ABSTRACT

Title of Dissertation: TOPICS IN HARMONIC ANALYSIS: THE HRT CONJECTURE AND SIGMA-DELTA QUANTIZATION

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In this thesis, we solve for special cases of the HRT Conjecture and provide an overview of $\Sigma\Delta$-Quantization.

The HRT conjecture has remained open ever since its introduction in 1996. Despite 25 years of concerted effort, even special cases of the conjecture with very stringent conditions are still unsolved. We solve special cases of the HRT conjecture for functions that are not smooth on a nonempty, bounded set and when the function has certain decay conditions.

We demonstrate how non-smooth functions that do not satisfy the HRT conjecture cannot be non-smooth at only a finite number of points or a bounded set of points. We introduce new definitions to quantify both the size and orientation of a discontinuity and various techniques for the resulting calculations.

We also study how a function's failure to satisfy the HRT conjecture affects the function's behavior at infinity. In order to do this, extensions of the real number line are defined along with new topologies on these extensions. These new topologies allow us to describe several new notions of convergence.

TOPICS IN HARMONIC ANALYSIS: THE HRT CONJECTURE AND SIGMA-DELTA QUANTIZATION

by

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1 Introduction

1.1 Overview

We will discuss two topics in this thesis. The first topic is the HRT conjecture while the second topic is $\Sigma\Delta$-quantization. Our fundamental contributions concern the HRT conjecture so the introduction will focus overwhelmingly on expositing the HRT conjecture, its difficulties and the techniques we used on it. The $\Sigma\Delta$-quantization material has been dealt with in section 7.

1.2 Description of the HRT Conjecture

1.2.1 Statement

Let $\Lambda = \{(\alpha_k, \beta_k)\}_{k=1}^n \subset \mathbb{R}^2$ where each (α_k, β_k) is distinct. Whenever we write a set of this form it is assumed that all elements are distinct. We define a finite Gabor system to be

$$\mathcal{G}(f, \Lambda) = \{e^{i\alpha_k t} f(t - \beta_k)\}_{k=1}^n.$$

Given the ubiquitous usefulness of Gabor frames in harmonic analysis, it is natural to ask if such a set is linearly independent. We define the HRT conjecture.

Conjecture 1.1. *For all nonzero $f \in L^2(\mathbb{R})$ and for all $\Lambda = \{(\alpha_k, \beta_k)\}_{k=1}^n$,*

$$\{e^{i\alpha_k t} f(t - \beta_k)\}_{k=1}^n$$

is a linearly independent set.

This conjecture, introduced in 1996 by Heil, Ramanathan and Topiwala in [1], is straightforward to solve in several simple situations that we are about to elaborate, but has so far proven intractable for even slightly more general cases. A variety of exceedingly technical methods have been applied from functional analysis, number theory and harmonic analysis, but the problem remains unsolved. It is unsolved even in special cases of the conjecture when the functions are very smooth and have fast decay or the sets, Λ, have four points.

In general, there are two methods for proving the HRT conjecture: The function approach and the point approach. The function approach is to take some function, f, and to demonstrate for all Λ whether or not $\{e^{i\alpha_k t} f(t-\beta_k)\}_{k=1}^n$ is linearly independent. We formally state this in the following definition.

Definition 1.2. *Let f be a complex valued function on $\mathbb{R}$. We say that f satisfies the HRT conjecture if for all $\{(\alpha_k, \beta_k)\}_{k=1}^n \subset \mathbb{R}^2$,*

$$\{e^{i\alpha_k t} f(t-\beta_k)\}_{k=1}^n$$

is a linearly independent set.

Note that we have placed no assumptions on f. We will in several places discuss functions that are not necessarily members of $L^2(R)$, but still satisfy the HRT in the above sense.

Alternatively, the point approach to the HRT conjecture is to take some set of points, Λ, and show that for all nonzero functions $f \in L^2(\mathbb{R})$, $\{e^{i\alpha_k t} f(t-\beta_k)\}_{k=1}^n$ is linearly independent. We state this in the following definition.

Definition 1.3. *Let $\Lambda = \{(\alpha_k, \beta_k)\}_{k=1}^n \subset \mathbb{R}^2$. We say that Λ satisfies the HRT conjecture if for all $f \in L^2(\mathbb{R})\backslash\{0\}$,*

$$\{e^{i\alpha_k t} f(t-\beta_k)\}_{k=1}^n$$

is a linearly independent set.

Note that while in definition 1.2, there is no restriction on the sort of functions we draw from, there is a restriction on which functions we use in definition 1.3. The reason is that every set of points, $\Lambda = \{(\alpha_k, \beta_k)\}_{k=1}^n$, will fail to satisfy the HRT conjecture for some function that is not a member of $L^2(\mathbb{R})$. To take a simple example, it will be demonstrated below that $\{(0,0),(0,2\pi)\}$ satisfies the

HRT conjecture when we deal with $L^2(\mathbb{R})$ functions, but this fails for functions in general. Let $f(t) = \sin(t)$. Then,

$$\{f(t), f(t-2\pi)\} = \{\sin(t), \sin(t-2\pi)\} = \{\sin(t), \sin(t)\}$$

which is a linearly dependent set.

The following is a hybrid statement that will help us to describe special cases of the conjecture when we have assumptions on both the function and the set of points.

Definition 1.4. *Let f be a complex valued function and $\Lambda = \{(\alpha_k, \beta_k)\}_{k=1}^n \subset \mathbb{R}^2$. We say that (f, Λ) satisfies the HRT conjecture if*

$$\{e^{i\alpha_k t} f(t-\beta_k)\}_{k=1}^n$$

is a linearly independent set.

Here is a sample of some sets of points and functions for which the HRT conjecture has been solved.

Theorem 1.5. *Let $f \in L^2(\mathbb{R})$ be nonzero and $\Lambda \subset \mathbb{R}^2$. (f, Λ) satisfies the HRT conjecture in the sense of definition 1.4 if any of the following hold.*

1. *f is compactly supported [1].*
2. *$f(t) = p(t)e^{-\pi t^2}$ where p is a polynomial. [1].*
3. *$\lim_{x \to x \to \infty} |g(x)| e^{cx \log(x)} = 0$ for all $c > 0$ [2].*
4. *f is ultimately positive and $\Lambda = \{(\alpha_k, \beta_k)\}_{k=1}^n$ is such that $\{\beta_k\}_{k=1}^n$ is independent over the rational numbers [3].*
5. *$\Lambda \subset A(\mathbb{Z}^2) + z$ where A is a full rank 2×2 matrix and $z \in \mathbb{R}^2$ [4].*
6. *Λ has four elements and two of the fours points lie on a line and the other two lie on a parallel line [5],[6].*

7. *f is Schwartz and Λ has four elements and three points lie on a line [5].*

8. *Λ is collinear [1].*

9. *Λ has N points, $N-1$ are collinear and equispaced [1].*

10. *f is real-valued and $\Lambda = \{(0,0),(0,1),(s,0),(a,b)\}$ where at least of a,b and ab is rational [7].*

Here are classes of functions for which the HRT conjecture remains unsolved.

Conjecture 1.6. *Let $f \in L^2(\mathbb{R})$ be a complex valued function. In the sense of definition 1.2, it is unknown whether f satisfies the HRT conjecture given any of the following conditions:*

1. *f is ultimately positive.*

2. *f is a Schwartz class function.*

3. *f has faster than exponential decay in the sense that*

$$\lim_{t\to\infty} e^{-at} f(t) \quad \textit{for all } a \in [0,\infty).$$

Here are classes of sets of points for which the HRT conjecture remains unsolved.

Conjecture 1.7. *Let $\Lambda = \{(\alpha_k,\beta_k)\}_{k=1}^n \subset \mathbb{R}^2$. In the sense of definition 1.3, it is unknown whether Λ satisfies the HRT conjecture given any of the following conditions:*

1. *Λ is a set of four points.*

2. *Λ is a subset of two parallel lines.*

3. *All but one point of Λ is a subset of a line.*

4. *All but two points of Λ are a subset of a line and equidistant.*

5. *$\{(\beta_k)\}_{k=1}^n$ is independent over the rationals.*

1.2.2 Basic Results

Let us begin examining some basic results on the HRT conjecture. This will help to illuminate not only what has been learned, but the limitations of the techniques so far used.

We define the Fourier transform of a function f to be the following.

Definition 1.8. *Let $f \in L^2(\mathbb{R})$. The Fourier transform of f is*

$$\hat{f}(\gamma) = \int_{\mathbb{R}} f(t) e^{-i\gamma t} dt.$$

Given this definition, we have the following identities.

Proposition 1.9. *Let $f \in L^2(\mathbb{R})$, $\alpha, \beta \in \mathbb{R}$. Then*

$$\widehat{e^{i\alpha t} f(t)}(\gamma) = \hat{f}(\gamma - \alpha)$$

and

$$\widehat{f(t - \beta)}(\gamma) = e^{i\beta\gamma} \hat{f}(\gamma).$$

Suppose that $\Lambda = \{(\alpha_k, 0)\}_{k=1}^n$. If Λ does not satisfy the HRT conjecture, then we have that there exists $f \in L^2(\mathbb{R})$ such that

$$\{e^{i\alpha_k t} f(t - \beta_k)\}_{k=1}^n$$

is linearly dependent. This implies that there exists $\{c_k\}_{k=1}^n \subset \mathbb{C}$ not all zero such that

$$\sum_{k=1}^n c_k e^{i\alpha_k t} f(t - \beta_k) = 0.$$

We can rewrite this as

$$f(t) \sum_{k=1}^n c_k e^{i\alpha_k t} = 0.$$

Since $\sum_{k=1}^n c_k e^{i\alpha_k t} = 0$ only a countable number of times, we have that $f = 0$ almost everywhere. By utilizing the Fourier transform, we can use this case to solve $\Lambda = \{(0, \beta_k)\}_{k=1}^n$ since

$$0 = \hat{0} = \sum_{k=1}^{n} \widehat{c_k f(t - \beta_k)}(\gamma) = \sum_{k=1}^{n} c_k e^{i\beta_k \gamma} \hat{f}(\gamma).$$

Difficulties start to arise only when we examine cases with both translation and modulation. Why is this? Part of the reason is that the Fourier transform does not in general reduce the problem to a simpler form. We see that if f is such that

$$\sum_{k=1}^{n} c_k e^{i\alpha_k t} f(t) = 0,$$

then taking the Fourier transform gives us

$$0 = \sum_{k=1}^{n} c_k \widehat{e^{-i\alpha_k t} f(t - \beta_k)}(\gamma) = \sum_{k=1}^{n} c_k e^{i\alpha_k \beta_k} e^{i\beta_k \gamma} \hat{f}(\gamma - \alpha_k).$$

This doesn't improve our situation in this particular case, because all we have done is shuffle the constants, but it does suggest one potential technique for solving the HRT: reduction of one (f, Λ) to another related (f_1, Λ_1). In the above, we have already seen how the Fourier transform reduced $(f, \{(\alpha_k, \beta_k)\}_{k=1}^n)$ to $(\hat{f}, \{(-\beta_k, \alpha_k)\}_{k=1}^n$ and in particular how it solved $\{(0, \beta_k)\}_{k=1}^n$ by reducing it to $\{(\beta_k, 0)\}_{k=1}^n$. The more general technique takes advantage of metaplectic transforms in order to reduce one case of the HRT conjecture to another.

The metaplectic transforms are the area-preserving affine transformations of phase space and they preserve the linear independence of $\{e^{i\alpha_k t} f(t - \beta_k)\}_{k=1}^n$. If the pair $(f, \{(\alpha_k, \beta_k)\}_{k=1}^n)$ satisfies the HRT conjecture, then the following also satisfy the HRT conjecture:

1. Time translation and horizontal translations:$(f(t - \beta), \{(\alpha_k, \beta_k + \beta)\}_{k=1}^n)$
2. Modulations and vertical translations: $(e^{i\alpha t} f, \{(\alpha_k + \alpha, \beta_k)\}_{k=1}^n)$.
3. Rescalings and dilation: $(D_r(f)(t) = |r|^{\frac{1}{2}} f(rt), \{(\frac{\alpha_k}{r}, r\beta_k)\}_{k=1}^n)$.
4. Modulation by a linear chirp and shears: $(e^{i\alpha t^2} f(t), \{(\alpha_k + 2\alpha\beta_k, \beta_k)\}_{k=1}^n)$.

5. The Fourier transform and rotation by $\pi/2$: $(\hat{f}, \{(-\beta_k, \alpha_k)\}_{k=1}^n)$.

These fundamental transformations can be utilized together to reach the following proposition.

Proposition 1.10. *Let $f \in L^2(\mathbb{R})\backslash\{0\}$ be a complex valued function and $\Lambda = \{(\alpha_k, \beta_k)\}_{k=1}^n \subset \mathbb{R}^2$. Let $\{e^{i\alpha_k t} f(t-\beta_k)\}_{k=1}^n$ be a linearly independent set. For every 2×2 matrix, M, with determinant one, and $(\alpha, \beta) \in \mathbb{R}^2$ there exists a nonzero $g \in L^2(\mathbb{R})$ such that (f, Λ) satisfies the HRT conjecture if and only if $(g, M\Lambda + (\alpha, \beta))$ satisfies the HRT conjecture.*

This allows us to generalize the technique used to solve the pure translation case to all Λ that are a subset of a line.

Proposition 1.11. *Let $\Lambda \subset \mathbb{R}^2$ be a subset of a line. Then, Λ satisfies the HRT conjecture.*

Proof. Since Λ is a subset of a line, there exists a matrix M and (α, β) such that $M\Lambda + (\alpha, \beta)$ is a subset of $\{(x, 0) | \quad x \in \mathbb{R}\}$. By proposition 1.10, we have that there exists a nonzero $g \in L^2(\mathbb{R})$ such that (f, Λ) satisfies the HRT conjecture if and only if $(g, M\Lambda + (\alpha, \beta) = \{(\alpha'_k, 0)\})$ satisfies the HRT conjecture. We have already seen that

$$\{e^{i\alpha'_k t} g(t)\}_{k=1}^n$$

is a linearly independent set. Thus, (f, Λ) satisfies the HRT conjecture for all nonzero $f \in L^2(\mathbb{R})$. □

A difficulty that is incurred while using metaplectic transforms is that they in general and shears in particular do not preserve crucial features of f. Decay, continuity, real range, symmetry and other important features can be lost. Nonetheless, metaplectic transformations are of great use while proving the HRT conjecture for a given class of points where no assumptions have been made on the functions.

Now we turn from the point formulation of the HRT to the function formulation of the HRT and prove some simple cases that certain functions fail to satisfy the HRT. It is straightforward to demonstrate that functions with bounded support will satisfy the HRT.

Proposition 1.12. *Let f be a nonzero function with bounded support. Then, for all $\{(\alpha_k, \beta_k)\}_{k=1}^n$,*

$$\{e^{i\alpha_k t} f(t - \beta_k)\}_{k=1}^n \tag{1}$$

is linearly independent.

Proof. Let f be as above. Suppose that there exists $\Lambda = \{\alpha_k, \beta_k\}_{k=1}^n$ such that (1) is linearly dependent. Thus, there exists $\{c_k\}_{k=1}^n \subset \mathbb{R}$ not all zero such that

$$\sum_{k=1}^{n} c_k e^{i\alpha_k t} f(t - \beta_k) = 0. \tag{2}$$

Without loss of generality, we assume that all c_k are nonzero and that $\beta_k \leqslant \beta_{k+1}$. We rewrite (2) as

$$\sum_{j=1}^{m} p_j(t) f(t - \beta_j^*) = \sum_{k=1}^{n} c_k e^{i\alpha_k t} f(t - \beta_k) = 0$$

where $\{\beta_j^*\} - j = 1^m$ are the unique translations of Λ and p_j are the corresponding exponential polynomials. Since f has bounded support, the support has a least upper bound, $M \in \mathbb{R}$. Consider the interval $[M - (\beta_2 - \beta_1), M]$. Let $t \in [M, M + \beta_2 - \beta_1]$. Applying (2), we have that

$$-p_1(t) f(t + \beta_1^*) = \sum_{j=2}^{n} p_k(t) f(t + \beta_k^*) = 0,$$

because for all $1 \leqslant j \leqslant n$, we have that

$$t + (\beta_j^* - \beta_1^*) > t + (\beta_2^* - \beta_1^*) \geqslant M + (\beta_2^* - \beta_1^*) \geqslant M.$$

Thus,

$$p_1(t)f(t + \beta_1^*) = 0, \quad \text{for all } t \in [M - (\beta_2 - \beta_1), M].$$

$p_1(t) = 0$ for at most a finite number of points so f is 0 almost everywhere on $[M - (\beta_2 - \beta_1), M]$. But this would demand that M was not the lowest upper bound on the support of f. Thus f has no lowest upper bound on its support, demanding that the support is an empty set. □

We would like to generalize this to other forms of decay like faster than exponential decay, but a major stumbling block for doing this is the case of non-unique translations. We will find again later that nonunique translations will cause difficulties in the techniques used in this dissertation but for radically different reasons.

1.3 Results

1.3.1 Non-smoothness and the HRT

If (f, Λ) fails to satisfy the HRT conjecture, then we have that there exists $\{c_k\}_{k=1}^n$ not all zero, such that

$$\sum_{k=1}^{n} c_k e^{i\alpha_k t} f(t - \beta_k) = 0.$$

Since this equation relates the value of a function to later and earlier values of the function, it occurred to the author to ask "What if the function is not smooth at some point?" Since the set of smooth functions is closed under addition, we have that if a function lacked smoothness at a specific value, then that lack of smoothness is perpetuated to later and earlier values of the function. This suggests that f is not smooth at perhaps an infinite number of points. In contrast to the standard approach of asking "If our function is sufficiently smooth, will the HRT be satisfied?" We have asked "If our function does not satisfy the HRT conjecture and is not smooth, what further properties does this function have?" We ask this, because if we have a function that is not smooth but fails to have the additional properties, then we are forced to conclude that f satisfies the HRT conjecture.

Let us demonstrate with a simple example. Let f be a real-valued function and $\Lambda = \{(0,0), (\alpha_1, \beta_1), (\alpha_2, \beta_2)\}$ where $0 < \beta_1 < \beta_2$ and $0 \neq \alpha_1, \alpha_2$ and f be such that

$$f(t) - e^{i\alpha_1 t} f(t - \beta_1) - e^{i\alpha_2 t} f(t - \beta_2) = 0.$$

Rewriting the above, we have that

$$f(t) = e^{i\alpha_1 t} f(t - \beta_1) + e^{i\alpha_2 t} f(t - \beta_2). \tag{3}$$

Suppose f has a discontinuity at 0. Then, the right hand side must have a discontinuity at 0 also. This demands that at least one of terms of the sum is discontinuous which in turn demands that f is discontinuous at at least one point in the set $\{\beta_1, \beta_2\}$. Additionally, we can also isolate either of the other two terms to attain

$$e^{i\alpha_1 t} f(t - \beta_1) = -f(t) + e^{i\alpha_2 t} f(t - \beta_2)$$

and

$$e^{i\alpha_2 t} f(t - \beta_2) = -f(t) + e^{i\alpha_1 t} f(t - \beta_1).$$

Since the the left hand side of each equation is discontinuous at β_1 and β_2, respectively, an identical argument shows us that there exists at least one discontinuity on each of the sets $\{-\beta_1, \beta_2 - \beta_1\}$ and $\{-\beta_2, \beta_1 - \beta_2\}$.

We see that (3) can also be be applied to the newly discovered discontinuities. More specifically, for every discontinuity t_0, we can find a new discontinuity on the sets $\{t_0 + \beta_1, t_0 + \beta_2\}$ and $\{t_0 - \beta_2, t_0 + \beta_1 - \beta_2\}$ which are sets either strictly greater or strictly smaller than t_0. By an inductive argument we can demonstrate that f must have an infinite and unbounded set of discontinuities.

This suggest that if a function only has a finite number of discontinuities or a bounded set of discontinuities, that would demand that it satisfies the HRT.

A major problem with this technique arises when one allows nonunique translations. Let's take an example where two of the shifts are the same: $\Lambda = \{(0,0),(1,0),(0,1)\}$. Let f be such that

$$f(t) - e^{it}f(t) - f(t-1) = 0.$$

We can rewrite this as

$$(1-e^{it})f(t) = f(t-1). \tag{4}$$

If f is discontinuous at some point in $\{n\pi\}_{n\in\mathbb{Z}}$, then $f(t-1)$ is not necessarily discontinuous at that same point. For example, $f(t) = H(t)g(t)$ where g is a continuous function and $H = \mathbb{1}_{[0,\infty)}$. Note that

$$(1-e^{it})f(t)|_{t=0} = (1-e^{it})H(t)g(t)|_{t=0} =$$

$$(1-e^{i0})H(0)g(0) = (1-1)(1)g(0) = 0.$$

Clearly, f is discontinuous at 0, but we see that

$$\lim_{t\to 0}|(1-e^{it})f(t) - (1-e^{it})f(t)|_{t=0}| =$$

$$\lim_{t\to 0}|(1-e^{it})H(t)g(t) - (1-e^{it})H(t)g(t)|_{t=0}| = \lim_{t\to 0}|(1-e^{it})H(t)g(t)| =$$

$$\lim_{t\to 0}|H(t)||g(t)||1-e^{it}| \leqslant \lim_{t\to 0}|1||g(0)||1-e^{it}| = 0.$$

Thus, in spite of the fact that $f(t)$ is discontinuous at 0, $(1-e^{it})f(t)$ is not discontinuous. This suggest a serious problem for the propagation of discontinuities in a function that does not satisfy the HRT conjecture.

One technique to handle this difficulty is to not merely consider discontinuity but non-smoothness in general. Returning to our last example, notice that

$$\lim_{h\to 0}\frac{(1-e^{ih})f(t)-(1-e^{it})f(t)|_{t=0}}{h}=\lim_{h\to 0}\frac{(1-e^{ih})f(h)}{h}=$$

$$\lim_{h\to 0}\frac{(ih+h^2g_0(h))f(h)}{h}=\lim_{h\to 0}\frac{ihf(h)}{h}+\lim_{h\to 0}\frac{h^2g_0(h))f(h)}{h}=$$

$$\lim_{h\to 0}if(h)+\lim_{h\to 0}hg_0(h))f(h)=\lim_{h\to 0}iH(h)g(h).$$

This last limit does not exist, so even though f was made continuous at 0 by multiplication by the exponential polynomial, $(1-e^{it}$, it was not made differentiable. This suggests that if a function is not smooth at a point, multiplication by an exponential polynomial will not make it smooth. We show in section 2 that even though we cannot propagate discontinuities, we can propagate non-smoothness.

This technique culminates in theorem 2.22 and is explicated in particular important cases in corollaries 2.23, 2.24, 2.25 2.26. We summarize these results in the theorem below.

Theorem 1.13. *Let f be a continuous complex valued function and A be the set of all points on f that are not infinitely differentiable. If A is nonempty and has Beurling density of zero, then f satisfies the HRT conjecture.*

Another way to handle the difficulty that nonunique translations introduces is to avoid the problem altogether: assume the translation are unique. This allows us to propagate the discontinuities, although it curtails which Λ we can examine. On the other hand, if we also assume that f is real, then we will be able propagate even further discontinuities.

To demonstrate this, let us return to our original example:

$$(f,\{(0,0),(\alpha_1,\beta_2),(\alpha_2,\beta_2)\})$$

where $0 < \beta_1 < \beta_2$ and $0 < \alpha_1 \neq \alpha_2$. Let f be real and such that

$$f(t) = e^{i\alpha_1 t} f(t - \beta_1) + e^{i\alpha_2 t} f(t - \beta_2).$$

We see that if f has a jump discontinuity at $t_0 \neq 0$, then the graph has a jump along the real axis. This demands that the right hand side also has a jump in the real direction. But at t_0, $e^{i\alpha_1 t} f(t - \beta_1)$ and $e^{i\alpha_2 t} f(t - \beta_2)$ do not have discontinuities in the real direction for all but a countable number of points. Thus, both $e^{i\alpha_1 t} f(t - \beta_1)$ and $e^{i\alpha_2 t} f(t - \beta_2)$ are discontinuous and in such a way that their sum is discontinuous along the real axis.

In order to generalize this insight, we define the notions of discrepancy and the axis of discrepancy. The latter notion is rather difficult to make calculations directly for so further notions like condensation, jumps and jump phase are defined in order to simplify the calculations. This results in the following theorem.

Theorem 1.14. *Let $\Lambda = \{(\alpha_k, \beta_k)\}_{k=1}^n$ be an HRT constellation where each α_k is unique and $\{\beta_k\}_{k=1}^n$ is rationally independent. Let f be a real valued function that has a discontinuity on D_Λ. If there is an interval, $[a, b]$, of length larger than $\beta_n - \beta_1$ such that f is discontinuous only on a finite number of times on that interval, then (f, Λ) satisfies the HRT conjecture.*

Note that D_Λ is a subset of $\mathbb{R}$ whose complement is countable.

In an attempt to consider the discontinuities of unbounded functions, we define the flat extended complex plane. This extension of the complex plane is distinct from the Riemann Sphere in that instead of adding a single infinite point, we define a ring of infinities, each one with their own modulation. The construction is similar to that of the extended real line in that the extended real line has both a positive and negative infinity and the flat extended complex plane includes such points as $i\infty$, $-i\infty$, and $\frac{1+i}{\sqrt{2}}$.

1.3.2 Periodic Infinity

We also considered how the end behavior of f is affected by a failure to satisfy the HRT conjecture. In previous treatments, the question was primarily phrased as "If our function has sufficiently fast decay, will the HRT be satisfied?" We asked "Given our function fails to decay and does not satisfy the HRT conjecture, what properties does this imply for our function?"

Just as in non-smoothness, we created a new set of analytic tools in order to describe the end behavior of a function in a way that was conducive to study of the HRT conjecture. The foundational concept in this approach is the notion of periodic convergence. We define the notion in terms of a new topology constructed on an extended version of the real number line in $\mathbb{R}_\alpha$. In essence, we say that a sequence $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R} \subset \mathbb{R}_\alpha$ converges to ∞_θ in $\mathbb{R}_\alpha$ if for all $M \in [0, \infty)$ and for all $\epsilon > 0$, there exists $N \geqslant 1$ such that

$$t_n > M \quad \text{and} \quad |e^{i\theta} - e^{i\alpha t_n}| < \epsilon \text{for all } n \geqslant N.$$

Using this notion of periodic convergence, we apply our notion of condensation to see what values of f recur on sequences that converge periodically. This development culminates in theorem 6.50 reproduced as follows.

Theorem 1.15. *Let $\{(\alpha_k, \beta_k)\}_{k=1}^{n} \subset \mathbb{R}^2$ and $\{c_k\}_{k=1}^{n}$ be not all zero. Let f be a real valued function such that there exists M such that $f([M, \infty))$ is bounded and*

$$\sum c_k e^{i\alpha_k t} f(t - \beta_k) = 0.$$

Then,

$$0 \in con_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f) \textit{for all } \boldsymbol{\theta} \in L^d.$$

Our application of this theorem to the HRT conjecture leads to the following theorem.

Theorem 1.16. *Let f be a real valued function, $\Lambda = \{\alpha_k, \beta_k\}_{k=1}^{n}$ and $\{c_k\}_{k=1}^{n}$*

be not all zero. If there exists $\epsilon > 0$*,* $N \in \mathbb{N}$ *such that for all* $t \in [N, \infty)$

$$f(t_0) \notin B_0(\epsilon),$$

then, (f, Λ) *satisfies the HRT conjecture.*

2 Propagation of Non-Smoothness

We will be approaching the HRT from a pointwise perspective in this section so we will assume that our function f is continuous.

Definition 2.1. *Let f be a continuous complex valued function. We say that f satisfies the HRT conjecture if for all $\{(\alpha_k, \beta_k)\}_{k=1}^n \subset \mathbb{R}^2$,*

$$\{e^{i\alpha_k t} f(t - \beta_k)\}_{k=1}^n$$

is a linearly independent set.

We define $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$.

Proposition 2.2. *Let f be a continuous complex valued function. f satisfies the HRT conjecture if and only if for all $\{(\alpha_k, \beta_k)\}_{k=1}^n$, for all $\{c_k\}_{k=1}^n \subset \mathbb{C}^*$, there exists $t \in \mathbb{R}$ such that*

$$\sum_{k=1}^n c_k e^{i\alpha_k t} f(t - \beta_k) \neq 0.$$

Proof. Let f be as above. Suppose that f satisfies the HRT conjecture. Let $\{(\alpha_k, \beta_k)\}_{k=1}^n$ and $\{c_k\}_{k=1}^n \subset \mathbb{C}^*$. Then, there exists $t \in \mathbb{R}$ such that

$$\sum_{k=1}^n c_k e^{i\alpha_k t} f(t - \beta_k) \neq 0.$$

We have demonstrated the forward direction.

Suppose that f does not satisfy the HRT conjecture. Then, there exists $\{(\alpha_k, \beta_k)\}_{k=1}^n$ such that

$$\{e^{i\alpha_k t} f(t - \beta_k)\}_{k=1}^n$$

is linearly dependent. This means there exists $\{c_k\}_{k=1}^{n_0} \subset \mathbb{C}$ where not all c_k are zero such that

$$\sum_{k=1}^n c_k e^{i\alpha_k t} f(t - \beta_k) = 0 \quad \text{for all } t \in \mathbb{R}. \tag{5}$$

Let $\{k_l\}_{l=1}^{n_0}$ be all indices such that $c_{k_l} \neq 0$. Applying this to (5), we have that

$$\sum_{l=1}^{n_0} c_{k_l} e^{i\alpha_{k_l} t} f(t-\beta_{k_l}) = \sum_{k=1}^{n} c_k e^{i\alpha_k t} f(t-\beta_k) = 0 \quad \text{for all } t \in \mathbb{R}. \tag{6}$$

Since $\{(\alpha_{k_l}, \beta_{k_l})\}_{l=1}^{n_0}$, $\{c_{k_l}\}_{l=1}^{n_0} \subset \mathbb{C}^*$ and (6) holds, we have demonstrated the backward direction. □

Definition 2.3. *Let* $\Xi = \{(c_k, \alpha_k, \beta_k)\}_{k=1}^{n} \subset \mathbb{C} \times \mathbb{R}^2$, $\{c_k\}_{k=1}^{n} \subset \mathbb{C}^*$ *and* Ξ *is nonempty. We call such a* Ξ *an HRT configuration.*

Definition 2.4. *Let* f *be a continuous complex valued function and* Ξ *be an HRT configuration. We say that* f *satisfies the* Ξ*-configuration if there exists* $t \in \mathbb{R}$ *such that*

$$\sum_{k=1}^{n} c_k e^{i\alpha_k t} f(t-\beta_k) \neq 0$$

and that f *does not satisfy the* Ξ*-configuration if*

$$\sum_{k=1}^{n} c_k e^{i\alpha_k t} f(t-\beta_k) = 0 \quad \textit{for all } t \in \mathbb{R}.$$

Proposition 2.5. *Let* f *be a continuous complex valued function.* f *satisfies the HRT conjecture if and only if for all HRT configurations,* Ξ*,* f *satisfies the* Ξ*-configuration.*

Proof. We will be using the equivalent condition in proposition 2.2 to characterize satisfaction and dissatisfaction of the HRT conjecture.

Suppose that f satisfies the HRT conjecture. Let $\Xi = \{(c_k, \alpha_k, \beta_k)\}_{k=1}^{n}$ be an HRT configuration. By definition of an HRT configuration, we have that $\{(\alpha_k, \beta_k)\}_{k=1}^{n}$ and $\{c_k\}_{k=1}^{n} \subset \mathbb{C}^*$. Since f satisfies the HRT conjecture, we have that there exists $t \in \mathbb{R}$ such that

$$\sum_{k=1}^{n} c_k e^{i\alpha_k t} f(t-\beta_k) \neq 0.$$

Thus, f satisfies the Ξ-configuration.

Suppose that f does not satisfy the HRT conjecture. Then, there exists $\{(\alpha_k, \beta_k)\}_{k=1}^n$ and there exists $\{c_k\}_{k=1}^n \subset \mathbb{C}^*$,

$$\sum_{k=1}^{n} c_k e^{i\alpha_k t} f(t-\beta_k) = 0 \quad \text{for all } t \in \mathbb{R}. \tag{7}$$

Define $\Xi = \{(c_k, \alpha_k, \beta_k)\}_{k=1}^n$. We have that each (α_k, β_k) is distinct and $\{c_k\}_{k=1}^n \subset \mathbb{C}^*$. Thus, Ξ is an HRT configuration and by (7), f does not satisfy the Ξ-configuration. □

Lemma 2.6. *Let Ξ be an HRT configuration. There exists*

$$\{\Xi_j = \{(c_k^l, \alpha_j^l, \beta_j^*)\}_{l=1}^{n_j}\}_{j=1}^{n_0}$$

such that $\beta_{j_1}^* \neq \beta_{j_2}^*$ if $j_1 \neq j_2$,

$$\bigcup_{j=1}^{n_0} \Xi_j = \Xi \quad \text{and} \quad \Xi_{j_1} \cap \Xi_{j_2} = \varnothing \quad \text{for all } j_1 \neq j_2.$$

We call $\{\Xi_j\}_{j=1}^n$ the translation subsets of Ξ and $\{\beta_k^*\}_{k=1}^{n_0}$ the unique translations of Ξ.

Proof. Let Ξ be as above. Define $\{\beta_j^*\}_{j=1}^{n_0} = \{\beta_k\}_{k=1}^n$ where each β_j^* is distinct. Define

$$\Xi_j = \{(c_k, \alpha_k, \beta_k) \in \Xi | \quad \beta_k = \beta_j^*.\}$$

Let $(c_k, \alpha_k, \beta_k) \in \Xi$. Then, there exists j such that $\beta_k = \beta_j^*$. This demands that $(c_k, \alpha_k, \beta_k) \in \Xi_j$. Since this is true of all elements of Ξ, $\Xi = \bigcup_{j=1}^{n_0} Xi_j$.

Let $j_1 \neq j_2$. Then, $\beta_{j_1}^* \neq \beta_{j_2}^*$ which demands that for all $1 \leqslant l \leqslant n_{j_1}$, $\alpha_{j_1}^l \notin \Xi_{j_2}$. Thus, $\Xi_{j_1} \cap \Xi_{j_2} = \varnothing$. □

Lemma 2.7. *Let Ξ be an HRT configuration, $\{\beta_k^*\}_{k=1}^{n_0}$ be the unique translations of Ξ and $\{\Xi_j\}_{j=1}^{n_0}$ be the translation subsets of Ξ. Let f not satisfy the Ξ-*

configuration. Then,

$$\sum_{j=1}^{n_0} p_j(t) f(t-\beta_j^*) = 0 \quad \textit{for all } t \in \mathbb{R}.$$

where $p_j(t) = \sum_{l=1}^{n_l} c_j^l e^{i\alpha_j^l t}$*. We call the set,* $\{p_j(t)\}_{j=1}^{n_0}$*, the translation polynomials of* Ξ*.*

Proof. Let f be as above. Then, we can write

$$0 = \sum_{k=1}^{n} c_k e^{i\alpha_k t} f(t-\beta_k) = \sum_{j=1}^{n_0} \sum_{l=1}^{n_l} c_j^l e^{i\alpha_j^l t} f(t-\beta_j^*) = \sum_{j=1}^{n_0} p_j(t) f(t-\beta_j^*).$$

We can break up the sum in this way, because lemma 2.6 gives us

$$\bigcup_{j=1}^{n_0} \Xi_j = \Xi \quad \text{and} \quad \Xi_{j_1} \cap \Xi_{j_2} = \varnothing \quad \text{for all } j_1 \neq j_2.$$

□

Lemma 2.8. *Let* f *be a continuous complex valued function that is not* m*-times continuously differentiable at* t_0*. Let* $g \in C^\infty(\mathbb{R})$ *and* $g(t_0) = 0$*. Then,* $(1+g)f$ *is not* m*-times continuously differentiable at* t_0*.*

Proof. Let f and g be as above. Without loss of generality, we may assume that $t_0 = 0$. Suppose that $0 \leqslant l < m$ is the highest integer such that f is l-times continuously differentiable. Then,

$$[(1+g)f]^{(l)} = \sum_{k=0}^{l} \binom{l}{k} (1+g)^{(l-k)} f^{(k)} =$$

$$(1+g)f^{(l)} + \sum_{k=0}^{l-1} \binom{l}{k} (1+g)^{(l-k)} f^{(k)} =$$

$$f^{(l)} + f^{(l)}g + \sum_{k=0}^{l-1} \binom{l}{k} g^{(l-k)} f^{(k)}. \tag{8}$$

The third term in (8) is a sum of the products of continuously differentiable functions, so it is continuously differentiable. Thus, the differentiability of

$[(1+g)f]^{(l)}$ is entirely dependent on the differentiability of $f^{(l)} + f^{(l)}g$.

Note that

$$\lim_{h\to 0} \frac{f^{(l)}(0+h)g(0+h) - f^{(l)}(0)g(0)}{h} = \lim_{h\to 0} \frac{f^{(l)}(h)g(h) - f^{(l)}0}{h} =$$

$$\lim_{h\to 0} \frac{f^{(l)}(h)g(h)}{h} = \lim_{h\to 0} f^{(l)}(h)\frac{g(h)-0}{h} = \lim_{h\to 0} f^{(l)}(h)\frac{g(0+h)-g(0)}{h} =$$

$$\lim_{h\to 0} f^{(l)}(h)\frac{g(0+h)-g(0)}{h} \lim_{h\to 0} = f^{(l)}(0)g'(0).$$

This demands that

$$[(1+g)f]^{(l+1)}(0) = [f^{(l)} + f^{(l)}g]^{(1)}|_{t=0} =$$

$$\lim_{h\to 0} \frac{1}{h}\left[f^{(l)}(0+h) + f^{(l)}(0+h)g(0+h) - f^{(l)}(0) - f^{(l)}(0)g(0)\right] =$$

$$\lim_{h\to 0} \frac{1}{h}\left[f^{(l)}(0+h) - f^{(l)}(0)\right] + \lim_{h\to 0} \frac{1}{h}\left[f^{(l)}(0+h)g(0+h) - f^{(l)}(0)g(0)\right] =$$

$$\lim_{h\to 0} \frac{1}{h}\left[f^{(l)}(0+h) - f^{(l)}(0)\right] + f^{(l)}(0)g'(0). \tag{9}$$

The first term of (9) does not exist so $[(1+g)f]^{(l+1)}(0)$ does not exist. Thus, $(1+g)f$ is not n times continuous differentiable at 0. □

Lemma 2.9. *Let f be a continuous complex valued function that is not m-times continuously differentiable at t_0. Then, $(t-t_0)^n f$ is not $(m+n)$-times continuously differentiable at t_0.*

Proof. Let f be as above. Without loss of generality, we assume that $t_0 = 0$. We proceed by induction. For the base case of $l = 0$, we have by assumption that $t^0 f$ is not $(m+0)$-times continuously differentiable.

