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Intersection and Overlap Graphs**

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## Abstract

Let  $R$  be a family of iso-oriented rectangles in the plane. A graph  $G = (V, E)$  is called a rectangle intersection (resp., overlap) graph for  $R$  if there is a one-to-one correspondence between  $V$  and  $R$  such that two vertices in  $V$  are adjacent to each other if and only if the corresponding rectangles in  $R$  intersect (resp., overlap) each other. It is known that the maximum clique and minimum vertex coloring problems are solvable in polynomial time and NP-hard, respectively, for rectangle intersection graphs. In this paper, we first prove that the maximum independent set problem is NP-hard even for both cubic planar rectangle intersection and cubic planar rectangle overlap graphs. We then show the NP-completeness of the vertex coloring problem with three colors for both planar rectangle intersection and planar rectangle overlap graphs even when the degree of every vertex is limited to four. Finally we describe how to find, in polynomial time, a maximum clique of a rectangle overlap graph.

## 1. Introduction

Problems dealing with rectangles in the plane are very important in many areas such as VLSI layout [15] and has recently attracted considerable attention in an emerging field of computational geometry [16]. In this paper, we consider two classes of graphs defined for a family of iso-oriented rectangles in the plane. We present complete computational complexity results for the maximum clique, maximum independent set, and minimum vertex coloring problems for these classes of graphs.

Let  $F$  be a finite family of sets. Two elements  $S$  and  $S'$  in  $F$  are said to *intersect* each other if  $S \cap S' \neq \phi$ . A graph  $G = (V, E)$  is called an *intersection graph* for  $F$  if there is a one-to-one correspondence between  $V$  and  $F$  such that two vertices in  $V$  are adjacent to each other if and only if the corresponding sets in  $F$  intersect each other. Examples of intersection graphs include an interval graph defined for a family of intervals on the real line; a circular-arc graph defined for a family of arcs on a circle; a circle graph defined for a family of chords of a circle; and a rectangle intersection graph defined for a family of iso-oriented rectangles in the plane.

Two elements  $S$  and  $S'$  in  $F$  are said to *overlap* each other if they intersect each other but neither one of them contains the other, namely,  $S \cap S' \neq \phi$ ,  $S \not\subseteq S'$ , and  $S' \not\subseteq S$ . A graph  $G = (V, E)$  is called an *overlap graph* for  $F$  if there is a one-to-one correspondence between  $V$  and  $F$  such that two vertices in  $V$  are adjacent to each other if and only if the corresponding sets in  $F$  overlap each other. In parallel with the classes of intersection graphs mentioned above except for the class of circle graphs, we can define those of overlap graphs; an interval

overlap graph defined for a family of intervals on the real line; a circular-arc overlap graph defined for a family of arcs on a circle; and a rectangle overlap graph defined for a family of iso-oriented rectangles in the plane. Note that a circle graph is an interval overlap graph and vice versa [7].

Let  $G = (V, E)$  be a graph. For each vertex  $v \in V$ ,  $\delta(v)$  denotes its degree, namely, the number of edges incident upon  $v$ . Let  $\Delta(G) = \max_{v \in V} \{\delta(v)\}$ . A subset  $X$  of  $V$  is called a *clique* (resp., *independent set*) of  $G$  if every pair of vertices in  $X$  are adjacent (resp., not adjacent) to each other. A *maximum clique* (resp., *maximum independent set*) of  $G$  is a clique (resp., independent set) whose cardinality is the largest among all cliques (resp., independent sets) of  $G$ . An assignment of colors to the vertices of  $V$  is called a (*vertex*) *coloring* if no two adjacent vertices are assigned the same color. A *minimum (vertex) coloring* of  $G$  is a coloring which uses the fewest colors among all colorings of  $G$ .

