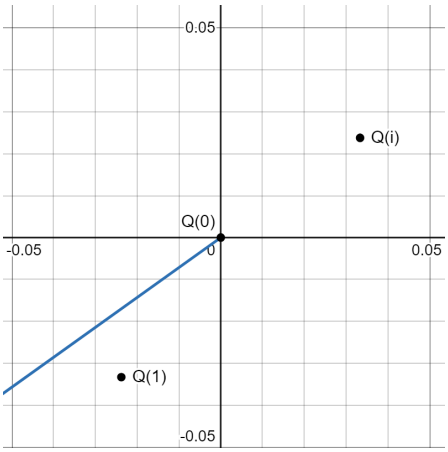
From here,

$$\arg q(\gamma(u))+\arg\gamma'(u)=\theta,$$

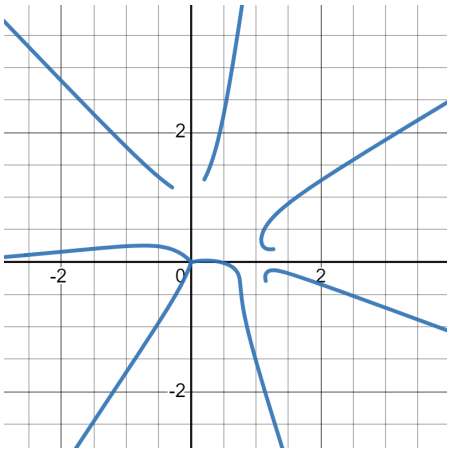
$$\arg(q(\gamma(u))-\arg(\gamma(u)^{\frac{1}{2}k(n-2)})+\arg(\gamma(u)^{\frac{1}{2}k(n-2)})+\arg\gamma'(u)=\theta,$$

and therefore

$$\lim_{u\to\infty}\Big(\arg(q(\gamma(u))-\arg(\gamma(u)^{\frac{1}{2}k(n-2)})\Big)+\lim_{u\to\infty}\Big(\arg(\gamma(u)^{\frac{1}{2}k(n-2)})+\arg\gamma'(u)\Big)=\theta. \tag{3.42}$$

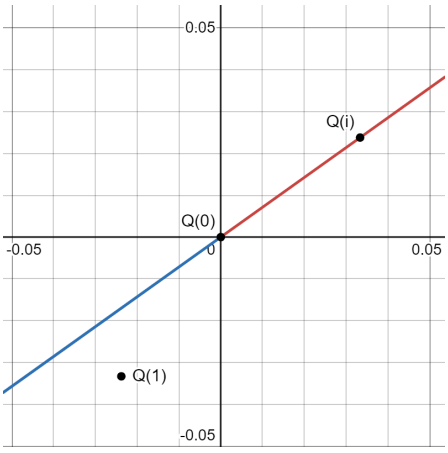


Mapped by Q



Preimages

Figure 3.9: $f(w) = w(w-1)(w-i), n = 6$



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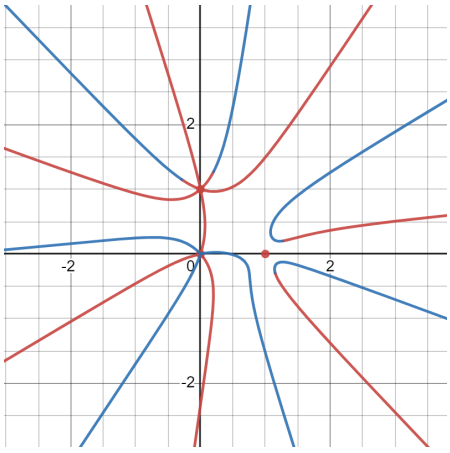


Preimages

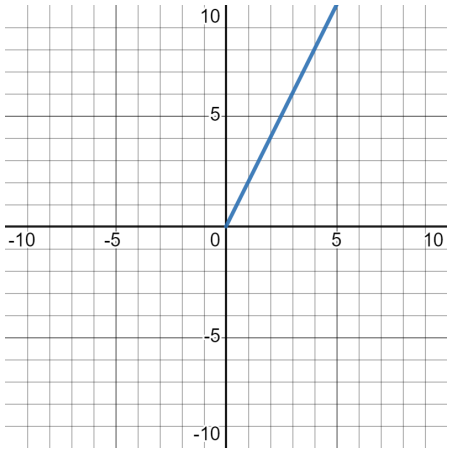
Figure 3.10: $f(w) = w(w-1)(w-i), n = 6$

Because of (3.41), it follows from (3.42) that

$$\lim_{u\to\infty}\Big(\arg(\gamma(u)^{\frac{1}{2}k(n-2)}) + \arg\gamma'(u)\Big) = \theta,$$

which means

$$\lim_{u\to\infty}\Big(\arg\left[(\gamma(u)^{\frac{1}{2}k(n-2)})\gamma'(u)\right]\Big) = \theta,$$



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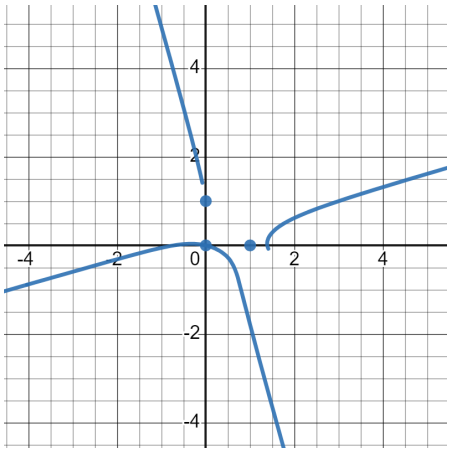
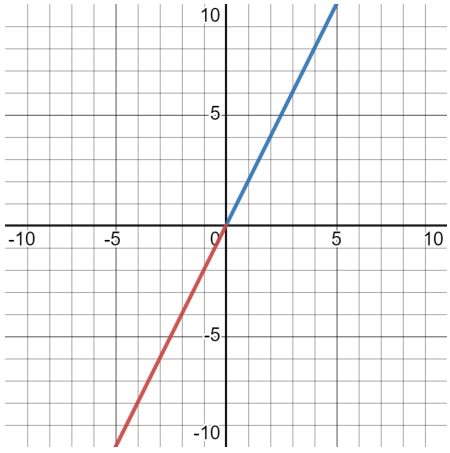
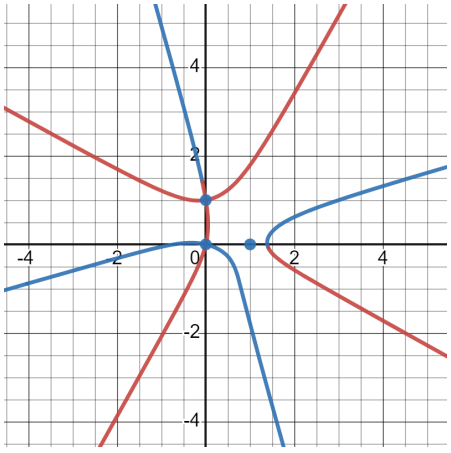


Preimages

Figure 3.11: $f(w) = w(w-1)(w-i), n = 4$

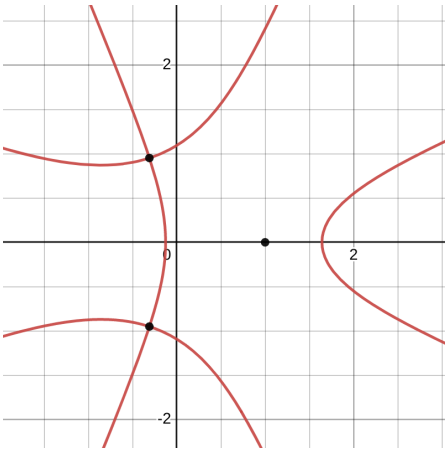


Mapped by Q

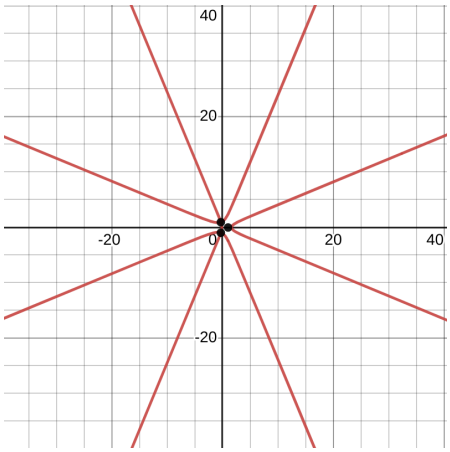


Preimages

Figure 3.12: $f(w) = w(w-1)(w-i), n = 4$



$k = 3, n = 4$, zoom in



$k = 3, n = 4$, zoom out

Figure 3.13: Two views of the preimages of a straight line

and therefore,

$$\lim_{u\to\infty}\left(\arg\left((\gamma(u)^{\frac{1}{2}k(n-2)+1})\right)'\right) = \theta.$$

If the tangent to the curve $(\gamma(u)^{\frac{1}{2}k(n-2)+1})$ has a fixed limiting direction then the curve itself has an asymptote with the same direction $\mathrm{e}^{i\theta}$. Then $\gamma(u)$ has an asymptote in the direction of one of

$$\exp\left(i\frac{\theta+2j\pi}{\frac{1}{2}k(n-2)+1}\right), \quad 0 \le j \le k\frac{n-2}{2}.$$

This Lemma is illustrated in Figures 3.9 and 3.10 with $n = 6$, and in Figures 3.11 and 3.12 with $n = 4$. We use the same polynomial, $f(w) = w(w-1)(w-i)$ (so $k = 3$), and we choose $Q(w)$ so as to satisfy $Q(0) = 0$. In both cases the ray shown has vertex $Q(0)$ and it does not contain $Q(i)$ although the line which contains the ray does contain $Q(i)$. Near 0 which is a zero of f, $q \approx w^{\frac{n-2}{2}}$ so $Q \approx w^{\frac{n}{2}}$. In Figures 3.10 and 3.12 the preimages of rays at an angle of π are curves forming an angle of $\frac{2\pi}{n}$ at 0 (and also at i).

□

Remark 3.12.3. We think of the zeros of $f(w)$ as being close to 0, i.e.,we think of the polynomial $f(w)$ as a perturbation of w^k. In the unperturbed case, the metric is given by $|q(w)|^2|dw|^2 = |w|^{k(n-2)}|dw|^2$ on $\mathbb{C} \setminus \{0\}$. Any ray starting at 0 gives rise to an SL which is not compact. There are no compact SL spheres. If we introduce a perturbation and move the zeros of f slightly away from 0 (separating them into k distinct zeros) we get compact SL spheres generated by geodesic paths (geodesic in the metric $|q|^2|dw|^2$) between zeros. Outside of a compact set the structure is asymptotic to that of the unperturbed case. See Figure 3.13.

3.12.2 n odd

Lemma 3.12.4. *Let $\theta \in [0, 2\pi)$ and $(w_0, z_0) \in Y_1$ be fixed, and let*

$$r_{Q_1(w_0,z_0),\theta} \quad \textit{and} \quad r_{Q_2(w_0,-z_0),\theta+\pi}$$

be rays in $\mathbb{C}$ contained in the same line through the origin. Then

$$\left(\pi \circ Q^{-1}\right)\left(r_{Q_1(w_0),\theta}\right)$$

and

$$\left(\pi \circ Q^{-1}\right)\left(r_{Q_2(w_0),\theta+\pi}\right)$$

coincide and are asymptotic to rays $r_{0,\phi}$, where

$$\phi \in \left\{ \frac{2\theta + 2j\pi}{k(n-2)+2} \ : \ 0 \leq j \leq k(n-2)+1 \right\}.$$

Proof. Let $\gamma(u) : [0, \infty) \to \mathbb{C}$ be such that $\gamma(0) = w_0$ and

$$Q_1(\gamma(u)) = r_{Q_1(w_0),\theta}.$$

As in Lemma 3.12.2, from equation (3.40) it follows that for u large,

$$f(\gamma(u)) \sim (\gamma(u))^k \tag{3.43}$$

Similarly,

$$q^2(\gamma(u)) = (f(\gamma(u)))^{n-2} \sim ((\gamma(u))^{k(n-2)}.$$

Consequently,

$$\lim_{u\to\infty} \Big(\arg(q^2(\gamma(u))) - \arg(\gamma(u)^{k(n-2)}) \Big) = 0. \tag{3.44}$$

From the equation (3.44):

$$\lim_{u\to\infty} \Big(\arg(q_1(\gamma(u))) - \arg(\gamma(u)^{\frac{k(n-2)}{2}}) \Big) = 0, \tag{3.45}$$

and

$$\lim_{u\to\infty} \Big(\arg(q_2(\gamma(u))) - (\arg(\gamma(u)^{\frac{k(n-2)}{2}}) + \pi) \Big) = 0 \tag{3.46}$$

Now, since by assumption,

$$Q_1(\gamma(u)) = Q_1(w_0) + r\mathrm{e}^{i\theta},$$

and

$$Q_2(\gamma(u)) = Q_2(w_0) + r\mathrm{e}^{i(\theta+\pi)},$$

we have

$$\frac{d}{du}Q_1(\gamma(u)) = \frac{d}{du}\Big[Q_1(w_0) + r\mathrm{e}^{i\theta}\Big]$$

$$q_1(\gamma(u))\gamma'(u) = \frac{dr}{du}\mathrm{e}^{i\theta}. \tag{3.47}$$

and

$$\frac{d}{du}Q_2(\gamma(u)) = \frac{d}{du}\Big[Q_2(w_0) + r\mathrm{e}^{i(\theta+\pi)}\Big]$$

$$q_2(\gamma(u))\gamma'(u) = \frac{dr}{du}\mathrm{e}^{i(\theta+\pi)}, \tag{3.48}$$

From equation (3.47)

$$\arg q_1(\gamma(u)) + \arg\gamma'(u) = \theta$$

$$\arg(q_1(\gamma(u)) - \frac{1}{2}k(n-2)\arg(\gamma(u)) + \frac{1}{2}k(n-2)\arg(\gamma(u)) + \arg\gamma'(u) = \theta,$$

and therefore

$$\lim_{u\to\infty}\Big(\arg(q_1(\gamma(u)) - \arg(\gamma(u)^{\frac{1}{2}k(n-2)})\Big) + \lim_{u\to\infty}\Big(\arg(\gamma(u)^{\frac{1}{2}k(n-2)}) + \arg(\gamma'(u))\Big) = \theta.$$

Consequently, because of (3.45),

$$\lim_{u\to\infty}\Big(\arg(\gamma(u))^{\frac{1}{2}k(n-2)} + \arg\gamma'(u)\Big) = \theta. \tag{3.49}$$

Now, similarly, it follows from equation (3.48) that

$$\arg q_2(\gamma(u)) + \arg \gamma'(u) = \theta + \pi,$$

which means

$$\arg(q_2(\gamma(u)) - \frac{1}{2}k(n-2)\arg(\gamma(u)) - \pi + \frac{1}{2}k(n-2)\arg(\gamma(u)) + \pi + \arg\gamma'(u) = \theta + \pi,$$

and therefore,

$$\lim_{u\to\infty}\Big(\arg(q_2(\gamma(u)) - (\arg(\gamma(u)^{\frac{1}{2}k(n-2)}) + \pi)\Big) + \lim_{u\to\infty}\Big(\arg(\gamma(u)^{\frac{1}{2}k(n-2)}) + \arg(\gamma'(u))\Big) = \theta.$$

Now, because of (3.46), the same conclusion (equation (3.49)) as in the first case follows:

$$\lim_{u\to\infty}\Big(\arg(\gamma(u))^{\frac{1}{2}k(n-2)} + \arg\gamma'(u)\Big) = \theta.$$

We modify the expression (3.49) to get

$$\lim_{u\to\infty}\Big(\arg\left[(\gamma(u))^{\frac{1}{2}k(n-2)}\gamma'(u)\right]\Big) = \theta,$$

and therefore,

$$\lim_{u\to\infty}\Big(\arg\left((\gamma(u))^{\frac{1}{2}k(n-2)+1}\right)'\Big) = \theta.$$

If the tangent to the curve $(\gamma(u))^{\frac{1}{2}k(n-2)+1}$ has a fixed limiting direction then the curve itself has an asymptote with the same direction which is $e^{i\theta}$. Then $\gamma(u)$ has an asymptote in the direction

of one of

$$\exp\Big(i\frac{2\theta+2j\pi}{k(n-2)+2}\Big), \quad 0\le j\le k(n-2)+1.$$

□

Chapter 4: Examples

We discuss in detail three examples:

(i) In subsection 4.1, X_f^n is defined by the polynomial $f = w^6 - 1$ and the chosen dimension is odd, $n = 3$.

(ii) In subsection 4.2, X_f^n is defined by the polynomial $f = w^6 - 1$ and the chosen dimension is even, $n = 4$.

(iii) In subsection 4.3, X_f^n is defined by the polynomial $f = w(w-1)(w-(\frac{1}{2}+\frac{1}{4}i))(w-(\frac{1}{2}+\frac{1}{5}i))$ and the chosen dimension is $n = 4$.

Of course any choice of a polynomial $f \in \mathcal{P}_{simple}$, and any choice of dimension, n, provides an example. Shapere and Vafa [1, pp. 19-20] present the same example in dimension 3. We prove the claims that they make. In addition, we analyze the case $n = 4$. The two are representative of what would happen for other values of n. Thus the conclusion (for any pair of zeros of f there exists a path with those endpoints which gives rise to an SL) is valid for any value of n.

In general, for any dimension n and any $k > 1$, $f = w^k - 1$ provides an example where for all pairs of vertices paths giving rise to SL spheres exist.

In example 4.3 we have a special situation when the images of some of the elements of $V(f)$ under Q coincide. The example illustrates Theorem 3.8.6.

To find curves which give rise to SLs we first find the primitive function Q for $q = f(w)^{\frac{n-2}{2}}$ (Definition 3.1). Next, for a pair of zeros of f, w_j, w_ℓ, we find $Q(w_j)$ and $Q(w_\ell)$ and construct a straight line through the two points. The inverse images by Q of the straight line are curves in the which give rise to SLs. If one of the preimages connects w_j and w_ℓ then it gives rise to a compact SL which is diffeomorphic to an n-sphere.

For n even, it is simple to find Q since q is a polynomial. For n odd, the procedure still works but we just need to be a little more careful (since the square root function is not an entire function).

4.1 Example 1

Let

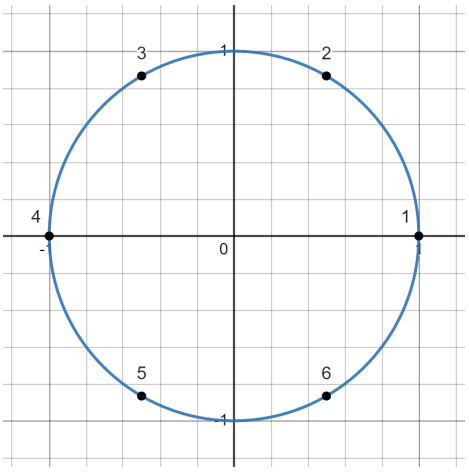
$$f(w) = w^6 - 1$$

and let

$$X_f^3 = \{(w, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \, : \, z_1^2 + z_2^2 + z_3^2 = f(w)\}.$$

It is convenient that $q = (w^6 - 1)^{\frac{1}{2}}$ is holomorphic in the disk of radius 1 around the origin since it enables us to use the power series expansion of q at 0 for all calculations. In Lemma 4.1.1 we show that Q maps vertices of the regular hexagon $w_1 \ldots w_6$ to vertices of a regular hexagon. In Lemma 4.1.2 we prove that certain rays are mapped by Q to rays.

Following that, we prove, in a series of propositions, the existence of paths (which give rise to SLs) between any pair of zeros of f. Thanks to symmetries of a regular polygon, it suffices to do this for each 'type' of pair of zeros.



