TECHNICAL RESEARCH REPORT

Harmonic Functions and Inverse Conductivity Problems on Networks

by Carlos A. Berenstein and Soon-Yeong Chung

TR 2003-16

ω-HARMONIC FUNCTIONS AND INVERSE CONDUCTIVITY PROBLEMS ON NETWORKS.

CARLOS A. BERENSTEIN AND SOON-YEONG CHUNG

ABSTRACT. In this paper, we discuss the inverse problem of identifying the connectivity and the conductivity of the links between adjacent pair of nodes in a network, in terms of an input-output map. To do this we introduce an elliptic operator Δ_ω and an ω-harmonic function on the graph, with its physical interpretation been the diffusion equation on the graph, which models an electric network. After deriving the basic properties of ω-harmonic functions, we prove the solvability of (direct) problems such as the Dirichlet and Neumann boundary value problems. Our main result is the global uniqueness of the inverse conductivity problem for a network under a suitable monotonicity condition.

Introduction

A network represents a way of interconnecting any pair of users or nodes by means of some meaningful links. Thus, it is quite natural that its structure can be represented, at least in a simplified form, by a connected graph whose vertices represent nodes and whose edges represent their links. When we have some problem on a part of the network or when we are in need of finding such problem, it is almost impossible to investigate the whole network, since the network may be too vast and its structure or connectivity too complicated.

From the graph theoretical point of view, problems involving graph identification have been among the most important and famous open problems in graph theory ([BH]). Most of the work on this subject has concentrated on spectral graph theory, on the realization of graphs with given distances, and on the reconstruction of graphs from vertex deleted subgraphs (see [B2], [C1], [C2], [C3], [CO], [CL], [CvDGT], [CvDS] and [HY]). Thus far, spectral theory has been one of the most significant tools used studying graphs, and it has led to noteworthy progress in the study along these questions. But, as it is well known, graphs are not in general completely characterized by their spectra (see [CvDGT], p. 66).

In this paper another method to study the graph identification problem will be introduced, a discrete version of the inverse conductivity problem.

The inverse conductivity problem original aim was to identify the conductivity coefficient in continuous media from boundary measurements, such as Dirichlet data, Neumann data, or their appropriate combinations.

2000 *Mathematics Subject Classification.* Primary 05C40, 35R30 ; Secondary 94C12.

Key words and phrases. discrete Laplacian, inverse conductivity problem, diffusion equation.

The work of Prof. Chung was supported by the LG-foundation in 2002-2003. Prof. Berenstein acknowledges support from grants NSF-DMS0070044 and ARO-DAAD19-01-1-0494.

The discrete or finite nature of graphs makes working on graphs basically easier than investigating these problems in the continuous case. On the other hand, their discrete nature of also gives rise to several disadvantages. For example, solutions of the Laplace equation (introduced in Section 2) have neither the local uniqueness property nor is their uniqueness guaranteed by the Cauchy data, contrary to the continuous case, where they are the most important mathematical tools used to study the inverse conductivity problem and related problems.

The purpose of this paper is to give a discrete analogue of the inverse conductivity problem studied as studied in a number of publications, such as [A], [Ca], [I], [IP], [KS]and [SU]. To do this we introduce an elliptic operator on the graph, the ω - Laplacian Δ_ω and interpret it as a diffusion equation on the graph modeled by the electric network. Since little has been so far studied about partial differential equations on graphs, we will establish several useful properties of Δ_ω, which are essential to solve the inverse problem.

The inverse problem we study is to identify the connectivity of the nodes and the conductivity of the edges between each adjacent pair of nodes. We begin by proving the following global uniqueness result for the inverse conductivity problem in a network satisfying the monotonicity condition:

Theorem. *Let ω_1 and ω_2 be weights with $\omega_1 \le \omega_2$ on $\overline{S} \times \overline{S}$ and $f_1,\ f_2 : \overline{S} \to \mathbb{R}$ be functions satisfying that for $j = 1,\ 2,$*

$$\begin{cases} \Delta_{\omega_j} f_j(x) = 0,\ x \in S, \\ \frac{\partial f}{\partial_{\omega_j} n}(z) = \psi(z),\ z \in \partial S, \\ \int_S f_j \ d_{\omega_j} = K \end{cases}$$

for a given function $\psi : \partial S \to \mathbb{R}$ with $\int_{\partial S} \psi = 0$ and for a suitably chosen number $K > 0$.

If we assume that

(i) $\omega_1(z, y) = \omega_2(z, y)$ *on* $\partial S \times \overset{\circ}{\partial S}$,
(ii) $f_1|_{\partial S} = f_2|_{\partial S}$,

then we have

$$f_1 = f_2 \text{ on } \overline{S}$$

and

$$\omega_1 = \omega_2 \quad \text{on} \overline{S} \times \overline{S}.$$

The second conclusion $\omega_1 = \omega_2$ above is exactly what we want to have. In fact, it shows not only whether or not, each pair of two nodes is connected by a link, but also how nice the link is.

Both the condition $\omega_1 \le \omega_2$ above (so called, the monotonicity condition) and the condition $\int_S f_j \ d_{\omega_j} = K$ (so called the normalization condition) will be shown to be essential by giving counterexamples. In fact, even in the continuous case, some form of monotonicity has also been considered (see [I], [Ca] and [A]).

We organized this paper as follows: First, we discuss calculus on graphs in Section 1 and in Section 2 we introduce ω-harmonic functions on graphs and some good properties of them, which are useful later and for further study. In fact, those properties are interesting by themselves in authors' opinion.

In Section 3, we discuss the direct problems such as the Dirichlet BVP and Neumann BVP, and give a physical interpretation of Δ_ω. Besides, additional useful properties of ω-harmonic functions will be introduced.

In the final Section 4, we prove the global uniqueness result of inverse problem under the monotonicity condition. Ahead of its proof, we derive an discrete version of the Dirichlet principle, which is an essential tool for the proof of the main theorem.

After the authors completed this paper, Professor Gunter Uhlmann informed the authors that Morrow with his group published a series of papers (see [MC1], [MC2],[MMC], [MI] and [MIC]) on the inverse problem of the networks. But their results were concentrated on the networks of special types such as circular networks or integer lattices. Moreover, their approaches would not work for the networks of general type.

1. **Calculus on Weighted Graphs**

We shall begin with some definitions of graph theoretic notions frequently used throughout this paper.

By a *graph* $G = G(V,E)$ we mean a finite set V of *vertices* with a set E of two-element subsets of V (whose elements are called *edges*). The set of vertices and edges of a graph G are some times denoted by $V(G)$ and $E(G)$, or simply V and E, respectively. But conventionally, we denote either $x \in V$ or $x \in G$ the fact that x is a vertex in G.

A graph G is said to be *simple* if it has neither multiple edge nor loop and G is said to be *connected* if for every pair of vertices x and y there exist a sequence (termed a *path*) of vertices $x = x_0, x_1, x_2, \cdots, x_{n-1}, x_n = y$ such that x_{j-1} and x_j are connected by an edge (termed *adjacent*) for $j = 1, 2, \cdots, n$.

A graph $S = S(V', E')$ is said to be a *subgraph* of $G(V,E)$ if $V' \subset V$ and $E' \subset E$. In this case, we call G a *host graph* of S. If E' consists of all the edges from E which connect the vertices in V', then S is called an *induced* subgraph. It is easy to see that every induced subgraph of a connected graph is also connected.

A *weighted (undirected) graph* is a graph $G(V,E)$ associated with a *weight* function $\omega : V \times V \to [0, \infty)$ satisfying

(i) $\omega(x,x) = 0, \quad x \in V,$
(ii) $\omega(x,y) = \omega(y,x), \quad \text{if } x \sim y,$
(iii) $\omega(x,y) = 0 \quad \text{if and only if} \quad \{x,y\} \notin E.$

Here, $x \sim y$ means that two vertices x and y are connected (adjacent) by an edge in E. In the case, $\{x,y\}$ denotes the edge connecting the vertices x, y.

In particular, a weight function ω satisfying

$$\omega(x,y) = 1, \quad \text{if } x \sim y$$

is called the *standard* weight on G. The physical meaning of the weight function will be discussed later in Section 3.

The *degree* $d_\omega x$ of a vertex x in a weighted graph $G(V,E)$ with a weight ω is defined to be

$$d_\omega x := \sum_{y \in V} \omega(x,y).$$

Throughout this paper, all the subgraphs are assumed to be induced subgraphs of a host graph, which is simple and connected, with a weight, and a function on a graph is understood as a function defined only on the set of vertices.

The integration of a function $f : G \to \mathbb{R}$ on a graph $G = G(V, E)$ is defined by

$$\int_G f d_\omega \quad \text{(or simply } \int_G f) := \sum_{x \in V} f(x) d_\omega x.$$

We shall now define the directional derivative of a function $f : G \to \mathbb{R}$. For each x and $y \in V$ we define

$$D_{\omega,y} f(x) := [f(y) - f(x)] \sqrt{\frac{\omega(x,y)}{d_\omega x}} \ .$$

The gradient ∇_ω of function f is defined to be a vector

$$\nabla_\omega f(x) := \left(D_{\omega,y} f(x)\right)_{y \in V} ,$$

which is indexed by the vertices $y \in V$. Then it is easy to see that

$$\begin{aligned}
\int_G |\nabla_\omega f(x)|^2 &= \sum_{x \in V} |\nabla_\omega f(x)|^2 \, d_\omega x \\
&= \sum_{x \in V} \sum_{y \in V} |f(y) - f(x)|^2 \, \omega(x,y) \\
&= 2 \sum_{\{x,y\} \in E} |f(y) - f(x)|^2 \, \omega(x,y),
\end{aligned}$$

which is called the energy of f on G.

For a subgraph S of a graph $G = G(V, E)$ the (vertex) *boundary* ∂S of S to be set of all vertices $z \in V$ not in S but adjacent to some vertex in S, i.e.

$$\partial S := \{z \in V | z \sim y \text{ for some } y \in S\}$$

and we define the *inner boundary* $\overset{\circ}{\partial} S$ is defined by

$$\overset{\circ}{\partial} S := \{y \in S | y \sim z \text{ for some } z \in \partial S\} .$$

Also, by $\overline{S}$ we denote a graph whose vertices and edges are in $S \cup \partial S$.