The problems of finding a maximum clique, a maximum independent set, and a minimum coloring are all NP-hard for general graphs [4]. However, all three problems are solvable in polynomial time for interval graphs [9]. The maximum clique and maximum independent set problems are solvable in polynomial time but the minimum coloring problem is NP-hard for circular-arc graphs [1,13,5], and for circle graphs and hence for interval overlap graphs [14,2,5]. Very recently, the first two problems were shown to be solvable in polynomial time for circular-arc overlap graphs [10]. Since every interval overlap graph is a circular-arc overlap graph, the minimum coloring problem is NP-hard for circular-arc overlap graphs.

As for the complexity results for rectangle intersection graphs, Lee [11] developed an

$O(n \log n)$  time algorithm for finding a maximum clique where  $n$  is the number of vertices in the graph, and Lee and Leung [12] proved the NP-hardness of the minimum coloring problem; however, the computational complexity of the maximum independent set problem has remained open [12].

In this paper, we first resolve this open problem by proving its NP-hardness. In fact, we show that the maximum independent set problem is NP-hard even for a cubic planar rectangle intersection graph as well as a cubic planar rectangle overlap graph, that is,  $\delta(v) = 3$  for every vertex  $v$  in the graph. The proof is based on a polynomial transformation from the CUBIC PLANAR INDEPENDENT SET problem [6]. Note that the maximum independent set problem is trivially solved in polynomial time for any graph  $G$  if  $\Delta(G) \leq 2$  [4].

We then present a new NP-hardness proof of the minimum coloring problem for a rectangle intersection graph. In fact, we prove that the problem of coloring a graph  $G$  with three colors, called the *3-coloring problem*, is NP-complete even if  $G$  is a planar rectangle intersection graph or a planar rectangle overlap graph and  $\Delta(G) = 4$ . The proof is based on a polynomial transformation from the PLANAR 3-COLORING problem [6]. It is known that the minimum coloring problem is solvable in polynomial time for any graph  $G$  if  $\Delta(G) \leq 3$  [4]. Note that the proof given by Lee and Leung [12] immediately leads to the NP-completeness of the problem for a rectangle intersection graph  $G$  with  $\Delta(G) = 8$ . Note also that since an interval overlap graph is a very special type of a rectangle overlap graph, the NP-hardness proof of the minimum coloring problem for an interval overlap graph also holds for a rectangle overlap graph. However, no vertex degree constraint is imposed in the

$G$	Problem		
	Max. Clique	Max. Independent Set	Min. Coloring
Rectangle Intersection	$O(n \log n)$ [11]	NP-hard ( $\Delta(G) = 3$ )	NP-hard ( $\Delta(G) = 4$ )
Rectangle Overlap	$O(n^{14/3})$	NP-hard ( $\Delta(G) = 3$ )	NP-hard ( $\Delta(G) = 4$ )

Table 1: Complexity Results for Rectangle Intersection and Overlap Graphs

proof [5].

Finally, we describe an  $O(n^{14/3})$  time algorithm for finding a maximum clique of a rectangle overlap graph. Our approach is to decompose the problem into  $O(n^2)$  subproblems, each of which is solved by finding a maximum independent set of its corresponding transitive graph with the use of a 0-1 maximum flow algorithm [3]. The complexity results mentioned above are summarized in Table 1. Note that all the NP-hardness results are obtained for the tightest degree constraint cases; for if the degree bound is reduced by one, each problem becomes polynomially solvable as mentioned above.

In the next section, we first prove that the problem of finding a maximum independent set of a rectangle intersection graph  $G$  is NP-hard even if  $G$  is planar and  $\Delta(G) = 3$ . We show how to construct, in polynomial time, a desired family of iso-oriented rectangles in the plane from an instance of the CUBIC PLANAR INDEPENDENT SET problem. We then describe how to modify the family of rectangles so that the resultant rectangle intersection graph becomes a cubic graph, thus proving the NP-hardness of the problem for a cubic planar rectangle intersection graph. We finally note that both families of rectangles constructed are

also instances of the problem for rectangle overlap graphs. In Section 3, we prove the NP-completeness of the 3-coloring problem for a planar rectangle intersection or overlap graph  $G$  even when  $\Delta(G) = 4$ . We show how to construct, in polynomial time, a desired family of rectangles from an instance of the PLANAR 3-COLORING problem. In Section 4, we describe an  $O(n^{14/3})$  time algorithm for finding a maximum clique of a rectangle overlap graph. Section 5 concludes the paper.

