ABSTRACT

Title of dissertation: PARABOLIC HIGGS BUNDLES AND THE DELIGNE-SIMPSON PROBLEM FOR LOXODROMIC CONJUGACY CLASSES IN $PU(n, 1)$

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In this thesis we study the Deligne-Simpson problem of finding matrices $A_j \in C_j$ such that $A_1 A_2 \ldots A_k = I$ for $k \geq 3$ fixed loxodromic conjugacy classes $C_1, \ldots, C_k$ in $PU(n, 1)$. Solutions to this problem are equivalent to representations of the k punctured sphere into $PU(n, 1)$, where the monodromy around the punctures are in the C_j. By Simpson's correspondence [27], irreducible such representations correspond to stable parabolic $U(n, 1)$-Higgs bundles of parabolic degree 0. A parabolic $U(n, 1)$-Higgs bundle can be decomposed into a parabolic $U(1, 1)$-Higgs bundle and a $U(n-1)$ bundle by quotienting out by the rank $n-1$ kernel of the Higgs field. In the case that the $U(1, 1)$-Higgs bundle is of loxodromic type, this construction can be reversed, with the added consequence that the stability conditions of the resulting $U(n, 1)$-Higgs bundle are determined only by the kernel of Φ, the number of marked points, and the degree of the $U(1, 1)$-Higgs bundle. With this result, we prove our main theorem, which says that when the log eigenvalues of lifts $\widetilde{C}_j$ of the C_j to $U(n, 1)$ satisfy the inequalities in [4] for the existence of a stable parabolic

bundle, then there is a stable parabolic $U(n,1)$-Higgs bundle whose monodromies around the marked points are in $\widetilde{C}_j$. This new approach using Higgs bundle techniques generalizes the result of Falbel and Wentworth in [12] for fixed loxodromic conjugacy classes in $PU(2,1)$.

This new result gives sufficient, but not necessary, conditions for the existence of an irreducible solution to the Deligne-Simpson problem for fixed loxodromic conjugacy classes in $PU(n,1)$. The stability assumption cannot be dropped from our proof since no universal characterization of unstable bundles exists. In the last chapter, we explore what happens when we change the weights of the stable kernel in the special case of three fixed loxodromic conjugacy classes in $PU(3,1)$. Using the techniques from [11], [12], and [25], we can show that our construction implies the existence of many other solutions to the problem.

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by

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Dedication

This thesis is dedicated to Darryl Ruppert and Mary Maschal.

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Chapter 1: Introduction

Recall Horn's Problem: given two hermitian matrices A and B with known spectra, what can we say about the spectrum of their sum, $A + B$ (see for example [13], [21])? Similarly, consider the multiplicative version of this problem: given two unitary matrices $A, B \in U(n)$, with known eigenvalues, what can we say about the eigenvalues of their product AB? In this thesis, we consider the generalization of this problem to other subgroups $G \subset GL(n, \mathbb{C})$, known as the Deligne-Simpson Problem: given conjugacy classes $C_1, \ldots C_k$ in G, what are necessary and sufficient conditions on the C_i such that there exists matrices $A_i \in C_i$ such that $A_1 \ldots A_k = I$?

Solutions to this problem are often of geometric interest, such as the moduli of polygonal linkages [18], [19], [20], [30]. For example, consider the case when $G = SU(2)$. We can use the three sphere S^3 as a model for $SU(2)$, where a matrix in $SU(2)$ corresponds to a point on the sphere. In this context, the conjugacy class of matrix is represented by the distance between the point it represents and the identity in S^3. Solutions to the Deligne-Simpson problem then correspond to different configurations of n-sided polygons on the sphere with given side lengths. Necessary and sufficient conditions are then given by the triangle inequalities on S^3.

More generally, the problem already has a complete solution when $G = U(n)$.

This was first proved in the case $G = U(2)$ by Biswas in [3], which he later generalized to $G = U(n)$. Independent solutions were given by Belkale in [2] and Agnihotri and Woodward in [1]. A conjugacy class in $U(n)$ is determined by the eigenvalues of any matrix in the conjugacy class. Necessary and sufficient conditions are given as a set of affine inequalities involving the logarithms of the eigenvalues. For simplicity, we state the case when $n = 2$:

Theorem 1.0.1. *[1], [2], [4], [3]*

Let $S = \{1, 2, \ldots, k\}$, and denote the log eigenvalues defining a conjugacy class C_s in $U(2)$ by $0 \leq \alpha_1^s < \alpha_2^s < 1$. Assume $\Sigma_{s \in S}(\alpha_1^s + \alpha_2^s)$ is an odd (respectively even) integer, say $2N$ (respectively $2N + 1$). Then there is a matrix $A_j \in C_j$ such that $A_1 \ldots A_k = I$ if and only if for every $D \subset S$ of size $2j$ (resp. $2j + 1$), where j is a non-negative integer, the following inequality holds:

$$-N - j + \Sigma_{s \in D} \alpha_2^s + \Sigma_{s \in S-D} \alpha_1^s < 0.$$

In [28], Simpson gives necessary and sufficient conditions for existence to a solution to the Deligne-Simpson problem in the case $G = SL(n, \mathbb{C})$ (see also the work of Crawley-Boevey on this problem in [9]). However, whereas the conditions in the $U(n)$ case depend directly on the eigenvalues of the individual conjugacy classes, the eigenvalues for the $SL(n, \mathbb{C})$ case don't matter at all! Instead, conditions are given in terms of the multiplicity of the eigenvalues and the sizes of the corresponding Jordan blocks. For fixed conjugacy classes $C_1, \ldots, C_k$ in $SL(n, \mathbb{C})$, let r_i denote the minimum rank of a matrix $A_i - \alpha$, where $A_i \in C_i$ and $\alpha \in \mathbb{C}$. Then:

Theorem. *[28]*

Suppose C_k is regular (i.e. semi-simple with distinct eigenvalues). Then there exists a solution $A_1 \dots A_k = I$ with $A_i \in C_i$ if and only if $\Sigma \dim(C_i) \geq 2n^2 - 2$ and $r_1 + \dots + r_{k-1} \geq n - 1$.

While these results are quite different, both use the following topological reinterpretation of the linear algebra problem. Let $S = \{p_1, \dots, p_k\}$ be a finite collection of points on the Riemann Sphere $\mathbb{P}^1$. Setting $X = \mathbb{P}^1 - S$, the fundamental group $\pi_1(X)$ has the following presentation:

$$\pi_1(X) =< \gamma_1, \dots \gamma_k | \gamma_1 \gamma_2 \dots \gamma_k = 1 >$$

where γ_i is a loop around the point p_i. Then a solution to the Deligne-Simspon Problem is equivalent to the existence of a representation $\rho : \pi_1(X) \to G$ with $\rho(\gamma_i) \in C_i$.

When $G = U(n)$, the Mehta-Seshadri Theorem [22] gives an equivalence of categories between the category of irreducible representations $\pi_1(X) \to U(n)$ and the category of stable parabolic vector bundles of rank n and parabolic degree 0 on $\mathbb{P}^1$. The problem of constructing a representation ρ with $\rho(\gamma_i) \in C_i$ then becomes one of constructing a stable parabolic vector bundle whose weights at a marked point p_i are determined by the eigenvalues defining C_i. The inequalities in theorem 2.4.1 come from the stability condition on the parabolic vector bundle.

When $G = SL(n, \mathbb{C})$, Simpson's Correspondence [27] gives an equivalence of categories between the category of stable filtered local systems of degree 0 and the category of stable parabolic Higgs bundles of rank n and parabolic degree 0 on $\mathbb{P}^1$.

With the solution in the $U(n)$ and $SL(n, \mathbb{C})$ cases understood, it is natural to ask what happens for other real forms, such as $PU(p,q)$, of $GL(n,\mathbb{C})$. There are already some results in this case.

In [25], Paupert studies products of elliptic isometries in $PU(2,1)$. Multiplying both sides by A_3^{-1}, the problem of finding a solution $A_1A_2A_3 = I$ with $A_i \in C_i$ becomes $A_1A_2 = A_3^{-1}$. For fixed C_1, C_2, the Deligne-Simpson problem then becomes one of studying the product map $\mu : C_1 \times C_2 \to PU(2,1)$. Let C_1, C_2 be elliptic conjugacy classes in $PU(2,1)$. Then C_j can be represented by a matrix A_j of the form:

$$A_j = \begin{bmatrix} 1 & & \\ & e^{2\pi i\theta_j^1} & \\ & & e^{2\pi i\theta_j^2} \end{bmatrix} \tag{1.1}$$

where $0 \leq \theta_j^1 < \theta_j^2 < 1$. Thus the space of elliptic conjugacy classes can be identified with $\mathbb{T}^2/S_2$, the two-torus modulo the action of the symmetric group S_2. Then Paupert proves the following concerning the product map $\widetilde{\mu}$:

Theorem. *Let C_1, C_2 be two elliptic conjugacy classes in $PU(2,1)$, represented by matrices A_j of the form in equation 1.1, and let*

$$\widetilde{\mu} : (C_1 \times C_2) \cap \{elliptics\} \to \mathbb{T}^2/S_2.$$

Then $\widetilde{\mu}$ is onto if and only if

$$\begin{cases} \theta_1^1 - 2\theta_1^2 + \theta_2^1 - 2\theta_2^2 \geq 1 \\ 2\theta_1^1 - \theta_1^2 + 2\theta_2^1 - \theta_2^2 \geq 3 \end{cases}$$

The key point of the proof is that the image of $\widetilde{\mu}$ is composed of "reducible walls" and "irreducible chambers." The reducible walls are found by computing the image under $\widetilde{\mu}$ of pairs (A, B) generating a reducible subgroup of $PU(2,1)$. These walls bound irreducible chambers, which are completely empty or completely full. The theorem can be interpreted as giving conditions on C_1 and C_2 so that every chamber is full.

In [12], Falbel and Wentworth consider products of loxodromic isometries. They prove that when the C_i are loxodromic conjugacy classes in $PU(n,1)$, the configurations of reducible walls are especially simple when $n = 1, 2$. In particular, the complement of the reducible walls is nonempty and connected. This gives:

Theorem. *Let $C_1, \ldots, C_k$, $k \geq 3$ be arbitrary conjugacy classes of loxodromic elements of $PU(n,1)$, for $n = 1, 2$. Then there exists $A_i \in C_i$ such that $A_1 \ldots A_k = I$.*

However, the proof does not generalize to larger n. As we shall see later, the complement of the reducibles is disconnected when $n \geq 3$.

In this thesis, we consider the case of $k \geq 3$ fixed loxodromic conjugacy classes in $PU(n,1)$ for $n \geq 3$. As we shall see later, the fact that $PU(n,1)$ is a rank 1 group is important (see for example Theorem 2 in [12]). The restriction to loxodromic conjugacy classes is reasonable for two reasons. First, the elliptic case seems to be much more difficult. In particular, Simpson's Correspondence between filtered local systems and parabolic Higgs bundles cannot completely identify the conjugacy class of an elliptic monodromy. Second, purely loxodromic representations often correspond to interesting geometric structures. For example, when Σ is a closed surface of genus $g \geq 2$, Teichmueller space is identified with discrete, faithful, and purely loxodromic representations $\pi_1(\Sigma) \to PSL(2, \mathbb{R}) = SU(1,1)$, up to conjugation.

Discrete, faithful, purely loxodromic representations $\pi_1(\Sigma) \to SU(2,1)$ are related to quasi-Fuchsian representations and complex hyperbolic structures, the space of which admit a coordinate system which is a direct generalization of Fenchel-Nielsen coordinates of Teichmueller space [24].

Similar to [28] and [4], we interpret solutions to the Deligne-Simpson problem as representations $\pi_1(\mathbb{P}^1 - \{p_1 \ldots, p_k\}) \to PU(n,1)$. By Simpson's Correspondence (Theorem 2.5.3), irreducible such representations correspond to stable parabolic Higgs bundles.

The use of Higgs bundles and Simpson's Correspondence is attractive for two reasons. First, this strategy has already proved successful when $G = SL(n, \mathbb{C})$ and $G = U(n)$. Additionally, we expect the solution for loxodromic classes in $PU(n,1)$ to somehow be a combination of these two cases. In particular, it would be interesting to see the extent to which the eigenvalues defining a loxodromic conjugacy class matter. Second, the topology of the $PU(p,q)$ character variety $Hom(\pi_1(X), U(p,q))//U(p,q)$ has been well-studied in the literature using Higgs bundle techniques. In particular, the case where X is compact and genus $g \geq 2$ is handled in [33], [34], [6], [32]. The case where X is a punctured Riemann surface of genus $g \geq 1$ was handled in [14]. Our main theorem builds on the latter result by guaranteeing in some cases that the $PU(n,1)$ character variety is nonempty for a punctured Riemann surface X of genus 0. An obvious next question then is to ask about the topology of these spaces, for which having a Higgs bundle description of the elements would prove useful.

Constructing irreducible representations $\pi_1(\mathbb{P}^1 - \{p_1 \ldots, p_k\}) \to PU(n,1)$ is equivalent to constructing stable parabolic Higgs bundles $(\mathcal{E}, \Phi)$ of the following form:

$$\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$$

$$\Phi = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}.$$

Information about the conjugacy class (namely the eigenvalues) of the monodromy around a marked point p_j can be found by studying the parabolic structure and the residue of Φ at p_j. In chapter 3, we study the stability properties of Higgs bundles of the above form. A stable parabolic Higgs bundle corresponding to a representation with loxodromic monodromies has in particular $b, c \neq 0$. As a result, when $n \geq 2$, $\mathcal{E}$ can be decomposed into a parabolic $U(1,1)$-Higgs bundle $(\widetilde{\mathcal{E}}, \widetilde{\Phi})$ and a rank $n-1$ parabolic bundle $\mathcal{S}$. Here $\mathcal{S}$ is the kernel of b, and $\widetilde{\mathcal{E}}$ is obtained by quotienting $\mathcal{E}$ by $\mathcal{S}$. Finally we show that you can reverse the process by finding an extension of $\widetilde{\mathcal{E}}$ by $\mathcal{S}$ such that $\widetilde{\Phi}$ lifts and has nice additional property. This property allows us to directly compute the invariant subbundles of Φ, which is important for computing the stability conditions of $(\mathcal{E}, \Phi)$.

The above decomposition of a $PU(n,1)$-Higgs bundle into a $U(1,1)$-Higgs bundle and a rank $n-1$ parabolic bundle suggests the solution for loxodromic conjugacy classes in $PU(n,1)$ should be related to the solution for $G = U(n-1)$. The main result of this thesis is that assuming some of the data defining the fixed loxodromic conjugacy classes satisfy the conditions in Theorem 1.0.1, then solutions to the Deligne-Simpson problem exist.

To be more precise, let $C_1, C_2, \ldots, C_k$ be k fixed loxodromic conjugacy classes in $PU(n,1)$. Lift each C_i to a conjugacy class $\widetilde{C}_i$ in $U(n,1)$, represented by a diagonal matrix of the following form:

$$A_j = \begin{bmatrix} r_j & & & & \\ & r_j^{-1} & & & \\ & & exp(2\pi i\alpha_j^1) & & \\ & & & \ddots & \\ & & & & exp(2\pi i\alpha_j^{n-1}) \end{bmatrix} \tag{1.2}$$

where r is a positive real number and $0 \leq \alpha_1^i < \ldots < \alpha_{n-1}^i < 1$. In Chapter 4, we prove the following:

Theorem 1. *(Main Theorem)*

Assume the α_l^j's defining the conjugacy classes $\widetilde{C}_j$ satisfy $\Sigma\alpha_l^j \in \mathbb{Z}$ and the strict $U(n-1)$ inequalities in [4]. Then there exists a stable parabolic $U(n,1)$-Higgs bundle $(\mathcal{E}, \Phi)$ with $k \geq 3$ marked points such that:

- *the filtration of $\mathcal{E}_{p_j}$ has a rank 2 jump at 0 and rank 1 jumps at each of $\alpha_1^j, \ldots, \alpha_{n-1}^j$;*
- *the residue of Φ at p_j has eigenvalues $\pm i\log(r)/4\pi$, and 0 with multiplicity $n-1$.*

By reinterpreting these stable parabolic $U(n,1)$-Higgs bundles as irreducible representations (technically filtered local systems with necessarily trivial filtration), we have the following corollary:

Corollary 2. *Let C_i be given loxodromic conjugacy classes in $PU(n,1)$, lifted to $\widetilde{C}_i$ to $U(n,1)$ as in 1.2. If the α_j^i satisfy the $U(n-1)$ inequalities in 2.4.1, then there is a matrix $A_i \in C_i$ such that $A_1 \dots A_k = 1$.*

In particular, our theorem generalizes the result in [12] for $PU(2,1)$. Furthermore, the Higgs bundle techniques we use give a new approach not used in any of [12], [25], and [28].

Theorem 1 gives sufficient conditions for the existence of an irreducible solution. In fact, we can see in the proof in Chapter 4 that the requirement that the kernel $\mathcal{S}$ of the Higgs field be stable is not necessary. However, this stability assumption is required for our proof. While stable bundles vary nicely in families, this is not the case for unstable bundles, which can fail to be stable in many interesting ways. As a result, we lack the tools required to approach the problem from the point of view of finding necessary conditions.

In chapter 5, we explore this issue further in the special case of three fixed conjugacy classes in $PU(3,1)$. Following [11], [12], and [25], we consider the product map $\mu : C_1 \times C_2 \to PU(3,1)$. We restrict μ to pairs (A,B) with loxodromic product, and then project onto $\mathcal{C}_{lox} = \mathbb{T}/S_2 \times (1,\infty)$, the space of loxodromic conjugacy classes. The image, as in [12] and [25], decomposes into a set of reducible walls and irreducible chambers. We first compute the set of reducible walls, and then consider the question of which chambers are full. Starting with a fixed stable parabolic $U(3,1)$-Higgs bundle $(\mathcal{E},\Phi)$ coming from Theorem 1, we examine how the stability changes as we vary the weights. As a result, we can show that many chambers are full, and therefore that Theorem 1 guarantees a solution to the Deligne-Simpson problem in many more cases.

Chapter 2: Background

In this chapter, we briefly outline the material which we use in the rest of this paper. In section 2.1, we define the indefinite unitary groups $PU(p,q)$, and give a classification of the conjugacy classes in the group $PU(n,1)$ of isometries of complex hyperbolic space.

In section 2.2, we give an overview of the Riemann-Hilbert Correspondence between local systems and flat bundles, and the Nonabelian Hodge Correspondence between Higgs bundles and flat bundles. We conclude the section with a discussion about $U(p,q)$-Higgs bundles.

In section 2.3, we define parabolic bundles and state the Mehta-Seshadri Theorem, which gives an equivalence of categories between the category of representations of the fundamental group of the punctured surface into $U(n)$ with the category of stable parabolic vector bundles of parabolic degree 0. As an example, we give a solution to the Deligne-Simpson problem in the case of three conjugacy classes in $SU(2)$. The section concludes with a statement of the general result in the case when $G = U(2)$.

Finally, in section 2.5, we define parabolic Higgs bundles and filtered local systems, and give a statement of Simpson's Correspondence between filtered local systems and parabolic Higgs bundles, simultaneously generalizing both the Non-abelian Hodge Correspondence and the Mehta-Seshadri Theorem. Since we are interested in using this correspondence to construct representations, we conclude the section with a careful discussion of the correspondence between the residues of the Higgs field acting on the parabolic structure at a marked point and the residue of the monodromy around the marked point.

2.1 The group $PU(n,1)$

Let $V^{p,q}(\mathbb{C})$ denote the $p+q=n$-dimensional complex vector space together with the Hermitian form of type (p,q):

$$\langle z, w\rangle = w^* J z$$
$$J = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}.$$

Definition 2.1.1. *Define the group*

$$U(p,q) = \{g \in GL(n,\mathbb{C}) \mid \langle gz, gw\rangle = \langle z, w\rangle\}.$$

For the rest of this section, we'll restrict our attention to the group $U(n,1)$.

2.1.1 Conjugacy classes in $U(n,1)$

Here we'll briefly review the different types of conjugacy classes in $U(n,1)$, as described in [7].

For $z \in V^{n,1}$, define $\Phi(z) = \langle z, z\rangle$.