The (outward) *normal derivative* $\frac{\partial f}{\partial_\omega n}(z)$ at $z \in \partial S$ is defined to be

$$\frac{\partial f}{\partial_\omega n}(z) := \sum_{y \in S} [f(z) - f(y)] \cdot \frac{\omega(z,y)}{d'_\omega z} \ ,$$

where $d'_\omega z = \sum_{y \in S} \omega(z,y)$.

The *ω-Laplacian* Δ_ω of a function $f : G \to \mathbb{R}$ on a graph G is defined to be

$$\Delta_\omega f(x) := -\sum_{y \in V} \left(D_y^2 f\right)(x), \quad x \in V.$$

In other words,

$$\Delta_\omega f(x) := \sum_{y\in V}[f(y)-f(x)]\cdot\frac{\omega(x,y)}{d_\omega x},\quad x\in V. \tag{1.1}$$

For notations, notions and conventions we refer to [C1] and [CvDS].

Remark 1.1. (i) The discrete Laplacian on graphs can be found in several places, such as [C1], [CvDS], [B1]. But the ω-Laplacian defined above is not exactly the same as the one considered in those references. In fact, the definition used here will give us an advantage of a more consistent treatment in Section 2.

(ii) The first derivatives and gradient in a discrete sense have not been introduced so far precisely in the literature, as far as the authors know. But the first derivative $D_{\omega,y}$ defined above may still be unsatisfactory in a sense that Leibniz' rule does not hold. In spite of this defect, it will be seen later that it has the appropriate physical meaning and works very well with respect to calculus on graphs.

In what follows, a function f defined on $\overline{S}$ may be understood as a function on its host graph G such that $f=0$ on $G\setminus\overline{S}$, if necessary.

Theorem 1.2. *Let S be a subgraph of a host graph G. Then for any pair of functions $f:\overline{S}\to\mathbb{R}$ and $h:\overline{S}\to\mathbb{R}$, we have*

$$2\int_{\overline{S}}h(-\Delta_\omega f)=\int_{\overline{S}}\nabla_\omega h\cdot\nabla_\omega f. \tag{1.2}$$

Proof. A direct use of the definitions mentioned above gives

$$\begin{aligned}
2\int_{\overline{S}}h(-\Delta_\omega f) &= 2\sum_{x\in\overline{S}}h(x)[-\Delta_\omega f(x)]d_\omega x\\
&= -2\sum_{x\in\overline{S}}h(x)\Big\{\sum_{y\in V(G)}[f(y)-f(x)]\omega(x,y)\Big\}\\
&= 2\sum_{x\in\overline{S}}\sum_{y\in\overline{S}}h(x)[f(x)-f(y)]\omega(x,y)\\
&= \sum_{x\in\overline{S}}\sum_{y\in\overline{S}}h(x)[f(x)-f(y)]\omega(x,y)+\sum_{x\in\overline{S}}\sum_{y\in\overline{S}}h(y)[f(y)-f(x)]\omega(x,y)\\
&= \sum_{x\in\overline{S}}\sum_{y\in\overline{S}}\Big\{[f(y)-f(x)]\sqrt{\omega(x,y)}\Big\}\cdot\Big\{[h(y)-h(x)]\sqrt{\omega(x,y)}\Big\}\\
&= \sum_{x\in\overline{S}}\{\nabla_\omega f(x)\cdot\nabla_\omega h(x)\}d_\omega x\\
&= \int_{\overline{S}}\nabla_\omega f\cdot\nabla_\omega h. \quad\blacksquare
\end{aligned}$$

The above theorem yields many useful formulas such as the Green theorem.

Corollary 1.3. *Under the same hypotheses as above we have the following identities:*

(i)

$$2\int_{\overline{S}} f(-\Delta_\omega f) = \int_{\overline{S}} |\nabla_\omega f|^2 .$$

(ii)

$$\int_{\overline{S}} h\Delta_\omega f = \int_{\overline{S}} f\Delta_\omega h.$$

(iii) *(Green's formula)*

$$\int_S (f\Delta_\omega h - h\Delta_\omega f) = \int_{\partial S} (f\frac{\partial h}{\partial_\omega n} - h\frac{\partial f}{\partial_\omega n}).$$

Proof. (i) is trivial and (ii) can be easily obtained by the symmetry in (1.2). We prove (iii). In view of (ii) we have

$$\begin{aligned} 0 &= \int_{\overline{S}} [f\Delta_\omega h - h\Delta_\omega f] \\ &= \int_S [f\Delta_\omega h - h\Delta_\omega f] + \int_{\partial S} [f\Delta_\omega h - h\Delta_\omega f] \end{aligned}$$

Then, since S is the induced subgraph, it follows that $\omega(z, y) = 0$ for all z and $y \in \partial S$ and

$$\begin{aligned} \int_S [f\Delta_\omega h - h\Delta_\omega f] &= \int_{\partial S} [h\Delta_\omega f - f\Delta_\omega h] \\ &= \sum_{z\in\partial S} [h(z)\Delta_\omega f(z) - f(z)\Delta_\omega h(z)] d_\omega z \\ &= \sum_{z\in\partial S}\sum_{y\in S} \Big\{ h(z)\big[f(y) - f(z)\big]\omega(z, y) - f(z)\big]\big[h(y) - h(z)\big]\omega(z, y) \Big\} \\ &= \sum_{z\in\partial S} \Big[h(z)\{ -\frac{\partial f}{\partial_\omega n}\} + f(z)\frac{\partial h}{\partial_\omega n}(z) \Big] d_\omega z \\ &= \int_{\partial S} [f\frac{\partial h}{\partial_\omega n} - h\frac{\partial f}{\partial_\omega n}] \quad \blacksquare \end{aligned}$$

In the continuous case, the followings are well-known formula :

$$\Delta(fg) = f\Delta g + 2\nabla f \cdot \nabla g + g\Delta f$$

$$\int_\Omega \nabla f \cdot \nabla g + \int_\Omega f\Delta g = \int_{\partial\Omega} f\frac{\partial g}{\partial n}$$

Here, we introduce a discrete analogue of these formula.

Theorem 1.4. *Under the same hypothesis as in Theorem 1.2, we have following identities hold :*

(i)

$$\Delta_\omega(fh) = f\Delta_\omega h + \nabla_\omega f \cdot \nabla_\omega h + h\Delta_\omega f$$

(ii)

$$\int_S \nabla_\omega f \cdot \nabla_\omega h + \int_S [f\Delta_\omega h + h\Delta_\omega f] = \int_{\partial S} \frac{\partial(fh)}{\partial_\omega n}$$

Proof. (i) can be obtained by an elementary manipulation. Using now (i) and Theorem 1.2, (iii) with $h \equiv 1$ we obtain (ii). ■

2. ω - Harmonic Functions

In this section we will discuss the functional properties of functions which satisfy the equation

$$\Delta_\omega f(x) := \sum_{y \in \overline{S}} [f(y) - f(x)] \frac{\omega(x,y)}{d_\omega x} = 0. \tag{2.1}$$

For a subgraph S with boundary $\partial S \neq \phi$ of a host graph G with a weight ω we say that a function $f : \overline{S} \to \mathbb{R}$ is *ω-harmonic* on S if it satisfies (2.1) for all $x \in S$, i.e.

$$f(x) = \frac{1}{d_\omega x} \sum_{y \in \overline{S}} f(y)\omega(x,y), \quad x \in S.$$

This implies that the value of f at x is given by a weighted average of the values of f at its neighboring vertices. From this point of view, we can clearly expect the following result to be true:

Theorem 2.1 (Minimum and Maximum Principle). *Let S be a subgraph of a host graph G with a weight ω and $f : \overline{S} \to \mathbb{R}$ be a function.*

(i) *If $\Delta_\omega f(x) \geq 0, x \in S$ and f has a maximum at a vertex in S, then f is constant.*
(ii) *If $\Delta_\omega f(x) \leq 0, x \in S$ and f has a minimum at a vertex in S, then f is constant.*
(iii) *If $\Delta_\omega f(x) = 0, x \in S$ and f has either a minimum or maximum in S, then f is constant.*
(iv) *If $\Delta_\omega f(x) = 0, x \in S$ and f is constant on the boundary ∂S, then f is constant.*

Proof. (ii) can be done in a similar way as in (i). (iii) and (iv) are easily obtained from (i) and (ii).
We prove (i). Assume that f has a maximum at a vertex $x_0 \in S$. Then

$$f(x_0) \geq f(y), \quad y \in \overline{S} \tag{2.2}$$

and

$$f(x_0) \leq \sum_{y \in \overline{S}} f(y) \frac{\omega(x_0,y)}{d_\omega x_0} \tag{2.3}$$

Suppose that there exists $y_0 \in \overline{S}$ such that $x_0 \sim y_0$ and $f(x_0) \neq f(y_0)$, i.e. $f(x_0) > f(y_0)$ in view of (2.2). Then it follows from (2.3) that

$$\begin{aligned} f(x_0) &\leq \sum_{\substack{y \in \overline{S} \\ y \neq y_0}} \frac{f(y)\omega(x_0, y)}{d_\omega x_0} + \frac{f(y_0)\omega(x_0, y_0)}{d_\omega x_0} \\ &< \sum_{\substack{y \in \overline{S} \\ y \neq y_0}} \frac{f(x_0)\omega(x_0, y)}{d_\omega x_0} + \frac{f(x_0)\omega(x_0, y_0)}{d_\omega x_0} \\ &= f(x_0), \end{aligned}$$

which implies that $f(x_0) = f(y)$ for all $y \in \overline{S}$ such that $y \sim x_0$. Now for any $x \in \overline{S}$, there exists a path

$$x_0 \sim x_1 \sim x_2 \sim \cdots \sim x_{n-1} \sim x_n = x,$$

since S is connected. By the applying the same argument as above inductively we see that $f(x_0) = f(x)$. ■

The following is an easy consequence of the above theorem

Corollary 2.2. *Under the same hypotheses as in Theorem 2.1, the following statements are true:*

(i) *If* $\Delta_\omega f \geq 0$ *on* S *and* $f|_{\partial S} \leq 0$ (< 0), *then* $f \leq 0$ (< 0) *on* S.
(ii) *If* $\Delta_\omega f \leq 0$ *on* S *and* $f|_{\partial S} \geq 0$ (> 0), *then* $f \geq 0$ (> 0) *on* S.