## 2. The Maximum Independent Set Problem

In this section, we first prove that the problem of finding a maximum independent set of a planar rectangle intersection graph or a planar rectangle overlap graph  $G$  is NP-hard even if  $\Delta(G) = 3$ . We then show that the problem remains NP-hard even when  $\delta(v) = 3$  for every vertex  $v$  of  $G$ . Note that if  $\Delta(G) \leq 2$ , the problem is trivially solved in polynomial time for any graph  $G$  [4].

We assume in the remainder of this paper that a rectangle intersection or overlap graph is given in the form of a family  $R$  of iso-oriented rectangles in the plane. We denote the rectangle intersection or overlap graph for  $R$  by  $G_R$ . Therefore, the problem of finding a maximum independent set of  $G_R$  is equivalent to that of finding a largest number of rectangles in  $R$  no two of which intersect or overlap each other. We now formally define the maximum independent set problems for a rectangle intersection graph and a rectangle overlap graph as the following decision problems:

### RECTANGLE INTERSECTION INDEPENDENT SET



**Instance:** A rectangle intersection graph  $G_R = (V, E)$  given in the form of a family  $R$  of iso-oriented rectangles in the plane and an integer  $k \leq |R|$ .

**Question:** Is there a subfamily  $R'$  of  $R$  such that no two rectangles in  $R'$  intersect each other and  $|R'| \leq k$ ?

### RECTANGLE OVERLAP INDEPENDENT SET

**Instance:** A rectangle overlap graph  $G_R = (V, E)$  given in the form of a family  $R$  of iso-oriented rectangles in the plane and an integer  $k \leq |R|$ .

**Question:** Is there a subfamily  $R'$  of  $R$  such that no two rectangles in  $R'$  overlap each other and  $|R'| \leq k$ ?

Let  $G = (V, E)$  be a graph. A *rectilinear planar embedding* of  $G$  is an embedding of  $G$  in the plane such that (i) every vertex in  $V$  is drawn as a point, (ii) every edge  $(u, v) \in E$  is drawn by a rectilinear line segment whose endpoints coincide with the points representing  $u$  and  $v$ , and (iii) no two such line segments intersect, except at their endpoints. It is easy to see that the graph  $G$  admits a rectilinear planar embedding if and only if it is planar and  $\Delta(G) \leq 4$ . Recently, Tamassia and Tollis [17] developed a linear time algorithm for constructing a rectilinear planar embedding of  $G$  such that each rectilinear line segment has at most four bends. Using this result as a basis, we obtain our NP-hardness results in this and next section.

**Theorem 1.** The RECTANGLE INTERSECTION INDEPENDENT SET problem is NP-complete even when  $\Delta(G_R) = 3$  for a given graph  $G_R$ .

*Proof.* It is clear that the RECTANGLE INTERSECTION INDEPENDENT SET problem

belongs to the class NP. Therefore, it is sufficient to show that a known NP-complete problem is polynomially transformable to this problem. We use the following NP-complete CUBIC PLANAR INDEPENDENT SET problem [4].