Definition 2.1.2. *For $z \in V^{n,1}$, define $\Phi(z) = \langle z, z \rangle$. Then for a subspace $W \subset V^{n,1}$, we say W is:*

- *hyperbolic if $\Phi|_W$ is non-degenerate and indefinite*
- *elliptic if $\Phi|_W$ is positive definite*
- *parabolic if $\Phi|_W$ is degenerate.*

Following [15], we define n-dimensional complex hyperbolic space as the set of negative lines in $\mathbb{C}^{n,1}$:

$$H^n(\mathbb{C}) = \{z \,|\, \Phi(z) < 0\}/\mathbb{C}^* = U(n,1)/U(n) \times U(1).$$

We can also define the boundary of $H^n(\mathbb{C})$ as the set of null lines in $\mathbb{C}^{n,1}$:

$$\partial H^n(\mathbb{C}) = \{z \,|\Phi(z) = 0\}/\mathbb{C}^*.$$

$PU(n,1)$ acts transitively and effectively by isometries under the Bergman metric on $H^n(\mathbb{C})$. Elements $g \in PU(n,1)$ extend to conformal transformations of boundary $\partial H^n(\mathbb{C})$. By the Brouwer fixed-point theorem, any $g \in PU(n,1)$ has at least one fixed point in $H^n(\mathbb{C}) \cup \partial H^n(\mathbb{C})$. This gives the following classification of elements in $PU(n,1)$:

Definition 2.1.3. *An isometry $g \in PU(n,1)$ of $H^n(\mathbb{C})$ is:*

- *elliptic if g has (at least) one fixed point in $H^n(\mathbb{C})$*
- *parabolic if g has exactly one fixed point on the boundary of $H^n(\mathbb{C})$*
- *loxodromic if g has exactly two fixed points on the boundary.*

The following alternate Hermitian forms are useful for describing elliptic and loxodromic elements, respectively:

$$J_e = \begin{bmatrix} I_n & 0 \\ 0 & -1 \end{bmatrix}, \quad J_l = \begin{bmatrix} I_{n-1} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Using J_e, any elliptic isometry in $U(n,1)$ is conjugate to a matrix of the following form:

$$\begin{bmatrix} U & 0 \\ 0 & \lambda \end{bmatrix}$$

where $U \in U(n)$, $\lambda \in U(1)$. The eigenvalue λ is said to be of "negative type" and the eigenvalues of U are said to be of "positive type." The conjugacy class of an elliptic isometry is determined by its groups of positive type and negative type eigenvalues.

Using J_l, any loxodromic isometry in $U(n,1)$ is conjugate to a matrix of the following form:

$$\begin{bmatrix} U & 0 & 0 \\ 0 & \lambda r & 0 \\ 0 & 0 & \lambda r^{-1} \end{bmatrix}$$

where $U \in U(n-1)$, $\lambda \in U(1)$, and $r > 1$.

2.2 Flat bundles, representations, and Higgs bundles

In this section, we briefly outline the Riemann Hilbert Correspondence between flat bundles and representations, and the Nonabelian Hodge Correspondence between flat bundles and Higgs bundles, in the case of a compact Riemann surface X of genus ≥ 2. We conclude this section with a description of $U(p,q)$-Higgs bundles.

2.2.1 Flat bundles and representations

Definition 2.2.1. *A flat bundle on X is a vector bundle E on X together with a connection ∇ with vanishing curvature: $\nabla \wedge \nabla = 0$.*

Definition 2.2.2. *A local system L is a vector bundle given by constant transition functions. Fixing a base point x, a local system is determined by its monodromy representation $\pi_1(X, x) \to L_x$.*

For a flat bundle (E, ∇), parallel translation of ∇ along a path γ only depends on the homotopy class of γ. Thus parallel translation gives the holonomy representation $\pi_1(X) \to GL(n, \mathbb{C})$.

Given a representation $\rho : \pi_1(X) \to GL(n, \mathbb{C})$, we can construct a flat bundle (E, ∇) on X as follows. Let $\widetilde{X}$ be the universal cover of X. Then letting $E = \widetilde{X} \times \mathbb{C}^n/(x, v) \sim (\gamma(x), \rho(\gamma)v)$, the differential d descends to a connection ∇ on E which is flat.

These two constructions are inverse to one another, and give the following:

Theorem 2.2.3. *(Riemann-Hilbert Correspondence)*

The above construction gives an analytic homeomorphism between the character variety $Hom(\pi_1(X), GL(n, \mathbb{C}))/GL(n, \mathbb{C})$ and the space of isomorphism classes of flat bundles (E, ∇).

2.2.2 Higgs bundles

Definition 2.2.4. *A Higgs bundle on X is a pair $(\mathcal{E}, \Phi)$ where $\mathcal{E} \to X$ is a holomorphic vector bundle, Φ is a holomorphic section of $Hom(\mathcal{E}, \mathcal{E} \otimes \mathcal{K}_X)$, and $\mathcal{K}_X \to X$ is the canonical bundle.*

Definition 2.2.5. *A subbundle $\mathcal{S} \subset \mathcal{E}$ is Φ-invariant if $\Phi(\mathcal{S}) \subset \mathcal{S} \otimes \mathcal{K}_X$.*

Definition 2.2.6. *The slope $\mu(\mathcal{E})$ of a holomorphic vector bundle is defined to be*

$$\mu(\mathcal{E}) = deg(\mathcal{E})/rank(\mathcal{E})$$

Definition 2.2.7. *A Higgs bundle $(\mathcal{E}, \Phi)$ is (semi-) stable if for every proper Φ-invariant holomorphic subbundle $\mathcal{S} \subset \mathcal{E}$, $\mu(\mathcal{E}) < (\leq)\mu(\mathcal{E})$.*

2.2.3 Hitchin's equations and the Nonabelian Hodge Correspondence

Let (E, Φ) be a Higgs bundle. If we fix an hermitian metric h on E, then $\nabla = d_A + \Phi + \Phi^*$ defines a connection on E. This is our candidate flat connection. However, its curvature $F_\nabla = F_A + [\Phi, \Phi^*]$ may not vanish. Thus our Higgs bundle corresponds to a flat bundle precisely when Hitchin's equations are satisfied:

$$F_A + [\Phi, \Phi^*] = 0 \tag{2.1}$$

$$\bar{\partial}_{\mathcal{E}}(\Phi) = 0 \tag{2.2}$$

The obstruction to finding Higgs bundles which solve Hitchin's equations is exactly the stability condition for Higgs bundles defined previously, as evidenced by the following proposition:

Proposition 2.2.8. *If* $(\mathcal{E}, \Phi)$ *is a Higgs bundle satisfying* 2.1*, then* $(\mathcal{E}, \Phi)$ *is polystable.*

However, the converse of this also true:

Theorem 2.2.9. *[17], [29] If* (E, Φ) *is polystable, then it admits a metric satisfying equation 2.1.*

Let (E, ∇) be a flat bundle. If we again fix an hermitian metric h, then the flat connection ∇ splits uniquely as $\nabla = d_A + \Psi$, where d_A is a metric connection and Ψ is an hermitian 1-form. In a local frame $\{s_i\}$, Ψ is defined by the equation

$$\langle \Psi s_i, s_j \rangle = \langle \nabla s_i, s_j \rangle + \langle s_i, \nabla s_j \rangle - d \langle s_i, s_j \rangle .$$

Ψ is hermitian, so we can write $\Psi = \Phi + \Phi^*$ for some endomorphism-valued 1-form Φ. However, Φ may not be holomorphic. In fact, Φ is holomorphic iff $d_A^* \Psi = 0$ (we also require $d_A \Psi = 0$, which we get for free since $\nabla = d_A + \Psi$ is assumed to be flat). In this case, Hitchin's equations become:

$$F_A + \frac{1}{2}[\Psi, \Psi] = 0 \tag{2.3}$$

$$d_A(\Psi) = 0 \tag{2.4}$$

$$d_A^*(\Psi) = 0 \tag{2.5}$$

Intuitively, an hermitian metric is a specification of which holomorphic frames are unitary. As such, h is a section of the $GL(n, \mathbb{C})/U(n)$ bundle naturally associated to E. In this way, we can view an hermitian metric h as a $\pi_1(X)$-equivariant map $h : \tilde{X} \to GL(n, \mathbb{C})/U(n)$.

More explicitly, let $h : \tilde{X} \to GL(n,\mathbb{C})/U(n)$ be such an equivariant map, where $GL(n,\mathbb{C})/U(n)$ corresponds to the set of positive definite hermitian matrices. For a section s of E, think of s as a $\pi_1(X)$-equivariant map $s : \tilde{X} \to \mathbb{C}^n$. Then define $||s||_h^2 : \tilde{X} \to \mathbb{C}$ by

$$||s||_h^2(x) = \langle s(x), h(x)s(x)\rangle_{\mathbb{C}^n}$$

$||s||_h^2$ is easily seen to be equivariant, and hence descends to X, giving an hermitian metric.

In the other direction, if s_1 and s_2 are sections of E with corresponding equivariant maps $\tilde{s}_1, \tilde{s}_2 : \tilde{X} \to \mathbb{C}^n$, then for $x \in X$ and $\tilde{x} \in \tilde{X}$ sitting above x, we can define $h : \tilde{X} \to GL(n,\mathbb{C})/U(n)$ by

$$\langle s_1(x), s_2(x)\rangle_h = \langle \tilde{s}_1(\tilde{x}), h(\tilde{x})\tilde{s}_2(\tilde{x})\rangle_{\mathbb{C}^n}.$$

Equivariance of h follows from the equivariance of $\tilde{s}_1$, $\tilde{s}_2$.

Define the energy density of h to be

$$\mathcal{E}(h) = \frac{1}{2}\int_M |dh|^2\omega.$$

The hermitian metric corresponding to h is said to be harmonic if h is a critical point of $\mathcal{E}$ (i.e. if h is a harmonic map). We have the following lemma:

Lemma 2.2.10. $E(h) = 2||\Psi||^2$.

From this it is easy to compute the Euler-Lagrange equations $d_A^*\Psi = 0$. We conclude that Φ defined above is holomorphic if and only if h is a harmonic metric, and the problem of relating flat bundles to Higgs bundles becomes one of finding harmonic metrics.

Theorem 2.2.11. *[8], [10] (E, ∇) admits a harmonic metric iff ∇ is semisimple.*

Let $\mathcal{M}^0_{Dol}$ be the moduli space of polystable Higgs bundles of degree 0. Define $\mathcal{M}_B = Hom(\pi_1(X), GL(n, \mathbb{C}))//GL(n, \mathbb{C})$ to be the character variety (or Betti moduli space). Above we defined maps from each moduli space to the other. By Theorems 2.2.9 and 2.2.11, these maps are inverse to one another. Thus we have the following theorem:

Theorem 2.2.12. *Nonabelian Hodge Correspondence, [8], [10], [17], [29]*

The correspondence above defines a homeomorphism between the Betti moduli space $\mathcal{M}^0_B(X)$ and the Dolbeault moduli space $\mathcal{M}^0_D(X)$.

Example Let $\mathcal{K}_X^{\frac{1}{2}}$ be a fixed square root of the canonical bundle $\mathcal{K}_X$. Define $\mathcal{E} = \mathcal{K}_X^{\frac{1}{2}} \oplus \mathcal{K}_X^{-\frac{1}{2}}$. Define the Higgs field Φ by

$$\Phi = \begin{bmatrix} 0 & q \\ 1 & 0 \end{bmatrix},$$

where

$$1 \in H^0(X, (\mathcal{K}_X^{\frac{1}{2}})^* \otimes \mathcal{K}_X^{-\frac{1}{2}} \otimes \mathcal{K}_X) = H^0(X, \mathcal{O})$$

$$q \in H^0(X, (\mathcal{K}_X^{-\frac{1}{2}})^* \otimes \mathcal{K}_X^{\frac{1}{2}} \otimes \mathcal{K}_X) = H^0(X, \mathcal{K}_X^2).$$

Then the Higgs bundle $(\mathcal{E}, \Phi)$ is stable of degree 0, and so corresponds to an irreducible representation $\pi_1(X) \to GL(n, \mathbb{C})$. Actually, this family of Higgs bundles corresponds to the uniformizing representations $\pi_1(X) \to SL(2, \mathbb{R})$, and the quadratic differential q gives coordinates for Teichmueller space.

2.2.4 $U(p,q)$-Higgs bundles

We've seen above how stable vector bundles of degree 0 correspond to irreducible representations of $\pi_1(X)$ into $U(n)$, and stable Higgs bundles of degree 0 correspond to irreducible representations into $GL(n,\mathbb{C})$. We can also consider other non-compact real forms of $GL(n,\mathbb{C})$, namely the groups $U(p,q)$, where $p+q=n$ (see for example [6], [14], [31]).

The maximal compact subgroup of $U(p,q)$ is $U(p)\times U(q)$. The Lie algebra $\mathfrak{u}(p,q)$ has a Cartan decomposition

$$\mathfrak{u}(p,q)=\mathfrak{u}(p)\oplus\mathfrak{u}(q)\oplus\mathfrak{m}$$

where $\mathfrak{m}$ is the set of matrices of the form

$$\begin{bmatrix} 0 & A \\ -\bar{A}^t & 0 \end{bmatrix}$$

A metric in this case is a reduction of structure group $h:\widetilde{X}\to U(p,q)/U(p)\times U(q)$. Any such reduction uniquely splits the flat connection $\nabla=d_A+\Psi$, where d_A is a $U(p)\times U(q)$ metric connection and Ψ is a one-form with values in $\mathfrak{m}$. The splitting of the d_A gives a holomorphic splitting of the holomorphic bundle $\mathcal{E}$. Writing $\Psi=\Phi+\Phi^*$, we have the following:

Definition 2.2.13. *A $U(p,q)$-Higgs bundle is a Higgs bundle $(\mathcal{E},\Phi)$ of the form*

$$\mathcal{E}=\mathcal{V}\oplus\mathcal{W}$$

$$\Phi=\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$$

where $b \in H^0(X, Hom(\mathcal{W}, \mathcal{V}))$ *and* $c \in H^0(X, Hom(\mathcal{V}, \mathcal{W}))$.

2.3 Parabolic vector bundles and the Mehta-Seshadri Theorem

Now we begin to introduce a generalization of the Nonabelian Hodge Correspondence to punctured Riemann Surfaces. For the rest of this chapter, let X be any compact Riemann surface, and $D = p_1 + \ldots + p_k$ the reduced divisor consisting of the marked points p_j. The main result of this section is a correspondence between representations $\pi_1(X - D) \to U(n)$ on the punctured surface and parabolic vector bundles on the compact surface.

Let $\mathcal{E} \to X$ be a holomorphic vector bundle. Adopting the notation in [5], we have:

Definition 2.3.1. *A parabolic structure on E consists of weighted flags*

$$\mathcal{E}_p = \mathcal{E}_1(p) \supset \mathcal{E}_2(p) \supset \ldots \supset \mathcal{E}_l(p) \supset \mathcal{E}_{l+1}(p) = 0$$

$$0 \leq \alpha_1(p) < \alpha_2(p) < \ldots < \alpha_l(p) < 1$$

for each $p \in D$.

A morphism of parabolic vector bundles is a morphism of holomorphic vector bundles which preserves the parabolic structure at every point $p \in D$. Formally, if $\Phi : \mathcal{E}^1 \to \mathcal{E}^2$ is a holomorphic map between vector bundles, then Φ is parabolic if whenever $\alpha_i^1(p) \leq \alpha_j^2(p)$, we have $\Phi(\mathcal{E}_i^1(p)) \subset \mathcal{E}_{j+1}^2(p)$, again for all $p \in D$. Denote the set of all parabolic morphisms by $H^0(ParHom(\mathcal{E}_*^1, \mathcal{E}_*^2))$.

Additionally, we say Φ is *strongly parabolic* if it is nilpotent with respect to the flag at every marked point. Equivalently, Φ is strongly parabolic if whenever

$\alpha_i^1(p) < \alpha_j^2(p)$, we have $\Phi(\mathcal{E}_i^1(p)) \subset \mathcal{E}_{j+1}^2(p)$. We denote the set of all strongly parabolic morphisms by $H^0(SParHom(\mathcal{E}_*^1, \mathcal{E}_*^2))$.

2.3.1 Statement of theorem

The parabolic degree and parabolic slope of a parabolic bundle are defined as follows:

$$pdeg(E) = deg(E) + \sum_{p_i} \sum_j \alpha_j(p_i) rk(E_j(p_i)/E_{j-1}(p_i))$$
$$p\mu(E) = \frac{pdeg(E)}{rk(E)}$$

A parabolic bundle is called stable (resp. semistable) if for every proper subbundle $S \subset E$ with the induced parabolic structure satisfies $p\mu(S) < (\leq) p\mu(E)$.

Theorem 2.3.2. *(Mehta-Seshadri Theorem) [22]*

There is an equivalence of categories between the category of irreducible representations $\gamma : \pi_1(X - D) \to U(n)$ and stable parabolic bundles $\mathcal{E}$ of rank n and parabolic degree 0 on X.

Moreover, fixing the conjugacy class C_j of the monodromy around p_i fixes the parabolic structure on $\mathcal{E}$. More explicitly, let C_j be represented by the following diagonal matrix:

$$A = \begin{bmatrix} exp(2\pi i \alpha_1^j) & 0 & \cdots & 0 \\ 0 & exp(2\pi i \alpha_2^j) & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & exp(2\pi i \alpha_n^j) \end{bmatrix}$$

where $0 \leq \alpha_1^j \leq \alpha_2^j \leq \ldots \leq \alpha_n^j < 1$. The jumps in the parabolic structure of $\mathcal{E}$ occur at the $\alpha^j s$, and they are of rank equal to the multiplicity of the corresponding eigenvalue.

The correspondence between the eigenvalues defining the conjugacy class of the monodromy at p_i and the parabolic structure of $\mathcal{E}$ at p_i is very important. For us, this means the problem of constructing irreducible solutions to the Deligne-Simpson problem when $G = U(n)$ is equivalent to constructing stable parabolic bundles with a certain parabolic structure.

Example We'll use the Mehta-Seshadri Theorem to give necessary and sufficient conditions for the existence of a solution to the Deligne-Simpson problem for 3 fixed conjugacy classes in $SU(2)$. As mentioned in the introduction, these conditions can also be interpreted as the triangle inequalities on the three sphere S^3.

Let C_1, C_2, and C_3 be three fixed conjugacy classes in $SU(2)$. Then C_j can be represented by a matrix of the form

$$\begin{bmatrix} e^{2\pi i \alpha_j} & 0 \\ 0 & e^{2\pi i (1-\alpha_j)} \end{bmatrix}$$

where $0 < \alpha_j < 1 - \alpha_j < 1$. From the discussion in the introduction, the existence of $A_j \in C_j$ such that $A_1 A_2 A_3 = I$ is equivalent to a representation $\rho : \pi_1(\mathbb{P}^1 - \{p_1, p_2, p_3\}) \to SU(2)$ with $\rho(\gamma_j) \in C_j$. By the Mehta-Seshadri Theorem, this is equivalent to the existence of a parabolic vector bundle of parabolic degree 0, whose weights at the marked point p_j are $\alpha_j < 1 - \alpha_j$. We'll also require that the representation be irreducible, which implies that the parabolic bundle be stable.

Notice that $\Sigma_j(\alpha_j + 1 - \alpha_j) = 3$. Therefore, since we want to construct a bundle with parabolic degree 0, the degree of the underlying holomorphic bundle must be -3.

To determine the underlying holomorphic bundle $\mathcal{E}$, we note the following two facts. First, every holomorphic line bundle on $\mathbb{P}^1$ is determined up to isomorphism by its degree. If $\mathcal{L}$ is a line bundle on $\mathbb{P}^1$ of degree d, we can unambiguously write $\mathcal{L} = \mathcal{O}(d)$. Second, Grothendieck's Lemma (see [16]) says that every vector bundle on $\mathbb{P}^1$ decomposes as a direct sum of line bundles, which is unique up to permutation of the factors. These two facts plus the requirement that $\mathcal{E}$ has degree -3 means we can write $\mathcal{E} = \mathcal{O}(d) \oplus \mathcal{O}(-d-3)$. The stability condition can be interpreted as all subbundles of $\mathcal{E}$ having strictly negative parabolic degree.

First, we must have $d = -1$ (or equivalently $d = -2$). Suppose for a contradiction that $d \geq 0$. Then the line subbundle $\mathcal{L} = \mathcal{O}(d)$ has nonnegative parabolic degree:

$$pdeg(\mathcal{L}) \geq deg(\mathcal{L}) = d \geq 0.$$

Since the parabolic degree of $\mathcal{E}$ is 0, $\mathcal{E}$ is necessarily unstable. The argument for $d \leq -3$ is similar, where $\mathcal{L} = \mathcal{O}(-d-3)$ has strictly positive degree. We conclude that $d = -1$ is the only possibility, and therefore $\mathcal{E} = \mathcal{O}(-1) \oplus \mathcal{O}(-2)$.