Corollary 2.3. *(1) If two functions* f *and* g *on* $\overline{S}$ *satisfy*

$$\Delta_\omega f = 0 \text{ and } \Delta_\omega g \geq 0$$

on S, *then* $g|_{\partial S} \leq f|_{\partial S}$ *implies* $g \leq f$ *on* S.
(2) If a function $f : \overline{S} \to \mathbb{R}$ *satisfies*

$$\Delta_\omega f(x) = 0, \quad x \in S$$

and $|f|$ *has a maximum in* S, *then* f *is constant.*

In the continuous case, it is well known that a local maximum principle holds for a harmonic function in an open subset $\Omega \subset \mathbb{R}^n$. But it is not hard to see that the local maximum principle is no longer true in general in our case. Moreover, the local uniqueness principle does not hold in general. As a matter of fact, it is rather natural to expect that such discrepancies are caused by the discrete nature of graphs.

A nonempty subset Γ of vertices of a subgraph $\overline{S}$ is said to be a surface in $\overline{S}$ if $\Gamma = \partial T$ for a subgraph T whose vertices belong to S. In this case, we denote by $\mathring{\Gamma}$ the inner boundary $\mathring{\partial} T$. For each vertex $z \in \Gamma$ and $x \in \mathring{\Gamma}$ we define

$$d'_\omega z := \sum_{y \in \mathring{\Gamma}} \omega(y, z) \quad \text{(inward degree)}$$

and

$$d''_\omega x := \sum_{z \in \Gamma} \omega(x,z) \quad \text{(outward degree)}.$$

In addition, for a function f on $\overline{S}$ we write

$$\int_\Gamma f(z) d'_\omega z = \sum_{z\in\Gamma} f(z) d'_\omega z \quad \text{(inward integral)}$$

and

$$\int_{\mathring{\Gamma}} f(x) d''_\omega x = \sum_{x\in\mathring{\Gamma}} f(x) d''_\omega x \quad \text{(outward integral)}$$

We use these notions to obtain the following interesting properties of ω-harmonic functions.

Theorem 2.4. *Let S be a subgraph of a host graph with weight ω and let $f : \overline{S} \to \mathbb{R}$. Then f is ω-harmonic on S, i.e., for all $x \in S$,*

$$\Delta_\omega f(x) = 0, \tag{2.4}$$

if and only if for every surface Γ in $\overline{S}$

$$\int_\Gamma f(z) d'_\omega z = \int_{\mathring{\Gamma}} f(y) d''_\omega y \tag{2.5}$$

Proof. Let $x \in S$ and $\Gamma_x = \{y \in \overline{S} | x \sim y\}$. Then Γ_x is a surface in $\overline{S}$ and $\mathring{\Gamma}_x = \{x\}$. Since $d_\omega x = d''_\omega x$ on $\mathring{\Gamma}_x$ and $d'_\omega z = \omega(x,z)$, (2.5) implies

$$f(x) d_\omega x = \sum_{z\in\Gamma_x} f(z)\omega(x,z),$$

which implies (2.4) immediately
Assume now that (2.4) holds and let Γ be a surface in $\overline{S}$ such that $\Gamma = \partial T$ for a subgraph $T \subset S$. We use Green's formula (Corollary 1.3, (ii)) to obtain

$$\begin{aligned} 0 &= \int_T \Delta_\omega f \\ &= \int_\Gamma \frac{\partial f}{\partial_\omega n} \\ &= \sum_{z\in\Gamma} \frac{\partial f}{\partial_\omega n}(z) d'_\omega z \\ &= \sum_{z\in\Gamma}\sum_{y\in\mathring{\Gamma}} [f(z) - f(y)]\omega(z,y). \end{aligned} \tag{2.6}$$

Then it follows that

$$\sum_{z\in\Gamma}\sum_{y\in\mathring{\Gamma}} f(z)\omega(z,y) = \sum_{z\in\Gamma}\sum_{y\in\mathring{\Gamma}} f(y)\omega(z,y)$$

or, equivalently

$$\sum_{z\in\Gamma} f(z)\Big[\sum_{y\in\overset{\circ}{\Gamma}}\omega(z,y)\Big] = \sum_{y\in\overset{\circ}{\Gamma}} f(y)\Big[\sum_{z\in\Gamma}\omega(z,y)\Big],$$

which yields (2.5). ■

In view of (2.6) we obtain the edge version of Theorem 2.4, the so called dual theorem, as follows :

Corollary 2.5. *Under the same conditions as in Theorem 2.4 , the formula (2.5) is equivalent to*

$$\sum_{\{x,y\}\in E(\Gamma,\overset{\circ}{\Gamma})} [f(z)-f(y)]\omega(z,y) = 0$$

where $E(\Gamma,\overset{\circ}{\Gamma})$ *denotes the set of all edges joining a vertex in* Γ *and a vertex in* $\overset{\circ}{\Gamma}$.

For two vertices x and y in a connected graph, the distance $d(x,y)$ between x and y is the number of edges in a shortest path joining x and y.

For a vertex x_0 in a subgraph S we write

$$\Gamma_j(x_0) := \{y\in\overline{S}| \quad d(x_0) = j\}\,, j = 0,1,2,\cdots$$

which is called a neighborhood of x_0 with radius j.

Then the following is a variant of Theorem 2.4:

Corollary 2.6. *Let* S *and* f *be the same as in Theorem 2.4. Then* f *is* ω*-harmonic on* S *if and only if for every* $x_0\in S$

$$\int_{\Gamma_j(x_0)} f(x)d''_\omega x = \int_{\Gamma_{j+1}(x_0)} f(x)d'_\omega x \tag{2.7}$$

for each j *with* $\Gamma_j(x_0)\subset S$.

Proof. Letting $j=0$ in (2.7) we have the sufficiency. To prove the necessity, consider an induced subgraph T whose vertices are exactly those of $\bigcup_{k=0}^{j}\Gamma_k(x_0)$. Then it is easy to see that

$$\partial T = \Gamma_{j+1}(x_0) \text{ and } \overset{\circ}{\partial T}\subset \Gamma_j(x_0).$$

But a vertex x in $\Gamma_j(x_0)$, which does not belong to $\overset{\circ}{\partial T}$, does not make any contribution to the outer integral $\int_{\Gamma_j(x_0)} f(x)d''_\omega x$, since $d''_\omega x = 0$. Hence, condition (2.5) in Theorem 2.4 shows the condition is necessary. ■

The following is the dual version of the above corollary:

Corollary 2.7. *Under the same conditions as in Corollary 2.6 the formula (2.7) is equivalent to*

$$\sum_{\{x,y\}\in E\left(\Gamma_j(x_0),\Gamma_{j+1}(x_0)\right)} [f(x)-f(y)]\omega(x,y) = 0$$

where $E\big(\Gamma_j(x_0),\Gamma_{j+1}(x_0)\big)$ *denotes the set of all edges joining a vertex in* $\Gamma_j(x_0)$ *and a vertex in* $\Gamma_{j+1}(x_0)$

3. The Dirichlet and Neumann Boundary Value Problems. Direct Problems

We start this section with a physical interpretation of the ω-Laplace and ω-Poisson equations. Consider a host graph G with a weight ω and an (induced) subgraph S. For a surface Γ in $\overline{S}$ with $\Gamma = \partial T$ for some $T \subset S$ and $z \in \Gamma$, the flux of energy passing through z to its adjacent nodes in T is given by

$$-\sum_{y \sim z} [f(z) - f(y)] \cdot \frac{\omega(z,y)}{d'z} \tag{3.1}$$

where $d'z = \sum_{y \sim z, y \in T} \omega(z,y)$ and f is a potential function in a diffusion field on a network. (For example, an electrostatic field, a thermal field, or an elastic membrane.) Here, the weight $\omega(z,y)$ plays the role of the conductivity of the diffusion along the edge $\{z,y\}$. In fact, (3.1) is exactly $-\frac{\partial f}{\partial_\omega n}(z)$ on Γ by definition (see Section 1) and thus, by Green's formula we have

$$\int_T (-\Delta_\omega f) = \int_\Gamma \Big(-\frac{\partial f}{\partial_\omega n}\Big),$$

which is the flow across Γ.

On the other hand, assume that T gains (or loses) an amount of energy $\int_T g$ where g is the energy density. Then we have

$$\int_T (-\Delta_\omega f) = \int_T g$$

Therefore, since T is arbitrary, by taking T to be any single vertex $x \in S$ we obtain the vertex equation

$$-\Delta_\omega f(x) = g(x), \quad x \in S. \tag{3.2}$$

Thus, it is reasonable to say that the conductivity equation on a graph can be represented as in (3.2), where $\omega(x,y)$ corresponds to the edge conductivity on the edge x, y.

Following the work of Fan Chung and her collaborators [C1], [C2] and [CO], we will discuss first the equation (3.2) on a graph $G = G(V,E)$ with a weight ω and no boundary. We consider the matrix

$$\Delta_\omega(x,y) = \begin{cases} -1, & \text{if } x = y \\ \frac{\omega(x,y)}{d_\omega x}, & \text{if } x \sim y \\ 0, & \text{otherwise .} \end{cases}$$

Considering the function f as a $|V|$-dimensional vector, the equation (3.2) can be understood as a matrix linear equation. Let D denote the diagonal matrix with (x,x)-th entry having the value $d_\omega x$ for each x and $\mathcal{L}_\omega = D^{1/2}\Delta_\omega D^{-1/2}$. Then $(-\mathcal{L}_\omega)$ is a nonnegative definite symmetric matrix, so that it has the eigenvalues

$$\lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{N-1}$$

and the corresponding eigenfunctions

$$\Phi_0, \Phi_1, \Phi_2, \cdots \Phi_{N-1}, \tag{3.3}$$

which form an orthonormal basis for $\mathbb{R}^N$ in the sense that for each pair of distinct i and j

$$\sum_{x\in V} \Phi_i(x)\cdot\Phi_j(x) = 0,$$

while, for all j,

$$\sum_{x\in V} |\Phi_j(x)|^2 = 1.$$

Here, N denotes $|V|$, the number of vertices in G. Then it is easy (see [C1]) to show that $\lambda_0 = 0, \lambda_1 > 0$ and $\Phi_0(x) = \frac{\sqrt{d_\omega x}}{\sqrt{\mathrm{vol}(G)}}$, $x \in V$, and $\mathrm{vol}(G) := \sum_{x\in V} d_\omega x$.