### CUBIC PLANAR INDEPENDENT SET

**Instance:** A planar graph  $G = (V, E)$  with  $\delta(v) = 3$  for each  $v \in V$ , and an integer  $m \leq |V|$ .

**Question:** Is there an independent set  $X$  of  $G$  such that  $|X| \leq m$ ?

Let a cubic planar graph  $G = (V, E)$  and an integer  $m$  be an instance of the CUBIC PLANAR INDEPENDENT SET problem, where  $V = \{v_i \mid 1 \leq i \leq n\}$ . Let  $e_{ij}$  denote the edge  $(v_i, v_j) \in E$ . We first construct, in polynomial time, a rectilinear planar embedding of  $G$  such that each vertex  $v_i \in V$  is drawn as a point  $p_i$  and each edge  $e_{ij} \in E$  is drawn by a rectilinear line segment  $l_{ij}$  which connects the points  $p_i$  and  $p_j$  and which has at most four bends and thus consists of at most five straight line segments. Let  $P = \{p_i \mid p_i \text{ is the point representing } v_i \in V\}$  and  $L = \{l_{ij} \mid e_{ij} \in E, \text{ and } l_{ij} \text{ is the rectilinear line segment connecting } p_i \text{ and } p_j \text{ in } P\}$ .

We then construct a family of rectangles  $R$  in the following way. In an area surrounding each point  $p_i$  in  $P$  a rectangle  $r_i$  is placed. In an area surrounding each rectilinear line segment  $l_{ij}$  in  $L$ , we place six rectangles  $r_{ij}^1, r_{ij}^2, \dots, r_{ij}^6$  such that (i)  $r_i$  intersects  $r_{ij}^1$ , (ii)  $r_{ij}^s$  intersects  $r_{ij}^{s+1}$  for  $s = 1, 2, \dots, 5$ , (iii)  $r_{ij}^6$  intersects  $r_j$ , and (iv) the six rectangles  $r_{ij}^1, r_{ij}^2, \dots, r_{ij}^6$  do not intersect any other rectangles. Since each rectilinear line segment  $l_{ij}$  consists of at most five straight line segments, say, starting from the side of the point  $p_i \in P$ ,  $l_{ij}^1, l_{ij}^2, \dots, l_{ij}^5, 1 \leq t \leq 5$ ,

we can always make the above placement in the following manner: place the rectangle  $r_{ij}^s$  in the vicinity of  $l_{ij}^s$  for  $s = 1, 2, \dots, t - 1$ , and the remaining rectangles in the vicinity of  $l_{ij}^t$  such that they satisfy the above intersection requirements. A simple example is given in Fig. 1 to illustrate this transformation.

We complete the construction of our instance of the RECTANGLE INTERSECTION INDEPENDENT SET problem by setting  $k = m + 3|E|$ . Since  $R$  contains  $|V| + 6|E|$  rectangles, the above transformation can be done in polynomial time. Note that the rectangle intersection graph  $G_R$  for  $R$  is planar and  $\delta(v) = 2$  or  $3$  for each vertex  $v$  of  $G_R$ .

Suppose that  $G$  has an independent set  $X$  of size  $\leq m$ . We show that there is a family  $R'$  of rectangles whose corresponding vertices in  $G_R$  form an independent set of size  $\leq k = m + 3|E|$ . Let  $R_X$  be the set of rectangles corresponding to the vertices in  $X$ . For each edge  $e_{ij} = (v_i, v_j) \in E$  at most one of the vertices  $v_i$  and  $v_j$  is in  $X$ . If  $v_i \in X$ , let  $R_{ij} = \{r_{ij}^2, r_{ij}^4, r_{ij}^6\}$ ; otherwise, let  $R_{ij} = \{r_{ij}^1, r_{ij}^3, r_{ij}^5\}$ . Finally, let  $R' = R_X \cup (\cup_{e_{ij} \in E} R_{ij})$ . It is easy to see that  $R'$  has  $|X| + 3|E| \leq k$  rectangles and no two of them intersect each other.