Now we need to determine the parabolic structure on $\mathcal{E}$. First, we'll consider subbundles of $\mathcal{E}$. The only two we need to worry about are $\mathcal{O}(-1)$ and $\mathcal{O}(-2)$. Any other subbundle has strictly negative parabolic degree, and can be eliminated from consideration. More explicitly, let $\mathcal{L} = \mathcal{O}(d)$, with $d \leq -3$. Then:

$$pdeg(\mathcal{L}) = d + \Sigma(1 - \alpha_j) < d + 3 < 0.$$

There is a unique inclusion $\mathcal{O}(-1) \hookrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-2)$. Since $H^0(Hom(\mathcal{O}(-2), \mathcal{O}(-1))) = \mathbb{C}^2$, and $H^0(Hom(\mathcal{O}(-2), \mathcal{O}(-2))) = \mathbb{C}$, an embedding $\mathcal{O}(-2) \hookrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-2)$ is determined by three constants, up to scaling. These constants are fixed by specifying the image of the inclusion at two points (assuming the image is not entirely contained in the fiber of $\mathcal{O}(-1)$).

Now we are ready to consider the parabolic structure on $\mathcal{E}$. This is a choice of a flag $\mathcal{E}_1(p_j) \subset \mathcal{E}_{p_j}$ for each $j = 1, 2, 3$. The first observation to make is that $\mathcal{E}_1(p_j)$ should not be contained in the fiber of $\mathcal{O}(-1)$ at p_j for any j.

Suppose for a contradiction that $\mathcal{E}_1(p_1) = \mathcal{O}(-1)_{p_1}$. Without loss of generality, assume $\mathcal{E}_1(p_j) \neq \mathcal{O}(-1)_{p_j}$ for $j = 2, 3$. Then $pdeg(\mathcal{O}(-1)) = -1 + (1 - \alpha_1) + \alpha_2 + \alpha_3$. Additionally, any choice of $\mathcal{E}_1(p_2)$ and $\mathcal{E}_1(p_3)$ determines an embedding $\mathcal{O}(-2) \hookrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-2)$ whose image at p_j is $\mathcal{E}_1(p_j)$ for $j = 2, 3$. This subbundle has parabolic degree $pdeg(\mathcal{O}(-2)) = -2 + \alpha_1 + (1 - \alpha_2) + (1 - \alpha_3)$. The stability condition on $\mathcal{E}$ for this parabolic structure requires:

$$p\mu(\mathcal{E}) - p\mu(\mathcal{O}(-1) = \alpha_1 - \alpha_2 - \alpha_3 > 0$$

$$p\mu(\mathcal{E}) - p\mu(\mathcal{O}(-2)) = -\alpha_1 + \alpha_2 + \alpha_3 > 0$$

which is clearly impossible. The general case follows similarly, so we must have $\mathcal{E}_1(p_j) \neq \mathcal{O}(-1)_{p_j}$ for all j.

By a similar argument, the choice of flags $\mathcal{E}_1(p_j)$ must be generic, in the sense that for any embedding $\mathcal{O}(-2) \hookrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-2)$, the image at p_j is $\mathcal{E}_1(p_j)$ for at most two of $j = 1, 2, 3$. This requirement on the parabolic structure gives the following necessary and sufficient conditions for stability:

$$\begin{aligned} 1 - \alpha_1 - \alpha_2 - \alpha_3 &> 0 \\ \alpha_1 + \alpha_2 - \alpha_3 &> 0 \\ \alpha_1 - \alpha_2 + \alpha_3 &> 0 \\ -\alpha_1 + \alpha_2 + \alpha_3 &> 0 \end{aligned}$$

where each of the above inequalities come from the requirement that the parabolic degree of any subbundle of $\mathcal{E}$ must have strictly negative parabolic degree. The first inequality comes from the uniquely embedded $\mathcal{O}(-1)$. The last three inequalities come form 3 separate embeddings of $\mathcal{O}(-2)$, the fibers of which correspond to the flag $\mathcal{E}_1(p_j)$ at exactly two of the marked points p_j.

2.4 Existence of solutions to Deligne-Simpson for $G = U(n)$

The case of $k \geq 3$ conjugacy classes in $U(2)$ was handled by Biswas in [3]. This result was later generalized to $U(n)$ by Biswas in [4], with alternative proofs given by Agnihotri and Woodward in [1], and Belkale in [2]. For simplicity, we state the result from [3] on the existence of irreducible solutions to the Deligne-Simpson problem when $G = U(2)$. Later, we will not directly reference the inequalities when $n > 2$. Therefore, the reader need only be aware that they exist.

Let $C_1, \ldots, C_k$ be fixed regular conjugacy classes in $U(2)$. Then C_s can be represented by a matrix of the form

$$\begin{bmatrix} e^{2\pi i \alpha_1^s} & 0 \\ 0 & e^{2\pi i \alpha_2^s} \end{bmatrix}$$

where $0 \leq \alpha_i^s < \alpha_2^s < 1$. Then we have the following

Theorem 2.4.1. *[4], [3], [1], [2]*

Let $S = \{1, 2, \dots, k\}$, and assume $\Sigma_{s\in S}(\alpha_1^s + \alpha_2^s)$ is an odd (respectively even) integer, say $2N$ (respectively $2N+1$). Then there is a stable rank 2 parabolic bundle with parabolic weights $\{\alpha_1^s, \alpha_2^s\}$ at the marked point $p_s \in \mathbb{P}^1$ if and only if for every $D \subset S$ of size $2j$ (resp. $2j+1$), where j is a nonnegative integer, the following inequality holds:

$$-N - j + \Sigma_{s\in D}\alpha_2^s + \Sigma_{s\in S-D}\alpha_1^s < 0.$$

By the Mehta-Seshadri Theorem, such a stable parabolic bundle corresponds to an irreducible representation $\pi_1(\mathbb{P}^1 - \{p_1, \dots, p_k\}) \to U(n)$. The parabolic structure at p_j determines the eigenvalues (and hence the conjugacy class) of the monodromy around the marked point p_j. As such, we have the following Corollary from the Introduction:

Corollary 2.4.2. *Let $S = \{1, 2, \dots, k\}$, and denote the log eigenvalues defining a conjugacy class C_s in $U(2)$ by $0 \leq \alpha_1^s < \alpha_2^s < 1$. Assume $\Sigma_{s\in S}(\alpha_1^s + \alpha_2^s)$ is an odd (respectively even) integer, say $2N$ (respectively $2N+1$). Then there is a matrix $A_j \in C_j$ such that $A_1 \dots A_k = I$ if and only if for every $D \subset S$ of size $2j$ (resp. $2j+1$), where j is a non-negative integer, the following inequality holds:*

$$-N - j + \Sigma_{s\in D}\alpha_2^s + \Sigma_{s\in S-D}\alpha_1^s < 0.$$

2.5 Simpson's Correspondence

2.5.1 Filtered local systems

Definition 2.5.1. *A filtered local system is a local system L on $X - D$ together with a filtration $L_{p_i,\beta}$ of L_x. It is assumed that the filtration $L_{p_i,\beta}$ is preserved by the monodromy around p_i.*

The degree of a filtered local system is defined by:

$$deg(L) = \sum_{p_i} \sum_{\beta} \beta rk(L_{p_i,\beta}/L_{p_i,\beta+\epsilon}).$$

A filtered local system L is said to be stable if for any subsystem $M \subset L$ with the induced filtrations, we have

$$\frac{deg(M)}{rk(M)} < \frac{degL)}{rk(L)}$$

An important thing to keep in mind is the relationship between the stability of the filtered local system and the irreducibility of the associated representation. When the local system has the trivial filtration at every point, stability is equivalent to the irreducibility of the corresponding representation (since any subsystem in this case has filtered degree 0, stability asserts no subsystems exist).

2.5.2 Parabolic Higgs bundles

Definition 2.5.2. *A parabolic Higgs bundle is a parabolic vector bundle (E, E_{p_i,α_i}) together with a meromorphic map $\Phi : E \to E \otimes K$ with poles of order at most 1 at* *the marked points p_i. Φ is regular everywhere else. The residue of Φ at a marked point p_i is assumed to preserve the given filtration E_{p_i,α_i}.*

Fix local coordinates z vanishing at p_i and a local frame e_i. Then near p_i, Φ can be written

$$\Phi(e_i) = \Phi_{ij} e_j \frac{dz}{z} + (regular\, stuff)$$

The statement that the residue of Φ preserves the filtration means that the matrix (Φ_{ij}) preserves the filtration.

Equivalently, a parabolic Higgs bundle is a parabolic vector bundle (E, E_{p_i,α_i}) and a parabolic morphism $\Phi \in ParHom(E, E \otimes \mathcal{K}(D))$. With this interpretation, the residue of the Higgs field Φ is simply its value at a point p_i.

The parabolic degree and parabolic slope of a parabolic Higgs bundle are defined as follows:

$$pdeg(E) = deg(E) + \sum_{p_i} \sum \alpha_j(p_i) rk(E_j(p_i)/E_{j-1}(p_i))$$
$$p\mu(E) = \frac{pdeg(E)}{rk(E)}$$

A parabolic Higgs bundle is called stable (resp. semistable) if for every Φ-invariant subbundle $S \subset E$ with the induced parabolic structure satisfies $p\mu(S) < (\leq) p\mu(E)$.

2.5.3 Statement of theorem

Theorem 2.5.3. *(Simpson, [27]) There is a bijective correspondence between stable filtered local systems of degree 0 and stable parabolic Higgs bundles of degree 0.*

Additionally, there is a correspondence between the residues of the filtered local system and parabolic Higgs bundles at each p_i, as described in [27], [28]. To make this more explicit, let L be a filtered local system and (E, Φ) a parabolic Higgs bundle (both stable, of degree 0). Then we define

$$res_{p_i}(L) = \bigoplus_{\beta} res_{p_i}(L)_\beta$$

$$res_{p_i}(E) = \bigoplus_{\alpha} res_{p_i}(E)_\alpha,$$

where $res_{p_i}(L)_\beta$ and $res_{p_i}(E)_\alpha$ are the blocks in the associated graded of each filtration at the point p_i:

$$res_{p_i}(L)_\beta = Gr_\beta(L_{p_i}) = (L_{p_i,\beta}/L_{p_i,\beta+\epsilon})$$

$$res_{p_i}(E)_\alpha = Gr_\alpha(E_{p_i}) = (E_{p_i,\alpha}/E_{p_i.\alpha+\epsilon}).$$

Since the monodromy A_i around p_i is assumed to preserve the filtration of L_{p_i}, it acts by an endomorphism, which we denote by $res(A_i)$, on the associated graded $res_{p_i}(L)$. Similarly, since the "residue" of Φ is assumed to preserve the filtration of E_{p_i}, it acts by an endomorphism $res_{p_i}(\Phi)$ on the associated graded $res_{p_i}(E)$. The residue of L at p_i is defined to be the pair $(res_{p_i}(L), res(A_i))$. Similarly, the residue of (E, Φ) at p_i is the pair $(res_{p_i}(E), res_{p_i}(\Phi))$.

Moreover, $res(A_i)$ acts on the blocks $res_{p_i}(L)_\beta$, and $res(\Phi)$ acts on the blocks $res_{p_i}(E)_\alpha$. Therefore, we can decompose the blocks $res_{p_i}(L)_\beta$ and $res_{p_i}(E)_\alpha$ into the generalized eigenspaces of $res(A_i)$ and $res(\Phi)$. We have

$$res_{p_i}(L)_\beta = \bigoplus_\lambda res_{p_i}(L)_{\beta,\lambda} \tag{2.6}$$

$$res_{p_i}(E)_\alpha = \bigoplus_\tau res_{p_i}(E)_{\alpha,\tau} \tag{2.7}$$

where λ is a generalized eigenvalue of $res(A_i)$ with generalized eigenspace $res_{p_i}(L)_{\beta,\lambda} \subset res_{p_i}(L)_\beta$, and τ is a generalized eigenvalue of $res(\Phi)$ with generalized eigenspace $res_{p_i}(E)_{\alpha,\tau} \subset res_{p_i}(E)_\alpha$.

Finally, $res(A_i)$ acts on $res_{p_i}(L)_{\beta,\lambda}$ by a matrix $res(A_i)_{\beta,\lambda}$. Putting $res(A_i)_{\beta,\lambda}$ into Jordan normal form induces a partition $P_{p_i}^{\beta,\lambda}$ of its generalized eigenspace $res_{p_i}(L)_{\beta,\lambda}$. The collection $P_{p_i}^{\beta,\lambda}$ forms the "residue diagram" of $(res_{p_i}(L), res(A_i))$. The residue diagram $P_{p_i}^{\beta,\lambda}$ uniquely determines the residue $(res(A_i), res_{p_i}(L))$, up to isomorphism.

The same process applied to $res(\Phi)$ acting on $res_{p_i}(E)_{\alpha,\tau}$ associates to $(res(\Phi), res_{p_i}(E))$ its partition diagram $P_{p_i}^{\alpha,\tau}$. Again this uniquely determines $(res(\Phi), res_{p_i}(E))$ up to isomorphism.

In the correspondence in Theorem 2.5.3, the residue diagrams $P_{p_i}^{\beta,\lambda}$ and $P_{p_i}^{\alpha,\tau}$ of $(res(A_i), res_{p_i})L))$ and $(res(\Phi), res_{p_i}(E))$ are the same, up to the following change of labels:

$$(\alpha, \tau = b + ci) \mapsto (\beta = -2b, \lambda = exp(2\pi i\alpha - 4\pi c) \tag{2.8}$$

$$(\beta, \lambda) \mapsto \left(\alpha = \frac{1}{2\pi} log(\lambda), \tau = -\frac{\beta}{2} - i\frac{log|\lambda|}{4\pi}\right) \tag{2.9}$$

To describe the correspondence more simply, the weights and rank of each jump in the corresponding filtrations at p_i are the same, up to the change of labels in 2.8. The eigenvalues of A_i and the residue of Φ at p_i are the same and have the same multiplicities, up to the change described in 2.8. The sizes of the nilpotent parts of $res(A_i)$ and $res(\Phi)$ acting on the associated graded objects $res_{p_i}(L)$ and $res_{p_i}(E)$ are also the same.

Theorem 2.5.3 allows us to reinterpret our problem of constructing representations into one of constructing parabolic Higgs bundles. The requirement on the local monodromy around a given puncture becomes a requirement on the parabolic structure and the residue of Φ at p_i. Going in the other direction, if we know everything about our parabolic Higgs bundle (i.e. the parabolic structure, Jordan form of $res(\Phi)$), then we can use this to gain knowledge about the monodromy around p_i. In particular, we know everything about the eigenvalues of the local monodromy A_i. However, we may not know everything. If the filtration on the local system is nontrivial (i.e. if the eigenvalues of the residue of Φ have nonzero real part) then we only have partial information about the nilpotent part of A_i. This could also mean the underlying representation is reducible. Since we are only interested in semisimple conjugacy classes, we do not need to worry about the nilpotent pieces. However, we must keep in mind that a stable filtered local system may not correspond to an irreducible representation.

Chapter 3: Parabolic $PU(n,1)$-Higgs Bundles

In this chapter, we discuss the structure of the Higgs field that guarantee the monodromies of the corresponding representation are in $U(n,1)$. In particular, since we are interested in constructing representations with loxodromic monodromies, Simpson's Correspondence says the residue of Φ at the marked points must have two nonzero (in particular with nonzero imaginary part) eigenvalues. This requirement adds further restrictions to the structure of the Higgs field.

A parabolic $U(n,1)$-Higgs bundle will always have invariant subbundles when $n \geq 2$). The restrictions on the structure of the Higgs field put restrictions on the structure of the invariant subbundles. In particular, the subbundles of interest are the rank $n-1$ kernel of Φ, and a rank 2 subbundle corresponding to the nonzero eigenspaces of Φ.

Quotienting out by the kernel, we can break a parabolic $U(n,1)$-Higgs bundle into a parabolic $U(1,1)$-Higgs bundle and a rank $n-1$ parabolic vector bundle. The main result of this chapter is that we can put these pieces back together into a parabolic $PU(n,1)$-bundle in a way that allows us to easily compute the two invariant subbundles.

3.1 Parabolic $PU(n,1)$-Higgs bundles

Let $D = p_1 + \ldots + p_k$ be a fixed reduced divisor on $X = \mathbb{P}^1$. Following section 2.2.4, we have the following:

Definition 3.1.1. *A parabolic $PU(n,1)$-Higgs bundle is a parabolic Higgs bundle $(\mathcal{E}, \Phi)$ such that:*

- $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ *where* $rank(\mathcal{E}_1) = 1$ *and* $rank(\mathcal{E}_2) = n$
- *With respect to the above splitting of $\mathcal{E}$,*

$$\Phi = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}.$$

where $b : \mathcal{E}_2 \to \mathcal{E}_1 \otimes \mathcal{K}(D)$ and $c : \mathcal{E}_1 \to \mathcal{E}_2 \otimes \mathcal{K}(D)$.

These Higgs bundles are exactly the ones corresponding to representations $\pi_1(X - D) \to PU(n,1)$ [33], [34], [6], [14].

We are interested in constructing (irreducible) representations of $\pi_1(\mathbb{P}^1 - \{p_1, \ldots, p_k\})$ with loxodromic monodromies. From the above, we can determine if a given (stable) Higgs bundle has loxodromic monodromies by looking at its residues at the marked points.

Definition 3.1.2. *Let $(\mathcal{E}, \Phi)$ be a parabolic $PU(n,1)$-Higgs bundle. We say Φ is of loxodromic type if its residue at each marked point p_j has exactly two eigenvalues with nonzero imaginary parts.*

For the rest of the paper, we will only consider parabolic $PU(n,1)$-Higgs bundles of loxodromic type. This assumption has important implications regarding which types of invariant subbundles can occur.

3.2 Invariant subbundles

Let $(\mathcal{E}, \Phi)$ be a parabolic $PU(n,1)$-Higgs bundle of loxodromic type, where $\mathcal{E}$ and Φ the decomposition as in Definition 3.1.1. In this section, we determine all invariant subbundles of Φ. These correspond to the eigenspaces of Φ. The first corresponds to the 0 eigenvectors, i.e. the kernel of Φ:

3.2.1 Kernel of the Higgs Field

Lemma 3.2.1. *The kernel of Φ corresponds exactly to the kernel of $b : \mathcal{E}_2 \to \mathcal{E}_1 \otimes \mathcal{K}(D)$.*

Proof. Since Φ is of loxodromic type, $c : \mathcal{E}_1 \to \mathcal{E}_2 \otimes \mathcal{K}(D)$ is necessarily nonzero. Since $\mathcal{E}_1$ is a line bundle, c is generically injective. This implies $\ker(\Phi) = \ker(b)$. $\square$

Note that since $rank(\mathcal{E}_2) = n$ and $\mathcal{E}_1 = 1$, $rank(\ker(\Phi)) = n - 1$. As such, when $n > 2$, any subbundle of $\ker(\Phi)$ is also necessarily invariant. This will not be an issue, since for our main construction we will assume $\ker(\Phi)$ is stable, when guaranteed by [4]. In chapter 5, we will explore the case where $\ker(\Phi)$ is actually unstable.

3.2.2 Invariant rank-2 subbundles

There is one other important subbundle, corresponding to the nonzero eigenspace of the Higgs field. If we write the Higgs bundle $(\mathcal{E}, \Phi)$ as

$$\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$$
$$\Phi = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}.$$

then $\mathcal{F} = \mathcal{E}_1 \oplus \mathcal{L}$, where $\mathcal{L}$ is the saturation of $Im(c)$. In order to check the stability of $\mathcal{E}$, we need to compute the degree of $\mathcal{F}$. Since $\mathcal{E}_1$ is given, we only need to compute the degree of $\mathcal{L}$. This depends on the vanishing of the map c:

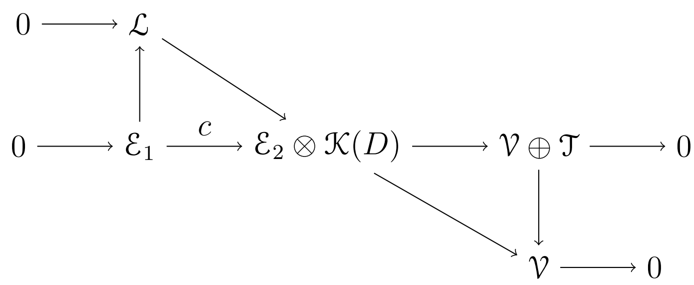
Lemma 3.2.2. *$Im(c) = \mathcal{E}_1((c)) \otimes \mathcal{K}(D)^*$, where (c) is the vanishing divisor of the section c. In particular, when c is non-vanishing, $Im(c) = \mathcal{E}_1 \otimes \mathcal{K}(D)^*$*

Proof. $\mathcal{E}_1$ and c fit in the following exact sequence of coherent sheaves:

$$0 \longrightarrow \mathcal{E}_1 \xrightarrow{\ c\ } \mathcal{E}_2 \otimes \mathcal{K}(D) \longrightarrow \mathcal{Q} \longrightarrow 0$$

If c vanishes, then the quotient sheaf $\mathcal{Q}$ is not locally free. On a Riemann surface, the stalk of the structure sheaf is a PID. As a consequence, any coherent sheaf splits as a direct sum of a locally free sheaf and a torsion sheaf. Therefore we can write $\mathcal{Q} = \mathcal{V} \oplus \mathcal{T}$, where $\mathcal{V}$ is locally free and $\mathcal{T}$ is a torsion sheaf supported on the vanishing divisor (c) of c.