In what follows, we occasionally use the notation $\langle\ ,\ \rangle_X$, defined by $\langle f, g\rangle_X = \sum_{x\in X} f(x)g(x)$ for simplicity. Now we have the following solvability result for the Poisson equation:

Theorem 3.1. *Let $G = G(V, E)$ be a graph with a weight ω and $f : G \to \mathbb{R}$ be a function. Then the equation*

$$\Delta_\omega f(x) = g(x), \quad x \in V \tag{3.4}$$

has a solution if and only if $\int_G g = 0$. In this case, the solution is given by

$$f(x) = a_0 + \langle \Gamma_\omega(x,\cdot), g\rangle_V, \quad x \in V \tag{3.5}$$

where a_0 is an arbitrary constant and

$$\Gamma_\omega(x,y) = \sum_{j=1}^{N-1} \Big(-\frac{1}{\lambda_j}\Big)\Phi_j(x)\Phi_j(y)\sqrt{\frac{d_\omega y}{d_\omega x}}, \quad x, y \in V. \tag{3.6}$$

Proof. Assume that $\int_G g = 0$. Then

$$\begin{aligned} \langle D^{1/2}g, \Phi_0\rangle &= \sum \sqrt{d_\omega x}g(x)\cdot\frac{\sqrt{d_\omega x}}{\sqrt{\mathrm{vol}G}} \\ &= \frac{1}{\sqrt{\mathrm{vol}G}}\int_G g \\ &= 0 \end{aligned}$$

where D is the diagonal matrix whose x-th diagonal entry is $d_\omega x$.

Consider the orthogonal expansion

$$\big(D^{1/2}f\big)(x) = \sum_{j=0}^{N-1} a_j\Phi_j(x), \quad x \in V$$

where $a_j = \langle D^{1/2}f, \Phi_j\rangle, j = 0, 1, 2, \cdots, N-1$. Then since $\mathcal{L}_\omega D^{1/2} = D^{1/2}\Delta_\omega$ and

$$\begin{aligned} -\lambda_j a_j &= \langle D^{1/2}f, \mathcal{L}_\omega\Phi_j\rangle \\ &= \langle \mathcal{L}_\omega D^{1/2}f, \Phi_j\rangle \\ &= \langle D^{1/2}g, \Phi_j\rangle, \end{aligned}$$

we have

$$a_j = \Big(-\frac{1}{\lambda_j}\Big)\langle D^{1/2}g, \Phi_j\rangle, \quad j = 1, 2, \cdots, N-1$$

and a_0 is an arbitrary constant. Hence

$$\sqrt{d_\omega x} f(x) = a_0 \frac{\sqrt{d_\omega x}}{\sqrt{\mathrm{vol} G}} + \sum_{j=1}^{N-1} \Big(-\frac{1}{\lambda_j} \Big) \Big[\sum_{y \in V} g(y) \Phi_j(y) \sqrt{d_\omega y} \Big] \Phi_j(x)$$

equivalently,

$$f(x) = \frac{a_0}{\sqrt{\mathrm{vol} G}} + \sum_{j=1}^{N-1} \Big(-\frac{1}{\lambda_j} \Big) \sum_{y \in V} g(y) \Phi_j(y) \frac{\sqrt{d_\omega y}}{\sqrt{d_\omega x}} \Phi_j(x)$$

which gives (3.6).
The proof of the converse is easy. ■

The matrix Γ_ω in (3.5) is called the Green function of Δ_ω.
The following corollary is a Liouville type theorem for ω-harmonic functions.

Corollary 3.2. *Under the same conditions as in Theorem 3.1, every solution f of*

$$\Delta_\omega f(x) = 0, \quad x \in V$$

is constant.

The following corollary describes all functions which are ω-harmonic except possibly on a given (singularity) set T.

Corollary 3.3. *Under the same conditions as in Theorem 3.1, let $T \subset V$. Then every solution to*

$$\Delta_\omega f(x) = 0, \quad x \in V \setminus T$$

can be represented as

$$f(x) = a_0 + \sum_{y \in T} \Gamma_\omega(x, y) \alpha(y), \quad x \in V \tag{3.7}$$

where a_0 is an arbitrary constant and

$$\alpha(y) = \Delta_\omega f(y), \quad y \in T.$$

In particular, if $T = \{x_0\}$, $x_0 \in V$, then (3.7) can be written as

$$f(x) = a_0 + \alpha_0 \Gamma_\omega(x, x_0), \quad x \in V$$

where $\alpha_0 = \Delta_\omega f(x_0)$.

Let us now turn to boundary value problems and their eigenvalues. For a subgraph S of a host graph G with a weight ω, the *Dirichlet eigenvalues* of $-\mathcal{L}_\omega = -D^{1/2} \Delta_\omega D^{-1/2}$ are defined to be the eigenvalues

$$\nu_1 \leq \nu_2 \leq \cdots \leq \nu_n$$

of the matrix $-\mathcal{L}_{\omega,S}$ where $\mathcal{L}_{\omega,S}$ is a submatrix of $\mathcal{L}_\omega$ with rows and columns restricted to those indexed by vertices in S and $n = |S|$. Let $\phi_1, \phi_2, \cdots, \phi_n$ be the functions on $\overline{S}$ such that for each $j = 1, 2, \cdots, n$,

$$\mathcal{L}_{\omega,S}\phi_j(x) = (-\nu_j)\phi_j(x), \quad x \in S \quad \text{and} \quad \phi_j|_{\partial S} = 0.$$

In fact, $\phi_1, \phi_2, \cdots, \phi_n$ are the eigenfunctions corresponding to $\nu_1 \le \nu_2 \le \cdots \le \nu_n$ and form an orthonormal basis for $\mathbb{R}^n$. Then it is easy to verify that the first eigenvalue $\nu_1 > 0$, (see for instance, [C1]).

One can follow now the standard procedure to define Green functions $\gamma_{\omega,S}$ as follows :

$$\gamma_{\omega,S}(x,y) = \sum_{j=1}^{|S|} \Big(-\frac{1}{\nu_j}\Big)\phi_j(x)\phi_j(y)\frac{\sqrt{d_\omega y}}{\sqrt{d_\omega x}}, \quad x, y \in S \tag{3.8}$$

Letting D_S stand for the diagonal matrix whose x-th entry is $d_\omega x$ for each $x \in S$ and setting $\Delta_{\omega,S} = D_S^{-1/2}\mathcal{L}_{\omega,S}D_S^{1/2}$, one can easily verify that

$$\gamma_{\omega,S}\Delta_{\omega,S} = \Delta_{\omega,S}\gamma_{\omega,S} = I \tag{3.9}$$

and

$$\Delta_{\omega,S}(x,y) = \sum_{j=1}^{|S|}(-\nu_j)\phi_j(x)\phi_j(y)\frac{\sqrt{d_\omega y}}{\sqrt{d_\omega x}}, \quad x, y \in S. \tag{3.10}$$

where I denotes the $|S|$-dimensional identity matrix.

The Dirichlet boundary value problem was solved by F.R.K.Chung in [C2], when the graph has the standard weight. (For the interested reader, despite some minor errota, the proof given there is correct.) We prove now the solvability of the Dirichlet boundary value problem for graphs with arbitrary weights using a different method.

Theorem 3.4. *Let S be a subgraph of a host graph with a weight ω and $\sigma : \partial S \to \mathbb{R}$ be a given function. Then the unique solution f to the Dirichlet boundary value problem (DBVP)*

$$\begin{cases} \Delta_\omega f(x) = 0, & x \in S, \\ f|_{\partial S} = \sigma \end{cases}$$

can be represented as

$$f(x) = -\langle \gamma_\omega(x,\cdot), B_\sigma \rangle_{y\in S}, \quad x \in S, \tag{3.11}$$

where

$$B_\sigma(y) = \sum_{z\in\partial S} \frac{\sigma(z)\omega(y,z)}{d_\omega y}, \quad y \in S \tag{3.12}$$

Proof. Let f be a solution of DBVP. Then

$$
\begin{aligned}
0 &= \sum_{y\in S}\gamma_{\omega,S}(x,y)\Delta_\omega f(y) \\
&= \sum_{j=1}^{|S|}\Big(-\frac{1}{\nu_j}\Big)\frac{\phi_j(x)}{\sqrt{d_\omega x}}\Big[\sum_{y\in S}\phi_j(y)\sqrt{d_\omega y}\Delta_\omega f(y)\Big] \\
&= \sum_{j=1}^{|S|}\Big(-\frac{1}{\nu_j}\Big)\frac{\phi_j(x)}{\sqrt{d_\omega x}}\Big[\int_S (D_S^{-1/2}\phi_j)\cdot\Delta_\omega f\Big] \\
&= \sum_{j=1}^{|S|}\Big(-\frac{1}{\nu_j}\Big)\frac{\phi_j(x)}{\sqrt{d_\omega x}}\Big[\int_S f\cdot\Delta_\omega(D_S^{-1/2}\phi_j) \\
&\quad + \int_{\partial S}\Big\{(D_S^{-1/2}\phi_j)\cdot\frac{\partial f}{\partial_\omega n} - f\cdot\frac{\partial}{\partial_\omega n}(D_S^{-1/2}\phi_j)\Big\}\Big].
\end{aligned}
\tag{3.13}
$$