Zeros of $f = w^6 - 1$ (labeled 1 - 6)

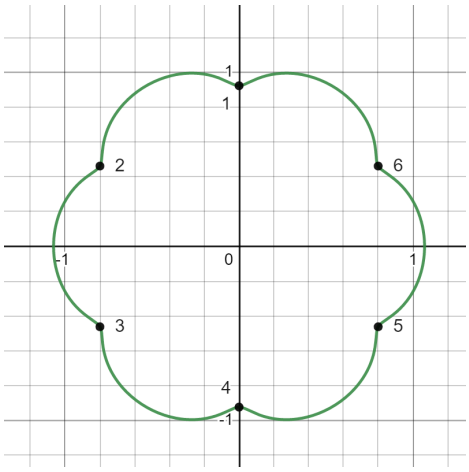
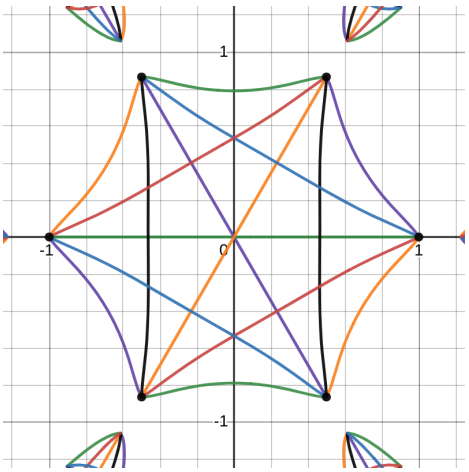
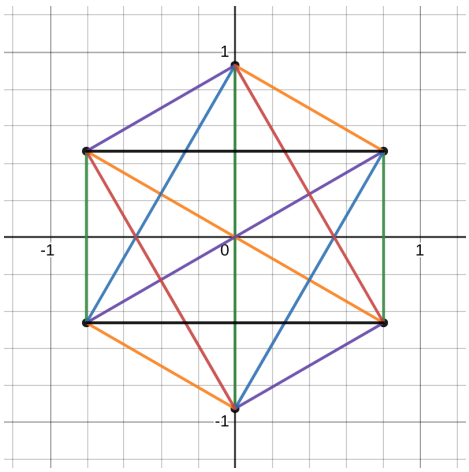


Image under Q (labeled 1 - 6)

Figure 4.1: Unit disc with the zeros of $f(w) = w^6 - 1$ and the image under Q.



Paths in $\mathcal{A}_{j\ell}$, which give rise to SLs, exist for any pair $w_j, w_\ell \in V(f)$.



Images under Q of the paths in $\mathcal{A}_{j\ell}$, on the left, are straight line segments.

Figure 4.2: Example 1. $n = 3$, $f(w) = w^6 - 1$
Note: $Q(w)$ approximated by the first two terms of its expansion, $i(w - \frac{1}{14}w^7)$, for this figure.

Lemma 4.1.1. *Let $w_0, \ldots, w_5$ be the roots of the polynomial f. Then $Q(w_0), \ldots, Q(w_5)$ are vertices of a regular hexagon and $|Q(w_j)| < 1$, $j = 0, \ldots, 5$.*

Proof. Expand q into a series at 0 (a choice is made for the branch of the square root but that is fine, as we saw in subsection 3.5):

$$q(w) = (w^6 - 1)^{\frac{1}{2}} = i(1 + (-w^6))^{\frac{1}{2}} = i \sum_{m=0}^{\infty} \binom{\frac{1}{2}}{m} (-w^6)^m.$$

The radius of convergence is 1 and the series converges absolutely on the closed disk of radius 1. Integrate q term by term to get an expression for Q:

$$Q(w) = i \sum_{m=0}^{\infty} \binom{\frac{1}{2}}{m} (-1)^m \frac{w^{6m+1}}{6m+1},$$

choosing the constant of integration so that $Q(0) = 0$.

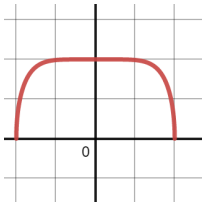
Let w_j be a zero of f. Then

$$Q(w_j) = i \sum_{m=0}^{\infty} \binom{\frac{1}{2}}{m} (-1)^m \frac{w_j^{6m+1}}{6m+1} = i w_j \sum_{m=0}^{\infty} \binom{\frac{1}{2}}{m} (-1)^m \frac{1}{6m+1} = w_j Q(1).$$

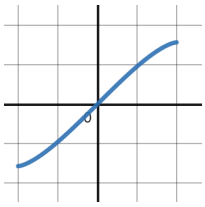
We just need to find $Q(1)$ to know $Q(w_j)$ for any zero w_j of f. $\frac{Q(1)}{i}$ is a positive real number smaller than 1. ($Q(1) = i\frac{\sqrt{\pi}\Gamma(\frac{7}{6})}{2\Gamma(\frac{5}{3})} \approx 0.910744i$. This calculation is by Wolfram Alpha.)

(Without calculating the value of $Q(1)$ we can still conclude that $|Q(1)| < 1$. The graph of $(1 - x^6)^{\frac{1}{2}}$ is shown on the left in Figure 4.3. Its integral passing through the origin looks approximately like the curve on the right in the same Figure. The slope is close to 1 over a large part of the interval $(-1, 1)$ and the slope decreases gradually to 0 when approaching 1 or -1. Thus the value at 1 is (somewhat) smaller than 1.)

□



$(1-x^6)^{\frac{1}{2}}$



Integral curve

Figure 4.3: for Lemma 4.1.1

Lemma 4.1.2. *The rays $re^{l\frac{\pi}{6}i}, r \in \mathbb{R}_+$ are mapped by Q to the rays $iRe^{l\frac{\pi}{6}i}, R \in \mathbb{R}_+$, $l \in \mathbb{Z}$.*

Proof. • For l even:

$$\begin{aligned} Q(re^{l\frac{\pi}{6}i}) &= i\sum_{m=0}^{\infty}\binom{\frac{1}{2}}{m}(-1)^m\frac{(re^{l\frac{\pi}{6}i})^{6m+1}}{6m+1} \\ &= ie^{l\frac{\pi}{6}i}\sum_{m=0}^{\infty}\binom{\frac{1}{2}}{m}(-1)^m\frac{r^{6m+1}}{6m+1} \\ &= ie^{l\frac{\pi}{6}i}Q(r) \in ie^{l\frac{\pi}{6}i}\mathbb{R}_+. \end{aligned}$$

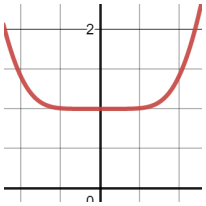
For l even, $|Q(re^{l\frac{\pi}{6}i})| < r$, as observed at the end of the proof of Lemma 4.1.1.

• For l odd:

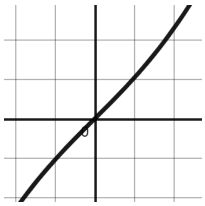
$$\begin{aligned} Q(re^{l\frac{\pi}{6}i}) &= i\sum_{m=0}^{\infty}\binom{\frac{1}{2}}{m}(-1)^m\frac{(re^{l\frac{\pi}{6}i})^{6m+1}}{6m+1} \\ &= ie^{l\frac{\pi}{6}i}\sum_{m=0}^{\infty}\binom{\frac{1}{2}}{m}(-1)^{2m}\frac{r^{6m+1}}{6m+1} \\ &= ie^{l\frac{\pi}{6}i}\int(1+r^6)^{\frac{1}{2}}dr. \end{aligned}$$

Claim: $|Q(re^{l\frac{\pi}{6}i})| > r$ for l odd.

Proof: See in Figure 4.4 on the left the graph of $(1+x^6)^{\frac{1}{2}}$. The slope of the tangent line to the integral curve is close to 1 and increases towards 1. The integral curve through the origin stays above the line $y = x$ for $x > 0$.



$(1+x^6)^{\frac{1}{2}}$



Integral curve

Figure 4.4: for Lemma 4.1.2

□

Proposition 4.1.3. *There exist compact SLs for paths between pairs of zeros of f which are opposite of each other:* 1 *and* -1, $\mathrm{e}^{\frac{\pi}{3}i}$ *and* $\mathrm{e}^{\frac{4\pi}{3}i}$, $\mathrm{e}^{\frac{2\pi}{3}i}$ *and* $\mathrm{e}^{\frac{5\pi}{3}i}$.

Proof. The straight lines which are connecting the pairs of opposite zeros of f are mapped into straight lines by Q by Lemma 4.1.2. Therefore the line segments between them give rise to compact SLs.

□

Lemma 4.1.4. *For any two adjacent vertices in the regular hexagon formed by the roots of f there exists a path joining them such that the corresponding compact Lag is special.*

Proof. By Theorem 3.10.2 it suffices to show that there exists a path, γ, between 0 and $\mathrm{e}^{\frac{\pi}{3}i}$ such that with the closed loop consisting of the path $Q(\gamma)$ and the straight line segment between the points $Q(1)$ and $Q(\mathrm{e}^{\frac{\pi}{3}i})$ does not contain in its interior any of the images of the zeros of the polynomial $w^6 - 1$.

Claim: We can take γ to be the path along the unit circle.

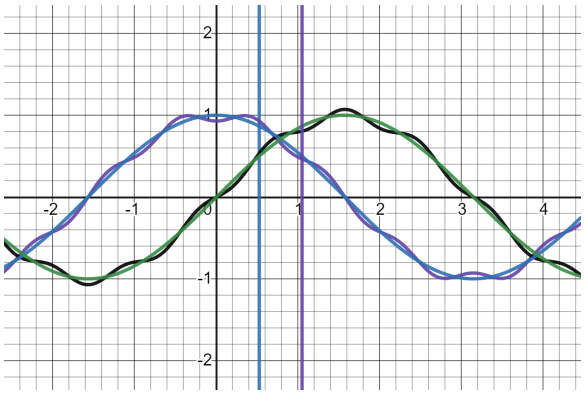


Figure 4.5: $\cos x - \frac{1}{14}\cos(7x)$ and $\sin x - \frac{1}{14}\sin(7x)$ superimposed on $\cos x$ and $\sin x$

Proof: Let $e^{\phi i}$ be a point on the unit circle. Then

$$Q(e^{\phi i}) = i(e^{\phi i} - \frac{1}{14}e^{7\phi i} - \frac{1}{208}e^{13\phi i} - \dots).$$

In Figure 4.5 we can see that the graphs of $\cos x - \frac{1}{14}\cos(7x)$ and $\sin x - \frac{1}{14}\sin(7x)$ stay near $\cos x$ and $\sin x$. $Q(\gamma)$ maps the points in the I, II, III, IV quadrant to points in the III, IV, I, II quadrant respectively. In Figure 4.1 , $Q(1)$ is the point of intersection of the closed curve with the positive imaginary axis and $Q(e^{\frac{\pi}{3}i})$ is in the II quadrant with argument $\frac{2\pi}{3}$.

□

The last type of path exists for the same reason:

Lemma 4.1.5. *For any two non-adjacent and not diametrically opposite vertices in the regular hexagon formed by the roots of f there exists a path joining them such that the corresponding compact Lag is special.*

The existence of one of the paths of each type implies the existence of all the paths of the given type due to symmetries. Therefore, in this example the maximal number of paths, $\frac{k(k-1)}{2}$ exists:

Theorem 4.1.6. *For any two zeros of $w^6 - 1$ there exists a path joining them which gives rise to an SL submanifold of $\{(z_1, z_2, z_3, w) \in \mathbb{C}^4 \,:\, z_1^2 + z_2^2 + z_3^2 = w^6 - 1\}$ (see Figure 4.2).*

4.2 Example 2

Let

$$f(w) = w^6 - 1$$

and let

$$X_f^4 = \{(w, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \,:\, z_1^2 + z_2^2 + z_3^2 + z_4^2 = f(w)\}.$$

In this case, as always when n is even, primitive function Q is a polynomial. In Lemma 4.2.1 we show that Q maps $w_1, \ldots w_6 \in V(f)$ to vertices of a regular hexagon. In Lemma 4.2.2 we prove that certain straight lines are mapped to straight lines by Q. (This is analogous to Lemma 4.1.2 of the $n = 3$ example.)

Following that, as in the subsection 4.1, we prove in a series of lemmas that for each 'type' of pair of zeros of f, a path which connects them and gives rise to an SL exists.

Now,

$$q(w) = f(w) = w^6 - 1 \quad \text{and} \quad Q(w) = \frac{1}{7}w^7 - w.$$

Lemma 4.2.1. *Q maps the zeros of f to vertices of a regular hexagon.*

Proof. $Q(w) = \frac{1}{7}w^7 - w = w\left(\frac{1}{7}w^6 - 1\right)$.

$$Q\left(e^{k\frac{2\pi}{6}}\right) = e^{k\frac{2\pi}{6}}\left(\frac{1}{7}\left(e^{k\frac{2\pi}{6}}\right)^6 - 1\right) = e^{k\frac{2\pi}{6}}\left(\frac{1}{7} - 1\right) = -\frac{6}{7}e^{k\frac{2\pi}{6}}$$

□

Lemma 4.2.2. *Q maps the straight lines $\arg w = l\frac{\pi}{6}$ to straight lines.*

Proof.

$$Q(r\mathrm{e}^{j\frac{2\pi}{6}i}) = r\mathrm{e}^{j\frac{2\pi}{6}i}(\frac{1}{7}(r\mathrm{e}^{j\frac{2\pi}{6}i})^6 - 1) = \mathrm{e}^{j\frac{2\pi}{6}i}(\frac{1}{7}r^7 - r), \quad l = 2j,$$

and

$$Q(r\mathrm{e}^{(2j+1)\frac{\pi}{6}i}) = r\mathrm{e}^{(2j+1)\frac{1\pi}{6}i}(\frac{1}{7}(r\mathrm{e}^{(2j+1)\frac{1\pi}{6}i})^6 - 1) = \mathrm{e}^{(2j+1)\frac{1\pi}{6}i}(-\frac{1}{7}r^7 - r), \quad l = 2j+1.$$

Note that in both cases (l even or odd), points on the line $\arg = l\frac{\pi}{6}$ inside the unit circle are mapped too the line $\arg = l\frac{\pi}{6} + \pi$.

□

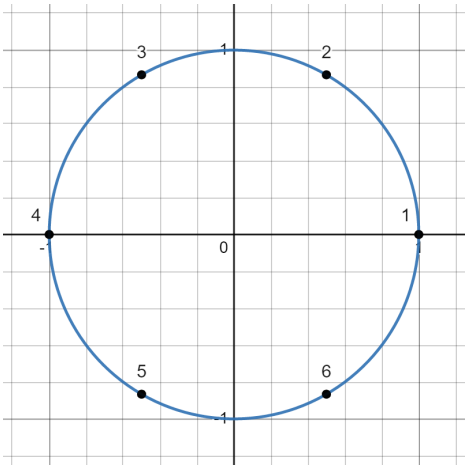
When $l = 2j$ the points on the unit circle are mapped to points of modulus $\frac{6}{7}$ and those with $l = 2j+1$ are mapped to points of modulus $\frac{8}{7}$. See in Figure 4.6 the image of the unit disc under Q. The root $w_j = \mathrm{e}^{j\frac{2\pi}{6i}}$ is mapped to the point on the boundary nearest to $-w_j$:

$$Q(\mathrm{e}^{j\frac{2\pi}{6}i}) = -\frac{6}{7}\mathrm{e}^{j\frac{2\pi}{6}i}, \quad 0 \leq j \leq 5.$$

For an arbitrary angle $\phi \in (0, \frac{\pi}{6})$,

$$\arg\left(Q(r\mathrm{e}^{\phi i})\right) = \arg\left(\frac{1}{7}r^7\mathrm{e}^{7\phi i} - r\mathrm{e}^{\phi i}\right) \approx \phi + \pi,$$

and



Zeros of $f = w^6 - 1$ (labeled 1 - 6)

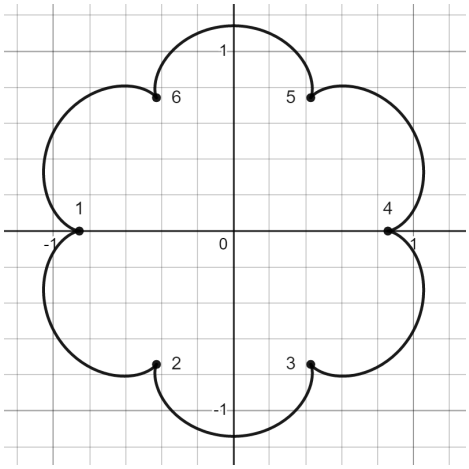
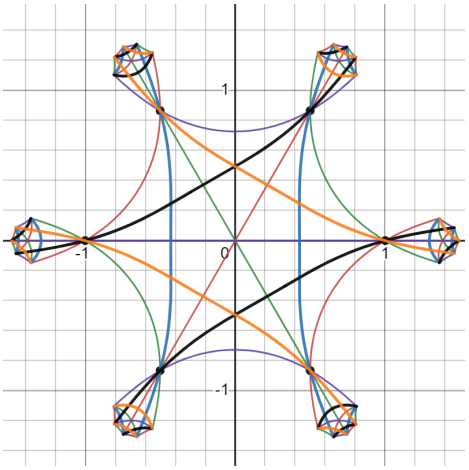
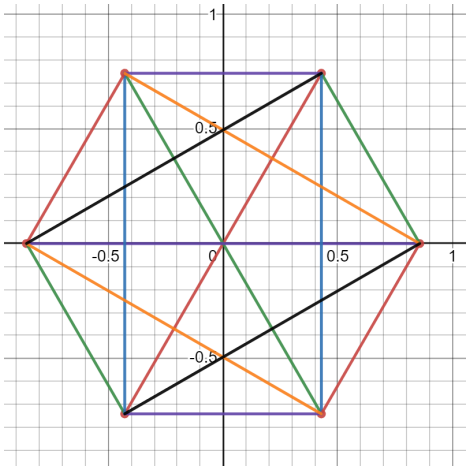


Image under Q (labeled 1 - 6)

Figure 4.6: Unit disc with the zeros of $f(w) = w^6 - 1$ and the image under Q.



Paths in $\mathcal{A}_{j\ell}$, which give rise to SLs, exist for any pair $w_j, w_\ell \in V(f)$.



Images under Q of the paths in $\mathcal{A}_{j\ell}$, on the left, are straight line segments.

Figure 4.7: Example 2. $n = 4$, $f(w) = w^6 - 1$.

$$\left|Q(re^{\phi i})\right| = \left|\frac{1}{7}r^7e^{7\phi i} - re^{\phi i}\right| \in \left[\frac{6}{7}r, \frac{8}{7}r\right].$$

It follows that the interior of the unit disc maps into the interior of the flower shape in Figure 4.6 and the boundary maps onto the boundary. Points with $\arg = l\frac{\pi}{6}$ map into points with $\arg = \pi + l\frac{\pi}{6}$. Other points are mapped to points with argument which differs by a small amount from the $\pi+$ the argument of the original.

Theorem 4.2.3. *For any two zeros of $w^6 - 1$ there exists a path joining them which gives rise to an SL submanifold of $\{(z_1, z_2, z_3, z_4, w) \in \mathbb{C}^5 : z_1^2 + z_2^2 + z_3^2 + z_4^2 = w^6 - 1\}$ (see Figure 4.7).*

Proof. As in the previous example, $n = 3$, in order to show the existence of paths between consecutive vertices we note that the image of the arc of the unit circle between two consecutive vertices of the hexagon whose vertices are the zeros of $w^6 - 1$, is an arc of the flower shape and that the area it encloses together with the straight line segment between the endpoints does not contain the image under Q of any zero of $w^6 - 1$.

For a path connecting two vertices adjacent to the same third vertex, take the image under Q of the straight line segment between them. Now, its image under Q is not contained in a straight line, and together with the straight line segment connecting the images under Q of the two vertices, contains no image of another zero of $w^6 - 1$. This is because we know that the points on the same ray which are closer to the origin are mapped by Q to points closer to the origin.

For diametrically opposite vertices, the straight line segment between them is mapped to the straight line segment between their images so the straight line segment is the desired path. □

4.3 Example 3

In this example the polynomial f is chosen in such a way that its antiderivative Q has two double roots. (We are choosing $n = 4$ so that $q = f^{\frac{n-2}{2}} = f$ and $Q' = q = f$.) This represents the special situation in which for two roots w_1 and w_2 of f, $Q(w_1) = Q(w_2)$ (See Theorem 3.8.6).