We want to compute the saturation $\mathcal{L}$ of the image of $\mathcal{E}_1$ in $\mathcal{E}_2 \otimes \mathcal{K}(D)$ of $\mathcal{E}_1$. The projection of $\mathcal{V} \oplus \mathcal{T} \to \mathcal{V}$ onto $\mathcal{V}$ induces a map $\mathcal{E}_2 \otimes \mathcal{K}(D)$. $\mathcal{L}$ is the kernel of this map, and sits inside the following commutative diagram:



Computing determinants, we see that

$$\det(\mathcal{E}_2 \otimes \mathcal{K}(D)) = \det(\mathcal{E}) \otimes \det(\mathcal{V}) \otimes \det(\mathcal{T})$$
$$= \det(\mathcal{L}) \otimes \det(\mathcal{V})$$

It follows that $\mathcal{L} = \mathcal{E}_1 \otimes \mathcal{O}((c))$. Tensoring with $\mathcal{K}(D)'$, we conclude that $Im(c) = \mathcal{L} \otimes \mathcal{K}(D)^* = \mathcal{E}_1 \otimes \mathcal{O}((c)) \otimes \mathcal{K}(D)^*$. □

3.3 Elementary reductions

We can reduce the problem of constructing $U(n,1)$-Higgs bundles into constructing a parabolic $U(1,1)$-Higgs bundle and a parabolic $U(n-1)$ bundle. The $U(1,1)$ piece determines the nonzero eigenvalues of the Higgs field and the $U(n-1)$ piece determines the kernel. The $U(n,1)$-Higgs bundle is constructed by taking an appropriate extension.

Let $(\mathcal{E}, \Phi)$ be a $U(n,1)$-Higgs bundle of loxodromic type:

$$\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$$

$$\Phi = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$$

$$b, c \neq 0$$

Then the invariant subbundle corresponding to the kernel of Φ is the kernel of b: $\mathcal{S} = \ker(b) \subset \mathcal{E}_2$. This fits in an exact sequence:

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{Q} \longrightarrow 0$$

We can construct a $U(1,1)$-Higgs bundle $(\widetilde{\mathcal{E}} = \mathcal{E}_1 \oplus \mathcal{Q}, \widetilde{\Phi})$, where $\widetilde{\Phi}$ has the following form:

$$\widetilde{\Phi} = \begin{bmatrix} 0 & \widetilde{b} \\ \widetilde{c} & 0 \end{bmatrix}. \tag{3.1}$$

The $\widetilde{b}$ and $\widetilde{c}$ determining $\widetilde{\Phi}$ are obtained in the following way. First $\widetilde{c}$ is determined by the following diagram:

$$\begin{array}{ccccccccc} & & & & \mathcal{E}_1 \otimes \mathcal{K}(D)^* & & & & \\ & & & & \downarrow c & \searrow \widetilde{c} & & & \\ 0 & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{E}_2 & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \end{array}$$

Put slightly differently, $\widetilde{c} \in H^0(\mathcal{E}_1^* \otimes \mathcal{Q} \otimes \mathcal{K}(D)$ is the image of $c \in H^0(\mathcal{E}_1^* \otimes \mathcal{E}_2 \otimes \mathcal{K}(D)$ in the following exact sequence in cohomology:

$$0 \longrightarrow H^0(\mathcal{E}_1^* \otimes \mathcal{S} \otimes \mathcal{K}(D)) \longrightarrow H^0(\mathcal{E}_1^* \otimes \mathcal{E}_2 \otimes \mathcal{K}(D)) \longrightarrow H^0(\mathcal{E}_1^* \otimes \mathcal{Q} \otimes \mathcal{K}(D))$$

$\widetilde{b}$ is slightly harder to obtain. For this, we have the following exact sequence in cohomology:

$$0 \longrightarrow H^0(\mathcal{Q}^* \otimes \mathcal{E}_1 \otimes \mathcal{K}(D)) \longrightarrow H^0(\mathcal{E}_2^* \otimes \mathcal{E}_1 \otimes \mathcal{K}(D)) \longrightarrow H^0(\mathcal{S}^* \otimes \mathcal{E}_1 \otimes \mathcal{K}(D))$$

Recall that $b \in H^0(\mathcal{E}_2^* \otimes \mathcal{E}_1 \otimes \mathcal{K}(D))$. $\widetilde{b}$ should be the preimage of b in the group $H^0(\mathcal{Q}^* \otimes \mathcal{E}_1 \otimes \mathcal{K}(D))$. This might not always be possible. Consider the image of b in $H^0(\mathcal{S}^* \otimes \mathcal{E}_1 \otimes \mathcal{K}(D))$. This is the morphism $\mathcal{L} \to \mathcal{E}_1 \otimes \mathcal{K}(D)$ obtained by restricting b to the subbundle $\mathcal{S}$. However, recall that $\mathcal{S}$ was defined to be the kernel of b. Therefore b restricts to the zero map on $\mathcal{S}$, and hence its image in $H^0(\mathcal{S}^* \otimes \mathcal{E}_1 \otimes \mathcal{K}(D))$ is 0. Thus b is the image of an element, which we name $\widetilde{b}$, in $H^0(\mathcal{Q}^* \otimes \mathcal{E}_1 \otimes \mathcal{K}(D))$. This proves the following lemma:

Lemma 3.3.1. *Let* $(\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2, \Phi = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix})$ *be a parabolic* $U(n,1)$*-Higgs bundle of loxodromic type. Then quotienting* $\mathcal{E}_2$ *by the kernel* $\mathcal{S}$ *of* b *decomposes* $(\mathcal{E}, \Phi)$ *into a parabolic* $U(1,1)$*-Higgs bundle of loxodromic type and a parabolic* $U(n-1)$ *bundle.*

Now we wish to invert the above process. Given a parabolic $U(n-1)$ bundle $\mathcal{S}$ and a parabolic $U(1,1)$-Higgs bundle $(\mathcal{E}, \Phi)$, how can we construct a parabolic $U(n,1)$-Higgs bundle $(\widetilde{E}, \widetilde{\phi})$? First, assume $(\mathcal{E}, \Phi)$ decompose as follows:

$$\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{Q}$$

$$\Phi = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$$

$$b, c \neq 0$$

Together with $\mathcal{S}$, we will show how to construct a parabolic $U(n,1)$-Higgs bundle $(\widetilde{E}, \widetilde{\phi})$ of the following form:

$$\widetilde{\mathcal{E}} = \mathcal{E}_1 \oplus \mathcal{E}_2$$
$$\widetilde{\Phi} = \begin{bmatrix} 0 & \widetilde{b} \\ \widetilde{c} & 0 \end{bmatrix}$$
$$\widetilde{b}, \widetilde{c} \neq 0$$

$\mathcal{E}_2$ is given by taking some extension of $\mathcal{S}$ by $\mathcal{Q}$:

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{Q} \longrightarrow 0$$

For now, we do not fix the extension class $\beta \in H^1(\mathcal{Q}^* \otimes \mathcal{L})$. The existence and uniqueness of the extension $\widetilde{b}$ of b follows immediately from the following lemma:

Lemma 3.3.2. *Given an extension $\mathcal{E}$ of a bundle $\mathcal{Q}$ by a bundle $\mathcal{S}$ and a map of bundles $b : \mathcal{Q} \to \mathcal{F}$, there exists a unqiue extension $\widetilde{b} : \mathcal{E} \to \mathcal{F}$ of b.*

Proof. In the long exact sequence of cohomology, we have in particular

$$0 \longrightarrow H^0(\mathcal{Q}^* \otimes \mathcal{F}) \xrightarrow{\alpha} H^0(\mathcal{E}^* \otimes \mathcal{F})$$

The (necessarily unique) extension $\widetilde{b}$ of b is the image of b under the injective map α.

□

On the other hand, c is slightly more difficult to obtain. Consider the following long exact sequence:

$$H^0(\mathcal{E}_1^* \otimes \mathcal{E}_2 \otimes \mathcal{K}(D)) \longrightarrow H^0(\mathcal{E}_1^* \otimes \mathcal{Q} \otimes \mathcal{K}(D)) \longrightarrow H^1(\mathcal{E}_1^* \otimes \mathcal{S} \otimes \mathcal{K}(D))$$

The section $c \in H^0(\mathcal{E}_1^* \otimes \mathcal{Q} \otimes \mathcal{K}(D))$, and we want to pull it back to a section $\widetilde{c} \in H^0(\mathcal{E}_1^* \otimes \mathcal{E}_2 \otimes \mathcal{K}(D))$. This is only possible if the image of c in $H^1(\mathcal{E}_1^* \otimes \mathcal{L} \otimes \mathcal{K}(D))$ is 0 (see Lemma 3.4.1).

To further complicate matters, just finding an extension for which c lifts is not enough. By Lemma 3.2.2, the vanishing of the lift $\widetilde{c}$ has a direct effect on the stability of $\widetilde{\mathcal{E}}$. As such, we must find an extension for which c lifts to a nonvanishing element $\widetilde{c} \in H^0(\mathcal{E}_1^* \otimes \mathcal{E}_2 \otimes \mathcal{K}(D))$. We'll tackle this issue in the next section.

3.4 Useful lemmas about extensions

To construct a $U(n,1)$-Higgs bundle out of a $U(n-1)$ bundle and a $U(1,1)$-Higgs bundle, we need to identify for which extensions $H^1(\mathcal{Q}^* \otimes \mathcal{S})$ the map $\widetilde{c}$ can be lifted. The following lemma due to Narasimhan and Ramanan in [23] does just that.

Lemma 3.4.1. *(Narasimhan-Ramanan) Let* $0 \to \mathcal{S} \to \mathcal{E} \to \mathcal{Q} \to 0$ *be a short exact sequence of holomorphic vectors bundles, and* $c : \mathcal{F} \to \mathcal{Q}$ *a map. Then the space of extensions of* $\mathcal{S}$ *by* $\mathcal{Q}$ *for which c lifts to a map* $\mathcal{F} \to \mathcal{E}$ *corresponds to the kernel of the map* $H^1(\mathcal{Q} * \otimes \mathcal{S}) \to H^1(\mathcal{F}^* \otimes \mathcal{S})$ *induced by c.*

Proof. Precomposition by c induces the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{Q}^* \otimes \mathcal{S} & \longrightarrow & \mathcal{Q}^* \otimes \mathcal{E} & \longrightarrow & \mathcal{Q}^* \otimes \mathcal{Q} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}^* \otimes \mathcal{S} & \longrightarrow & \mathcal{F}^* \otimes \mathcal{E} & \longrightarrow & \mathcal{F}^* \otimes \mathcal{Q} & \longrightarrow & 0 \end{array}$$

Taking cohomology, we get the following:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^0(\mathcal{Q}^* \otimes \mathcal{Q}) & \longrightarrow & H^1(\mathcal{Q}^* \otimes \mathcal{S}) & \longrightarrow & H^1(\mathcal{Q}^* \otimes \mathcal{E}) & \longrightarrow & \cdots \\ & & \downarrow c_* & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H^0(\mathcal{F}^* \otimes \mathcal{Q}) & \xrightarrow{\delta} & H^1(\mathcal{F}^* \otimes \mathcal{S}) & \longrightarrow & H^1(\mathcal{F}^* \otimes \mathcal{E}) & \longrightarrow & \cdots \end{array}$$

A necessary and sufficient condition for a lift of c to exist is that δc be 0. But c is the image of the identity under c_*. By commutativity of the diagram, $\delta(c)$ is the image of the extension class $\beta \in H^1(\mathcal{Q}^* \otimes \mathcal{S})$. Thus c can be lifted if and only if β is in the kernel of $c_* : H^1(\mathcal{Q}^* \otimes \mathcal{S}) \to H^1(\mathcal{F}^* \otimes \mathcal{Q})$. □

Lemma 3.4.2. *If $\mathcal{F}$ and $\mathcal{Q}$ above are line bundles and $c : \mathcal{F} \to \mathcal{Q}$ is a nonzero morphism, then the map $H^1(\mathcal{Q}^* \otimes \mathcal{S}) \to H^1(\mathcal{F}^* \otimes \mathcal{S})$ is surjective.*

Proof. $\mathcal{F}$ and $\mathcal{Q}$ fit into the following short exact sequence, where T is a torsion sheaf:

$$0 \longrightarrow \mathcal{F} \xrightarrow{c} \mathcal{Q} \longrightarrow \mathcal{T} \longrightarrow 0$$

Dualizing and tensoring with $\mathcal{S}$, we have

$$0 \longrightarrow \mathcal{Q}^* \otimes \mathcal{S} \xrightarrow{c_*} \mathcal{F}^* \otimes \mathcal{S} \longrightarrow \mathcal{T}' \otimes \mathcal{S} \longrightarrow 0$$

where $\mathcal{T}'$, and hence $\mathcal{T}' \otimes \mathcal{S}$, is torsion. In the long exact sequence, this induces

$$H^1(\mathcal{Q}^* \otimes \mathcal{S}) \xrightarrow{c_*} H^1(\mathcal{F}^* \otimes \mathcal{S}) \longrightarrow H^1(\mathcal{T}' \otimes \mathcal{S}) \longrightarrow 0$$

$\mathcal{T}' \otimes \mathcal{S}$ is torsion, and thus $H^1(\mathcal{T}' \otimes \mathcal{S}) = 0$. Therefore, the map $c_* : H^1(\mathcal{Q}^* \otimes \mathcal{S}) \to H^1(\mathcal{F}^* \otimes \mathcal{S})$ induced by c is surjective.

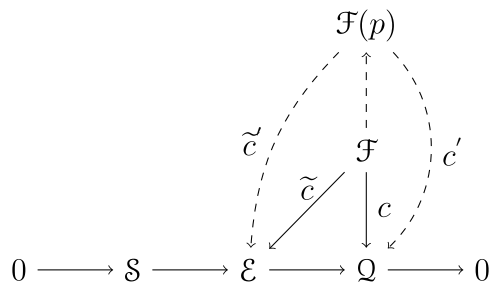
□

Lemma 3.4.3. *Let $p \in \mathbb{P}^1$ be a point at which $c \in H^0(\mathcal{F}^* \otimes \mathcal{Q})$ vanishes. Then*

- *The extensions* $\mathcal{E}$ *of* $\mathcal{Q}$ *by* $\mathcal{S}$ *for which* c *lifts to a* $\widetilde{c} \in H^0(\mathcal{F}^* \otimes \mathcal{E})$ *which vanishes at* p *are in the kernel of the map* $H^1(\mathcal{Q} * \otimes \mathcal{S}) \to H^1(\mathcal{F}(p)^* \otimes \mathcal{S})$ *induced by* c*.*

- *There is a natural inclusion of* $\ker(H^1(\mathcal{Q} * \otimes \mathcal{S}) \to H^1(\mathcal{F}(p)^* \otimes \mathcal{S}))$ *into* $\ker(H^1(\mathcal{Q} * \otimes \mathcal{S}) \to H^1(\mathcal{F}^* \otimes \mathcal{S}))$.

Proof. Suppose c lifts to a map $\widetilde{c}$ which vanishes at p. Then c and $\widetilde{c}$ factor through the inclusion $\mathcal{F} \to \mathcal{F}(p)$, giving the following commutative diagram:



Thus a lift $\widetilde{c}$ of c which vanishes at p is equivalent to a lift $\widetilde{c}'$ of c'. By Lemma 3.4.1, these extensions are contained in $\ker(H^1(\mathcal{Q} * \otimes \mathcal{S}) \to H^1(\mathcal{F}(p)^* \otimes \mathcal{S}))$.

Next we show that $\ker(H^1(\mathcal{Q} * \otimes \mathcal{S}) \to H^1(\mathcal{F}(p)^* \otimes \mathcal{S}))$ is naturally included in $\ker(H^1(\mathcal{Q} * \otimes \mathcal{S}) \to H^1(\mathcal{F}^* \otimes \mathcal{S}))$. Consider the following short exact sequence:

$$0 \longrightarrow \mathcal{F} \xrightarrow{\ i\ } \mathcal{F}(p) \longrightarrow \mathcal{O}|_p \longrightarrow 0$$

where $\mathcal{F}$ is included in $\mathcal{F}(p)$ as the subsheaf of sections vanishing at p. Dualizing and tensoring with $\mathcal{S}$, we have

$$0 \longrightarrow \mathcal{F}(p)^* \otimes \mathcal{S} \xrightarrow{\ i_*\ } \mathcal{F}^* \otimes \mathcal{S} \longrightarrow \mathcal{S}|_p \longrightarrow 0$$

Taking cohomology, we have by Lemma 3.4.2 the following commutative diagram:

$$\begin{array}{ccccc} H^1(\mathcal{F}(p)^* \otimes \mathcal{S}) & \xrightarrow{i_*} & H^1(\mathcal{F}^* \otimes \mathcal{S}) & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \\ H^1(\mathcal{Q}^* \otimes \mathcal{S}) & \xrightarrow{=} & H^1(\mathcal{Q}^* \otimes \mathcal{S}) & & \end{array}$$

The containment of $\ker(H^1(\mathcal{Q} * \otimes \mathcal{S}) \to H^1(\mathcal{F}(p)^* \otimes \mathcal{S}))$ into $\ker(H^1(\mathcal{Q} * \otimes \mathcal{S}) \to H^1(\mathcal{F}^* \otimes \mathcal{S}))$ follows from a simple diagram chase, proving the lemma. $\square$

When $\ker(H^1(\mathcal{Q}^* \otimes \mathcal{S}) \to H^1(\mathcal{F}(p)^* \otimes (S)))$ is strictly contained in $\ker(H^1(\mathcal{Q}^* \otimes \mathcal{S}) \to H^1(\mathcal{F}^* \otimes \mathcal{S}))$, then we can find extensions $\mathcal{E}$ of $\mathcal{Q}$ by $\mathcal{S}$ such that the map $c : \mathcal{F} \to \mathcal{Q}$ has a lift $\widetilde{c} : \mathcal{F} \to \mathcal{E}$ which is nonvanishing at p. If the containment is not strict, the existence of a nonvanishing lift is measured by $H^0(\mathcal{F}^* \otimes \mathcal{S})$. By adding an element in the image of $H^0(\mathcal{F}^* \otimes \mathcal{S})$ to the lift $\widetilde{c}$, we can find new lift which does not vanish.

Proposition 3.4.4. *Given a vector bundle $\mathcal{S}$, line bundles $\mathcal{Q}$ and $\mathcal{F}$, and a nonzero morphism $c : \mathcal{F} \to \mathcal{Q}$, there exists an extension of $\mathcal{Q}$ by $\mathcal{S}$ such that c has a lift $\widetilde{c}$ that is nonvanishing.*

Proof. The lift $\widetilde{c}$ of c can only vanish along the divisor of c. First, we'll pick an extension. We have the following commutative diagram, with exact top row:

$$\begin{array}{ccccccccc} H^0(\mathcal{F}^* \otimes \mathcal{S}) & \xrightarrow{\gamma} & \mathcal{S}|_p & \xrightarrow{\delta} & H^1(\mathcal{F}(p)^* \otimes \mathcal{S}) & \xrightarrow{i_*} & H^1(\mathcal{F}^* \otimes \mathcal{S}) & \longrightarrow & 0 \\ & & & & {\scriptstyle c'_*}\uparrow & & \uparrow{\scriptstyle c_*} & & \\ & & & & H^1(\mathcal{Q}^* \otimes \mathcal{S}) & \xrightarrow{id} & H^1(\mathcal{Q}^* \otimes \mathcal{S}) & & \end{array}$$

Note that the maps γ and δ only depend on $\mathcal{F}$ and $\mathcal{S}$, and not the extension class $\beta \in H^1(\mathcal{Q}^* \otimes \mathcal{S})$. Let p be a point in the divisor of c. Then there are two cases:

Case 1: The map γ is not surjective

If γ is not surjective, then there is an element $\alpha \in \mathcal{S}_p$ whose image $\delta(\alpha) \in H^1(\mathcal{F}(p)^* \otimes \mathcal{S})$ is not zero. By exactness, $i_*(\delta(\alpha)) = 0$. On the other hand, c'_* is surjective, and therefore there is a $\beta \in H^1(\mathcal{Q}^* \otimes \mathcal{S})$ such that $c'_*(\beta) = \delta(\alpha)$. By commutativity, $c_*(\beta) = i_*(\delta(\alpha)) = 0$. By Lemma 3.4.1, β is an extension for which c lifts to a an element $\widetilde{c}$. On the other hand, $c'_*(\beta)$ is nonzero, so by Lemma 3.4.3 the lift $\widetilde{c}$ is nonvanishing at p.