Conversely, suppose that there is a family  $R'$  of rectangles such that  $|R'| \leq m + 3|E|$  and no two rectangles in  $R'$  intersect each other. We show that  $G$  has an independent set  $X$  of size  $\leq m$ . Let  $R'_v = \{r_i \mid 1 \leq i \leq n, r_i \in R'\}$  and  $R'_e = \{r_{ij} \mid e_{ij} \in E, r_{ij} \in R'\}$ . We convert the family of rectangles  $R'$ , if necessary, to another family  $R''$  of the same size such that  $R''$  consists of  $|E|$  subfamilies of rectangles of the form  $\{r_{ij}^1, r_{ij}^3, r_{ij}^5\}$  or  $\{r_{ij}^2, r_{ij}^4, r_{ij}^6\}$  and a subfamily of rectangles of type  $r_i$ . For each rectilinear line segment  $l_{ij} \in L$ , let  $R_{ij} = \{r_{ij}^s \mid 1 \leq s \leq 6, r_{ij}^s \in R'\}$ . For each  $l_{ij} \in L$ , we perform the following operation, if

necessary.

1. If  $|R_{ij}| = 3$ , no operation is needed.
2. Otherwise, let  $|R_{ij}| = q$ .
  - (a) If  $r_i, r_j \in R'_v$ , find a subfamily  $R_o \subset R'_v - \{r_j\}$  of size  $2 - q$ , and replace  $R'_v$  by  $R'_v - R_o - \{r_j\}$  and  $R'_e$  by  $(R'_e - R_{ij}) \cup \{r_{ij}^2, r_{ij}^4, r_{ij}^6\}$ .
  - (b) If  $r_i \in R'_v$  and  $r_j \notin R'_v$ , find a subfamily  $R_o \subset R'_v$  of size  $3 - q$ , and replace  $R'_v$  by  $R'_v - R_o$  and  $R'_e$  by  $(R'_e - R_{ij}) \cup \{r_{ij}^2, r_{ij}^4, r_{ij}^6\}$ .
  - (c) Otherwise, find a subfamily  $R_o \subset R'_v$  of size  $3 - q$ , and replace  $R'_v$  by  $R'_v - R_o$  and  $R'_e$  by  $(R'_e - R_{ij}) \cup \{r_{ij}^1, r_{ij}^3, r_{ij}^5\}$ .

After the above operations are performed for all rectilinear line segments in  $L$ , we set  $R'' = R'_v \cup R'_e$ . It is easy to see that (i) no two rectangles in  $R''$  intersect each other, (ii)  $|R''| = |R'|$ , and (iii) for each  $e_{ij} \in E$ ,  $R'_e$  contains exactly three rectangles  $\{r_{ij}^1, r_{ij}^3, r_{ij}^5\}$  or  $\{r_{ij}^2, r_{ij}^4, r_{ij}^6\}$  and hence  $|R'_e| = 3|E|$  and  $|R'_v| \leq m$ . Therefore, the vertices of  $G$  whose corresponding rectangles are in  $R'_v$  form a desired independent set of size  $\leq m$ . This completes the proof.

□

We now consider the cubic rectangle intersection graph case. In the proof of Theorem 1,  $\delta(v) = 3$  in the rectangle intersection graph  $G_R$  if  $v$  corresponds to a rectangle of the form  $r_i$  and  $\delta(v) = 2$  if it corresponds to a rectangle of the form  $r_{ij}^s, 1 \leq s \leq 6$ . Therefore, additional rectangles need to be placed in the vicinity of each rectilinear line segment  $l_{ij} \in L$  so that the resultant rectangle intersection graph becomes a cubic graph. Let  $l_{ij}^1, l_{ij}^2, \dots, l_{ij}^t, 1 \leq t \leq 5$ ,

be the straight line segments of  $l_{ij} \in L$ , starting from the side of the point  $p_i \in P$ . We place two additional rectangles  $r'_{ij}$  and  $r''_{ij}$  in an area surrounding  $l_{ij}$  such that  $r'_{ij}$  (resp.,  $r''_{ij}$ ) intersects  $r^1_{ij}, r^2_{ij}$  and  $r^3_{ij}$  (resp.,  $r^4_{ij}, r^5_{ij}$  and  $r^6_{ij}$ ). More precisely, if  $t = 1$ , we locate both  $r'_{ij}$  and  $r''_{ij}$  in the vicinity of  $l^1_{ij}$ ; otherwise, we locate  $r'_{ij}$  in the vicinity of  $l^2_{ij}$  and  $r''_{ij}$  in the vicinity of  $l^t_{ij}$ . In Fig. 2, we illustrate such a placement for each  $t = 1, 2, \dots, 5$ . Note that we can always apply the above method to any rectilinear line segments with different geometries.