Let

$$f(w) = w^4 - (2 + \frac{9}{20}i)w^3 + (\frac{6}{5} + \frac{27}{40}i)w^2 - (\frac{1}{5} + \frac{9}{40}i)w = w(w-1)(w-(\frac{1}{2} + \frac{1}{4}i))(w-(\frac{1}{2} + \frac{1}{5}i)).$$

Then

$$Q(w) = \frac{1}{5}w^5 - (\frac{1}{2} + \frac{9}{80}i)w^4 + (\frac{2}{5} + \frac{9}{40}i)w^3 - (\frac{1}{10} + \frac{9}{80}i)w^2 = \frac{1}{5}w^2(w-1)^2(w-(\frac{1}{2} + \frac{9}{16}i)).$$

The polynomial f has four roots: $w_1 = 0$, $w_2 = 1$, $w_3 = \frac{1}{2} + \frac{1}{4}i$, and $w_4 = \frac{1}{2} + \frac{1}{5}i$.

Lemma 4.3.1. *There are three SL spheres in X_f^4 defined by paths connecting roots of the polynomial f. All three SLs have constant phase equal to $\frac{\pi}{2}$.*

Proof. First we evaluate Q at each of the zeros of f:

$$Q(w_1) = Q(w_2) = 0,$$

$$Q(w_3) = \frac{1}{5}((\frac{1}{2} + \frac{1}{4}i)(-\frac{1}{2} + \frac{1}{4}i))^2(\frac{1}{2} + \frac{1}{4}i - (\frac{1}{2} + \frac{9}{10}i)) = -\frac{13i}{1024},$$

$$Q(w_4) = \frac{1}{5}((\frac{1}{2} + \frac{1}{5}i)(-\frac{1}{2} + \frac{1}{5}i))^2(\frac{1}{2} + \frac{1}{5}i - (\frac{1}{2} + \frac{9}{10}i)) = -\frac{7 \cdot 29^2 i}{500000}.$$

Therefore, $Q(w_1)$, $Q(w_2)$, $Q(w_3)$, and $Q(w_4)$ are all contained in the line $\mathrm{Re}(Q(w)) = 0$. For

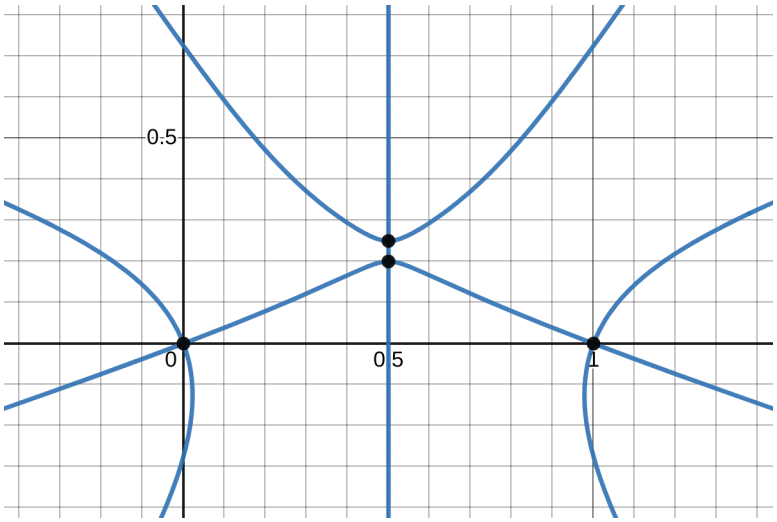


Figure 4.8: Preimage of the line $\mathrm{Re}(Q(w)) = 0$

a path γ such that θ_γ =const. and which connects w_1 and w_2, $Q(\gamma)$ is a loop which starts and ends at the origin and is contained in $\mathrm{Re}(Q(w)) = 0$. The extreme point of that loop satisfies $Q'(w) = 0$. Since $Q' = f$, the line segment passes through the image by Q of another zero of f. That is why the SL defined by a curve joining two such roots necessarily splits into a union of two SL spheres with a common point.

There are three SLs, corresponding to the paths: from $(0, 0)$ to $(\frac{1}{2}, \frac{1}{5})$, from $(\frac{1}{2}, \frac{1}{5})$ to $(1, 0)$, and from $(\frac{1}{2}, \frac{1}{5})$ to $(\frac{1}{2}, \frac{1}{4})$.

All three have the same phase $\theta = \frac{\pi}{2}$. There is no SL sphere corresponding to a path between $(0, 0)$ and $(1, 0)$, but there is a constant phase union of two Lag spheres with a common point. The same holds for the path between $(0, 0)$ and $(\frac{1}{2}, \frac{1}{4})$, and for the path between $(1, 0)$ and $(\frac{1}{2}, \frac{1}{4})$. See Figure 4.8. □

4.4 Limit as $n \to \infty$

If we fix a polynomial $f \in \mathcal{P}_{simple}$, we can construct X_f^n for any n. Even though $n \in \mathbb{N}$, the equation

$$0 = \frac{d}{du}\theta_\gamma = \frac{d}{du} \arg \Big[\big(f(\gamma(u))\big)^{\frac{n-2}{2}} \gamma'(u) \Big], \tag{4.1}$$

which characterizes paths γ such that Lag L_γ is special, can be considered when the parameter n is allowed to take non-integer values, $n \in [1, \infty)$.

Integral curves of the equation (4.1) change continuously with n because the dependence on the parameter n is continuous.

In this section we find the equation which defines the limit of those paths as $n \to \infty$.

Lemma 4.4.1. *The limit of the solution to the equation (4.1) when $n \to \infty$ satisfies the following equation:*

$$\arg \Big[f(\gamma) \Big] = \text{const.}$$

Proof. Let F be a primitive function for f, $F' = f$. Suppose that u is the arclength parameter for γ and that s is the arclength parameter for $F(\gamma)$. From:

$$\theta_\gamma = \arg \Big[\gamma'(u) \big(f(\gamma(u))\big)^{\frac{n-2}{2}} \Big] = \arg \Big[\big(\gamma'(u) f(\gamma(u))\big)^{\frac{n-2}{2}} \Big] + \arg \Big[\big(\gamma'(u)\big)^{\frac{4-n}{2}} \Big],$$

after differentiating by u we get:

$$\frac{d\theta}{du} = \frac{n-2}{2} \frac{d}{du} \arg \Big[\frac{d}{du} F(\gamma) \Big] + \frac{4-n}{2} \frac{d}{du} \arg \Big[\frac{d}{du} \gamma \Big].$$

In the first term we replace $\frac{d}{du}$ by $\frac{ds}{du}\frac{d}{ds}$, and for the second term we use the fact that the curvature of a curve is the rate of change with respect to the arclength of the argument of the tangent vector

$$\frac{d\theta}{du} = \frac{n-2}{2}\frac{ds}{du}\frac{d}{ds}\arg\left[\frac{ds}{du}\frac{d}{ds}F(\gamma)\right] + \frac{4-n}{2}\kappa(\gamma).$$

Now use the fact that $\frac{ds}{du} = |f(\gamma)|$ (as in (3.12)),

$$\frac{d\theta}{du} = \frac{n-2}{2}\kappa(F(\gamma))|f(\gamma)| + \frac{4-n}{2}\kappa(\gamma).$$

SL has $\theta = \text{const.}$, which implies

$$(n-4)\kappa(\gamma) = (n-2)\kappa(F(\gamma))|f(\gamma)|.$$

Note that $n = 4 \Leftrightarrow \kappa(F(\gamma)) = 0$ which is in accordance with the characterization of paths giving rise to SLs by the equation $\kappa(Q(\gamma)) = 0$, since for $n = 4$, $Q = F$.

Consider now the limit:

$$\lim_{n\to\infty}\kappa(\gamma) = \lim_{n\to\infty}\frac{n-2}{n-4}\kappa(F(\gamma))|f(\gamma)|.$$

It follows that the limit curve is defined by:

$$\kappa(\gamma) = \kappa(F(\gamma))|f(\gamma)|. \tag{4.2}$$

We can rewrite the last equation (4.2) in terms of derivatives by u, where we assume that u is the

arclength of γ and s is the arclength of $F(\gamma)$:

$$\begin{aligned}\frac{d}{du}\arg\left[\gamma'\right] &= \frac{d}{ds}\arg\left[\frac{d}{ds}F(\gamma)\right]\frac{ds}{du}\\ &= \frac{d}{du}\arg\left[\frac{d}{du}F(\gamma)\frac{du}{ds}\right]\\ &= \frac{d}{du}\arg\left[\frac{d}{du}F(\gamma)\right]\\ &= \frac{d}{du}\arg\Big[f(\gamma)\gamma'\Big].\end{aligned}$$

Therefore,

$$\frac{d}{du}\Big(\arg\left[f(\gamma)\gamma'\right]-\arg\left[\gamma'\right]\Big)=0.$$

So,

$$\frac{d}{du}\Big(\arg\left[(f(\gamma)\gamma')(\gamma')^{-1}\right]\Big)=0.$$

In conclusion, the limit curve satisfies:

$$\arg\left[f(\gamma)\right]=\text{const.}$$

□

Lemma 4.4.2. *The limit curves pass through points where $f'(w)=0$.*

Proof. The limiting path γ is in $\mathcal{A}_{j\ell}$. Its image, $f(\gamma)$, starts and ends at the same point, 0, since $f(w_j)=f(w_\ell)=0$. It goes out on a straight line and comes back along the same straight line. At the point where $|f(\gamma)|=\max$ there is a singularity: the tangent vector changes its direction by π. Therefore, $f'=0$ at that point, as $\frac{d}{du}(f(\gamma))=f'(\gamma)\gamma'$ and $\gamma'\neq 0$. □

4.4.1 Examples - $n \to \infty$

We will find the limiting positions of paths γ such that L_γ is SL for the examples presented earlier in this chapter. Since Examples 4.1 and 4.2 are based on the same polynomial f, they are, naturally, treated together in the limit.

•· Let $f = w^6 - 1$. For any value of n there exist paths between any two zeros of f which give rise to SLs in X_f^n, as we have seen in examples 4.1 and 4.2 for $n = 3$ and $n = 4$. We can now find the limiting positions of the paths with that property.

By Lemma 4.4.1, $f(\gamma)$ is a straight line through the origin, thus defined by an equation of the form:

$$\mathrm{Im}(f) = k\big(\mathrm{Re}(f)\big)$$

or

$$\mathrm{Re}(f) = 0.$$

Writing in terms of the real variables x, y, where $w = x + iy$:

$$6x^5y - 20x^3y^3 + 6xy^5 = k\big(x^6 - 15x^4y^2 + 15x^2y^4 - y^6 - 1\big). \tag{4.3}$$

We need to find k such that the point $(1, 0)$ is on the curve defined by (4.3). By Lemma 4.4.2, any such line passes through the image under f of a zero of f'. As

$$f'(w) = 6w^5,$$

$w = 0$ is the only zero of f'. $f(0) = -1$. The slope of the line passing through $(0, 0)$ and $(-1, 0)$

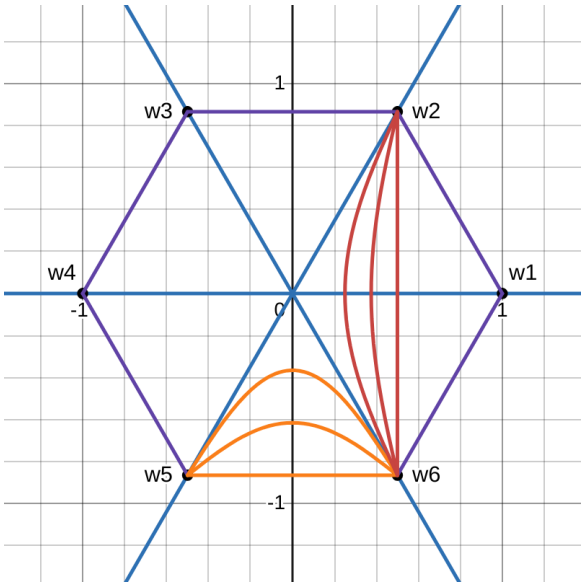


Figure 4.9: $n \to \infty$

is 0, which implies $k = 0$. For $k = 0$, the equation (4.3) becomes:

$$6x^5y - 20x^3y^3 + 6xy^5 = 0,$$

which can be factored as:

$$2xy(\sqrt{3}x - y)(\sqrt{3}x + y)(x - \sqrt{3}y)(x + \sqrt{3}y) = 0.$$

The graph is a union of six lines, three of which contain roots of f.

In Figure 4.9 we illustrate that the path between w_5 and w_6 has the limit $\overline{w_5O} \cup \overline{Ow_6}$ (union of two straight line segments), and the path between w_6 and w_2 has the limit $\overline{w_6O} \cup \overline{Ow_2}$.

•· In Example 4.3 where $n = 4$ only $3(= \deg f - 1)$ paths which give rise to SLs exist. In the limit, there are three paths as well. They are constructed as paths γ satisfying $\arg\big[f(\gamma)\big] =$ const., thus preimages under f of straight lines through the origin. To determine which lines, we

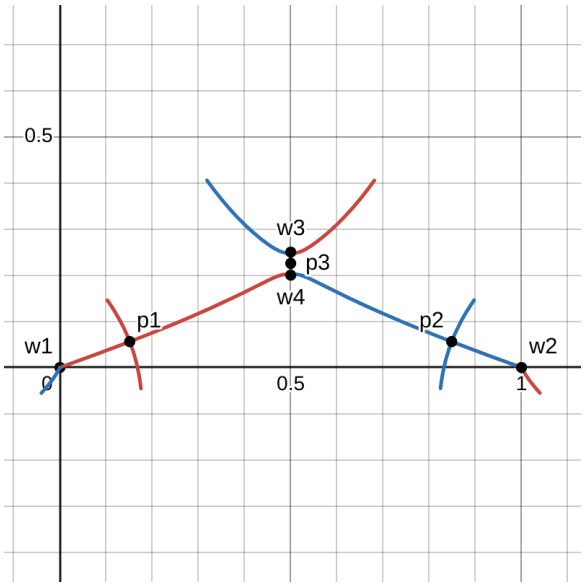


Figure 4.10: $n \to \infty$

find images of zeros of f', $\{p_1, p_2, p_3\} = V(f')$, since each limit curve contains a point from $V(f')$. Then we construct straight line segments from the origin to each of $f(p_1)$, $f(p_2)$, $f(p_3)$ and we find their preimages (pictured in Figure 4.10).

Chapter 5: Equivariant Lagrangians

Harvey and Lawson [4, Section III.3.B.] present a family of examples of Lagrangians in $\mathbb{C}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ which are invariant under the diagonal action of $SO(n)$. We describe the Harvey–Lawson construction and we explain how their Lagrangians are related to Lagrangians in Milnor fibers which we study in this thesis.

5.1 Harvey–Lawson construction

In this section we summarize [4, Section III.3.B.].

Let $\rho : \mathbb{R}_{>0} \to \mathbb{R}_{>0},\ a \mapsto \rho(a)$ be a C^1 function. Let

$$L := \Big\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^n \ :\ |\mathbf{x}|\mathbf{y} = |\mathbf{y}|\mathbf{x} \text{ and } |\mathbf{y}| = \rho(|\mathbf{x}|)\Big\}. \tag{5.1}$$

Proposition 5.1.1. *L given by (5.1) is a Lag submanifold of $\mathbb{C}^n$ with the standard symplectic structure.*

Proof. Let ϕ be a primitive function for ρ. Then

$$\frac{\partial}{\partial x_j}\phi(|\mathbf{x}|) = \frac{\rho(|\mathbf{x}|)}{|\mathbf{x}|}x_j$$

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be given by

$$F(\mathbf{x}) := \frac{\rho(|\mathbf{x}|)}{|\mathbf{x}|}\mathbf{x} = \Big(\frac{\partial}{\partial x_1}\phi(|\mathbf{x}|), \frac{\partial}{\partial x_2}\phi(|\mathbf{x}|), \ldots, \frac{\partial}{\partial x_n}\phi(|\mathbf{x}|)\Big).$$

By definition, L is the graph of $F = \nabla\phi(|\mathbf{x}|)$:

$$L = \Big\{\big(\mathbf{x}, \frac{\rho(|\mathbf{x}|)}{|\mathbf{x}|}\mathbf{x}\big) \,:\, \mathbf{x} \in \mathbb{R}^n\Big\} = \Big\{\big(\mathbf{x}, F(\mathbf{x})\big) \,:\, \mathbf{x} \in \mathbb{R}^n\Big\}.$$

It follows that L is Lag which we can verify explicitly. TL_p, $p \in L$ is generated by vectors:

$$v_j = \Big(0, \ldots, 1, \ldots, 0, 0, \frac{\partial^2}{\partial x_j \partial x_1}\phi(|\mathbf{x}|), \ldots, \frac{\partial^2}{\partial x_j \partial x_n}\phi(|\mathbf{x}|)\Big).$$

Evaluate the symplectic form on a pair of tangent vectors:

$$\sum_{m=1}^{n} dx_m \wedge dy_m(v_j, v_k) = \frac{\partial^2}{\partial x_j \partial x_k}\phi(|\mathbf{x}|) - \frac{\partial^2}{\partial x_k \partial x_j}\phi(|\mathbf{x}|) = 0,$$

as ρ is C^1 by assumption and thus ϕ is C^2.

□

Harvey–Lawson prove the following theorem [4, III.3.B, Theorem 3.5]. The way the theorem is stated here corresponds to the definition of SLs which we use, namely that the Lagrangian angle is a constant function. In Harvey–Lawson's *Calibrated Geometries* SLs are defined by $\mathrm{Im}\Omega|_L = 0$.

Theorem 5.1.2. *$L \subset \mathbb{C}^n$ given by (5.1) is an SL if and only if*

$$\{(a + i\rho(a))^n \,:\, a \in \mathbb{R}\}$$

is contained in a straight line in the complex plane.

Proof. Let F_* be the differential of the map F. The matrix F_* has entries $h_{j\ell}$:

$$h_{j\ell} = \frac{\partial}{\partial x_j}\left(\frac{\rho(|\mathbf{x}|)}{|\mathbf{x}|}x_l\right).$$

L is an SL if and only if the argument of the determinant of the matrix $I + iF_*$ is constant. Harvey–Lawson find this determinant explicitly [4, p. 99] by finding the eigenvalues of F_*.

Let $a = |\mathbf{x}|$. It is straightforward to verify that F_* has eigenvalue

$$\frac{\rho(a)}{a} + a\frac{d}{da}\left(\frac{\rho(a)}{a}\right) = \frac{d\rho}{da}$$

with eigenvector $\mathbf{x}$ and that the only other eigenvalue is

$$\frac{\rho(a)}{a}$$

which has multiplicity $n-1$ and eigenspace $\mathbf{x}^\perp$ (hyperplane perpendicular to $\mathbf{x}$). Now,

$$\det\left(I + iF_*\right) = \left(1 + i\frac{\rho(a)}{a}\right)^{n-1}\left(1 + i\frac{d\rho(a)}{da}\right).$$

To find the phase of the Lag L, observe that

$$\det\left(I + iF_*\right) = \left(1 + i\frac{\rho(a)}{a}\right)^{n-1}\left(1 + i\frac{d\rho(a)}{da}\right) = \frac{1}{a^{n-1}}\frac{d}{da}\Big(a + i\rho(a)\Big)^n.$$

Since $a \in \mathbb{R}$,

$$\arg\det\left(I + iF_*\right) = \arg\frac{d}{da}\Big(a + i\rho(a)\Big)^n.$$

The argument is constant if and only if $\big(a+i\rho(a)\big)^n$ is contained in a straight line. This completes the proof.

□

5.2 Relationship to Milnor fibers

Consider the Milnor Fiber X_f^n (Definition 2.1.13) for $f = w^2$. We normally require the polynomial f to have simple zeros only in order to obtain a smooth manifold. In this case X_f^n is not smooth everywhere. It has a singular point at $(0, 0, \ldots 0)$.

Let γ be a curve in $\mathbb{C}$. Then (Definition 2.1.19)

$$L_\gamma = \{(w, \mathbf{z}) \in X_f^n \,:\, w \in \gamma,\, \mathbf{z} \in \mathbb{C}^n,\, \mathbf{r} \in S^{n-1} \subset \mathbb{R}^n,\, \mathbf{z} = \mathbf{r}\sqrt{f(\gamma)}\}.$$

Let Π be the projection map:

$$\Pi : \mathbb{C}^{n+1} \to \mathbb{C}^n,$$

$$\Pi : (w, \mathbf{z}) \mapsto \mathbf{z}.$$

Theorem 5.2.1. *(i) Given L, as defined in (5.1), there exists a curve γ such that*

$$L = \Pi(L_\gamma).$$

(ii) Conversely, given a path γ, there is a subdivision of γ, $\gamma = \gamma_1.\gamma_2.\ldots\gamma_p$ such that for each segment γ_j, $\Pi(L_{\gamma_j})$ is a Lag in $\mathbb{C}^n$.