As such, $\ker(c'_*)$ is strictly contained inside $\ker(c_*)$. In particular, there is a nonempty open subset $U_p \subset \ker(c_*)$ of extensions for which c lifts to a $\widetilde{c}$, which does not vanish at p.

Case 2: γ is surjective

If γ is surjective, then δ is identically 0, and i_* is an isomorphism. In this case, $\ker(c'_*)$ and $\ker(c_*)$ are exactly equal. Thus we expect any lift $\widetilde{c}$ to vanish at p, regardless of which extension we choose. However, $H^0(\mathcal{F}^* \otimes \mathcal{S})$ is necessarily nonzero, and hence a lift $\widetilde{c}$ is unique only up to addition with an element in $H^0(\mathcal{F}^* \otimes \mathcal{S})$. Since γ is surjective, $V_p = \gamma^{-1}(\mathcal{S}|_p - \{0\})$ is a nonempty open subset of $H^0(\mathcal{F}^* \otimes \mathcal{S})$. The sum of $\widetilde{c}$ plus any element in V_p is necessarily nonvanishing at p.

Finishing the proof of Proposition 3.4.4

Now, let $\{p_1, \ldots, p_n\}$ be the set of points where c vanishes. Further, assume (up to relabelling) that $\{p_1, \ldots, p_k\}$ are the points for which a in the above diagram is not surjective. Then $U = \cap_1^k U_{p_i}$ is nonempty (since it is the complement of a finite union of positive codimension closed subsets of $\ker(c_*)$), and consists of extensions for which c lifts to an element $\widetilde{c}$ is nonvanishing at p_i, for $1 \leq i \leq k$. Additionally, $V = \cap_{k+1}^n V_{p_i}$ is nonempty an consists of sections in $H^0(\mathcal{F}^* \otimes \mathcal{S})$ which are non vanishing at p_i, for $k+1 \leq i \leq n$. Picking an element $\gamma \in V$, $\gamma + \widetilde{c}$ is a lift of c which does not vanish at any p_i, completing the proof.

□

Putting everything together, we have the following corollary :

Corollary 3.4.5. *Let* $(\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{Q}, \Phi = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix})$ *be a parabolic* $U(1,1)$*-Higgs bundle of loxodromic type and let* $\mathcal{S}$ *be a rank* $n-1$ *parabolic vector bundle. Then there is an extension* $\mathcal{E}_2$ *of* $\mathcal{Q}$ *by* $\mathcal{S}$ *such that* c *lifts to a* $\widetilde{c} \in H^0(\mathcal{E}_1^* \otimes \mathcal{E}_2 \otimes \mathcal{K}(D))$ *which is non-vanishing.*

Chapter 4: Deligne-Simpson for $PU(n, 1)$

In Chapter 3, we described the structure of a parabolic Higgs bundle corresponding to a representation into $PU(p, q)$. In the case of $PU(n, 1)$, we described the additional structure of the Higgs field required to guarantee the monodromies around the punctures are loxodromic. For such a Higgs field, there are two important invariant subbundles, the rank $n - 1$ kernel and a rank 2 subbundle corresponding to the non-zero eigenspace. In particular, the degree of the rank 2 subbundle depends on the vanishing of the map $c : \mathcal{E}_1 \to \mathcal{E}_2$.

Finally, we showed that a parabolic $PU(n, 1)$-Higgs bundle can be decomposed into a parabolic $U(1, 1)$-Higgs bundle of loxodromic type and a parabolic $U(n - 1)$ bundle, corresponding to the kernel of Φ. The main result of chapter 3, Proposition 3.4.4, says that we can put pieces back together with c nonvanishing, thus minimizing the degree of the rank 2 subbundle.

In this chapter, we prove our Main Theorem with the aid of the above. The construction of the kernel of Φ comes from 2.4.1, so we are left to construct the $U(1, 1)$ piece. First, we show that for $k \geq 3$ fixed loxodromic conjugacy classes in $U(1, 1)$, there is a corresponding stable parabolic $U(1, 1)$-Higgs bundles. As a corollary, solutions to the Deligne-Simpson problem for loxodromic conjugacy classes in $U(1,1)$ always exist. Second, we combine these two pieces to prove our Main Theorem regarding the existence of stable parabolic $U(n,1)$-Higgs bundles.

4.1 Constructing stable parabolic $U(1,1)$-Higgs bundles

Let C_1, C_2, C_3 be fixed loxodromic conjugacy classes in $U(1,1)$. The conjugacy class C_j can be represented by matrix of the form

$$A_j = \begin{bmatrix} exp(2\pi i\beta_j)r_j & 0 \\ 0 & exp(2\pi i\beta_j)r_j^{-1} \end{bmatrix}. \tag{4.1}$$

By Simpson's Correspondence and the discussion in section 2.5.3, constructing a solution to the Deligne-Simpson problem for C_1, C_2, C_3 is equivalent to constructing a stable parabolic $U(1,1)$-Higgs bundles (not necessarily of degree 0) satisfying:

- the filtration of $\mathcal{E}_{p_j}$ has one rank 2 jump at β_j;
- the residue of Φ acts on $Gr_{\beta_j}(E_{p_j})$ with eigenvalues $\pm i\log(r_j)/4\pi$.

We should say something about why the eigenvalues of the residue should be $\pm i\log(r_j)/4\pi$. By Simpson's Correspondence, the imaginary parts of eigenvalues of the residue of Φ at the marked points determine the norms of the eigenvalues of the monodromy around the marked points. In particular, if the eigenvalue of $res(\Phi) = b + ci$, then the corresponding eigenvalue of the monodromy has norm $exp(-4\pi c)$. Since we are interested in constructing representations with loxodromic monodromies, then it follows from equation 4.1 that the eigenvalues of the monodromy have norms r, r^{-1}. As such, the (imaginary parts of) the eigenvalues of $res(\Phi)$ should be $\pm i\log(r_j)/4\pi$.

Since we are interested in representations into $PU(n,1)$, we don't require that the Higgs bundle have parabolic degree 0. If (E,Φ) has nonzero degree, then after tensoring with an appropriate line bundle (with trivial parabolic structure), we can assume $-2 < pdeg(E) \leq 0$. The trick (following [28]) is to then add an additional marked point p, distinct from $\{p_1, p_2, p_3\}$. The filtration at this point is given by a single rank 2 jump of weight $\alpha = \frac{-pdeg(E)}{2}$ at p. This gives a new parabolic bundle, which is stable iff E is, of parabolic degree 0. This new bundle gives a representation of $\pi_1(\mathbb{P}^1 - \{p_1, p_2, p_3, p\})$ i.e. matrices $A_j \in C_j$ such that $A_1 A_2 A_3 M = 1$, where M is the monodromy around the new point p. It is not hard to see that M must be the scalar matrix $M = exp(\frac{-2}{3}\pi i pdeg(E))I$, and so we end up with a representation $\pi_1(\mathbb{P}^1 - \{p_1, p_2, p_3\}) \to PU(1,1)$.

In this section, we prove the following:

Proposition 4.1.1. *Let C_1, C_2, and C_3 be loxodromic conjugacy classes in $U(1,1)$, represented by the matrix A_j of the form given in equation* (4.1) *. Then there exists a stable parabolic $U(1,1)$-Higgs bundle with three marked points $D = p_1 + p_2 + p_3$ with the following properties:*

- *The parabolic structure at p_j consists of a single rank-2 jump with weight β_j*
- *The residue of Φ at p_j has eigenvalues $\pm i \log(r_j)/4\pi$.*

On the representation side, we have the following easy corollary:

Corollary 4.1.2. *Given loxodromic conjugacy classes C_1, C_2, and C_3 in $PU(1,1)$, there is a matrix $A_j \in C_j$ such that $A_1 A_2 A_3 = I$.*

Let $(\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2, \Phi)$ be a stable $U(1,1)$-Higgs bundle. Possibly tensoring with a line bundle (with trivial parabolic structure), we may assume $\mathcal{E}_1 = \mathcal{O}$ and $\mathcal{E}_2 = \mathcal{O}(d)$. With respect to this splitting of $\mathcal{E}$, we have

$$\Phi = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$$

where $b \in H^0(\mathcal{O}(d)^* \otimes \mathcal{O} \otimes \mathcal{K}(D))) \simeq H^0(\mathcal{O}(1-d))$ and $c \in H^0(\mathcal{O}^* \otimes \mathcal{O}(d) \otimes \mathcal{K}(D))) \simeq H^0(\mathcal{O}(1+d))$. Since we require Φ to be of loxodromic type, each of b and c must be nonzero at each p_j, and therefore $-1 \leq d \leq 1$.

4.1.1 Constructing the Higgs field

Here we show how to construct a $U(1,1)$-Higgs field with appropriate residues. From the above, we have 3 separate cases to consider: given $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(d)$, we have the cases $d = -1, 0, 1$.

We will build the components b and c of the Higgs field out of global sections of $H^0(\mathcal{E}_2^* \otimes \mathcal{E}_1 \otimes \mathcal{K}(D))$ and $H^0(\mathcal{E}_1^* \otimes \mathcal{E}_2 \otimes \mathcal{K}(D))$. In order to verify our new Higgs field has residues with appropriate eigenvalues, we will need to fix a choice of a local frame and a choice of coordinates at every marked point p_j. The following lemma says the eigenvalues of $res(\Phi)_{p_j}$ do not depend on these choices:

Lemma 4.1.3. *The eigenvalues of the residue of the Higgs field does not depend on the choice of local frames and local coordinates.*

Proof. Changing frames or coordinates corresponds to conjugating the Higgs field by an invertible matrix, which preserves the eigenvalues. □

By Simpson's Correspondence and the discussion above, we want the residue of Φ to have eigenvalues with imaginary parts $\pm i\log(r_j)/4\pi$. For a $PU(n,1)$-Higgs field, the residues have the form:

$$\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$$

The characteristic polynomial of such a matrix is $p(\lambda) = \lambda^2 - bc$, and therefore the eigenvalues are $\pm\sqrt{bc}$. As such we'll construct a Φ whose residue at p_j has determinant $-\left(\log(r_j)/4\pi\right)^2$.

Case 1: $d = \pm 1$

We'll focus on the case when d=1. The other case will follow a similar argument. We have

$$b \in H^0(\mathcal{O}(1)^* \otimes \mathcal{K}(D)) = H^0(\mathcal{O})$$

$$c \in H^0(\mathcal{O}^* \otimes \mathcal{O}(1) \otimes \mathcal{K}(D)) = H^0(\mathcal{O}(2)).$$

Let $\{s_1, s_2, s_3\}$ be a basis of $H^0(\mathcal{O}(2))$ such that s_j is nonzero at p_j and zero at the remaining marked points. Let $b \in H^0(\mathcal{O})$ be any fixed nonzero constant and $c = xs_1 + ys_2 + zs_3 \in H^0(\mathcal{O}(2)$. Fixing local frames and local coordinates at each p_j, denote the residue of s_j by r_j. Then we want to solve the following linear equations in x, y, z:

$$br_1x = -\gamma_1^2$$

$$br_2y = -\gamma_2^2$$

$$br_3z = -\gamma_3^2$$

where $\gamma_j = \log(r_j)/4\pi$. This is, of course, trivial. Using these to define Φ gives us a Higgs field with the required residues.

Note that if we write $E_2 = \mathcal{O}(-1)$, we can also produce a suitable Φ by mimicking the above construction with a and c switched.

Case 2: $d = 0$

Now we have

$$b \in H^0(\mathcal{O}^* \otimes \mathcal{K}(D)) = H^0(\mathcal{O}(1))$$

$$c \in H^0(\mathcal{O}^* \otimes \mathcal{K}(D)) = H^0(\mathcal{O}(1)).$$

Let $\{s_1, s_2\}$ be a basis for $H^0(\mathcal{O}(1))$, where s_1 vanishes at p_1 and s_2 vanishes at p_2. Then setting $b = ws_1 + xs_2$ and $c = ys_1 + zs_2$. . Fixing local frames and local coordinates at each p_j, label the residues of s_1 at p_2, p_3 by r_{12} and r_{13} and the residues of s_2 at p_1, p_3 by r_{21} and r_{23}. All the residues are necessarily nonzero. Then we want to solve the following quadratic equations in w, x, y, and z:

$$yzr_{21}^2 = -\gamma_1^2$$

$$wxr_{12}^2 = \gamma_2^2$$

$$wxr_{13}^2 + (wz + xy)r_{13}r_{23} + yzr_{23}^2 = \gamma_3^2.$$

Note that, since the γ_i's and the r_{ij}'s are nonzero, w, x, y, and z, must also be nonzero. For simplicity, write $a = \frac{-\gamma_1^2}{r_{21}^2}$ and $b = \frac{-\gamma_2^2}{r_{12}^2}$. Then $w = \frac{b}{x}$ and $y = \frac{a}{z}$. Substituting this into the third equation, we want to solve:

$$br_{13}^2 + (b\frac{z}{x} + a\frac{x}{z})r_{13}r_{23} + ar_{23}^2 = -\gamma_3^2$$

Finally, substituting $u = \frac{z}{x}$ and multiplying by u, we need to solve the following quadratic equation in u:

$$br_{13}r_{23}u^2 + (\gamma_3^2 + br_{13}^2 + ar_{23}^2)u + ar_{13}r_{23} = 0.$$

Of course, by the quadratic formula, this equation has at least one solution. Since $qr_{13}r_{23}$ is nonzero, the solution u is nonzero. Writing $z = ux$ and fixing a nonzero x, we can find w and y solving the given equations.

4.1.2 Stability

To show the Higgs bundles we constructed above are stable, we must compute the invariant subbundles and show the slope condition is satisfied. Since a and c are both nonzero, an invariant line subbundle $\mathcal{L}$ must be a subsheaf of both summands of $\mathcal{E}$. For $d \neq 0$, this gives

$$p\mu(\mathcal{L}) = min(0, d) + \sum \beta_j$$
$$p\mu(\mathcal{E}) = \frac{d}{2} + \sum \beta_j$$

As such , we have

$$p\mu(\mathcal{E}) - p\mu(\mathcal{L}) = \frac{d}{2} - min(0, d) = \frac{1}{2} > 0$$

The $d = 0$ case is slightly trickier. The issue is that $\mathcal{O}$ could be invariant, in which case the bundle is semistable. This happens when the only zero of b and c agree, and so the Higgs bundles we've constructed are generically stable.

4.1.3 Existence of stable $U(1,1)$-Higgs bundles

The above construction for $d = 1$ generalizes to an arbitrary number of marked points/conjugacy classes. The general version of Proposition 4.1.1 is the following:

Theorem 4.1.4. *Let $C_1, \ldots, C_k$ $(k \geq 3)$ be fixed loxodromic conjugacy classes in $U(1,1)$, represented by a matrix A_j of the form given in equation 4.1. Then there exists a stable parabolic $U(1,1)$-Higgs bundle with three marked points $D = p_1 + \ldots + p_k$ with the following properties:*

- *The parabolic structure at p_j consists of a single rank-2 jump with weight β_j*
- *The residue of Φ at p_j has eigenvalues $\pm i \log(r_j)/4\pi$.*

Proof. Let $d = k - 2$. Then $(\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(d), \Phi)$ with Φ constructed similar to the $d = 1$ case above is a stable parabolic $U(1,1)$-Higgs bundle with the required properties. □

Again, we have the following easy corollary:

Corollary 4.1.5. *Let $C_1, \ldots, C_k$ $(k \geq 3)$ be fixed loxodromic conjugacy classes in $U(1,1)$. Then there is a matrix $A_j \in C_j$ such that $A_1 \ldots A_k = I$. This solution is irreducible, in the sense that the A_j do not all preserve a nonzero subspace $V \subset \mathbb{C}^n$*

Proof. By Simpson's Correspondence, the stable parabolic $U(1,1)$-Higgs bundle in Theorem 4.1.4 corresponds to a stable filtered local system. Since by construction the real parts of the eigenvalues of $res(\Phi)_j$ are zero, the filtration on the local system is trivial. As such, the stable filtered local system is really an irreducible representation $\pi_1(\mathbb{P}^1 - \{p_1, \ldots, p_k\}) \to PU(1,1)$, whose monodromy around p_j is A_j. $\square$

The construction in the case $d = 0$ does not, to my knowledge, generalize easily. For the purposes of proving the main theorem, the result in theorem 4.1.4 will suffice.

4.2 Main Theorem for $PU(n, 1)$

Let $C_1, C_2, \ldots, C_k$ be k fixed loxodromic conjugacy classes in $PU(n, 1)$. Lift each C_j to a conjugacy class $\widetilde{C}_j$ in $U(n, 1)$, represented by a diagonal matrix of the following form:

$$A_j = \begin{bmatrix} r_j & & & & \\ & r_j^- 1 & & & \\ & & e^{2\pi i \alpha_j^1} & & \\ & & & \ddots & \\ & & & & e^{2\pi i \alpha_j^{n-1}} \end{bmatrix} \tag{4.2}$$

where r is a positive real number and $0 \leq \alpha_1^j \leq \ldots \leq \alpha_{n-1}^j < 1$.

Theorem 4.2.1. *Main Theorem: Assume the α^js defining the conjugacy classes $\widetilde{C}_j$ satisfy $\Sigma_{j,k}\alpha_k^j \in \mathbb{Z}$ and the strict $U(n-1)$ inequalities in Theorem [4]. Then there exists a stable parabolic $U(n, 1)$-Higgs bundle $(\mathcal{E}, \Phi)$ with $n \geq 3$ marked points such that:*

- *the filtration of $\mathcal{E}_{p_j}$ has a rank 2 jump at 0 and rank 1 jumps at each of $\alpha_1^i, \ldots, \alpha_{n-1}^i$*

- *the residue of* Φ *at* p_j *has eigenvalues* $\pm i\log(r)4\pi$*, and* 0 *with multiplicity* $n-1$*.*

Note that these conditions are sufficient, but not necessary. It is possible that there are still solutions to the Deligne-Simpson problem when the $U(n-1)$ inequalities are not satisfied.

4.3 Proof of Main Theorem

4.3.1 Construction

Following the procedure outlined in sections 3.3 and 3.4, we can build a parabolic $U(n,1)$-Higgs bundle out of a rank $n-1$ parabolic bundle and a parabolic $U(1,1)$-Higgs bundle. The construction is as follows.

First, we construct the kernel of our Higgs field. Since the α^i_j satisfy the $U(n-1)$ inequalities, there exists by Theorem 2.4.1 a stable rank $n-1$ parabolic bundle $\mathcal{S}$, with weights at p_j given by the α^j_k.

Second, we need to ensure that the residues of our Higgs field have the appropriate eigenvalues. By theorem 4.1.4, there exists a (necessarily stable) parabolic $U(1,1)$-Higgs bundle $(\mathcal{E},\Phi)$ of parabolic degree $k-2$, such the eigenvalues of the residue of Φ at p_j are $\pm i\log(r)/4\pi$. Writing $\mathcal{E}=\mathcal{E}_1\oplus\mathcal{Q}$, Φ has the following form:

$$\Phi = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$$

We will build our new Higgs bundle $(\widetilde{\mathcal{E}}=\mathcal{E}_1\oplus\mathcal{E}_2,\widetilde{\Phi})$. With respect to the splitting of $\widetilde{\mathcal{E}}$, $\widetilde{\Phi}$ has the following form:

$$\widetilde{\Phi} = \begin{bmatrix} 0 & \widetilde{b} \\ \widetilde{c} & 0 \end{bmatrix}$$

$\mathcal{E}_2$ is constructed by picking a special extension of $\mathcal{Q}$ by $\mathcal{S}$. No matter which extension we pick, $b : \mathcal{Q} \to \mathcal{E}_1 \otimes \mathcal{K}(D)$ extends to a morphism $\widetilde{b} : \mathcal{E}_2 \to \mathcal{E}_1 \otimes \mathcal{K}(D)$ by Lemma 3.3.2. However, we still need to pick an extension for which $c : \mathcal{E}_1 \to \mathcal{Q} \otimes \mathcal{K}(D)$ lifts to $\widetilde{c} : \mathcal{E}_1 \to \mathcal{E}_2 \otimes \mathcal{K}(D)$. For stability reasons, we must also minimize the vanishing of the lift $\widetilde{c}$. By Proposition 3.4.4, there is an extension such that c lifts and is non-vanishing.