We proceed to the induction step. Our induction assumption is that $t^n f$ is not $(m+n)$-times continuously differentiable at t_0. We show that $t^{n+1} f$ is not $(m+n+1)$-times continuously differentiable at t_0. Suppose that $0 \leqslant l < m+n$ is the highest integer such that $t^n f$ is l-times continuously differentiable. Define

$g(t) = t^n f(t)$. Thus,

$$(t^{n+1}f)^{(l)} = ((t)(t^n f))^{(l)} = (tg)^{(l)} = \sum_{k=0}^{l} \binom{l}{k} t^{(k)} g^{(l-k)} =$$

$$\sum_{k=0}^{1} \binom{l}{k} t^{(k)} g^{(l-k)} = ng^{(l-1)} + tg^{(l)}.$$

Thus,

$$(t^{n+1}f)^{(l+1)} = \left(ng^{(l-1)} + tg^{(l)}\right)^{(1)} =$$

$$\lim_{h\to 0} \frac{[ng^{(l-1)}(t+h) + (t+h)g^{(l)}(t+h)] - [ng^{(l-1)}(t) + tg^{(l)}(t)]}{h} =$$

$$\lim_{h\to 0} \frac{[ng^{(l-1)}(t+h) - ng^{(l-1)}(t)] + [(t+h)g^{(l)}(t+h) - tg^{(l)}(t)]}{h} =$$

$$\lim_{h\to 0} \frac{ng^{(l-1)}(t+h) - ng^{(l-1)}(t)}{h} + \lim_{h\to 0} \frac{(t+h)g^{(l)}(t+h) - tg^{(l)}(t)}{h} =$$

$$ng^{(l)}(t) + \lim_{h\to 0} \frac{hg^{(l)}}{h} + t \lim_{h\to 0} \frac{g^{(l)}(t+h) - g^{(l)}(t)}{h} =$$

$$(n+1)g^{(l)}(t) + t \lim_{h\to 0} \frac{g^{(l)}(t+h) - g^{(l)}(t)}{h}. \tag{10}$$

The first term is continuous at t_0 by our induction assumption so the continuity of $(t^{n+1}f)^{(l+1)}$ at 0 is entirely contingent upon the continuity at 0 of the second term. If the second term is not continuous at 0, then $(t^{n+1}f)^{(l+1)}$ is not continuous and we have proven that $t^{n+1}f$ is no more than l-times continuously differentiable. Since $l < n+m < n+m+1$, we would have proven the induction step. Suppose that the second term is continuous. Then,

$$(t^{n+2}f)^{(l+1)}(0) = \left((n+1)g^{(l)}(t) + t \lim_{h\to 0} \frac{g^{(l)}(t+h) - g^{(l)}(t)}{h}\right)^{(1)}\Bigg|_{t=0} =$$

$$\lim_{h_1\to 0} \frac{1}{h_1}\left[\left[(n+1)g^{(l)}(h_1) + h_1 \lim_{h\to 0} \frac{g^{(l)}(h_1+h) - g^{(l)}(h_1)}{h}\right] - \right.$$

$$\left.\left[(n+1)g^{(l)}(0) + 0 \lim_{h\to 0} \frac{g^{(l)}(0+h) - g^{(l)}(0)}{h}\right]\right] =$$

$$\lim_{h_1\to 0} \frac{(n+1)g^{(l)}(h_1) - (n+1)g^{(l)}(0)}{h_1} + \lim_{h_1\to 0} \frac{h_1}{h_1} \lim_{h\to 0} \frac{g^{(l)}(h_1+h) - g^{(l)}(h_1)}{h} =$$

$$(n+1)g^{l+1}(0) + \lim_{h_1 \to 0} g^{l+1}(h_1) = (n+2)g^{l+1}.$$

By the induction assumption, g^{l+1} is discontinuous at 0 so $(t^{n+2}f)^{(l+1)}(0)$ is discontinuous. we have proven that $t^{n+1}f$ is no more than $(l+1)$-times continuously differentiable. Since $l+1 < n+m+1$, we have proven the induction step. □

Lemma 2.10. *Let $m, n \in \mathbb{R}$ and f be a continuous complex valued function that is not m-times continuously differentiable at t_0. Let p be an analytic function such that there exists $0 \leqslant k < n$ such that $p^{(k)}(t_0) \neq 0$. Then, $p(t)f(t)$ is not $(n+m)$-times continuously differentiable at t_0.*

Proof. Let f and p be as above. Since p is analytic, there exists $\{c_l\}_{l=0}^{\infty} \subset \mathbb{C}$ such that

$$p(t) = \sum_{l=0}^{\infty} c_l t^l.$$

Let k_0 be the largest integer in $\{0, 1, ..., n-1\}$ such that $p^{(l)}(t_0) = 0$ for all $1 \leqslant l \leqslant k_0$. By our conditions on the derivatives of p, we have that $c_l = 0$ for all $0 \leqslant l < n$ and $c_l \neq 0$ for all $n \leqslant l$. This grants us

$$p(t) = \sum_{l=0}^{\infty} c_l t^l = \sum_{l=k_0}^{\infty} c_l t^l = t^{k_0} \sum_{l=k_0}^{\infty} c_l t^{l-k_0} = t^{k_0} \sum_{l=0}^{\infty} c_{k_0+l} t^l =$$

$$t^{k_0}\left(c_n + \sum_{l=1}^{\infty} c_{k_0+l} t^l\right) = c_{k_0} t^{k_0}\left(1 + t\sum_{l=1}^{\infty} \frac{c_{k_0+l}}{c_{k_0}} t^{l-1}\right) = c_{k_0} t^{k_0}(1+h(t))$$

where $h(t) = t\sum_{l=1}^{\infty} \frac{c_{k_0+l}}{c_{k_0}} t^{l-1}$. Clearly, h is infinitely differentiable and $h(0) = 0$. and By lemma 2.8, we have that $(1+h)f$ is not m-times continuously differentiable at t_0. By lemma 2.9, we have that $t^{(k_0)}(1+h)f$ is not $(n+k_0)$-times continuously differentiable at t_0 which implies that it is not $(n+m)$-times continuously differentiable. Thus, $p(t)f(t) = c_{k_0} t^{k_0}(1+h(t))f(t)$ is not $(n+m)$-times continuously differentiable at t_0. □

Lemma 2.11. *Let $\{c_k\}_{k=1}^{n} \subset \mathbb{C}$ be not all zero and $\{\alpha_k\}_{k=1}^{n} \subset \mathbb{R}$ where each α_k*

is distinct. Define

$$p(t) = \sum c_k e^{i\alpha_k t}.$$

Then, for all $t \in \mathbb{R}$, there exists $0 \leqslant l \leqslant n-1$ such that $p^{(l)}(t) \neq 0$.

Proof. We proceed by contradiction. Suppose that there existed a t_0 such that $p^{(k)}(t_0) = 0$ for all $0 \leqslant l \leqslant n-1$. Then, we have that

$$0 = p^{(l)}(t_0) = \left(\sum_{k=1}^{n} c_k e^{i\alpha_k t}\right)^{(l)}(t_0) = \sum c_k (i\alpha_k)^l e^{i\alpha_k t_0}, \tag{11}$$

for all $0 \leqslant l \leqslant n-1$. Define

$$A = \begin{pmatrix} e^{i\alpha_1 t_0} & e^{i\alpha_2 t_0} & \cdots & e^{i\alpha_n t_0} \\ i\alpha_1 e^{i\alpha_1 t_0} & i\alpha_2 e^{i\alpha_2 t_0} & \cdots & i\alpha_n e^{i\alpha_n t_0} \\ \vdots & \vdots & \ddots & \vdots \\ (i\alpha_1)^{n-1} e^{i\alpha_1 t_0} & (i\alpha_2)^{n-1} e^{i\alpha_2 t_0} & \cdots & (i\alpha_n)^{n-1} e^{i\alpha_n t_0} \end{pmatrix} \text{ and } v = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

Because of the n equations in (11), we can write the matrix equation, Av $=$ 0. We also know that because $\{e^{i\alpha_k t}\}_{k=1}^{n}$ is a linearly independent set, the Wronskian $W(e^{i\alpha_1 t}, ..., e^{i\alpha_n t}) \neq 0$ for all $t \in \mathbb{R}$. We see that

$$\det(A) = W(e^{i\alpha_1 t}, ..., e^{i\alpha_n t})|_{t_0} \neq 0.$$

This demonstrates that A is nonsingular and taking this fact together with $Av = 0$, we have that $v = 0$, contradicting our assumption that $\{c_k\}_{k=1}^{n}$ is not all zero. Thus, we reject our assumption and conclude that for all $t \in \mathbb{R}$, there exists $0 \leqslant l \leqslant n-1$ such that $p^{(l)}(t) \neq 0$. □

Lemma 2.12. *Let f be a continuous complex valued function that is not infinitely continuously differentiable at t_0. Let Ξ be a HRT configuration and f not satisfy the Ξ-configuration. Let $\{\beta_j\}_{j=1}^{n_0}$ be the unique translations of Ξ. Then, for all $1 \leqslant j \leqslant n_0$, f is not infinitely differentiable on at least one element of the set $\{t_0 + \beta_j - \beta_l\}_{j=1, j\neq k}^{n_0}$ if that set is nonempty.*

Proof. Let m be an integer for which f is not m-times differentiable at t_0. Let $n = \#(\Xi)$ Fix $1 \leqslant l \leqslant n_0$. Let $\{p_j(t)\}_{j=1}^{n_0}$ be the translation polynomials of f. Since f does not satisfy the Ξ-configuration, we can write

$$-p_l(t)f(t-\beta_l) = \sum_{j=1, j\neq l}^{n} p_k(t)f(t-\beta_j^*). \tag{12}$$

Notice that p_l has fewer than n_0 terms in its sum so lemma 2.11 demands that for all $t \in \mathbb{R}$, there exists $0 \leqslant l \leqslant n-1$ such that $p^{(l)}(t) \neq 0$. Therefore, we can apply lemma 2.10 to see that the left hand side of (12) is not $(n+m)$-times differentiable at $t_0 + \beta_l$. Therefore, the right hand side of (12) is not infinity differentiable at t_0. Thus, at least one of $\{p_k(t)f(t-\beta_k)\}_{k=1}^{n_0}$ is not infinitely differentiable. Let k_0 be the index of said function. Since $p_{k_0}(t)$ is infinitely differentiable, we have that $f(t-\beta_{k_0})$ is not infinitely differentiable at $t_0 + \beta_l - \beta_{k_0}$. □

Lemma 2.13. *Let f be a continuous complex valued function that is not infinitely continuously differentiable at t_0. Let Ξ be a HRT configuration and $\{\beta_k\}_{k=1}^{n}$ be the unique translation of Ξ. Let $n \geqslant 2$. Let f not satisfy the Ξ-configuration. Then, there exists $t_1 \in (t_0 + (\beta_2^* - \beta_1), t_0 + (\beta_n - \beta_1)]$ and $t_{-1} \in [t_0 - (\beta_n - \beta_1), t_0 - (\beta_n - \beta_{n-1}^*))$ such that f is not infinitely differentiable at t_1 and t_{-1}.*

Proof. Let f, Ξ and $\{\beta_k\}_{k=1}^{n}$ be as above. We see that $\{t_0 + (\beta_k - \beta_1)\}_{l=0}^{n} \subset (t_0, t_0 + (\beta_n - \beta_1)]$ and $\{t_0 + (\beta_k - \beta_1)\}_{l=0}^{n-1} \subset [t_0 - (\beta_n - \beta_1), t_0)$. By lemma 2.12, there exists $t_1 \in \{t_0 + \beta_k - \beta_0\}_{l=0}^{n}$ and $t_{-1} \in \{t_0 + \beta_k - \beta_0\}_{l=0}^{n-1}$ such that f is not infinitely differentiable. □

Lemma 2.14. *Let f be a continuous complex valued function that is not infinitely continuously differentiable. Let Ξ be an HRT configuration with unique translations $\{\beta_k\}_{k=1}^{n}$. Let $n \geqslant 2$. Let f not satisfy the Ξ-configuration. Then, for every closed interval, $[a,b]$, such that $b - a \geqslant \beta_n - \beta_1$, there exists $t \in [a,b]$ such that f is not infinitely differentiable.*

Proof. Since f is not infinitely differentiable, there exists t_0 such that f is not infinitely differentiable at that point. Define inductively the following sequence $\{t_n\}_{n=0}^{\infty}$. We have that t_n is a point at which f is not infinitely continuously differentiable. By lemma 2.13, we have that there exists a point in $[t_n + (\beta_2 - \beta_1), t_n + (\beta_n - \beta_1)]$ such that f is not infinitely differentiable. We will call this point t_{n+1}. Let $M > t_0$. By the Archimedean property, there exists $n \in \mathbb{N}$ such that $t_0 + n\beta_2 > M$. Thus,

$$t_n \geqslant t_{n-1} + \beta_2 \geqslant t_0 + n\beta > M.$$

So the set of all points that are not infinitely differentiable has no upper bound and for all $M \in \mathbb{R}$, $\{t_n\}_{n=0}^{\infty} \cap (\infty, M]$ is a finite set.

Define inductively the following sequence $\{t_{-n}\}_{n=0}^{\infty}$. We have that t_{-n} is a point at which f is not infinitely continuously differentiable. By lemma 2.13, we have that there exists a point in $[t_{-n} - (\beta_n - \beta_1), t_{-n} + (\beta_n - \beta_{n-1}))$ such that f is not infinitely differentiable. We will call this point t_{-n-1}. Let $M < t_0$. By the Archimedean property, there exists $n \in \mathbb{N}$ such that $t_0 - n\beta_2 < M$. Thus

$$t_{-n} \leqslant t_{-n+1} - \beta_{n-1} \leqslant t_0 - n\beta_{n-1} < M.$$

So, the set of all points that are not infinitely differentiable has no lower bound and $\{t_{-n}\}_{n=0}^{\infty} \cap [M, \infty])$ is a finite set. Thus, the set $\{t_n\}_{n=-\infty}^{\infty}$ has no upper or lower bound and for all $a, b \in \mathbb{R}$,

$$\#\Big(\{t_n\}_{n=-\infty}^{\infty} \cap [a, b]\Big) = \#\Big(\{t_n\}_{n=0}^{\infty} \cup \{t_{-n}\}_{n=2}^{\infty}) \cap [a, b]\Big) =$$

$$\#\Big(\{t_n\}_{n=0}^{\infty} \cap [a, b]) \cup (\{t_n\}_{n=-\infty}^{-1} \cap [a, b]\Big) =$$

$$\#\Big(\{t_n\}_{n=0}^{\infty} \cap [a, b]\Big) + \#\Big(\{t_n\}_{n=-\infty}^{-1} \cap [a, b]\Big) \leqslant$$

$$\#\Big(\{t_n\}_{n=0}^{\infty} \cap (-\infty, b]\Big) + \#\Big(\{t_n\}_{n=-\infty}^{-1} \cap [a, \infty)\Big) < \infty$$

which is to say that there are at most only finite number of $\{t_n\}_{n=-\infty}^{\infty}$ in any interval $[a, b]$.

Suppose that there existed an interval $[a, b]$ such that $b - a \geqslant \beta_n - \beta_1$. Since $\{t_n\}_{n=-\infty}^{\infty}$ has no lower bound, there must exist an $M > 0$ such that

$$\{t_n\}_{n=-\infty}^{\infty} \cap [a - M, a) = \varnothing.$$

We know that there are only finitely many elements in that intersection. Let t_{n_0} be the largest such element. We know that t_{n_0+1} is such that $0 < t_{n_0+1} - t_{n_0} \leqslant \beta_n - \beta_1$. This gives us $t_{n_0+1} > t_{n_0}$ and $t_{n_0+1} \in (t_{n_0}, \infty)$. It also gives us

$$t_{n_0+1} = t_{n_0} + (t_{n_0+1} - t_{n_0}) \leqslant t_0 + (\beta_n - \beta_1) \leqslant$$

$$a + (\beta_n - \beta_1) \leqslant a + (b - a) = b.$$

This shows us that $t_{n_0+1} \in (\infty, b]$. So, we have that $t_{n_0+1} \in (t_{n_0}, b]$. We see that $t_{n_0+1} \in (t_{n_0}, a)$, because that would make it an element of $\{t_n\}_{n=-\infty}^{\infty} \cap [a - M, a)$ that is greater than the greatest element of that set, t_0. Thus, $t_{n_0+1} \in [a, b]$. □

Definition 2.15. *Let $A \subset \mathbb{R}$. We say that A is a tattered set if for all M, there exists $a, b \in \mathbb{R}$ such that $b - a \geqslant M$ and $[a, b] \subset A^C$.*

Proposition 2.16. *Let $A \subset \mathbb{R}$. If there exists $M > 0$ such that for all closed intervals $[a, b]$ with length less than M and such that $A \cap [a, b] = \varnothing$, then A is not tattered.*

Proof. The condition on A is the direct logical negation of the definition of a tattered set. □

Proposition 2.17. *Let $A \subset \mathbb{R}$. If A is bounded, then A is tattered.*

Proof. Let B be the upper bound of A. Then, for all $M > 0$, $[B+1, B+1+m] \subset A^C$. □

Corollary 2.18. *Let $A \subset \mathbb{R}$. If A is finite, then A is tattered.*

Proof. Since A is finite, A is also bounded. By proposition 2.17, A is a tattered set. □

Proposition 2.19. *The subset of a tattered set is tattered.*

Proof. Let $B \subset A \subset \mathbb{R}$ where A is a tattered set. For all $M > 0$, we have that there exists an interval $[a, b]$ such that $b - a \geqslant M$

$$[a, b] \subset A^C \subset B^C.$$

Thus, B^C is a tattered set. □

We note that the empty set is a tattered set.

Definition 2.20. *Let $A \subset \mathbb{R}$. We define the lower Buerling density of A to be*

$$D^-(A) = \liminf_{h\to\infty} \frac{\min_{x\in\mathbb{R}} \#(A \cap [x, x + h])}{h}.$$

Proposition 2.21. *Let $A \subset \mathbb{R}$. A is tattered if and only if $D^-(A) = 0$.*

Proof. Let A be tattered. Fix $h > 0$. Then, there exists a closed interval $[a, b]$ of length greater than $b - a > h$ such that $[a, b] \subset A^C$ or $[a, b] \cap A = \varnothing$. Thus,

$$\frac{\min_{x\in\mathbb{R}} \#(A \cap [x, x + h])}{h} \leqslant \frac{\#(A \cap [a, a + h])}{h} \leqslant \frac{\#(\varnothing)}{h} = 0.$$

Since this is true of all $h > 0$, we have that

$$D^-(A) = \liminf_{h\to\infty} \frac{\min_{x\in\mathbb{R}} \#(A \cap [x, x + h])}{h} = \liminf_{h\to\infty} 0 = 0.$$

We have proven the forward direction.

Let A not be tattered. Then, there exists $M_0 > 0$ such that for all closed intervals, [a,b], such that $b - a \geqslant M$, $A \cap [a, b] \neq \varnothing$. Let $h > 0$. Then, for all $x \in \mathbb{R}$, we have that we can cover $[x, x + h]$ by $\lfloor \frac{h}{M} \rfloor$ disjoint intervals of length M. Each of those intervals has at least 1 element of A in it, so there are at least $\lfloor \frac{h}{M} \rfloor$ in every interval of the form $[x, x+h]$. Thus, Since each o Thus for all $x \in \mathbb{R}$,

$$D^{-}(A) = \liminf_{h\to\infty} \frac{\min_{x\in\mathbb{R}} \#(A \cap [x, x+h])}{h} \geqslant \liminf_{h\to\infty} \frac{\lfloor \frac{h}{M} \rfloor}{h} = \lim_{h\to\infty} \frac{\lfloor \frac{h}{M} \rfloor}{h} = \frac{1}{M} > 0.$$

We have proven the backward direction. □

Theorem 2.22. *Let f be a continuous complex valued function and A is the set of all points on which f is not infinitely differentiable. If A is a nonempty tattered set, then f satisfies the HRT conjecture.*

Proof. Let f and A be as above. Assume that f does not satisfy the HRT conjecture. Then by proposition 2.5, there exists an HRT configuration, Ξ, such that f does not satisfy the Ξ-configuration. It is well known that if Ξ has only one unique translation, then the Ξ-configuration is satisfied for all nonzero f. Thus, Ξ must have at least two unique translations.

Let $\{\beta_k\}_{k=0}^{n}$ be the unique translations of Ξ. Since A is nonempty, lemma 2.14 demands that for all intervals of length $\beta_n - \beta_1$, there exists $t \in A$. By proposition 2.16, A is not a tattered set, contradicting our assumption that A is a tattered set. We reject our assumption and conclude that f satisfies the HRT conjecture. □

Corollary 2.23. *Let f be a continuous complex valued function that is not infinitely differentiable only on a bounded nonempty set. Then, f satisfies the HRT conjecture.*

Proof. By proposition 2.17, we have that the set on which f is not infinitely differentiable is a tattered set. By theorem 2.22, f satisfies the HRT conjecture. □

Corollary 2.24. *Let f be a continuous complex valued function. that is not infinitely differentiable only on a finite set. Then, f satisfies the HRT conjecture.*

Proof. By proposition 2.18, we have that the set on which f is not infinitely differentiable is a tattered set. By theorem 2.22, f satisfies the HRT conjecture. □

Corollary 2.25. *Let f be a continuous complex valued function and A be the set of all points for which f is not infinitely differentiable. If A is nonempty and $D^{-}(A) = 0$, then f satisfies the HRT conjecture.*

Proof. Suppose that $D^{-}(A) = 0$. By proposition 2.21, we have that A is a tattered set. By theorem 2.22, f satisfies the HRT conjecture. □

Corollary 2.26. *Let $\{f_n\}_{n=-\infty}^{\infty} \subset C^{\infty}(\mathbb{R})$ and $\{t_n\}_{n=-\infty}^{\infty} \subset \mathbb{R}$ be a tattered set with $t_n < t_{n+1}$. Define the function $f(t) = f_n(t)$ if $t \in [t_n, t_{n+1})$. If there exists $n \in \mathbb{Z}$ such that f is not infinitely differentiable at t_n, then f satisfies the HRT conjecture.*

Proof. Let f and $\{t_n\}_{n=\infty}^{\infty}$ be as above. Since each $f_n \in \mathbb{C}^{\infty}$, we have that the only possible points on which f is not infinitely differentiable is $\{t_n\}_{n=-\infty}^{\infty}$. By proposition 2.19, we have that the set of all points on which f is not infinitely differentiable is a tattered set. By 2.22, we have that f satisfies the HRT conjecture. □

3 Basic Propositions

3.1 Discrepancy

Definition 3.1. *The discrepancy of the complex valued function $f(t)$ at t_0 is*

$$disc_{t_0}(f) = \lim_{\delta \to 0} \sup_{x,y \in [t_0-\delta, t_0+\delta]} |f(x) - f(y)|.$$

Note that $\mathrm{disc}_{t_0}(f)$ may take on any value in $[0, \infty]$. This notion of discrepancy is entirely distinct from the one found in [8].

Example 3.2. *Fix a, b. Define*

$$f(t) = \begin{cases} a & x < 0 \\ b & x \geqslant 0. \end{cases}$$

We will show that $disc_0(f(x)) = |b - a|$.

Proof. We have that

$$\mathrm{disc}_0(f(x)) = \lim_{\delta \to 0} \sup_{x,y \in [t_0-\delta, t_0+\delta]} |f(x) - f(y)| \leqslant \sup_{x,y \in [t_0-1, t_0+1]} |b - a| = |b - a|.$$

We also have that

$$\mathrm{disc}_0(f(x)) = \lim_{\delta \to 0} \sup_{x,y \in [t_0-\delta, t_0+\delta]} |f(x) - f(y)| \geqslant$$

$$\lim_{\delta \to 0} |f(t_0 + \delta) - f(t_0 - \delta)| = \lim_{\delta \to 0} |b - a| = |b - a|.$$

The above inequalities demand that $\mathrm{disc}_0(f(x)) = |b - a|$. □

Example 3.3.

$$disc_0\left(\frac{1}{x}\right) = \infty.$$

Proof. We see that

$$\operatorname{disc}_0(\frac{1}{x}) = \lim_{\delta\to 0} \sup_{x,y\in[t_0-\delta,t_0+\delta]} |\frac{1}{x} - \frac{1}{y}| \geqslant \lim_{\delta\to 0} \frac{1}{\delta} - \frac{1}{-\delta} = \lim_{\delta\to 0} \frac{2}{\delta} = \infty.$$

□

Proposition 3.4. *Let f be a complex valued function. f has a discontinuity at t_0 if and only if $disc_{t_0}(f) > 0$.*

Proof. First, we will show the forward direction. Let f have a discontinuity at t_0. Then there exists a sequence $\{t_n\}_{n=1}^{\infty}$ convergent to t_0 such that $f(t_n)$ fails to converge to $f(t_0)$. This failure of convergence demands that there exists a fixed $\epsilon > 0$ such that for all $N \geqslant 1$, there exists $n_N \geqslant N$ such that $|f(t_0) - f(t_n)| > \epsilon$. Fix δ. The convergence of $\{t_n\}_{n=1}^{\infty}$ demands that there exists N sufficiently large that $|t_0 - t_n| < \delta$ for all $n \geqslant N$. Thus,

$$\sup_{x,y\in[t_0-\delta,t_0+\delta]} |f(x) - f(y)| \geqslant |f(t_{n_N}) - f(t_0)| \geqslant \epsilon.$$

Since this is true of all δ,

$$\operatorname{disc}_{t_0}(f) = \lim_{\delta\to 0} \sup_{x,y\in[t_0-\delta,t_0+\delta]} |f(x) - f(y)| \geqslant \epsilon > 0.$$

Now, we prove the other direction. Assume $\operatorname{disc}_{t_0}(f) > 0$. Let $\operatorname{disc}_{t_0}(f) = \epsilon_0 > 0$. This means that

$$\lim_{n\to\infty} \sup_{x,y\in[t_0-1/n,t_0+1/n]} |f(x) - f(y)| = \lim_{\delta\to 0} \sup_{x,y\in[t_0-\delta,t_0+\delta]} |f(x) - f(y)| = \epsilon_0. \quad (13)$$

From (13), we see that for all n there exists $x_n, y_n \in [t_0 - 1/n, t_0 + 1/n]$ such that $|f(x) - f(y)| > \epsilon/2$. Define

$$t_n = \begin{cases} x_{(n+1)/2}, & \text{if } n \text{ is odd} \\ y_{n/2}, & \text{if } n \text{ is even.} \end{cases}$$

We see that $|t_n - t_0| < 2/(n-1)$ for all $n > 1$ and thereby $\{t_n\}_{n=1}^{\infty}$ is a convergent sequence. For all $k \geqslant 1$,

$$|f(t_{2k-1}) - f(t_{2k})| = |f(x_k) - f(y_k)| > \epsilon/2.$$

Thus, the sequence $\{f(t_k)\}$ fails to be Cauchy and thereby convergent despite the convergence of $\{t_n\}$, demonstrating that f is not a continuous function. □

Corollary 3.5. *Let f be a complex valued function. f is continuous at t_0 if and only if $disc_{t_0}(f) = 0$.*

Proof. If f is continuous at t_0 then proposition 3.4 demands that $\text{disc}_{t_0}(f)$ is zero. If $\text{disc}_{t_0}(f)$ is zero then proposition 3.4 demands that f is continuous at t_0. □

Proposition 3.6. *Let f be a complex-valued function and $c \in \mathbb{R}$. Then*

$$disc_{t_0}(cf(t)) = |c| \; disc_{t_0}(f(t)).$$

Proof. We see that

$$\text{disc}_{t_0}(cf) = \lim_{\delta\to 0} \sup_{x,y\in[t_0-\delta,t_0+\delta]} |cf(x) - cf(y)| =$$

$$\lim_{\delta\to 0} \sup_{x,y\in[t_0-\delta,t_0+\delta]} |c||f(x) - f(y)| =$$

$$|c| \lim_{\delta\to 0} \sup_{x,y\in[t_0-\delta,t_0+\delta]} |f(x) - f(y)| = |c|\text{disc}_{t_0}(f).$$

□

Proposition 3.7. *Let f and g be complex valued functions and $disc_{t_0}(f) \geqslant disc_{t_0}((g)$. Then*

$$disc_{t_0}(f) - disc_{t_0}(g) \leqslant disc_{t_0}(f+g) \leqslant disc_{t_0}(f) + disc_{t_0}(g).$$

Proof. We show the first inequality. By definition 3.1,

$$\mathrm{disc}_{t_0}(f+g) = \lim_{\delta\to 0} \sup_{x,y\in[t_0-\delta,t_0+\delta]} |[f(x)+g(x)]-[f(y)+g(y)]| \geqslant$$

$$\lim_{\delta\to 0} \sup_{x,y\in[t_0-\delta,t_0+\delta]} |f(x)-f(y)| - |g(x)-g(y)| \geqslant$$

$$\lim_{\delta\to 0} \sup_{x,y\in[t_0-\delta,t_0+\delta]} |f(x)-f(y)| - \lim_{\delta\to 0} \sup_{x,y\in[t_0-\delta,t_0+\delta]} |g(x)-g(y)| =$$

$$\mathrm{disc}_{t_0}(f) - \mathrm{disc}_{t_0}(g).$$

The second inequality is shown by a similar procedure. We have that

$$\mathrm{disc}_{t_0}(f+g) = \lim_{\delta\to 0} \sup_{x,y\in[t_0-\delta,t_0+\delta]} |[f(x)+g(x)]-[f(y)+g(y)]| \leqslant$$

$$\lim_{\delta\to 0} \sup_{x,y\in[t_0-\delta,t_0+\delta]} |f(x)-f(y)| + |g(x)-g(y)| \leqslant$$

$$\lim_{\delta\to 0} \sup_{x,y\in[t_0-\delta,t_0+\delta]} |f(x)-f(y)| + \lim_{\delta\to 0} \sup_{x,y\in[t_0-\delta,t_0+\delta]} |g(x)-g(y)| =$$

$$\mathrm{disc}_{t_0}(f) + \mathrm{disc}_{t_0}(g).$$

□

Corollary 3.8. *Let f and g be complex valued functions with g continuous at t_0. Then $disc_{t_0}(f+g) = disc_{t_0}(f)$.*

Proof. By proposition 3.7, we have

$$\mathrm{disc}_{t_0}(f+g) \leqslant \mathrm{disc}_{t_0}(f) + \mathrm{disc}_{t_0}(g) = \mathrm{disc}_{t_0}(f) + 0 = \mathrm{disc}_{t_0}(f)$$

and

$$\mathrm{disc}_{t_0}(f+g) = \mathrm{disc}_{t_0}(f+g) + 0 = \mathrm{disc}_{t_0}(f+g) + \mathrm{disc}_{t_0}(-g) \geqslant$$

$$\mathrm{disc}_{t_0}(f+g-g) = \mathrm{disc}_{t_0}(f).$$

Putting the above inequalities together, we have that

$$\text{disc}_{t_0}(f+g) = \text{disc}_{t_0}(f).$$

□

Proposition 3.9. *$disc_{t_0}(f)$ is a pseudonorm on the set of all complex valued functions.*

Proof. Proposition 3.7 gives us the triangle inequality and proposition 3.6 gives us absolute scalability. Thus we have satisfied the two properties of a pseudonorm. □

Proposition 3.10. *Let V be the set of all complex valued functions on $\mathbb{R}$. Define the binary operation $d_{t_0} : V \times V \to \mathbb{R}$ by*

$$d_{t_0}(f,g) = disc_{t_0}(f-g).$$

d_{t_0} is a pseudometric.

Proof. Since the constant function 0 is continuous, proposition 3.5 demands

$$d_{t_0}(f,f) = \text{disc}_{t_0}(f-f) = \text{disc}_{t_0}(0) = 0.$$

d_{t_0} is symmetric as evinced by

$$d_{t_0}(f,g) = \text{disc}_{t_0}(f-g) = \lim_{\delta\to 0} \sup_{x,y\in[t_0-\delta,t_0+\delta]} |[f(x)-g(x)]-[f(y)-g(y)]| =$$

$$\lim_{\delta\to 0} \sup_{x,y\in[t_0-\delta,t_0+\delta]} |[f(y)-g(y)]-[f(x)-g(x)]| = \text{disc}_{t_0}(g-f) = d_{t_0}(g,f).$$

Proposition 3.7 gives us

$$d_{t_0}(f,g) = \text{disc}_{t_0}(f-g) = \text{disc}_{t_0}((f-h)-(g-h)) \leqslant$$

$$\text{disc}_{t_0}(f-h) + \text{disc}_{t_0}(g-h) = d_{t_0}(f,h) + d_{t_0}(g,h).$$

We have shown that d_{t_0} has all three properties of a pseudometric. $\square$

3.2 Axis of Discrepancy

Definition 3.11. *The axis of discrepancy of $f(t)$ at t_0, called $axis_{t_0}(f)$, is the subset of $\{e^{i\theta} | \theta \in [0, 2\pi)\}$ where each element, $e^{i\theta_0}$, is such that there exists a real number $\epsilon > 0$ and two sequences $\{x_n\}_{n=1}^{\infty}$,$\{y_n\}_{n=1}^{\infty}$ convergent to t_0 such that*

$$|f(x_n) - f(y_n)| \geqslant \epsilon, \quad \textit{for all } n,$$

and

$$\lim_{n\to\infty} \frac{f(x_n) - f(y_n)}{|f(x_n) - f(y_n)|} = e^{i\theta_0}.$$

In the pair of sequences, $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$, in the definition above we demand that there exists $\epsilon > 0$ such that,

$$|f(x_n) - f(y_n)| \geqslant \epsilon, \quad \text{for all } n.$$

The following example demonstrates why we make this requirement.

Example 3.12. *Define*

$$f(t) = \begin{cases} te^{i/t}, & t \neq 0 \\ 0, & t = 0. \end{cases}$$

Even though $f(t)$ is continuous at $t = 0$, we have for all $\theta \in [0, 2\pi)$, there exists sequences $\{x_n\}_{n=1}^{\infty}$,$\{y_n\}_{n=1}^{\infty}$ convergent to 0 and such that

$$\lim_{n\to\infty} \frac{f(x_n) - f(y_n)}{|f(x_n) - f(y_n)|} = e^{i\theta}.$$

Proof. Let t_n converge to 0. Fix $\epsilon > 0$. Then there exists N sufficiently large that $|t_n| = |t_n - 0| < \epsilon$ for all $n \geqslant N$. Thus

$$|0 - f(t_n)| = |t_n e^{i/t_n}| = |t_n||e^{i/t_n}| = |t_n| < \epsilon \quad \text{for all } n \geqslant N.$$

Since this is true of all ϵ, f is continuous at 0.

Define

$$\left\{x_n = \frac{1}{\theta + n2\pi}\right\}_{n=1}^{\infty} \quad \text{and} \quad \{y_n = 0\}_{n=1}^{\infty}.$$

Both sequences obviously converge to 0, but we also have that

$$\lim_{n\to\infty} \frac{f(x_n) - f(y_n)}{|f(x_n) - f(y_n)|} = \lim_{n\to\infty} \frac{f(\frac{1}{\theta+n2\pi}) - f(0)}{|f(\frac{1}{\theta+n2\pi}) - f(0)|} =$$

$$\lim_{n\to\infty} \frac{(\theta + n2\pi)^{-1} e^{i(\theta+n2\pi)} - 0}{|(\theta + n2\pi)^{-1} e^{i(\theta+n2\pi)} - 0|} = \lim_{n\to\infty} \frac{e^{i(\theta+n2\pi)}}{|e^{i(\theta+n2\pi)}|} = \lim_{n\to\infty} e^{i\theta} = e^{i\theta}$$

for all $\theta \in [0, 2\pi)$. □

The direct procedure for finding the axis of discrepancy of some function, f, at some point, t_0, generally consists of two steps. First, for each point $e^{i\theta_0}$ in the axis of discrepancy, find two sequences $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ with three properties: they converge to t_0, $|x_n - y_n| \geqslant \epsilon > 0$ for all n and

$$\frac{f(x_n) - f(y_n)}{|f(x_n) - f(y_n)|} \to e^{i\theta_0} \quad \text{as} \quad n \to \infty.$$

Second, show that no other elements of $\{e^{i\theta} | \theta \in [0, 2\pi)\}$ belong to $\text{axis}_{t_0}(f)$. As can be plainly seen, this is a rather cumbersome procedure, but we feel it necessary to exhibit it before supplanting it by other techniques later.

Example 3.13. *Fix $a, b \in \mathbb{R}$ such that $a < b$. Define*

$$f(t) = \begin{cases} a & x < 0 \\ b & x \geqslant 0. \end{cases}$$

We have that $axis_{t_0}(f) = \{-1, 1\}$.

Proof. Define the sequences $\{x_n = 1/n\}_{n=1}^{\infty}$ and $\{y_n = -1/n\}_{n=1}^{\infty}$. We see that these sequences converge to zero,

$$\lim_{n\to\infty} |f(x_n) - f(y_n)| = \lim_{n\to\infty} |b - a| = |b - a| > 0$$

and

$$\lim_{n\to\infty} \frac{f(x_n) - f(y_n)}{|f(x_n) - f(y_n)|} = \lim_{n\to\infty} \frac{b-a}{|b-a|} = \lim_{n\to\infty} 1 = 1.$$

Thus $1 \in \text{axis}_{t_0}(f)$.

Define $\{x_n = -1/n\}_{n=1}^{\infty}$ and $\{y_n = 1/n\}_{n=1}^{\infty}$. We have that these sequences converge to zero,

$$\lim_{n\to\infty} |f(x_n) - f(y_n)| = \lim_{n\to\infty} |a - b| = |a - b|,$$

and

$$\lim_{n\to\infty} \frac{f(x_n) - f(y_n)}{|f(x_n) - f(y_n)|} = \lim_{n\to\infty} \frac{a-b}{|a-b|} = \frac{a-b}{|a-b|} = -1.$$

Thus $-1 \in \text{axis}_{t_0}(f)$. So $\{-1, 1\} \subseteq \text{axis}_{t_0}(f)$.