Here, we have used Green's formula from the Corollary 1.3. On the other hand, one can show that

$$\Delta_\omega\big(D_S^{-1/2}\phi_j\big)(x) = (-\nu_j)\big(D_S^{-1/2}\phi_j\big)(x), \quad x\in S,$$

since $\phi_j = 0$ on ∂S and

$$\sum_{j=1}^{|S|}\phi_j(x)\phi_j(y)\sqrt{\frac{d_\omega y}{d_\omega x}} = \delta(x,y), \quad x,\ y\in S,$$

where δ denotes the Kronecker delta. From these identities we can conclude that

$$
\begin{aligned}
&\sum_{j=1}^{|S|}\Big(-\frac{1}{\nu_j}\Big)\frac{\phi_j(x)}{\sqrt{d_\omega x}}\Big[\int_S f\cdot\Delta_\omega\big(D_S^{-1/2}\phi_j\big)\Big] \\
&= \sum_{j=1}^{|S|}\frac{\phi_j(x)}{\sqrt{d_\omega x}}\cdot\Big[\sum_{y\in S} f(y)\cdot\frac{\phi_j(x)}{\sqrt{d_\omega x}}\cdot d_\omega y\Big] \\
&= \sum_{y\in S} f(y)\Big[\sum_{j=1}^{|S|}\phi_j(x)\phi_j(y)\sqrt{\frac{d_\omega y}{d_\omega x}}\ \Big] \\
&= f(x).
\end{aligned}
$$

Hence, from the equality (3.13) and the fact that $\phi_j = 0$ on ∂S, we have

$$\begin{aligned}
f(x) &= \sum_{j=1}^{|S|} \big(-\frac{1}{\nu_j}\big)\frac{\phi_j(x)}{\sqrt{d_\omega x}} \int_{\partial S} \Big[f \cdot \frac{\partial}{\partial_\omega n}(D_S^{-1/2}\phi_j)\Big] \\
&= \sum_{j=1}^{|S|} \big(-\frac{1}{\nu_j}\big)\frac{\phi_j(x)}{\sqrt{d_\omega x}} \Big[\sum_{z\in\partial S} f(z) \cdot \frac{\partial}{\partial_\omega n}(D_S^{-1/2}\phi_j)(z)\cdot dz\Big] \\
&= \sum_{j=1}^{|S|} \big(-\frac{1}{\nu_j}\big)\frac{\phi_j(x)}{\sqrt{d_\omega x}} \sum_{z\in\partial S} \sigma(z)dz\Big[\sum_{y\in S}\Big\{\frac{\phi_j(z)}{\sqrt{d_\omega z}} - \frac{\phi_j(y)}{\sqrt{d_\omega y}}\Big\}\frac{\omega(z,y)}{d_\omega z}\Big] \\
&= -\sum_{j=1}^{|S|} \big(-\frac{1}{\nu_j}\big)\frac{\phi_j(x)}{\sqrt{d_\omega x}} \sum_{y\in S}\phi_j(y)\sqrt{d_\omega y}\Big(\sum_{z\in\partial S}\frac{\sigma(z)\omega(z,y)}{d_\omega y}\Big) \\
&= -\sum_{y\in S}\gamma_{\omega,S}(x,y)B_\sigma(y) \\
&= -\langle \gamma_{\omega,S}(x,\cdot), B_\sigma\rangle_S
\end{aligned}$$

for each $x \in S$.

The desired uniqueness result now follows easily from Theorem 2.1. ■

Remark 3.5. (i) The identity (3.11) can be rewritten as

$$f(x) = \sum_{j=1}^{|S|}\frac{1}{\nu_j}\sum_{y\in S}\Big[\sum_{z\in\partial S}\frac{\sigma(z)\omega(y,z)}{d_\omega y}\Big]\phi_j(y)\phi_j(x)\sqrt{\frac{d_\omega y}{d_\omega x}}, \quad x \in S.$$

In fact, B_σ is a function on S depending only on the value of σ on ∂S and $B_\sigma(y) = 0$ for $y \in S\setminus \mathring{\partial S}$. On the other hand, two different boundary conditions σ_1 and σ_2 may give rise to the same solution whenever $B_{\sigma_1} = B_{\sigma_2}$.

(ii) (3.11) can be understood as a matrix multiplication by

$$f = -\gamma_{\omega,S}\cdot B_\sigma \text{ on } S \tag{3.14}$$

or, equivalently,

$$\Delta_{\omega,S} f = -B_\sigma \text{ on } S \tag{3.15}$$

in view of (3.8). The relation(3.15) enables us to identify uniquely the boundary values from a ω-harmonic function f with $\Delta_\omega f = 0$ on S.

Now we characterize the ω-harmonic functions with a set of singularities in a subgraph with nonempty boundary.

Theorem 3.6. *Let S be a subgraph of a graph with weight ω and $T \subset S$. Then every $f : \overline{S} \to \mathbb{C}$ satisfying*

$$\Delta_\omega f(x) = 0, \quad x \in S \setminus T$$

can be uniquely represented as

$$f(x) = h(x) + \sum_{y\in T} \gamma_{\omega,S}(x,y)\beta(y), \quad x \in \overline{S}, \tag{3.16}$$

where h is a ω-harmonic function on S satisfying $h|_{\partial S} = f|_{\partial S}$ and $\beta(y) = \Delta_\omega f(y), \quad y \in T$.

Proof. The uniqueness is easy, by Theorem 2.1. Now let $\beta(y) := \Delta_\omega f(y), y \in T$. Then we have

$$\Delta_\omega f(x) = \begin{cases} 0, & x \in S \setminus T, \\ \beta(x), & x \in T. \end{cases}$$

Define, for $x \in \overline{S}$,

$$f_1(x) := \sum_{y\in T} \gamma_{\omega,S}(x,y)\beta(y),$$

and

$$h(x) := f(x) - f_1(x).$$

Then $h|_{\partial S} = f|_{\partial S}$ and for each $x \in S$,

$$\begin{aligned} \Delta_\omega h(x) &= \Delta_\omega f(x) - \Delta_\omega \Big[\sum_{y\in T} \sum_{j=1}^{|S|} \big(-\frac{1}{\nu_j} \frac{\phi_j(x)}{\sqrt{d_\omega x}} \cdot \phi_j(y)\sqrt{d_\omega y}\beta(y) \Big] \\ &= \Delta_\omega f(x) - \sum_{y\in T} \sum_{j=1}^{|S|} \frac{\phi_j(x)}{\sqrt{d_\omega x}} \cdot \Big[\phi_j(y)\sqrt{d_\omega y}\beta(y) \Big] \\ &= \Delta_\omega f(x) - \sum_{y\in T} \delta(x,y)\beta(y) \\ &= 0, \end{aligned}$$

which completes the proof. ■

Remark 3.7. (i) In particular, if $T = \{x_0\}$, $x_0 \in S$, then (3.16) can be written simply as

$$f(x) = h(x) + \gamma_{\omega,S}(x, x_0)\beta(x_0),$$

where $\beta(x_0) = \Delta_\omega f(x_0)$.

(ii) In fact, in view of (3.16) and Theorem 3.4, the solution to the nonhomogeneous DBVP

$$\begin{cases} \Delta_\omega f(x) &= g(x), \quad x \in S, \\ f|_{\partial S} &= \sigma \end{cases}$$

can be represented by

$$f(x) = -\langle \gamma_{\omega,S}(x,\cdot), B_\sigma \rangle_S + \langle \gamma_{\omega,S}(x,\cdot), g \rangle_S.$$

Now we will discuss the Neumann boundary value problem (NVBP). First, we recall Green's formula

$$\int_S \Delta_\omega f = \int_{\partial S} \frac{\partial f}{\partial_\omega n}.$$

Hence, if there exists a solution to

$$\begin{cases} \Delta_\omega f = g & \text{on } S, \\ \frac{\partial f}{\partial_\omega n} = \psi & \text{on } \partial S, \end{cases}$$

then by Green's formula it is necessary that $\int_S g = \int_{\partial S} \psi$.

Theorem 3.8. *Let S be a subgraph of a host graph G with a weight ω and let $f : \overline{S} \to \mathbb{R}$, $g : S \to \mathbb{R}$, and $\psi : \partial S \to \mathbb{R}$ be functions with $\int_{\partial S} \psi = \int_S g$. Then the solution to the NBVP*

$$\begin{cases} \Delta_\omega f(x) = g(x), & x \in S, \\ \frac{\partial f}{\partial_\omega n}(z) = \psi(z), & z \in \partial S \end{cases}$$

is given by

$$f(x) = a_0 + \langle \Gamma_\omega(x, \cdot), g \rangle_S - \langle \Gamma_\omega(x, \cdot), \psi \rangle_{\partial S},$$

where Γ_ω is the Green function of Δ_ω on the graph $\overline{S}$ as a new host graph of S and a_0 is an arbitrary constant.

Proof. We rewrite (NBVP) as

$$\begin{cases} \sum_{y \in \overline{S}} \left[f(y) - f(x) \right] \frac{\omega(x,y)}{d_\omega x} = g(x), & x \in S, \\ \sum_{y \in S} \left[f(y) - f(z) \right] \frac{\omega(y,z)}{d'_\omega z} = -\psi(z), & z \in \partial S. \end{cases} \tag{3.17}$$

To solve the system(3.17), consider $\overline{S}$ as a new host graph with the weight ω and with no boundary. Then S is still a subgraph of $\overline{S}$. (In fact, we should note here that if we regard $\overline{S}$ as a subgraph of G, then its boundary $\partial\overline{S}$ may not be empty.) Then, for each $z \in \partial S$, the inner degree $d'_\omega z$ is equal to $d_\omega z$ in this new graph $\overline{S}$, since the induced subgraph has no edges between the vertices on ∂S. Hence the equation (3.17) can be written as

$$\begin{cases} \sum_{y \in V_0} \left[f(y) - f(x) \right] \frac{\omega(x,y)}{d_\omega x} = g(x), & x \in S, \\ \sum_{y \in V_0} \left[f(y) - f(z) \right] \frac{\omega(y,z)}{d'_\omega z} = -\psi(z), & z \in \partial S, \end{cases} \tag{3.18}$$

where V_0 is the set of vertices in $\overline{S}$. Hence (3.18) is equivalent to

$$\sum_{y \in V_0} \left[f(y) - f(x) \right] \frac{\omega(x,y)}{d_\omega x} = \Psi(x), \quad x \in \overline{S}, \tag{3.19}$$

where

$$\Psi(x) = \begin{cases} g(x), & x \in S, \\ -\psi(x), & x \in \partial S. \end{cases}$$

Therefore, (NBVP) is equivalent to

$$\Delta_\omega f(x) = \Psi(x), \quad x \in \overline{S}.$$

Thus, it follows from Theorem 3.1 that

$$\begin{aligned} f(x) &= a_0 + \langle \Gamma_\omega(x,\cdot), \Psi \rangle_{x\in V_0} \\ &= a_0 + \sum_{y\in V_0} \Gamma_\omega(x,y)\Psi(y) \\ &= a_0 + \sum_{y\in S} \Gamma_\omega(x,y)g(y) - \sum_{z\in\partial S} \Gamma_\omega(x,z)\psi(z) \\ &= a_0 + \langle \Gamma_\omega(x,\cdot), g\rangle_S - \langle \Gamma_\omega(x,\cdot), \psi\rangle_{\partial S}, \end{aligned}$$

where a_0 is an arbitrary constant. This completes the proof. ■

Remark 3.9. The solution to (NBVP) is uniquely determined by the Neumann data ψ on ∂S up to an additive constant. Thus, we get a unique solution if we prescribe the value of f at some vertex in S or, for example, if we seek the solution f with $\int_S f =$ (a given constant).