4.3.2 The parabolic structure

We haven't mentioned the parabolic structure on $\widetilde{\mathcal{E}}$. However, both the $U(1,1)$ bundle $\mathcal{E}$ and the $U(n-1)$ bundle $\mathcal{S}$ come equipped with a parabolic structure, and hence induce a parabolic structure on the extension $\mathcal{E}_2$. However, this induced structure is not in general unique. All parabolic extensions of $\mathcal{Q}$ by $\mathcal{S}$ are classified by elements of $H^1(ParHom(\mathcal{Q}, \mathcal{S}))$.

Lemma 4.3.1. *Let $\mathcal{S}$ and $\mathcal{Q}$ be two parabolic vector bundles. If $\mathcal{Q}$ has the trivial parabolic structure, then $ParHom(\mathcal{Q}, \mathcal{S}) = Hom(\mathcal{Q}, \mathcal{S})$.*

Proof. Since the parabolic structure on $\mathcal{Q}$ is trivial, every morphism $\mathcal{Q} \to \mathcal{S}$ is parabolic, and hence $ParHom(\mathcal{Q}, \mathcal{S}) = Hom(\mathcal{Q}, \mathcal{S})$. □

In our case, we have only fixed the extension $\mathcal{E}_2$. However, since by construction $\mathcal{Q}$ has trivial parabolic structure, the parabolic structure on $\mathcal{S}$ uniquely determines the parabolic structure on $\mathcal{E}_2$, and thus on $\mathcal{E}$.

4.3.3 Stability

Finally we must determine if the Higgs bundle $(\widetilde{\mathcal{E}}, \widetilde{\Phi})$ is stable. From the discussion above, there are two invariant subbundles. The first is the kernel of b, which by construction is $\mathcal{S}$. The second invariant subbundle is the rank-2 subbundle. By Lemma 3.2.2, this is $\mathcal{V} = \mathcal{E}_1 \oplus (\mathcal{E}_1((\widetilde{c})) \otimes \mathcal{K}(D)^*)$. Since in our construction $\widetilde{c}$ doesn't vanish, we have $\mathcal{V} = \mathcal{E}_1 \oplus (\mathcal{E}_1 \otimes \mathcal{K}(D)^*)$.

When $n > 2$, $rank(\mathcal{S}) > 1$, and since $\mathcal{S}$ is the kernel of $\widetilde{\Phi}$, any subbundle of $\mathcal{S}$ is contained in the kernel, and hence invariant. However, $\mathcal{S}$ is a stable parabolic bundle, as constructed by theorem 2.4.1. As such, if $\mathcal{F}$ is any subbundle of $\mathcal{S}$, then $p\mu(\mathcal{F}) < p\mu(\mathcal{S})$. Thus we need only verify that $p\mu(\mathcal{S}) < p\mu(\mathcal{E})$.

By construction, $pdeg(\mathcal{E}_1) = 0$, and $pdeg(\mathcal{Q}) = k - 2$. By the above, 3.2.2, $pdeg(\mathcal{V}) = 2 - k$. We compute the parabolic degree of $\widetilde{\mathcal{E}}$ below:

$$
\begin{aligned}
pdeg(\widetilde{\mathcal{E}}) &= pdeg(\mathcal{E}_1) + pdeg(\mathcal{E}_2) \\
&= pdeg(\mathcal{E}_1) + pdeg(\mathcal{Q}) + pdeg(\mathcal{S}) \\
&= k - 2 + pdeg(\mathcal{S}) \\
&= k - 2
\end{aligned}
$$

Finally, we compare the degrees of these subbundles to the degree of $\widetilde{\mathcal{E}}$:

$$
\begin{aligned}
p\mu(\widetilde{\mathcal{E}}) - p\mu(\mathcal{S}) &= \frac{k-2}{n+1} - 0 = \frac{k-2}{n+1} > 0 \\
p\mu(\widetilde{\mathcal{E}}) - p\mu(\mathcal{V}) &= \frac{k-2}{n+1} - \frac{2-k}{2} > 0
\end{aligned}
$$

We conclude that the parabolic $U(n,1)$ bundle $\widetilde{\mathcal{E}}$ is stable.

Remark Note that in equation 4.2, we pick a specific lift of a loxodromic conjugacy class in $PU(n,1)$ to a conjugacy class in $U(n,1)$. There are many other choices, for example:

$$A_j = \begin{bmatrix} r_j e^{2\pi i \beta_j} & & & & \\ & r_j^{-1} e^{2\pi i \beta_j} & & & \\ & & e^{2\pi i \alpha_j^1} & & \\ & & & \ddots & \\ & & & & e^{2\pi i \alpha_j^{n-1}} \end{bmatrix}$$

where β_j is nonzero. For the purposes of stating the theorem, which lift we choose is not important. In fact, the stability of the kernel does not depend on which lift we choose (but the condition that the weights sum to an integer does). However, recall that the Higgs field we construct must be parabolic with respect to the flags chosen at the marked points. The weights on the $U(1,1)$ bundle are the β_j's. If these are larger than the α_j^k's, then the morphism c constructed above, and hence Φ, is not parabolic.

4.4 Deligne-Simpson and Representations

Reinterpreting the existence of a stable Higgs bundle in terms of the corresponding filtered local system, we have the following the corollary:

Corollary 4.4.1. *Let C_j be given loxodromic conjugacy classes in $PU(n,1)$, lifted to $\widetilde{C}_j$ to $U(n,1)$ as in equation 4.2. If the α_k^j satisfy the $U(n-1)$ inequalities, then* *there is a matrix $A_j \in C_j$ such that $A_1 \dots A_k = 1$.*

Proof. By Simpson's Correspondence, the stable parabolic $U(n,1)$-Higgs bundle in Theorem 4.2.1 corresponds to a stable filtered local system L.

The filtration on L is determined by the real parts of the eigenvalues of $res(\Phi)_j$. The nonzero eigenvalues of $res(\Phi)_j$ come the $U(1,1)$ bundle $\mathcal{E}$, as determined by 4.1.4. By construction the real parts of the eigenvalues of $res(\Phi)_j$ are zero, and hence the filtration on L is trivial. As such, the stable filtered local system L is really an irreducible representation $\pi_1(\mathbb{P}^1 - \{p_1, \dots, p_k\}) \to PU(n,1)$, whose monodromy around p_j is A_j.

□

4.5 Return to $PU(2,1)$

The final remark is that our result generalizes the result of Falbel and Wentworth in [12] $PU(2,1)$ case. We provide a separate proof, similar to that of theorem 4.2.1, but with one key exception. In this case, we can remove the requirement that the weights sum to an integer.

Theorem 4.5.1. *Let $C_, \dots C_k$ with $k \geq 3$ be fixed loxodromic conjugacy classes in $PU(2,1)$, represented by a matrix of the following form:*

$$\begin{bmatrix} r_j & & \\ & r_j^- 1 & \\ & & e^{2\pi i \alpha_j} \end{bmatrix}$$

Then there exists a stable parabolic $U(n,1)$-Higgs bundle with $n \geq 3$ marked points *and parabolic structure at* p_j *consisting of weights* $0, 0, \alpha_j$*, where the residues of the Higgs field at* p_j *have eigenvalues* $\pm i \log(r)/4\pi$*, and* 0 *with multiplicity* 1.

Proof. The only difference from the proof of theorem 4.2.1 is how we choose the kernel, which in this case is a line bundle we label $\mathcal{L} = \mathcal{O}(n)$. Once we pick n, the parabolic structure on $\mathcal{L}$ is obvious, and the parabolic line bundle $\mathcal{L}$ is automatically stable. Choose n such that

$$-1 < \Sigma_j \alpha_j + n \leq 0.$$

Finally, we check that the stability conditions on $\mathcal{E}$ are satisfied.

$$\begin{aligned} pdeg(\mathcal{E}) - 3pdeg(\mathcal{L}) &= (k - 2 + \Sigma_j \alpha_j + n) - 3\Sigma_j \alpha_j - 3n \\ &= k - 2 - 2\Sigma_j \alpha_j - 2n \\ &> k - 2 > 0 \end{aligned}$$

By Lemma 3.2.2, the invariant rank 2 bundle is $\mathcal{V} = \mathcal{O} \oplus \mathcal{K}(D)^* = \mathcal{O} \oplus \mathcal{O}(2-k)$.

$$\begin{aligned} pdeg(\mathcal{E}) - \frac{3}{2} pdeg(\mathcal{V}) &= (k - 2 + \Sigma_j \alpha_j + n) - \frac{3}{2}(k-2) \\ &= \frac{5}{2}k - 3 + \Sigma_j \alpha_j + n \\ &> \frac{5}{2}k - 4 \\ &> 0. \end{aligned}$$

We conclude that $\mathcal{E}$ is stable. □

As before, we can reinterpret the stable Higgs bundles we've constructed as filtered local systems with trivial filtration, i.e. a representation $\pi_1(\mathbb{P}^1 - \{p_1, \ldots, p_k\}) \to PU(2,1)$:

Corollary 4.5.2. *Let $C_, \dots C_k$ with $k \geq 3$ be fixed loxodromic conjugacy classes in $PU(2,1)$. Then there exists matrices $A_j \in C_j$ such that $A_1 \dots A_k = I$.*

s

Chapter 5: Solutions to the Deligne-Simpson Problem for $PU(3,1)$

Our main theorem says that when some of the data defining a collection of loxodromic conjugacy classes in $U(n,1)$ satisfies the $U(n-1)$ inequalities in 2.4.1, then there is an irreducible solution to Deligne-Simpson problem. But we've actually proven that solutions exist in many other cases as well. In this chapter, we study the $PU(3,1)$ case with three fixed loxodromic conjugacy classes more closely.

In section 5.1, we review some material regarding the product map $\mu : C_1 \times C_2 \to PU(3,1)$. First, we restrict μ to pairs (A,B) with loxodromic product. Then we construct a new map, $\widetilde{\mu}$, which is the projection of the set of loxodromics in $PU(3,1)$ onto $\mathcal{C}_{lox}$, the space of loxodromic conjugacy classes. The main result is that the image of $\widetilde{\mu}$ consists as a collection of reducible walls and irreducible chambers. Computing the differential $d\widetilde{\mu}$, we can show that $\widetilde{\mu}$ is a submersion at a pair (A,B) generating an irreducible subgroup. As a consequence, any chamber is either completely empty or completely full. This technique was used in [11] and [26] to study the $U(n)$ case, and used in [25] and [12] to study the elliptic and loxodromic case for $PU(2,1)$, respectively.

In section 5.2 we consider the different types of reducible subgroups a pair (A,B) can generate. With this information in mind, we compute the reducible walls in $\mathcal{C}_{lox} = \mathbb{T}^2/S_2 \times (1, \infty)$.

Finally, in section 5.3, we use the Higgs bundles constructed in chapter 4 to show that many of the chambers are full. The idea is that although there is no universal description of an unstable parabolic bundle, we can start with a stable parabolic bundle with known parabolic structure, and study how stability changes as we vary the weights. In particular, we have the Harder-Narasimhan filtration of an unstable bundle, which in rank 2 says that any destabilizing line subbundle of a rank 2 unstable parabolic bundle is unique. In terms of the inequalities determining stability given in Theorem 2.4.1, exactly one of them becomes invalid. By keeping track of which inequality is invalid as we vary the weights, we can show that certain walls are in the image of a pair (A, B) generating an irreducible. As a result, we conclude that many chambers are full.

5.1 The product map μ

Fix two loxodromic conjugacy classes C_1 and C_2 in $PU(3,1)$. Then the group-valued moment map μ is defined as follows:

$$\mu : C_1 \times C_2 \to PU(3,1)$$

$$(A, B) \mapsto AB$$

Since in our case we are interested in $(A, B) \in C_1 \times C_2$ with loxodromic product, we restrict μ to $(A, B) \in C_1 \times C_2$ whose product is loxodromic. This map is equivariant with respect to conjugation by elements of $PU(3,1)$ in each factor. Thus our restricted map descends to map onto $\mathcal{C}_{lox}$, the space of loxodromic conjugacy classes:

$$\widetilde{\mu} : (C_1 \times C_2) \bigcap \mu^{-1}(loxodromics) \to \mathcal{C}_{lox}$$

For $PU(3,1)$, a loxodromic conjugacy class can be identified (up to scaling) by two angles, $\alpha^1 < \alpha^2$, and a real number $r > 1$. Thus $\mathcal{C}_{lox} \simeq \mathbb{T}^2/S_2 \times (1, \infty)$. As an example, consider a conjugacy class C in $PU(3,1)$. Then C is represented by a matrix of the following form:

$$A = \begin{bmatrix} r & 0 & 0 & 0 \\ 0 & r^{-1} & 0 & 0 \\ 0 & 0 & e^{2\pi i \alpha_1} & 0 \\ 0 & 0 & 0 & e^{2\pi i \alpha_2} \end{bmatrix}, \quad \alpha_1 \leq \alpha_2, \quad r > 1$$

Then the conjugacy class C is identified by the point $(\alpha_1, \alpha_2, r) \in \mathbb{T}^2/S_2 \times (1, \infty)$.

The $(1, \infty)$-factor is not important in these circumstances, and we will usually suppress it, and study the $\mathbb{T}^2/S_2$ portion instead.

The next result concerns the image of pairs (A, B) generating an irreducible subgroup. Since it involves computing the differential of the map $\widetilde{\mu}$, we must be careful to ensure the source and target of $\widetilde{\mu}$ are smooth manifolds. To this end, we further restrict the map $\widetilde{\mu}$ to (A, B) with regular loxodromic product. Smoothness is guaranteed by the following proposition. Following the proof in [15] of the well known fact that the regular elliptic elements in $PU(n,1)$ form an open set, we have the following:

Proposition 5.1.1. *The regular loxodromic elements in $PU(n,1)$ form an open subset.*

Proof. Let $g \in PU(n,1)$ be a regular loxodromic element. Then by definition, we can represent g by an element $\widetilde{g} \in U(n,1)$ with distinct eigenvalues. The elements in $GL(n,\mathbb{C})$ with distinct eigenvalues forms an open set, which when intersected with $U(n,1)$ gives an open neighborhood of $\widetilde{g}$. The projection onto $PU(n,1)$ gives an open neighborhood of g. □

We have the following proposition, adapted from [11] and [25] to our situation:

Proposition 5.1.2. *Let C_1, C_2 be two fixed loxodromic conjugacy classes, and $(A,B) \in C_1 \times C_2$ such that AB is loxodromic and the subgroup generated by A, B irreducible. Then the differential of $\widetilde{\mu}$ is surjective at (A,B), and hence $\widetilde{\mu}$ is locally surjective at (A,B).*

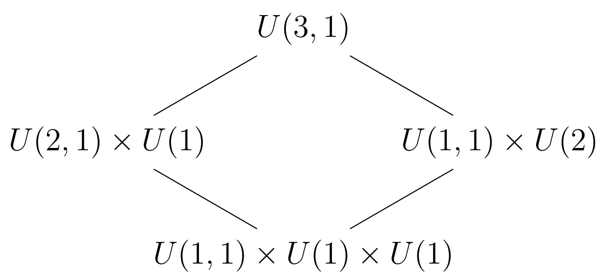
This follows immediately from the following lemma, since $PU(n,1)$ has trivial center. (see for example [25], [11]):

Lemma 5.1.3. $Im(d_{(A,B)}\mu) = \zeta(A,B)^{\perp}AB$

The application is that the image of $\widetilde{\mu}$ is divided into pieces by the image of reducible points (called walls). The interiors of these walls (called chambers) are, by the proposition, either completely full or completely empty. The main theorem actually constructs irreducible solutions on one of the walls in our picture, and thus implies a large number of chambers are full.

5.2 Description of reducible walls for $PU(3,1)$

Let $(A, B) \in C_1 \times C_2$ generate a reducible subgroup of $PU(3,1)$. In this case, reducible can be interpreted as A and B both preserving a fixed subspace of $\mathbb{C}^4$. Depending on the type of subspace preserved, (A, B) (after lifting to $U(n,1)$) generates a subgroup of one of the groups in the following diamond:



Remark While there are other possibilities, such as when A and B are simultaneously diagonalizable, we may assume without loss of generality that A, B generate one of the above subgroups. The missing cases result in reducible walls which give a relationship between the translation lengths (the r's) of A, B, and C. In the proof and statement of Theorem 4.2.1, the translation length r plays no role in the stability of the resulting bundle, and hence plays no role in the existence of irreducible solutions to $AB = C$.

More explicitly, if $AB = C$ is irreducible, then by Theorem 2.5.3 there is a stable parabolic $PU(3,1)$-Higgs bundle $(\mathcal{E}, \Phi)$ with three marked points, where the monodromy around each marked point $\{p_a, p_B, p_C\}$ are in the conjugacy class of A, B, and C^{-1}, respectively. Then the results of section 3.3 allow us to break $(\mathcal{E}, \Phi)$ into a parabolic $U(1,1)$-Higgs bundle $(\widetilde{\mathcal{E}}, \widetilde{\Phi})$ and a parabolic rank 2 bundle $\mathcal{S}$. In fact, $\mathcal{E}$ is, up to tensoring by a line bundle, one of the 3 possibilities in section 4.1.

Thus we can replace the Higgs field $\widetilde{\Phi}$ with a new Higgs field $\widetilde{\Phi}'$, whose residue at p_C has different eigenvalues. Putting $(\widetilde{\mathcal{E}}, \widetilde{\Phi}')$ and $\mathcal{S}$ back together via the results in section 3.4, we a new, necessarily stable, parabolic $PU(3,1)$-Higgs bundle such that the conjugacy class of the monodromy around the point p_C is the same as C^{-1} except with a different translation length.

5.2.1 $U(1,1) \times U(1) \times U(1)$ walls

There are two "vertices" (really, it's two lines, but we're ignoring the $(1,\infty)$ factor) corresponding to groups $< A, B >$ where A,B both preserve two linearly independent 1-dim elliptic subspaces.

Proposition 5.2.1. *Let A, B be two loxodromic elements in $PU(3,1)$, which preserve two 1 dimensional independent elliptic subspaces of $\mathbb{C}^4$. Then $\{\gamma_1, \gamma_2\}$ are given by one of the following:*

- $$\begin{cases} \gamma_1 = \alpha_1 + \beta_1 (mod 1) \\ \gamma_2 = \alpha_2 + \beta_2 (mod 1) \end{cases}$$
- $$\begin{cases} \gamma_1 = \alpha_1 + \beta_2 (mod 1) \\ \gamma_2 = \alpha_2 + \beta_1 (mod 1) \end{cases}$$

Proof. We use the following decomposition of $AB = C$:

$$\begin{bmatrix} A' & 0 & 0 \\ 0 & e^{2\pi i \alpha_1} & 0 \\ 0 & 0 & e^{2\pi i \alpha_2} \end{bmatrix} \begin{bmatrix} B' & 0 & 0 \\ 0 & e^{2\pi i \beta_1} & 0 \\ 0 & 0 & e^{2\pi i \beta_2} \end{bmatrix} = \begin{bmatrix} C' & 0 & 0 \\ 0 & e^{2\pi i \psi_1} & 0 \\ 0 & 0 & e^{2\pi i \psi_2} \end{bmatrix}$$

where $A', B', C' \in U(1,1)$ are loxodromic. By either Theorem 4.1.4, Theorem 4.2.1, or [12], C' ranges over all possible loxodromic conjugacy classes. Diagonalizing C', we have:

$$C' = \begin{bmatrix} r_C e^{2\pi i \psi} & 0 \\ 0 & r_C^{-1} e^{2\pi i \psi} \end{bmatrix}$$

Finally, we compute γ_1, γ_2:

$$\gamma_1 = \psi_1 - \psi = \alpha_1 + \beta_1$$

$$\gamma_2 = \psi_2 - \psi = \alpha_2 + \beta_2$$

This gives the first case of the result. The second case follows from the above, interchanging the roles of β_1 and β_2. □

Note that in the above proof, r_C is allowed to range over all possible values. Thus the image of $< A, B >$ consists of the whole line $(\gamma_1, \gamma_2) \times (1, \infty)$.