Proof. (i) Let $\mathbf{z} \in L$. Let a and $\rho(a)$ be as in 5.1. Let

$$w := a + i\rho(a)$$

define a curve $\gamma \subset \mathbb{C}$. Then

$$z_j = x_j + i\frac{\rho(|\mathbf{x}|)}{|\mathbf{x}|}x_j = \frac{x_j}{|\mathbf{x}|}\big(|\mathbf{x}| + i\rho(|\mathbf{x}|)\big).$$

Therefore,

$$\sum_{j=1}^{n} z_j^2 = \Big(|\mathbf{x}| + i\rho(|\mathbf{x}|)\Big)^2 \frac{1}{|\mathbf{x}|^2}\sum_{j=1}^{n} x_j^2 = w^2$$

and clearly $w = |\mathbf{x}| + i\rho(|\mathbf{x}|) \in \gamma$.

(ii) Let $(w, z_1, \ldots, z_n) \in L_\gamma$ for some curve γ. Then

$$z_j = r_j w, \quad j = 1, \ldots, n.$$

From there,

$$x_j = r_j u \quad \text{and} \quad y_j = r_j v,$$

so

$$u^2 = \sum x_j^2 \quad \text{and} \quad v^2 = \sum y_j^2.$$

Let $r = |u|$ and $\rho(r) = |v|$. The path γ may not be the graph of a function ρ, but there exists a

concatenation of paths $\gamma_1 \dots . \gamma_p = \gamma$ such that γ_l is the graph of a function ρ_l for each l.

Since

$$y_j = r_j v = \frac{x_j}{u} v = \frac{\rho_l(r)}{r} x_j$$

$$\Pi(L_{\gamma_l}) = \left\{ (z_1, \dots z_n) \, : \, r y_j = \rho_l(r) x_j, \, j = 1, \dots n \right\} =: L_l.$$

The last expression defines a Lag in $\mathbb{C}^n$ as in the Harvey–Lawson construction (5.1).

□

5.3 Arbitrary f

Suppose now that X_f^n is given for some arbitrary f. Let γ be a path in $\mathbb{C}$. With a choice of a branch of the complex square root function, $\sqrt{f(\gamma)}$ is a path in the complex plane. It is a concatenation of paths which are graphs of functions $\rho_j = \rho_j(a)$. ($a = \text{Re}\left[\sqrt{f(\gamma)}\right]$, $\rho(a) = \text{Im}\left[\sqrt{f(\gamma)}\right]$).

For each ρ_j we have a Lag in $\mathbb{C}^n$ by the construction given in (5.1).

Lemma 5.3.1. *Let X_f^n be a Milnor fiber. Let $L_\gamma \subset X_f^n$ be a Lag submanifold. Then there exists a concatenation of paths $\gamma_1 \cdot \gamma_2 \cdot \dots \cdot \gamma_m = \gamma$ such that $\Pi(L_{\gamma_j})$ is a Lag submanifold in $\mathbb{C}^n$ of the type given in (5.1).*

Proof. Without loss of generality and for ease of notation assume that $f(\gamma)$ satisfies the following:

- There is a choice of a branch of $\sqrt{f(\gamma)}$ contained in the first quadrant of the complex plane.
- The imaginary part of this choice of $\sqrt{f(\gamma)}$ is a function of the real part.

Let

$$a := \mathrm{Re}\sqrt{f(\gamma)}, \quad \text{and} \quad \rho := \rho(a) = \mathrm{Im}\sqrt{f(\gamma)}.$$

Now, construct a Lag in $\mathbb{C}^n$ according to (5.1):

$$L = \Big\{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^n \,:\, |\mathbf{x}|\mathbf{y} = |\mathbf{y}|\mathbf{x} \text{ and } |\mathbf{y}| = \rho(|\mathbf{x}|)\Big\}.$$

Identifying $\mathbb{R}^n \times \mathbb{R}^n$ with $\mathbb{C}^n$ we can write for $(\mathbf{x},\mathbf{y}) \in L$:

$$(\mathbf{x},\mathbf{y}) = \left(\frac{|\mathbf{y}|\mathbf{x}}{|\mathbf{y}|}\,,\,\frac{|\mathbf{x}|\mathbf{y}}{|\mathbf{x}|}\right) = \left(\frac{1}{|\mathbf{y}|} + \frac{i}{|\mathbf{x}|}\right)|\mathbf{y}|\mathbf{x} = \left(1 + \frac{|\mathbf{y}|}{|\mathbf{x}|}i\right)\mathbf{x} = \left(1 + \frac{\rho(a)}{a}i\right)\mathbf{x}.$$

On the other hand

$$L_\gamma = \Big\{(w,\mathbf{z}) \,:\, \sum_{j=1}^{n} z_j^2 = f(\gamma)\,,\, \mathbf{z} = \mathbf{r}\sqrt{f(\gamma)}\,,\, \mathbf{r} \in S^{n-1}\Big\},$$

and, for $(w,z) \in L_\gamma$,

$$\mathbf{z} = \mathbf{r}\sqrt{f(\gamma)} = \mathbf{r}\left(\mathrm{Re}\Big(\sqrt{f(\gamma)}\Big) + i\,\mathrm{Im}\Big(\sqrt{f(\gamma)}\Big)\right) = (a + i\,\rho(a))\,\mathbf{r} = \left(1 + i\,\frac{\rho(a)}{a}\right)a\mathbf{r}.$$

Thus, for $(w,\mathbf{z}) \in L_\gamma$,

$$\Pi(w,\mathbf{z}) = \mathbf{z} = \left(1 + i\frac{\rho}{a}\right)a\mathbf{r} \in L,$$

with $\mathbf{x} = a\mathbf{r}$, so $\Pi(L_\gamma) \subset L$.

But also $L \subset \Pi(L_\gamma)$ since for some fixed r, $\rho(r)$ is also fixed and by construction,

$$a + i\rho = \sqrt{f(\gamma(u))}$$

for some u which means that $(a\mathbf{r}, \rho\mathbf{r}) \in L$ for all $\mathbf{r} \in \mathbb{R}^n$ such that $|\mathbf{r}| = 1$. Then

$$\Pi((w, (a + i\rho)\mathbf{r})) = (a\mathbf{r}, \rho\mathbf{r}).$$

□

Remark 5.3.2. (Special Lagrangians)

By Theorem 5.1.2, the condition for a Harvey–Lawson Lag in $\mathbb{C}^n$ to be special, with $a + i\rho(a) = \sqrt{f(\gamma)}$ and f arbitrary, becomes: the graph of

$$(f(\gamma))^{\frac{n}{2}}$$

is contained in a straight line.

On the other hand, the condition for L_γ to be an SL is: the graph of

$$Q(\gamma)$$

is contained in a straight line.

Let $\gamma = \gamma(t)$. Then the slopes of $(f(\gamma))^{\frac{n}{2}}$ and $Q(\gamma)$ are different unless $f'(\gamma)$ is real, since

$$\frac{d}{dt}(\frac{2}{n}f(\gamma))^{\frac{n}{2}} = (f(\gamma))^{\frac{n-2}{2}} f'(\gamma)\gamma',$$

and

$$\frac{d}{dt}Q(\gamma) = (f(\gamma))^{\frac{n-2}{2}}\gamma'.$$

In conclusion, L_γ and L are special for the same choice of γ if and only if $f'(\gamma) \subset \mathbb{R}e^{i\alpha}$, for some fixed $\alpha \in \mathbb{R}$.

5.4 Equivariant Lagrangian submanifolds

Equivariant Lag submanifolds in [32] are defined using a very similar construction as the Harvey–Lawson Lags defined by (5.1). As the formulation is slightly different, we will give here the definition according to [32, Section 2].

Definition 5.4.1. *Let α be a regular curve in the complex plane: $\alpha(s) = u(s) + iv(s)$, $(u, v) : (-\delta, \delta) \to \mathbb{C}$, $\delta > 0$. Let $\mathbf{r} \in S^{n-1} \subset \mathbb{R}^n$. Let the map $F : (-\delta, \delta) \times S^{n-1} \to \mathbb{C}^n$ be given by*

$$F(s, \mathbf{r}) := \big(u(s)\mathbf{r}, v(s)\mathbf{r}\big).$$

Then F is an equivariant submanifold of $\mathbb{C}^n$ and α is called the profile curve of F.

Set $a = |u(s)\mathbf{r}| = |u(s)|$ and $\rho = |v(s)\mathbf{r}| = |v(s)|$. Then

$$|\mathbf{x}|\mathbf{y} = |u(s)|v(s)\mathbf{r},$$

and

$$|\mathbf{x}|\mathbf{y} = |v(s)|u(s)\mathbf{r}.$$

The condition in (5.1) is satisfied up to a $\pm$ sign, therefore the construction in [32, Section 2] is

more general but it coincides with (5.1) when $u(s)v(s) > 0$.

5.5 Equivariant Lagrangian spheres

There are no embedded Lag spheres in $\mathbb{C}^n$. For a particular choice of the curve γ in Definition 5.4.1 the resulting Lag is an immersed sphere.

Savas-Halilaj–Smoczyk [32, Lemma 2.3] characterize equivariant Lagrangian immersions of the sphere S^n in $\mathbb{C}^n$: If $F : S^n \to \mathbb{C}^n$ is a real equivariant Lagrangian immersion then its profile curve can be parametrized by $w : [-\pi, \pi] \to \mathbb{C}$, with w real analytic, $w^{-1}(0) = \{-\pi, 0, \pi\}$, and $w(-s) = -w(s)$, for all $s \in [-\pi, \pi]$.

Of particular interest is a special case of this construction known as the Whitney sphere:

$$\mathcal{W}_{R,n} : S^n \to \mathbb{R}^{2n}$$
$$\mathcal{W}_{R,n} : (x_0, x_1, \dots, x_n) \mapsto \frac{R}{1 + x_0^2}(x_1, \dots, x_n; x_0x_1, \dots, x_0x_n).$$

It is an immersed Lag sphere, embedded except for:

$$(\pm 1, 0, \dots, 0) \mapsto (0, \dots, 0; 0, \dots, 0)$$

Remark 5.5.1. There are no embedded Lag spheres in $\mathbb{C}^n$ with standard symplectic structure (Gromov [33]).

Chapter 6: Almost Lagrangian Mean Curvature Flow

6.1 Introduction

6.1.1 The flow of $Q(\gamma)$

In this section we determine the equation governing the flow of the curve $Q(\gamma)$ induced by the almost Lagrangian mean curvature flow of the Lagrangian sphere L_γ in the Milnor fiber X_f^n.

Definition 6.1.1. *Almost mean curvature vector for a Lag submanifold L of an almost Calabi-Yau manifold (X, J, ω, Ω), is the vector $J\nabla_L\theta$, where θ is the Lagrangian angle function on L.*

Thomas–Yau [3, Section 6, equation (6.5)] show:

Lemma 6.1.2. *Let L_γ be a Lag sphere in X_f^n. Then the almost mean curvature vector $J\nabla_L\theta \in NL$, is the lift to TX_f^1 of the following vector in $\mathbb{C}$:*

$$\frac{4|f(\gamma)|}{4|f(\gamma)| + |f'(\gamma)|^2}\left(\kappa(\gamma) + \frac{2-n}{2}\vec{n}(log|f(\gamma)|)\right)\vec{n},$$

where $\vec{n}$ is the unit normal vector, and $\kappa(\gamma)$ is the (scalar) curvature of the curve γ.

Instead of studying the flow of the path γ by this vector, as is done by Thomas–Yau [3], we wish to study the corresponding flow of the curve $Q(\gamma)$, where Q is defined as a primitive function of $q(w) = (f(w))^{\frac{n-2}{2}}$. The first step is to determine the equation governing this flow.

Theorem 6.1.3. *The almost mean curvature flow of the Lag L_γ induces the flow of $Q(\gamma)$ by the vector*

$$\frac{4|f|^{n-1}}{4|f|+|f'|^2}\vec{\kappa}(Q).$$

Proof. Recall that a basis of tangent vectors to L_γ at a point $(w, \sqrt{f}r_1, \ldots, \sqrt{f}r_n)$ consists of a basis of tangent vectors of the cross-subsectional S^{n-1} and a tangent vector which projects down to the copy of $\mathbb{C}$ corresponding to w as γ':

$$\gamma'\left(\partial_w + \frac{f'(\gamma)}{2f(\gamma)}\sum z_i\partial_{z_i}\right).$$

As

$$\theta = \arg\left(\gamma' f^{\frac{n-2}{2}}\right) = \arg\left(\gamma' q\right) = \arg\left(Q'\right)$$

is constant on the cross-subsection, the gradient of θ is the lift to L of the vector:

$$\frac{d\theta}{du}\frac{\gamma'}{|\gamma'|^2\left(1+\frac{|f'|^2}{4|f|}\right)} = \frac{d\theta}{du}\frac{\vec{t}}{|\gamma'|\left(1+\frac{|f'|^2}{4|f|}\right)}.$$

$J\nabla\theta$ is then the lift of

$$\frac{d\theta}{du}\frac{\vec{n}}{|\gamma'|\left(1+\frac{|f'|^2}{4|f|}\right)}. \tag{6.1}$$

Recall that we have shown that $\frac{d}{du}\theta = \kappa(Q)|q||\gamma'|$. Combining with (6.1), the flow vector is:

$$\kappa(Q)|q|\frac{1}{1+\frac{|f'|^2}{4|f|}}\vec{n}.$$

The corresponding flow of the curve $Q(\gamma)$ is then

$$\frac{d}{dt}Q(\gamma) = q(\gamma)\frac{d}{dt}\gamma = q\kappa(Q)|q|\frac{1}{1+\frac{|f'|^2}{4|f|}}\vec{n}.$$

Evaluate now the tangent and normal vectors to $Q(\gamma)$:

$$\frac{d}{du}Q(\gamma) = q(\gamma)\gamma'(u) = q(\gamma)|\gamma'(u)|\vec{t},$$

$$i\frac{d}{du}Q(\gamma) = iq(\gamma)|\gamma'(u)|\vec{t} = q(\gamma)|\gamma'(u)|\vec{n}.$$

So, denoting by $\vec{T}$ and $\vec{N}$ the unit tangent and normal vectors to the curve $Q(\gamma)$ we have

$$\vec{T} = \mathrm{e}^{i\arg(q)}\vec{t},$$

and

$$\vec{N} = \mathrm{e}^{i\arg(q)}\vec{n}.$$

And therefore,

$$\frac{d}{dt}Q = \kappa(Q)|q|^2\frac{1}{1+\frac{|f'|^2}{4|f|}}\vec{N}.$$

Substituting back $q = f^{\frac{n-2}{2}}$ and writing $\vec{\kappa}(Q)$ for $\kappa(Q)\vec{N}$, the almost mean curvature flow of L_γ causes $Q(\gamma)$ to flow by

$$\frac{4|f|^{n-1}}{4|f|+|f'|^2}\vec{\kappa}(Q).$$

□

6.2 Generalized curve shortening flow

6.2.1 Introduction

The curve shortening flow is given by an initial curve $\gamma(u,0)$ evolving over time according to:

$$\frac{\partial}{\partial t}\gamma(u,t) = \vec{\kappa}(\gamma(u,t)),$$

where $\vec{\kappa}(\gamma(u,t))$ is the curvature vector (pointing in the direction of the center of curvature). The curve shortening flow has been studied by many authors (Gage, Hamilton, Grayson, Huisken, and others see [5], [7], [19]). Gage–Hamilton showed that a convex closed curve shrinks to a round point in finite time. Grayson showed that a non-convex closed curve under the flow becomes convex (and then shrinks to a round point). Huisken also considered a flow of a curve which is not closed. We employ some of the techniques of these authors to study the generalized mean curve shortening flow:

Definition 6.2.1. *An initial curve $\gamma(u,0)$ evolves by generalized curve shortening flow if*

$$\frac{\partial}{\partial t}\gamma(u,t) = \phi(x(u,t),y(u,t))\vec{\kappa}(\gamma(u,t)),$$

where $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a non-negative smooth function with a discrete set of zeros.

The curve shortening flow is obtained as a special case of the generalized curve shortening flow choosing $\phi \equiv 1$. We are primarily interested in a generalized curve shortening flow of an open curve with endpoints at zeros of the function ϕ.

6.2.2 Evolution equations

We begin with several useful lemmas. The derivations are analogous to the ones in Gage-Hamilton [5] where they studied the curve shortening flow ($\phi \equiv 1$).

6.2.2.1 Notation

Two different parametrizations for the curve γ are used: the parameter u which takes values in a fixed interval, $u \in [0,1]$ for every time t, and the parameter s (s_t) which is the arc length of γ_t and depends on the time parameter, t.

Let $v := \sqrt{(\frac{\partial x}{\partial u})^2 + (\frac{\partial y}{\partial u})^2}$ be the speed of the curve at time t and let $\vec{t}$ and $\vec{n}$ be the unit tangent and normal vectors. Let $\theta = \arg(\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s})$.

6.2.2.2 Lemmas

Lemma 6.2.2.

$$\frac{\partial \vec{t}}{\partial u} = v\kappa\vec{n}, \qquad \frac{\partial \vec{n}}{\partial u} = -v\kappa\vec{t}.$$

Proof.

$$\frac{\partial \vec{t}}{\partial u} = \frac{\partial \vec{t}}{\partial s}\frac{ds}{du} = \kappa\vec{n}v$$
$$\frac{\partial \vec{n}}{\partial u} = \frac{\partial \vec{n}}{\partial s}\frac{ds}{du} = -\kappa\vec{t}v$$

□

Lemma 6.2.3.

$$\frac{\partial v}{\partial t} = -\phi\kappa^2 v.$$

Proof.

$$\begin{aligned} v^2 &= (\frac{\partial x}{\partial u})^2 + (\frac{\partial y}{\partial u})^2 \\ 2v\frac{\partial v}{\partial t} &= 2\frac{\partial x}{\partial u}\frac{\partial^2 x}{\partial t \partial u} + 2\frac{\partial y}{\partial u}\frac{\partial^2 y}{\partial t \partial u} \end{aligned}$$

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{1}{v}\left[\frac{\partial x}{\partial u}\frac{\partial}{\partial u}\frac{\partial x}{\partial t} + \frac{\partial y}{\partial u}\frac{\partial}{\partial u}\frac{\partial y}{\partial t}\right] \\ &= \frac{1}{v}(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}) \cdot \frac{\partial}{\partial u}(\phi\kappa)\vec{n} \\ &= \vec{t} \cdot (\frac{\partial(\phi\kappa)}{\partial u}\vec{n} + \phi\kappa\frac{\partial \vec{n}}{\partial u}) \\ &= \vec{t} \cdot \phi\kappa(-v\kappa\vec{t}) \\ &= -\phi\kappa^2 v. \end{aligned}$$

□

Let $L_{[a,b]}(t)$ be the length of the segment of the curve defined by $u \in [a, b] \subset [0, 1]$ at time t.

Lemma 6.2.4. *The length of any segment of the curve decreases over time.*

Proof.

$$L_{[a,b]} = \int_a^b v du,$$

$$\frac{\partial L}{\partial t} = \int_a^b \frac{\partial v}{\partial t} du = -\int_a^b \phi\kappa^2 v du = -\int_a^b \phi\kappa^2 ds \leq 0.$$

The length is strictly decreasing unless the interval has length zero or the curvature is zero on the whole interval $[a, b]$. □

Lemma 6.2.5.

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s} = \frac{\partial}{\partial s}\frac{\partial}{\partial t} + \phi\kappa^2\frac{\partial}{\partial s}.$$

Proof.