4. **Inverse Problems**

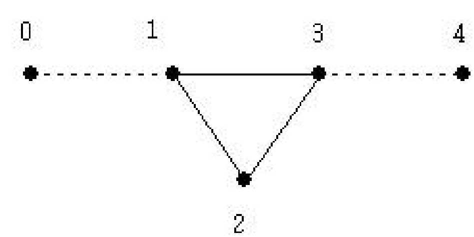
In the previous section, we have seen that for a function function $\psi : \partial S \to \mathbb{R}$ with $\int_{\partial S} \psi = 0$ the Neumann boundary value problem

$$(NBVP)\begin{cases} \Delta_\omega f(x) = 0, & x \in S, \\ \frac{\partial f}{\partial_\omega n}(z) = \psi(z), & z \in \partial S \end{cases}$$

has a unique solution up to an additive constant. Therefore, the Dirichlet data $f|_{\partial S}$, $z \in \partial S$ is well-defined up to an additive constant.

In this section, we will discuss the inverse conductivity problem on the network (graph) S with nonempty boundary, which consists in recovering the conductivity (connectivity or weight) ω of the graph by using, the so called input-output map, for example by using the Dirichlet data induced by the Neumann data (Neumann-to-Dirichlet map), with one boundary measurement.

In order to deal with this inverse problem , we need at least to know or be given the boundary data such as $f(z)$, $\frac{\partial f}{\partial_\omega n}(z)$ for $z \in \partial S$ and ω near the boundary. So it is natural to assume that $f|_{\partial S}$, $\frac{\partial f}{\partial_\omega n}|_{\partial S}$ and $\omega|_{\partial S \times \mathring{\partial S}}$ are known (given or measured). But even though we are given all these data on the boundary, we are not be guaranteed, in general, to be able to identify the conductivity ω uniquely. To illustrate this we consider a graph S whose vertices are $\{1, 2, 3\}$ and $\partial S = \{0, 4\}$ as follows:



with the weight

$$\omega(0,1)=1,\ \omega(0,k)=0\quad(k=2,\ 3,\ 4),$$

and

$$\omega(3,4)=1,\ \omega(k,4)=0\quad(k=0,\ 1,\ 2).$$

Let $f:\overline{S}\to\mathbb{R}$ be functions satisfying $\Delta_\omega f(k)=0,\ k=1,2,3$. Assume that

$$f(0)=0,\ f(1)=1,\ f(3)=3,\ f(4)=4,\ f(2)=\text{(unknown)}.$$

Thus, since $\overset{\circ}{\partial}S=\{1,3\}$, the boundary data $f|_{\partial S}$, $\frac{\partial f}{\partial_\omega n}|_{\partial S}$ and $\omega|_{\partial S\times\overset{\circ}{\partial}S}$ are known.

In fact,

$$\frac{\partial f}{\partial_\omega n}(0)=f(0)-f(1)=-1,$$
$$\frac{\partial f}{\partial_\omega n}(4)=f(4)-f(3)=1.$$

The problem is to determine

$$\omega(1,2)=x,\quad \omega(2,3)=y,\quad \omega(1,3)=z,\quad \text{and}\quad f(2).$$

From $\Delta_\omega f(k)=0,\ k=1,2,3$, we have

$$\begin{aligned} f(1) &= \frac{f(0)+xf(2)+3z}{1+x+z}=1,\\ f(2) &= \frac{xf(1)+yf(3)}{x+y},\\ f(3) &= \frac{zf(1)+yf(2)+f(4)}{z+y+1}=3. \end{aligned}$$

This system is equivalent to

$$\begin{cases} x(y-1)+y(x-1)+2z(x+y)=0,\\ f(2)=\frac{x+3y}{x+y}. \end{cases}\tag{4.1}$$

This system has infinitely many solutions. For instance, assume $z=0$, that is, the two vertices 1 and 3 are not adjacent. Then (4.1) is reduced to

$$\begin{cases} \frac{1}{x}+\frac{1}{y}=2,\\ f(2)=\frac{x+3y}{x+y}. \end{cases}\tag{4.2}$$

It is easy to see that there are infinitely many pairs (x,y) of nonnegative numbers satisfying the first equation in (4.2), so that $f(2)$ is undetermined as a result.

In view of the above example, in order to determine the weight ω uniquely we need some more information than just $f|_{\partial S}$, $\frac{\partial f}{\partial_\omega n}|_{\partial S}$ and $\omega|_{\partial S\times\overset{\circ}{\partial}S}$. To motivate the main theorem we impose in this example the additional constraints that

$$x\geq 1,\ y\geq 1 \text{ and } z\geq 0\tag{4.3}$$

in (4.1). Then the equation (4.1) yields a unique triple of solution $x = 1,\ y = 1,\ z = 0$ and $f(2) = 2$.

As a matter of fact, even the inverse conductivity problem of a diffusion equation in a bounded open subset $\Omega \subset \mathbb{R}^n$

$$P[a;u] := \begin{cases} div[a(x)\nabla u(x)] = 0, & x \in \Omega, \\ u|_{\partial\Omega} = \sigma \end{cases} \tag{4.4}$$

has been studied under some additional constraints besides Dirichlet and Neumann data (see [A], [BF], [Ca], [I], [IP], and [SU]). In particular, in [A] and [I] it is shown that there is a global uniqueness result under the condition that

(i) $a_1 = a_2$ near $\partial\Omega$, and $a_1 \le a_2$ in Ω,
(ii) $\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n}$ on $\partial\Omega$,
(iii) $\int_\Omega u_1 = \int_\Omega u_2 = 0$,

where $P[a_j;u_j] = 0,\ j = 1,2$ in (4.4).

Now we are in a position to state the first main theorem of this paper.

Theorem 4.1. *Let ω_1 and ω_2 be weights with $\omega_1 \le \omega_2$ on $\overline{S}\times\overline{S}$ and $f_1,\ f_2 : \overline{S} \to \mathbb{R}$ be functions satisfying that*

$$\begin{cases} \Delta_{\omega_j} f_j(x) = 0, & x \in S, \\ \frac{\partial f}{\partial_{\omega_j} n}(z) = \psi(z), & z \in \partial S \end{cases}$$

for a given function $\psi : \partial S \to \mathbb{R}$ with $\int_{\partial S} \psi = 0$ and $j = 1,\ 2$.

If we assume that

(i) $\omega_1(z,y) = \omega_2(z,y)$ *on* $\partial S \times \overset{\circ}{\partial S}$,
(ii) $f_1|_{\partial S} = f_2|_{\partial S}$,

then we have

(i) $f_1 = f_2$ *on* $\overline{S}$,
(ii) $\omega_1(x,y) = \omega_2(x,y)$ *whenever* $f_1(x) \neq f_1(y)$, *or* $f_2(x) \neq f_2(y)$.

To prove this result we adapt the method of energy functionals, extensively used for nonlinear partial differential equations. For a function $\sigma : \partial S \to \mathbb{R}$ we define a functional by

$$I_\omega[h] := \int_{\overline{S}} [\frac{1}{4}|\nabla_\omega h|^2 - hg] \tag{4.5}$$

for every function h in the set

$$A := \{h : \overline{S} \to \mathbb{R}| \quad h|_{\partial S} = \sigma\}, \tag{4.6}$$

which is called the admissible set. In the continuous case, the well known Dirichlet's principle states that the energy minimizer in the admissible set is a solution of the Dirichlet boundary value problem. We derive here the discrete version of Dirichlet's principle as follows :

Theorem 4.2 (Dirichlet's principle). *Assume that $f : \overline{S} \to \mathbb{R}$ is a solution to*

$$\begin{cases} -\Delta_\omega f = g & \text{on } S, \\ f|_{\partial S} = \sigma. \end{cases} \tag{4.7}$$

Then

$$I_\omega[f] = \min_{h \in A} I_\omega[h]. \tag{4.8}$$

Conversely, if $f \in A$ satisfies (4.8), then f is the solution 0f (4.7), and the only one.