5.2.2 $U(1,1) \times U(2)$ walls

When A, B preserve a common 2-dimensional elliptic subspace, we have the following:

Proposition 5.2.2. *Let A, B be two loxodromic elements $PU(3,1)$ with angles $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$ respectively. If the group generated by A,B preserves a 2-dimensional elliptic subspace, then the angle pair $\{\gamma_1, \gamma_2\}$ of $C = AB$ lie on one of the following segments:*

- $I(\rho) = 2$: $\gamma_1 + \gamma_2 = \alpha_1 + \alpha_2 + \beta_1 + \beta_2$

$$\begin{cases} \gamma_2 > max(\alpha_1 + \beta_2, \alpha_2 + \beta_1) \\ \gamma_2 < min(\alpha_1 + \beta_1 + 1, \alpha_2 + \beta_2) \end{cases}$$

- $I(\rho) = 3$: $\gamma_1 + \gamma_2 = \alpha_1 + \alpha_2 + \beta_1 + \beta_2 - 1$

$$\begin{cases} \gamma_2 > max(\alpha_1 + \beta_1, \alpha_2 + \beta_2 - 1) \\ \gamma_2 < min(\alpha_1 + \beta_2, \alpha_2 + \beta_1) \end{cases}$$

- $I(\rho) = 4$: $\gamma_1 + \gamma_2 = \alpha_1 + \alpha_2 + \beta_1 + \beta_2 - 2$

$$\begin{cases} \gamma_2 > max(\alpha_1 + \beta_2, \alpha_1 + \beta_2) - 1 \\ \gamma_2 < min(\alpha_1 + \beta_1, \alpha_2 + \beta_2 - 1) \end{cases}$$

In this situation, $AB = C$ reduces to:

$$\begin{bmatrix} A' & 0 \\ 0 & A'' \end{bmatrix} \begin{bmatrix} B' & 0 \\ 0 & B'' \end{bmatrix} = \begin{bmatrix} C' & 0 \\ 0 & C'' \end{bmatrix} \tag{5.1}$$

where $A', B', C' \in U(1,1)$ are loxodromic, and $A'', B'', C'' \in U(2)$. Me may assume the subgroup $< A'', B'' >$ of $U(2)$ is irreducible. The case that it is reducible was handled in the previous section. In order to prove this proposition, we will need two results. The first is Theorem 2.4.1, regarding the existence of stable parabolic bundles of rank 2. Fix k conjugacy classes C_1, C_2, ..., C_k in $U(2)$, each represented by a matrix of the form:

$$A_s = \begin{bmatrix} e^{2\pi i \alpha_1^j} & 0 \\ 0 & e^{2\pi i \alpha_2^s} \end{bmatrix} \quad \alpha_1^s \leq \alpha_2^s$$

For convenience, we include the Theorem again here:

Theorem 5.2.3. *(Biswas) [3]*

Let $S = \{1, 2, \ldots, k\}$, and assume $\Sigma_{s\in S}(\alpha_1^s + \alpha_2^s)$ is an odd (respectively even) integer, say $2N$ (respectively $2N+1$). Then there is a stable rank 2 parabolic bundle with parabolic weights $\{\alpha_1^s, \alpha_2^s\}$ at the marked point $p_s \in \mathbb{P}^1$ if and only if for every $D \subset S$ of size $2j$ (resp. $2j+1$), where j is a nonnegative integer, the following inequality holds:

$$-N - j + \Sigma_{s\in D}\alpha_2^s + \Sigma_{s\in S-D}\alpha_1^s < 0.$$

The second result, from [11], concerns the index of a representation. For a representation $\rho : \pi_1(\mathbb{P}^1 - \{p_1, \ldots, p_k\}) \to U(2)$, the index $I(\rho)$ is the sum of the angles defining the conjugacy class of the monodromy around each marked point p_j. From the perspective of parabolic bundles, the representation is the sum of the parabolic weights. In either case, we have $I(\rho) = \Sigma_{s\in S}(\alpha_1^s + \alpha_2^s)$.

Proposition 5.2.4. *(Falbel, Wentworth) [11]*

For any representation $\rho : \pi_1(\mathbb{P}^1 - \{p_1, \ldots, p_k\}) \to U(n)$, we have

$$2 - N_0(\rho) \le I(\rho) \le 2(k-1) + N_0(\rho) - N_1(\rho)$$

where $N_0(\rho)$ is the number of trivial representations appearing in the decomposition of ρ into irreducibles, and $N_1(\rho)$ is the total multiplicity of 0 among the α_j^s.

We can now return to the proof of Proposition 5.2.2.

Proof. Given the reduction in equation 5.1, we need to study the $U(2)$ piece, i.e. the equation $A''B'' = C''$. Note that this setup is slightly different than in Theorem 5.2.3 and Proposition 5.2.4. To apply these theorems to our case, simply consider the representation ρ corresponding to the equation $A''B''(C'')^{-1} = I$. Represent the conjugacy class of A'', B'', and C'' by $\{\alpha_1, \alpha_2\}$, $\{\beta_1, \beta_2\}$, and $\{\gamma_1, \gamma_2\}$ respectively, with $\alpha_1 \leq \alpha_2$, etc., as usual. Then using $(C'')^{-1}$ amounts to using $1 - \gamma_2 \leq 1 - \gamma_1$ in the inequalities in Theorem 5.2.3 and Proposition 5.2.4.

Since we only need to worry about irreducible solutions, Proposition 5.2.4 says the possible values of $I(\rho)$ are 2, 3, or 4. We'll handle the case when $I(\rho) = 3$. The other two cases follow similarly.

$I(\rho) = 3$ gives the following:

$$\gamma_1 + \gamma_2 = \alpha_1 + \alpha_2 + \beta_1 + \beta_2 - 1 \tag{5.2}$$

This equation is only valid, of course, when an irreducible representation ρ actually exists. Necessary and sufficient conditions for the existence of ρ are given by Theorem 5.2.3. The inequalities take the following form:

$$\alpha_1 + \beta_1 < \gamma_2$$

$$\alpha_2 + \beta_2 < 1 + \gamma_2$$

$$\alpha_2 + \beta_1 < 1 + \gamma_1$$

$$\alpha_1 + \beta_2 < 1 + \gamma_1$$

Equation 5.2 gives a linear relationship between γ_1 and γ_2, depending on the α_j's and the β_j's. In the future, we will think of these walls as giving γ_1 as a function of γ_2. For this particular case, we have $\gamma_1 = \alpha_1 + \alpha_2 + \beta_1 + \beta_2 - \gamma_2 - 1$. In this case, the above inequalities, together with $0 \leq \gamma_2 < 1$, give the domain of this function.

Substituting γ_1 into the above inequalities, we have:

$$\gamma_2 > \alpha_1 + \beta_1$$

$$\gamma_2 > \alpha_2 + \beta_2 - 1$$

$$\gamma_2 < \alpha_1 + \beta_2$$

$$\gamma_2 < \alpha_2 + \beta_1$$

This completes the proof for the case $I(\rho) = 3$. The other two cases follow similarly. □

For fixed loxodromic conjugacy classes C_1 and C_2, at least one and at most two of the equations in Proposition 5.2.2 have no solutions. To best understand the geometry of these walls, we temporarily relax the restriction $0 \leq \gamma_2 \leq \gamma_1 < 1$ to $0 \leq \gamma_1 - \gamma_2 \leq 1$. Then the wall is the shortest line segment of slope -1 line connecting the two vertices given in 5.2.1. When two equations in 5.2.2 have solutions, the image of this line segment in the affine chart for $\mathbb{T}^2/S_2$ is disconnected, as seen in figure 5.1. When only one equation has solutions, the segment in $\mathbb{T}^2/S_2$ is connected, as in 5.2

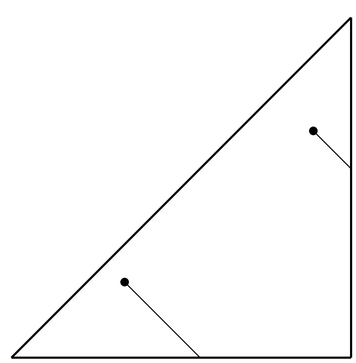


Figure 5.1: An example of a disconnected $P(U(1,1) \times U(2))$ wall

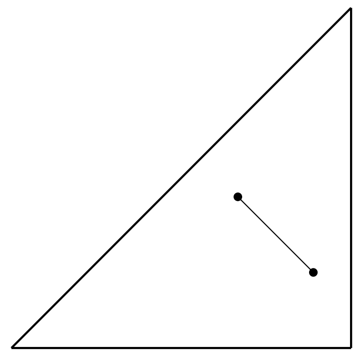


Figure 5.2: An example of a connected $P(U(1,1) \times U(2))$ wall

5.2.3 $U(2,1) \times U(1)$ walls

In this section we consider the case were A, B preserve a common 1-dimensional elliptic subspace.

Proposition 5.2.5. *Let A, B be two loxodromic elements in $PU(3,1)$ with angles $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$ respectively. If the group generated by A, B preserves a 1-dimensional elliptic subspace, then the angles $\{\gamma_1, \gamma_2\}$ of $AB = C$ are given by one of the following lines:*

- $\gamma_1 = 3\gamma_2 + \alpha_1 + \beta_1 - 3\alpha_2 - 3\beta_2 (mod 1)$
- $\gamma_1 = 3\gamma_2 + \alpha_1 + \beta_2 - 3\alpha_2 - 3\beta_1 (mod 1)$
- $\gamma_1 = 3\gamma_2 + \alpha_2 + \beta_1 - 3\alpha_1 - 3\beta_2 (mod 1)$
- $\gamma_1 = 3\gamma_2 + \alpha_2 + \beta_2 - 3\alpha_1 - 3\beta_1 (mod 1)$

where $0 \leq \gamma_2 \leq \gamma_1 < 1$.

Proof. We'll show how to derive the first equation. The argument for the other three is similar, and simply involves permuting the angles of A and B among the four possibilities.

Since A, B preserve a common elliptic subspace, $AB = C$ can be written in the following form:

$$\begin{bmatrix} A' & 0 \\ 0 & e^{2\pi i \alpha_2} \end{bmatrix} \begin{bmatrix} B' & 0 \\ 0 & e^{2\pi i \beta_2} \end{bmatrix} = \begin{bmatrix} C' & 0 \\ 0 & e^{2\pi i \psi_3} \end{bmatrix}$$

where $A', B', C' \in U(2,1)$ are loxodromic. By Theorem 4.2.1, C' ranges over all possible conjugacy classes of loxodromic elements. If we write

$$C' = \begin{bmatrix} re^{2\pi i \psi_1} & 0 & 0 \\ 0 & r^{-1}e^{2\pi i \psi_1} & 0 \\ 0 & 0 & e^{2\pi i \psi_2} \end{bmatrix},$$

then diagonalizing A and B, we have the following three equalities involving γ_1 and γ_2:

$$\gamma_1 = \psi_2 - \psi_1$$

$$\gamma_2 = \psi_3 - \psi_1 = \alpha_2 + \beta_2 - \psi_1$$

$$\alpha_1 + \beta_1 = 2\psi_1 + \psi_2$$

where each equation is taken mod 1. The third equation comes from the requirement that $det(A'B') = det(C')$. The derivation of the equation for the wall is as follows:

$$\alpha_1 + \beta_1 = 2\psi_1 + \psi_2$$

$$\alpha_1 + \beta_1 = 3\psi_1 + \gamma_1$$

$$\gamma_1 = -3\psi_1 + \alpha_1 + \beta_1$$

$$\gamma_1 = 3\gamma_2 + \alpha_1 + \beta_1 - 3\alpha_2 - 3\beta_2.$$

The remaining three equations can be derived similarly, by interchanging the roles of α_1 and α_2, β_1 and β_2 in each of the four possible ways. □

Though the above proof follows similarly to the corresponding result in [25] for elliptic classes in $PU(2,1)$, there is one key difference. For two elliptic conjugacy classes C_1, C_2 in $PU(1,1)$, there are restrictions on the third elliptic conjugacy class C_3, for there to be a solution $AB = C$. The result is that the $U(1,1) \times U(1)$ walls start at one of the totally reducible vertices and emanate either to the right or left. Thus there are many different possible configurations for the $U(1,1) \times U(1)$ walls. For fixed loxodromic conjugacy classes, there are no such restrictions. As a result, the walls given in 5.2.5 are in the image of $\widetilde{\mu}$ for all $0 \leq \gamma_1 \leq \gamma_2 < 1$.

We should say something about the geometry of these walls. As an example, consider the first and fourth equations. Plugging $\gamma_2 = \alpha_2 + \beta_2$ into the first wall, we get the point $(\alpha_2 + \beta_2, \alpha_1 + \beta_1)$ on the wall (compare this to our vertices in Proposition 5.2.1. Plugging $\gamma_2 = \alpha_1 + \beta_1$ into the fourth equation, we get the point $(\alpha_2 + \beta_2, \alpha_1 + \beta_1)$ on the wall. But these are the same point with our normalization $\gamma_1 \leq \gamma_2$. Accounting for this, the first and fourth wall give line segments in $\mathbb{T}^2/S_2$, one of slope 3 and the other of slope 1/3, intersecting at exactly one point, namely one of the two vertices.

With this understood and combining with Propositions 5.2.1 and 5.2.2, examples for possible configurations of walls are given in figures 5.3 and 5.4:

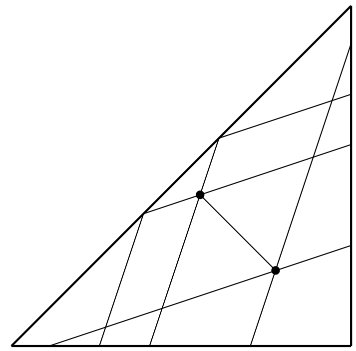


Figure 5.3: A configuration of reducible walls with connected $U(1,1) \times U(2)$ wall

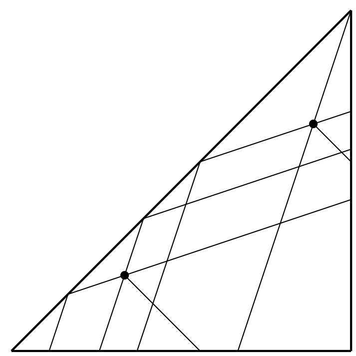


Figure 5.4: A configuration of reducible walls with disconnected $U(1,1) \times U(2)$ wall

5.3 Determining which chambers are full

In the proof of Theorem 4.2.1, we construct stable parabolic $U(n,1)$-Higgs bundles with stable kernel. Though the condition that the kernel be stable is not necessary, a general rank-(n-1) parabolic bundle can be unstable in many interesting ways. However, if we have a stable rank 2 bundle with known parabolic structure, we can use the Harder-Narasimhan filtration to observe how stability changes as we vary the weights of the bundle.

As a direct consequence to Theorem 4.2.1, we have:

Proposition 5.3.1. *The $U(1,1) \times U(2)$ walls in Proposition 5.2.2 are in the image* *of $\widetilde{\mu}$.*

Proof. This is a direct application of our main theorem, Theorem 4.2.1. □

As an application of this proposition, since the image is of $\widetilde{\mu}$ is open at irreducibles, any chambers intersecting with the $U(1,1) \times U(2)$ wall is full. See figure 5.5:

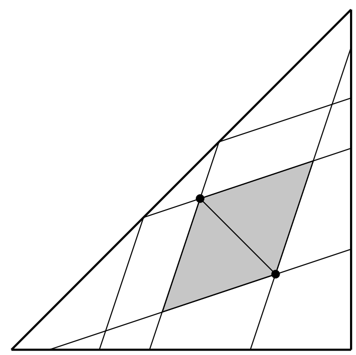


Figure 5.5: The $U(1,1) \times U(2)$ wall is in the image of irreducibles under $\widetilde{\mu}$. Hence the shaded chambers are full.

5.3.1 Stability conditions for the parabolic $U(3,1)$-Higgs bundles with three marked points

In this section, we explore the stability conditions for the parabolic $U(3,1)$-Higgs bundles with three marked points constructed in Theorem 4.2.1. In particular, we lift the requirements that the weights sum to an integer and the kernel be stable, and give new set of inequalities determining stability. However, it will still be difficult to determine for which weights the inequalities satisfied. As such, we will instead start with a bundle from Theorem 4.2.1, which gives a solution to $AB = C$ whose image lies on the $U(1,1)\times U(2)$ wall. Leaving the underlying parabolic bundle fixed, we'll then study what happens to stability as we move the weights along the wall towards the $U(1,1)\times U(1)\times U(1)$ vertices. From here we'll study how stability changes along the $U(2,1)\times U(1)$ walls.

As we move along the first wall and onto a vertex, the kernel $\mathcal{S}$ of Φ changes from stable to semi-stable. Furthermore, as we move from the vertices onto the $U(2,1)\times U(1)$ walls, $\mathcal{S}$ becomes unstable. The Harder-Narasimhan filtration for parabolic bundles tells us that a destabilizing line subbundle of a rank 2 bundle is unique. In terms of the inequalities determining the stability of $\mathcal{S}$, exactly one is violated when $\mathcal{S}$ is unstable. By keeping track of which inequality is violated as we change walls, we can easily find more walls which are in the image of $\widetilde{\mu}$.

Before going further, we should mention again that the computation and characterization of the reducible walls were performed with respect to the problem $AB = C$, whereas the construction of the $U(n,1)$-Higgs bundles was performed with respect to the problem $ABC = 1$. Thus $\gamma_1 \leq \gamma_2$ in the $AB = C$ problem becomes $(1-\gamma_2) \leq (1-\gamma_1)$. Despite possible confusion, we also label the new weights $(1-\gamma_2) \leq (1-\gamma_1)$ by γ_1 and γ_2, respectively.

Let $(\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2, \Phi)$ be a stable parabolic $U(3,1)$-Higgs bundle with stable kernel, as constructed in Theorem 4.2.1.

Proposition 5.3.2. *The $U(1,1)\times U(1)\times U(1)$ vertices in Proposition 5.2.1 are in the image of an irreducible solutions to $AB = C$ by $\widetilde{\mu}$.*

Proof. Allow the weights of the parabolic structure on $\mathcal{E}$ to move along the $U(1,1)\times$

$U(2)$ wall towards the vertices. When we reach the vertices, the kernel $\mathcal{S}$ of Φ becomes semi-stable. Note however, that we have changed neither the degree of $\mathcal{S}$, nor the filtration at each marked point. As such, $\mathcal{S}$ with the new weights is semi-stable but remains non-split. It follows that $\mathcal{E}$ with the new weights is stable, and the vertices are in the image of an irreducible under $\widetilde{\mu}$. □

At first, this does not seem to give enough information to fill in chambers incident with the vertex, since the image of $\widetilde{\mu}$ is only guaranteed to be open at irreducibles with regular loxodromic product (where $\widetilde{\mu}$ is smooth). When the vertex lies on the line $\gamma_1 = \gamma_2$, the regularity assumption is violated. On the other hand, stability is an open condition, and hence we can further perturb the weights on $\mathcal{E}$ slightly to find irreducibles near the vertex with regular product, thus guaranteeing the chambers are full.

As we move away from the $U(1,1) \times U(2)$ wall, we'll lose the assumption that the weights add to an integer. In order to determine the Harder-Narasimhan filtration of $\mathcal{S}$ as we move away from this wall and along the $U(2,1) \times U(1)$ wall, we need stability conditions for this more general case. These are given in the following proposition whose proof is almost exactly the same as in [3]. The integer d in the proposition is determined by the index of the wall we begin on. The index can take values 2, 3, and 4 for which $\mathcal{S}$ has the structure $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, $\mathcal{O}(-1) \oplus \mathcal{O}(-2)$, and $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$ respectively. However the result is true for all $d \leq 0$:

Proposition 5.3.3. *Let $\mathcal{S} = \mathcal{O}(d) \oplus \mathcal{O}(d-1)$ be a rank 2 parabolic bundle with three marked points and weights $0 \leq \alpha_1 \leq \alpha_2 < 1$, $0 \leq \beta_1 \leq \beta_2 < 1$ and $0 \leq \gamma_1 \leq \gamma_2 < 1$.*

Then $\mathcal{S}$ is stable if and only if the following inequalities hold:

- $-1 - \alpha_1 - \beta_1 - \gamma_1 + \alpha_2 + \beta_2 + \gamma_2 > 0$
- $1 + \alpha_1 + \beta_1 - \gamma_1 - \alpha_2 - \beta_2 + \gamma_2 > 0$
- $1 + \alpha_1 - \beta_1 + \gamma_1 - \alpha_2 + \beta_2 - \gamma_2 > 0$
- $1 - \alpha_1 + \beta_1 + \gamma_1 + \alpha_2 - \beta_2 - \gamma_2 > 0$

Let $\mathcal{S} = \mathcal{O}(d) \oplus \mathcal{O}(d)$ be a rank-2 parabolic bundle with three marked points and weights as above. Then $\mathcal{S}$ is stable if and only if the following inequalities hold:

- $\alpha_1 - \beta_1 - \gamma_1 - \alpha_2 + \beta_2 + \gamma_2 > 0$
- $-\alpha_1 + \beta_1 - \gamma_1 + \alpha_2 - \beta_2 + \gamma_2 > 0$
- $-\alpha_1 - \beta_1 + \gamma_1 + \alpha_2 + \beta_2 - \gamma_2 > 0$
- $2 + \alpha_1 + \beta_1 + \gamma_1 - \alpha_2 - \beta_2 - \gamma_2 > 0$

Proof. See [3] for more information. For our purposes, the decomposition of $\mathcal{S}$ into line bundles is fixed, and one of the above cases. Additionally the parabolic structure is fixed, and generic. We'll illustrate the final steps for the case $\mathcal{S} = \mathcal{O}(d) \oplus \mathcal{O}(d)$.