Since the range of $f(t)$ is $\{a, b\}$, we have that

$$\{f(x) - f(y)| \quad x, y \in \mathbb{R}\} = \{0, a-b, b-a\}.$$

This implies that

$$\frac{f(x) - f(y)}{|f(x) - f(y)|} \tag{14}$$

equals either 1 or -1 for the values on which it is defined. Thus any limit on a sequence of numbers of the form (14) will have to be either -1 or 1. So $\text{axis}_{t_0}(f) \subseteq \{-1, 1\}$. We conclude that $\text{axis}_{t_0}(f) = \{-1, 1\}$. □

Example 3.14. *Fix* $\theta_0 \in [0, 2\pi)$. *Define*

$$f(t) = \begin{cases} 0 & x < 0 \\ e^{i\theta_0} & x \geqslant 0. \end{cases}$$

Then, $axis_{t_0}(f) = \{e^{i\theta_0}, e^{i(\theta_0+\pi)}\}$.

Proof. Define the sequences $\{x_n = 1/n\}_{n=1}^{\infty}$ and $\{y_n = -1/n\}_{n=1}^{\infty}$. We have that these sequences converge to zero,

$$\lim_{n\to\infty} |f(x_n) - f(y_n)| = \lim_{n\to\infty} |e^{i\theta_0} - 0| = 1,$$

and

$$\lim_{n\to\infty} \frac{f(x_n) - f(y_n)}{|f(x_n) - f(y_n)|} = \lim_{n\to\infty} \frac{e^{i\theta_0} - 0}{|e^{i\theta_0} - 0|} = \frac{e^{i\theta_0}}{1} = e^{i\theta_0}.$$

Thus, $e^{i\theta_0} \in \text{axis}_{t_0}(f)$.

Define $\{x_n = -1/n\}_{n=1}^{\infty}$ and $\{y_n = 1/n\}_{n=1}^{\infty}$. We have that these sequences converge to zero,

$$\lim_{n\to\infty} |f(x_n) - f(y_n)| = \lim_{n\to\infty} |0 - e^{i\theta_0}| = 1,$$

and

$$\lim_{n\to\infty} \frac{f(x_n) - f(y_n)}{|f(x_n) - f(y_n)|} = \lim_{n\to\infty} \frac{0 - e^{i\theta_0}}{|0 - 1|} = \frac{-e^{i\theta_0}}{1} = e^{i(\theta_0+\pi)}.$$

Thus $e^{i(\theta_0+\pi)} \in \text{axis}_{t_0}(f)$. So $\{e^{i\theta_0}, e^{i(\theta_0+\pi)}\} \subseteq \text{axis}_{t_0}(f)$.

Since the range of $f(t)$ is $\{0, e^{i\theta_0}\}$, we have that

$$\{f(x) - f(y) | x, y \in \mathbb{R}\} = \{0, e^{i\theta_0}, -e^{i\theta_0}\}.$$

This implies that

$$\frac{f(x) - f(y)}{|f(x) - f(y)|} \tag{15}$$

equals either $e^{i\theta_0}$ or $-e^{i\theta_0}$ for the values on which (15) is defined. Thus any limit on a sequence of numbers of the form (15) will have to be either $e^{i\theta_0}$ or $-e^{i\theta_0}$. So $\text{axis}_{t_0}(f) \subseteq \{e^{i\theta_0}, -e^{i\theta_0} = e^{i(\theta_0)}\}$. We conclude that $\text{axis}_{t_0}(f) = \{e^{i\theta_0}, e^{i(\theta_0)}\}$.

□

Example 3.15. *Let* $f(t) = e^{i/t}$ *for* $t \neq 0$ *and* $f(t) = 1$ *at* $t = 0$. *Then* $axis_0(f)$ *equals the unit circle.*

Proof. Fix $\theta_0 \in [0, 2\pi)$. Let

$$x_n = \frac{1}{\theta_0 + 2\pi n} \quad \text{and} \quad y_n = \frac{1}{\theta_0 - \pi - 2\pi n}.$$

Clearly, $x_n, y_n \to 0$. We also have that

$$|f(x_n) - f(y_n)| = |f(\frac{1}{\theta_0 + 2\pi n}) - f(\frac{1}{\theta_0 - \pi - 2\pi n})| = |e^{i(\theta_0+2\pi n)} - e^{i(\theta_0-\pi-2\pi n)}| =$$

$$|e^{i\theta_0} - e^{i(\theta_0-\pi)}| = |e^{i\theta_0} + e^{i\theta_0}| = 2|e^{i\theta_0}| = 2,$$

which gives

$$\lim_{n\to\infty} |f(x_n) - f(y_n)| = 2.$$

Finally, we have

$$\lim_{n\to\infty} \frac{f(x_n) - f(y_n)}{|f(x_n) - f(y_n)|} = \lim_{n\to\infty} \frac{e^{i(\theta_0+2\pi n)} - e^{i(\theta_0-\pi-2\pi n)}}{2} = \frac{2e^{i\theta_0}}{2} = e^{i\theta_0}.$$

So, $e^{i\theta_0} \in \text{axis}_0(f)$, and thereby, all the unit circle belongs to this axis of discrepancy. □

Proposition 3.16. *Let f be a complex valued function. If $e^{i\theta_0} \in axis_{t_0}(f)$ then $e^{i(\theta_0+\pi)} \in axis_{t_0}(f)$.*

Proof. Let $e^{i\theta_0} \in \text{axis}_{t_0}(f)$. Then, there exists a real number $\epsilon_0 > 0$ and two sequences $\{x_n\}_{n=1}^{\infty}$,$\{y_n\}_{n=1}^{\infty}$ convergent to t_0 such that

$$|f(x_n) - f(y_n)| \geqslant \epsilon_0, \quad \text{for all } n$$

and

$$\lim_{n\to\infty} \frac{f(x_n) - f(y_n)}{|f(x_n) - f(y_n)|} = e^{i\theta_0}.$$

Define $x_n^{\pi} = y_n$ and $y_n^{\pi} = x_n$. We see that x_n^{π} and y_n^{π} are convergent to t_0,

$$|f(x_n^{\pi}) - f(y_n^{\pi})| = |f(y_n) - f(x_n)| \geqslant \epsilon_0, \quad \text{for all } n,$$

and

$$\lim_{n\to\infty} \frac{f(x_n^\pi) - f(y_n^\pi)}{|f(x_n^\pi) - f(y_n^\pi)|} = \lim_{n\to\infty} \frac{f(y_n) - f(x_n)}{|f(y_n) - f(x_n)|} =$$

$$-\lim_{n\to\infty} \frac{f(x_n) - f(y_n)}{|f(x_n) - f(x_n)|} = -e^{i\theta_0} = e^{i(\theta_0+\pi)}.$$

Thus $e^{i(\theta_0+\pi)} \in \text{axis}_{t_0}(f)$. □

By an abuse of notation, if the axis of discrepancy is a set with two elements $\{x_0, -x_0\}$, then we will refer to the axis as x_0.

So that we can more effectively describe the variety of limits possible on the complex plane, we will define an extended form of the complex plane, but not as a Riemann sphere where there is only one infinite point.

3.3 Flat Extended Complex Plane

Definition 3.17. *We formally define the infinity ring to be the set of elements*

$$\mathbb{T}_\infty = \{e^{i\theta}\infty | \quad \theta \in [0, 2\pi)\}.$$

Definition 3.18. *We define the flat extended complex plane, $\mathbb{C}^f$, by*

$$\mathbb{C}^f = \mathbb{C} \cup \mathbb{T}_\infty.$$

Elements in $\mathbb{T}_\infty$ are called infinite and elements in $\mathbb{C}$ are called finite.

If $a, b \in \mathbb{C}$, the algebra is the same as in $\mathbb{C}$. We partially define the operations on the infinite elements of $\mathbb{C}^f$ in the following way:

1. *If a is finite, then*

$$e^{i\theta}\infty + a = e^{i\theta}\infty, \qquad \frac{a}{e^{i\theta}\infty} = 0;$$

2. $$\frac{e^{i\theta}\infty}{e^{i\phi}\infty} = e^{i(\theta-\phi)};$$

3.

$$|e^{i\theta}\infty| = \infty.$$

Addition between infinite elements and multiplication between finite and infinite elements are left undefined.

We note for future use that if a is finite then

$$\frac{e^{i\theta}\infty - a}{|e^{i\theta}\infty - a|} = \frac{e^{i\theta}\infty}{|e^{i\theta}\infty|} = \frac{e^{i\theta}\infty}{\infty} = e^{i\theta}. \tag{16}$$

We denote the closed unite ball in the complex plane as $\overline{B_1(0)})$.

The following subsets of $\mathbb{C}^f$ will be useful in defining a topology on $\mathbb{C}^f$.

Definition 3.19. *Let $r > 0$ and $\alpha, \beta \in \mathbb{R}$. Define the wedge of radius r from α to β to be*

$$W_r^{\alpha,\beta} = \left\{se^{i\theta}\middle| \quad s \in (r, \infty], \theta \in (\alpha, \beta)\right\}.$$

Note that $W_r^{\alpha,\beta} \cap \mathbb{C}$ is an open subset of $\mathbb{C}$. This demands that if O is an open subset then

$$O \cap W_r^{\alpha,\beta} = (O \cap \mathbb{C}) \cap W_r^{\alpha,\beta} = O \cap (W_r^{\alpha,\beta} \cap \mathbb{C}).$$

Thus, $O \cap W_r^{\alpha,\beta}$ is open in $\mathbb{C}$.

Also, note that we did not assume that $\alpha < \beta$. If $\alpha \geqslant \beta$, $W_r^{\alpha,\beta} = \varnothing$.

Proposition 3.20. *Let $r_1, r_2 > 0$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0, 2\pi)$ be such that $\alpha_1 \leqslant \beta_1$ and $\alpha_2 \leqslant \beta_2$. Define $\alpha = \max(\alpha_1, \alpha_2)$, $\beta = \min(\beta_1, \beta_2)$ and $r = \max(r_1, r_2)$. We have that*

$$W_{r_1}^{\alpha_1,\beta_1} \cap W_{r_2}^{\alpha_2,\beta_2} = W_r^{\alpha,\beta}.$$

Proof. We have that

$$W_{r_1}^{\alpha_1,\beta_1} \cap W_{r_2}^{\alpha_2,\beta_2} =$$

$$\left\{se^{i\theta}\middle| \quad s \in (r_1, \infty], \theta \in (\alpha_1, \beta_1)\right\} \cap \left\{se^{i\theta}\middle| \quad s \in (r_2, \infty], \theta \in (\alpha_2, \beta_2)\right\} =$$

$$\left\{se^{i\theta}\middle| \quad s \in (r,\infty], \theta \in (\alpha,\beta)\right\} = W_r^{\alpha,\beta}.$$

□

Proposition 3.21. *Define the following collection of sets*

$$\mathcal{B} = \{O| \quad O \text{ is open in } \mathbb{C}\} \cup \{W_r^{\alpha,\beta}| \quad \alpha,\beta \in \mathbb{R}, r \geqslant 0\}.$$

$\mathcal{B}$ is a basis on $\mathbb{C}^f$.

Proof. If $x \in \mathbb{C}^f \backslash \{0\}$ then $x \in W_0^{0,3\pi} \in \mathcal{B}$. If $x = 0$, then $x \in B_1(0) \in \mathcal{B}$. Thus, all elements of $\mathbb{C}$ are contained in some element of $\mathcal{B}$.

If B_1 and B_2 are both open sets in $\mathbb{C}$ then $B_1 \cap B_2$ is an open set in $\mathbb{C}$ and $B_1 \cap B_2 \in \mathcal{B}$. If B_1 is an open set and B_2 is a wedge set, then $B_1 \cap B_2$ is an open set in $\mathbb{C}$ and $B_1 \cap B_2 \in \mathcal{B}$. If both B_1 and B_2 are wedge sets, their intersection is also a wedge set and $B_1 \cap B_2 \in \mathcal{B}$. Thus, for all $B_1, B_2 \in \mathcal{B}$, $B_1 \cap B_2 \in \mathcal{B}$.

Let $B_1, B_2 \in \mathcal{B}$ have nonempty intersection. We see that if $x \in B_1 \cap B_2$, then $x \in B_1 \cap B_2 \in \mathcal{B}$. We have shown that $\mathcal{B}$ is a basis. □

We define the topology of $\mathbb{C}^f$ to be the topology induced by the basis $\mathcal{B}$.

Proposition 3.22. *The topology of $\mathbb{C}$ and the subspace topology $\mathbb{C}^f \cap \mathbb{C}$ are equivalent.*

Proof. Let O be an open set in $\mathbb{C}$. Then, $O \in \mathcal{B}$ and $O = O \cap \mathbb{C}$ is open in the subspace topology of $\mathbb{C}^f \cap \mathbb{C}$.

Let O_1 be an open set in the subspace topology $\mathbb{C}^f \cap \mathbb{C}$. Then there exists an open set $O_2 \subseteq \mathbb{C}^f$ such that $O_2 \cap \mathbb{C} = O_1$. O_2 equals the union of some collection of basis elements, $\{O_j\}_{j \in J_1} \cup \{W_j\}_{j \in J_2}$ where O_j is an open subset of $\mathbb{C}$, W_j is a wedge set and J_1 and J_2 are index sets. Thus,

$$O_1 = O_2 \cap \mathbb{C} = \left(\left(\bigcup_{j \in J_1} O_j\right) \cup \left(\bigcup_{j \in J_2} W_j\right)\right) \cap \mathbb{C} =$$

$$\left(\bigcup_{j \in J_1} O_j \cap \mathbb{C}\right) \cup \left(\bigcup_{j \in J_2} W_j \cap \mathbb{C}\right) = \left(\bigcup_{j \in J_1} O_j\right) \cup \left(\bigcup_{j \in J_2} W_j \cap \mathbb{C}\right).$$

$W_j \cap \mathbb{C}$ is open in $\mathbb{C}$, so O_1 is the union of a collection of open sets, making O_1 open in $\mathbb{C}$. $\square$

Definition 3.23. *We say that a real nonnegative sequence, $\{x_n\}_{n=1}^{\infty}$, converges to infinity or*

$$\lim_{n\to\infty} |x_n| = \infty$$

if for all $M > 0$ there exists N such that

$$|x_n| \geqslant M, \quad \textit{for all } n \geqslant N.$$

This definition will primarily be applied to the absolute value of complex sequences.

We will use the following criterion to determine if a sequence in $\mathbb{C}^f$ converges to an infinite element.

Proposition 3.24. *Let $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{C}^f$ be a sequence. We can write $x_n = r_n e^{i\theta_n}$ where $r_n = |x_n|$ and $\theta_n \in \mathbb{T}_{2\pi}$. $\{x_n\}_{n=1}^{\infty}$ converges to $e^{i\theta}\infty$ if and only if*

$$\lim_{n\to\infty} r_n = \infty \quad \textit{and} \quad \lim_{n\to\infty} \theta_n = \theta.$$

Proof. Let $\{x_n = r_n e^{i\theta_n}\}_{n=1}^{\infty} \subseteq \mathbb{C}^f$ be such that

$$\lim_{n\to\infty} r_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \theta_n = \theta.$$

Let $B \in \mathcal{B}$ be such that $e^{i\theta}\infty \in B$. B is either an open subset of $\mathbb{C}$ or a wedge set. Since B has an infinite element, $B = W_r^{\alpha,\beta}$. There exists a sufficiently large N_1 such that for all $n \geqslant N_1$, $r_n > r$. Define $\epsilon = \max(|\theta - \alpha|, |\theta - \beta|)$. There exists sufficiently large N_2 such that for all $n \geqslant N_2$, $|\theta_n - \theta| < \epsilon$. Thus,

$$r_n e^{i\theta_n} \in \overline{B_r(0)}^C \quad \text{and} \quad r_n e^{i\theta_n} \in W_0^{\theta-\epsilon,\theta+\epsilon} \subset W_0^{\alpha,\beta}$$

for all $n \geqslant N = \max(N_1, N_2)$. Thus,

$$x_n = r_n e^{i\theta_n} \in \overline{B_r(0)}^C \cap W_0^{\alpha,\beta} = W_r^{\alpha,\beta} \quad \text{for all } n \geqslant N.$$

Since this is true of all basis elements, we have that $\{x_n\}_{n=1}^{\infty}$ converges to $e^{i\theta}\infty$.

Now to prove the other direction. Let $\{x_n = r_n e^{i\theta_n}\}_{n=1}^{\infty} \subseteq \mathbb{C}^f$ converge to $e^{i\theta}\infty$. Fix $\epsilon > 0$ and $M \in \mathbb{N}$. We have for all basis elements, B, containing $e^{i\theta}\infty$, there exists an $N \geqslant 1$, for all $n \geqslant N$, such that $x_n \in B$. $W_r^{\theta-\epsilon,\theta+\epsilon}$ is a basis element that contains $e^{i\theta}\infty$. Let N_0 be such that for all $n \geqslant N$,

$$x_n = r_n e^{i\theta_n} \in W_r^{\theta-\epsilon,\theta+\epsilon} = \overline{B_r(0)}^C \cap W_0^{\theta-\epsilon,\theta+\epsilon}.$$

So, for all $n \geqslant N$,

$$r_n \in \overline{B_r(0)}^C \quad \text{and} \quad \theta_n \in (\theta - \epsilon, \theta + \epsilon).$$

Since this is true of all $m \in \mathbb{N}$ and all $\epsilon > 0$, we have that

$$\lim_{n\to\infty} r_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \theta_n = \theta.$$

□

The following proposition will be important for the calculation of limits to infinite numbers.

Proposition 3.25. *Let $a \neq b \in \mathbb{C}^f$ where at least one of the two is finite. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ converge to a and b respectively. Then*

$$\lim_{n\to\infty} \frac{a_n - b_n}{|a_n - b_n|} = \frac{a-b}{|a-b|}.$$

Proof. Let $a, b \in \mathbb{C}$. Since a_n and b_n both converge in the conventional sense,

$$\lim_{n\to\infty} a_n - b_n = a - b.$$

Since a and b are distinct, the above limit is nonzero and so is

$$\lim_{n\to\infty} |a_n - b_n| = |a - b|.$$

Thus the limit can pass through the quotient in the following since limits exist for both the numerator and denominator and the limit of the denominator is nonzero:

$$\lim_{n\to\infty} \frac{a_n - b_n}{|a_n - b_n|} = \lim_{n\to\infty} a_n - b_n \Big/ \lim_{n\to\infty} |a_n - b_n| = \frac{a - b}{|a - b|}.$$

Now we turn to the case where at least one of a, b is infinite. Without loss of generality, we assume it is $b = e^{i\theta}\infty$. Fix $\epsilon > 0$. We know that

$$\lim_{n\to\infty} \frac{b_n}{|b_n|} = e^{i\theta}.$$

This implies that there is N_1 sufficiently large that

$$\left| \frac{b_n}{|b_n|} - e^{i\theta} \right| < \frac{\epsilon}{3}, \quad \text{for all } n \geqslant N_1.$$

Let $M > 0$ be large enough that $\frac{2|a|}{M-2|a|} < \epsilon/3$. Let N_2 be sufficiently large such that for all $n \geqslant N_2$, $|b_n| > M$. Let N_3 be sufficiently large such that for all $n \geqslant N_3$, $|a_n| < 2|a|$. Define $N = \max(N_1, N_2, N_3)$. Then if $n \geqslant N$,

$$\left| \frac{a_n + b_n}{|a_n + b_n|} - e^{i\theta} \right| \leqslant \left| \frac{a_n}{|a_n + b_n|} \right| + \left| \frac{b_n}{|a_n + b_n|} - e^{i\theta} \right| \leqslant$$

$$\frac{2|a|}{M - 2|a|} + \left| \frac{b_n}{|a_n + b_n|} - e^{i\theta} \right| < \frac{\epsilon}{3} + \left| \frac{b_n}{|a_n + b_n|} - e^{i\theta} \right| =$$

$$\frac{\epsilon}{3} + \left| \frac{b_n}{|a_n + b_n|} - \frac{b_n}{|b_n|} + \frac{b_n}{|b_n|} - e^{i\theta} \right| \leqslant \frac{\epsilon}{3} + \left| \frac{b_n}{|a_n + b_n|} - \frac{b_n}{|b_n|} \right| + \left| \frac{b_n}{|b_n|} - e^{i\theta} \right| <$$

$$\frac{\epsilon}{3} + |b_n| \left| \frac{1}{|a_n + b_n|} - \frac{1}{|b_n|} \right| + \frac{\epsilon}{3} \leqslant |b_n| \left| \frac{|b_n| - (|a_n + b_n|)}{|b_n||a_n + b_n|} \right| + \frac{2\epsilon}{3} =$$

$$|b_n|\left|\frac{a_n}{|b_n|(|a_n+b_n|)}\right| + \frac{2\epsilon}{3} \leqslant \left|\frac{a_n}{|a_n+b_n|}\right| + \frac{2\epsilon}{3} = \frac{|a_n|}{|a_n+b_n|} + \frac{2\epsilon}{3} \leqslant$$

$$\frac{|a_n|}{|b_n|-|a_n|} + \frac{2\epsilon}{3} < \frac{2|a|}{M-2|a|} + \frac{2\epsilon}{3} \leqslant \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon.$$

Thus,

$$\lim_{n\to\infty} \frac{a_n+b_n}{|a_n+b_n|} = e^{i\theta}.$$

□

Proposition 3.26. *Let a be finite and fix $e^{i\theta}\infty$. If there exists sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that*

$$\lim_{n\to\infty} a_n = a \quad \textit{and} \quad \lim_{n\to\infty} b_n = e^{i\theta}\infty,$$

then

$$\lim_{n\to\infty}(a_n+b_n) = e^{i\theta}\infty.$$

Proof. Fix $M > 0$. Since $|b_n|$ converges to infinity, we have that there exists N_1 such that

$$|b_n| > M + 2|a|, \quad \text{for all } n \geqslant N_1.$$

There exists N_2 sufficiently large that $|a_n| < 2|a|$ for all $n \geqslant N_2$. Thus

$$|a_n+b_n| = |b_n| - |a_n| > M + 2|a| - 2|a| = M, \quad \text{for all } n \geqslant \max(N_1, N_2).$$

Thus, $|a_n + b_n|$ converges to infinity. Since a is finite, proposition 3.25 demands that

$$\lim_{n\to\infty} \frac{a_n+b_n}{|a_n+b_n|} = e^{i\theta}.$$

Therefore, we have that

$$\lim_{n\to\infty}(a_n+b_n) = e^{i\theta}\infty.$$

□

Proposition 3.27. *Every complex sequence has a limit on the flat extended* *complex plane.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a complex sequence. If it is bounded, then the sequence is contained by some closed ball of radius M, and by Bolzano-Weierstrass has a convergent subsequence.

If the sequence is unbounded, then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $|x_{n_k}| > k$ and thereby

$$\lim_{n\to\infty} |x_{n_k}| = \infty.$$

Consider the sequence

$$\left\{\frac{x_{n_k}}{|x_{n_k}|}\right\}_{k=1}^{\infty}.$$

This is a sequence in the unit circle. The unit circle is a closed and bounded set, so there is a convergent subsequence which we will call $\{y_j = x_{n_{k_j}}\}_{j=1}^{\infty}$. Since y_j is a subsequence of x_{n_k}, there exists θ_0 such that

$$\lim_{j\to\infty} |y_j| = \infty \quad \text{and} \quad \lim_{n\to} \frac{y_j}{|y_j|} = e^{i\theta_0}.$$

Thus, we have that every complex sequence converges to some element of the flat extended complex plane. □

3.4 Condensation

Definition 3.28. *We define the condensation points of f at t_0, called $con_{t_0}(f)$, to be the set of all points $x \in \mathbb{C}^f$ such that there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that*

$$\lim_{n\to\infty} t_n = t_0 \quad \textit{and} \quad \lim_{n\to\infty} f(t_n) = x.$$

Example 3.29. *Since the set of complex points of the form $q_1 + iq_2$ where $q_1, q_2 \in \mathbb{Q}$ is countable, we can enumerate it by $\{x_n\}_{n=1}^{\infty}$. Define the sequence $\{y_n = x_l\}_{n=1}^{\infty}$ where l is the smallest positive number in the set*

$$\left\{n - \sum_{k=1}^{m} k\right\}_{m=1}^{\infty},$$

which is to say that $\{y_n\}_{n=1}^{\infty}$ *is the sequence* $x_1, x_1, x_2, x_1, x_2, x_3, \ldots$. *Define*

$$f(t) = \begin{cases} y_n, & t \in (\frac{1}{n+1}, \frac{1}{n}], \quad n \in \mathbb{N} \\ 0, & \textit{otherwise.} \end{cases}$$

We have that

$$con_0(f) = \mathbb{C}.$$

Proof. Let $x \in \mathbb{C}$. Then, since the set of all complex number with rational real and imaginary parts is dense in $\mathbb{C}$, there exists a sequence $\{p_k + iq_k\}_{k=1}^{\infty}$ that converges to x. For all k, $p_k + iq_k$ recurs infinitely in $\{y_n\}_{n=1}^{\infty}$ there exists a subsequence $\{y_{n_k} = p_k + iq_k\}_{k=1}^{\infty}$. We see that

$$\lim_{k\to\infty} f\left(\frac{1}{n_k}\right) = \lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} p_k + iq_k = x.$$

This demands that $x \in \text{con}_0(f)$ and thereby $\mathbb{C} \subset \text{con}_0(f)$. ☐

Proposition 3.30. *Let* f *be a complex valued function. Then,*

$$con_{t_0}(f) = \bigcap_{n=1}^{\infty} \overline{f([t_0 - 1/n, t_0 + 1/n])}.$$

Proof. Let $x \in \text{con}_{t_0}(f)$. Then, there exists $\{t_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} t_n = t_0 \quad \text{and} \quad \lim_{n\to\infty} f(t_n) = x.$$

We will take a subsequence $\{t_{n_k}\}_{k=1}^{\infty}$ such that $|t_0 - t_{n_k}| < \frac{1}{k}$. Then, $f(t_{n_k}) \in f([t_0 - 1/m, t_0 + 1/m])$ for all $k \geqslant m$. x is the limit of $\{t_{n_k}\}_{k=m}^{\infty}$ so

$$x \in \overline{\{f(t_{n_k})\}_{k=m}^{\infty}} \subseteq \overline{f([t_0 - 1/m, t_0 + 1/m])}.$$

Since this is true for all m,

$$x \in \bigcap_{m=1}^{\infty} \overline{f([t_0 - 1/m, t_0 + 1/m])}$$

which immediately implies that

$$\mathrm{con}_{t_0}(f) \subseteq \bigcap_{n=1}^{\infty} \overline{f([t_0 - 1/n, t_0 + 1/n])}.$$

Now to prove the other direction. Let $x \in \overline{f([t_0 - 1/n, t_0 + 1/n])}$ for all $n \geqslant 1$. Then for all n, there exists $t_n \in \overline{f([t_0 - 1/n, t_0 + 1/n])}$ such that

$$|f(t_n) - x| < \frac{1}{n}.$$

Thus, we both have that

$$\lim_{n\to\infty} t_n = t_0 \quad \text{and} \quad \lim_{n\to\infty} f(t_n) = x.$$

Thus $x \in \mathrm{con}_{t_0}(f)$ and

$$\bigcap_{n=1}^{\infty} \overline{f([t_0 - 1/n, t_0 + 1/n])} \subseteq \mathrm{con}_{t_0}(f).$$

We have the statement. □

Proposition 3.31. *Let f be a complex valued function. Then,*

$$f(t_0) \in con_{t_0}(f).$$

Proof. Let $\{t_n = t_0\}_{n=1}^{\infty}$. Then,

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} t_0 = t_0 \quad \text{and} \quad \lim_{n\to\infty} f(t_n) = \lim_{n\to\infty} f(t_0) = f(t_0).$$

Thus, $f(t_0) \in \mathrm{con}_{t_0}(f)$. □

Proposition 3.32. *Let f be a complex valued function. f is continuous at t_0 if and only if*

$$con_{t_0}(f) = \{f(t_0)\}.$$

Proof. Assume that f is continuous at t_0. By proposition 3.31, $f(t_0) \in \text{con}_{t_0}(f)$. Let $x_0 \in \text{con}_{t_0}(f)$. We have that there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} t_n = t_0 \quad \text{and} \quad \lim_{n\to\infty} f(t_n) = x_0.$$

From the continuity of f at t_0, we have that

$$x_0 = \lim_{n\to\infty} f(t_n) = f(t_0).$$

Since this is true of all elements of $\text{con}_{t_0}(f)$, $\{f(t_0)\} = \text{con}_{t_0}(f)$.

Now to prove the opposite direction. Suppose that

$$\text{con}_{t_0}(f) = \{f(t_0)\}.$$

Let $\{t_n\}_{n=1}^{\infty}$ be a sequence convergent to t_0. Then by proposition 3.27, there exists a subsequence $\{t_{n_k}\}_{k=1}^{\infty}$ the converges to some element of $\mathbb{C}^f$, x. Clearly, $x \in \text{con}_{t_0}(f) = \{f(t_0)\}$ making $x = f(t_0)$. So, every convergent subsequence of $\{t_n\}_{n=1}^{\infty}$ is such that it converges to $f(t_0)$. Thus,

$$\lim_{n\to\infty} f(t_n) = f(t_0).$$

□

Proposition 3.33. *Let f be a complex valued function. f is continuous at t_0 if and only if $con_{t_0}(f)$ has only one element.*

Proof. Let f be continuous at t_0. By proposition 3.32, $\text{con}_{t_0}(f) = \{f(t_0)\}$ and we have proven the forward direction.

Let f be discontinuous at t_0. By proposition 3.31, $f(0) \in \text{con}_{t_0}(f))$. Since

f is discontinuous, there exists $\{t_n\}_{n=1}^{\infty}$ the converges to t_0, but $\{f(t_n)\}_{n=1}^{\infty}$ does not converge to $f(t_0)$. Thus, there exists $\epsilon > 0$ for all $k \geqslant 1$, there exists $n_k \geqslant k$ such that

$$|f(t_{n_k}) - f(t_0)| \geqslant \epsilon.$$

By proposition 3.27, $\{t_{n_k}\}_{k=1}^{\infty}$ has a limit, x, but the above demonstrates that $x \neq f(t_0)$. Thus, $x \in \text{con}_{t_0}(f)\backslash\{f(0)\}$. Thus, we have at least two elements in $\text{con}_{t_0}(f)$ when f is discontinuous. □

Corollary 3.34. *Let f be a complex valued function. Then $con_{t_0}(f)$ is closed.*

Proof. By proposition 3.30, we have that

$$\text{con}_{t_0}(f) = \bigcap_{n=1}^{\infty} \overline{f([t_0 - 1/n, t_0 + 1/n])}.$$

The closure of a set is closed so $\overline{f([t_0 - 1/n, t_0 + 1/n])}$ is closed. The infinite intersection of closed sets is closed so $\text{con}_{t_0}(f)$ is closed. □

Proposition 3.35. *Let f be a complex valued function, $t_0 \in \mathbb{R}$ and $C \in \mathbb{C}$. Then,*

$$con_{t_0}(Cf) = C\,con_{t_0}(f).$$

Proof. Let $x \in \text{con}_{t_0}(Cf)$. Then there exists $\{t_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} t_n = t_0 \quad \text{and} \quad \lim_{n\to\infty} Cf(t_n) = x.$$

This grants us

$$\lim_{n\to\infty} f(t_n) = \frac{C}{C}\lim_{n\to\infty} f(t_n) = \frac{1}{C}\lim_{n\to\infty} Cf(t_n) = \frac{1}{C}x.$$

Since $\{t_n\}_{n=1}^{\infty}$ converges to t_0, $\frac{x}{C} \in \text{con}_{t_0}(f)$ and $x \in C\text{con}_{t_0}(f)$. Thus, $\text{con}_{t_0}(Cf) \subset C\text{con}_{t_0}(f)$.

Let $x \in C\text{con}_{t_0}(f)$. Then, $\frac{x}{C} \in \text{con}_{t_0}(f)$ and there exists $\{t_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} t_n = t_0 \quad \text{and} \quad \lim_{n\to\infty} f(t_n) = \frac{x}{C}.$$

Thus

$$\lim_{n\to\infty} Cf(t_n) = C \lim_{n\to\infty} f(t_n) = C\frac{x}{C} = x.$$

We conclude that $x \in \text{con}_{t_0}(Cf)$ and $C\text{con}_{t_0}(f) \subset \text{con}_{t_0}(Cf)$. □

3.5 Jumps

Definition 3.36. *We define the jumps of f at t_0 to be*

$$\textit{jump}_{t_0}(f) = \{x - y \big| \quad x, y \in \textit{con}_{t_0}(f) \cap \mathbb{C}\} \cup (\textit{con}_{t_0}(f)/\mathbb{C}).$$

An alternative definition to $\text{jump}_{t_0}(f)$ that was considered was the following.

$$\text{jump}^*_{t_0}(f)\{x - y\big| \quad x, y \in \text{con}_{t_0}(f) \quad (x, y) \in \mathbb{C}^f \times \mathbb{C}^f \backslash \mathbb{T}_\infty \times \mathbb{T}_\infty\}. \tag{17}$$

In other words, $\text{jump}^*_{t_0}(f)$ is the set of all differences between $x, y \in \text{con}_{t_0}(f)$ where at least one of x and y is finite. This definition is problematic in the case that all condensation points are infinite. We believe that definition 3.36 gives a more alternative result in the following definition.

Example 3.37. *Let $f(t) = 1/t$. Then,*

$$\textit{jump}_{t_0}(f) = \{\infty, -\infty\} \quad \textit{and} \quad \textit{jump}^*_{t_0}(f) = \varnothing.$$

Proof. We will first calculate $\text{con}_0(f)$. Since f is real, $\text{con}_0(f) \subset \mathbb{R}$. Let $\{t_n\}_{n=1}^{\infty}$ be a sequence convergent to 0. Fix $M > 0$. Then there exists N sufficiently large that for all $n \geqslant N$,

$$|t_n| = |t_n - 0| < \frac{1}{M}.$$

Then,

$$|f(t_n)| > |f(\frac{1}{M})| = 1/|\frac{1}{M}| = M, \quad \text{for all } n \geqslant N.$$

Since this is true of all $M > 0$, we have that $\text{con}_0(f) \subset \mathbb{R}^C$. This demands that $\text{con}_0(f) \subset \{\infty, -\infty\}$. Since

$$\lim_{n\to\infty} \frac{1}{n} \quad \text{and} \quad \lim_{n\to\infty} f(\frac{1}{n}) = \lim_{n\to\infty} n = \infty,$$

$\infty \in \text{con}_0(f)$. Since

$$\lim_{n\to\infty} \frac{-1}{n} \quad \text{and} \quad \lim_{n\to\infty} f(\frac{-1}{n}) = \lim_{n\to\infty} -n = -\infty,$$

$-\infty \in \text{con}_0(f)$. Thus,

$$\text{con}_0(f) = \{\infty, -\infty\}.$$

Since $\text{con}_0(f)$ is composed only of infinite elements,

$$\text{jump}^*_{t_0}(f) = \left\{x - y \middle| \quad x, y \in \text{con}_{t_0}(f) \quad (x,y) \in \mathbb{C}^f \times \mathbb{C}^f \backslash \mathbb{T}_\infty \times \mathbb{T}_\infty \right\} = \varnothing.$$

On the other hand,

$$\text{jump}_{t_0}(f) = \left\{x - y \middle| \quad x \neq y \in \text{con}_{t_0}(f) \cap \mathbb{C}\right\} \cup (\text{con}_{t_0}(f)/\mathbb{C}) =$$

$$\varnothing \cup \{\infty, -\infty\} = \{\infty, -\infty\}.$$

□

Proposition 3.38. *Let f be a complex valued function and $con_{t_0}(f)$ is bounded. Then, $jump_{t_0}(f)$ is closed.*

Proof. Fix $x \in \overline{\text{jump}_{t_0}(f)}$. Then, there exists $\{x_n\}_{n=1}^\infty \subseteq \text{jump}_{t_0}(f)$ convergent to x. Since $x_n \in \text{jump}_{t_0}(f)$, there exists $a_n, b_n \in \text{con}_{t_0}(f)$ such that $x_n = a_n - b_n$. Since $\text{con}_{t_0}(f)$ is bounded, $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ have subsequences, $\{a_{n_k}\}_{k=1}^\infty$ and $\{b_{n_k}\}_{k=1}^\infty$, convergent to some $a, b \in \mathbb{C}$. Since $\text{con}_{t_0}(f)$ is closed, $a, b \in \text{con}_{t_0}(f)$.

We have that

$$x = \lim_{n\to\infty} x_n = \lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty}(a_{n_k} - b_{n_k}) =$$

$$\lim_{k\to\infty} a_{n_k} - \lim_{k\to\infty} b_{n_k} = a - b \in \text{jump}_{t_0}(f).$$

□

This is not necessarily the case when $\text{con}_{t_0}(f)$ has infinite elements.

Example 3.39. *Define*

$$x_n = \begin{cases} 2^n, & n = 2k, \quad k \in \mathbb{N} \\ 2^n + i\frac{n+1}{n}, & n = 2k+1, \quad k \in \mathbb{N}. \end{cases}$$

Define the sequence $y_n = x_l$ where l is the smallest natural number in the set

$$\left\{n - \sum_{k=1}^{m} k\right\}_{m=1}^{\infty}.$$

For clarity, we mention that the first ten y_n are $x_1, x_1, x_2, x_1, x_2, x_3, x_1, x_2, x_3, x_4$. Define

$$f(t) = \begin{cases} y_n, & t = \frac{1}{n}, \quad n \in \mathbb{N} \\ 0, & \textit{otherwise.} \end{cases}$$

We have that

$$\textit{jump}_0(f) = \{2^n | n \in \mathbb{N}\} \cup \{-2^n | n \in \mathbb{N}\} \cup \{2^n + i\frac{n}{n+1} - 2^m | n, m \in \mathbb{N}\} \cup$$

$$\{2^n - 2^m - i\frac{m}{m+1} | n, m \in \mathbb{N}\} \cup \{2^n - 2^m | n, m \in \mathbb{N}\}.$$

This is not a closed set since $i \in \overline{\textit{jump}_0(f)}$, but $i \notin \textit{jump}_0(f)$.