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s} = \frac{\partial}{\partial t}\frac{1}{v}\frac{\partial}{\partial u} = -\frac{1}{v^2}\frac{\partial v}{\partial t}\frac{\partial}{\partial u} + \frac{1}{v}\frac{\partial^2}{\partial t\partial u} = -\frac{1}{v^2}(-\phi\kappa^2 v)\frac{\partial}{\partial u} + \frac{1}{v}\frac{\partial}{\partial u}\frac{\partial}{\partial t} = \phi\kappa^2\frac{\partial}{\partial s} + \frac{\partial}{\partial s}\frac{\partial}{\partial t}.$$

□

Lemma 6.2.6.

$$\frac{\partial\vec{t}}{\partial t} = \frac{\partial(\phi\kappa)}{\partial s}\vec{n}, \qquad \frac{\partial\vec{n}}{\partial t} = -\frac{\partial(\phi\kappa)}{\partial s}\vec{t}.$$

Proof.

$$\begin{aligned}\frac{\partial\vec{t}}{\partial t} &= \frac{\partial}{\partial t}\frac{\partial}{\partial s}(x, y) = (\frac{\partial}{\partial s}\frac{\partial}{\partial t} + \phi\kappa^2\frac{\partial}{\partial s})(x, y)\\ &= \frac{\partial}{\partial s}(\phi\kappa\vec{n}) + \phi\kappa^2\vec{t} = \frac{\partial(\phi\kappa)}{\partial s}\vec{n} + \phi\kappa\frac{\partial\vec{n}}{\partial s} + \phi\kappa^2\vec{t}\\ &= \frac{\partial(\phi\kappa)}{\partial s}\vec{n},\end{aligned}$$

$$\begin{aligned}\frac{\partial\vec{n}}{\partial t} &= \frac{\partial}{\partial t}\frac{\partial}{\partial s}(-y, x) = (\frac{\partial}{\partial s}\frac{\partial}{\partial t} + \phi\kappa^2\frac{\partial}{\partial s})(-y, x)\\ &= \frac{\partial}{\partial s}(-\phi\kappa\vec{t}) + \phi\kappa^2\vec{n} = -\frac{\partial(\phi\kappa)}{\partial s}\vec{t} - \phi\kappa\frac{\partial\vec{t}}{\partial s} + \phi\kappa^2\vec{n}\\ &= -\frac{\partial(\phi\kappa)}{\partial s}\vec{t}.\end{aligned}$$

□

Recall that $\theta = \arg(\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s})$.

Lemma 6.2.7.

$$\frac{\partial \theta}{\partial t} = \frac{\partial(\phi\kappa)}{\partial s}, \qquad \frac{\partial \theta}{\partial s} = \kappa.$$

Proof.

$$\vec{t} = (\cos\theta, \sin\theta).$$

We know that on one hand,

$$\frac{\partial \vec{t}}{\partial t} = \frac{\partial(\phi\kappa)}{\partial s}\vec{n}.$$

On the other hand,

$$\frac{\partial \vec{t}}{\partial t} = (-\sin\theta\frac{\partial\theta}{\partial t}, \cos\theta\frac{\partial\theta}{\partial t}) = \frac{\partial\theta}{\partial t}\vec{n}.$$

We conclude that

$$\frac{\partial \theta}{\partial t} = \frac{\partial(\phi\kappa)}{\partial s}.$$

Similarly,

$$\frac{\partial \vec{t}}{\partial s} = \frac{\partial\theta}{\partial s}\vec{n} = \kappa\vec{n}.$$

□

Lemma 6.2.8.

$$\frac{\partial \kappa}{\partial t} = \frac{\partial^2(\phi\kappa)}{\partial s^2} + \phi\kappa^3.$$

Proof.

$$\frac{\partial \kappa}{\partial t} = \frac{\partial}{\partial t}(\frac{\partial\theta}{\partial s}) = \frac{\partial}{\partial s}\frac{\partial}{\partial t}\theta + \phi\kappa^2\frac{\partial}{\partial s}\theta = \frac{\partial^2(\phi\kappa)}{\partial s^2} + \phi\kappa^3$$

□

For any fixed value of $u \in [0,1]$, the trajectory of a point on the initial curve is parametrized by t. Denote by σ the arc length along the trajectory. The speed along the trajectory is $\sqrt{(\frac{\partial x}{\partial t})^2 + (\frac{\partial y}{\partial t})^2} = \phi|\kappa|$.

Lemma 6.2.9. *The curvature of the trajectory is*

$$\kappa_t = \frac{1}{\phi|\kappa|}\frac{\partial(\phi\kappa)}{\partial s} = \frac{1}{\phi|\kappa|}\frac{\partial\theta}{\partial t}.$$

Proof. The unit tangent vector of the trajectory is:

$$(\frac{\partial x}{\partial \sigma}, \frac{\partial y}{\partial \sigma}) = \text{sgn}(\kappa)\vec{n} = \text{sgn}(\kappa)(-\frac{\partial y}{\partial s}, \frac{\partial x}{\partial s}).$$

$$\arg(\frac{\partial x}{\partial \sigma}, \frac{\partial y}{\partial \sigma}) = \arg(\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}) + \text{sgn}(\kappa)\frac{\pi}{2} = \theta + \text{sgn}(\kappa)\frac{\pi}{2}.$$

$$\frac{\partial(\theta + \text{sgn}(\kappa)\frac{\pi}{2})}{\partial\sigma} = \frac{\partial\theta}{\partial t}\frac{dt}{d\sigma} = \frac{\partial(\phi\kappa)}{\partial s}\frac{1}{\phi|\kappa|}.$$

□

6.2.3 Shrinking of area

We define a function, $A(t)$, which is closely related to an area enclosed by the curve. Assume that the variation of the direction of the tangent vector of the curve is less than π. Start with one of the endpoints of the curve γ and connect it to the first inflection point of the curve.

Proceed to construct a polygonal path by connecting the inflection points in order. Connect the last inflection point to the other endpoint of the curve γ. Between two consecutive vertices of the polygonal path, the curvature of the curve is either positive or negative and is zero at the vertices except at the end points where the limit of the curvature is not necessarily zero. If there is a segment of the curve which is contained in a straight line then take any point in the interior of the segment for the construction of the polygonal path.

Given two consecutive vertices of the polygonal path, p and q, let

$$A_{pq}(t) = \frac{1}{2} \oint_{\gamma^*_{pq}} (xdy - ydx),$$

where γ^*_{pq} is the closed curve formed by traversing the segment of the curve γ from p to q (which we denote by γ_{pq}) and the straight line segment from q to p.

$A_{pq}(t)$ is related to the area enclosed between the curve and the corresponding polygonal path at time t. It equals the area if the segment of the curve between p and q doesn't intersect the cord pq and if the curvature of γ_{pq} is non-negative. If the curvature of γ_{pq} is non-positive and the segment of the curve between p and q doesn't intersect the cord pq then it equals the negative of the area. If the segment of γ contained in γ_{pq} intersects the cord pq then part(s) of the area are added more than once so $|A_{pq}(t)|$ is greater than the area.

We can calculate the rate of change of $A_{pq}(t)$ using the results developed in the preceding lemmas.

Lemma 6.2.10. *In the generalized curve shortening flow of a path connecting two zeros of the*

function ϕ,

$$\frac{\partial A_{pq}(t)}{\partial t} = -\int_{\gamma_{pq}} \phi\kappa ds.$$

Proof. The integral defining A_{pq} splits into two parts, one along the cord and one along the curve. The integral along the curve is from p to q and the integral along the cord is from q to p.

$$A_{pq} = \frac{1}{2}\oint_{\gamma^*_{pq}} (xdy - ydx) = \frac{1}{2}\int_{\gamma_{pq}} (xdy - ydx) + \frac{1}{2}\int_{\text{cord}_{qp}} (xdy - ydx).$$

Along the cord, we can parametrize the line as follows:

$$x = x_q + u(x_p - x_q)$$

$$y = y_q + u(y_p - y_q).$$

And so, the integral along the cord can be calculated as:

$$\begin{aligned}\frac{1}{2}\int_0^1 (x\frac{\partial y}{\partial u} - y\frac{\partial x}{\partial u})du &= \frac{1}{2}\int_0^1 [(x_q + u(x_p - x_q))(y_p - y_q) - (y_q + u(y_p - y_q))(x_p - x_q)]du \\ &= \frac{1}{2}\int_0^1 (x_q y_p - y_q x_p)du = \frac{1}{2}(x_q y_p - y_q x_p).\end{aligned}$$

Therefore A_{pq} is:

$$A_{pq} = \frac{1}{2}\left[(x_q y_p - y_q x_p) + \int_{y_p}^{y_q} xdy - \int_{x_p}^{x_q} ydx\right].$$

Differentiate by t:

$$\begin{aligned}
\frac{\partial}{\partial t}A_{pq} =& \frac{1}{2}\frac{\partial}{\partial t}\left[(x_q, y_q)\cdot(y_p, -x_p) + \int_{y_p}^{y_q} x dy - \int_{x_p}^{x_q} y dx\right] \\
=& \frac{1}{2}[\phi_q\kappa_q\vec{n}\cdot(-y_p, x_p) + \phi_p\kappa_p\vec{t}\cdot(x_q, y_q)] \\
&+ \frac{1}{2}\left[\int_{y_p}^{y_q}\frac{\partial}{\partial t}(x dy) + x_q\frac{\partial y_q}{\partial t} - x_p\frac{\partial y_p}{\partial t}\right] - \frac{1}{2}\left[\int_{x_p}^{x_q}\frac{\partial}{\partial t}(y dx) + y_q\frac{\partial x_q}{\partial t} - y_p\frac{\partial x_p}{\partial t}\right] \\
=& \frac{1}{2}\left[\int_{u_p}^{u_q}\frac{\partial}{\partial t}[xv\frac{\partial y}{\partial s}]du - \int_{u_p}^{u_q}\frac{\partial}{\partial t}[yv\frac{\partial x}{\partial s}]du\right] \\
=& \frac{1}{2}\left[-\int_{u_p}^{u_q}\frac{\partial}{\partial t}[v(x,y)\cdot\vec{n}]du\right] \\
=& -\frac{1}{2}\int_{u_p}^{u_q}[-\phi\kappa^2 v(x,y)\cdot\vec{n} + v\kappa\phi\vec{n}\cdot\vec{n} - v\frac{\partial(\phi\kappa)}{\partial s}(x,y)\cdot\vec{t}\,]du \\
=& -\frac{1}{2}\left[\int_{u_p}^{u_q}\phi\kappa v du - \int_{u_p}^{u_q}\phi\kappa^2(x,y)\cdot\vec{n}v du - \int_{s_p}^{s_q}\frac{\partial(\phi\kappa)}{\partial s}(x,y)\cdot\vec{t}\,ds\right]
\end{aligned}$$

Integrate by parts the last integral:

$$\begin{aligned}
\int_{s_p}^{s_q}(x,y)\cdot\vec{t}\,\frac{\partial(\phi\kappa)}{\partial s}ds &= [\phi\kappa(x,y)\cdot\vec{t}\,]|_{s_p}^{s_q} - \int_{s_p}^{s_q}\phi\kappa[\vec{t}\cdot\vec{t} + (x,y)\cdot\kappa\vec{n}]ds \\
&= -\int_{u_p}^{u_q}[\phi\kappa v + \phi\kappa^2 v(x,y)\cdot\vec{n}]du.
\end{aligned}$$

Therefore,

$$\frac{\partial}{\partial t}A_{pq} = -\int_{u_p}^{u_q}\phi\kappa v du = -\int_{s_p}^{s_q}\phi\kappa ds.$$

□

For $\kappa > 0$ the integration is in the counterclockwise direction and A_{pq} is the area enclosed by the curve segment between p and q and the cord pq. If $\kappa < 0$ then the integration is in the

clockwise direction so the area enclosed by the curve and the cord is equal to $-A_{pq}$. Therefore we can write the rate of change (in both cases) as:

$$\frac{\partial}{\partial t}|A_{pq}| = -\int_{u_p}^{u_q} \phi|\kappa|vdu = -\int_{s_p}^{s_q} \phi|\kappa|ds.$$

Let $A(t) = \sum |A_{pq}|$. $A(t)$ is a measure of how much γ_t deviates from a straight line segment, or a polygonal line. An immediate consequence of the Lemma is:

Corollary 6.2.11.

$$\frac{\partial}{\partial t}A(t) = -\int_{\gamma(t)} \phi|\kappa|vdu = -\int_{\gamma(t)} \phi|\kappa|ds.$$

Thus, as long as the flow exists, $A(t) \searrow$.

6.2.4 Curvature decrease

In this section we continue to focus on the curve segments on which the curvature is only non-negative or only non-positive. For a pair of consecutive inflection points, p, q, we look at the ratio $\frac{d}{l}$ of the distance of the points p and q along the cord and along the curve. This is the distance comparison principle of Huisken [19]. The idea is that if the curvature were to blow up on a segment of the curve then

$$\frac{d}{l} \searrow 0.$$

Let p and q be two consecutive inflection points and consider the ratio

$$\frac{d}{l},$$

where

$$d = \sqrt{(x_q - x_p)^2 + (y_q - y_p)^2}$$

is the length of the cord connecting p and q, and

$$l = \int_{u_p}^{u_q} v du$$

is the length of the segment of the curve between p and q.

Lemma 6.2.12. *Consider the generalized curve shortening flow of a path between two zeros of the function ϕ. Then $(\frac{d}{l})|_{pq}$ increases with time for p and q two neighboring vertices of the polygonal path.*

Proof. Let $p(t_0)$ and $q(t_0)$ be two consecutive vertices of the polygonal path at time t_0, $p(t) = \gamma(u_p, t)$ and $q(t) = \gamma(u_q, t)$. Then, at time t_0, $\phi_q \vec{\kappa}_q = \phi_p \vec{\kappa}_p = 0$. Differentiate by t the function $\left(\frac{d}{l}\right)_{u_p u_q}$.

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{d}{l}\right)_{u_p u_q} &= \frac{1}{l}\frac{\partial d}{\partial t} - \frac{d}{l^2}\frac{\partial l}{\partial t} \\
&= \frac{1}{l}\frac{1}{d}\left((x_q - x_p)(\frac{\partial x_q}{\partial t} - \frac{\partial x_p}{\partial t}) + (y_q - y_p)(\frac{\partial y_q}{\partial t} - \frac{\partial y_p}{\partial t})\right) - \frac{d}{l^2}\frac{\partial}{\partial t}\int_{u_p}^{u_q} v du \\
&= \frac{1}{l}\frac{\vec{d}}{d} \cdot (\phi_q \vec{\kappa}_q - \phi_p \vec{\kappa}_p) - \frac{d}{l^2}\int_{u_p}^{u_q} (-\phi \kappa^2 v) du \\
&= \frac{d}{l^2}\int_{s_p}^{s_q} \phi \kappa^2 ds.
\end{aligned}$$

Consequently we also have,

$$\frac{\partial}{\partial t}\left(\sum\left(\frac{d}{l}\right)_{u_p u_q}\right) = \sum\left(\left(\frac{d}{l^2}\right)_{u_p u_q}\int_{u_p}^{u_q}\phi\kappa^2 ds\right) \geq 0,$$

□

Let Γ be one of the segments of the curve with curvature non-negative or non-positive. For pairs of points $p, q \in \Gamma$, $\frac{d}{l} : \Gamma \times \Gamma \to (0, 1]$, where for $p = q$ we define $\left(\frac{d}{l}\right)_{(p,p)} = \lim_{q\to p}\left(\frac{d}{l}\right)|_{(p,q)} = 1$. Note that for now, we are considering a curve for fixed t.

Lemma 6.2.13. *If the function $\frac{d}{l}$ has a local minimum at $(p, q) \in \Gamma \times \Gamma$, then*

$$\frac{1}{l}\frac{\vec{d}}{d} \cdot (\vec{\kappa}_q - \vec{\kappa}_p) \geq 0.$$

Proof. Suppose that $\frac{d}{l}$ has a local minimum at (p, q). Without loss of generality we may assume that the curve is parametrized by arc length. Make a small variation of $\frac{d}{l}$ at the point p.

$$\left(\frac{d}{l}\right)|_{(p_\epsilon,q)},$$

where

$$\begin{aligned} d|_{(p_\epsilon,q)} &= \sqrt{(x(s_q,t) - x(s_p+\epsilon,t))^2 + (y(s_q,t) - y(s_p+\epsilon,t))^2} \\ &= \sqrt{(x_q - x_{p_\epsilon})^2 + (y_q - y_{p_\epsilon})^2}, \end{aligned}$$

and

$$l|_{(p_\epsilon,q)} = \int_{s_p+\epsilon}^{s_q} ds.$$

Differentiate by ϵ:

$$\begin{aligned}
\frac{d}{d\epsilon}\left(\frac{d}{l}\right)|_{(p_\epsilon,q)} &= \frac{1}{l}\frac{d}{d\epsilon}d - \frac{d}{l^2}\frac{d}{d\epsilon}l \\
&= \frac{1}{l}\frac{1}{d}\left((x_q - x_{p_\epsilon})(-x'_{p_\epsilon}) + (y_q - y_{p_\epsilon})(-y'_{q_\epsilon})\right) + \frac{d}{l^2} \\
&= -\frac{1}{l}\frac{\vec{d}}{d}\cdot\vec{T_{p_\epsilon}} + \frac{d}{l^2}.
\end{aligned}$$

The derivative at $\epsilon = 0$ equals 0 if and only if:

$$\frac{\vec{d}}{d}\cdot\vec{T_p} = \frac{d}{l}.$$

$\frac{\vec{d}}{d}$ is the unit vector of the cord pq and $\vec{t_p}$ is the unit tangent vector to the curve at p, both in the direction from p to q, so

$$\frac{\vec{d}}{d}\cdot\vec{T_p} = cos(\vec{t_p}, \vec{d}).$$

Similar calculations using a small variation of the point q yield:

$$\frac{\vec{d}}{d}\cdot\vec{T_q} = cos(\vec{t_q}, \vec{d}) = \frac{d}{l}.$$

If (p, q) is a local minimum, the second derivative of a variation is non-negative at (p, q). For this calculation, consider a small variation of both p and q:

$$\left(\frac{d}{l}\right)|_{(p_\epsilon,q_\epsilon)},$$

where

$$\begin{aligned} d|_{(p_\epsilon,q)} &= \sqrt{(x(s_q-\epsilon,t)-x(s_p+\epsilon,t))^2+(y(s_q-\epsilon,t)-y(s_p+\epsilon,t))^2} \\ &= \sqrt{(x_{q_\epsilon}-x_{p_\epsilon})^2+(y_{q_\epsilon}-y_{p_\epsilon})^2}, \end{aligned}$$

and

$$l|_{(p_\epsilon,q_\epsilon)} = \int_{s_p+\epsilon}^{s_q-\epsilon} ds.$$

Differentiate by ϵ twice ($'$ is used here for the derivative by ϵ for simplicity):

$$\left(\frac{d}{l}\right)'' = \frac{1}{l}d'' - 2\frac{1}{l^2}l'd' + (\frac{2}{l^2}l'^2 - \frac{1}{l^2}l'')d = \frac{1}{l}d'' + 4\frac{1}{l^2}d' + 8\frac{1}{l^2}d,$$

since $l' = -2$ and $l'' = 0$. Now,

$$d' = \frac{1}{d}((x_{q_\epsilon}-x_{p_\epsilon})(-x'_{q_\epsilon}-x'_{p_\epsilon}) + (y_{q_\epsilon}-y_{p_\epsilon})(-y'_{q_\epsilon}-y'_{p_\epsilon})) = -\frac{\vec{d}}{d}\cdot(\vec{T_{q_\epsilon}}+\vec{T_{p_\epsilon}}),$$

and

$$\left(\frac{\vec{d}}{d}\right)' = -\frac{1}{d^2}d'\vec{d} + \frac{1}{d}(-x'_{q_\epsilon}-x'_{p_\epsilon}, -y'_{q_\epsilon}-y'_{p_\epsilon}) = -\frac{1}{d^2}d'\vec{d} - \frac{1}{d}(\vec{T_{q_\epsilon}}+\vec{T_{p_\epsilon}}),$$

and so,

$$d'' = (\frac{1}{d^2}d'\vec{d} + \frac{1}{d}(\vec{T_{q_\epsilon}}+\vec{T_{p_\epsilon}}))\cdot(\vec{T_{q_\epsilon}}+\vec{T_{p_\epsilon}}) - (\frac{\vec{d}}{d})\cdot(-\vec{\kappa}_{q_\epsilon}+\vec{\kappa_{p_\epsilon}}).$$

At $\epsilon = 0$, we have

$$d' = -2\frac{d}{l},$$

and

$$\begin{aligned} d'' &= \left(-\frac{2}{l}\frac{\vec{d}}{d} + \frac{1}{d}(\vec{T_q} + \vec{T_p}) \right) \cdot (\vec{T_q} + \vec{T_p}) + (\frac{\vec{d}}{d}) \cdot (\vec{\kappa}_q - \vec{\kappa}_p) \\ &= -4\frac{d}{l^2} + \frac{1}{d}\left(\frac{2d}{l}\right)^2 + (\frac{\vec{d}}{d}) \cdot (\vec{\kappa}_q - \vec{\kappa}_p) \\ &= \frac{\vec{d}}{d} \cdot (\vec{\kappa}_q - \vec{\kappa}_p). \end{aligned}$$

Finally,

$$\left(\frac{d}{l}\right)'' |_{\epsilon=0} = \left(\frac{1}{l}d'' + 4\frac{1}{l^2}d' + 8\frac{1}{l^2}d \right) |_{\epsilon=0} = \frac{1}{l}\frac{\vec{d}}{d} \cdot (\vec{\kappa}_q - \vec{\kappa}_p).$$

Therefore, if for $p, q \in \gamma$, $(\frac{d}{l})|_{pq}$ has a local minimum then

$$\frac{1}{l}\frac{\vec{d}}{d} \cdot (\vec{\kappa}_q - \vec{\kappa}_p) \geq 0 \tag{6.2}$$

which means that the angle between $\vec{d}$ and $\vec{\kappa}_q - \vec{\kappa}_p$ is not greater than $\frac{\pi}{2}$.