Proof. Let h be a function in A. Then making use of (1.2) in Theorem 1.2 we have

$$\begin{aligned} 0 &= \int_{\overline{S}} (-\Delta_\omega f - g)(f - h) \\ &= \int_{\overline{S}} \big[(-\Delta_\omega f)(f-h) - g(f-h)\big] \\ &= \int_{\overline{S}} \big[\frac{1}{2}\nabla_\omega f \cdot \nabla_\omega (f-h) - g(f-h)\big] \\ &= \frac{1}{2}\int_{\overline{S}} |\nabla_\omega f|^2 - \frac{1}{2}\int_{\overline{S}} \nabla_\omega f \cdot \nabla_\omega h - \int_{\overline{S}} g(f-h). \end{aligned}$$

Hence

$$\begin{aligned} \int_{\overline{S}} \big[\frac{1}{2}|\nabla_\omega f|^2 - gf\big] &= \int_{\overline{S}} \big[\frac{1}{2}\nabla_\omega f \cdot \nabla_\omega h - gh\big] \\ &\le \frac{1}{2}\sum_{x \in \overline{S}}\sum_{y \in \overline{S}} \big|\big[f(y) - f(x)\big]\cdot\big[h(y) - h(x)\big]\big| \cdot \omega(x,y) - \int_{\overline{S}} gh \\ &\le \frac{1}{2}\sum_{x \in \overline{S}}\sum_{y \in \overline{S}} \frac{\big[f(y)-f(x)\big]^2 + \big[h(y)-h(x)\big]^2}{2} \cdot \omega(x,y) - \int_{\overline{S}} gh \\ &= \frac{1}{4}\int_{\overline{S}} |\nabla_\omega f|^2 + \frac{1}{4}\int_{\overline{S}} |\nabla_\omega h|^2 - \int_{\overline{S}} gh, \end{aligned}$$

where we used the triangular inequality

$$|ab| \le \frac{a^2 + b^2}{2}, \quad a, b \in \mathbb{R}.$$

Thus, it follows that

$$\int_{\overline{S}} \big[\frac{1}{4}|\nabla_\omega f|^2 - gf\big] \le \int_{\overline{S}} \big[\frac{1}{4}|\nabla_\omega h|^2 - gh\big],$$

which implies

$$I_\omega[f] \le I_\omega[h], \quad h \in A.$$

Since $f \in A$, we have

$$\min_{h \in A} I_\omega[h] = I_\omega[f].$$

Now we prove the converse. Let T be a subset of vertices in S and

$$\chi_T(x) = \begin{cases} 1, & x \in T \\ 0, & \text{otherwise.} \end{cases}$$

Then $f + \tau\chi_T \in A$ for each real number τ, since $\chi_T = 0$ on ∂S. Define

$$i(\tau) := I_\omega[f + \tau\chi_T], \quad \tau \in \mathbb{R}.$$

Then

$$\begin{aligned} i(\tau) &= \int_{\overline{S}} [\frac{1}{4}|\nabla_\omega f + \tau\nabla_\omega\chi_T|^2 - (f + \tau\chi_T)g] \\ &= \frac{1}{4}\int_{\overline{S}} |\nabla_\omega f|^2 + 2\tau\nabla_\omega f\cdot\nabla_\omega\chi_T + \tau^2|\nabla_\omega\chi_T|^2 - \int_{\overline{S}} (f + \tau\chi_T)g \end{aligned}$$

Note that the scalar function $i(\tau)$ has a minimum at $\tau = 0$ and thus $\frac{di}{d\tau}(0) = 0$. That is,

$$\begin{aligned} 0 &= \frac{1}{2}\int_{\overline{S}} \nabla_\omega f\cdot\nabla_\omega\chi_T - \int_{\overline{S}} \chi_T\cdot g \\ &= \int_{\overline{S}} [\chi_T(-\Delta_\omega f - g)] \\ &= \sum_{x\in T} [-\Delta_\omega f(x) - g(x)]\, d_\omega x \end{aligned}$$

In particular, taking $T = \{x\}, x \in S$, we obtain

$$-\Delta_\omega f(x) - g(x) = 0,$$

which is the required result. The uniqueness follows from Theorem 3.4. ■

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1.

(i) Let $\sigma : \partial S \to \mathbb{R}$ be the function defined by

$$\sigma(z) = f_1(z) = f_2(z), \quad z \in \partial S,$$

using the hypothesis (ii). Define

$$I_{\omega_1}[h] := \frac{1}{4}\int_{\overline{S}} |\nabla_{\omega_1} h|^2 d_{\omega_1}$$

for every h in the admissible set

$$A = \{h : \overline{S} \to \mathbb{R} | \quad h|_{\partial S} = \sigma\}.$$

Then, by virtue of Theorem 1.2 we have

$$\begin{aligned} I_{\omega_1}[h] &= \frac{1}{2}\int_{\overline{S}} h(-\Delta_{\omega_1} h) d_{\omega_1} \\ &= \frac{1}{2}\int_S h(-\Delta_{\omega_1} h) d_{\omega_1} + \frac{1}{2}\int_{\partial S} h(-\Delta_{\omega_1} h) d_{\omega_1}. \end{aligned}$$

Moreover, by the coincidence of the Dirichlet and Neumann data we can see that the boundary ∂S and the inner boundary $\overset{\circ}{\partial} S$ are well-defined independently of the values of the weights ω_1, ω_2 and, moreover, for $z \in \partial S$

$$d_{\omega_1} z = \sum_{y \in \overset{\circ}{\partial} S} \omega_1(z,y) = \sum_{y \in \overset{\circ}{\partial} S} \omega_2(z,y) = d_{\omega_2} z, \tag{4.9}$$

$$\begin{aligned} \Delta_{\omega_1} f_1(z) &= \sum_{y \in \overset{\circ}{\partial} S} [f_1(y) - f_1(z)] \frac{\omega_1(z,y)}{d_{\omega_1} z} \\ &= \sum_{y \in \overset{\circ}{\partial} S} [f_2(y) - f_2(z)] \frac{\omega_2(z,y)}{d_{\omega_2} z} \\ &= \Delta_{\omega_2} f_2(z). \end{aligned} \tag{4.10}$$

Then, it follows from the condition $\omega_1 \le \omega_2$ that

$$\begin{aligned} I_{\omega_1}[f_1] &= \frac{1}{2} \int_{\partial S} f_1(-\Delta_{\omega_1} f_1) d_{\omega_1} \\ &= \frac{1}{2} \int_{\partial S} f_2(-\Delta_{\omega_2} f_2) d_{\omega_1} \\ &= \frac{1}{2} \int_{S} f_2(-\Delta_{\omega_2} f_2) d_{\omega_2} + \frac{1}{2} \int_{\partial S} f_2(-\Delta_{\omega_2} f_2) d_{\omega_2} \\ &= \frac{1}{2} \int_{\overline{S}} f_2(-\Delta_{\omega_2} f_2) d_{\omega_2} \\ &= \frac{1}{4} \int_{\overline{S}} |\nabla_{\omega_2} f_2|^2 d_{\omega_2} \\ &= \frac{1}{4} \sum_{x \in \overline{S}} \sum_{y \in \overline{S}} \left[f_2(x) - f_2(y)\right]^2 \omega_2(x,y) \\ &\ge \frac{1}{4} \sum_{x \in \overline{S}} \sum_{y \in \overline{S}} \left[f_2(x) - f_2(y)\right]^2 \omega_1(x,y) \\ &= \frac{1}{4} \int_{\overline{S}} |\nabla_{\omega_1} f_2|^2 d_{\omega_1} \\ &= I_{\omega_1}[f_2] \end{aligned}$$

Using Dirichlet's principle (Theorem 4.4) one sees that $f_1 = f_2$ on $\overline{S}$.

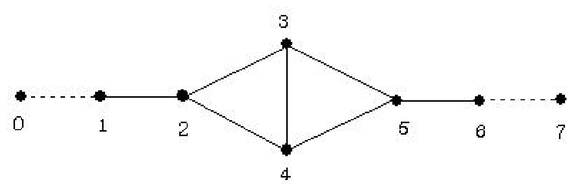
(ii) In the proof of (i) we actually have proved that $I_{\omega_1}[f_1] = I_{\omega_1}[f_2]$. In other words, taking $f := f_1 = f_2$ on $\overline{S}$

$$\sum_{x \in \overline{S}} \sum_{y \in \overline{S}} \left[f(x) - f(y)\right]^2 \omega_2(x,y) = \sum_{x \in \overline{S}} \sum_{y \in \overline{S}} \left[f(x) - f(y)\right]^2 \omega_1(x,y),$$

or, equivalently
$$\sum_{x\in\overline{S}}\sum_{y\in\overline{S}}[f(x)-f(y)]^2\cdot[\omega_2(x,y)-\omega_1(x,y)]=0.$$
Therefore, we have
$$[f(x)-f(y)]^2\cdot[\omega_2(x,y)-\omega_1(x,y)]=0,$$
for all $x\in\overline{S}$ and $y\in\overline{S}$. This gives (ii). ■

Remark 4.3. In Theorem 4.1 above, if $f:=f_1=f_2$ is injective on S then we are able to get $\omega_1=\omega_2$ on $\overline{S}\times\overline{S}$. For example, if S is the path P_n on n vertices with arbitrary weight ω, then it is not hard to see that every nonconstant ω-harmonic function f on P_n is strictly monotonic and hence all the weights are identified. But, in general, most graphs, even with the standard weight do not admit an injective solution to the DBVP or NBVP. Therefore, it will be quite interesting to figure out a pair of graphs and weights which admits an injective solution to the DBVP or NBVP.

To develop an idea to improve Theorem 4.1 we consider a graph $S=\{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$ with $\partial S=\{0,\ 7\}$ as follows:



Suppose that ω_1 is the standard weight and ω_2 is the weight given by $\omega_1=\omega_2$ except only $\omega_2(3,\ 4)=k,\ k\geq 1$. Then $\omega_1\leq\omega_2$ throughout the graph $\overline{S}$ and $\omega_1=\omega_2$ except on the edge $\{3,4\}$. Now define a function $f:\overline{S}\to\mathbb{R}$ as
$$f(0)=a,\ f(1)=a-\alpha,\ f(2)=a-2\alpha,\ f(3)=f(4)=\frac{(a+b)-(\alpha+\beta)}{2},$$
$$f(5)=b-2\beta,\ f(6)=b-\beta,\ f(7)=b,$$
where a, b, α and β are arbitrary real numbers. Then it is easy to verify that f satisfies both the equations
$$\Delta_{\omega_1}f(x)=0=\Delta_{\omega_2}f(x),\quad x\in S.$$
Here, we note that f is uniquely determined by the boundary data
$$f(0)=a,\quad f(7)=b,$$
$$\frac{\partial f}{\partial_\omega n}(0)=f(0)-f(1)=\alpha,\quad \frac{\partial f}{\partial_\omega n}(7)=f(7)-f(6)=\beta$$
and each value $f(x)$ is determined regardless of the value $\omega_2(3,4)=k$. This implies that we cannot identify the weight $\omega_2(3,\ 4)=k$ even with all possible boundary data a, b, α and β. To

derive a key idea to identify $\omega_2(3,\ 4) = k$, we take $a > 0$, $b > 0$ so that $f(0) > 0$ and $f(7) > 0$. By a direct calculation (or, using Corollary 2.2) we see that

$$f(m) > 0,\ m = 0,\ 1,\ 2,\ \cdots,\ 7.$$

Suppose that f satisfies the relation

$$\int_S f d_{\omega_1} = \int_S f d_{\omega_2}\ . \tag{4.11}$$

Then, since

$$\int_S f d_{\omega_1} = 2f(1) + 3f(2) + 3f(3) + 3f(4) + 3f(5) + 2f(6)$$

and

$$\int_S f d_{\omega_2} = 2f(1) + 3f(2) + (2+k)f(3) + (2+k)f(4) + 3f(5) + 2f(6),$$

it follows that

$$k\big[f(3) + f(4)\big] = f(3) + f(4),$$

which gives $k = 1$. Therefore, in order to identify the weight over all edges we need to impose an additional condition such as (4.11).