Proof. We first calculate $\text{con}_{t_0}(f)$. We have that

$$f([t_0 - \frac{1}{n}, t_0 + \frac{1}{n}]) = \{0\} \cup \{2^n | n \in \mathbb{N}\} \cup \{2^n + i | n \in \mathbb{N}\}.$$

By proposition 3.30,

$$\text{con}_0(f) = \bigcap_{n=1}^{\infty} f([t_0 - \frac{1}{n}, t_0 + \frac{1}{n}]) = \bigcap_{n=1}^{\infty} \{0\} \cup \{2^n | n \in \mathbb{N}\} \cup \{2^n + i | n \in \mathbb{N}\} =$$

$$\{0\} \cup \{2^n | n \in \mathbb{N}\} \cup \{2^n + i | n \in \mathbb{N}\} = A_0 \cup A_1 \cup A_2$$

where

$$A_0 = \{0\}, \quad A_1 = \{2^n\}_{n=1}^{\infty} \quad \text{and} \quad A_2 = \{2^n + i\}_{n=1}^{\infty}.$$

Note that

$$\text{jump}_0(f) = \{x - y | x, y \in \text{con}_0(f)\} = \{x - y | x, y \in A_0 \cup A_1 \cup A_2\} =$$

$$\bigcup_{m,n=0}^{2} \{x - y | x \in A_m, y \in A_n\}.$$

To calculate $\text{jump}_0(f)$, we will calculate individually each of the nine sets in the union above:

1. $$\{x - y | x \in A_0, y \in A_0\} = \{0\};$$

2. $$\{x - y | x \in A_0, y \in A_1\} = \{-2^n | n \in \mathbb{N}\};$$

3. $$\{x - y | x \in A_0, y \in A_2\} = \{-2^n - i \frac{n+1}{n} \frac{n}{n+1} | n \in \mathbb{N}\};$$

4. $$\{x - y | x \in A_1, y \in A_0\} = \{2^n | n \in \mathbb{N}\};$$

5. $$\{x - y | x \in A_1, y \in A_1\} = \{2^n - 2^m | n, m \in \mathbb{N}\};$$

6.

$$\{x-y|x\in A_1, y\in A_2\} = \{2^n - 2^m - i\frac{m}{m+1}|n,m\in\mathbb{N}\};$$

7.

$$\{x-y|x\in A_2, y\in A_0\} = \{2^n + i\frac{n}{n+1}|n\in\mathbb{N}\};$$

8.

$$\{x-y|x\in A_2, y\in A_1\} = \{2^n + i\frac{n}{n+1} - 2^m|n,m\in\mathbb{N}\};$$

9.

$$\{x-y|x\in A_2, y\in A_2\} = \{2^n - 2^m|n,m\in\mathbb{N}\}.$$

Taking the above together, we have that

$$\text{jump}_0(f) = \{\pm 2^n|n\in\mathbb{N}\}\cup\{\pm 2^n \pm i\frac{n}{n+1})|n\in\mathbb{N}\}\cup\{2^n - 2^m|n,m\in\mathbb{N}\}\cup$$

$$\{2^n + i\frac{n}{n+1} - 2^m|n,m\in\mathbb{N}\}\cup\{2^n - 2^m - i\frac{m}{m+1}|n,m\in\mathbb{N}\}.$$

We see that $i\notin \text{jump}_0(f)$. At the same time, we see that the sequence

$$\{t_n = i\frac{n}{n+1} = 2^n + i\frac{n}{n+1} - 2^n\}\subseteq \text{jump}_0(f)$$

and that

$$\lim_{n\to\infty} t_n = i\notin \text{jump}_0(f).$$

Thus, $\text{jump}_0(f)$ is not closed. □

Corollary 3.40. *Let f be a complex-valued function. Then if $\text{jump}_{t_0}(f) = \{0\}$ then f is continuous at t_0.*

Proof. Let $\text{jump}_{t_0}(f) = \{0\}$. Let $x, y\in \text{con}_{t_0}(f)$. Then, $x-y\in\text{jump}_{t_0}(f) = \{0\}$. Thus, $x = y$ and all elements of $\text{con}_{t_0}(f)$ are the same. Thus, $\text{con}_{t_0}(f)$ a set with only one element. Consider the sequence $\{t_n = t_0\}_{n=1}^{\infty}$. Clearly, this sequence converges to t_0 and the sequence $\{f(t_n) = f(t_0)\}_{n=1}^{\infty}$ converges to $f(t_0)$ making

$f(t_0)$ an element in the singleton $\text{con}_{t_0}(f)$. Thus, $\{f(t_0)\} = \text{con}_{t_0}(f)$ which by proposition (3.32), demands that f is continuous at t_0. □

Proposition 3.41. *Let f be a complex-valued function. If $z \in jump_{t_0}(f)$ then $-z \in jump_{t_0}(f)$.*

Proof. Let $z \in \text{jump}_{t_0}(f)$. Then there exists $x, y \in \text{con}_{t_0}(f)$ such that $z = x - y$. Thus $-z = y - x \in \text{jump}_{t_0}(f)$. □

3.6 Jump Phase

Definition 3.42. *We define the jump phase of f at t_0 to be*

$$phase_{t_0}(f) = \left\{ \frac{x}{|x|} \,\middle|\, x \in jump_{t_0}(f)/\{0\} \right\}.$$

Proposition 3.43. *Let f be a complex-valued function. Then,*

$$phase_{t_0}(f) \subseteq axis_{t_0}(f).$$

Proof. Let $x \in \text{phase}_{t_0}(f)$. Thus, there exists $y \in \text{jump}_{t_0}(f)\backslash\{0\}$ such that $x = y/|y|$. y may be either finite or infinite. Assume that y is finite. Then, there exists $a, b \in \text{con}_{t_0}(f)$ such that

$$y = b - a$$

and at least one of a, b is finite. There exists sequences $\{t_n\}_{n=1}^{\infty}$, $\{s_n\}_{n=1}^{\infty}$ convergent to t_0 such that

$$\lim_{n\to\infty} f(t_n) = a \quad \text{and} \quad \lim_{n\to\infty} f(s_n) = b.$$

There exists N_1, N_2 sufficiently large that if $n_1 > N_1$ or $n_2 > N_2$, then

$$|f(t_{n_1}) - a| < \frac{|b-a|}{4} \quad \text{and} \quad |f(s_{n_2}) - b| < \frac{|b-a|}{4}.$$

Thus we have that for all $n \geqslant N = \max(N_1, N_2)$,

$$|f(t_n) - f(s_n)| = |f(t_n) - b + b - a + af(s_n)| \geqslant |b - a| - |f(t_n) - b| - |a - f(s_n)| >$$

$$|b - a| - \frac{|b-a|}{4} - \frac{|b-a|}{4} = \frac{|b-a|}{2} > 0.$$

We also have that

$$\lim_{n\to\infty} \frac{f(t_n) - f(s_n)}{|f(t_n) - f(s_n)|} = \frac{b-a}{|b-a|} = \frac{y}{|y|} = x.$$

The sequences $\{t_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ are the necessary ones to demonstrate that $x \in \text{axis}_{t_0}(f)$.

Let y be infinite. We can write $y = e^{i\theta_0}\infty$ for some $\theta_0 \in [0, 2\pi)$. We have that there exists $\{t_n\}_{n=1}^{\infty}$ convergent to t_0 such that

$$\lim_{n\to\infty} e^{i\theta_0}\infty.$$

Define the sequence $\{s_n\}_{n=1}^{\infty}$ such that s_n equals the first t_k such that $|f(t_k)| > (n+1)|f(t_n)| + 1$. Since $\{s_n\}_{n=1}^{\infty}$ is a subsequence of $\{t_n\}_{n=1}^{\infty}$, it converges to t_0. We have that

$$|f(s_n) - f(t_n)| > |f(s_n)| - |f(t_n)| > n|f(t_n)| + 1 - |f(t_n)| \geqslant 1,$$

for all $n \geqslant 1$. Fix $\epsilon > 0$. Let N_1 be sufficiently large that for all $n \geqslant N_1$, $\frac{1}{n} < \frac{\epsilon}{3}$. Let N_2 be sufficiently large that for all $n \geqslant N_2$,

$$\left| e^{i\theta_0} - \frac{f(s_n)}{|f(s_n)|} \right|.$$

Then, for all $n \geqslant \max(N_1, N_2)$,

$$\left| e^{i\theta_0} - \frac{f(s_n) - f(t_n)}{|f(s_n) - f(t_n)|} \right| \leqslant \left| e^{i\theta_0} - \frac{f(s_n)}{|f(s_n) - f(t_n)|} \right| + \left| \frac{f(t_n)}{|f(s_n) - f(t_n)|} \right| \leqslant$$

$$\left|e^{i\theta_0} - \frac{f(s_n)}{|f(s_n) - f(t_n)|}\right| + \left|\frac{f(t_n)}{|(n+1)f(t_n) + 1 - f(t_n)|}\right| \leqslant$$

$$\left|e^{i\theta_0} - \frac{f(s_n)}{|f(s_n)|} + \frac{f(s_n)}{|f(s_n)|} - \frac{f(s_n)}{|f(s_n) - f(t_n)|}\right| + \left|\frac{f(t_n)}{n|f(t_n)|}\right| <$$

$$\left|e^{i\theta_0} - \frac{f(s_n)}{|f(s_n)|}\right| + \left|\frac{f(s_n)}{|f(s_n)|} - \frac{f(s_n)}{|f(s_n) - f(t_n)|}\right| + \frac{\epsilon}{3} \leqslant$$

$$\frac{\epsilon}{3} + |f(s_n)|\left|\frac{1}{|f(s_n)|} - \frac{1}{|f(s_n) - f(t_n)|}\right| + \frac{\epsilon}{3} \leqslant$$

$$|f(s_n)|\left|\frac{|f(s_n) - f(t_n)| - |f(s_n)|}{|f(s_n)||f(s_n) - f(t_n)|}\right| + \frac{2\epsilon}{3} = \frac{||f(s_n) - f(t_n)| - |f(s_n)||}{|f(s_n) - f(t_n)|} + \frac{2\epsilon}{3} \leqslant$$

$$\frac{|f(t_n)|}{|f(s_n) - f(t_n)|} + \frac{2\epsilon}{3} \leqslant \frac{|f(t_n)|}{|(n+1)f(t_n) + 1 - f(t_n)|} + \frac{2\epsilon}{3} < \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon.$$

So we have that

$$\lim_{n\to\infty} \frac{f(s_n) - f(t_n)}{|f(s_n) - f(t_n)|} = e^{i\theta_0} = \frac{e^{i\theta_0}\infty}{|e^{i\theta_0}\infty|} = \frac{y}{|y|} = x.$$

Since the pair of sequences above satisfy the necessary properties, we have that $x \in \text{axis}_{t_0}(f)$. Now, that we have demonstrated the proposition for both when y is finite or infinite, we are finished. □

It does happen that $\text{axis}_{t_0}(f)$ is a proper subset of

$$\left\{\frac{x-y}{|x-y|}\,\middle|\, x \neq y \in \text{con}_{t_0}(f)\right\}$$

as demonstrated in the example below.

Example 3.44. *Define*

$$f(t) = \begin{cases} 1 + e^{i/t}, & t \in [\frac{1}{2n\pi}, \frac{1}{(2n+1)\pi}], \quad n \in \mathbb{N}^0 \\ 0, & \textit{otherwise.} \end{cases}$$

We have that

$$\textit{phase}_{t_0}(f) = \{e^{i\theta} | \theta \in [0, 2\pi)\}/\{i, -i\}$$

which is not a closed set.

Proof. We see that

$$f([t_0 - \frac{1}{n}, t_0 + \frac{1}{n}]) = \{1 + e^{it} | t \in [0, \pi]\} \quad \text{for all } n \geqslant 1.$$

By proposition 3.30, we have that

$$\text{con}_{t_0}(f) = \bigcap_{n=1}^{\infty} f([t_0 - \frac{1}{n}, t_0 + \frac{1}{n}]) =$$

$$\bigcap_{n=1}^{\infty} \{1 + e^{it} | t \in [0, 2\pi)\} = \{1 + e^{it} | t \in [0, \pi)\}.$$

Lets find some elements of $\text{jump}_{t_0}(f)$ so as to determine $\text{phase}_{t_0}(f)$. Since 0 and $1 + e^{it}$ for $t \in [0, \pi)$ are elements of $\text{con}_{t_0}(f)$, we see that

$$\{1 + e^{it} | t \in [0, \pi)\} \cup \{-1 - e^{it} | t \in (0, \pi)\} =$$

$$\{(1 + e^{it}) - 0 | t \in [0, \pi)\} \cup \{0 - (1 + e^{it}) | t \in [0, \pi)\} \subseteq \text{jump}_{t_0}(f).$$

Let θ be the phase of $1 + e^{it} = 1 + \cos(t) + i\sin(t)$. Then $\tan(\theta) = \frac{1+\cos(t)}{\sin(t)}$. Note that

$$\left\{ \frac{1 + \cos(t)}{\sin(t)} \middle| t \in [0, \frac{\pi}{2}) \right\} = [0, \infty)$$

We have that

$$\{\theta \in [0, 2\pi) | \quad \theta \text{ is the phase of } 1 + e^{it}, t \in [0, \pi/2))\} =$$

$$\left\{ \arctan\left(\frac{1 + \cos(t)}{\sin(t)}\right) \middle| t \in [0, \frac{\pi}{2}) \right\} = \arctan([0, \infty) = [0, \frac{\pi}{2}).$$

This demands that

$$[0, \frac{\pi}{2}) = \{e^{i\theta} | \quad \{\theta \in [0, 2\pi) | \text{where } \theta \text{ is the phase of } 1 + e^{it}, t \in [0, \pi/2)\} =$$

$$\left\{ \frac{1+e^{it}}{|1+e^{it}|} \middle| t \in [0,\pi) \right\} \subseteq \text{phase}_{t_0}(f).$$

Similar calculations on $\{-1-e^{it}|t \in (0,\pi)\} \subseteq \text{jump}_{t_0}(f)$ demonstrate that $[\pi, \frac{3\pi}{2}) \subseteq \text{phase}_{t_0}(f)$.

Since 2 and $1+e^{it}$ for $t \in [0,\pi)$ are elements of $\text{con}_{t_0}(f)$, we see that

$$\{-1+e^{it}|t \in [0,\pi)\} \cup \{1-e^{it}|t \in (0,\pi)\} =$$

$$\{(1+e^{it})-2|t \in [0,\pi)\} \cup \{2-(1+e^{it})|t \in [0,\pi)\} \subseteq \text{jump}_{t_0}(f).$$

Similar calculations as were done above demonstrate that $(\frac{\pi}{2},\pi] \cup (\frac{3\pi}{2}, 2\pi] \subseteq \text{phase}_{t_0}(f)$. Thus

$$\{e^{i\theta}|\theta \in [0,2\pi)\}/\{i,-i\} \subseteq \text{phase}_{t_0}(f).$$

Let $x, y \in \text{con}_{t_0}(f) = \{1+e^{it}|t \in [0,\pi)\}$. If $x = y$ then x - y = 0. If $x \neq y$ then $x = 1+e^{i\theta}$ and $y = 1+e^{i\phi}$ where $\theta \neq \phi$. Then

$$x - y = (1+e^{i\theta}) - (1+e^{i\phi}) = e^{i\theta} - e^{i\phi}.$$

This implies that

$$\text{Re}(x-y) = \cos(\theta) - \cos(\phi) \neq 0.$$

So no element of $\text{jump}_{t_0}(f)$ is purely imaginary and thereby no element of $\text{jump}_{t_0}(f)$ has modulus equal to $e^{i\frac{\pi}{2}}$ or $e^{i\frac{3\pi}{2}}$. Thus

$$e^{i\frac{\pi}{2}}, e^{i\frac{3\pi}{2}} \notin \text{phase}_{t_0}(f).$$

Thus

$$\{e^{i\theta}|\theta \in [0,2\pi)\}/\{i,-i\} = \text{phase}_{t_0}(f).$$

□

Proposition 3.45. *Let f be a complex-valued function. Then if 0 is an isolated point in $jump_{t_0}(f)$ then $phase_{t_0}(f)$ is closed.*

Proof. Since 0 is an isolated point in $\text{jump}_{t_0}(f)$, there exists $\epsilon > 0$ such that $B_\epsilon(0) \cap (\text{jump}_{t_0}(f)/\{0\}) = \varnothing$. Thus we have that

$$\text{jump}_{t_0}(f) \cap B_\epsilon(0)^C = \text{jump}_{t_0}(f)/(\text{jump}_{t_0}(f) \cap B_\epsilon(0)) = \text{jump}_{t_0}(f)/\{0\}.$$

We note that $\text{jump}_{t_0}(f)/\{0\}$ is closed since $\text{jump}_{t_0}(f)$ is closed by proposition 3.38 and $\text{jump}_{t_0}(f) \cap B_\epsilon(0)^C$ is the intersection of two closed sets.

The function $T(x) = \frac{x}{|x|}$ defined on $\mathbb{C}/\{0\}$ is a continuous function. On the set $\mathbb{C}/B_\epsilon(0)$, T is uniformly continuous. Thus the image of any closed set under T is also closed. By definition of phase we have that

$$T(\text{jump}_{t_0}(f)/\{0\}) = \left\{ \frac{x}{|x|} \middle| x \in \text{jump}_{t_0}(f) \right\} = \text{phase}_{t_0}(f)$$

which gives us

$$T(\text{jump}_{t_0}(f) \cap B_\epsilon(0)^c) = T(\text{jump}_{t_0}(f)/\{0\}) = \text{phase}_{t_0}(f).$$

Thus, $\text{phase}_{t_0}(f)$ is a closed set. □

3.7 Local Boundedness

Definition 3.46. *A function is locally bounded at t_0 if there exists $\delta > 0$ and $M \geqslant 0$*

$$|f(x)| < M, \quad \textit{for all } x \in (t_0 - \delta, t_0 + \delta).$$

Proposition 3.47. *The following are equivalent:*

1. *f is locally bounded at t_0.*

2. *$disc_{t_0}(f)$ is finite.*

3. $con_{t_0}(f)$ *is bounded.*

4. $jump_{t_0}(f)$ *is bounded.*

Proof. We first show that (1) $\Leftrightarrow$ (2). Let f be locally bounded at t_0. Then there exists $\delta_0 > 0$ and $M \geqslant 0$

$$|f(x)| < M, \quad \text{for all } x \in (t_0 - \delta, t_0 + \delta).$$

This implies that

$$\sup_{x,y \in [t_0-\delta, t_0+\delta]} |f(x)-f(y)| \leqslant \sup_{x \in [t_0-\delta, t_0+\delta]} |f(x)| + \sup_{y \in [t_0-\delta, t_0+\delta]} |f(y)| \leqslant M+M = 2M$$

for all $\delta < \delta_0$. Thus,

$$\lim_{\delta \to \infty} \sup_{x,y \in [t_0-\delta, t_0+\delta]} |f(x) - f(y)| \leqslant \lim_{\delta \to \infty} 2M = 2M.$$

We see that $\text{disc}_{t_0}(f) < 2M$.

Let f not be locally bounded at t_0. Then for all $\delta > 0$ and for all $M > 0$, there exists $t \in (t_0 - \delta, t_0 + \delta)$, such that $|f(t) \geqslant M$.

Fix $M \geqslant 0$ and $\delta > 0$. Let $t_1 \in (t_0 - \delta, t_0 + \delta)$ be such that $f(t_1) = M_1 > M$. Let $t_2 \in (t_0 - \delta, t_0 + \delta)$ be such that $f(t_2) > 2M_1$. This implies that

$$\sup_{x,y \in [t_0-\delta, t_0+\delta]} |f(x) - f(y)| \geqslant |f(t_2) - f(t_1)| > |2M_1 - M_1| = M_1 > M.$$

Since this is true of all M, we have that

$$\sup_{x,y \in [t_0-\delta, t_0+\delta]} |f(x) - f(y)| = \infty.$$

Since this is true of all δ, we have that

$$\text{disc}_{t_0}(f) = \lim \delta \to \infty \sup_{x,y \in [t_0-\delta, t_0+\delta]} |f(x) - f(y)| = \infty.$$

By contraposition, we have that if $\text{disc}_{t_0}(f) = M$ be finite, then f is locally bounded at t_0.

Now, we show that (1) $\Leftrightarrow$ (3). Let f be locally bounded at t_0. Then there exists $M > 0$ and $N \geqslant 2$ such that $|f(t)| < M$ for all $t \in (t_0 - \frac{1}{N-1}, t_0 + \frac{1}{N-1})$. This gives us

$$\text{con}_{t_0}(f) = \bigcap_{n=1}^{\infty} f(t_0 - \frac{1}{n}, t_0 + \frac{1}{n}) = \bigcap_{n=N}^{\infty} f(t_0 - \frac{1}{n}, t_0 + \frac{1}{n}) \subseteq \bigcap_{n=N}^{\infty} B_M(t_0) = B_M(t_0)$$

where $B_M(t_0)$ is the open ball of radius M centered at t_0. Thus $\text{con}_{t_0}(f)$ is bounded.

Let f not be locally bounded at t_0. Then for all M, for all $n \geqslant 1$, there exists $t_n^M \in (t_0 - \frac{1}{n}, t_0 + \frac{1}{n})$ such that $|f(t_n)| > M$. Consider the sequence $\{t_n^n\}_{n=1}^{\infty}$. It converges to t_0 and $|f(t_n^n)| > n$ for all n which means that $\{|f(t_n^n)|\}_{n=1}^{\infty}$ converges to infinity. By proposition 3.31, $\{f(t_n^n)\}_{n=1}^{\infty}$ must have a subsequence, $\{f(t_{n_k}^{n_k})\}_{k=1}^{\infty}$ convergent to some element $a \in \mathbb{C}^f$. Since $\{|f(t_n^n)|\}_{n=1}^{\infty}$ converges to infinity, a must be infinite. Thus $\text{con}_{t_0}(f)$ has an infinite element and cannot be bounded. By contraposition, if $\text{con}_{t_0}(f)$ is bounded then f is locally bounded at t_0.

Finally, we show that (3) $\Leftrightarrow$ (4). Let $\text{con}_{t_0}(f)$ be bounded. Let $x \in \text{jump}_{t_0}(f)$. Then there exists $a, b \in \text{con}_{t_0}(f)$ such that $x = a - b$. Since $\text{con}_{t_0}(f)$ is bounded, there exists M greater than the absolute value of any element of $\text{con}_{t_0}(f)$. Thus

$$|x| = |b - a| \leqslant |b| + |a| = 2M.$$

Since this is true of every element of $\text{jump}_{t_0}(f)$, $\text{jump}_{t_0}(f)$ is bounded.

Let $\text{jump}_{t_0}(f)$ be bounded. Then all elements of $\text{jump}_{t_0}(f)$ are finite and there exists $M > 0$ such that $\text{jump}_{t_0}(f) \subseteq B_M(t_0)$. From proposition 3.31, $\text{con}_{t_0}(f)$ is nonempty. Fix $x_0 \in \text{con}_{t_0}(f)$ which we know to be a finite element.

Then for all $y \in \text{con}_{t_0}(f)$,

$$|y| = |y - x_0 + x_0| \leqslant |y - x_0| + |x_0| \leqslant M + |x_0|.$$

Thus $\text{con}_{t_0}(f) \subseteq B_{M+|x_0|}(0)$ and $\text{con}_{t_0}(f)$ is thereby bounded.

Let $\text{con}_{t_0}(f)$ be bounded. Then there exists $M > 0$ such that for all elements of $x \in \text{con}_{t_0}(f)$, $|x| < M$. Since for all elements of $z \in \text{jump}_{t_0}(f)$, there exists $x, y \in \text{con}_{t_0}(f)$ such that $z = x - y$. Thus

$$|z| = |x - y| \leqslant |x| + |y| < M + M = 2M.$$

Thus all elements of $\text{jump}_{t_0}(f)$ are bounded by $2M$. □

We define addition between two sets A and B to be

$$A + B = \{a + b| \quad a \in A, b \in B\}.$$

We also define addition between a set A and a complex number β to be

$$A + \beta = A + \{\beta\} = \{a + \beta| \quad a \in A\}.$$

Proposition 3.48. *Let f and g be complex valued functions. If $con_{t_0}(f)$ and $con_{t_0}(g)$ are locally bounded at t_0, then*

$$con_{t_0}(f + g) \subseteq con_{t_0}(f) + con_{t_0}(g).$$

Proof. Let $x \in \text{con}_{t_0}(f + g) \cap \mathbb{C}$. Then, there exists $\{t_n\}_{n=1}^{\infty}$ convergent to t_0 such that

$$\lim_{n\to\infty} t_n = t_0 \quad \text{and} \quad \lim_{n\to\infty} \big(f(t_n) + g(t_n)\big) = x.$$

Since , there exists $M_1, M_2 \geqslant 0$ $\delta_1, \delta_2 > 0$ such that if $t \in B_{\delta_1}(t_0)$ then $f(t) \in B_{M_1}(0)$ and if $t \in B_{\delta_2}(t_0)$ then $f(t) \in B_{M_2}(0)$. Thus the sets $\{f(t_n)\}_{n=1}^{\infty}$ and

$\{g(t_n)\}_{n=1}^{\infty}$ are subsets of a closed bounded set and thereby both have limit points in x_1, x_2. There are subsequences $\{f(t_{n_k})\}_{k=1}^{\infty}$ and $\{g(t_{n_k})\}_{k=1}^{\infty}$ which converge to x_1 and x_2. Since the subsequences of convergent sequences have the same limit as the sequence, we have that

$$\lim_{k\to\infty} t_{n_k} = t_0, \quad \lim_{k\to\infty} f(t_k) = x_1 \quad \text{and} \quad \lim_{k\to\infty} f(t_k) = x_2.$$

Thus, $x_1 \in \text{con}_{t_0}(f)$ and $x_2 \in \text{con}_{t_0}(g)$. We conclude that

$$x_0 = \lim_{n\to\infty} f(t_n) + g(t_n) = \lim_{k\to\infty} f(t_{n_k}) + g(t_{n_k}) = x_1 + x_2 \in \text{con}_{t_0}(f) + \text{con}_{t_0}(g).$$

□

Proposition 3.49. *Let f be a complex valued function. If f is locally bounded at t_0, then*

$$phase_{t_0}(f) = axis_{t_0}(f).$$

Proof. Given proposition 3.43, we only need to prove that

$$\text{axis}_{t_0}(f) \subseteq \text{phase}_{t_0}(f).$$

Let $e^{i\theta_0} \in \text{axis}_{t_0}(f)$. By definition of the axis of discrepancy, there exists a real number $\epsilon > 0$ and two sequences $\{t_n\}_{n=1}^{\infty}$,$\{s_n\}_{n=1}^{\infty}$ convergent to t_0 such that

$$|f(t_n) - f(s_n)| \geqslant \epsilon, \quad \text{for all } n$$

and

$$\lim_{n\to\infty} \frac{f(t_n) - f(s_n)}{|f(t_n) - f(s_n)|} = e^{i\theta_0}.$$

Since f is locally bounded at t_0, the sequences $\{f(t_n)\}_{n=1}^{\infty}$ and $\{f(s_n)\}_{n=1}^{\infty}$ are bounded. This demands that there exists convergent subsequences $\{f(t_{n_k})\}_{k=1}^{\infty}$ convergent to some $x \in \mathbb{C}$. Also since f is locally bounded, $\{f(s_{n_k})\}_{k=1}^{\infty}$ is bounded and there exists another subsequence $\{f(s_{n_{k_j}})\}_{j=1}^{\infty}$ that converges to some element $y \in \mathbb{C}$. Both $x_{n_{k_j}}$ and $y_{n_{k_j}}$ converge to t_0 and are such that

$$\lim_{j\to\infty} f(t_{n_{k_j}}) = x \quad \text{and} \quad \lim_{j\to\infty} f(t_{n_{k_j}}) = y.$$

Thus, $x, y \in \text{con}_{t_0}(f)$. We also have that

$$|f(t_{n_{k_j}}) - f(s_{n_{k_j}})| \geqslant \epsilon \text{for all } j \in \mathbb{N},$$

so $x \neq y$, and

$$\frac{x-y}{|x-y|} = \frac{\lim_{n\to\infty}(f(t_{n_{k_j}}) - f(s_{n_{k_j}})}{\lim_{n\to\infty}|f(t_{n_{k_j}}) - f(s_{n_{k_j}})|} = \lim_{n\to\infty} \frac{f(t_{n_{k_j}}) - f(s_{n_{k_j}}}{|f(t_{n_{k_j}}) - f(s_{n_{k_j}})|} =$$

$$\lim_{n\to\infty} \frac{f(t_n) - f(s_n)}{|f(t_n) - f(s_n)|} = e^{i\theta_0}.$$

We see that

$$x - y \in \text{jump}_{t_0}(f) \backslash \{0\}.$$

This demands that

$$e^{i\theta} = \frac{x-y}{|x-y|} \in \{\frac{z}{|z|}| \quad z \in \text{jump}_{t_0}(f)\backslash\{0\}\} = \text{phase}_{t_0}(f).$$

Thus,

$$\text{axis}_{t_0}(f) \subseteq \text{phase}_{t_0}(f).$$

□

3.8 Calculation of the Axis of Discrepancy

Proposition 3.50. *If f has a discontinuity at t_0 then, $\text{axis}_{t_0}(f)$ is nonempty.*

Proof. By proposition 3.33, $\text{con}_{t_0}(f)$ has at least two distinct elements, x and

y. Then, $x - y \in \text{jump}_{t_0}(f)\backslash\{0\}$. This grants us

$$\frac{x-y}{|x-y|} \in \text{phase}_{t_0}(f) \subset \text{axis}_{t_0}(f).$$

Thus, $\text{axis}_{t_0}(f)$ is nonempty. ☐

Proposition 3.51. *If f is continuous at t_0, then $axis_{t_0}(f) = \varnothing$.*

Proof. Since f is continuous, $\text{con}_{t_0}(f) = \{f(t_0)\}$. This demands that

$$\text{jump}_{t_0}(f) = \{x - y| \quad x, y \in \text{con}_{t_0}(f)\} = \{x - y| \quad x, y \in \{f(t_0)\}\} = \{0\}.$$

Since f is continuous at t_0, it is locally bounded. By propositions 3.49, we have that

$$\text{axis}_{t_0}(f) = \text{phase}_{t_0}(f) = \varnothing.$$

☐

Proposition 3.52. *Let f be a complex valued function. Then the following are equivalent:*

(1) f is discontinuous at t_0.

(2) $disc_{t_0}(f) > 0$.

(3) $axis_{t_0}(f) \neq \varnothing$.

Proof. (1) $\iff$ (2) by proposition 3.4, (1) $\implies$ (3) by proposition 3.50 and (3) $\implies$ (2) by proposition 3.51. ☐

Proposition 3.53. *Let f and g be a complex valued functions locally bounded at t_0. Let g be continuous at t_0. Then,*

$$con_{t_0}(f + g) = con_{t_0}(f) + g(0).$$

Proof. By propositions 3.48 and 3.32, we have that

$$\mathrm{con}_{t_0}(f+g) \subseteq \mathrm{con}_{t_0}(f) + \mathrm{con}_{t_0}(g) = \mathrm{con}_{t_0}(f) + \{g(0)\}.$$

We also have that

$$\mathrm{con}_{t_0}(f)+\{g(0)\} = \mathrm{con}_{t_0}(f+g-g)+\{g(0)\} \subseteq \mathrm{con}_{t_0}(f+g)-\mathrm{con}_{t_0}(g)+\{g(0)\} =$$

$$\mathrm{con}_{t_0}(f+g) - \{g(0)\} + \{g(0)\} = \mathrm{con}_{t_0}(f+g)$$

Thus,

$$\mathrm{con}_{t_0}(f+g) = \mathrm{con}_{t_0}(f) + \{g(0)\} = \mathrm{con}_{t_0}(f) + g(0).$$

□

Proposition 3.54. *Let f and g be a complex valued functions be locally bounded at t_0. Let g be continuous at t_0. Then, $axis_{t_0}(f+g) = axis_{t_0}(f)$.*

Proof. By proposition 3.53, we have that

$$\mathrm{jump}_{t_0}(f+g) = \{x-y| \quad x,y \in \mathrm{con}_{t_0}(f+g)\} =$$

$$\{x-y| \quad x,y \in \mathrm{con}_{t_0}(f)+g(t_0)\} = \{(x+g(t_0))-(y+g(t_0))| \quad x,y \in \mathrm{con}_{t_0}(f)\} =$$

$$\{x-y| \quad x,y \in \mathrm{con}_{t_0}(f)\} = \mathrm{jump}_{t_0}(f+g).$$

Since f is locally bounded proposition 3.49 gives us

$$\mathrm{axis}_{t_0}(f+g) = \mathrm{phase}_{t_0}(f+g) = \{\frac{x}{|x|}| \quad x \in \mathrm{jump}_{t_0}(f+g)\} =$$

$$\{\frac{x}{|x|}| \quad x \in \mathrm{jump}_{t_0}(f)\} = \mathrm{phase}_{t_0}(f) = \mathrm{axis}_{t_0}(f).$$

□

Proposition 3.55. *Let f be a complex-valued function locally bounded at t_0.*

and $z \in \mathbb{C}/\{0\}$. *We can write* $z = re^{i\phi}$ *where* $r > 0$ *and* $\phi \in [0, 2\pi)$. *Then,*

$$axis_{t_0}(zf) = e^{i\theta} axis_{t_0}(f).$$

Proof. By proposition 3.35, we have that

$$\text{jump}_{t_0}(f) = \{x - y| \quad x, y \in \text{con}_{t_0}(zf)\} = \{x - y| \quad x, y \in z\text{con}_{t_0}\} =$$

$$\{z(x - y)| \quad x, y \in \text{con}_{t_0}\} = z\{(x - y)| \quad x, y \in \text{con}_{t_0}\} = z\text{jump}_{t_0}(f).$$

Since f is locally bounded, proposition 3.49 grants us

$$\text{axis}_{t_0}(zf) = \text{phase}_{t_0}(zf) = \left\{\frac{x}{|x|}\right| \quad x \in \text{jump}_{t_0}(zf)\Big\} = \left\{\frac{x}{|x|}\right| \quad x \in z\text{jump}_{t_0}(f)\Big\} =$$

$$\left\{\frac{zx}{|zx|}\right| \quad x \in \text{jump}_{t_0}(f)\Big\} = \left\{e^{i\theta}\frac{x}{|x|}\right| \quad x \in \text{jump}_{t_0}(f)\Big\} =$$

$$e^{i\theta}\left\{\frac{x}{|x|}\right| \quad x \in \text{jump}_{t_0}(f)\Big\} = e^{i\theta}\text{phase}_{t_0}(f).$$

□

Proposition 3.56. *Let* f *be a real valued function locally bounded at* t_0 *that has a discontinuity at* t_0. *Then the axis of discrepancy will be* $\{1, -1\}$.

Proof. Let $x \in \text{con}_{t_0}(f)$. Then, there exists $\{t_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} t_n = t_0 \quad \text{and} \quad \lim_{n\to\infty} f(t_n) = x.$$

Since f is real valued $\{f(t_n)\}_{n=1}^{\infty} \subset \mathbb{R}$ and $x_0 \in \mathbb{R}$. Thus, $\text{con}_{t_0}(f) \subset \mathbb{R}$. This in turn demands that

$$\text{jump}_{t_0}(f) = \{x - y| \quad x, y \in \text{con}_{t_0}(f)\} \subset \{x - y| \quad x, y \in \mathbb{R}\} = \mathbb{R}.$$

By proposition (3.49), we have that

$$\text{axis}_{t_0}(f) = \text{phase}_{t_0}(f) = \{\frac{x}{|x|}| \quad x \in \text{jump}_{t_0}(f)\backslash\{0\}\} \subset$$

$$\{\frac{x}{|x|}| \quad x \in \mathbb{R}\backslash\{0\}\} = \{1, -1\}.$$

By proposition 3.33, the discontinuity of f demands that $\text{con}_{t_0}(f)$ has at least two distinct elements, $x, y \in \mathbb{R}\backslash\{0\}$. We see that $x-y \neq 0$ and $y-x \neq x-y$ so $\text{jump}_{t_0}(f)$ has at least two elements. Without loss of generality, we assume that $x - y$ is positive. This demands that $y - x$ is negative. We see that

$$\{-1, 1\} = \left\{\frac{x-y}{|x-y|}, \frac{y-x}{|y-x|}\right\} \subset \text{phase}_{t_0}(f) = \text{axis}_{t_0}(f).$$

□

Lemma 3.57. *Let f be a real valued function that is locally bounded at t_0. Let g be a function that is continuous at t_0 and $g(t_0) = 0$. Then, $con_{t_0}(gf) = \{0\}$.*

Proof. There exists a $M > 0$ and $\delta > 0$ such that $|f((t_0 - \delta, t_0 + \delta))| < M$. Let $\{t_n\}_{n=1}^{\infty}$ converge to t_0. Then,

$$\lim_{n\to\infty} |f(t_n)g(t_n)| \leqslant M \lim_{n\to\infty} |g(t_0)| = 0.$$

This demands that for all sequences $\{t_n\}_{n=1}^{\infty}$ convergent to t_0,

$$\lim_{n\to\infty} f(t_n) = 0.$$

We see that $\text{con}_{t_0}(gf) = \{0\}$. □

Proposition 3.58. *Let f be a real valued function that is locally bounded at t_0. If $e^{i\alpha t}f(t)$ has a discontinuity at t_0, then*

$$axis_{t_0}(f) = \{e^{i\alpha t_0}, -e^{i\alpha t_0}\}.$$

Proof. Since $e^{i\alpha t} - e^{i\alpha t_0}$ is continuous and equal to 0 at t_0, proposition (3.57) demands that

$$\mathrm{con}_{t_0}(e^{i\alpha t}f(t)) = \mathrm{con}_{t_0}((e^{i\alpha t} - e^{i\alpha t_0})f(t) + e^{i\alpha t_0}f(t)) =$$

$$\mathrm{con}_{t_0}(e^{i\alpha t_0}f(t)) = e^{i\alpha t_0}\mathrm{con}_{t_0}(f).$$

□

4 Discontinuity and HRT

4.1 Real HRT Conjecture

For this section, we will exclusively deal with real valued functions. We state the real version of the function formulation of the HRT in the following.

Definition 4.1. *Let f be a real valued function. We say that f satisfies the real HRT conjecture if for all $\{(\alpha_k, \beta_k)\}_{k=1}^n \subset \mathbb{R}^2$,*

$$\{e^{i\alpha_k t} f(t - \beta_k)\}_{k=1}^n$$

is a linearly independent set over the real numbers.

We assume without loss of generality that β_k are increasing, although not perhaps strictly increasing.