□

Lemma 6.2.14. *If the function $\frac{d}{l}$ has a local minimum at $(p, q) \in \Gamma \times \Gamma$, then*

$$\frac{\partial}{\partial t}(\frac{d}{l})|_{pq} > 0.$$

Proof. From (6.2) if $\vec{d} \cdot \vec{\kappa}_p > 0$ then also $\vec{d} \cdot \vec{\kappa}_p > 0$. This configuration is not possible with κ not changing sign somewhere along the path from p to q. Similarly, if $\vec{d} \cdot \vec{\kappa}_p < 0$ then the curve

cannot reach q without κ changing its sign somewhere. It remains possible that $\vec{d} \cdot \vec{\kappa}_p = 0$. In that case, the calculation of Lemma 6.2.12 yields,

$$\frac{\partial}{\partial t}\left(\frac{d}{l}\right)_{u_p u_q} = \frac{1}{l}\frac{\vec{d}}{d} \cdot (\phi_q \vec{\kappa}_q - \phi_p \vec{\kappa}_p) - \frac{d}{l^2}\int_{u_p}^{u_q} (-\phi\kappa^2 v) du \geq \frac{d}{l^2}\int_{s_p}^{s_q} \phi\kappa^2 ds > 0.$$

□

Proposition 6.2.15. *Given a simple smooth path γ between two points in $V(\phi)$, the limit of the generalized mean curvature flow,*

$$\frac{\partial}{\partial t}\gamma(u,t) = \phi(x(u,t), y(u,t))\vec{\kappa}(\gamma(u,t)),$$

is a straight line segment or a union of straight line segments.

Proof. From Lemma 6.2.10 the area A_{pq} decreases and from Lemma 6.2.14 the curvature along γ_{pq} decreases. As long as the curvature does not blow up and is not 0 everywhere, the flow exists. Therefore, we conclude that the limit of the flow is a straight line segment or a polygonal line - union of straight line segments. □

6.2.5 Convergence of the flow

Theorem 6.2.16. *Let $\phi : \mathbb{C} \to \mathbb{R}_{\geq 0}$ have exactly two zeros. The generalized mean curvature flow of a path between the zeros of ϕ exists for all time and converges to the straight line segment between the zeros of ϕ.*

Proof. The total number of inflexion points can never increase during the flow and in fact it decreases, just as it does in the usual curve shortening flow [7]. By Lemmas 6.2.12 and 6.2.14

the curvature does not blow up so there is no singularity of the flow. As the area $A(t)$ decreases with time (Corollary 6.2.11), and $A(t) > 0$ implies that the flow can be continued, the flow exists for all time and converges to the straight line segment connecting the two fixed points. □

Remark 6.2.17. If ϕ has more than two zeros than it is possible that a zero of ϕ is in the region enclosed by the initial path and the straight line segment between the endpoints. Clearly, in this situation the path is prevented from flowing to the straight line segment. In the following section we show that depending on n the path reaches the third zero in finite or infinite time.

6.3 Flow near the fixed points

As in the curve shortening flow (flow by curvature) the number of intersection points of curves cannot increase during the flow (avoidance principle). In order to understand the flow near a fixed point, we study a flow of a curve that remains self-similar. We can use curves whose flow we understand as barriers for a more arbitrary curve and draw conclusions about its flow.

6.3.1 Flow by $\phi\vec{\kappa}$

We look more closely at the flow near one of the fixed points. WLOG we can assume that the fixed point is the origin. Let r be the distance of a point $(x, y) \in \mathbb{R}^2$ to the origin.

We will gradually increase the generality of the function ϕ. We start with the simplest case, $\phi = mr^\alpha$, for $m =$ const., then proceed with ϕ being radially symmetric, and finally, we drop the requirement of radial symmetry.

6.3.1.1 $\phi = \mathbf{mr}^{\alpha}$

Let $m > 0$ be a constant and let $\alpha > 0$ be fixed. We study the flow by the vector $mr^{\alpha}\vec{\kappa}$ of a circle with the center at the origin.

Let $\gamma_0 : x^2 + y^2 = r_0^2$ be the initial curve and let $\gamma_t : x^2 + y^2 = r^2$ be the result of the flow of γ_0 at time t. Let $A(t)$ denote the area enclosed by γ_t. By Corollary 6.2.11,

$$\frac{dA}{dt} = -\int_{\gamma_t} \phi|\kappa| = -\int_{\gamma_t} mr^{\alpha-1} = -2\pi m r^{\alpha}.$$

On the other hand,

$$A = r^2\pi \quad \text{so} \quad \frac{dA}{dt} = 2r\pi\frac{dr}{dt}.$$

Therefore

$$\frac{dr}{dt} = -mr^{\alpha-1}. \tag{6.3}$$

We can solve the differential equation (6.3) distinguishing three different cases:

- $\alpha = 2$

$$\frac{dr}{r} = -m dt \quad \text{so} \quad r = r_0 \mathrm{e}^{-mt}$$

The flow exists for all time and $\lim_{t\to\infty} r = 0$, i.e. in the limit the circle shrinks to a point.

- $\alpha > 2$

$$\frac{dr}{r^{\alpha-1}} = -mdt$$

therefore

$$r^{2-\alpha} = (\alpha-2)mt + r_0^{2-\alpha} \geq r_0^{2-\alpha} > 0, \quad \text{for all} \quad t \in [0,\infty).$$

From here,

$$r = \frac{1}{\left((\alpha-2)mt + r_0^{2-\alpha}\right)^{\frac{1}{\alpha-2}}}.$$

The flow exists for all time and $\lim_{t\to\infty} r = 0$, i.e. in the limit the circle gradually shrinks to a point.

- $\alpha < 2$

$$r^{2-\alpha} = -(2-\alpha)mt + r_0^{2-\alpha} \geq 0 \quad \text{iff} \quad t \leq \frac{r_0^{2-\alpha}}{(2-\alpha)m}.$$

The flow exists for $t \in [0, \frac{r_0^{2-\alpha}}{(2-\alpha)m}]$. At $t = \frac{r_0^{2-\alpha}}{(2-\alpha)m}$ the circle has shrunk to a point.

6.3.1.2 $\phi = \mathbf{r}^{\alpha}\mathbf{c}(\mathbf{r})$

Here we assume that the flow is still radially symmetric and that $c(r) > 0$ and $\lim_{r\to 0} c(r) = c > 0$. Due to the radial symmetry, a circle remains a circle under the flow. Let us start again with the circle $\gamma_0 : x^2 + y^2 = r_0^2$, and let $\gamma_t : x^2 + y^2 = r$ be the result of the flow at time t.

On the interval $[0, r_0]$ the function $c(r)$ has a minimum, m, and a maximum, M.

Denote by γ_t^M the flow of γ_0 by $Mr^{\alpha}\vec{\kappa}$ and by γ_t^m the flow of γ_0 by $mr^{\alpha}\vec{\kappa}$.

The curve γ_t is sandwiched between the curves γ_t^m and γ_t^M. Both of the latter curves have the same dependence on α for their limit behavior. Therefore the same holds for γ_t, i.e., if $\alpha \geq 2$

then the flow exists for all time and in the limit the circle shrinks to a point, and if $\alpha < 2$ the flow exists for $t \in [0, T)$ and at $t = T$ there is a singularity: the circle has shrunk to a point

6.3.1.3 $\phi = \mathbf{r}^{\alpha}\mathbf{c}(\mathbf{x}, \mathbf{y})$

Here we assume that $c(x, y) > 0$ is sufficiently well-behaved so that $\lim_{r\to 0} c(x, y) = c > 0$ and that $m(r) := \min_{x^2+y^2=r^2} c(x, y)$ and $M(r) := \max_{x^2+y^2=r^2} c(x, y)$ are continuous, $r \in [0, r_0]$.

Denote by γ_t^M the flow of γ_0 by $M(r)r^{\alpha}\vec{\kappa}$ and by γ_t^m the flow of γ_0 by $m(r)r^{\alpha}\vec{\kappa}$. These two flows are radially symmetric.

The curve γ_t is sandwiched between the curves γ_t^m and γ_t^M. Again both of the latter curves have the same dependence on α for their limit behavior. Therefore the same holds for γ_t, i.e., if $\alpha \geq 2$ then the flow exists for all time and in the limit the circle shrinks to a point, and if $\alpha < 2$ the flow exists for $t \in [0, T)$ and at $t = T$ there is a singularity: the circle has shrunk to a point.

In conclusion, under the flow by $r^{\alpha}c(x, y)\vec{\kappa}$, a circle centered at the origin shrinks to a point in finite time if $\alpha < 2$ and in infinite time if $\alpha \geq 2$.

Remark 6.3.1. Lotay-Gonçalo [31] develop a very similar approach to the flow when proving a version of the Thomas–Yau conjecture in their setting. The similarity comes from the fact, which they prove, that SLs are inverse images of straight lines under a certain map. Their setting is a hyperKähler manifold of dimension four (which is Calabi–Yau) with an action of the circle group. There are several examples which they investigate. In general, there are finitely many points over which the action is not free and the straight line segments between those points determine SL spheres. The flow of curves which is related to the flow of Lagrangian spheres is,

in our terminology, a generalized mean curvature flow.

6.3.2 Example: Orthogonal pencils of circles

Let $\{(x-a)^2+y^2=a^2 | a \in \mathbb{R}\}$ and $\{x^2+(y-b)^2=b^2 | b \in \mathbb{R}\}$ be a pair of orthogonal parabolic pencils of circles passing through the origin. We will show that there exists a flow such that first family of circles is the family of curves stemming from the flow and that the second family consists of trajectories of individual points.

1. We will first parametrize the circles in the first pencil in such a way that the parameter stays constant along any trajectory.

2. We can choose the speed of the flow freely and we will make a choice which will enable us to compare the flow of circles to the flow by $\phi\vec{\kappa}$.

Lemma 6.3.2. *Circles in the family a can be parametrized by $u \in [0,1)$ so that $u =$ const. defines a circle in the family b.*

Proof. Let $r = 2a\cos\phi$ and $r = 2b\sin\phi$ be two circles, one from each of the two pencils. The point of intersection (other than the origin) is given by the equation

$$2a\cos\phi = 2b\sin\phi,$$

from which, for given a and b, ϕ can be determined.

Suppose we fix b and let $a > 0$ vary. Then for any a

$$b = a\cot\phi$$

is constant along the circle b.

Define

$$u := \begin{cases} \frac{1}{\pi}(\operatorname{arccot}(a\cot\phi) + \frac{\pi}{2}), & b > 0 \\ \\ \frac{1}{\pi}(\operatorname{arccot}(a\cot\phi) - \frac{\pi}{2}), & b < 0. \end{cases}$$

Note that $b > 0$ corresponds to a first quadrant intersection point and $b < 0$ to a fourth quadrant intersection point, assuming that $a > 0$. □

Now we assume that $a = a(t)$ is shrinking over time and look at the corresponding flow vector.

Lemma 6.3.3. *When a circle in the family a changes over time, $a = a(t)$, then the flow vector is given by*

$$\frac{\partial}{\partial t}(x, y) = -\frac{2U^2\dot{a}}{a^2 + U^2}\vec{n}.$$

Proof. We can work in the first quadrant so both a and b are assumed positive and

$$u = \frac{1}{\pi}(\operatorname{arccot}(a\cot\phi) + \frac{\pi}{2}).$$

From here,

$$\cot\phi = \frac{1}{a}\cot(\pi u - \frac{\pi}{2}).$$

A point $(x.y)$ on the circle a can be expressed in terms of a and u.

$$\begin{aligned}(x,y) &= 2a\cos\phi(\cos\phi,\sin\phi)\\ &= 2a(\cos^2\phi,\cos\phi\sin\phi)\\ &= 2a(\frac{\cot^2\phi}{1+\cot^2\phi},\frac{\cot\phi}{1+\cot^2\phi}).\end{aligned}$$

Define $U := \cot(\pi u - \frac{\pi}{2})$ so that $\cot\phi = \frac{1}{a}U$. U is constant along b. Now

$$\begin{aligned}(x,y) &= 2a(\frac{\cot^2\phi}{1+\cot^2\phi},\frac{\cot\phi}{1+\cot^2\phi})\\ &= 2a(\frac{\frac{1}{a^2}U^2}{1+\frac{1}{a^2}U^2},\frac{\frac{1}{a}U}{1+\frac{1}{a^2}U^2})\\ &= 2a(\frac{U^2}{a^2+U^2},\frac{aU}{a^2+U^2})\\ &= \frac{2aU}{a^2+U^2}(U,a).\end{aligned}$$

Differentiate by t the last expression:

$$\begin{aligned}\frac{\partial}{\partial t}(x,y) &= 2U\frac{\dot{a}(a^2+U^2)-2a^2\dot{a}}{(a^2+U^2)^2}(U,a)+\frac{2aU}{(a^2+U^2)}(0,\dot{a})\\ &= \frac{2U\dot{a}}{(a^2+U^2)^2}(U(U^2-a^2),a(U^2-a^2)+a(U^2+a^2))\\ &= \frac{2U^2\dot{a}}{(a^2+U^2)^2}(U^2-a^2,2aU).\end{aligned}$$

The length of this vector is

$$|\frac{\partial}{\partial t}(x,y)| = \frac{2U^2|\dot{a}|}{(a^2+U^2)^2}\sqrt{(U^2-a^2)^2+(2aU)^2} = \frac{2U^2|\dot{a}|}{a^2+U^2}.$$

For $a(t) \to 0$ this is $\sim 2|\dot{a}|$. □

Lemma 6.3.4. *The self similar flow of a circle with radius $a = a(t) = a_0 \mathrm{e}^{-2t}$ is equivalent to the flow by the vector $r^2\vec{\kappa}$.*

Proof. If $a = a_0 \mathrm{e}^{-2t}$ then $\dot{a} = -2a$. The point $(2a, 0)$ flows along the trajectory which is a straight line, $\frac{\partial}{\partial t}(2a, 0) = (-4a, 0)$. On the other hand, the vector $r^2\vec{\kappa}$ at that point has length $(2a)^2 \frac{1}{a} = 4a$ which is exactly the same.

At a point which is different from $(2a, 0)$ we can also calculate and compare the two vectors. Substituting $U = a \cot \phi$ we get that the length of the flow vector at the point $2a(t)(\cos \phi, \sin \phi)$ is

$$|\frac{\partial}{\partial t}(x, y)| = \frac{2U^2|\dot{a}|}{a^2 + U^2} = \frac{2a^2 \cot^2 \phi |\dot{a}|}{a^2 + a^2 \cot^2 \phi} = 2|\dot{a}| \cos^2 \phi.$$

On the other hand, the vector $r^2\vec{\kappa}$ at that point has length $(2a \cos \phi)^2 \frac{1}{a} = 4a \cos^2 \phi = -2\dot{a} \cos^2 \phi$, which is exactly the same.

□

6.4 Flow of the curve $Q(\gamma)$

In 6.1.3 we have shown that the curve $Q(\gamma)$ flows by the vector

$$\frac{4|f|^{n-1}}{4|f| + |f'|^2}\vec{\kappa}(Q)$$

when the Lag L_γ flows by almost Lagrangian mean curvature flow. The endpoints of $Q(\gamma)$ remain fixed during the flow as they are images of points in $V(f)$. The flow of the curve $Q(\gamma)$ is similar to mean curvature flow since the function $\frac{4|f|^{n-1}}{4|f|+|f'|^2}$ is bounded on compact sets. That is why we

concentrate next on studying the flow near the fixed points. We will use the results of Section 6.3.

6.4.1 Flow near the fixed points

Lemma 6.4.1. *Suppose that L^t_γ is ALMCF, $\gamma^t \in \mathcal{A}_{j\ell}$, and let $Q(\gamma^t)$ be the corresponding generalized mean curvature flow. Then the flow vector of a point p on $Q(\gamma^t)$ near $Q(w_j)$ is asymptotically*

$$r^{2-2/n}\vec{\kappa},$$

where r is the distance of p from $Q(w_j)$ and $\vec{\kappa}$ is the curvature vector of $Q(\gamma^t)$ at p.

Proof. Since $Q' = f^{\frac{n-2}{2}}$ and f vanishes to order 1 at 0, the order of vanishing of Q' is $\frac{n-2}{2}$ which means that $Q(w) - Q(0)$ vanishes to order $\frac{n-2}{2} + 1 = \frac{n}{2}$ and $|Q(w) - Q(0)| \sim |w - 0|^{\frac{n}{2}}$. Now, near 0, $|f| \sim C|w| \sim C|Q(w) - Q(0)|^{\frac{2}{n}}$ so

$$\frac{4|f|^{n-1}}{4|f| + |f'|^2} \sim C|Q(w) - Q(0)|^{2-\frac{2}{n}}.$$

□

This means that our flow is similar to the flow by $r^\alpha\vec{\kappa}$ with $\alpha < 2$. We have shown that circles centred at the origin shrink to a point in finite time under this flow.

Corollary 6.4.2. *Let L_{γ^t} be ALMCF. Then it is possible for the flow to have a finite time singularity.*

Proof. Suppose that $\gamma^t \in \mathcal{A}_{j\ell}$. By Lemma 6.4.1 if $Q(\gamma^t)$ is near $Q(w_m)$, for some $w_m \in V(f)$ it is possible for $Q(\gamma^T)$ for some $T > t$ to reach the point $Q(w_m)$ since circles around that point

can shrink in finite time to $Q(w_m)$. □

6.4.2

The setting in the following proposition can be thought of as $\mathrm{Im}(Q)$.

Proposition 6.4.3. *Suppose that a simple smooth path $\gamma \equiv \gamma_0$ connecting two zeros of ϕ, z_j and z_ℓ, is flowing under generalized mean curvature flow, $\frac{\partial}{\partial t}\gamma_t = \phi\vec{\kappa}$. Suppose further that the area enclosed by γ and the straight line segment $\overline{z_j z_\ell}$ contains exactly one other zero of ϕ, z_m of order α.*

(i) If z_j, z_ℓ and z_m are not colinear, or are colinear and z_m is not between z_j and z_ℓ, then the generalized mean curvature flow of γ converges to $\overline{z_j z_m} \cup \overline{z_m z_\ell}$.

(ii) If z_j, z_ℓ and z_m are colinear and z_m is between z_j and z_ℓ, then the generalized mean curvature flow of γ converges to $\overline{z_j z_\ell}$.

In addition, if $\alpha \geq 2$ and $z_m \notin \gamma$, then $z_m \notin \gamma_t$, for $t < \infty$.

Proof. (i) Away from the points fixed by the flow (zeros of ϕ) the flow is governed by a parabolic equation. γ_t remains smooth and the flow continues until $\kappa \equiv 0$. Since γ can't flow to $\overline{z_j z_\ell}$ without encountering z_m, the flow converges to the path from z_j to z_ℓ consisting of straight line segments $\overline{z_j z_m}$ and $\overline{z_m z_\ell}$.

(ii) In this case $\overline{z_j z_m} \cup \overline{z_m z_\ell} = \overline{z_j z_\ell}$.