Since there are three marked points, the only possible destabilizing subbundles are $\mathcal{L} = \mathcal{O}(d)$ and $\mathcal{L} = \mathcal{O}(d-1)$.

An embedding $\mathcal{O}(d) \hookrightarrow \mathcal{S}$ is determined (up to scale) by two parameters. Since the flags are generic, any embedding $\mathcal{O}(d) \hookrightarrow \mathcal{S}$ coincides with the flag at at most one marked point. The requirement that $p\mu(\mathcal{S}) - p\mu(\mathcal{O}(d)) > 0$ gives the first three inequalities.

An embedding $\mathcal{O}(d-1) \hookrightarrow \mathcal{S}$ on the other hand is determined (up to scale) by four parameters. As such, there is a unique embedding $\mathcal{O}(d-1) \hookrightarrow \mathcal{S}$ which coincides with the flag at every marked point. The requirement $p\mu(\mathcal{S}) - p\mu(\mathcal{O}(d-1)) > 0$ gives the third inequality. □

Let $(\widetilde{\mathcal{E}} = \mathcal{O} \oplus \mathcal{O}(1), \widetilde{\Phi})$ be a (necessarily stable) parabolic $U(1,1)$-Higgs bundle of loxodromic type. Let $\mathcal{S}$, of either the form $\mathcal{O}(d) \oplus \mathcal{O}(d)$ or $\mathcal{O}(d) \oplus \mathcal{O}(d-1)$, be a non-split rank 2 parabolic vector bundle, with weights as in propositon 5.3.3. Then our main construction builds a new parabolic $U(3,1)$-Higgs bundle $(\mathcal{E} = \mathcal{O} \oplus \mathcal{E}_2, \Phi)$ from $(\widetilde{E}, \widetilde{\Phi})$ and $\mathcal{E}$. The following proposition combines the analysis in the proof of Theorem 4.2.1 with the generalized inequality in Proposition 5.3.3 to give the stability conditions for $(\mathcal{E}, \Phi)$.

Proposition 5.3.4. *Let $\mathcal{S}$ be of the form $\mathcal{O}(d) \oplus \mathcal{O}(d)$ with the usual weights, and $\mathcal{V} = \mathcal{O} \oplus \mathcal{O}(-1)$ the other invariant rank 2 subbundle. Then $\mathcal{E}$ is stable if and only if the following six inequalities are satisifed:*

- $1 > pdeg(\mathcal{S})$
- $pdeg(\mathcal{S}) > -3$
- $1 - 2d + \alpha_1 + \beta_2 + \gamma_2 - 3\alpha_2 - 3\beta_1 - 3\gamma_1 > 0$
- $1 - 2d + \alpha_2 + \beta_1 + \gamma_2 - 3\alpha_1 - 3\beta_2 - 3\gamma_1 > 0$
- $1 - 2d + \alpha_2 + \beta_2 + \gamma_1 - 3\alpha_1 - 3\beta_1 - 3\gamma_2 > 0$
- $5 - 2d + \alpha_1 + \beta_1 + \gamma_1 - 3\alpha_2 - 3\beta_2 - 3\gamma_2 > 0$

Let $\mathcal{S}$ be of the form $\mathcal{O}(d) \oplus \mathcal{O}(d-1)$ with the usual weights. Then $\mathcal{E}$ is stable if and only if the following six inequalities are satisifed:

- $1 > pdeg(\mathcal{S})$
- $pdeg(\mathcal{S}) > -3$
- $-2d + \alpha_2 + \beta_2 + \gamma_2 - 3\alpha_1 - 3\beta_1 - 3\gamma_1 > 0$
- $4 - 2d + \alpha_1 + \beta_1 + \gamma_2 - 3\alpha_2 - 3\beta_2 - 3\gamma_1 > 0$
- $4 - 2d + \alpha_1 + \beta_2 + \gamma_1 - 3\alpha_2 - 3\beta_1 - 3\gamma_2 > 0$
- $4 - 2d + \alpha_2 + \beta_1 + \gamma_1 - 3\alpha_1 - 3\beta_2 - 3\gamma_2 > 0$

Proof. The proof follows similarly to that of Proposition 5.3.3, except in the context of the $PU(3,1)$-Higgs bundle $(\mathcal{E}, \Phi)$ constructed from $\mathcal{S}$ and a $U(1,1)$-Higgs bundle $(\mathcal{O} \oplus \mathcal{O}(1), \widetilde{\Phi})$. There are two additional inequalities, coming from the requirement that $p\mu(\mathcal{E}) - p\mu(\mathcal{S}) > 0$, and similarly for the other invariant rank-2 bundle $\mathcal{V}$. The other four inequalities come from the stability requirement applied to line subbundles of the kernel $\mathcal{S}$, as in 5.3.3. □

Notice that unlike in the $U(2)$ case, the parameter d plays a role in stability. Also, the last four inequalities in each case resemble the equations giving the $U(2,1) \times U(1)$ walls in Proposition 5.2.5.

5.3.2 Moving along $U(2,1) \times U(1)$ walls

Now we're ready to study the stability of our $PU(3,1)$-Higgs bundle $(\mathcal{E}, \Phi)$ as we move away from the vertices and onto the $U(2,1) \times U(1)$ walls. At this point, the combinatorics become a bit difficult to manage.

First, we begin on a $U(1,1) \times U(2)$ wall, where our main theorem guarantees the existence of a stable $PU(3,1)$-Higgs bundle with stable kernel. The kernel $\mathcal{S}$ of Φ is determined the index, for which there are three possibilities. Then, varying the weights of the bundle but nothing else, we move toward a vertex, for which there are two choices. Next, the vertices given in Proposition 5.2.1 do not account for the normalization $\gamma_2 > \gamma_1$, resulting in two possibilities. Finally, it could be the case that the $U(1,1) \times U(2)$ wall is of disconnected type. We'll show the details completely one case. The others follow similarly.

$I(\rho) = 3$ case:

Let's study the case where the index of the $U(1,1) \times U(2)$ wall is 3. In this case, $\mathcal{S} = \mathcal{O}(-1) \oplus \mathcal{O}(-2)$. By Proposition 5.2.2, the wall is given by the equation $\gamma_1 + \gamma_2 = \alpha_1 + \alpha_2 + \beta_1 + \beta_2 - 1$, where $\gamma_2 > max(\alpha_1 + \beta_1, \alpha_2 + \beta_2 - 1)$ and $\gamma_2 < min(\alpha_1 + \beta_2, \alpha_2 + \beta_1$.

First, we'll vary the weight γ_2 in the negative direction. The kernel $\mathcal{S}$ becomes semi-stable when $\gamma_2 = max(\alpha_1 + \beta_1, \alpha_2 + \beta_2 - 1)$. In particular, we'll study the case where $max(\alpha_1 + \beta_1, \alpha_2 + \beta_2 - 1) = \alpha_2 + \beta_2 - 1$. Combining with Proposition 5.2.1, the vertex is given by:

$$\begin{cases} \gamma_1 = \alpha_1 + \beta_1 \\ \gamma_2 = \alpha_2 + \beta_2 - 1 \end{cases}$$

If we look at the conditions for stability in Proposition 5.3.3, the inequality $1 + \alpha_1 + \beta_1 - \gamma_1 - \alpha_2 - \beta_2 + \gamma_2 > 0$ becomes equality. Continuing to move γ_2 in the negative direction, this inequality becomes invalid. Recall that this inequality comes from a particular embedding $\mathcal{O}(-2) \hookrightarrow \mathcal{S}$. Therefore, this is the (unique!) destabilizing line subbundle of $\mathcal{S}$ that we need to worry about. The corresponding inequality in Proposition 5.3.4 is, given that $d = -1$, $6+\alpha_1+\beta_1+\gamma_2-3\alpha_2-3\beta_2-3\gamma_1 > 0$. Verifying the stability of $(\mathcal{E}, \Phi)$ is then one of verifying only this inequality and $pdeg(\mathcal{S}) > -3$, coming from $p\mu(\mathcal{E}) - p\mu(\mathcal{V}) > 0$. The second inequality is, of course, trivially valid.

The first $U(2,1)\times U(1)$ wall intersecting this vertex is $\gamma_1 = 3\gamma_2+\alpha_1+\beta_1-3\alpha_2-3\beta_2$. Note that this equation is taken $mod(1)$. Thus we can think of it as the weights satisfying some integer relation, in particular $\alpha_1+\beta_1-\gamma_1+-3\alpha_2-3\beta_2+3\gamma_2 \in \mathbb{Z}$. In particular, since this wall intersects the index 3 wall at the vertex, we can determine this integer by plugging in the values of γ_1, γ_2 at the vertex, we can determine this integer:

$$\begin{aligned}
&\alpha_1 + \beta_1 - \gamma_1 + -3\alpha_2 - 3\beta_2 + 3\gamma_2 \\
=&\alpha_1 + \beta_1 - (\alpha_1 + \beta_1) - 3\alpha_2 - 3\beta_2 + 3(\alpha_2 + \beta_2 - 1) \\
=& -3
\end{aligned}$$

Finally, we convert from the $AB = C$ problem to the $ABC = I$ problem by replacing γ_1 with $1 - \gamma_2$ and γ_2 with $1 - \gamma_1$. This gives:

$$\alpha_1 + \beta_1 + \gamma_2 - 3\alpha_2 - 3\beta_2 - 3\gamma_1 = -5$$

We can easily confirm then that $6 + \alpha_1 + \beta_1 + \gamma_2 - 3\alpha_2 - 3\beta_2 - 3\gamma_1 > 0$.

Finally, as we move along $\gamma_1 = 3\gamma_2 + \alpha_1 + \beta_1 - 3\alpha_2 - 3\beta_2$, $1 + \alpha_1 + \beta_1 - \gamma_1 - \alpha_2 - \beta_2 + \gamma_2 > 0$ is only invalid on half the wall. In particular, plugging in γ_1, the values of γ_2 for which the inequality is invalid are given by the inequality:

$$\gamma_2 > \alpha_2 + \beta_2 - 1$$

Now we consider the other wall passing through the vertex given in equation 5.3.2, which has formula $\gamma_1 = 3\gamma_2 + \alpha_2 + \beta_2 - 3\alpha_1 - 3\beta_1$. Actually, viewed as a function γ_1 of γ_2, this passes through $(\gamma_2, \gamma_1) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2 - 1)$, which does not satisfy$\gamma_1 < \gamma_2$, so we swap γ_1 and γ_2. Then in $\mathbb{T}^2/S_2$, the wall has equation $\gamma_2 = 3\gamma_1 + \alpha_2 + \beta_2 - 3\alpha_1 - 3\beta_1$. This can again be interpreted as the weights satisfying an integer relationship. In this case, we have:

$$3\gamma_1 + \alpha_2 + \beta_2 - 3\alpha_1 - 3\beta_1 - \gamma_2 = -1$$

Applying the transformation from $AB = C$ to $ABC = I$, we end up with the following relation:

$$\gamma_1 = -3 + 3\alpha_1 + 3\beta_1 + -\alpha_2 - \beta_2 + 3\gamma_2$$

Unlike the previous case, this does not reduce the inequality $6 + \alpha_1 + \beta_1 + \gamma_2 - 3\alpha_2 - 3\beta_2 - 3\gamma_1 > 0$ in a nice way. However, plugging in γ_1 to the inequality:

$$6 + \alpha_1 + \beta_1 + \gamma_2 - 3\alpha_2 - 3\beta_2 - 3\gamma_1 = 12 - 8\alpha_1 - 8\beta_1 - 8\gamma_2 + 6\alpha_2 + 6\beta_2$$

Now, $\alpha_2 > \alpha_1$ and $\beta_2 > \beta_1$, and $\alpha_j, \beta_j, \gamma_j < 1$, so finally

$$\begin{aligned} 12 - 8\alpha_1 - 8\beta_1 - 8\gamma_2 + 6\alpha_2 + 6\beta_2 &> 12 - 2\alpha_1 - 2\beta_1 - 8\gamma_2 \\ &> 0 \end{aligned}$$

This is valid, complimentary to the last wall, for $\gamma_2 < \alpha_2 + \beta_2 - 1$ (this is the direction where $\mathcal{O}(d-1) \hookrightarrow \mathcal{S}$ from above is still the unique destabilizing subbundle).

The last important thing to mention is that this analysis is valid as long as $\gamma_1 < \gamma_2$. When we hit the outer wall $\gamma_1 = \gamma_2$, the parabolic structure of the kernel $\mathcal{S}$ changes. Assuming γ_1, γ_2 are the weights at the marked point p, the parabolic structure changes from having two rank 1 jumps, at γ_1 and γ_2, to having a single rank 2 jump at $\gamma_1 = \gamma_2$. Unfortunately, the analysis past the $\gamma_1 = \gamma_2$ wall since we don't know what the filtration should look like on the other side.

The computations for other choices of index/vertex are similar. Geometrically, the picture is as follows. Passing through each vertex are two $U(2,1) \times U(1)$ walls. The above computations show that the portion of the walls moving away from the vertex are in the image of an irreducible under $\widetilde{\mu}$. Examples are shown in figures 5.6 and 5.7.

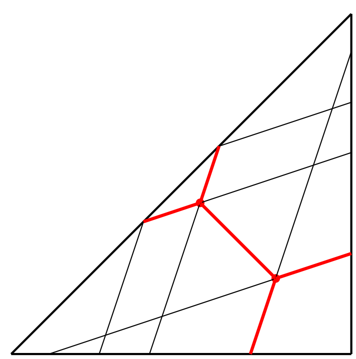


Figure 5.6: An example showing walls (highlighted in red) which are in the image of an irreducible under $\widetilde{\mu}$

Finally, in the case that the $U(1,1) \times U(2)$ wall is connected, we can say a little bit more. From figure 5.6 for example a pair of $U(2,1) \times U(1)$ walls intersecting different vertices with reciprocal slopes could intersect each other. It would be interesting to consider the image of $\widetilde{\mu}$ at this intersection point.

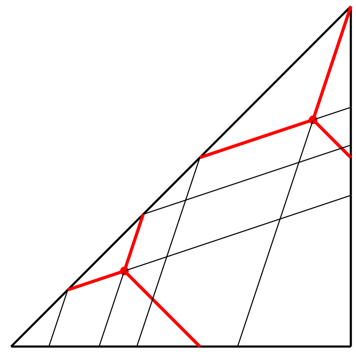


Figure 5.7: An example showing walls (highlighted in red) which are in the image of an irreducible under $\widetilde{\mu}$

As we change the weights away from a vertex and along a $U(2,1) \times U(1)$ with angle less than $\frac{\pi}{2}$ off the $U(1,1) \times U(2)$ wall, it is difficult to determine which, if any subbundles of $\mathcal{S}$ are destabilizing. In fact, none of them are! If we analyze the inequalities in Proposition 5.3.3 in a way similar to how we determined the bounds in the proof of Proposition 5.2.2, we can see that $\mathcal{S}$ remains stable if γ_1, γ_2 remain in between the two lines with slope 1, passing through the vertices. See figure 5.8 for an example.

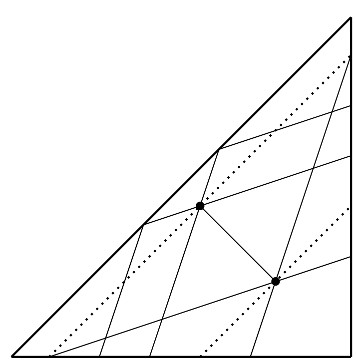


Figure 5.8: The kernel $\mathcal{S}$ remains stable for γ_1, γ_2 in between the dotted lines

Since $\mathcal{S}$ remains stable within this region, the inequalities determining the stability of $\mathcal{E}$ (from Proposition 5.3.4) are:

$$pdeg(\mathcal{S}) > -3$$

$$pdeg(\mathcal{S}) < 1$$

The first inequality is trivially valid. The second is not as obvious. We are interested in verifying that $pdeg(\mathcal{S}) < 1$ at the intersection of the two $U(2,1) \times U(1)$ walls of slope $\frac{1}{3}$ from the top vertex and slope 3 from the bottom vertex. Suppose for a contradiction that prior reaching this intersection point that $pdeg(\mathcal{S}) \geq 1$. Then in particular there is a choice of γ_1, γ_2 where $pdeg(\mathcal{S}) = 1$ (an integer!) and $\mathcal{S}$ is stable. This implies in particular that there is another nonempty $U(1,1) \times U(2)$ wall, contradicting our assumption that there is one connected $U(1,1) \times U(2)$ wall.

Therefore, when the $U(1,1) \times U(2)$ wall is connected, the intersections (when they exist) of $U(2,1) \times U(1)$ walls of reciprocal slope from opposite vertices are in the image of an irreducible under $\widetilde{\mu}$.

This argument does not work when the $U(1,1) \times U(2)$ wall is disconnected. Of course the contradiction above does not apply, but more importantly each piece of the wall only has one vertex. The $U(2,1) \times U(1)$ walls emanating from the two vertices still intersect, but are unrelated from the perspective of our higgs bundle construction. Therefore we would need a new strategy to show these intersections are in the image of an irreducible.

5.4 Conclusion

Finally we combine the descriptions of reducible walls in Propositions 5.2.1, 5.2.2, and 5.2.5; the results regarding image of $\widetilde{\mu}$ in Propositions 5.3.1 and 5.3.2, and the analysis along $U(2,1) \times U(1)$ walls in section 5.3.2. Some examples of what we know are summarized in figures 5.9, 5.10, and 5.11.

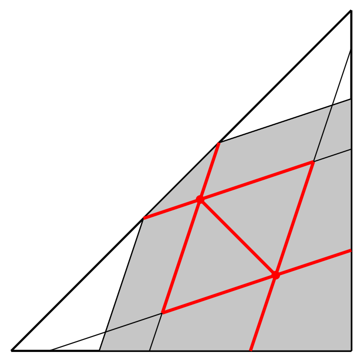


Figure 5.9: Since the red walls are in the image of an irreducible with regular product, the chambers they bound are full.

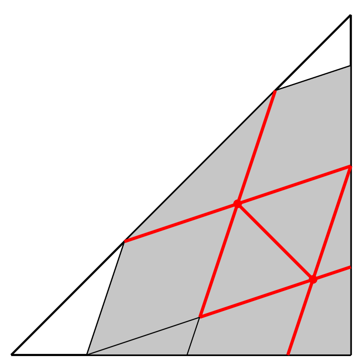


Figure 5.10: Since the red walls are in the image of an irreducible with regular product, the chambers they bound are full.

At this point, we have exhausted all of our tools for studying these diagrams. For diagrams of connected type, it is difficult to determine what happens when we

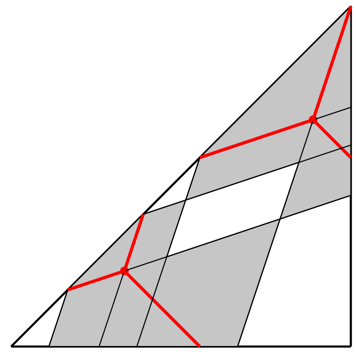


Figure 5.11: Since the red walls are in the image of an irreducible with regular product, the chambers they bound are full.

pass through the outer $\gamma_1 = \gamma_2$ wall, since we do not know what the parabolic structure of our Higgs bundle looks like on the other side of the wall. For diagrams of disconnected type, there is also the issue of what happens when $U(2,1) \times U(1)$ walls intersect. Our Higgs bundle techniques do not apply in this case, since each wall corresponds to Higgs bundles of different starting index. The Higgs bundles sitting above these intersections are therefore unrelated.

While there are still chambers which we cannot at this point guarantee are full, we have also failed to construct any examples of fixed loxodromic conjugacy classes in $PU(3,1)$ for which there is no solution. New insights are therefore required to proceed further.

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