Proposition 4.2. *Let f be a real valued function. f satisfies the real HRT conjecture if and only if f is such that for all $\{(\alpha_k, \beta_k)\}_{k=1}^n \subset \mathbb{R}^2$, for all $\{c_k\}_{k=1}^n \subset \mathbb{R}^*$,*

$$\sum_{k=1}^n c_k e^{i\alpha_k t} f(t - \beta_k) \neq 0.$$

Proof. Let f satisfy the real HRT conjecture. Let $\{(\alpha_k, \beta_k)\}_{k=1}^n \subset \mathbb{R}^2$ and $\{c_k\}_{k=1}^n \subset \mathbb{R}^*$. From the linear independence of $\{e^{i\alpha_k t} f(t - \beta_k)\}_{k=1}^n$, we have that

$$\sum_{k=1}^n c_k e^{i\alpha_k t} f(t - \beta_k) \neq 0.$$

Let f not satisfy the real HRT conjecture. Then there exists $\{(\alpha_k, \beta_k)\}_{k=1}^n \subset \mathbb{R}^2$ such that $\{e^{i\alpha_k t} f(t - \beta_k)\}_{k=1}^n$ is linearly independent. This means that there exists $\{c_k\}_{k=1}^n \subset \mathbb{R}$ where not all elements equal zero such that

$$\sum_{k=1}^n c_k e^{i\alpha_k t} f(t - \beta_k) = 0.$$

Let $\{k_j\}_{j=1}^m$ be all indices such that $c_{k_j} \neq 0$. We have that this is a nonempty set. Thus,

$$\sum_{j=1}^{m} c_{k_j} e^{i\alpha_{k_j} t} f(t-\beta_{k_j}) = 0$$

where each coefficient is nonzero. □

Definition 4.3. *Let* $\Xi = \{(c_k, \alpha_k, \beta_k)\}_{k=1}^n \subset \mathbb{R} \times \mathbb{R}^2$ *where each* (α_k, β_k) *is distinct,* $\{c_k\}_{k=1}^n \subset \mathbb{R}^*$ *and* Ξ *is nonempty. We call such a* Ξ *a real HRT configuration.*

Proposition 4.4. *Let* f *be a real valued function.* f *satisfies the real HRT conjecture if and only if for all real* Ξ*-configurations,* f *satisfies the* Ξ*-configuration.*

Proof. Let f be a real valued function. Suppose f satisfies the real HRT conjecture. Let $\Xi = \{(c_k, \alpha_k, \beta_k)\}_{k=1}^n$ be a real HRT configuration. By proposition 4.2, we have that

$$\sum_{k=1}^{n} c_k e^{i\alpha_k} f(t-\beta_k) \neq 0.$$

Thus, f satisfies the Ξ-configuration.

Let f not satisfy the real HRT conjecture. By proposition 4.2, there exists $\{(\alpha_k, \beta_k)\}_{k=1}^n \subset \mathbb{R}^2$ and $\{c_k\}_{k=1}^n \subset \mathbb{R}^*$ such that

$$\sum_{k=1}^{n} c_k e^{i\alpha_k t} f(t-\beta_k) = 0.$$

We see that $\Xi = \{(c_k, \alpha_k, \beta_k)\}_{k=1}^n$ is a real HRT configuration. Thus, f does not satisfy the Ξ-configuration. □

Definition 4.5. *Let* $\Lambda = \{(\alpha_k, \beta_k)\}_{k=1}^n \subset \mathbb{R}^2$ *be an HRT constellation and* $\Xi = \{(c_k, \alpha_k, \beta_k)\}_{k=1}^n$ *be a real HRT configuration. We call* Ξ *a real* Λ*-set.*

4.2 Propagation of Discontinuities

Definition 4.6. *Let* $\Lambda = \{(\alpha_k, \beta_k)\}_{k=1}^n \subset \mathbb{R}^2$ *be an HRT constellation. Define*

$$D_\Lambda = \left\{ \sum_{l>k=0}^{n} \frac{\pi m_{l,k}}{\alpha_l - \alpha_k} + \sum_{k=1}^{n} m'_k \beta_k \,\middle|\, \quad m_{l,k}, m'_k \in \mathbb{Z} \right\}^C.$$

We call this set the Λ-domain.

Proposition 4.7. *Let $\Lambda = \{(\alpha_k, \beta_k)\}_{k=1}^n \subset \mathbb{R}^2$ be an HRT constellation. D_Λ is closed under any β_k-shift.*

Proof. Fix $1 \leqslant k_0 \leqslant n$. We see that

$$(D_\Lambda + \beta_{k_0})^C = (D_\Lambda)^C + \beta_{k_0} =$$

$$\{\sum_{l>k=0}^{n} \frac{\pi m_{l,k}}{\alpha_l - \alpha_k} + \sum_{k=1}^{n} m'_k \beta_k | \quad m_{l,k}, m'_k \in \mathbb{Z}\} + \beta_{k_0} =$$

$$\{\sum_{l>k=0}^{n} \frac{\pi m_{l,k}}{\alpha_l - \alpha_k} + (m'_{k_0} + 1)\beta_{k_0} + \sum_{k=1, k\neq k_0}^{n} m'_k \beta_k | \quad m_{l,k}, m'_k \in \mathbb{Z}\} = D_\Lambda^C.$$

Thus, $D_\Lambda + \beta_{k_0} = D_\Lambda$. □

Proposition 4.8. *Let $\Lambda = \{(\alpha_k, \beta_k)\}_{k=1}^n \subset \mathbb{R}^2$ be an HRT constellation. Then,*

$$e^{i\alpha_k t} \notin \{e^{i\alpha_l t}, -e^{i\alpha_l t}\} \quad \textit{for all } k \neq l, \textit{ for all } t \in D_\Lambda.$$

Proof. Suppose that there exists $l \neq k$ and $t \in \mathbb{R}$ such that

$$e^{i\alpha_k t} \notin \{e^{i\alpha_l t}, -e^{i\alpha_l t}\}.$$

Then,

$$e^{i(\alpha_k - \alpha_k)t} = \pm 1.$$

This demands that there exists $n \in \mathbb{Z}$ such that $(\alpha_k - \alpha_k)t = n\pi$. From this we see that

$$t = \frac{\pi n}{\alpha_k - \alpha_k} \notin D_\Lambda.$$

Thus, if $t \in D_\Lambda$, then

$$e^{i\alpha_k t} \notin \{e^{i\alpha_l t}, -e^{i\alpha_l t}\}.$$

□

Lemma 4.9. *Let $\Lambda = \{(\alpha_k, \beta_k)\}_{k=1}^n$ be an HRT constellation where each α_k and each β_k is unique. Let $\Xi = \{(c_k, \alpha_k, \beta_k)\}_{k=1}^n$ be a real Λ-set. Let f be a real valued function that does not satisfy the Ξ-configuration and have a discontinuity at $t_0 \in D_\Lambda$. Then, for all $1 \leqslant k \leqslant n$, f has at least two other discontinuities on the set $\{t_0 - \beta_k + \beta_l\}_{l=0,l\neq k}^n$.*

Proof. Fix $1 \leqslant k \leqslant n$ and $t_0 \in \mathbb{R}$. Consider $t_k = t_0 + \beta_{k_0}$. Since f does not satisfy the Ξ-configuration, we have that

$$\sum_{l=1}^{n} c_l e^{i\alpha_l t} f(t - \beta_l) = 0 \quad \text{for all } t \in \mathbb{R}.$$

We rewrite the above as

$$f(t - \beta_k) = \sum_{l=1, l\neq k}^{n} \frac{c_l}{c_k} e^{i(\alpha_l - \alpha_k)t} f(t - \beta_l). \tag{18}$$

We note that

$$\text{axis}_{t_0}(f(t - \beta_{k_0})) = \text{axis}(\sum_{l=1, l\neq k}^{n} \frac{c_l}{c_k} e^{i(\alpha_l - \alpha_k)t} f(t - \beta_l)). \tag{19}$$

The discontinuity at t_{k_0} on the left hand side of (18) demands a discontinuity at t_{k_0} on the right hand side. Since the linear combination of functions continuous at the point t_{k_0} is itself continuous at t_{k_0}, at least one of the functions $e^{i\alpha_l t} f(t - \beta_l)$ is discontinuous at t_{k_0}.

Suppose that only one such function in the sum on the right hand side of (18) is discontinuous at t_k. Let its index be l_0. We rewrite (18) with this function separated from the sum in the following.

$$f(t - \beta_k) = \frac{c_{l_0}}{c_k} e^{i(\alpha_{l_0} - \alpha_k)t} f(t - \beta_{l_0}) + \sum_{l\neq k, l_0}^{n} \frac{c_l}{c_k} e^{i(\alpha_l - \alpha_k)t} f(t - \beta_l). \tag{20}$$

Consider the axis of discrepancy at t_k for both sides of this function. Because the left hand side is fully real, we have by proposition 3.56 that

$$\text{axis}_{t_0}(f(t-\beta_k)) = \{1,-1\}.$$

On the right-hand side of (20), we have the sum of a function discontinuous at t_k that is the modulation of a real-valued function and the sum of functions continuous at t_k. By propositions 3.54 and 3.58, we have that

$$\text{axis}_{t_0}\left(\frac{c_{l_0}}{c_k}e^{i(\alpha_{l_0}-\alpha_{k_0})t}f(t-\beta_{l_0}) + \sum_{l\neq k,l_0}^{n}\frac{c_l}{c_k}e^{i(\alpha_l-\alpha_{k_0})t}f(t-\beta_l)\right) =$$

$$\text{axis}_{t_0}\left(\frac{c_{l_0}}{c_k}e^{i(\alpha_{l_0}-\alpha_{k_0})t}f(t-\beta_{l_0})\right) = e^{i(\alpha_{l_0}-\alpha_{k_0})t_{k_0}}\{1,-1\} =$$

$$\{e^{i(\alpha_{l_0}-\alpha_{k_0})t_{k_0}}, -e^{i(\alpha_{l_0}-\alpha_{k_0})t_{k_0}}\}.$$

Since $t_k \in D_\Lambda$, proposition (4.8) demands that

$$\text{axis}_{t_0}(f) \cap \text{axis}_{t_0}(\frac{c_{l_0}}{c_k}e^{i(\alpha_{l_0}-\alpha_k)t}f(t-\beta_{l_0}) + \sum_{l\neq k,l_0}^{n}\frac{c_l}{c_k}e^{i(\alpha_l-\alpha_k)t}f(t-\beta_l)) =$$

$$\{1,-1\} \cap \{e^{i(\alpha_{l_0}-\alpha_{k_0})t_{k_0}}, -e^{i(\alpha_{l_0}-\alpha_{k_0})t_{k_0}}\} = \varnothing.$$

This means that

$$\text{axis}_{t_0}(f) \neq \text{axis}_{t_0}(\frac{c_{l_0}}{c_k}e^{i(\alpha_{l_0}-\alpha_k)t}f(t-\beta_{l_0}) + \sum_{l\neq k,l_0}^{n}\frac{c_l}{c_k}e^{i(\alpha_l-\alpha_k)t}f(t-\beta_l)).$$

This contradicts (19). We reject our assumption that there is only one function discontinuous at t_{k_0} in the linear combination in (18). Let the distinct indices l_1, l_2 correspond to those two functions. Since $e^{i\alpha_k}$ is continuous and never zero, we have that $f(t-\beta_{k_{l_1}})$ and $f(t-\beta_{k_{l_2}})$ are discontinuous at $t_k = t_0 + \beta_k$. We conclude that f is discontinuous at $t_0 + \beta_k - \beta_{l_1}$ and $t_0 + \beta_k - \beta_{l_2}$. □

Corollary 4.10. *Let $\Lambda = \{(\alpha_k, \beta_k)\}_{k=1}^n$ be an HRT constellation where each α_k is unique and each β_k is unique. Let Ξ be a real Λ-set. Let f not satisfy* *the Ξ-configuration and have a discontinuity at* $t_0 \in D_\Lambda$. *Then,* f *has at least two discontinuities in each of the intervals* $[t_0 - \beta_n + \beta_1, t_0 - \beta_n + \beta_{n-1}]$ *and* $[t_0 + \beta_2 - \beta_1, t_0 + \beta_n - \beta_1]$.

Proof. By lemma 4.9, f has at least two discontinuities on the set $D^- = \{t_0 - \beta_n + \beta_l\}_{l=0,l\neq k}^n$. Since β_k is strictly increasing $-\beta_n + \beta_1$ and $-\beta_n + \beta_{n-1}$ are the least and greatest elements of D^- so $D^- \subseteq [-\beta_n + \beta_1, -\beta_n + \beta_{n-1}]$. Thus, f is discontinuous at at least two points of $[-\beta_n + \beta_1, -\beta_n + \beta_{n-1}]$.

By lemma 4.9, f has at least two discontinuities on the set $D^+ = \{t_0 - \beta_1 + \beta_l\}_{l=0,l\neq k}^n$. Since β_k is strictly increasing $-\beta_1 + \beta_2$ and $-\beta_1 + \beta_n$ are the least and greatest elements of D^+ so $D^+ \subseteq [\beta_2 - \beta_1, \beta_n - \beta_1$. Thus, f is discontinuous at at least two points of $[\beta_2 - \beta_1, \beta_n - \beta_1$. □

Theorem 4.11. *Let* $\Lambda = \{(\alpha_k, \beta_k)\}_{k=1}^n$ *be an HRT constellation where each* α_k *and each* β_k *is unique. Let* Ξ *be a real* Λ*-set. Let* f *not satisfy the* Ξ*-configuration and have a discontinuity at* $t_0 \in D_\Lambda$. *Then* f *has a discontinuity in every interval of length* $\beta_n - \beta_0$.

Proof. We proceed by contradiction. Suppose that there exists an interval $[a, b]$ with $b - a > \beta_n - \beta_0$.

Suppose that $t_0 < a$. Let D^- be the set of all points on which f is discontinuous and that are less than a. Since $t_0 < a$, D^- is nonempty and has a supremum. Let $t_1 \in D^-$ be such that

$$|t_1 - \sup(D^-)| < \min\left(\frac{\beta_2 - \beta_1}{2}, \frac{(b-a) - (\beta_n - \beta_1)}{2}\right).$$

We have by corollary 4.10 that f has at least one discontinuity on the interval $[t_0 + \beta_2 - \beta_1, t_0 + \beta_n - \beta_1]$. Call it t_2. t_2 cannot be smaller than a, since then it would belong to D^- while at the same time being greater than the supremum of D^- as demonstrated by

$$t_2 \geqslant t_1 + \beta_2 - \beta_1 > \sup(D^-) - \frac{\beta_2 - \beta_1}{2} + \beta_2 - \beta_1 =$$

$$\sup(D^-) + \frac{\beta_2 - \beta_1}{2} > \sup(D^-).$$

So $t_2 \geqslant a$. t_2 cannot be greater than b since

$$t_2 < t_1 + (\beta_n - \beta_1) < t_1 + (b - a) < a + (b - a) = b.$$

So $a \geqslant t_2 \geqslant b$ which implies that there exists a discontinuity within the interval $[a, b]$ violating our assumption that there exists an interval longer than β on which f has no discontinuities. Applying the identical argument to $f(-t)$, we can show that having a discontinuity larger than the interval b demands the existence of a discontinuity on the interval $[a, b]$. Thus, for every interval longer than $\beta_n - \beta_1$, f has a discontinuity.

□

Definition 4.12. *Let f be a complex valued function and $t_0 \in \mathbb{R}$. We denote*

We define the (f, t_0)-lattice to be

$$L_{t_0}^f = \{(m_1, ..., m_n) \in \mathbb{Z}^n | \quad f(\textstyle\sum_{k=1}^n \beta_k m_k) \text{ is discontinuous}\}.$$

We define the (f, t_0)-mesh to be

$$M_{t_0}^f = \{t_0 + \sum_{k=1}^n m_k \beta_k | \quad (m_1, ..., m_n) \in L_{t_0}(f)\}.$$

Corollary 4.13. *Let $\Lambda = \{(\alpha_k, \beta_k)\}_{k=1}^n$ be an HRT constellation where each α_k and each β_k is unique. Let Ξ be a real Λ-set. Let f not satisfy the Ξ-configuration and have a discontinuity at $t_0 \in D_\Lambda$. If $m = (m_1, ..., m_n) \in L_{t_0}^f$ then $L_{t_0}^f$ also contains at least two elements of*

$$L_m^j = \{m - e_j + e_l\}_{l=0, l \neq j}^n = \{(m_1, ..., m_j - 1, ..., m_k + 1, ..., m_n)\}_{l=0, l \neq j}^n$$

for all $1 \leqslant j \leqslant m$.

Proof. Let $m = (m_1, ..., m_n) \in L_{t_0}^f$. Then, there f is discontinuous on $x_0 =$

$t_0 + \sum_{k=1}^{n} m_k\beta_k \in M_{t_0}^f$. By Lemma 4.9 the discontinuity of f at x_0 implies that for all $1 \leqslant j \leqslant n$ there exists two other discontinuities on the set

$$\{x_0 - \beta_j + \beta_l = t_0 + \sum_{k=1}^{n} m_k\beta_k - \beta_j + \beta_l\}_{l=0,l\neq j}^{n}.$$

This implies that for all $1 \leqslant j \leqslant n$, $L_{t_0}^f$ contains at least two elements of the set

$$L_m^j\{m + e_j - e_l\}_{l=0,l\neq j}^{n} = \{(m_1, ..., m_j - 1, ..., m_k + 1, ..., m_n)\}_{l=0,l\neq j}^{n}.$$

□

Theorem 4.14. *Let $\Lambda = \{(\alpha_k, \beta_k)\}_{k=1}^{n}$ be an HRT constellation where each α_k is unique and $\{\beta_k\}_{k=1}^{n}$ is rationally independent. Let Ξ be a real Λ-set. Let f not satisfy the Ξ-configuration and have a discontinuity at $t_0 \in D_\Lambda$. Then, $f(t)$ has an infinite number of discontinuities in every interval of length greater than or equal to $\beta_n - \beta_1$.*

Proof. We proceed by contradiction. Fix the interval $[a, b]$ with length larger than $\beta_n - \beta_1$. We know from theorem 4.11 that there exists at least one discontinuity in $[a, b]$ of the form

$$t_0 + \sum_{k=1}^{n} m_k\beta_k, \quad \text{where } m_k \in \mathbb{Z}. \tag{21}$$

Assume that in the interval $[a, b]$ there are a finite number, N, of discontinuities of this form. Define the function g on this set of discontinuities in $[a, b]$ of this form by

$$g(t_0 + \sum_{k=1}^{n} m_k\beta_k) = -m_1 + \sum_{k=2}^{n} m_k.$$

g is well defined since $\{\beta_k\}_{k=1}^{n}$ is rationally independent, making every representation above unique.

Since we have assumed that the set of discontinuities on $[a, b]$ of the form

(21) is finite, g has a maximum, M_0, and maximizer, t_1. By corollary 4.10, there exists distinct $1 < l_1, l_2 \leqslant n$ such that f is discontinuous on $t_1 + \beta_{l_1}$ and $t_1 + \beta_{l_2}$ since l_1 and l are distinct, we know one of these will not be β_n. Without loss of generality, assume it is β_{l_1}. We see that

$$g(t_1 + \beta_{l_1} - \beta_1) = g\left(t_0 + \sum_{k=1}^{n} m_k\beta_k + \beta_{l_1} - \beta_1\right) =$$

$$g\left(t_0 + (m_1 - 1)\beta_1 + (m_l + 1)\beta_{l_1} + \sum_{k=2, k\neq l_1}^{n} m_k\beta_k\right) =$$

$$-(m_1 - 1) + (m_l + 1) + \sum_{k=2, k\neq l}^{n} m_k =$$

$$-m_1 + \sum_{k=2}^{n} m_k + 2 = M_0 + 2 > M_0.$$

This implies that $t_1 + \beta_l$ does not belong to $[a, b]$ since $g(t_1 + \beta_l) > g(t_1)$ which is the max of g in $[a, b]$. So $b < s_0 + \beta_l$.

By corollary 4.10, for any discontinuity of the form (21) there exists discontinuities of the form

$$t_0 + \sum_{k=1}^{n} m_k\beta_k - \beta_n + \beta_l, \quad \text{where } m_k \in \mathbb{Z}$$

for at least two distinct l. Since they are distinct, at least one of them does not equal 1. Inductively define s_k by

$$s_k = \begin{cases} t_1, & k = 0 \\ s_{k-1} - \beta_n + \beta_{l_k}, & k > 0. \end{cases}$$

where β_{l_k} is one of the discontinuities that does not equal β_1 and must exist by the above. We see that

$$g(s_0) = g(t_1) = M_0 + 2$$

and

$$g(s_{k+1}) = g(s_k - \beta_n + \beta_{l_k}) = g(s_k) - 1 + 1 = g(s_k).$$

Taking the above equalities together, we have that

$$g(s_k) = M_0 + 2.$$

By definition of s_k, we have that

$$s_k = s_{k-1} - \beta_n + \beta_{l_k} \leqslant s_{k-1} - \beta_n + \beta_{n-1}.$$

By induction, we have that

$$s_k \leqslant s_0 - k(\beta_n - \beta_{n-1}).$$

There exists k_0 such that $s_{k_0} \leqslant s_0 - k_0(\beta_n - \beta_{n-1}) \leqslant b$. Since s_k is a strictly decreasing sequence of isolated points, there exists a greatest k, called k_1, such that $s_{k_1} \leqslant b$. This demands that $s_{k_1-1} > b$. Given that $b - a \leqslant \beta_n - \beta_0$, we have that

$$s_{k_1} = s_{k_1-1} - \beta_n + \beta_{l_k} \geqslant s_{k_1-1} - \beta_n + \beta_2 > s_{k_1-1} - \beta_n + \beta_1 >$$

$$b - \beta_n + \beta_1 = b - (b - a) = a.$$

So, $a < s_{k_1} \geqslant b$ and $s_{k_1} \in [a, b]$. So s_{k_1} is in $[a, b]$, of the form in (21) and $g(s_{k_1}) = M_0 + 2$ which is larger than M_0, the maximum of g. This is a contradiction so, we reject our assumption that there are only finitely many discontinuities of the form (21) in the interval $[a, b]$. Thus, f has an infinite number of discontinuities in any interval of length larger than $\beta_n - \beta_0$.

□

Theorem 4.15. *Let $\Lambda = \{(\alpha_k, \beta_k)\}_{k=1}^n$ be an HRT constellation where each α_k is unique and $\{\beta_k\}_{k=1}^n$ is rationally independent. Let f be a real valued function* *that has a discontinuity on D_Λ. If there is an interval, $[a, b]$, of length larger than $\beta_n - \beta_1$ such that f is discontinuous only on a finite number of times on that interval, then (f, Λ) satisfies the HRT conjecture.*

Proof. Let f be as above. Suppose that (f, Λ) did not satisfy the real HRT conjecture. By proposition (4.4), there exists a real Ξ-configuration that is a real Λ-set that f does not satisfy. By theorem 4.14, no such interval of length $\beta_n - \beta_1$ with finite discontinuities exists. By contraposition, (f, Λ) satisfies the HRT conjecture. □

5 HRT on Real Functions

Lemma 5.1. *Let $a, b \in \mathbb{R}$ be nonzero and such that for all $q \in \mathbb{Q}$, $\frac{a}{b} \neq \mathbb{Q}$. The set*

$$A_0 = \{na + mb| \quad n \in \mathbb{Z}/\{0\}, m \in \mathbb{N}\}$$

is dense in $\mathbb{R}$.

Proof. Define $\alpha = \frac{b}{a}$. Clearly, the density of A_0 is equivalent to a nonzero multiple of A_0,

$$A = \left\{n + m\alpha \middle| \quad n \in \mathbb{Z}/\{0\}, m \in \mathbb{N}\right\}.$$

Note, that if A is dense then so is the set in question. We have that α and $m\alpha \notin \mathbb{N}$ for all $m \neq 0$. So, if $n_1 + m_1\alpha = n_2 + m_2\alpha$, then $(m_1 - m_2)\alpha = n_1 - n_2$ which is only possible when $m_1 - m_2 = 0$ and $n_1 - n_2 = 0$. Thus every $n + m\alpha$ is a distinct number.

Fix $\epsilon > 0$. For all $m \in \mathbb{N}$, there exists at least one $n_m \in \mathbb{Z}\backslash\{0\}$ such that $n_m + m\alpha \in [0, 2]$. Since the m is distinct in every $n_m + m\alpha$, each $n_m + m\alpha$ is distinct. Since there is an infinite number of distinct $n_m + m\alpha \in [0, 2]$, there exists $m_1, m_2 \in \mathbb{N}$ such that $|(n_{m_1} + m_1\alpha) - (n_{m_2} + m_2\alpha)| < \epsilon$. By renaming, let $m_1 > m_2$. Define $n_0 = n_{m_1} - n_{m_2}$ and $m_0 = m_1 - m_2$. We see that $|n_0 - m_0\alpha| = |(n_{m_1} + m_1\alpha) - (n_{m_2} + m_2\alpha)| < \epsilon$ and since $m_0 = m_1 - m_2 > 0$, $n_0 + m_0\alpha \in A$.

Let $x \in \mathbb{R}$. If $n_0 + \alpha m_0 < 0$, define

$$A_x^\epsilon = \{\lceil x\rceil + k(n_0 + \alpha m_0) = (\lceil x\rceil + kn_0) + km_0\alpha\}_{k=1, k\neq \frac{-\lceil x\rceil}{n_0}}^{\infty}.$$

Clearly, $(\lceil x\rceil + kn_0) \in \mathbb{Z}$ and $km_0 \in \mathbb{N}$, so $A_x^\epsilon \subset A$. Each element of A_x^ϵ is equally spaced by $(n_0 + \alpha m_0) < \epsilon$, except for $-\frac{\lceil x\rceil}{n_0}m_0$ so every element in $(-\infty, \lceil x\rceil]$ is less than 2ϵ away from an element of $A_x^\epsilon \subset A$. Thus A is dense in $\mathbb{R}$. [Fix above to be below] Since $k \neq \frac{-\lceil x\rceil}{n_0}$, $(\lceil x\rceil + kn_0) \in \mathbb{Z}\backslash\{0\}$. Since $k, m_0 > 0$, $km_0 \in \mathbb{N}$. This shows us that if $a \in A_x^\epsilon$, then $a \in A$. Each element of A_x^ϵ is equally spaced by $(n_0 + \alpha m_0) < \epsilon$, except for $-\frac{[x]}{n_0}m_0$ so every element in $[\lfloor x \rfloor, \infty]$ is less than 2ϵ away from an element of $A_x^\epsilon \subset A$. Since this is true of all $\epsilon > 0$, A is dense in $\mathbb{R}$.

If $n_0 + \alpha m_0 > 0$, define

$$A_x^\epsilon = \{\lfloor x \rfloor + k(n_0 + \alpha m_0) = (\lfloor x \rfloor + kn_0) + km_0\alpha\}_{k=1, k \neq \frac{-\lfloor x \rfloor}{n_0}}^{\infty}.$$

Since $k \neq \frac{-\lfloor x \rfloor}{n_0}$, $(\lfloor x \rfloor + kn_0) \in \mathbb{Z}\backslash\{0\}$. Since $k, m_0 > 0$, $km_0 \in \mathbb{N}$. This shows us that if $a \in A_x^\epsilon$, then $a \in A$. Each element of A_x^ϵ is equally spaced by $(n_0 + \alpha m_0) < \epsilon$, except for $-\frac{[x]}{n_0}m_0$ so every element in $[\lfloor x \rfloor, \infty]$ is less than 2ϵ away from an element of $A_x^\epsilon \subset A$. Since this is true of all $\epsilon > 0$, A is dense in $\mathbb{R}$.

Since A is dense in $\mathbb{R}$ in both cases, we have the statement. □

Theorem 5.2. *Let f be a real valued continuous function. Let $\{\alpha_k\}_{k=3}^4, \{\beta_k\}_{k=1}^4 \subset \mathbb{R}$ where $\alpha_1 = \alpha_2$ and $\frac{\alpha_3}{\alpha_4} \notin \mathbb{Q}$ and $\frac{\alpha_3\beta_4}{\pi} \notin \mathbb{Q}$. Let $c_1, c_2, c_3, c_4 \in \mathbb{R}/\{0\}$. If*

$$c_1 f(t) + c_2 f(t - \beta_2) + c_3 e^{i\alpha_3 t} f(t - \beta_3) + c_4 e^{i\alpha_4 t} f(t - \beta_4) = 0, \tag{22}$$

then $f = 0$.

Proof. We will use an inductive argument to show that

$$f(t) = 0 \quad \text{for all } t \in \{\tfrac{2\pi n}{\alpha_3} + m\beta_4\}_{n \in \mathbb{Z}/\{0\}, m \in \mathbb{N}}$$

and then use the continuity of f to deduce that $f = 0$.

We will begin with the base case, $m = 0$. Note that $c_1 f(t) + c_2 f(t - \beta_2)$ is real for all $t \in \mathbb{R}$. We can rewrite (22) as

$$c_1 f(t) + c_2 f(t - \beta_2) + c_3 e^{i\alpha_3 t} f(t - \beta_3) = -c_4 e^{i\alpha_4 t} f(t - \beta_4). \tag{23}$$

Let $t \in \{\frac{2\pi n}{\alpha_3}\}_{n \in \mathbb{Z}}$. Then, $e^{i\alpha_3 t} f(t - \beta_3) = f(t - \beta_3)$ which is real. So the left hand side of (23) is real for all $t \in \{\frac{2\pi n}{\alpha_3}\}_{n\in\mathbb{Z}}$. Since $\frac{\alpha_4}{\alpha_3} \notin \mathbb{Q}$, $e^{i\alpha_4 t} = \exp(i\alpha_4(\frac{2\pi n}{\alpha_3}))$ clearly has a nonzero imaginary component for all $n \in \mathbb{Z}\backslash\{0\}$. This demands that the right hand side of (23) is not real if $f(t-\beta_4)$ is not zero so $f(t-\beta_4)$ must equal 0. Thus,

$$f(t) = 0 \quad \text{for all } t \in \{\tfrac{2\pi n}{\alpha_3}\}_{n\in\mathbb{Z}/\{0\}}.$$

We now move to the induction step. Fix $m \in \mathbb{N}$. Assume that

$$f(t) = 0 \quad \text{for all } t \in \{\tfrac{2\pi n}{\alpha_3} - m\beta_3\}_{n\in\mathbb{Z}/\{0\}}. \tag{24}$$

Let $t \in \{\frac{2\pi n}{\alpha_3} - (m+1)\beta_3\}_{n\in\mathbb{Z}/\{0\}}$. By (3), we have that

$$f(t-\beta_4) = f\left(\frac{2\pi n}{\alpha_3} + (m+1)\beta_4 - \beta_4\right) = f\left(\frac{2\pi n}{\alpha_3} + m\beta_4\right) = 0.$$

Using this, we can rewrite (23) as

$$c_1 f(t) + c_2 f(t-\beta_2) = -c_3 e^{i\alpha_3 t} f(t) \quad \text{for all } t \in \{\tfrac{2\pi n}{\alpha_3} - (m+1)\beta_3\}_{n\in\mathbb{Z}/\{0\}}. \tag{25}$$

Notice that

$$e^{i\alpha_3 t} = \exp(i\alpha_3\left(\frac{2\pi n}{\alpha_3} + (m+1)\beta_4\right)) = \exp(i(2\pi n + (m+1)\alpha_3\beta_4)).$$

Since we have assumed that $\frac{\alpha_3\beta_4}{\pi} \notin \mathbb{Q}$, we have that $e^{i\alpha_3 t}$ has a nonzero imaginary component.. Since the left hand side of (25) is real, the right hand side must be real and thereby $f(t) = 0$. Thus

$$f(t) = 0 \quad \text{for all } t \in \{\tfrac{2\pi n}{\alpha_3} + (m+1)\beta_4\}_{n\in\mathbb{Z}/\{0\}}.$$

We can now induce that

$$f(t) = 0 \quad \text{for all } t \in \{\tfrac{2\pi n}{\alpha_3} + m\beta_4\}_{n\in\mathbb{Z}/\{0\}, m\in\mathbb{N}}.$$

Since $\frac{\alpha_3\beta_4}{\pi} \notin \mathbb{Q}$, we have that $\frac{2\pi}{\alpha_3}$ and β_4 are relatively irrational. Lemma 5.1 demands that the above set is dense in $\mathbb{R}$. Since f is continuous, we have that $f = 0$. □

Theorem 5.3. *Let $f \in L^2(\mathbb{R})\backslash\{0\}$ be complex valued. Let $1 \leqslant m \leqslant n$, $\{\alpha_k, \beta_k\}_{k=1}^n \subset \mathbb{R}^2$ where $\alpha_k = \alpha_1$ for all $1 \leqslant k \leqslant m-1$, $\alpha_k = \alpha_n + 1$ for all $m \leqslant k \leqslant n$, $\alpha_1 \neq \alpha_m$. There does not exist any $\{c_k\}_{k=1}^n \subset \mathbb{R}\backslash\{0\}$ such that*

$$\sum_{k=1}^{n} c_k e^{i\alpha_k t} f(t-\beta_k) = e^{i\alpha_1 t}\sum_{k=1}^{m-1} c_k f(t-\beta_k) + e^{i\alpha_m t}\sum_{k=m}^{n} c_k f(t-\beta_k) = 0. \quad (26)$$

Proof. Let $f \in L^2(\mathbb{R})$ be real valued and satisfy (26) for some fixed $\{c_n\}_{k=1}^n \in \mathbb{R}/\{0\}$. Dividing both sides of (26) by $e^{i\alpha_1 t}$, we have that

$$\sum_{k=1}^{m-1} c_k f(t-\beta_k) = -e^{i(\alpha_m-\alpha_1)t}\sum_{k=m}^{n} c_k f(t-\beta_k). \quad (27)$$

The left hand side (27) is purely real while the right hand side is non-real for all $t \in \mathbb{R}\backslash\{\frac{\pi}{\alpha_m}\}_{n\in\mathbb{Z}}$ if $\sum_{k=m}^n c_k f(t-\beta_k) \neq 0$. Thus $\sum_{k=m}^n c_k f(t-\beta_k) = 0$ for almost all $t \in \mathbb{R}$. It is well-known that f must be identically zero. □

6 Periodic End behavior

6.1 Alpha Ring

Definition 6.1. *We formally define the points*

$$\infty_\theta, \quad \text{for all } \theta \in \mathbb{R} \text{ and } \quad \infty_\theta = \infty_{\theta+2\pi n} \quad \text{for all } n \in \mathbb{Z}.$$

We call each of these points a periodic infinity and denote the set of all periodic infinities as

$$\Theta = \{\infty_\theta| \quad \theta \in [0, 2\pi)\}.$$

We define intervals on Θ to be

$$(\theta_1, \theta_2)_\infty = \{\infty_\theta| \quad \theta \in (\theta_1, \theta_2)\}.$$

Proposition 6.2. *Let $0 < \delta \leqslant \pi$, $\theta_1 \leqslant \theta_2 \in \mathbb{R}$. Define*

$$a_3 = \max(\theta_1 - \delta_1, \theta_2 - \delta_2), \quad b_3 = \min\Big(\theta_1 + \delta_1, \theta_2 + \delta_2\Big),$$

$$a_4 = \max\Big(\theta_1 + 2\pi - \delta_1, \theta_2 - \delta_2\Big), \quad b_4 = \min\Big(\theta_1 + 2\pi + \delta_1, \theta_2 + \delta_2\Big),$$

$$\theta_3 = \frac{a_3 + b_3}{2}, \quad \delta_3 = \frac{b_3 - a_3}{2} \quad \text{and} \quad \theta_4 = \frac{a_4 + b_4}{2} \quad \delta_4 = \frac{b_4 - a_4}{2}.$$

Then,

$$(\theta_1 - \delta_1, \theta_1 + \delta_1)_\infty \cap (\theta_2 - \delta_2, \theta_2 + \delta_2)_\infty = (\theta_3 - \delta_3, \theta_3 + \delta_3)_\infty \cup (\theta_4 - \delta_4, \theta_4 + \delta_4)_\infty.$$

Proof.

$$(\theta_1 - \delta_1, \theta_1 + \delta_1)_\infty \cap (\theta_2 - \delta_2, \theta_2 + \delta_2)_\infty =$$

$$\Big((\theta_1 - \delta_1, \theta_1 + \delta_1)_\infty \cap (\theta_2 - \delta_2, \theta_2 + \delta_2)_\infty\Big) \cap (\theta_2 - \delta_2, \theta_1 + \delta_1)_\infty \cup$$

$$\Big((\theta_1 - \delta_1, \theta_1 + \delta_1)_\infty \cap (\theta_2 - \delta_2, \theta_2 + \delta_2)_\infty\Big) \cap (2\pi + \theta_1 - \delta_1, \theta_2 + \delta_2)_\infty =$$

$$(a_3, b_3) \cup (a_4, b_4) = (\theta_3 - \delta_3, \theta_3 + \delta_3)_\infty \cup (\theta_4 - \delta_4, \theta_4 + \delta_4)_\infty.$$

□

Definition 6.3. *The α-ring is the set*

$$\mathbb{R}_\alpha = \mathbb{R} \cup \Theta$$

endowed with the topology induced by the basis, $\mathcal{B}_\alpha$, defined below.

Definition 6.4. *Let $\theta \in \mathbb{R}$, $0 < \delta \geqslant \pi$ and $N \in \mathbb{N}$. We define a periodic interval to be*

$$P^{\alpha,\theta}_{\delta,N} = (\theta - \delta, \theta + \delta)_\infty \cup \bigcup_{n=N}^{\infty} \left(\frac{2\pi n + \theta - \delta}{\alpha}, \frac{2\pi n + \theta + \delta}{\alpha} \right).$$

We note that for all periodic intervals, $P^{\alpha,\theta}_{\delta,N}$, $P^{\alpha,\theta}_{\delta,N} \cap \mathbb{R}$ is open. We see this from

$$P^{\alpha,\theta}_{\delta,N} \cap \mathbb{R} = \bigcup_{n=N}^{\infty} \left(\frac{2\pi n + \theta - \delta}{\alpha}, \frac{2\pi n + \theta + \delta}{\alpha} \right)$$

which is a union of open intervals and thereby open.