If $\alpha \geq 2$ we saw in subsection 6.3.1 that circles take an infinite amount of time to shrink to a point. Disjoint curves will remain disjoint by the avoidance principle. Therefore γ cannot reach z_m in a finite amount of time. (Figure 6.1)

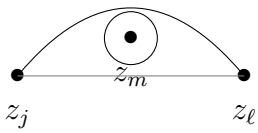


Figure 6.1: Circle shrinks to a point in infinite time if $\alpha \geq 2$

□

Proposition 6.4.4. *Suppose that a simple smooth path $\gamma \equiv \gamma_0$ connecting two zeros of ϕ, z_j and z_ℓ, is flowing under generalized mean curvature flow, $\frac{\partial}{\partial t}\gamma_t = \phi\vec{\kappa}$. If the flow does not converge to $\overline{z_j z_\ell}$ then the area enclosed by γ and the straight line segment $\overline{z_j z_\ell}$ contains at least one other zero of ϕ, z_m.*

Proof. As curvature decreases during the flow (Lemmas 6.2.12 and 6.2.14), and the flow continues as long as the curvature is not identically zero, the limit of the flow is a straight line segment or a polygonal line.

Claim: If the limit is a polygonal line, any vertex of the polygonal line different from zj, z_ℓ must be a zero of ϕ.

Proof of the Claim: Suppose that z_m is a vertex of the polygonal line and that $\phi(z_m) \neq 0$. Then there is a closed disc around z_m such that $\phi \geq M$ on it, for some $M > 0$. Also, as the limit is a polygonal line, near z_m, $|\kappa| > C$, for some $C > 0$. Therefore the speed along the trajectory leading to z_m is bounded from below in a small disc. It follows that the point z_m is reached in a finite amount of time. In a finite amount of time the curve cannot reach a stage where it is not C^1 since away from the zeros of ϕ the flow is governed by a parabolic equation and regularity can only increase. But then the flow continues past the point z_j as $(\phi|\kappa|)(z_j) > 0$. Contradiction. □

Now, to apply this to Milnor fibers, recall that (Theorem 6.1.3) the flow of $Q(\gamma)$ is given

by:

$$\frac{\partial}{\partial t}Q(\gamma) = \frac{4|f(\gamma)|^{n-1}}{4|f(\gamma)| + |f'(\gamma)|^2}\vec{\kappa}(Q(\gamma)).$$

Lemma 6.4.5. *Suppose that L_{γ_t} is ALMCF, $\gamma_t \in \mathcal{A}_{j\ell}$. Then there exists a function $\phi : U \to \mathbb{R}_{>0}$, where U is an open set containing $\bigcup_t Q(\gamma_t) \setminus \{Q(w_j), Q(w_\ell)\}$), satisfying:*

$$\phi(Q(w)) = \frac{4|f(w)|^{n-1}}{4|f(w)| + |f'(w)|^2},$$

$w \in \gamma_t$ for some t.

Proof. The only difficulty is that for each $z \in \mathbb{C} \setminus Q(V(f))$ there exist multiple $w \in \mathbb{C}$ such that $Q(w) = z$. But, away from the set $Q(V(f))$, Q is locally invertible and, consequently, the function ϕ is well-defined.

□

Theorem 6.4.6. *Let L_γ be a positive Lagrangian sphere in the Milnor fiber X_f^n. Assume $\phi(L_\gamma) = 0$. If L_γ satisfies:*

$$[\phi(L_1), \phi(L_2)] \nsubseteq (\inf \theta_{L_\gamma}, \sup \theta_{L_\gamma}),$$

or

$$\int_{L_\gamma} |\Omega| dV_{g_{L_\gamma}} \leq \Big| \int_{L_{\gamma_1}} \Omega|_{L_{\gamma_1}} \Big| + \Big| \int_{L_{\gamma_2}} \Omega|_{L_{\gamma_2}} \Big|$$

for all graded connect sums $L_1 \# L_2$ of Lagrangian spheres in the same Hamiltonian deformation class as L_γ, then almost mean curvature flow for L exists for all time and converges to a spe-

cial Lagrangian in its Hamiltonian deformation class. The special Lagrangian is unique in its Hamiltonian deformation class.

Proof. It is sufficient to prove that the induced flow of $Q(\gamma)$ converges to the straight line segment between its endpoints. Each of the two conditions separately implies that $Q(\gamma)$ does not enclose any other $Q(w_m)$, $w_m \in V(f)$ and since these conditions continue to hold throughout the flow $Q(\gamma^t)$ will flow towards the straight line segment without becoming singular. □

6.5 Advantage of our method

- We formulate the Thomas–Yau conjecture [3] for almost Calabi–Yau manifolds precisely. The angle condition remains the same as in the original conjecture but the volume condition looks different. (Conjecture 1.2.5)

- We add the condition to the conjecture that the phase variation of the original Lagrangian is less than π. This condition is assumed by all authors investigating (almost) Lagrangian mean curvature flow. We do not need a stronger assumption for the phase variation which was needed in the Thomas–Yau approach [3] to proving the conjecture in Milnor fibers. The assumption needed a correction (phase variation less than $\frac{n\pi}{2(n-1)}$) for the argument in the proof to succeed (Dellatorre [2]).

- Thomas–Yau use two flows of curves, one with fixed endpoints, the other a closed curve with two points fixed throughout the flow [2, Equations (5.60), (5.61)]. One of the issues that requires the extra assumption on the variation of the phase is to prove that the curve does not make a kink of more than π [2, Lemma 5.6.2].

- In contrast, the curve we study has θ as the angle of its tangent with the real axis. Since the range of θ during the flow is contained in the initial range which is less than π, there is no possibility for our curve to have a kink of more than π.

- Our flow is a a generalized curve shortening flow as a consequence of the differential equations governing the flow (1.9) since the length of the curve is $C \int_{L_{\gamma^t}} |\Omega| dV_{g_{L_\gamma}}$. Angenent (Angenent [8], [9]) studied a very general flow of closed curves which is uniformly parabolic. Our flow differs from Angenent's:

  · It has a discrete set of points in the plane which remain fixed.

  · We are primarily (but not solely) interested in the flow of paths with fixed endpoints. Angenent investigates the flow of closed curves.

Chapter 7: Thomas–Yau Conjecture

7.1 Thomas–Yau Conjecture

In this chapter we develop an approach for more general almost Calabi–Yau (AC–Y). A variant of the Thomas–Yau conjecture is formulated as Conjecture 7.1.1.

Conjecture 7.1.1. *Let (X, J, ω, Ω) be almost Calabi–Yau. Let $L \subset X$ be an orientable, graded, compact Lag submanifold. Let $\theta : L \to [\theta_{min}, \theta_{max}] \subset (-\pi, \pi]$ be a grading on L and assume that $\theta_{max} - \theta_{min} < \pi$.*

There exists an SL in the homotopy class of L if either one of the following two conditions holds for all Lags L_1 and L_2 such that $[L] = [L_1 \# L_2]$:

$$\int |\Omega| \, dV_{g_L} < \Big| \int_{L_1} \Omega|_{L_1} \Big| + \Big| \int_{L_2} \Omega|_{L_2} \Big|, \tag{7.1}$$

or

$$\Big[\arg \int_{L_1} \Omega|_{L_1}, \arg \int_{L_2} \Omega|_{L_2} \Big] \not\subseteq (\theta_{min}, \theta_{max}). \tag{7.2}$$

First we will introduce some notation and then we will define a curve, γ_L, associated with the Lag L.

We will use the following notation for level, sublevel, and superlevel sets for the function

$\theta : L \to [\theta_{min}, \theta_{max}] \subset \mathbb{R}$:

$$L_\tau := \{p \in L \,:\, \theta(p) = \tau\}, \tag{7.3}$$

$$M_\theta := \bigcup_{\tau \leq \theta} L_\tau, \tag{7.4}$$

$$N_\theta := \bigcup_{\tau \geq \theta} L_\tau. \tag{7.5}$$

Let

$$A_L(\theta) := \int_{M_\theta} \Omega|_L. \tag{7.6}$$

Using the function $A_L(\theta)$ we proceed to define a curve, γ_L, in the following way:

Consider the image of $A_L : [\theta_{min}, \theta_{max}] \to \mathbb{C}$. It is a path in $\mathbb{C}$ which is continuous unless for some value $\theta_0 \in [\theta_{min}, \theta_{max}]$,

$$\int_{L_{\theta_0}} \Omega|_L > 0, \text{ i.e. } \dim L_{\theta_0} = n.$$

In that case there is a jump discontinuity since

$$A_L(\theta_0) = \lim_{\theta \to \theta_0^-} A_L(\theta) + \int_{L_{\theta_0}} \Omega|_L > \lim_{\theta \to \theta_0^-} A_L(\theta).$$

We close the gap with a straight line segment which necessarily has direction $e^{i\theta_0}$ and length $\int_{L_{\theta_0}} |\Omega| dV_{g_L}$.

(7.7)

Definition 7.1.2. *Let γ_L be the curve in the complex plane associated with a Lag L in an almost Calabi–Yau manifold (X, J, ω, Ω) in the following way: γ_L is given by the image of the map A (7.6) and, in case of discontinuities of the graph, straight line segments closing the gaps. Furthermore, let γ_L be "free" in the sense that any translation of the above defined γ_L is considered the same as γ_L (analogous to free vectors).*

Remark 7.1.3. Sometimes we will be considering the translation of γ_L by c, $\gamma_L + c$, for some $c \in \mathbb{C}$. This will be needed when looking at the almost mean curvature flow of L and possible singularities of the flow.

In Figure 7.1 we see an example of a curve associated with a Lag L. The first and the third segments are straight, corresponding to $\dim L_{\theta_{min}} = n$ and $\dim L_{\theta_0} = n$ for some $\theta_0 \in (\theta_{min}, \theta_{max})$.

Lemma 7.1.4. *Let (X, J, ω, Ω) be almost Calabi–Yau and let L be a Lag submanifold of X. Let the path γ_L be as in Definition 7.1.2. Then:*

(i) The end points of the path γ_L are (0,0) and $\int_L \Omega$.

(ii) The length of the path γ_L is equal to $\int_L |\Omega| \, dV_{g_L}$.

(iii) The path γ_L is convex.

(iv) If L is non-singular then γ_L is C^1.

Proof. For any level set L_θ,

$$\dim L_\theta \leq \dim L = n.$$

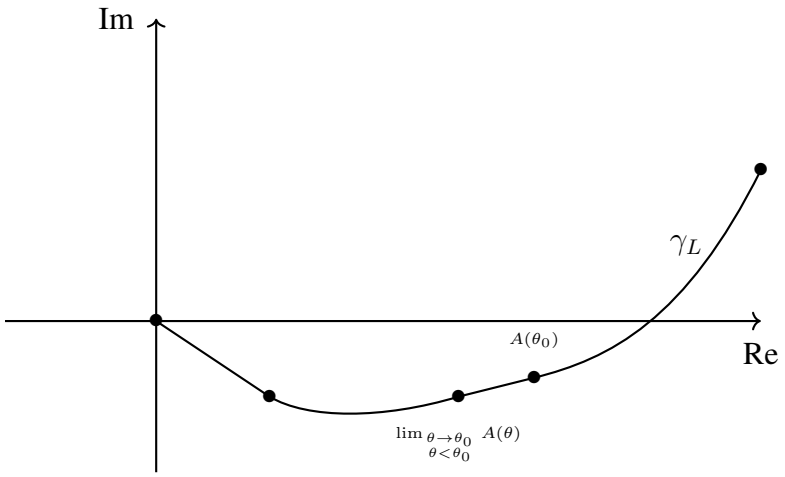


Figure 7.1: Curve γ_L

(i) If $\dim L_{\theta_{min}} < n$ then

$$A_L(\theta_{min}) = \int_{M_{\theta_{min}}} \Omega|_L = \int_{L_{\theta_{min}}} \Omega|_L = 0.$$

If $\dim L_{\theta_{min}} = n$ then the initial segment of γ_L is a straight line segment from $(0,0)$ to

$$\int_{L_{\theta_{min}}} \Omega|_L = e^{i\theta_{min}} \int_{L_{\theta_{min}}} |\Omega| dV_{g_L} \neq 0.$$

In any case γ_L starts at $(0,0)$. The other endpoint of γ_L is $\int_L \Omega|_L$ as

$$A_L(\theta_{max}) = \int_{M_{\theta_{max}}} \Omega|_L = \int_L \Omega|_L.$$

(ii) Suppose that $p \in L_\theta$ is a regular point for the function θ, i.e.,$\nabla\theta|_p \neq 0$. Then there exists a neighborhood of p in L, U, where $\nabla\theta \neq 0$. Let e_1 be the unit vector field in TU which is in the direction of $\nabla\theta|_q$ at every point $q \in U$. By [34, Theorem 9.22], there exist local coordinates such that $\nabla\theta$ is the first coordinate vector field. Gram-Schmidt algorithm can be applied to the

local coordinates to obtain an orthonormal frame of vector fields in TL_{U_1} for some $U_1 \subset U$:

$$\{e_1, e_2, \ldots, e_n\}.$$

$g(e_1, e_j) = 0, \forall j = 2, 3, \ldots n.$ $e_1 = \frac{1}{H}\nabla\theta$, where $H = |\nabla\theta| = |J\nabla\theta|$.

$$e_j(\theta) = d\theta(e_j) = g(\nabla\theta, e_j) = g(He_1, e_j) = 0$$

so $e_2, \ldots e_n$ are tangent to $L_{\theta(p)} \bigcap U_1$.

Let $f_1, \ldots f_n$ be the basis of 1-forms dual to $e_1, \ldots e_n$.

Ω restricted to L is a top level form. dV_{g_L} denotes the volume form on L which comes from the volume form associated with the metric on X with the property that when applied to an orthonormal basis of tangent vectors it returns the value 1. $\Omega|_L$ can be expressed in terms of $f_1, \ldots, f_n$ as:

$$\Omega|_L = e^{i\theta}|\Omega| dV_{g_L} = e^{i\theta}|\Omega| f_1 \wedge \cdots \wedge f_n. \tag{7.8}$$

Since $\nabla\theta = He_1$ the covectors dual to $\nabla\theta$ and e_1, $d\theta$ and f_1, satisfy:

$$d\theta = Hf_1.$$

Therefore in U_1 we have

$$\Omega|_L = e^{i\theta}|\Omega| f_1 \wedge \cdots \wedge f_n \tag{7.9}$$

$$= e^{i\theta}\frac{1}{H}|\Omega| d\theta \wedge f_2 \wedge \cdots \wedge f_n \tag{7.10}$$

$$= e^{i\theta}\frac{1}{H}|\Omega| d\theta \wedge dV_{g_{L_\theta}}. \tag{7.11}$$

1. Assume first that the set $\{p \in L \ : \ \nabla\theta|_p = 0\}$ does not contain a submanifold of dimension n of L. Then the value of the integral defining $A(\theta)$ is unchanged if the set of points where $\nabla\theta = 0$ is removed from L.

L is compact, therefore it can be covered by finitely many open sets in which, by (7.8) and (7.11), the volume form can be expressed as:

$$dV_{g_L} = \frac{1}{H} d\theta \wedge dV_{g_{L_\theta}}$$

and in which Fubini's theorem applies. Using a partition of unity we obtain:

$$A_L(\theta) = \int_{M_\theta} \Omega|_L = \int_{\theta_{min}}^{\theta} \Big(\int_{L_\tau} e^{i\tau} \frac{1}{H} |\Omega| \, dV_{g_{L_\tau}} \Big) \, d\tau.$$

The tangent vector to the curve γ_L, γ_L', is

$$\frac{d}{d\theta} A(\theta) = \frac{d}{d\theta} \int_{\theta_{min}}^{\theta} \Big(\int_{L_\tau} e^{i\tau} \frac{1}{H} \, |\Omega| \, dV_{g_{L_\tau}} \Big) \, d\tau \tag{7.12}$$

$$= \int_{L_\theta} e^{i\theta} \, \frac{1}{H} |\Omega| \, dV_{g_{L_\theta}} \tag{7.13}$$

$$= e^{i\theta} \int_{L_\theta} \frac{1}{H} \, |\Omega| \, dV_{g_{L_\theta}}. \tag{7.14}$$

The length of a segment of γ_L is

$$s = \int_{\theta_1}^{\theta_2} |\gamma_L'| d\theta \tag{7.15}$$

$$= \int_{\theta_1}^{\theta_2} \Big(\int_{L_\tau} \frac{1}{H} |\Omega| \, dV_{g_{L_\tau}} \Big) \, d\tau \tag{7.16}$$

$$= \int_{M_{\theta_2} \setminus M_{\theta_1}} |\Omega| \, dV_{g_L}. \tag{7.17}$$

2. Now suppose that the set of singular points of the function θ contains a submanifold of L of dimension n. Then $\theta = $ const. on any such submanifold. Let θ_0 be a value for which the level set L_{θ_0} contains a set of dimension n. Then $A(\theta)$ has a discontinuity at θ_0. There is a jump in the value of $A(\theta)$ which equals

$$\int_{L_{\theta_0}} \Omega_L = e^{i\theta_0} \int_{L_{\theta_0}} |\Omega| \, dV_{g_L},$$

and the length of a segment of the curve γ_L which is along a straight line corresponding to the discontinuity of $A(\theta)$ at θ_0 is the length of that vector:

$$\int_{L_{\theta_0}} |\Omega| \, dV_{g_L}.$$

In both situations the resulting expression is the same. Therefore the total length of γ_L is:

$$\int_{L} |\Omega| \, dV_{g_L}.$$

(iii) The tangent vector at $A_L(\theta)$ has the direction $e^{i\theta}$ by (7.12). When moving along γ_L from left to right, θ is increasing (or staying the same along straight line segments). Therefore, γ_L' is non-decreasing and hence γ_L is convex.

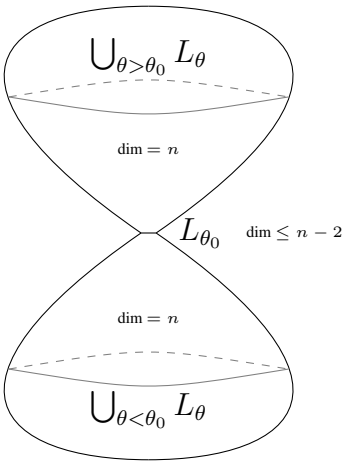


Figure 7.2: If $\dim L_{\theta_0} < n-1$ then L is singular

(iv) From the expression for $A'(\theta)$ (equation (7.12)), if for some value θ_0, $A'(\theta_0) = 0$, then $\dim L_{\theta_0} < n-1$. But then, since θ is continuous, L would be singular contrary to the assumption (see Figure 7.2).

□

Lemma 7.1.5. *If an SL exists in the homology class of the Lag L then the curve corresponding to the SL is the straight line segment connecting the endpoints of L_γ.*

Proof. $A_L(\theta_{max}) = \int_L \Omega|_L$. This integral has the same value for any Lag in the same homology class as L which means that the curve associated with any other Lag in the same class has the same end points (Lemma 7.1.4). The function θ is constant on an SL so the corresponding curve is the straight line segment connecting the origin with the endpoint of the curve γ_L. □

Lemma 7.1.6. *Suppose that $L \approx L_1 \# L_2$ where L_1 and L_2 are SLs in $[L_1]$ and $[L_2]$. Then $\gamma_{L_1 \cup L_2}$ is the union of two sides of the triangle with vertices*

$$0, \int_{L_1} \Omega|_{L_1}, \int_L \Omega|_L,$$

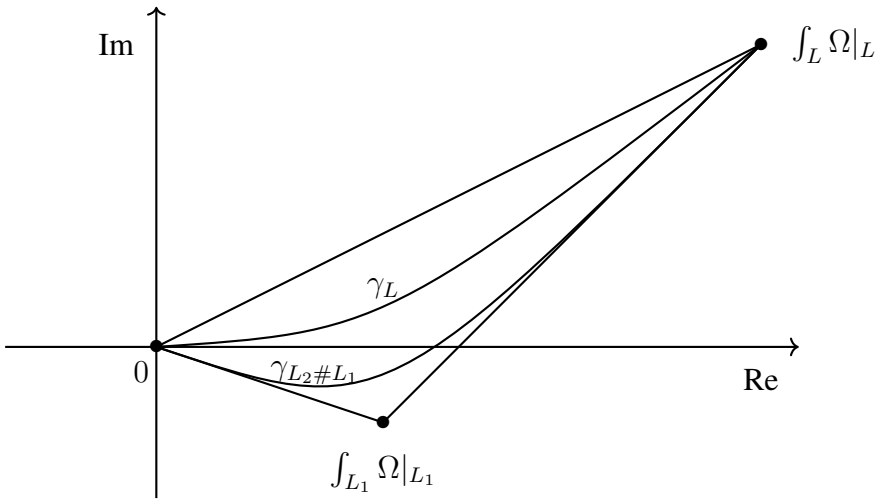


Figure 7.3: for Lemma 7.1.6

where the third side is $\overline{0 \int_L \Omega_L}$.