Now we return to the general situation. We know that for a function $\psi : \partial S \to \mathbb{R}$ with $\int_{\partial S} \psi = 0$ and $j = 1,\ 2$, the equation

$$\begin{cases} \Delta_{\omega_j} h_j(x) = 0,\ x \in S, \\ \frac{\partial h}{\partial_{\omega_j} n}(z) = \psi(z),\ z \in \partial S, \\ \int_S h_j\ d_{\omega_j} = 0 \end{cases} \tag{4.12}$$

has a unique pair of solution (h_1, h_2). Let

$$m_j = \min_{z \in \partial S}\ h_j(z),\ j = 1, 2 \tag{4.13}$$

and

$$m_0 = \max_{j=1,2}\ |m_j| \cdot \mathrm{vol}(S, \omega_j), \tag{4.14}$$

where $\mathrm{vol}(S, \omega_j) = \sum_{x \in S} d_{\omega_j} x$.

Motivated by the above example we refine Theorem 4.1 as follows:

Theorem 4.4. *Let ω_1 and ω_2 be weights with $\omega_1 \leq \omega_2$ on $\overline{S} \times \overline{S}$ and $f_1,\ f_2 : \overline{S} \to \mathbb{R}$ be functions satisfying that for each $j = 1,\ 2$,*

$$\begin{cases} \Delta_{\omega_j} f_j(x) = 0,\ x \in S, \\ \frac{\partial f}{\partial_{\omega_j} n}(z) = \psi(z),\ z \in \partial S \\ \int_S f_j\ d_{\omega_j} = K \end{cases} \tag{4.15}$$

for a given function $\psi : \partial S \to \mathbb{R}$ with $\int_{\partial S} \psi = 0$ and a given constant K with $K > m_0$. (Here, m_0 is the constant in (4.14)).

If we assume that

(i) $\omega_1(z,y) = \omega_2(z,y)$ *on* $\partial S \times \overset{\circ}{\partial S}$,
(ii) $f_1|_{\partial S} = f_2|_{\partial S}$,

then we have

$$f_1 \equiv f_2$$

and

$$\omega_1(x,y) = \omega_2(x,y)$$

for all x and y in $\overline{S}$.

Proof. We have already shown in Theorem 4.1 that $f_1 \equiv f_2$. Now, for each $j = 1,\ 2$, we choose a constant C_j so that $C_j \cdot \mathrm{vol}(S,\omega_j) = K$. Then, it follows that $C_j > |m_j|$ and, hence, $h_j(x) + C_j > 0,\ x \in S$ by the maximum principle (or, Corollary 2.2). Moreover, the function $\tilde{h}(x) := h_j(x) + C_j$ satisfies the equation (4.15). By the uniqueness of the solution we have

$$f_j(x) = \tilde{h}(x) = h_j(x) + C_j > 0,\ x \in S.$$

Let $f := f_1 = f_2$ on $\overline{S}$. Then it follows from the condition $\int_S f_1\, d_{\omega_1} = K = \int_S f_2\, d_{\omega_2}$ that

$$\sum_{x\in S} f(x) d_{\omega_1}(x) = \sum_{x\in S} f(x) d_{\omega_2}(x),$$

or, equivalently

$$\sum_{x\in S} f(x)\big[d_{\omega_2}(x) - d_{\omega_1}(x)\big] = 0.$$

Since $f(x) > 0$ and $d_{\omega_1}(x) \geq d_{\omega_1}(x)$ for all $x \in S$, we have

$$\begin{aligned} 0 &= d_{\omega_2}(x) - d_{\omega_1}(x) \\ &= \sum_{\in \overline{S}} \big[\omega_2(x,y) - \omega_2(x,y)\big]. \end{aligned}$$

Since $\omega_1(x,y) \leq \omega_2(x,y)$, we obtain

$$\omega_1(x,y) = \omega_2(x,y)$$

for all x and y in $\overline{S}$, which is the required. ■

Remark 4.5. In the above proof, the contributuion of the new condition that $\int_S f_j\, d_{\omega_j} = K > m_0$ is used only but to guarantee that $f_j(x) > 0,\ x \in S$. Hence, if we replace this condition by $f|_{\partial S} > 0,\ j = 1,\ 2$ in Theorem 4.4, we arrive at the same conclusion.

As seen in the study of the inverse conductivity problem in the continuous case (see, for instance, [A], [Ca], [I], [IP], [SU]) it would be worthwhile to prove the uniqueness under a condition weaker than the monotonicity condition $\omega_1 \leq \omega_2$ imposed the above. Moreover, it would also be interesting to consider a stability theorem for the same conductivity equation.

Acknowledgement: This paper was completed during the second author's visit to Department of Mathematics, University of Maryland during the 2002-2003 academic year. The authors are grateful to F. Faroozan and F. Gavilanez for many valuable comments and remarks.

References

[A] G. Alessandrini, Remark on a paper of Bellout and Friedman, Boll. Uni. Math. Ital. 3A, 7(1989), 243-250.

[BF] H. Bellout and A. Friedman, Identification problems in potential theory, Arch. Rational Mech. Analy. 101(1988), 143-160.

[B1] N. L. Biggs, Algebraic graph theory,(2nd ed), Cambridge University Press, Cambridge, 1993.

[B2] N. L. Biggs, Potential theory on distance-regular graphs, Combinatorics, probability and computing 2(1993), 243-255.

[B3] N. L. Biggs, Algebraic graph theory on graphs, Bull. London math. Soc. 29(1997), 641-682.

[BH] A. Bondy and R. L. Hemminger, Graph reconstruction-A survey, Jour. Comb. Th. 1(1977), 227-168.

[C1] F. R. K. Chung, Spectral graph theory, CBMS Regional Conference Series in Math. 92, Amer. Math. Soc. 1997.

[C2] F. R. K. Chung, Discrete Green's functions, J. Comb. Theory(A), 91(2001), 191-214.

[C3] F. R. K. Chung, Distance realization problem with applications to internet tomography, J. Comb. System. Soc., to appear.

[Ca] A. P. Calderon, On an inverse boundary value problem, Seminar on numerical analysis and its applications to continuum physics, Soc. Braz. Math. 1980, 65-73.

[CvDS] D. M. Cvetkovic, M. Doob and H.Sachs, Specra of Graphs, Theory and Applications, Acad. Press, New York, 1980.

[CvDGT] D. M. Cvetkovic, M. Doob, I. Gutman, and A. Torgasev, Recent Results in the Theory of Graph Specra, Ann. Discrete Math. 36, Elsevier Sci. Publ., North-Holland, New York, 1988.

[CL] F. R. K. Chung and R. P. Langlands, A Combinatorial Laplacian with vertex weights, J. Comb. Theory(A) 75(1996), 316-327.

[CO] F. R. K. Chung and K. Oden, Weighted graph Laplacians and isoperimetric inequalities, Pacific J. Math. 192(2000), 257-274.

[HY] S. L. Hakimi and S. S. Yau, Distance matrix of a graph and its realizability, Quart. Appl. Math. 22(1965), 305-317.

[I] V. Isakov, Inverse problem for partial differential equations, Springer-Verlag, New York, 1998.

[IP] V. Isakov and J. Powell, On the inverse conductivity problem with one measurement, Inverse Prob. 6(1990), 311-318.

[KS] H. Kang and J. K. Seo, Recent progress in the inverse conductivity problem with single measurement. Inverse problems and related topics (Kobe, 1998), 69–80, Chapman Hall/CRC Res. Notes Math., 419, Chapman Hall/CRC, Boca Raton, FL, 2000.

[MC1] J. A. Morrow and E. B. Curtis, Determining the resistors in a network, SIAM J. Appl. Math. 50(1990), 918-930.

[MC2] J. A. Morrow and E. B. Curtis, The Dirichlet to Neumann map for a resistor network, SIAM J. Appl. Math. 51(1991), 1011-1029.

[MMC] J. A. Morrow, E. Mooers and E. B. Curtis, Finding the conductors in circular networks from boundary measurements, RAIRO Model. Math. Anal. Numer. 28(1994), 781-814.

[MI] J. A. Morrow and D. Ingerman, On a characterization of the kernel of the Dirichlet-to-Neumann map for a planar region, SIAM J. Math. Appl. 29(1998), 106-115.

[MIC] J. A. Morrow, D. Ingerman and E. B. Curtis, Circula planar graphs and resistor networks, Linear Alagbra Appl. 283(1998), 115-150.

[SU] J. Sylvester and G.Uhlmann, A global uniqueness theorem for an inverse boundary value problem, Ann. Math. 125(1987), 153-169.

BERENSTEIN: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD20742, USA

CHUNG: DEPARTMENT OF MATHEMATICS, SOGANG UNIVERSITY, SEOUL 121-742, KOREA

CURRENT ADDRESS CHUNG: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD20742, USA

E-mail address: sychung@sogang.ac.kr

E-mail address: carlos@math.umd.edu