Proposition 6.5. *Let $\theta_1 \leqslant \theta_2 \in \mathbb{R}$, $0 < \delta_1, \delta_2 \leqslant \pi$ and $N_1, N_2 \in \mathbb{N}$. Define*

$$a_3 = \max\left(\frac{\theta_1 - \delta_1}{\alpha}, \frac{\theta_2 - \delta_2}{\alpha} \right), \quad b_3 = \min\left(\frac{\theta_1 + \delta_1}{\alpha}, \frac{\theta_2 + \delta_2}{\alpha} \right),$$

$$a_4 = \max\left(\frac{\theta_1 + 2\pi - \delta_1}{\alpha}, \frac{\theta_2 - \delta_2}{\alpha} \right), \quad b_4 = \min\left(\frac{\theta_1 + 2\pi + \delta_1}{\alpha}, \frac{\theta_2 + \delta_2}{\alpha} \right),$$

$$\theta_3 = \frac{a_3 + b_3}{2}, \quad \delta_3 = \frac{b_3 - a_3}{2}, \quad \theta_4 = \frac{a_4 + b_4}{2} \quad \delta_4 = \frac{b_4 - a_4}{2} \quad \text{and}$$

$$N = \max(N_1, N_2).$$

Then,

$$P^{\alpha,\theta_1}_{\delta_1,N_1} \cap P^{\alpha,\theta_2}_{\delta_2,N_2} = P^{\alpha,\theta_3}_{\delta_3,N} \cup P^{\alpha,\theta_4}_{\delta_4,N}.$$

Proof. We see that

$$P^{\alpha,\theta_1}_{\delta_1,N_1} \cap P^{\alpha,\theta_2}_{\delta_2,N_2} \cap \mathbb{R} =$$

$$\bigcup_{n=N_1}^{\infty}\left(\frac{2\pi n+\theta_1-\delta_1}{\alpha_1},\frac{2\pi n+\theta_1+\delta_1}{\alpha_1}\right)\cap\bigcup_{n=N_2}^{\infty}\left(\frac{2\pi n+\theta_2-\delta_2}{\alpha_2},\frac{2\pi n+\theta_2+\delta_2}{\alpha_2}\right)=$$

$$\bigcup_{n=N}^{\infty}\left(\frac{2\pi n+\theta_1-\delta_1}{\alpha_1},\frac{2\pi n+\theta_1+\delta_1}{\alpha_1}\right)\cap\left(\frac{2\pi n+\theta_2-\delta_2}{\alpha_2},\frac{2\pi n+\theta_2+\delta_2}{\alpha_2}\right)\cup$$

$$\bigcup_{n=N}^{\infty}\left(\frac{2\pi(n+1)+\theta_1-\delta_1}{\alpha_1},\frac{2\pi(n+1)+\theta_1+\delta_1}{\alpha_1}\right)\cap\left(\frac{2\pi n+\theta_2-\delta_2}{\alpha_2},\frac{2\pi n+\theta_2+\delta_2}{\alpha_2}\right)=$$

$$\bigcup_{n=N}^{\infty}(a_3,b_3)\cup\bigcup_{n=N}^{\infty}(a_4,b_4)=$$

$$(P^{\alpha,\theta_3}_{\delta_3,N}\cup P^{\alpha,\theta_4}_{\delta_4,N})\cap\mathbb{R}.$$

Taking this with proposition 6.2, we have

$$P^{\alpha,\theta_1}_{\delta_1,N_1}\cap P^{\alpha,\theta_2}_{\delta_2,N_2}=(P^{\alpha,\theta_1}_{\delta_1,N_1}\cap P^{\alpha,\theta_2}_{\delta_2,N_2}\cap\mathbb{R})\cup(P^{\alpha,\theta_1}_{\delta_1,N_1}\cap P^{\alpha,\theta_2}_{\delta_2,N_2}\cap\Theta)=$$

$$((P^{\alpha,\theta_3}_{\delta_3,N}\cup P^{\alpha,\theta_4}_{\delta_4,N})\cap\mathbb{R})\cup((P^{\alpha,\theta_3}_{\delta_3,N}\cup P^{\alpha,\theta_4}_{\delta_4,N})\cap\Theta)=P^{\alpha,\theta_3}_{\delta_3,N}\cup P^{\alpha,\theta_4}_{\delta_4,N}.$$

□

Proposition 6.6. *Define the following collection of sets*

$$\mathcal{B}_\alpha=\{P^{\alpha,\theta}_{\delta,N}|\quad\delta>0,N\in\mathbb{N},\theta\in\mathbb{R}\}\cup\{O\subset\mathbb{R}|\quad O\text{ is open}\}.$$

This is a basis on $\mathbb{R}_\alpha$*.*

Proof. If $x\in\mathbb{R}_\alpha$, then either $x\in\mathbb{R}$ or $x\in\Theta$. If $x\in\mathbb{R}$, then since $\mathbb{R}$ is open, $x\in\mathbb{R}\in\mathcal{B}_\alpha$. If $x\in\Theta$, then we can write $x=\infty_\theta$ and $\infty_\theta\in P^{\alpha,\theta}_{1,1}\in\mathcal{B}_\alpha$. Thus, all elements of $\mathbb{R}_\alpha$ belong to some element of $\mathcal{B}_\alpha$.

Let $B_1,B_2\in\mathbb{R}$ and $B_1\cap B_2\neq\varnothing$. Let $x\in B_1\cap B_2$. If $x\in\mathbb{R}$, then $x\in(B_1\cap B_2)\cap\mathbb{R}=(B_1\cap\mathbb{R})\cap(B_2\cap\mathbb{R})\in\mathcal{B}_\alpha$ because all open set of $\mathbb{R}$ belong to $\mathcal{B}_\alpha$. If $x\in\Theta$, we can write it as ∞_θ. Clearly, this demands that both B_1 and B_2 are periodic intervals so we can write $B_1=P^{\alpha,\theta_1}_{\delta_1,N_1}$ and $B_2=P^{\alpha,\theta_2}_{\delta_2,N_2}$. Without loss of generality, we assume that $\theta_1\geqslant\theta_2$. By proposition 6.5, there exists $P^{\alpha,\theta_3}_{\delta_3,N_3}$ and $P^{\alpha,\theta_4}_{\delta_4,N}$ such that $P^{\alpha,\theta_1}_{\delta_1,N_1}\cap P^{\alpha,\theta_2}_{\delta_2,N_2}=P^{\alpha,\theta_3}_{\delta_3,N}\cup P^{\alpha,\theta_4}_{\delta_4,N}$. Thus, x is an element of a periodic interval which is a subset of $B_1 \cap B_2$. We have satisfied both conditions of a basis. $\square$

We endow $\mathbb{R}_\alpha$ with the topology induced by the basis $\mathcal{B}_\alpha$.

Proposition 6.7. *The topology of* $\mathbb{R}$ *and the subspace topology on* $\mathbb{R}$ *induced by* $\mathbb{R}_\alpha$ *are equivalent.*

Proof. Let $O \subset \mathbb{R}$ be open in $\mathbb{R}$. Then, $O \in \mathcal{B}_\alpha$, making $O = O \cap \mathbb{R}$ open in the subspace topology.

Let $O \subset \mathbb{R}_\alpha \cap \mathbb{R}$ be open in the subspace topology induced by $\mathbb{R}_\alpha$. Thus, there exists open $O_1 \subset \mathbb{R}_\alpha$ such that $O_1 = O \cap \mathbb{R}$. Since O_1 is open in $\mathbb{R}_\alpha$, it is the union of some collection $\{B_j\}_{j \in J} \subset \mathcal{B}_\alpha$ where J is an index set. Thus,

$$O = O_1 \cap \mathbb{R} = \left(\bigcap_{j \in J} B_j \right) \cap \mathbb{R} = \bigcap_{j \in J} (B_j \cap \mathbb{R}).$$

Since B_j is either an open set in $\mathbb{R}$ or a periodic set, $B_j \cap \mathbb{R}$ is open. We conclude that O is the union of open sets in $\mathbb{R}$, making it open in $\mathbb{R}$. $\square$

Proposition 6.8. *Let* $\{t_n\}_{n=1}^\infty \subset \mathbb{R} \subset \mathbb{R}_\alpha$ *be a sequence. The following are equivalent:*

1. $\{t_n\}_{n=1}^\infty$ *converges to* ∞_θ.

2. *For all periodic intervals,* $P_{\delta,N}^{\alpha,\theta}$*, that contain* ∞_θ*, there exists* $N \in \mathbb{N}$ *such that*

$$x_n \in P_{\delta,N}^{\alpha,\theta}, \quad \textit{for all } n \geqslant N.$$

3.

$$\lim_{n\to\infty} t_n = \infty \quad \textit{and} \quad \lim_{n\to\infty} e^{i\alpha t_n} = e^{i\theta}.$$

Proof. First we prove (1) $\Leftrightarrow$ (2). It is well-known that a sequence in a topology induced by a basis, $\mathcal{B}$, will converge to some element, x, if and only if for all $B \in \mathcal{B}$ containing x, there exists $N \geqslant 1$, for all $n \geqslant N$ such that $t_n \in B$. $\mathcal{B}_\alpha$ is the basis of the topology of $\mathbb{R}_\alpha$. The only elements of $\mathcal{B}_\alpha$ that contain infinite elements are the periodic intervals. Thus, $\{x_n\}_{n=1}^\infty$ converges to ∞_θ if and only if for every periodic set, $P_{\delta,N}^{\alpha,\theta}$, containing ∞_θ, there exists $N \geqslant 1$ such that for all $n \geqslant N$, $x \in P_{\delta,N}^{\alpha,\theta}$.

Now, we prove (2) $\Rightarrow$ (3). Suppose that $\{t_n\}_{n=1}^\infty$ satisfies (2). Then, for all $M \in \mathbb{N}$, there exists $N \geqslant 1$ such that

$$t_n \in P_{1,M+\lceil|\theta+1/\alpha|\rceil}^{\alpha,\theta} \subset [M,\infty) \quad \text{for all } n \geqslant N.$$

This gives us $\lim_{n\to\infty} t_n = \infty$. We also have that for all $\epsilon > 0$, there exists $N \geqslant 1$ such that

$$t_n \in P_{\epsilon,1}^{\alpha,\theta} \subset [M,\infty) \quad \text{for all } n \geqslant N.$$

This gives us

$$|e^{i\alpha t_n} - e^{i\theta}|^2 = |e^{i(\alpha t_n-\theta)} - 1|^2 = \cos^2(\alpha t_n - \theta) - 2\cos(\alpha t_n - \theta) + 1 + \sin^2(\alpha t_n - \theta) =$$

$$2 - 2\cos(\alpha t_n - \theta) \leqslant 2 - 2(1 - (\alpha t_n - \theta)^2/2) = (\alpha t_n - \theta)^2 < \epsilon^2.$$

Thus, $\lim_{n\to\infty} e^{\alpha t_n} = e^{i\theta}$.

Finally, we prove (3) $\Rightarrow$ (2). Suppose that $\{t_n\}_{n=1}^\infty$ does not satisfy (2). Then, there exists $P_{\delta,M}^{\alpha,\theta}$ containing ∞_θ such that for all $N \geqslant 1$ there exists $n > N$ such that $t \notin P$. We see that there exists a subsequence $\{t_{n_k}\}_{k=1}^\infty$ such that $t_{n_k} \notin P_{\delta,M}^{\alpha,\theta}$. Suppose that $\lim_{n\to\infty} e^{i\alpha t_n} = e^{i\theta}$. Then, there exists K such that

$$t_{n_k} \in \bigcup_{m\in Z} \left(\frac{\theta + 2\pi m - \delta}{\alpha}, \frac{\theta + 2\pi m + \delta}{\alpha}\right), \quad \text{for all } k \geqslant K.$$

Since, $\{t_{n_k}\}_{k=1}^\infty \cap P_{\delta,M}^{\alpha,\theta} = \varnothing$, we have that

$$t_{n_k} \in \bigcup_{n<M} \left(\frac{\theta + 2\pi n - \delta}{\alpha}, \frac{\theta + 2\pi n + \delta}{\alpha}\right), \quad \text{for all } k \geqslant K.$$

Thus, $\{t_n\}_{n=1}^\infty$ does not converge to infinity and does not satisfy (3). $\square$

6.2 Alpha Rings

Definition 6.9. *Let* $\boldsymbol{\alpha} \in \mathbb{R}^n$*. We formally define the* $\boldsymbol{\alpha}$*-ring to be*

$$\Theta^n = \{(\infty_{\theta_1}, \infty_{\theta_2}, ..., \infty_{\theta_n}) | \quad \infty_{\theta_k} \in \Theta \quad \textit{for all } 1 \leqslant k \leqslant n\}.$$

Definition 6.10. *Let* $\boldsymbol{\alpha} \in \mathbb{R}^n$*. Define the* $\boldsymbol{\alpha}$*-periodic reals to be*

$$\mathbb{R}_{\boldsymbol{\alpha}} = \mathbb{R} \cup \Theta^n.$$

We endow this set with topology induced by the basis, $\mathcal{B}_{\boldsymbol{\alpha}}$*, defined below.*

Proposition 6.11. *Fix* $\boldsymbol{\alpha} \in \mathbb{R}^n$*. Define the collection of sets,* $\mathcal{B}_{\boldsymbol{\alpha}}$*, to be all sets,* O*, of the form*

$$O = \left(\mathbb{R} \cap \bigcap_{k=1}^{n} O_k\right) \cup \bigtimes_{k=1}^{n} (O_k \cap \Theta)$$

where $O_k \subset \mathbb{R}_{\alpha_k}$ *is open in* $\mathbb{R}_{\alpha_k}$ *for all* $1 \leqslant k \leqslant n$*.* $\mathcal{B}_{\boldsymbol{\alpha}}$ *is a basis on* $\mathbb{R}_{\boldsymbol{\alpha}}$*.*

Proof. Let $x \in \mathbb{R}_{\boldsymbol{\alpha}}$. By definition, if $x \in \mathbb{R}_{\boldsymbol{\alpha}}$, then $x \in \mathbb{R}$ or $x \in \Theta^n$.

Let $x \in \mathbb{R}$. Let $O \subset \mathbb{R}$ be an open set containing x. By proposition 6.7, we have that the topology $\mathbb{R}$ is a subset of $\mathbb{R}_\alpha$ so O is open in $\mathbb{R}_{\alpha_k}$ for all k making $x \in O = \mathbb{R} \cap \bigcap_{k=1}^n O \in \mathcal{B}_{\boldsymbol{\alpha}} \cup \varnothing \in \mathcal{B}$.

Let $x \in \Theta^n$. We can write $x = (\infty_{\theta_1}, \infty_{\theta_2}, ..., \infty_{\theta_n})$. For all k, there exists an open $O_k \subset \mathbb{R}_{\alpha_k}$ such that $\infty_{\theta_k} \in O_k$. Thus,

$$x \in \bigtimes_{k=1}^{n} (O_k \cap \Theta) \subset \left(\mathbb{R} \cap \bigcap_{k=1}^{n} O_k\right) \cup \bigtimes_{k=1}^{n} (O_k \cap \Theta)$$

which is an element of $\mathcal{B}_{\boldsymbol{\alpha}}$. We have satisfied the first property of a basis.

Let $O^1, O^2 \in \mathcal{B}_{\boldsymbol{\alpha}}$ have nonempty intersection. Let $x \in O^1 \cap O^2$. Since $O^1, O^2 \in \mathcal{B}_{\boldsymbol{\alpha}}$, there exists $\{O_k^1\}_{k=1}^n, \{O_k^2\}_{k=1}^n$ such that

$$O^1 = \left(\mathbb{R} \cap \bigcap_{k=1}^{n} O_k^1\right) \cup \bigtimes_{k=1}^{n} (O_k^1 \cap \Theta) \quad \text{and} \quad O^2 = \left(\mathbb{R} \cap \bigcap_{k=1}^{n} O_k^2\right) \cup \bigtimes_{k=1}^{n} (O_k^2 \cap \Theta).$$

We have that

$$O^1 \cap O^2 = \left(\left(\mathbb{R} \cap \bigcap_{k=1}^{n} O_k^1\right) \cup \bigtimes_{k=1}^{n} (O_k^1 \cap \Theta)\right) \cap \left(\left(\mathbb{R} \cap \bigcap_{k=1}^{n} O_k^2\right) \cup \bigtimes_{k=1}^{n} (O_k^2 \cap \Theta)\right) =$$

$$\left(\left(\mathbb{R} \cap \bigcap_{k=1}^{n} O_k^1\right) \cap \left(\mathbb{R} \cap \bigcap_{k=1}^{n} O_k^2\right)\right) \cup \left(\bigtimes_{k=1}^{n} (O_k^2 \cap \Theta) \cap \bigtimes_{k=1}^{n} (O_k^1 \cap \Theta)\right) =$$

$$\left(\mathbb{R} \cap \bigcap_{k=1}^{n} (O_k^1 \cap O_k^2)\right) \cup \left(\bigtimes_{k=1}^{n} (O_k^1 \cap O_k^2 \cap \Theta)\right) =$$

$$\left(\mathbb{R} \cap \bigcap_{k=1}^{n} O_k^3\right) \cup \left(\bigtimes_{k=1}^{n} (O_k^3 \cap \Theta)\right)$$

where $O_k^3 = O_k^1 \cap O_k^2$ for all k. Clearly, for all $1 \leqslant k \leqslant n$, O_k^3 is open in $\mathbb{R}_{\alpha_k}$ so $O^1 \cap O^2 \in \mathcal{B}_{\boldsymbol{\alpha}}$. If $x \in O^1 \cap O^2$, then $x \in O^1 \cap O^2 \in \mathcal{B}_{\boldsymbol{\alpha}}$. □

We endow $\mathbb{R}_{\boldsymbol{\alpha}}$ with the topology induced by the basis defined in proposition 6.11.

Proposition 6.12. *The subspace topology on* $\mathbb{R} \cap \mathbb{R}_{\boldsymbol{\alpha}}$ *is equivalent to the topology on* $\mathbb{R}$.

Proof. Let O be a subset of $\mathbb{R}$ under the subspace topology of $\mathbb{R}_{\boldsymbol{\alpha}}$. Then there exists open $O^1 \subset \mathbb{R}_{\boldsymbol{\alpha}}$ such that $\mathbb{R} \cap O^1 = O$. By the characteristics of a basis, for all $x \in O^1 \cap \mathbb{R}$ there exists $\{O_k^x\}_{k=1}^n$ such that O_k is open in $\mathbb{R}_{\alpha_k}$ and

$$O^x = \left(\mathbb{R} \cap \bigcap_{k=1}^{n} O_k\right) \cup \bigtimes_{k=1}^{n} (O_k \cap \Theta) \in \mathcal{B}_{\boldsymbol{\alpha}},$$

$x \in O^x \subset O^1$. Clearly, $x \in O^x \cap \mathbb{R}$ and by proposition 6.7, we have that $O^x \cap \mathbb{R}$ is open in $\mathbb{R}_{\alpha_k}$. We see that

$$O^1 = \bigcup_{x \in O^1} \{x\} \subset \bigcup_{x \in O} O^x \subset \bigcup_{x \in O} O = O.$$

Let O be an open subset of $\mathbb{R}$. By proposition 6.7, we have that O is also an open subset of $\mathbb{R}_{\alpha_k}$ for all k. Thus

$$O = \left(\mathbb{R} \cap \bigcap_{k=1}^{n} O\right) \cup \varnothing = \left(\mathbb{R} \cap \bigcap_{k=1}^{n} O\right) \cup \bigtimes_{k=1}^{n} (O \cap \Theta) \in \mathcal{B}_{\boldsymbol{\alpha}}.$$

All basis elements are open in the topology of $\mathbb{R}_{\boldsymbol{\alpha}}$ so O is open. □

Proposition 6.13. *Let $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_d)$, $\boldsymbol{\theta} = (\theta_1, ..., \theta_d) \in \mathbb{R}^d$ and $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R} \subset \mathbb{R}_{\boldsymbol{\alpha}}$. The sequence $\{t_n\}_{n=1}^{\infty}$ converges to $\infty_{\boldsymbol{\theta}}$ in $\mathbb{R}_{\boldsymbol{\alpha}}$ if and only if $\{t_n\}_{n=1}^{\infty}$ converges to ∞_{θ_k} in $\mathbb{R}_{\alpha_k}$ for all $1 \leqslant k \leqslant d$.*

Proof. Let $\boldsymbol{\alpha}$, $\boldsymbol{\theta}$ and $\{t_n\}_{n=1}^{\infty}$ be as above. Fix $1 \leqslant k \leqslant d$. Let $\{t_n\}_{n=1}^{\infty}$ not converge to ∞_{θ_k} in $\mathbb{R}_{\alpha_k}$. Let $O \in \mathcal{B}_{\boldsymbol{\alpha}}$ contain $\infty_{\boldsymbol{\theta}}$. Since O is a basis element, there exists open $O_l \subset \mathbb{R}_{\alpha_k}$ for all $1 \leqslant l \leqslant d$ such that

$$O = \left(\mathbb{R} \cap \bigcap_{l=1}^{n} O_l\right) \cup \bigtimes_{k=1}^{n} (O_k \cap \Theta) \subset O_k.$$

Since $\{t_n\}_{n=1}^{\infty}$ does not converge to ∞_{θ_k} we have that for all $N \geqslant 1$ there exists $n \geqslant N$, such that $t_n \notin O_k$. This demands that for all $N \geqslant 1$, there exists $n \geqslant N$ such that

$$t_n \notin O_k \supset \left(\mathbb{R} \cap \bigcap_{k=1}^{d} O_k\right) \cup \bigtimes_{k=1}^{d} (O_k \cap \Theta).$$

Thus, $\{t_n\}_{n=1}^{\infty}$ does not converge to $\infty_{\boldsymbol{\theta}}$.

Let $\{t_n\}_{n=1}^{\infty}$ converge to ∞_{θ_k} in $\mathbb{R}_{\alpha_k}$ for all $1 \leqslant k \leqslant d$. Let $O \in \mathcal{B}_{\boldsymbol{\alpha}}$ contain $\infty_{\boldsymbol{\theta}}$. Since O is a basis element, there exists open $O_k \subset \mathbb{R}_{\alpha_k}$ such that

$$O = \left(\mathbb{R} \cap \bigcap_{k=1}^{n} O_k\right) \cup \bigtimes_{k=1}^{n} (O_k \cap \Theta) \quad \text{and} \quad \infty_{\theta_k} \in O_k.$$

Since $\{t_n\}_{n=1}^{\infty}$ converges to ∞_{θ_k} for all $1 \leqslant k \leqslant n$, we have that for all O_k, there exists N_k such that for all $n \geqslant N_k$, $t_n \in O$. Let N_k be such that for all $n \geqslant N_k$, $t_n \in O_k$. Let $N = \max(N_1, ..., N_n)$. Thus, for all $n \geqslant N$, $t_n \in O_k$ and thereby

$t_n \in \bigcap_{k=1}^n O_k$. This demands that

$$t_n \in \left(\mathbb{R} \cap \bigcap_{k=1}^{d} O_k\right) \cup \bigtimes_{k=1}^{d} (O_k \cap \Theta) = O \quad \text{for all } n \geqslant N.$$

Since this is true of every basis element, $\{t_n\}_{n=1}^{\infty}$ converges to $\infty_{\boldsymbol{\theta}}$. $\square$

Corollary 6.14. *Let* $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_d), \boldsymbol{\theta} = (\theta_1, ..., \theta_d) \in \mathbb{R}^d$, $1 \leqslant d_0 \leqslant d$ *and* $\{n_k\}_{k=1}^{d_0} \subseteq \{n\}_{n=1}^{d}$. *Define* $\boldsymbol{\alpha}^1 = (\alpha_{n_1}, ..., \alpha_{n_{d_0}})$ *and* $\boldsymbol{\theta}^1 = (\theta_{n_1}, ..., \theta_{n_{d_0}})$. *Let* $\{t_n\}_{n=1}^{\infty}$ *be such that*

$$\lim_{n\to\infty} t_n = \infty_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}.$$

Then

$$\lim_{n\to\infty} t_n = \infty_{\boldsymbol{\theta}^1}^{\boldsymbol{\alpha}^1}.$$

Proof. By proposition 6.13, we have that $\{t_n\}_{n=1}^{\infty}$ converges to $\infty_{\theta_{n_k}}^{\alpha_{n_k}}$ in $\mathbb{R}_{\alpha_k}$ for all $1 \leqslant k \leqslant d_0$. By the same proposition, we have that $\{t_n\}_{n=1}^{\infty}$ converges to $\infty_{\boldsymbol{\theta}^1}^{\boldsymbol{\alpha}^1}$. $\square$

Corollary 6.15. *Let* $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_d), \boldsymbol{\theta} = (\theta_1, ..., \theta_d) \in \mathbb{R}^d$. *Let* $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}$ *such that*

$$\lim_{n\to\infty} t_n = \infty_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}.$$

Then,

$$\lim_{n\to\infty} t_n = \infty \quad \textit{and} \quad \lim_{n\to\infty} e^{i\alpha_k t_n} = e^{i\theta_k} \quad \textit{for all } 1 \leqslant k \leqslant d.$$

Proof. By proposition 6.13, we have that for all for all $1 \leqslant k \leqslant d$,

$$\lim_{n\to\infty} t_n = \infty_{\theta_k} \text{in } \mathbb{R}_{\alpha_k}.$$

By proposition 6.8,

$$\lim_{n\to\infty} t_n = \infty \quad \text{and} \quad \lim_{n\to\infty} e^{i\alpha t_n} = e^{i\theta_k} \quad \text{for all } 1 \leqslant k \leqslant d.$$

$\square$

Definition 6.16. *Let* $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_d) \in \mathbb{R}^d$*. We define*

$$P^{\boldsymbol{\alpha},\boldsymbol{\theta}}_{\delta,N} = \bigcap_{k=1}^{d} P^{\alpha_k,\theta_k}_{\delta,N}.$$

Proposition 6.17. *Let* $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_d) \in \mathbb{R}^d$*. The sequence* $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}$ *converges to* $\infty^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}}$ *if and only if for all* $\delta > 0$ *and* $N \geqslant 1$*, there exists* N_m *sufficiently large that if* $n \geqslant N_m$*, then* $t_n \in P^{\boldsymbol{\alpha},\boldsymbol{\theta}}_{\delta,N}$*.*

Proof. Let $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}$ converge to $\infty^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}}$. Let $\infty^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}} \in P^{\boldsymbol{\alpha},\boldsymbol{\theta}}_{\delta,N} = \bigcap_{k=1}^{d} P^{\alpha_k,\theta_k}_{\delta,N}$. By proposition 6.13, we have that $\{t_n\}_{n=1}^{\infty}$ converges to ∞_{θ_k} in $\mathbb{R}_{\alpha_k}$ for all k. By proposition 6.8, we have that for all k, there exists N_k, for all $n \geqslant N_k$ such that $t_n \in P^{\alpha_k,\theta_k}_{\delta,N}$. Define $N = \max(N_1, ..., N_d)$. Then, for all $n \geqslant N$, $t_n \in \bigcap_{k=1}^{d} P^{\alpha_k,\theta_k}_{\delta,N} = P^{\boldsymbol{\alpha},\boldsymbol{\theta}}_{\delta,N}$.

Let $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}$ be such that for all $\delta > 0$ and $N \geqslant 1$, there exists N_m sufficiently large that if $n \geqslant N_m$, then $t_n \in P^{\boldsymbol{\alpha},\boldsymbol{\theta}}_{\delta,N}$. Fix $1 \leqslant m \leqslant n$. Let $P^{\alpha_m,\theta_m}_{\delta,N}$ contain $\infty^{\alpha_m}_{\theta_m}$. We have that there exists $M \geqslant 1$ such that

$$t_n \in \bigcup_{k=1}^{d} P^{\alpha_k,\theta_k}_{\delta,N} \subset P^{\alpha_m,\theta_m}_{\delta,N} \quad \text{for all } n \geqslant M.$$

Since this is true for all $\delta > 0$ and $N \geqslant 1$, we have that $\lim_{n\to\infty} t_n = \infty_{\theta_m}$ for all $1 \leqslant m \leqslant d$. By proposition 6.13, we have that $\{t_n\}_{n=1}^{\infty}$ converges to $\infty^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}}$. □

6.3 Condensation at Infinity

Definition 6.18. *Let* f *be a complex valued function. Then, the condensation of* f *at infinity, called* $con_{\infty}(f)$*, is the set of all points* $x \in \mathbb{C}^f$ *such that there exists* $\{t_n\}_{n=1}^{\infty}$ *that*

$$\lim_{n\to\infty} t_n = \infty \quad \textit{and} \quad \lim_{n\to\infty} f(t_n) = x.$$

Proposition 6.19. *Let* f *be a complex valued function. Then* $con_{\infty}(f)$ *is nonempty.*

Proof. Let $\{t_n\}_{n=1}^{\infty}$ converge to infinity. By proposition 3.31, we have that the sequence $\{f(t_n)\}_{n=1}^{\infty}$ has a subsequence $\{f(t_{n_k})\}_{k=1}^{\infty}$ convergent to some $x_0 \in \mathbb{C}^f$. Thus,

$$\lim_{k\to\infty} t_{n_k} = \infty \quad \text{and} \quad \lim_{k\to\infty} f(t_{n_k}) = x_0$$

which is to say that $x_0 \in \text{con}_\infty(f)$. □

Definition 6.20. *Let f be a complex valued function. We say that f is bounded at infinity if there exists $M > 0$ such that f is bounded on $[M, \infty)$.*

Proposition 6.21. *Let f be a complex valued function that is bounded at infinity. Then, $con_\infty(f)$ is bounded.*

Proof. Let N be as above and let M be the bound on f for the interval $[N, \infty)$. Thus

$$\text{con}_\infty(f) = \bigcap_{n=1}^{\infty} \overline{f([n,\infty))} = \bigcap_{n=N}^{\infty} \overline{f([n,\infty))} \subseteq \overline{B_M(0)}.$$

Thus $\text{con}_\infty(f)$ is bounded. □

Corollary 6.22. *Let f be a complex valued function bounded at infinity. Then, $con_\infty(f) \cap \mathbb{C} \neq 0$.*

Proof. Let f be as above. By proposition 6.19 $\text{con}_\infty(f)$ is nonempty and by proposition 6.21, $\text{con}_\infty(f) \subset \mathbb{C}$. □

It will be useful to define the following function.

Definition 6.23. *Let f be a complex-valued function. By proposition 6.22, we are assured that $con_\infty(f)$ is nonempty. Let $x_0 \in con_\infty(f)$. Then,*

$$f_{x_0}^*(t) = \begin{cases} f(\frac{1}{|t|}) & t \neq 0 \\ x_0 & t = 0. \end{cases}$$

Lemma 6.24. *Let f be a complex-valued function bounded at infinity. Let $x_0 \in con_\infty(f)$. Then,*

$$con_\infty(f) = con_0(f_{x_0}^*).$$

Proof. Let $x \in \text{con}_\infty(f)$. Then there exists $\{t_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty} t_n = \infty \quad \text{and} \quad \lim_{n\to\infty} f(t_n) = x.$$

We see that

$$\lim_{n\to\infty} \frac{1}{t_n} = 0$$

and

$$\lim_{n\to\infty} f^*_{x_0}\left(\frac{1}{t_n}\right) = \lim_{n\to\infty} f\left(\frac{1}{|1/t_n|}\right) = \lim_{n\to\infty} f(|t_n|) = \lim_{n\to\infty} f(t_n) = x,$$

which is to say that $x \in \text{con}_0(f(\frac{1}{|t|})$.

Now to prove the opposite direction. Let $x \in \text{con}_0(f(1/|t|))$. Then, there exists $\{t_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty} t_n = 0 \quad \text{and} \quad \lim_{n\to\infty} f(\frac{1}{|t_n|}) = x.$$

Define $\{s_n = \frac{1}{|t_n|}\}_{n=1}^\infty$. Then,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{|t_n|} = \infty$$

and

$$\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} f(|1/t_n|) = x.$$

Thus, $x \in \text{con}_\infty(f)$. □

Proposition 6.25. *Let f be a complex valued function and $C \in \mathbb{C}$. Then,*

$$con_\infty(Cf) = C\, con_\infty(f).$$

Proof. Let $x_0 \in \mathrm{con}_\infty(f)$. Define

$$f^*(t) = \begin{cases} f(\frac{1}{|t|}) & t \neq 0 \\ x_0 & t = 0. \end{cases}$$

By lemma 6.24 and proposition 3.35, we have that

$$\mathrm{con}_\infty(Cf) = \mathrm{con}_0(Cf^*) = C\mathrm{con}_0(f^*) = C\mathrm{con}_\infty(f).$$

□

Proposition 6.26. *Let f and g be complex valued functions that are bounded at ∞. Then,*

$$con_\infty(f+g) \subset con_\infty(f) + con_\infty(g).$$

Proof. By proposition 3.47, $\mathrm{con}_\infty(f)$ and $\mathrm{con}_\infty(g)$ are bounded. Let $x_1 \in \mathrm{con}_\infty(f)$. Then there exists $\{t_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty} t_n = \infty \quad \text{and} \quad \lim_{n\to\infty} f(t_n) = x_1.$$

Since $\{g(t_n)\}_{n=1}^\infty$ is a bounded sequence, it has a limit point x_2. Let $\{g(t_{n_k})\}_{k=1}^\infty$ be a subsequence converging to x_2. We have that

$$\lim_{k\to\infty} t_{n_k} = \infty \quad \text{and} \quad \lim_{k\to\infty} g(t_{n_k}) = x_2$$

which gives us $x_2 \in \mathrm{con}_\infty(g)$. We also see that

$$\lim_{k\to\infty} t_{n_k} = \infty \quad \text{and} \quad \lim_{k\to\infty} (f+g)(t_{n_k}) = \lim_{k\to\infty} f(t_{n_k}) + \lim_{k\to\infty} g(t_{n_k}) = x_1 + x_2.$$

So $x_1 + x_2 \in \mathrm{con}_\infty(f+g)$. We are now allowed to define $f^*_{x_1}, g^*_{x_2}$ and $(f+g)^*_{x_1+x_2}$.

We see that

$$\begin{cases} (f+g)^*_{x_1+x_2}(t) = (f+g)\left(\frac{1}{t}\right) = f\left(\frac{1}{t}\right) + g\left(\frac{1}{t}\right) = f^*_{x_1}(t) + g^*_{x_2}(t), & t \neq 0 \\ (f+g)^*_{x_1+x_2}(t) = x_1 + x_2 = f^*_{x_1}(0) + g^*_{x_2}(0) = f^*_{x_1}(t) + g^*_{x_2}(t), & t = 0. \end{cases}$$

Thus, $(f+g)^*_{x_1+x_2} = f^*_{x_1} + g^*_{x_2}$ By lemma 6.27 and proposition 3.48,

$$\text{con}_\infty(f+g) = \text{con}_0((f+g)^*_{x_1+x_2}) = \text{con}_0(f^*_{x_1} + g^*_{x_2}) \subseteq$$

$$\text{con}_0(f^*_{x_1}) + \text{con}_0(g^*_{x_2}) = \text{con}_\infty(f) + \text{con}_\infty(g).$$

$\square$

Lemma 6.27. *Let f be a complex valued function. Then,*

$$con_\infty(f) = \bigcap_{n=0}^{\infty} \overline{f([n,\infty)}.$$

Proof. Let $x \in \text{con}_\infty(f)$. Then, there exists $\{t_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty} t_n = \infty \quad \text{and} \quad \lim_{n\to\infty} f(t_n) = x.$$

So, for all m, there exists N_m sufficiently large that if $n \geqslant N$ then $t_n > m$. Thus, for all $m \in \mathbb{N}$,

$$x \in \overline{\{f(t_n)\}_{n=N_m}^\infty} \subseteq \overline{f([m,\infty))}$$

which in turn demands that

$$x \in \bigcap_{m=0}^{\infty} \overline{f([m,\infty)}.$$

Let $x \in \bigcap_{n=0}^\infty \overline{f([n,\infty)}$. Because $x \in \overline{f([m,\infty)}$, we have that there exists $t_n \in [n,\infty]$ such that $|x - f(t_n)| < \frac{1}{n}$. By this means, we define the sequence $\{t_n\}_{n=1}^\infty$. We see that

$$\lim_{n\to\infty} t_n = \infty \quad \text{and} \quad \lim_{n\to\infty} f(t_n) = x.$$

Thus, $x \in \mathrm{con}_\infty(f)$. □

Proposition 6.28. *Let f be a complex valued function. Then, $con_\infty(f)$ is closed.*

Proof. By lemma 6.27, $\mathrm{con}_\infty(f)$ is the infinite intersection of closed sets. Thus, f is closed. □

Proposition 6.29. *Let f be a complex valued function. Then*

$$con_\infty(|f|) \subset [0, \infty].$$

Proof. Let $x \in \mathrm{con}_\infty(|f|)$. There exists $\{t_n\}_{n=1}^\infty$ convergent to infinity such that $\{|f(t_n)|\}_{n=1}^\infty$ converges to x. Since $\{|f(t_n)|\}_{n=1}^\infty \subseteq [0, \infty)$, we conclude that $x \in [0, \infty]$. □

Proposition 6.30. *Let $f \in L^2(\mathbb{R})$ be a complex valued function. Then,*

$$0 \in con_\infty(f).$$

Proof. Let $f \in L^2(\mathbb{R})$. By Markov's inequality, we have that

$$m(\{t \in [n, \infty)| \quad |f|^2(t) \geqslant \epsilon\}) \leqslant m(\{t \in \mathbb{R}| \quad |f|^2(t) \geqslant \epsilon\}) \leqslant \frac{||f||^2}{\epsilon}.$$

Since, $\frac{||f||^2}{\epsilon}$ is finite and $m([n, \infty))$ is infinite, we have that there exists $t_n \in [n, \infty)$ such that $|f(t_n)| < \epsilon$. We see that

$$\lim_{n\to\infty} t_n = \infty \quad \text{and} \quad \lim_{n\to\infty} f(t_n) = 0.$$

Thus, $0 \in \mathrm{con}_\infty(f)$. □

Proposition 6.31. *Let f be a continuous and bounded complex valued function. Then $con_\infty(f)$ is a connected set.*

Proof. Suppose $\text{con}_\infty(f)$ is disconnected. By definition, there exists $\varnothing \neq A, B \subset \text{con}_\infty(f)$ such that $A \cap \overline{B} = \varnothing = \overline{A} \cap B$ and $A \cup B = \text{con}_\infty(f)$. Clearly, both A and B are closed. By proposition 6.28, $\text{con}_\infty(f)$ is bounded and so are its subsets A and B. Thus, A and B are compact. The compactness and disjointedness of A and B demand that $\text{dist}(A, B) = r > 0$. Thus,

$$A^r = \bigcup_{a \in A} B_{r/4}(a) \quad \text{and} \quad B^r = \bigcup_{b \in B} B_{r/4}(b)$$

are open sets with nonempty intersection and such that $\text{dist}(A^r, B^r) = \frac{r}{2}$ which in turn makes them separate sets. Let $a_0 \in A$ and $b_0 \in B$. There exists sequences $\{t_n\}_{n=1}^\infty$ and $\{s_m\}_{m=1}^\infty$ convergent to ∞ and such that

$$\lim_{n \to \infty} f(t_n) = a_0 \quad \text{and} \quad \lim_{m \to \infty} f(s_m) = b_0.$$

There exists $N, M \geqslant 1$ such that for all $n \geqslant N$ and for all $m \geqslant M$, $|a_0 - f(t_n)| < \frac{r}{4}$ and $|b_0 - f(s_m)| < \frac{r}{4}$. Inductively define the following subsequences. Define the first elements as

$$n_1 = N \quad \text{and} \quad m_1 = M.$$

Define all later elements as

$$n_k = \min(\{n > n_{k-1} | t_n > s_{n_{k-1}}\}) \quad \text{and} \quad m_l = \min(\{m > m_{k-1} | s_m > t_{n_k}\}).$$

For simplicity, we rename the above sequences as $\{t_n\}_{n=1}^\infty$ and $\{s_n\}_{n=1}^\infty$. We see that

$$\{t_n\}_{n=1}^\infty \subseteq B_{r/4}(a_0) \subseteq A^r \quad \text{and} \quad \{s_n\}_{n=1}^\infty \subseteq B_{r/4}(b_0) \subseteq B^r.$$

and that $t_{n_k} < s_{m_k} < t_{n_{k+1}} < s_{n_{k+1}}$ for all $k \geqslant 1$.