Proof. See Figure 7.3. As L and $L_1\#L_2$ are in the same homology class, the associated paths, γ_L and $\gamma_{L_1\#L_2}$ begin at the same point and end at the same point. $L_1\#L_2$ is not uniquelly determined. It is a family of submanifolds which mostly coincide with L_1 in one part and L_2 in another part and there exists a smoothing part over which the connection is established between L_1 and L_2. The smoothing part of $L_1\#L_2$ can be made arbitrarily small so the singular manifold $L_1 \cup L_2$ satisfies

$$\int_{L_1} \Omega + \int_{L_2} \Omega = \int_L \Omega.$$

□

Remark 7.1.7. Lagrangian angle is a function on L. Its gradient with respect to the metric induced on L by the metric on X is a vector field on L:

$$\nabla\theta = \nabla_{g_L}\theta.$$

$J\nabla\theta$ is a vector field in NL. If X is Calabi–Yau then $J\nabla\theta$ is the mean curvature vector field of L. If X is almost Calabi–Yau but not Calabi–Yau then $J\nabla\theta$ is the mean curvature vector field of L in a metric which is conformal to the original metric with conformal factor $|\Omega|$. Such conformal change of metric is not a holomorphic change so it is preferable to remain in the original metric and be aware that the SLs are not volume minimizing.

7.1.1 Almost Lagrangian mean curvature flow

As defined in [2, p. 124] following [3, Section 6], $J\nabla_{g_L}\theta$ is called the almost mean curvature vector of the Lag L. (When the manifold containing L, X, which is almost Calabi–Yau, is actually Calabi–Yau, $|\Omega| \equiv 1$ and $J\nabla_{g_L}\theta$ is the mean curvature vector of L.)

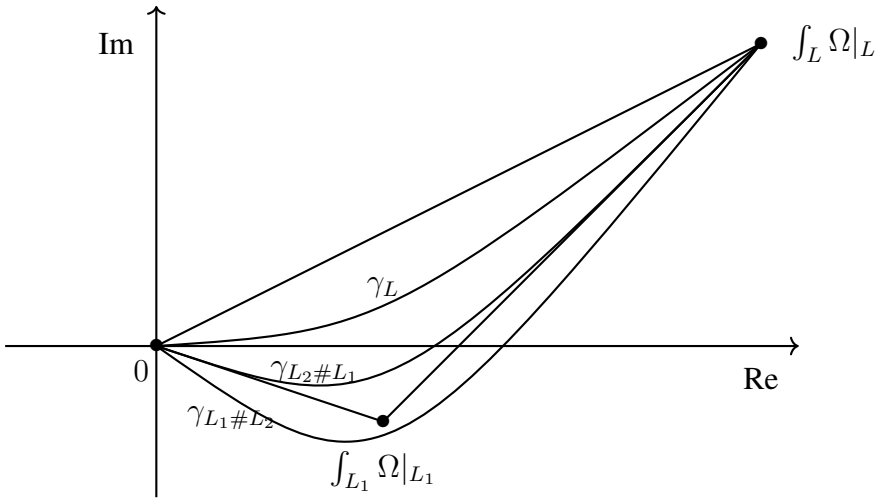


Figure 7.4: stable and unstable

We now examine the almost mean curvature flow of a compact graded Lag with phase variation less than π.

The main idea for the proof of the conjecture:

When L^t flows by ALMCF the corresponding curve γ_L^t remains convex, has fixed end-

points, and is contained in the region enclosed by the initial curve, γ_L^0 and the straight line segment connecting its endpoints. In the absence of singularities, the flow of γ_L^t converges to the straight line segment connecting its endpoints.

Now, if the flow has a singularity at $t = T$, possibly with $T = \infty$, then the singular set has dimension smaller than n and it the curve corresponding to L^T is well-defined and is the limit of curves γ_L^t for $t \to T$. Clearly, γ_L^T is contained in the region enclosed by γ_L^0 and the straight line segment connecting its endpoints.

For simplicity assume that L^T has two components which are non-singular (and therefore Lagrangian), L_1 and L_2. We will show that if there is a singularity in the flow then there exist Lags L_1 and L_2, resulting from L^t at the singular time $t = T$, such that the conditions (7.1) and (7.2) fail for them.

First we prove

Lemma 7.1.8. *If L^t is the flow of L by the vector $J\nabla_{g_L}\theta$ (ALMCF) then:*

(i) If $t_1 < t_2$ then $[\theta^{t_2}_{min}, \theta^{t_2}_{max}] \subset [\theta^{t_1}_{min}, \theta^{t_1}_{max}]$.

(ii) The length of γ_{L^t} decreases.

(iii) γ_{L^t} remains convex and non-singular.

(iv) $\frac{d}{dt}A(\theta) = 2ie^{i\theta}\int_{L^*_\theta} H|\Omega|\, dV_{g_{L_\theta}}$.

Proof. Equations governing the Lagrangian almost mean curvature flow are [2, Proposition 5.5.7., equations (5.53)]:

$$\frac{d}{d\theta} = \Delta\theta + \frac{\langle d\theta, d|\Omega|\rangle}{|\Omega|} \tag{7.18}$$

$$\frac{d}{dt}(|\Omega\,|dV_{g_L}) = -|d\theta|^2(|\Omega\,|dV_{g_L}) \tag{7.19}$$

(i) The interval $[\theta^t_{min}, \theta^t_{max}]$ decreases over time as the equation (7.18) is parabolic.

(ii) The length of γ_{L^t} is $\int_{L^t} |\Omega| dV_{g_L}$ by Lemma 7.1.4. By equation (7.19), the volume measured with respect to $|\Omega| dV_{g_L}$ decreases over time. Then the length of γ_{L^t} which equals that volume must decrease with time as well.

(iii) As long as the flow does not run into a singularity L^t is a smooth Lag and by Lemma 7.1.4 the curve γ_{L^t} is convex and non-singular.

(iv) We have the following calculation:

$$A(\theta) = \int_{M_\theta} \Omega|_L \tag{7.20}$$

$$\frac{d}{dt}A(\theta) = \frac{d}{dt}\int_{M_\theta} \Omega|_L \tag{7.21}$$

$$= \int_{M_\theta} \iota_{J\nabla\theta} d\Omega|_L + \int_{\partial M_\theta} \iota_{J\nabla\theta}\Omega|_L + \int_{M_\theta} \frac{d}{dt}\Omega|_L \tag{7.22}$$

$$= \int_{\partial M_\theta} \iota_{J\nabla\theta}\Omega|_L + \int_{M_\theta} d(\iota_{J\nabla\theta}\Omega)|_L \tag{7.23}$$

$$= 2i\int_{L_\theta} \iota_{\nabla\theta}\Omega|_L \tag{7.24}$$

$$= 2i\int_{L_\theta} \iota_{\nabla\theta}\left(e^{i\theta}|\Omega| dV_{g_L}\right) \tag{7.25}$$

$$= 2ie^{i\theta}\int_{L_\theta} H|\Omega| dV_{g_{L_\theta}}, \tag{7.26}$$

where we used in line (7.22) the Leibniz formula for the differentiation of the integral [35, (7.2)

p.624] and in line (7.23) Cartan's magic formula as well as the fact that Ω is closed [2, equation (5.55)]:

$$\frac{d}{dt}\Omega|_L = \mathcal{L}_{J\nabla\theta}\Omega|_L = d(\iota_{J\nabla\theta}\Omega|_L) + \iota_{J\nabla\theta}d\Omega|_L = d(\iota_{J\nabla\theta}\Omega|_L).$$

Line (7.26) follows since the contraction of the volume form dV_{g_L} with the vector field $\nabla\theta$ (which is normal to L_θ and is equal to $H\times$ (normal vector to L_θ)) results in $H\, dV_{g_{L_\theta}}$.

□

To the flow L^t we associate the corresponding curves γ_{L^t}. In Lemma 7.1.11 we evaluate the curvature vector of γ_{L^t}.

Corollary 7.1.9. *If the ALMCF of $L = L^0$, L^t, does not have a singularity for $t < \infty$ or $t = \infty$ then the corresponding flow of paths γ_L^t exists for all time and converges to a straight line segment.*

Proof. The flow of curves γ_L^t is a generalized mean curvature flow as in Chapter 6. □

Corollary 7.1.10. *If there is no singularity then the limit of the ALMCF L^t is an SL.*

Proof. The curve corresponding to the limit of ALMCF L^t is a straight line segment by Corollary 7.1.9 therefore the corresponding Lagrangian has constant phase, i.e., it is special. □

Lemma 7.1.11. *The curvature of γ_L^t at θ is*

$$\Big(\int_{L_\theta} \frac{1}{H}|\Omega|\, dV_{g_{L_\theta}}\Big)^{-1}.$$

Proof. By Lemma 7.1.4, the tangent vector to γ_{L^t} at $A(\theta)$ is

$$e^{i\theta}\int_{L_\theta}\frac{1}{H}|\Omega|dV_{g_{L_\theta}}.$$

The unit tangent vector is therefore $e^{i\theta}$. By Lemma 7.1.4 the arclength is

$$s=\int_{\theta_{min}}^{\theta}\Big(\int_{L_\tau}\frac{1}{H}|\Omega|\,dV_{g_{L_\tau}}\Big)\,d\tau.$$

Therefore

$$\frac{ds}{d\theta}=\int_{L_\theta}\frac{1}{H}|\Omega|\,dV_{g_{L_\theta}}.$$

To calculate the curvature, differentiate the unit tangent vector with respect to s:

$$\frac{d}{ds}e^{i\theta}=ie^{i\theta}\frac{d\theta}{ds}=ie^{i\theta}\Big(\int_{L_\theta}\frac{1}{H}|\Omega|\,dV_{g_{L_\theta}}\Big)^{-1}$$

□

Remark 7.1.12. From Lemmas 7.1.11 and 7.1.8, we see that γ_{L^t} is not evolving under mean curvature flow. γ_{L^t} is evolving under generalized mean curvature flow of Chapter 6 as the direction of the flow vector the same as the direction of the curvature vector, the flow vector vanishes at the ends of the path γ_{L^t}, the modulus of the flow vector is a bounded function in any compact set containing the flow.

Let L^t be the almost Lagrangian mean curvature flow of a compact graded Lag with phase variation less than π, $L\equiv L^0$. If L^t is non-singular for all t then the limit of the flow, L^∞, is an SL of phase $\arg\int_L\Omega|_L$ or a singular union of SLs.

Now, there may be a finite time singularity of the flow. Use the following notation:

- If the flow of L by the almost mean curvature flow has a finite time singularity let S denote the singular set.
- For $p \in L$, let $p^t \in L^t$ denote the flow of p at time t.

Lemma 7.1.13. *Suppose that the almost Lagrangian mean curvature flow has a singularity at time $t = T$, possibly infinite. Suppose that $L_T \setminus S$, where S is the singular set, is not connected. Let $L_1, \ldots, L_m$ be the closures of the connected components of $L_T \setminus S$. Then*

(i) *$L_1, \ldots L_m$ are Lag submanifolds of X.*

(ii) *$\int_{L_1} \Omega|_{L_1} + \cdots + \int_{L_m} \Omega|_{L_m} = \int_L \Omega|_L$.*

Proof. (i) The Lag condition is preserved by the flow.

(ii) Since $\int_{L^t} \Omega|_{L^t}$=const. and the singular set S has $\dim S < \dim L^T$.

□

Conjecture 7.1.14. *$L_1, \ldots L_m$ are non-singular.*

Proposition 7.1.15. *Let L^t be ALMCF of the Lag $L = L^0 \subset X$, (X, J, ω, Ω) almost C-Y. Suppose that there is a singularity of the flow at $t = T$. Let S denote the singular set. Let $L^T \setminus S = L_1 \cup L_2$. Then*

(i)

$$\int_L \big|\Omega\big| \, dV_{g_L} \geq \Big| \int_{L_1} \Omega|_{L_1} \Big| + \Big| \int_{L_2} \Omega|_{L_2} \Big|,$$

(ii)

$$[\phi(L_1), \phi(L_2)] \subseteq (\theta_{min}, \theta_{max}),$$

which means that (7.1) and (7.2) fail to be satisfied.

Proof. Let L^t be ALMCF of the Lagrangian $L = L^0 \subset X$. By Lemma 7.1.8 the flow L^t induces a flow of corresponding curves γ_L^t such that:

- The endpoints of γ_L are fixed by the flow.
- γ_L^t is contained in the region enclosed by γ_L^τ and the straight line segment connecting its endpoints, for $\tau < t$.

(i) From $L^T \setminus S = L_1 \cup L_2$ it follows that

$$\int_{L^T} |\Omega| \, dV_{g_{L^T}} = \int_{L_1} |\Omega| \, dV_{g_{L_1}} + \int_{L_2} |\Omega| \, dV_{g_{L_2}}.$$

In other words,

$$\text{length}(\gamma_L^T) = \text{length}(\gamma_{L_1}) + \text{length}(\gamma_{L_2}).$$

From Lemma 7.1.8 it follows that

$$\text{length}(\gamma_L^T) \leq \text{length}(\gamma_L).$$

In addition, we use

$$\Big| \int_{L_j} \Omega|_{L_j} \Big| \leq \int_{L_j} \big|\Omega\big| \, dV_{g_{L_j}},$$

for $j = 1, 2$. Therefore, we have

$$\int_L \big|\Omega\big| \, dV_{g_L} \geq \int_{L^T} \big|\Omega\big| \, dV_{g_{L^T}} = \int_{L_1} \big|\Omega\big| \, dV_{g_{L_1}} + \int_{L_2} \big|\Omega\big| \, dV_{g_{L_2}},$$

and

$$\int_{L_1} \left|\Omega\right| dV_{g_{L_1}} + \int_{L_2} \left|\Omega\right| dV_{g_{L_2}} \geq \Big| \int_{L_1} \Omega|_{L_1} \Big| + \Big| \int_{L_2} \Omega|_{L_2} \Big|,$$

and consequently

$$\int_{L} \left|\Omega\right| dV_{g_{L}} \geq \Big| \int_{L_1} \Omega|_{L_1} \Big| + \Big| \int_{L_2} \Omega|_{L_2} \Big|,$$

which means that the condition (7.1) fails to be satisfied.

(ii) Now we turn to showing that the condition (7.2) fails to be satisfied for L_1 and L_2.

At $t = T$ the range of θ^T, $[\theta^T_{min}, \theta^T_{max}] \subseteq [\theta_{min}, \theta_{max}]$, where the latter is the range of the Lagrangian angle for the initial Lagrangian L. Therefore, $\phi(L_1) \in [\theta^T_{min}, \theta^T_{max}] \subset [\theta_{min}, \theta_{max}]$ and $\phi(L_2) \in [\theta^T_{min}, \theta^T_{max}] \subset [\theta_{min}, \theta_{max}]$. It follows that

$$[\phi(L_1), \phi(L_2)] \subseteq (\theta_{min}, \theta_{max}).$$

This means that the condition (7.2) fails to be satisfied.

□

Remark 7.1.16. To continue the flow after the singular time, we start the flow again for each of the submanifolds $L_1, \dots L_m$.

Continuing the flow for each of the resulting Lags separately, we eventually get a set of SLs.

Each of the SLs in this decomposition of the original Lag corresponds to a homology class (generator of the homology group $H_n(X)$). Each of them is a sphere or a homology sphere.

The curve corresponding to this decomposition is a polygonal line having the same end-points as γ_L.

The conditions from Conjecture 7.1.1 translate into: the region enclosed by γ_L and the straight line segment connecting its endpoints does not contain any of the vertices of such a polygonal line, for any choice of SLs, provided that if the flow has a singularity then necessarily L is a connect sum of the non-singular parts resulting from the singularity. Since the flow of γ_L stays in that region for all time. it can never reach any of the vertices of the polygonal line and thus the flow will continue for all time and be a straight line segment in the limit, which corresponds to an SL in the homology (and homotopy) class of L.

Remark 7.1.17. The missing part for the completeness of the proof is to either show that L is in the same homotopy class as a connect sum of the non-singular parts or to show that this need not be the case. In the latter situation it would be important to understand the nature of a possible construction. Thomas–Yau [3, Introduction, last paragraph] thought that it is probably too restrictive to use only Lagrangian connect sums as necessary degenerations.

7.1.2 Milnor fibers

In this subsection we show that the method of section 7.1 is related to the method we used for Milnor fibers, especially if θ is monotone on γ.

We showed in Lemma 2.2.1 that for Lag spheres in Milnor fibers

$$\int_{L_\gamma} \Omega_f^n = \frac{n}{2} \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} \int_\gamma \left(\sqrt{f(\gamma(u))}\right)^{n-2} \gamma'(u) du.$$

We can also find the value of the integral over a part of the Lag L_γ corresponding to a segment of the path γ between $\gamma(u_1)$ and $\gamma(u_2)$:

Lemma 7.1.18.

$$\int_{L_{\gamma(u_1)\gamma(u_2)}} \Omega_f^n = \frac{n}{2}\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\Big(Q(\gamma(u_2)) - Q(\gamma(u_1)\Big).$$

Proof.

$$\begin{aligned}\int_{L_{\gamma(u_1)\gamma(u_2)}} \Omega_f^n &= \int_{L_{\gamma(u_1)\gamma(u_2)}} \frac{dz_1 \wedge \cdots \wedge dz_n}{f'(w)} \\ &= \int_{L_{\gamma(u_1)\gamma(u_2)}} \frac{1}{2}\left(\sqrt{f(w)}\right)^{n-2} dw \wedge \sum_{j=1}^{n}(-1)^{j-1} r_j dr_1 \wedge \cdots \wedge \widehat{dr_j} \wedge \cdots \wedge dr_n \\ &= \frac{1}{2}\int_{\gamma(u_1)\gamma(u_2)} \left(\sqrt{f(w)}\right)^{n-2} dw \int_{S^{n-1}} \Big(\sum_{j=1}^{n}(-1)^{j-1} r_j dr_1 \wedge \cdots \wedge \widehat{dr_j} \wedge \cdots \wedge dr_n\Big)\end{aligned}$$

By Stokes' Theorem

$$\begin{aligned}&\int_{r_1^2+\cdots+r_n^2=1} \Big(r_1 dr_2 \wedge \cdots \wedge dr_n + \cdots + (-1)^{n-1} dr_1 \wedge \cdots \wedge dr_{n-1}\Big) = \\ &\qquad = \int_{r_1^2+\cdots+r_n^2\le 1} d\Big(r_1 dr_2 \wedge \cdots \wedge dr_n + \cdots + (-1)^{n-1} dr_1 \wedge \cdots \wedge dr_{n-1}\Big) \\ &\qquad = \int_{r_1^2+\cdots+r_n^2\le 1} n dr_1 \wedge \cdots \wedge dr_n = \frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}.\end{aligned}$$

Therefore,

$$\int_{L_{\gamma(u_1)\gamma(u_2)}} \Omega_f^n = \frac{n}{2}\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} \int_{\gamma(u_1)\gamma(u_2)} \left(\sqrt{f(w)}\right)^{n-2} dw \tag{7.27}$$

$$= \frac{n}{2}\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} \int_{u_1}^{u_2} \left(\sqrt{f(\gamma(u))}\right)^{n-2} \gamma'(u) du \tag{7.28}$$

$$= \frac{n}{2}\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\Big(Q(\gamma(u_2)) - Q(\gamma(u_1)\Big). \tag{7.29}$$

□

Corollary 7.1.19. *If θ is increasing along γ from θ_{min} to θ_{max} then*

$$A_{L_\gamma}(\theta) = \frac{n}{2}\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\Big(Q(\gamma(\theta)) - Q(\gamma(\theta_{min}))\Big).$$

In conclusion, if θ is monotone then there is a homothety between curves $Q(\gamma)$ and A_{L_γ}. If θ is not monotone the total lengths of the two paths are related by the factor $\frac{n}{2}\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ as in the θ monotone case.

In Milnor fibers it is beneficial to use the procedure which we have developed for them since we can get explicitly the paths which give rise to SLs.

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