Since $f(t_n) \in A^r$, $f(s_n) \in B^r$, f is continuous and A^r and B^r are separated sets, there exists $u_n \in (t_n, s_n)$ such that $f(u_n) \notin A^r \cup B^r \supseteq \text{con}_\infty(f)$. $\{u_n\}_{n=1}^\infty$ converges to infinity since $u_n > t_n$ and $limnt_n = \infty$. $\{u_n\}_{n=1}^\infty$ is a bounded sequence since f is bounded by some $M > 0$. Thus, $f(u_n) \in \overline{B_M(0)} \cap (A^r \cup B^r)^C$ which is the intersection of two closed sets and therefore itself closed. Since $\overline{B_M(0)} \cap (A^r \cup B^r)^C$ is bounded, it is compact and $\{f(u_n)\}_{n=1}^{\infty}$ has a limit $x_0 \in \overline{B_M(0)}$ which demands that

$$x_0 \in \overline{B_M(0)} \cap (A^r \cup B^r)^C \subseteq (A^r \cup B^r)^C \subseteq (\mathrm{con}_{\infty}(f))^C. \tag{28}$$

But since x_0 is a limit point of $\{f(u_n)\}_{n=1}^{\infty}$, there exists a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} u_{n_k} = \infty \quad \text{and} \quad \lim_{k\to\infty} f(u_{n_k}) = x_0.$$

This demands that $x_0 \in \mathrm{con}_{\infty}(f)$ contradicting (28). We reject our assumption and conclude that $\mathrm{con}_{\infty}(f)$ is connected. □

Proposition 6.32. *Let $f \in L^2(\mathbb{R})$ be complex valued and continuous. If $x_0 \in$ $con_{\infty}(|f|)$, then $[0, x_0] \subseteq con_{\infty}(|f|)$.*

Proof. Since $x \in \mathrm{con}_{\infty}(|f|)$, there exists a sequence, $\{t_n\}_{n=1}^{\infty}$ convergent to infinity such that $\{f(t_n)\}_{n=1}^{\infty}$ converges to x. Since $f \in L^2(\mathbb{C})$, proposition 6.30 demands that $0 \in \mathrm{con}_{\infty}(|f|)$ which in turn demands that there exists a sequence, $\{s_n\}_{n=1}^{\infty}$ convergent to infinity such that $\{f(s_n)\}_{n=1}^{\infty}$ converges to 0.

Fix $y \in (0, x)$. This demands that there exists $\epsilon > 0$ such that $y \in (\epsilon, x - \epsilon)$. There exists N_1 such that if $n \geqslant N_1$, then $|x - f(t_n)| < \epsilon$ and in turn that $|f(t_n)| > x - \epsilon$. There exists N_2 such that if $n \geqslant N_2$, then $|f(t_n)| = |0 - f(t_n)| < \epsilon$. Thus if $n \geqslant N = \max N_1, N_2$, then we have that $y \in [f(s_n), f(t_n)]$. Since f is continuous, we have by the intermediate value theorem that for all $n \geqslant N$, there exists $r_n \in [f(s_n), f(t_n)]$ such that $f(r_n) = y$. Since $s_n < r_n$ for all $n \geqslant N$,

$$\lim_{n\to\infty} r_n \geqslant \lim_{n\to\infty} s_n = \infty,$$

and that

$$\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} y = y.$$

Thus $y \in \mathrm{con}_\infty(|f|)$. □

6.4 Alpha Condensation

Definition 6.33. *Let f be a complex valued function and $\boldsymbol{\alpha}, \boldsymbol{\theta} \in \mathbb{R}^d$. Then, the $\boldsymbol{\alpha}, \boldsymbol{\theta}$-condensation of f at ∞, called $con_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f)$, is the set of all points $x \in \mathbb{C}^f$ such that there exists $\{t_n\}_{n=1}^\infty$ such that*

$$\lim_{n\to\infty} t_n = \infty_{\boldsymbol{\theta}} \quad \text{and} \quad \lim_{n\to\infty} f(t_n) = x.$$

If $\alpha, \theta \in \mathbb{R}^1$ then we denote the (α, θ)-condensation of f at ∞ as $con_\theta^\alpha(f)$.

Lemma 6.34. *Let f be a complex valued function and $\boldsymbol{\alpha} = (\alpha^1, ..., \alpha^d) \in \mathbb{R}^d$. Then,*

$$con_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f) = \bigcap_{m=1}^{\infty} \bigcap_{M=1}^{\infty} \overline{f(P_{1/m,M}^{\boldsymbol{\alpha},\boldsymbol{\theta}})}.$$

Proof. Let $x \in \mathrm{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f)$. Then, there exists $\{t_n\}_{n=1}^\infty \subset \mathbb{R}$ such that

$$\lim_{n\to\infty} t_n = \infty_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}} \quad \text{and} \quad \lim_{n\to\infty} f(t_n) = x.$$

By proposition 6.17, we have that for all $m, M \geqslant 1$, there exists $N \geqslant 1$ such that

$$t_n \in P_{1/m,N}^{\boldsymbol{\alpha},\boldsymbol{\theta}} \quad \text{for all } n \geqslant N.$$

That implies that

$$x \in \overline{\{f(t_n)\}_{n=1}^\infty} \subset \overline{f(P_{1/m,M}^{\boldsymbol{\alpha},\boldsymbol{\theta}})} \quad \text{for all } m, N \geqslant 1.$$

Thus,

$$x \in \bigcap_{m=1}^{\infty} \bigcap_{M=1}^{\infty} \overline{f(P_{1/m,M}^{\boldsymbol{\alpha},\boldsymbol{\theta}})},$$

and thereby,

$$\mathrm{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f) \subset \bigcap_{m=1}^{\infty} \bigcap_{M=1}^{\infty} \overline{f(P_{1/m,M}^{\boldsymbol{\alpha},\boldsymbol{\theta}})}.$$

Let $x \in \bigcap_{m=1}^{\infty} \bigcap_{M=1}^{\infty} \overline{f(P_{1/m,M}^{\boldsymbol{\alpha},\boldsymbol{\theta}})}$. Thus,

$$x \in \bigcap_{m=1}^{n} \bigcap_{M=1}^{n} \overline{f(P_{1/m,M}^{\boldsymbol{\alpha},\boldsymbol{\theta}})} = \overline{f(P_{1/n,n}^{\boldsymbol{\alpha},\boldsymbol{\theta}})} \quad \text{for all } n \geqslant 1.$$

Thus, there exists $t_n \in f(P_{1/n,n}^{\boldsymbol{\alpha},\boldsymbol{\theta}})$ such that $|x - f(t_n)| < \frac{1}{n}$. We see that

$$\lim_{n\to\infty} t_n = \infty_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}} \quad \text{and} \quad \lim_{n\to\infty} f(t_n) = x.$$

Thus, $x \in \text{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f)$. $\square$

Lemma 6.35. *Let f be a complex valued function and $\boldsymbol{\alpha} = (\alpha^1, ..., \alpha^d) \in \mathbb{R}^d$. Then,*

$$con_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f) = \left(\bigcap_{m=1}^{\infty} \bigcap_{M=1}^{\infty} con_{\infty}(\mathbb{1}_{P_{1/m,M}^{\boldsymbol{\alpha},\boldsymbol{\theta}}}(t)f(t))\backslash\{0\} \right) \cup (con_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f) \cap \{0\}).$$

Proof. Let $x \in \text{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f)\backslash\{0\}$. Then, there exists $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}$ such that

$$\lim_{n\to\infty} t_n = \infty_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}} \quad \text{and} \quad \lim_{n\to\infty} f(t_n) = x \quad \text{for all } n \geqslant N.$$

This demands that $\lim_{n\to\infty} t_n = \infty$ and for all $m, M \geqslant 1$, we have that there exists N such that

$$t_n \in P_{1/m,M}^{\boldsymbol{\alpha},\boldsymbol{\theta}} \quad \text{for all } n \geqslant N.$$

This implies that

$$\lim_{n\to\infty} \mathbb{1}_{P_{1/m,M}^{\boldsymbol{\alpha},\boldsymbol{\theta}}}(t_n)f(t_n) = \lim_{n\to\infty} f(t_n) = x.$$

Since $x \neq 0$, we have that $x \in \text{con}_{\infty}(\mathbb{1}_{P_{1/m,M}^{\boldsymbol{\alpha},\boldsymbol{\theta}}}(t)f(t))\backslash\{0\}$ for all $m, M \geqslant 1$. Thus,

$$x \in \bigcap_{m=1}^{\infty} \bigcap_{M=1}^{\infty} \text{con}_{\infty}(\mathbb{1}_{P_{1/m,M}^{\boldsymbol{\alpha},\boldsymbol{\theta}}}(t)f(t))\backslash\{0\}.$$

This gives us

$$\operatorname{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f)\backslash\{0\} \subset \bigcap_{m=1}^{\infty} \bigcap_{M=1}^{\infty} \operatorname{con}_{\infty}(\mathbb{1}_{P_{1/m,M}^{\boldsymbol{\alpha},\boldsymbol{\theta}}}(t) f(t))\backslash\{0\}.$$

Let

$$x \in \bigcap_{m=1}^{\infty} \bigcap_{M=1}^{\infty} \operatorname{con}_{\infty}(\mathbb{1}_{P_{1/m,M}^{\boldsymbol{\alpha},\boldsymbol{\theta}}}(t) f(t))\backslash\{0\}.$$

Since $x \in \operatorname{con}_{\infty}(\mathbb{1}_{P_{1/m,M}^{\boldsymbol{\alpha},\boldsymbol{\theta}}}(t) f(t))\backslash\{0\}$, there exists $\{t_n^{m,M}\}_{n=1}^{\infty} \subset \mathbb{R}$ such that

$$\lim_{n\to\infty} t_n^{m,M} = \infty \quad \text{and} \quad \lim_{n\to\infty} f(t_n^{m,M}) = x.$$

Since $x \neq 0$, there exists $N^{m,M}$ such that

$$f(t_n^{m,M}) \neq 0 \quad \text{for all } n \geqslant N^{m,M}.$$

This demands that $t_n^{m,M} \in P_{1/m,M}^{\boldsymbol{\alpha},\boldsymbol{\theta}}$ for all $n \geqslant N^{m,M}$. This demands that

$$\lim_{n\to\infty} t_n^{m,M} = \infty_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}.$$

Thus, $x \in \operatorname{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f)$ and

$$\bigcap_{m=1}^{\infty} \bigcap_{M=1}^{\infty} \operatorname{con}_{\infty}(\mathbb{1}_{P_{1/m,M}^{\boldsymbol{\alpha},\boldsymbol{\theta}}}(t) f(t))\backslash\{0\} \subset \operatorname{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f)\backslash\{0\}$$

which in turn grants

$$\operatorname{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f) = \bigcap_{m=1}^{\infty} \bigcap_{M=1}^{\infty} \operatorname{con}_{\infty}(\mathbb{1}_{P_{1/m,M}^{\boldsymbol{\alpha},\boldsymbol{\theta}}}(t) f(t))\backslash\{0\}.$$

We conclude that

$$\operatorname{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f) = (\operatorname{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f)\backslash\{0\}) \cup (\operatorname{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f) \cap \{0\}) =$$

$$\left(\bigcap_{m=1}^{\infty} \bigcap_{M=1}^{\infty} \operatorname{con}_{\infty}(\mathbb{1}_{P_{1/m,M}^{\boldsymbol{\alpha},\boldsymbol{\theta}}}(t) f(t))\backslash\{0\}\right) \cup (\operatorname{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f) \cap \{0\}).$$

□

Proposition 6.36. *Let f be a complex valued function, $\boldsymbol{\theta} = (\theta_1, ..., \theta_n), \boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^n$ and $C \in \mathbb{C}$. Then,*

$$con_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(Cf) = C\, con_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f).$$

Proof. By lemma 6.34, we have that

$$\text{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(Cf) = \bigcap_{m=1}^{\infty} \bigcap_{M=1}^{\infty} \overline{Cf(P_{1/m,M}^{\boldsymbol{\alpha},\boldsymbol{\theta}})} =$$

$$C \bigcap_{m=1}^{\infty} \bigcap_{M=1}^{\infty} \overline{f(P_{1/m,M}^{\boldsymbol{\alpha},\boldsymbol{\theta}})} = C\text{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f).$$

□

Proposition 6.37. *Let f and g be a complex valued function that is bounded at infinity and $\boldsymbol{\theta} = (\theta_1, ..., \theta_n), \boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^n$. Then,*

$$con_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f + g) \subseteq con_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f) + con_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(g).$$

Proof. Let $x \in \text{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f + g)$. Then, there exists $\{t_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} t_n = \infty_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}} \quad \text{and} \quad \lim_{n\to\infty} f(t_n) + g(t_n) = x.$$

Since f and g are both bounded at infinity, there exists $B_1, B_2 > 0$ and $N_1, N_2 \geqslant 1$ such that

$$|f(t)| < B_1 \quad \text{for all } n \geqslant N_1 \quad \text{and} \quad |f(t)| < B_2 \quad \text{for all } n \geqslant N_2.$$

Thus, $\{f(t_n)\}_{n=1}^{\infty}$ has a convergent subsequence $\{f(t_{n_k})\}_{k=1}^{\infty}$ with limit x_1 and $\{g(t_{n_k})\}_{k=1}^{\infty}$ has a convergent subsequence $\{g(t_{n_{k_l}})\}_{l=1}^{\infty}$ convergent to x_2. We see that

$$x = \lim_{n\to\infty} f(t_n) + g(t_n) = \lim_{l\to\infty} f(t_{n_{k_l}}) + g(t_{n_{k_l}}) =$$

$$x_1 + x_2 \in \mathrm{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f) + \mathrm{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(g).$$

Since this is true of all $x \in \mathrm{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f+g)$, we have that

$$\mathrm{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f+g) \subseteq \mathrm{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f) + \mathrm{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(g).$$

□

Lemma 6.38. *Let f be a complex valued function bounded at infinity and $\boldsymbol{\theta} = (\theta_1, ..., \theta_n), \boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^n$. Let g be a uniformly continuous complex valued function and $g((\pi l + \theta_k)/\alpha_k) = 0$ for all $l \in \mathbb{N}$ and for all $1 \leqslant k \leqslant n$. Then,*

$$con_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(gf) = \{0\}.$$

Proof. Fix $\epsilon > 0$. Let $B > 0$ and $N_B \geqslant 0$ be such that for all $t \geqslant N_B$, $|f(t)| \leqslant B$. Because g is uniformly continuous, there exists $\delta > 0$ such that

$$|g(t) - g(s)| < \frac{\epsilon}{B} \quad \text{for all } t, s \in \mathbb{R} \text{ such that } |t - s| < \delta.$$

Let $x \in \mathrm{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(gf)$. Then, there exists $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}$ such that

$$\lim_{n\to\infty} t_n = \infty_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}} \quad \text{and} \quad \lim_{n\to\infty} g(t_n)f(t_n) = x.$$

This demands that there exists $N \geqslant 1$ such that

$$t_n \in P_{\delta,1}^{\boldsymbol{\alpha},\boldsymbol{\theta}} \quad \text{for all } n \geqslant N.$$

Since every $t \in P_{\delta,1}^{\boldsymbol{\alpha},\boldsymbol{\theta}}$ is less than δ away from $(\pi l_n + \theta_k)/\alpha_k$ for some $l_n \in \mathbb{N}$, we have that

$$|g(t_n)f(t_n)| = |g(t_n)||f(t_n)| < B|g(t_n)| = B|g(t_n) - 0| =$$

$$B|g(t_n) - g(\frac{\pi l_N + \theta}{\alpha})| < B\frac{\epsilon}{B} = \epsilon.$$

Thus $\text{con}^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}}(gf) \subset B_0(\epsilon)$. Since this is true of all ϵ, we have that

$$\text{con}^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}}(gf) \subset \{0\}.$$

Since $\text{con}^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}}(gf)$ is nonempty, we have that $\text{con}^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}}(gf) = \{0\}$. □

Proposition 6.39. *Let f be a complex valued function bounded at infinity and $\boldsymbol{\theta} = (\theta_1, ..., \theta_n), \boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n d) \in \mathbb{R}^d$. Then,*

$$con^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}}(e^{i\alpha_k t} f) = e^{i\theta_k} con^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}}(f) \quad \textit{for all } 1 \leqslant k \leqslant d.$$

Proof. Fix $1 \leqslant k \leqslant d$. Since $e^{i\alpha_k t} - e^{i\theta_k}$ is zero on $\frac{\theta_k + l\pi}{\alpha}$ for all $l \in \mathbb{N}$ and is uniformly continuous, we can apply lemma 6.38 to attain

$$\text{con}^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}}((e^{i\alpha_k} - e^{i\theta_k)t} f) = \{0\}.$$

This grants us

$$\text{con}^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}}(e^{i\alpha_k t} f) = \text{con}^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}}(e^{i\theta_k} f + (e^{i\alpha_k} - e^{i\theta_k)t} f) \subset$$

$$\text{con}^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}}(e^{i\theta_k} f) + \{0\} = e^{i\theta_k} \text{con}^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}}(f).$$

Let $x \in e^{i\theta_k} \text{con}^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}}(f)$. Clearly, there exists $y \in \text{con}^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}}(f)$ such that $y = e^{i\theta_k} x$. We have that there exists $\{t_n\}_{n=1}^{\infty}$ such that

$$\lim_{n \to \infty} t_n = \infty^{\boldsymbol{\theta}}_{\boldsymbol{\alpha}} \quad \text{and} \quad \lim_{n \to \infty} f(t_n) = y.$$

By lemma 6.13, this demands that

$$\lim_{n \to \infty} e^{it_n} = e^{i\theta_k}.$$

Thus,

$$\lim_{n \to \infty} e^{it_n} f(t_n) = \lim_{n \to \infty} e^{it_n} \lim_{n \to \infty} f(t_n) = e^{i\theta_k} y = x.$$

Thus, $x \in \mathrm{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(e^{i\alpha_k t} f)$. $\square$

Proposition 6.40. *Let f be a complex function. Then $con_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f)$ is a closed set.*

Proof. By lemma 6.34,

$$\mathrm{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f) = \bigcap_{m=1}^{\infty} \bigcap_{N=1}^{\infty} \overline{f(P_{1/m,N}^{\boldsymbol{\alpha},\boldsymbol{\theta}})}.$$

Since $\mathrm{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f)$ is an infinite intersection of closed sets, $\mathrm{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f)$ is closed. $\square$

Definition 6.41. *Let f be a complex valued function. Then the α-condensation graph of f is a subset of $(\times^n[0, 2\pi)) \times \mathbb{C}^f$ where each element is of the form*

$$(\boldsymbol{\theta}, x) \quad \textit{where} \quad x \in con_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f).$$

Proposition 6.42. *Let f be a complex valued function, $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_d), \boldsymbol{\theta} = (\theta_1, ..., \theta_d) \in \mathbb{R}^d$, $1 \leqslant d_0 \leqslant d$ and $\{n_k\}_{k=1}^{d_1} \subseteq \{n\}_{n=1}^{d}$. Define $\boldsymbol{\alpha}^1 = (\alpha_{n_1}, ..., \alpha_{n_{d_0}})$ and $\boldsymbol{\theta}^1 = (\theta_{n_1}, ..., \theta_{n_{d_0}})$. Then,*

$$con_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f) \subseteq con_{\boldsymbol{\theta}^1}^{\boldsymbol{\alpha}^1}(f).$$

Proof. Let $x \in \mathrm{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f)$. Then, there exists $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}$ such that

$$\lim_{n\to\infty} t_n = \infty_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}} \quad \text{and} \quad \lim_{n\to\infty} f(t_n) = x.$$

By proposition 6.13, t_n must also converge to $\infty_{\boldsymbol{\theta}^1}^{\boldsymbol{\alpha}^1}$. Thus,

$$\lim_{n\to\infty} t_n = \infty_{\boldsymbol{\theta}^1}^{\boldsymbol{\alpha}^1} \quad \text{and} \quad \lim_{n\to\infty} f(t_n) = x.$$

We conclude that $x \in \mathrm{con}_{\boldsymbol{\theta}^1}^{\boldsymbol{\alpha}^1}(f)$ and $\mathrm{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f) \subseteq \mathrm{con}_{\boldsymbol{\theta}^1}^{\boldsymbol{\alpha}^1}(f)$. $\square$

Corollary 6.43. *Let f be a complex valued function, $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_d), \boldsymbol{\theta} = (\theta_1, ..., \theta_d) \in \mathbb{R}^d$, $P = \left\{P_k = \{n_k^l\}_{k=1}^{d_l}\right\}_{l=1}^{d_0}$ be a partition of a subset of $\{n\}_{n=1}^{d}$.*

Define $\boldsymbol{\alpha}^l = (\alpha_{n_1^l}, ..., \alpha_{n_{d_l}^l})$ *and* $\boldsymbol{\theta}^1 = (\theta_{n_1^l}, ..., \theta_{n_{d_l}^l})$*. Then,*

$$con_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f) \subseteq \bigcap_{l=1}^{d_0} con_{\boldsymbol{\theta}^l}^{\boldsymbol{\alpha}^l}(f).$$

Proof. By applying proposition 6.42 to each element of the partition, $P = \left\{P_k = \{n_k^l\}_{k=1}^{d_l}\right\}_{l=1}^{d_0}$, we have that

$$\mathrm{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f) \subseteq \mathrm{con}_{\boldsymbol{\theta}^k}^{\boldsymbol{\alpha}^k}(f) \quad \text{for all } 1 \leqslant k \leqslant n.$$

Thus

$$\mathrm{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f) \subseteq \bigcap_{l=1}^{d_0} \mathrm{con}_{\boldsymbol{\theta}^l}^{\boldsymbol{\alpha}^l}(f).$$

□

Corollary 6.44. *Let f be a complex valued function,* $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_d), \boldsymbol{\theta} = (\theta_1, ..., \theta_d) \in \mathbb{R}^d$*. Then*

$$con_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f) \subseteq \bigcap_{k=1}^{d} con_{\theta_k}^{\alpha_k}(f).$$

Proof. This is an immediate application of corollary 6.43 using $\left\{\{n\}\right\}_{n=1}^{d}$ as the partition of $\{n\}_{n=1}^{d}$. □

Proposition 6.45. *Let f be a complex valued function,* $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_d), \boldsymbol{\theta} = (\theta_1, ..., \theta_d) \in \mathbb{R}^d$ *and* $\beta \in \mathbb{R}$*. Then*

$$con_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f(t-\beta)) = con_{\boldsymbol{\theta}+\beta}^{\boldsymbol{\alpha}}(f)$$

where $\boldsymbol{\theta} + \beta = (\theta_1 + \beta, ..., \theta_d + \beta)$*.*

Proof. Let $x \in \mathrm{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f(t-\beta))$. Then, there exists $\{t_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} t_n = \infty_{\boldsymbol{\alpha}}^{\boldsymbol{\theta}} \quad \text{and} \quad \lim_{n\to\infty} f(t_n - \beta) = x.$$

Define $\{s_n = t_n - \beta\}_{n=1}^{\infty}$. We see that

$$s_n = t_n - \beta \in \tau_\beta(P_{\delta,N}^{\boldsymbol{\alpha},\boldsymbol{\theta}}) = P_{\delta,N}^{\boldsymbol{\alpha},\boldsymbol{\theta}+\beta} \quad \text{for all } \delta > 0, N \geqslant 1.$$

This implies that $\lim_{n\to\infty} s_n = \infty_{\boldsymbol{\theta}+\beta}^{\boldsymbol{\alpha}}$ which demands that $x \in \text{con}_{\boldsymbol{\theta}+\beta}^{\boldsymbol{\alpha}}(f)$. This gives us

$$\text{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f(t-\beta)) \subset \text{con}_{\boldsymbol{\theta}+\beta}^{\boldsymbol{\alpha}}(f).$$

Let $x \in \text{con}_{\boldsymbol{\theta}+\beta}^{\boldsymbol{\alpha}}(f)$. Then, there exists $\{t_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} t_n = \infty_{\boldsymbol{\theta}+\beta}^{\boldsymbol{\alpha}} \quad \text{and} \quad \lim_{n\to\infty} f(t_n) = x.$$

Define $\{s_n = t_n + \beta\}_{n=1}^{\infty}$. We see that

$$s_n = t_n + \beta \in \tau_\beta(P_{\delta,N}^{\boldsymbol{\alpha},\boldsymbol{\theta}-\beta}) = P_{\delta,N}^{\boldsymbol{\alpha},\boldsymbol{\theta}} \quad \text{for all } \delta > 0, N \geqslant 1.$$

This implies that $\lim_{n\to\infty} s_n = \infty_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}$ which demands that

$$\lim_{n\to\infty} s_n = \infty_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}} \quad \text{and} \quad \lim_{n\to\infty} f(s_n-\beta) \lim_{n\to\infty} f(t_n) = x.$$

Thus, $x \in \text{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f(t-\beta))$. We conclude that

$$\text{con}_{\boldsymbol{\theta}+\beta}^{\boldsymbol{\alpha}}(f) \subset \text{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f(t-\beta))$$

and

$$\text{con}_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f(t-\beta)) = \text{con}_{\boldsymbol{\theta}+\beta}^{\boldsymbol{\alpha}}(f).$$

□

Lemma 6.46. *Let f be a complex valued function bounded at infinity, $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n), \boldsymbol{\theta} = (\theta_1, ..., \theta_n) \in \mathbb{R}^d$. Then,*

$$con_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}\left(\sum_{k=1}^{n} c_k e^{i\alpha_k t} f(t-\beta_k)\right) \subseteq \sum_{k=1}^{n} c_k e^{i\theta_k} con_{\boldsymbol{\theta}+\beta_k}^{\boldsymbol{\alpha}}(f).$$

where $\sum$ *is set addition.*

Proof. By propositions 6.37), (6.36, 6.39 and 6.45, we have that

$$\text{con}^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}}(\sum_{k=1}^{n} c_k e^{i\alpha_k t} f(t-\beta_k)) \subseteq \sum_{k=1}^{n} \text{con}^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}}(c_k e^{i\alpha_k t} f(t-\beta_k)) =$$

$$\sum_{k=1}^{n} c_k \text{con}^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}}(e^{i\alpha_k t} f(t-\beta_k)) = \sum_{k=1}^{n} c_k e^{i\theta_k} \text{con}^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}}(f(t-\beta_k)) =$$

$$\sum_{k=1}^{n} c_k e^{i\theta_k} \text{con}^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}+\beta_k}(f).$$

□

Theorem 6.47. *Let* $\Xi = \{c_k, \alpha_k, \beta_k\}_{k=1}^{d}$ *be an HRT configuration. Let* f *be a complex valued function bounded at infinity that does not satisfy the* Ξ*-configuration. Then,*

$$0 \in \sum_{k=1}^{d} c_k e^{i\theta_k} \, con^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}+\beta_k}(f).$$

Theorem 6.48. *Let* $\{t_n\}_{n=1}^{\infty}$ *converge to* $\infty^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}}$*. Since* f *does not satisfy the* Ξ*-configuration, we have that*

$$\lim_{n\to\infty} \sum_{k=1}^{d} c_k e^{i\alpha_k t} f(t_n - \beta_k) = \lim_{n\to\infty} 0 = 0.$$

The above and lemma 6.46 gives us

$$0 \in con^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}}(\sum_{k=1}^{n} c_k e^{i\alpha_k t} f(t-\beta_k)) \subseteq \sum_{k=1}^{n} c_k e^{i\theta_k} \, con^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}+\beta_k}(f).$$

Definition 6.49. *Define the set of nearly linear elements,* L^d*, to be all vectors,* $\boldsymbol{\theta} = (\theta_1, ..., \theta_d)$*, such that there exists* $1 \leqslant k \leqslant d$ *such that*

$$e^{i\theta_k} \notin \{e^{i\theta_j}, -e^{i\theta_j}\} \quad \textit{for all } j \neq k$$

and

$$e^{i\theta_j} \in \{e^{i\theta_l}, -e^{i\theta_l}\} \quad \textit{for all } j, l \neq k.$$

We call this k the abnormal index of $\boldsymbol{\theta}$ and we call the other indices the normal indices of $\boldsymbol{\theta}$.

Theorem 6.50. *Let $\Xi = \{c_k, \alpha_k, \beta_k\}_{k=1}^n$ be a real HRT configuration. Let f be a real valued function bounded at infinity that does not satisfy the Ξ-configuration. Then,*

$$0 \in con_{\boldsymbol{\theta}}^{\boldsymbol{\alpha}}(f) \textit{for all } \boldsymbol{\theta} \in L^d.$$

Proof. Let $\boldsymbol{\theta} \in L^d$ with abnormal index k_0 and K_1 is a normal index. We see that

$$\sum_{k=1,k\neq k_0}^{d} c_k e^{i\theta_k} \mathrm{con}_{\boldsymbol{\theta}+\beta_k}^{\boldsymbol{\alpha}}(f) = \sum_{k=1,k\neq k_0}^{d} c_k e^{i\theta_{k_1}} \mathrm{con}_{\boldsymbol{\theta}+\beta_k}^{\boldsymbol{\alpha}}(f) =$$

$$e^{i\theta_{k_1}} \sum_{k=1,k\neq k_0}^{d} c_k \mathrm{con}_{\boldsymbol{\theta}+\beta_k}^{\boldsymbol{\alpha}}(f) \subset e^{i\theta_{k_1}}\mathbb{R}$$

and

$$c_{k_0} e^{i\theta_{k_0}} \mathrm{con}_{\boldsymbol{\theta}+\beta_{k_0}}^{\boldsymbol{\alpha}}(f) \subset e^{i\theta_{k_0}}\mathbb{R}. \tag{29}$$

By theorem 6.47, we have that

$$0 \in \sum_{k=1}^{d} c_k e^{i\theta_k} \mathrm{con}_{\boldsymbol{\theta}+\beta_k}^{\boldsymbol{\alpha}}(f) = c_{k_0} e^{i\theta_{k_0}} \mathrm{con}_{\boldsymbol{\theta}+\beta_{k_0}}^{\boldsymbol{\alpha}}(f) + \sum_{k=1,k\neq k_0}^{d} c_k e^{i\theta_k} \mathrm{con}_{\boldsymbol{\theta}+\beta_k}^{\boldsymbol{\alpha}}(f) \subseteq$$

$$c_{k_0} e^{i\theta_{k_0}} \mathrm{con}_{\boldsymbol{\theta}+\beta_{k_0}}^{\boldsymbol{\alpha}}(f) + e^{i\theta_{k_1}}\mathbb{R}.$$

Taking the above and (29), we have that

$$0 \in \mathrm{con}_{\boldsymbol{\theta}+\beta_{k_0}}^{\boldsymbol{\alpha}}(f),$$

because if $a \in e^{i\theta_{k_0}}\mathbb{R}$, $b \in e^{i\theta_{k_1}}\mathbb{R}$ and $a + b = 0$, then $a, b = 0$. □

Theorem 6.51. *Let $\Xi = \{c_k, \alpha_k, \beta_k\}_{k=1}^n$ be an HRT configuration. Let f be a real valued function bounded at infinity that does not satisfy the Ξ-configuration. Then, for all $\boldsymbol{\theta} \in L^d$, for all $\epsilon > 0$, $\delta > 0$, $N \in \mathbb{N}$, there exists $t_0 \in$ such that*

$$f(t_0) \in B_0(\epsilon).$$

Proof. By theorem 6.50, we have that $0 \in \text{con}^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}+\beta_{k_0}}(f)$. Thus, there exists $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}$ such that

$$\lim_{n\to\infty} t_n = \infty^{\boldsymbol{\alpha}}_{\boldsymbol{\theta}} \quad \text{and} \quad \lim_{n\to\infty} f(t_n) = 0.$$

Thus, for all nonempty periodic intervals $P^{\boldsymbol{\alpha},\boldsymbol{\theta}}_{\delta,N}$ there exists a t_n such that

$$f(t_n) \in B_0(\epsilon).$$

□

7 Sigma Delta Quantization

For the sake of storing and transmitting various messages, it is impractical to represent any given function in the traditional, input-output fashion or a point by its decimal representation. Therefore, we instead represent the signal not in terms of its value at any given point but rather in terms of other relevant functions, i.e.,

$$x = \sum_n c_n e_n,$$

where e_n are other functions or points which constitute an alphabet by means of which we represent our signal, and c_n are real or complex coefficients.

This has simplified the problem of storage and transmission but not sufficiently as the coefficients are drawn from a continuous range. Quantization converts this continuous representation to a corresponding sum with coefficients, q_n, drawn from a discrete set. $\tilde{x}$ denotes the resulting finite sum of $\{e_n\}$ using q_n as coefficients. Naturally, the measure of the quality of the quantizer, the map between the continuous coefficients to their discrete approximations, is the approximation error $||x - \tilde{x}||$ where the norm is chosen for the situation. The most obvious construction for the quantizer, called pulse code modulation (PCM), is to choose q_n as close to c_n as possible. This method is optimal when our alphabet is orthonormal, since the independence of the elements makes better estimates impossible, but is less than optimal when our alphabet is redundant as PCM fails to take advantage of the resulting interdependence.

7.1 Basic Frame Theory

7.2 Introduction

In order to move further, we must have some understanding of the nature of the alphabets, particularly redundant alphabets, that will be used in our quantizations.

Definition 7.1. *A collection $F = \{e_n\}_{n\in\Lambda}$ in a Hilbert space H is a frame for H if there exists $0 < A \leqslant B < \infty$ such that*

$$A||x||^2 \leqslant \sum_{n\in\gamma} |\langle x, e_n\rangle|^2 \leqslant B||x||^2, \quad \text{for all } x \in H.$$

The constants A and B are called the frame bounds and the frame is tight when $A = B$.

Definition 7.2. *Let $e_n, n \in \Lambda$ be a frame for a Hilbert space H with frame bounds A and B. The analysis operator*

$$L : H \to l^2(\Lambda)$$
$$(Lx)_k = \langle x, e_k\rangle.$$

The operator $S = L^*L$ is called the frame operator, and it satisfies

$$AI \leqslant S \leqslant BI,$$

where I is the identity operator on H. S^{-1} is called the dual frame operator, and it satisfies

$$B^{-1}I \leqslant S^{-1} \leqslant A^{-1}I.$$

We may interpret L as a map that sends an element of H to its representation in the frame.

Note that the earlier frame bounds, A and B, here result in the injectivity and continuity of L, respectively.

Theorem 7.3. *Let $\{e_n\}_{n\in\Lambda}$ be a frame for H with the frame bounds A and B, and let S be the corresponding frame operator. Then $\{S^{-1}e_n\}_{n\in\Lambda}$ is a frame for H with frame bounds B^{-1} and A^{-1}. Further, for all $x \in H$*

$$x = \sum_{n\in\Lambda}\langle x, e_n\rangle(S^{-1}e_n) = \sum_{n\in\Lambda}\langle x, (S^{-1}e_n)\rangle e_n$$

with unconditional convergence of both sums.

Note that when the frame is tight with bound A then x is equal to its frame expansion divided by A.

Corollary 7.4. *A set of vectors $\{v_n\}_{n=1}^N$ in $H = \mathbb{R}^n$ or $\mathbb{C}^n$ is a tight frame with frame bound A if and only if its associated matrix L satisfies $S = L^*L = AI_d$.*

In the special case of finite unit norm tight frames, the frame constant is N/d. Notice that when we have a basis, our constant is one.

7.3 PCM and Bennett's White Noise

In the normalized tight frame $\{e_n\}_{n=1}^N$, $x \in \mathbb{R}^d$ has the frame expansion

$$x = \frac{d}{N}\sum_{n=1}^{N} x_n e_n, x_n = \langle x, e_n \rangle.$$

We define the PCM quantizer with step size δ as

$$q_n = q_n(x) = \begin{cases} \delta(\lceil x_n/\delta \rceil - 1/2), & \text{if } |x_n| < 1 \\ \delta(\lceil 1/\delta \rceil - 1/2), & \text{if } x_n \geqslant 1 \\ -\delta(\lceil 1/\delta \rceil - 1/2), & \text{if } x_n \leqslant -1. \end{cases}$$

The PCM quantizes x such that we get the immediate estimate

$$||x - \tilde{x}|| = \frac{d}{N}||\sum_{n=1}^{N} q_n e_n|| \leqslant (\frac{\delta}{2})(\frac{d}{N})\sum_{n=1}^{N} ||e_n|| = \frac{d\delta}{2}.$$

This error is far from optimal since it does not make use of the redundancy of the frame.

7.4 $\Sigma\Delta$ Algorithm

Before defining the algorithm, let us first define where we will draw our estimates from. The midrise quantization alphabet is defined,

$$A_K^\delta = \{(-K + 1/2)\delta, ..., (K - 1/2)\delta\}.$$

From it we select the coefficients of our alphabet for our estimate, the 2K-level midrise uniform scalar quantizer with step size δ,

$$Q(u) = \arg\min_{q \in A_K^\delta} |u - q|.$$

Definition 7.5. *Given the above, a permutation on n elements, p, and the $\{x_n\}_{n=1}^N$ to be quantized, the $\Sigma\Delta$ quantizer is defined*

$$u_n = u_{n-1} + x_{p(n)} - q_n$$

$$q_n = Q(u_{n-1} + x_{p(n)})$$

where u_0 is a preset constant, usually zero in this dissertation. q_n is the quantized sequence while u_n is an auxiliary sequence.

It is natural to ask what is the role played by the auxiliary sequence. In essence, u_n keeps track of how much we have over or underestimated up to that time by how negative or positive it is, respectively. To see this let us work through an example.

Example 7.6. *Suppose we wanted to quantize $\{1/2, 1/4, -1/8\}$ with $\delta = 2$, $K = 1$, our permutation os the identity and u_0 is set to 0. The PCM and $\Sigma\Delta$ quantizations are listed below:*

n	x_n	*PCM's* q_n	$\Sigma\Delta$*'s* q_n	u_n
0	-	-	-	*0*
1	*1/2*	*1*	*1*	*-1/2*
2	*1/4*	*1*	*-1*	*3/4*
3	*-1/8*	*-1*	*1*	*-5/8*

For the first quantization, we have agreement between the two methods, but the fact we have an overestimation is captured by the negativity of u_1*.* u_1 *in turn results in the next iteration of* $\Sigma\Delta$ *being an underestimate rather than PCM's overestimate. This underestimate results in a positive* u_2 *and an overestimation, in contrast to* $\Sigma\Delta$*'s underestimate.*

Notice that this discrepancy was only possible by decreasing the size of x_n*. This could not be done in perpetuity, since it would require an indefinitely precise digitization.*

So PCM and $\Sigma\Delta$ may differ radically, but in what situations is this useful? When our frame has high redundancy.

Example 7.7. *We wish to approximate* $(\frac{1}{2}, \frac{1}{\pi})$ *using the frame* $\{(\sin\frac{2\pi j}{10}, \cos\frac{2\pi j}{10}\}_{j=0}^{9}$ *and the quantization alphabet* $\{-1, 0, 1\}$*. The resulting estimates by the two methods are*

n	x_n	*PCM's* q_n	$\Sigma\Delta$*'s* q_n
1	*0.5514*	*1*	*1*
2	*0.5739*	*1*	*0*
3	*0.3772*	*0*	*1*
4	*0.0364*	*0*	*0*
5	*-0.3183*	*0*	*-1*
6	*-0.5514*	*-1*	*0*
7	*-0.5739*	*-1*	*-1*
8	*-0.3372*	*0*	*0*
9	*-0.0364*	*0*	*0*
10	*0.3183*	*0*	*0*

The difference in the L^2 *norms to* $\{x^n\}$ *is 0.1731 and 0.0435, clearly in favor of the* $\Sigma\Delta$ *estimate.*

But this method is actually a hindrance when we have a low redundancy frame such as an orthogonal basis.

Example 7.8. *Let us estimate the randomly generated vector* x *whose entries are between zero and ten with an integral quantizer. The two schemes are presented below.*

n	x_n	*PCM's* q_n	$\Sigma\Delta$*'s* q_n
1	*4.5054*	*5*	*5*
2	*0.8382*	*1*	*0*
3	*2.2898*	*2*	*3*
4	*9.1334*	*9*	*9*
5	*1.5238*	*2*	*1*
6	*8.2582*	*8*	*9*
7	*5.3834*	*5*	*5*
8	*9.9613*	*10*	*10*
9	*0.7818*	*1*	*1*
10	*4.4268*	*4*	*4*

The difference in the L^2 *norms to* $\{x^n\}$ *is 1.0219 and 1.6347 in favor of PCM. The ancillary term in* $\Sigma\Delta$ *resulted in an unnecessary bias in various quantizations, such as in steps two, three, five and six.*

This example demonstrates that the positivity or negativity will tend to bias us towards or against an under or overestimation, but it does not demonstrate why we want this. In fact, $\Sigma\Delta$ is at a disadvantage compared to PCM when our frame is a basis. The advantage only comes forward when we have redundancy.

7.5 Basic Error Estimates

Before we take our first error estimates, we will need to use the correct representation and introduce the notion of frame variation. We represent x with $\{e_n\}_{n=1}^N$. Thus,

$$x = \sum_{n=1}^{N} x_n S^{-1} e_n, \; x_n = \langle x, e_n \rangle.$$

Definition 7.9. *Let* $F = \{e_n\}_{n=1}^n$ *be a finite frame for* $\mathbb{R}^d$ *and let* p *be a permutation of* $\{1, 2, ..., n\}$*. The variation of the frame* F *with respect to* p *is defined as*

$$\sigma(F, p) = \sum_{n=1}^{N-1} ||e_{p(n)} - e_{p(n+1)}||.$$

Variation may roughly be interpreted as the amount of oscillation between the elements. Our first estimate will be done using $\{S^{-1}e_n\}$ as our alphabet.

Theorem 7.10. *Given the finite, normalized frame for* $\mathbb{R}^d$, $F = \{e_n\}_{n=1}^n$, *permutation* p, $|u_0| < \delta/2$, *and* $x \in \mathbb{R}^d$ *such that* $||x|| \leqslant (K - 1/2)\delta$, *the approximation error* $||x - \tilde{x}||$ *satisfies*

$$||x - \tilde{x}|| \leqslant ||S^{-1}||_{op}(\sigma(F,p)\frac{\delta}{2} + |u_N| + |u_0|).$$

Proof. We have that

$$x - \tilde{x} = \tilde{x} = \sum_{n=1}^{N}(x_{p(n)} - q_n)S^{-1}e_{p(n)}\} =$$

$$\sum_{n=1}^{N}(u_n - u_{n-1})S^{-1}e_{p(n)} =$$

$$\sum_{n=1}^{N} u_n S^{-1}(e_{p(n)} - e_{p(n+1)}) + u_N S^{-1}e_{p(N)} - u_0 S^{-1}e_{p(1)}.$$

Our bound on x gives us an identical bound on $\langle x, e_n \rangle$ so we have

$$||x - \tilde{x}|| \leqslant \sum_{n=1}^{N}\frac{\delta}{2}||S^{-1}||_{op} \leqslant ||e_{p(n)} - e_{p(n+1)}|| + |u_N|\ ||S^{-1}||_{op} + |u_0|\ ||S^{-1}||_{op} =$$

$$||S^{-1}||_{op}(\sigma(F,p)\frac{\delta}{2} + |u_0| + |u_N|).$$

□

We can further refine this estimate by considering the case of a tight frame of frame bound N/d and realizing the bounds on our ancillary sequence, the u_m. This results in $||S^{-1}||_{op} = ||\frac{d}{N}I||_{op} = d/N$ and $||u_n|| < \frac{\delta}{2}$. So our new bound is

$$||x - \tilde{x}|| \leqslant \frac{\delta d}{2N}(\sigma(F,p) + 2).$$

We can yet further refine our estimate by setting u_0 to 0 and imposing the zero sum condition on our frame, such as when roots of unity are used.

Theorem 7.11. *Given the $\Sigma\Delta$ scheme, a normalized tight frame of frame bound with zero sum condition N/d, $F = \{e_n\}_{n=1}^N$, a permutation p of N elements, and $||x|| \leqslant (k - 1/2)\delta$, the approximation error satisfies*

$$||x - \tilde{x}|| \leqslant \begin{cases} \frac{\delta d}{2N}\sigma(F,p), & \text{if } N \text{ even} \\ \frac{\delta d}{2N}(\sigma(F,p)+1), & \text{if } N \text{ odd.} \end{cases}$$

Proof. Using the iterative definition of our ancillary sequence and that $u_0 = 0$, we have

$$u_N = \sum_{n=1}^N x_n - \sum_{n=1}^N q_n.$$

The definition of x_n, linearity and the zero sum condition demand that

$$\sum_{n=1}^N x_n = \sum_{n=1}^N \langle x, e_n\rangle = \langle x, \sum_{n=1}^N e_n\rangle = 0.$$

Since each q_n is an odd integer multiple of $\delta/2$, the above demands that u_N is also. An odd sum of odd multiples is an odd multiple and an even sum of odd multiples is even. This combined with the bound $|u_N| \leqslant \delta/2$ demands $|u_N| = \delta/2$ when we have an odd frame and $u_N = 0$ when we have an even frame. This in conjunction with our approximation bound from theorem 7.10 gives the desired result. □

7.6 Families of Frames of Bound Variation

Looking at the bound obtained in theorem 7.11, one obvious way to decrease our error is to decrease δ and thereby have a higher resolution quantization alphabet. The major flaw in this approach is that in applications, such as Analog-to-Digital conversion, such increased accuracy is quite costly and thus infeasible.

The next obvious way is to increase N which is to say increase the redundancy of our frame. The major issue that that brings up is that this may increase our frame variation and thus undermine any gains from redundancy.

Thus, we must find families of frames of uniformly bounded variation. A clear candidate is the roots of unity.

Example 7.12. *For $N \geqslant 3$, define $R_N = \{e_n^N\}_{n=1}^N$ be the Nth roots of unity in $\mathbb{R}^d$. Note that this family satisfies the zero sum condition. Since the distance between any two adjacent roots is less than their angle apart, $||e_n - e_{n+1}|| \leqslant 2\pi/N$ and thereby*

$$\sigma(F, p) \leqslant 2\pi.$$

Applying this to theorem 7.11, we have

$$||x - \tilde{x}|| \leqslant \begin{cases} \frac{\delta}{N} 2\pi & \text{if } N \text{ even} \\ \frac{\delta}{N}(2\pi + 1) & \text{if } N \text{ odd.} \end{cases}$$

The most natural generalization of this example to n-dimensions is the harmonic frames.

Example 7.13. *The harmonic frame $H_N^d = e_j{}_{j=0}^{N-1}$ is defined differently for the odd or even dimensional case. If $d \geqslant 2$ is even, let*

$$e_j = \sqrt{\frac{2}{d}}[\cos\frac{2\pi j}{N}, \sin\frac{2\pi j}{N}, ..., \cos\frac{2\pi\frac{d-1}{2}j}{N}, \sin\frac{2\pi\frac{d-1}{2}j}{N}].$$

If $d \geqslant 2$ is odd, let

$$e_j = \sqrt{\frac{2}{d}}[1/\sqrt{2}, \cos\frac{2\pi j}{N}, \sin\frac{2\pi j}{N}, ..., \cos\frac{2\pi\frac{d-1}{2}j}{N}, \sin\frac{2\pi\frac{d-1}{2}j}{N}].$$

The frame variation when d is even is bounded by

$$\sqrt{\frac{d}{2}}\sigma(H_N^d, p) = \sqrt{\frac{d}{2}}\sum_{j=0}^{N} -2||e_j - e_j + 1||$$

$$= \sum_{j=0}^{n_2}[\sum_{k=1}^{d/2}(\cos(\frac{2\pi kj}{N}) - \cos(\frac{2\pi(j+1)}{N}))^2]$$

$$+\sum_{k=1}^{d/2}(\cos(\frac{2\pi kj}{N})-\cos(\frac{2\pi(j+1)}{N}))^2]^{\frac{1}{2}}$$

$$\leqslant \sum_{j=0}^{N-2}[2\sum_{k=1}^{d/2}(\frac{2\pi k}{N})^2]^{\frac{1}{2}} \leqslant 2\pi\sqrt{2}[\sum_{k=1}^{d/2}k^2]^{\frac{1}{2}} \leqslant$$

$$2\pi\sqrt{2}[\frac{d(d/2+1)(d+1)}{12}]^{\frac{1}{2}} \leqslant 2\pi\sqrt{\frac{d}{6}}(d+1).$$

Proceeding in a similar fashion when d is odd, we have

$$\sqrt{\frac{d}{2}}\sigma(H_N^d,p) \leqslant 2\pi\sqrt{\frac{d}{6}}(d+1).$$

Taken together, this inequalities yield

$$\sigma(H_N^d,p) \leqslant \frac{2\pi(d+1)}{\sqrt{3}}.$$

Applying theorem 7.11, we have

$$||x-\tilde{x}|| \leqslant \begin{cases} \frac{\delta d}{2N}\frac{2\pi(d+1)}{\sqrt{3}} & \textit{if } N \textit{ even} \\ \frac{\delta d}{2N}[\frac{2\pi(d+1)}{\sqrt{3}}+1] & \textit{if } N \textit{ odd.} \end{cases}$$

The significance of the order provided by our permutation has so far been unexplained; the order is central to the way in which $\Sigma\Delta$ estimates utilize redundancy.

Suppose we have a heavily redundant frame such as

$$\{(\cos(2\pi n/N), \sin(2\pi n/N))\}_{n=1}^{N}.$$

Each element will differ only slightly to the elements in adjacent indices. Recall that $\Sigma\Delta$ quantization using its ancillary sequence keeps a record of its total over or underestimation and that this will thereby bias the next estimation towards under or overestimation, respectively. This means that we will tend to overestimate coefficients for elements when we have recently been taking underestimates and vice versa. When we apply this to our frame with the identity permutation, this translates into tending to draw overestimates on elements when we have drawn underestimates on similar elements. The hope is that these contrary over and under estimates on similar elements will cancel out and that an ordering with minimal discrepancy uses $\Sigma\Delta$ to the greatest advantage.

Note that if one wanted to use the $\Sigma\Delta$ to the greatest disadvantage, one method would be to maximize discrepancy, or equivalently, have the nearest element to the negative of the former follow it in the sequence, such as in

$$\{(\cos(2\pi n/N + \pi(1-(-1)^n)), \sin(2\pi n/N + \pi(1-(-1)^n)))\}_{n=1}^{N}$$

where N is odd.

7.7 Refined Estimate

Numerical simulations suggest that our lower bound can be improved significantly for the even case.

Let $\{F_n = /e_n^N/_{n=1}^N\}$ be a family of normalized tight frames for $\mathbb{R}^d$, with $\{x_n^N\}_{n=1}^N$ the corresponding sequence of frame coefficients with respect to F_n Let $\{q_n^N\}_{n=1}^N$ be the quantization obtained from the $\Sigma\Delta$ scheme and $\{u_n^N\}_{n=1}^N$ the corresponding ancillary sequence. The resulting quantized expansion is

$$\tilde{x} = \frac{d}{N}\sum_{n=1}^{N} q_n^N e_n^N.$$

We can rewrite the approximation error to be

$$x - \tilde{x}_N = \frac{d}{N}\left(\sum_{n=1}^{N-1} u_n^N(e_n^N - e_{n+1}^N) + u_n^N e_n^N\right) =$$

$$\frac{d}{N}\left(sum_{n=1}^{N-2} v_n^N(f_n^N - f_{n+1}^N) + v_{N-1}^N f_{N-1}^N + u_n^N e_n^N\right)$$

where we have defined

$$f_n^N = e_n^N - e_{n+1}^N, v_n^N = \sum_{j=1}^{n} u_j^N v_0^N = 0.$$

Definition 7.14. *Let $f \in B_\Omega$ and let $\{z_j\}_{j=1}^{n*}$ be the finite set of zeros of f' contained in $[0,1]$. We say that $f \in M_\Omega$ if $f' \in L^\infty$, and if*

$$f''(z_j) \neq 0 \quad \textit{for all } 1 \leqslant j \leqslant n^*.$$

Definition 7.15. *Let $\{u_n\}_{n=1}^N \subset [-1/2, 1/2)$. The discrepancy of $\{u_n\}_{n=1}^N$ defined*

$$Disc(\{u_n\}_{n=1}^N) = \sup_{I \subset T} \left| \frac{\#(\{u_n\}_{n=1}^N \cap I)}{N} - |I| \right|.$$

The Erdős-Túran inequality provides estimates of discrepancy in terms of exponential sums:

$$Disc(\{u_k\}_{n=1}^j) \leqslant \frac{1}{K} + \frac{1}{j} \sum_{k=1}^{K} \frac{1}{k} \left| \sum_{n=1}^{j} e^{2\pi i k u_n} \right| \quad \text{for all } K.$$

Koksma's inequality states that any function $f : [-1/2, 1/2) \leftarrow R$ of bounded variation,

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(u_n) - \int_{-1/2}^{1/2} f(t) dt \right| \leqslant Var(f) Disc(u_{n\,n=1}^{\;N}).$$

Theorem 7.16. *Suppose $x \in \mathbb{R}^d$ satisfies $||x|| \leqslant (K - 1/2)\delta$, and $\{x_n\}_{n=1}^N$ be the frame coefficients with respect to F_N. If, for some $\Omega > 0$, there exists $h \in M_\Omega$ such that*

$$x_n^N = h(n/N) \quad \textit{for all } N,\ 1 \leqslant n \leqslant N,$$

and if N is sufficiently large, then

$$|v_n^N| \lesssim \delta \left(\frac{n}{N^{\frac{1}{4}}} + N^{\frac{3}{4}} \log N \right) \lesssim \delta N^{\frac{3}{4}} \log N.$$

The implicit constants are independent of N and δ, but they do depend on x and hence h. The value of what constitutes a sufficiently large N depends on δ.

Proof. Define $\tilde{u}_n^N = u_n^N/\delta$. Using Koksma's inequality, one has

$$|v_j^N| = \delta \left| \sum_{n=1}^{J} \tilde{u}_n^N \right| = j\delta \left| \frac{1}{j} \sum_{n=1}^{j} \tilde{u}_n^N - \int_{-1/2}^{1/2} y dy \right|$$

$$\leqslant j\delta Var(x) Disc(\{\tilde{u}_n^N\}_{n=1}^{j}).$$

□

Using the Erdős-Túran inequality, we can estimate $D_j^N = Disc(\{\tilde{u}_n^N\}_{n=1}^{j})$ as

$$D_j^N \leqslant \frac{1}{K} + \frac{1}{j} \sum_{k=1}^{K} \frac{1}{k} \left| \sum_{n=1}^{j} e^{2\pi i k \tilde{u}_n^N} \right| \quad \text{for all } K.$$

The problem is now reduced to estimating $\sum_{n=1}^{j} e^{2\pi i k \tilde{u}_n^N}$.

We know from [9],

$$\forall N, \exists \text{ analytic } X_N \in B_\Omega \text{ such that } u_n^N = X_N(n) \text{ modulo } [-\delta/2, \delta/2]$$

and

$$|X_N'(t) - h(t/N)| \lesssim 1/N.$$

Bernstein's inequality gives

$$|x_N''(t) - \frac{1}{N} h'(t/N)| \lesssim 1/N^2.$$

Let $\{z_j\}_{n=1}^{n*}$ be the set of all zeros of h in [0,1], and let $0 < \alpha < 1$ be a fixed constant to be specified later. Define the intervals I_j and J_j by

$$\forall j = 1, \ldots, n^*, \; I_j = [Nz_j - N^\alpha, Nz_j + N^\alpha],$$

$$\forall j = 1, \ldots, n^* - 1, \; J_j = [Nz_j + N^\alpha, Nz_{j+1} - N^\alpha]$$

and

$$J_0 = [1, Nz_1 - N^\alpha] \text{ and } J_{n*} = [Nz_{n*} + N^\alpha, N].$$

Note that for large enough N, the union of these I_j and J_j is $[1, N]$.

$$\forall n \in \mathbb{N} \cap J_j,\ \frac{1}{N^{1-\alpha}} \lesssim |h'(n/N)|.$$

Thus,

$$\forall n \in \mathbb{N} \cap J_j,\ \frac{k}{\delta N^{2-\alpha}} \lesssim \left|\frac{k}{\delta} X_N''(n)\right|.$$

Notice that since h belongs to L^∞,

$$\forall n \in \mathbb{N} \cap J_j,\ \left|\frac{k}{\delta} X_N'(n)\right| \lesssim \frac{k}{\delta}.$$

Using these last two inequalities, we have that for all $k \geqslant 1$,

$$\left|\sum_{n \in \mathbb{N} \cap J_j} e^{2\pi i k \tilde{u}_n^N}\right| = \left|\sum_{n \in \mathbb{N} \cap J_j} e^{2\pi i (k/\delta) X_N(n)}\right| \lesssim$$

$$(2k/\delta + 2)\left(\frac{\delta^{\frac{1}{2}} N^{1-\frac{\alpha}{2}}}{k^{1/2}} + 1\right) \lesssim \frac{k^{1/2} N^{1-\frac{\alpha}{2}}}{\delta^{1/2}} + \frac{k}{\delta}.$$

Using the trivial estimate,

$$\left|\sum_{\mathbb{N} \cap J_j}^{j} e^{2\pi i k \tilde{u}_n^N}\right| \leqslant 2N^\alpha.$$

We have that

$$\left|\sum_{n=1}^{j} e^{2\pi i k \tilde{u}_n^N}\right| \lesssim N^\alpha + \frac{k^{1/2} N^{1-\frac{\alpha}{2}}}{\delta^{1/2}} + \frac{k}{\delta}.$$

Set $\alpha = 3/4$ and $K = N^{1/4}$. By our earlier bound on D_j^N, if N is sufficiently large compared to δ, then

$$D_j^N \leqslant \frac{1}{4} + \frac{K^{1/2} N^{1-\frac{\alpha}{2}}}{\delta^{1/2} j} + \frac{K}{\delta j} \lesssim$$

$$\frac{1}{N^{\frac{1}{4}}} + \frac{N^{3/4}\log(N)}{j} + \frac{N^{3/4}}{\delta^{1/2}j} + \frac{n^{1/4}}{\delta j} \lesssim \frac{1}{N^{\frac{1}{4}}} + \frac{N^{3/4}\log(N)}{j}.$$

Applying this to our bound on v_n^N, we have

$$|v_n^N| \leqslant \frac{\delta_n}{N^{1/4}} + \delta N^{3/4}\log N \lesssim \delta N^{3/4}\log N.$$

Our result is reached when this is used on our initial identity of $x - \tilde{x}_N$:

Corollary 7.17. *Let $\{F_N\}_{N=d}^{\infty}$ be a family of normalized tight frames for $\mathbb{R}^d$, for which each $F_N = \{e_n^N\}_{n=1}^N$ satisfies the zero sum condition. Let $x \in \mathbb{R}^d$ satisfy $||x|| \leqslant (K-1/2)\delta$, let $\{x_n^N\}_{n=1}^N$ be the frame coefficients of x with respect to F_N, and suppose there exists $h \in M_\Omega, \Omega > 0$, such that*

$$\forall N \text{ and } 1 \leqslant n \leqslant N, \quad x_n^N = h(n/N).$$

Suppose that $f_n^N = e_n^N - e_{n+1}^N$ satisfies

$$||f_n^N|| \lesssim \frac{1}{N} \text{ and } ||f_n^N - f_{n+1}^N|| \lesssim \frac{1}{N^2} \quad \text{for all } N, \text{ for all } 1 \leqslant n \leqslant N$$

and set $u_0^N = 0$ in the following $\Sigma\Delta$ estimate. If N is even and sufficiently large, we have

$$||x - \tilde{x}_N|| \lesssim \frac{\delta \log N}{N^{5/4}}.$$

If N is odd and sufficiently large, we have

$$\frac{\delta}{N} \lesssim ||x - \tilde{x}_N|| \lesssim \frac{\delta d}{2N}(\sigma(F_N, p_N) + 1).$$

The implicit constants are independent of δ and N, but do depend on x and hence h.

Proof. By the previous theorem,

$$\left\|\frac{2}{N}\left(\sum_{n=1}^{N-2} v_n^N (f_n^N - f_{n+1}^N)\right)\right\| \lesssim \frac{\delta \log(N)}{n^{5/4}}.$$

If N is even, then

$$||x - \tilde{x}|| \lesssim \frac{\delta \log N}{N^{5/4}}.$$

If N is odd, then

$$\frac{\delta}{N} = \frac{2|u_N^N| \, ||e_N^N||}{N^{5/4}}.$$

By theorem 7.11, the above fact gives us our bound. □

7.8 Güntürk's Approximating a bandlimited function using very coarsely quantized data

The basic means by which we draw an approximation on a given π-bandlimited function $x(t)$ is by first sampling our function with

$$S_\lambda : C(\mathbb{R}) \to \mathbb{R}^{\mathbb{Z}}$$
$$(S_\lambda x)_n = x(\frac{n}{\lambda}).$$

We then quantize the coefficients with Q and reconstruct using an interpolator operator,

$$T_{\lambda,\phi} : l^\infty \to C^\infty(\mathbb{R})$$
$$s \mapsto \frac{1}{\lambda} \sum_{n \in \mathbb{Z}} s_n \phi(t - \frac{n}{\lambda}).$$

This results in our defining our estimate on x as

$$\tilde{x}_\lambda = T_{\lambda,\phi} Q S_\lambda.$$

Since ϕ is defined to be Schwartz, we have a uniform bound on the average of its shifts, for all $\lambda > \lambda_0$. This and the fact that the difference between $S(x) - Q_\delta S(x)$ is bounded in L^∞ by δ gives convergence of $\tilde{x}$ to x in L^∞ as $\delta \to 0$.

Of course, this convergence relies in costly quantizations, so again we turn to $\Sigma\Delta$. Defining u_n and q_n as before with our quantizer, $Q = Heaviside(t-1)$. After some work, we find that

$$||x - \tilde{x}_\lambda||_{L^\infty} \leqslant \frac{1}{\lambda}\mathrm{Var}(\phi).$$

Given various empirical estimates, $\Sigma\Delta$ is hoped to have the decay rate of $\lambda^{-3/2}$. [9] makes progress towards this goal with

Theorem 7.18. *For all $\epsilon > 0$, there exists a family $\{\phi_\lambda\}_{\lambda>1}$ of reconstruction kernels such that for all pi-bandlimited functions x with range in $[0,1]$, and for all t for which $x'(t) \neq 0$,*

$$|x(t) - \tilde{x}_\lambda(t)| \ll_{\epsilon, x'(t)} \lambda^{-4/3+\epsilon}.$$

We use the following results in order to attain this bound.

Lemma 7.19. *There exist two absolute constants $C_1, C_2 > 0$ such that for all t at which $x'(t) \neq 0$, and for all intervals I and numbers λ satisfying $I \subset [-C_1|x'(t)|, C_1|x'(t)|]$, and $\lambda > \max(|I|^{-1}, C_2|x'(t)|^{-1})$, one has*

$$d(t, I, \lambda) \ll \frac{1}{\lambda^{1/3}} + \frac{1}{|I|\sqrt{|x'(t)|}}\frac{1}{\lambda^{1/2}}.$$

Proposition 7.20. *For each $\lambda > 1$, there exists an analytic function X_λ such that*

$$u_n^\lambda = X_\lambda(n)\,(mod\ 1)$$

and

$$||X'_\lambda - x(\frac{\cdot}{\lambda})||_{L^\infty}.$$

Corollary 7.21. *There exists two absolute constants $C_1, C_2 > 0$ such that for all t at which $x'(t) \neq 0$, and for all τ and λ satisfying $|\frac{\tau}{\lambda} - t| \leqslant C_1|x'(t)|$ and $\lambda > C_2|x'(t)|^{-1}$,*

$$\frac{1}{2}\frac{|x'(t)|}{\lambda} \leqslant |X''_\lambda(\tau)| \leqslant \frac{3}{2}\frac{|x'(t)|}{\lambda}.$$

The former theorem can be improved significantly for constant functions in the following.

Theorem 7.22. *Given the assumptions of theorem 1, if $x(t)$ is a constant function, then*

$$||e_\lambda||_{L^\infty} \ll_x \lambda^{-2}(\log\lambda)^{2+\epsilon}.$$

7.9 Yilmaz's Stability analysis for several second-order $\Sigma\Delta$ methods of coarse quantizaion of by bandlimited functions

Let $0 < \beta < 1$, f as before, and define the first order finite (leaky) memory scheme as follows:

$$u_n = \beta u_{n-1} + f_n^\lambda - q_n^\lambda$$

$$q_n^\lambda = sign(\beta u_{n-1} + f_n^\lambda).$$

This is equivalent to the previous $\Sigma\Delta$ scheme when $\beta = 1$, but when we have $\beta < 1$, the memory of the previous over and underestimations is lost.

This could be interpreted as a recognition that later elements of the frame are related to previous elements of the frame but not perfectly. Thus although it may be useful to compensate for an earlier over or underestimation by biasing the next estimate to the opposite tendency, the dissimilarity of the frame elements makes the bias less relevant. Reducing the bias by a multiplication of β reflects this fact in the new procedure.

Let g be such that

$$\hat{g}(\xi) = \begin{cases} \frac{1}{\sqrt{2\pi}}, & \text{if } |\xi| \leqslant \Omega \\ 0, & \text{if } |\xi| \geqslant \lambda\Omega. \end{cases}$$

Theorem 7.23. *Let $f \in L^2(\mathbb{R})$ be bandlimited with $supp\hat{f} \subset [-\pi,\pi]$ and $||f||_{L^\infty} \leqslant 1$. Let g be a function satisfying the above with $\Omega = \pi$. Let the* *leakage factor be* $\beta = e^{-\frac{c}{\lambda}}$. *Assume that the sequence* (v_n) *generated by our finite memory scheme is bounded. Then if* (q_n^λ) *is the output of the first-order leaky* $\Sigma\Delta$*-quantizer given above, then*

$$|f(t) - \tilde{f}(t)| \leqslant \frac{||v||_{l^\infty}}{\lambda}(||g'||_{L^1} + cC_g)$$

where C_g *is as before with* $\Omega = \pi$, *and* $\tilde{f}(t) = \frac{1}{\lambda}\sum_n^\lambda g(t - \frac{n}{\lambda})$.

The second order finite memory scheme is defined as

$$u_n = \beta u_{n-1} + f_n^\lambda - q_n^\lambda$$

$$v_n = \beta v_{n-1} + u_n$$

$$q_n^\lambda = sign(F(\beta u_{n-1}, \beta v_{n-1})).$$

Theorem 7.24. *Let* f, g *and* β *be as in Theorem 7.11. Assume that* (v_n), *generated by the above scheme is bounded. Then if* (q_n^λ) *is the output of the second-order leaky* $\Sigma\Delta$*-quantizer given above, then*

$$|f(t) - \tilde{f}(t)| \leqslant \frac{||v||_{l^\infty}}{\lambda^2}(||g''||_{L^1} + 2c||g'||_{L^1} + 2c^2C_g),$$

where C_g *is as before and* $\tilde{f}(t) = \frac{1}{\lambda}\sum q_n^\lambda g(t - \frac{n}{\lambda})$.

We define the tri-level finite memory second-order $\Sigma\Delta$-quantizer as we did the finite memory scheme above except for the last line which we make

$$q_n^\lambda = m(\eta F(\beta u_{n-1}, \beta v_{n-1})).$$

where $\eta > 0$ is fixed, and F and η are chosen in such a manner that the sequences u and v stay bounded.

Proposition 7.25. *Consider the tri-level finite memory second-order* $\Sigma\Delta$*-quantizer with* $F(u, v) = u + \gamma v$. *Let the input sequence* (x_n) *be identically equal zero for* *all $n \geqslant N$ for some N, $q_N = 0$ and*

$$|u_{N-1}| < \frac{1-\beta}{2\eta\gamma\beta_\lambda^2}.$$

Then, $q_n = 0$ for all $n \geqslant N$.

Let us rewrite our sequence as

$$(u_n, v_n) = \begin{cases} S_l^{\delta_n}(u_{n-1}, v_{n-1}) = (u_{n-1} - \delta_n, u_{n-1} + v_{n-1} - \delta_n), & \text{if } q_n = 1 \\ S_l^{\delta_n}(u_{n-1}, v_{n-1}) = (u_{n-1} + \delta_n, u_{n-1} + v_{n-1} + \delta_n), & \text{if } q_n = -1 \end{cases}$$

$$q_n = sign(F(u_{n-1}, v_{n-1})),$$

which may be alternatively written as

$$(u_n, v_n) = S(u_{n-1}, v_{n-1}, \delta_n).$$

Define the functions

$$B_1(u) = \begin{cases} -\frac{1}{\delta_-}(u - \frac{\delta_-}{2})^2 + \frac{\delta_-}{8} + C, & \text{if } u \geqslant 0 \\ -\frac{1}{\delta_+}(u - \frac{\delta_+}{2})^2 + \frac{\delta_+}{8} + C, & \text{if } u < 0. \end{cases}$$

$$B_2(u) = \begin{cases} -\frac{1}{\delta_+}(u + \frac{\delta_+}{2})^2 - \frac{\delta_+}{8} - C, & \text{if } u \geqslant 0 \\ -\frac{1}{\delta_-}(u + \frac{\delta_-}{2})^2 - \frac{\delta_1}{8} - C, & \text{if } u < 0. \end{cases}$$

Define

$$R_1 = (u, v) : v \leqslant B_1(u), v \geqslant B_2(u), v \geqslant l(u),$$

$$R_2 = (u, v) : v \leqslant B_1(u), v \geqslant B_2(u), v \leqslant l(u) \quad \text{and}$$

$$R = R_1 \cup R_2.$$

Theorem 7.26. *Let $P_1 = (u_1, v_1)$ be the intersection point of the line L, defined* *by* $F(u,v) = u + \gamma v = 0$*, and* Γ_{B_1}*, i.e.* $P_1 = (L \cap \Gamma_{B_1})_<$*. Suppose*

$$u_0 + \delta_+ \leqslant u_1 \leqslant -\delta_+.$$

Then, $S_l^\delta(R_1) \subseteq R$*, for any* $\delta \in [\delta_-, \delta_+]$.

We will need the following conditions for the following theorems

$$C \geqslant 2\frac{1+\alpha}{1-\alpha}.$$

The range of γ for a given $C \geqslant 2\frac{1+\alpha}{1-\alpha}$ is:

$$\frac{1}{\gamma} \geqslant \frac{[2C(1-a^2)]^{1/2} + 2\alpha C}{2\{[2C(1-a^2)]^{1/2} - (1+\alpha)\}}$$

and

$$\frac{1}{\gamma} \leqslant \frac{C-(1+\alpha)}{1+\alpha}.$$

Theorem 7.27. *Let* S *be the mapping defined above with the rule* $F(u,v) = u + \gamma v$*. Suppose* C *and* γ *satisfy the conditions just stated for some* $\alpha < 1$*. Then, the set* R *is positively invariant under* $S(\cdot, c; \delta)$ *for any* $\delta \in [1-\alpha, 1+\alpha]$*. Equivalently,* $S(u,v,\delta) \in R$ *for any* $(u,v) \in R$ *and* $\delta \in [1-\alpha, 1+\alpha]$.

Corollary 7.28. *Let* (x_n) *be an arbitrary sequence such that* $|x_n| \leqslant \alpha < 1$*. Suppose* $(u_0, v_0) \in R$ *and* (u_n, v_n) *are obtained via the standard second order scheme with* $F(u,v) = u+\gamma v$*. If* C *and* γ *satisfy the above conditions,* $(u_n, v_n) \in R$ *for all* n*; thus* $|v_n| < C$ *for all* n.

Theorem 7.29. *Let* $F(u,v) = u + \gamma v$ *be a given quantization rule with* γ *satisfying the above conditions with strict inequalities for some* $C > 2\frac{1+\alpha}{1-\alpha}$*. Let* u_0 *be such that* $\delta_+ - \delta \leqslant B_1(u-\delta) - B_1(u-\delta_+)$ *and let* (u_1, v_1) *be the intersection of* L *with* Γ_{B_1}*, as in theorem 7.18. Take* ϵ *such that*

$$|\epsilon| < \min\{u_1(\alpha,\gamma,C) - u_0(\alpha,C) - (1+\alpha); -u_1(\alpha,\gamma,C) - (1+\alpha)\},$$

with

$$u_0(\alpha, C) = -[2C(1-a^2)]^{1/2}$$

and

$$u_1(\alpha, \gamma, C) = (1+\alpha)\left(\frac{\gamma+2}{2\gamma} - \left[(\frac{\gamma+2}{2\gamma})^2 + \frac{2C}{1+\alpha}\right]^{1/2}\right).$$

Then, the second-order $\Sigma\Delta$*-scheme with the quantization rule* $F^{\epsilon}(u, v) = u + \gamma v + \epsilon$ *is stable with the positively invariant set* R*.*

References

[1] C. Heil, J. Ramanathan, and P. Topiwala. Linear independence of time-frequency translates. *Proc. Amer. Math. Soc.*, 124(9):2787–2795, 1996.

[2] M. Bownik and D. Speegle. Linear independence of time-frequency translates of functions with faster than exponential decay. *Bull. Lond. Math. Soc.*, 45(3):554–566, 2013.

[3] J. Benedetto and A. Bourouihiya. The HRT conjecture for functions with certain behavior at infinity. *J. Geom. Anal.*, 25(2):226–254, 2015.

[4] P. A. Linnell. von Neumann algebras and linear independence of translates. *Proc. Amer. Math. Soc.*, 127(11):3269–3277, 1999.

[5] C. Demeter. Linear independence of time frequency translates for special configurations. *Math. Res. Lett.*, 17(4):761–779, 2010.

[6] C. Demeter and A. Zaharescu. Proof of the HRT conjecture for (2,2) configurations. *J. Math. Anal. Appl.*, 388(1):151–159, 2012.

[7] K. A. Okoudjou. Extension and restriction principles for the HRT conjecture. *J. Fourier Anal. Appl.*, 23(4):1874–1901, 2019.

[8] B. Chazelle. *The Discrepancy Method.* Cambridge University Press, Cambridge, MA, 2000.

[9] C. S. Güntürk. Approximating a bandlimited function using very coarsely quantized data: Improved error estimates in sigma-delta modulation. *J. Amer. Math. Soc.*, 17(1):229–242, 2003.

[10] L. Auslander and R. Tolimieri. Radar ambiguity functions and group theory. *SIAM Journal on Mathematical Analysis*, 16(3):577–601, 1985.

[11] J. Benedetto and W. Czaja. *Integration and Modern Analysis.* Birkhäuser, Cambridge, MA, 2009.

[12] J. Benedetto, Powell A., and Yilmaz O. Sigma-Delta quantization and finite frames. *IEEE Inform. Theory*, 52:1990–2005, 2006.

[13] M. Bownik and D. Speegle. Linear independence of time-frequency translates in $\mathbb{R}^d$. *J. Geom. Anal.*, 26(3):1678–1692, 2016.

[14] Heil C. Linear Independence of Finite Gabor Systems. In C. Heil, editor, *Harmonic analysis and applications*, pages 171–206. Birkhäuser Boston, Boston, MA, 2006.

[15] C. Wilcox. *The Synthesis Problem for Radar Ambiguity Functions*, volume 1. Springer-Verlag, 1991.

[16] C. Demeter and Z.S. Gautam. On the finite linear independence of lattice Gabor systems. *Proc. Amer. Math. Soc.*, 141(5):1735–1747, 2013.

[17] H. F. Bueckner. *Signals having the same ambiguity functions.* Technical Report 67-C-456, General Electric, Research and Development Center, Schnectady, NY, 1967.

[18] J. Munkres. *Topology.* Prentice Hall, Upper Saddle River, NJ, 1974.

[19] J. J. Benedetto and O. Oktay. Pointwise comparison of pcm and sigma-delta quantization. *Constructive Approximation*, pages 131–158, 2010.

[20] P. Jaming. Phase retrieval techniques for radar ambiguity problems. *Journal of Fourier Analysis and Applications*, 5:313–333, 1999.

[21] W. Liu. Short proof of the HRT conjecture for almost every (1, 3) configuration. *J. Fourier Anal. Appl.*, 25(4):1350–1360, 2019.

[22] J. Benedetto. *Harmonic Analysis and Applications.* CRC Press, Boca Raton, FL, 1997.

[23] P. Fitzpatrick and H. Royden. *Real Analysis.* Pearson, New York, NY, 2010.