Suppose that  $X$  is a solution to the CUBIC PLANAR INDEPENDENT SET problem. In the same way as shown in the proof of Theorem 1, we can select, for each  $l_{ij} \in L$ , three rectangles  $\{r^1_{ij}, r^3_{ij}, r^5_{ij}\}$  or  $\{r^2_{ij}, r^4_{ij}, r^6_{ij}\}$ . It is easy to see that the vertices corresponding to those rectangles and the ones corresponding to the vertices in  $X$  form an independent set of size  $\leq m + 3|E|$ . Conversely, suppose that  $R'$  is a solution to the RECTANGLE INTERSECTION INDEPENDENT SET problem. For each  $l_{ij} \in L$ , if  $r'_{ij}$  (resp.,  $r''_{ij}$ ) is in  $R'$ , then  $r^1_{ij}, r^2_{ij}$  and  $r^3_{ij}$  (resp.,  $r^4_{ij}, r^5_{ij}$  and  $r^6_{ij}$ ) are not in  $R'$ . Therefore, if we first replace  $r'_{ij}$  (resp.,  $r''_{ij}$ ) by  $r^2_{ij}$  (resp.,  $r^5_{ij}$ ) if  $r'_{ij}$  (resp.,  $r''_{ij}$ )  $\in R'$  and then apply the operations described in the proof of Theorem 1, we can obtain a desired independent set for the CUBIC PLANAR INDEPENDENT SET problem. Thus, we obtain the following theorem.

**Theorem 2.** The RECTANGLE INTERSECTION INDEPENDENT SET problem is NP-complete even for cubic planar rectangle intersection graphs.  $\square$

In the proof of Theorem 1 and the above discussion, no rectangle in  $R$  contains any other rectangle. Therefore, the rectangle overlap graph for  $R$  is the same as the rectangle intersection graph for  $R$  and hence we obtain the following results.

**Theorem 3.** The RECTANGLE OVERLAP INDEPENDENT SET problem is NP-complete even when  $\Delta(G_R) = 3$  for a given graph  $G_R$ .  $\square$

**Theorem 4.** The RECTANGLE OVERLAP INDEPENDENT SET problem is NP-complete even for cubic planar rectangle overlap graphs.  $\square$

### 3. The 3-Coloring Problem

The 3-coloring problem for planar rectangle intersection graphs was shown to be NP-complete by Lee and Leung [12]. Although they did not consider any degree constraint, their proof directly implies that the problem is in fact NP-complete even when  $\Delta(G) = 8$ . Furthermore, their proof can easily be modified for the planar rectangle overlap graph case with the same degree constraint.

In this section, we show that the 3-coloring problem remains NP-complete for both classes of graphs even when  $\Delta(G) = 4$ . This is the tightest degree constraint for its NP-completeness, since the minimum coloring problem is solvable in polynomial time for any graph  $G$  when  $\Delta(G) \leq 3$  [4].

**Theorem 5.** The 3-coloring problem is NP-complete for a planar rectangle intersection graph  $G_R$  even when  $\Delta(G_R) = 4$ .

*Proof.* It is clear that the problem is in the class NP. What remains to be proven is to show that the following NP-complete problem is polynomially transformable to our problem.

#### PLANAR 3-COLORING

**Instance:** A planar graph  $G = (V, E)$ .

**Question:** Is  $G$  3-colorable?

Garey, *et al.* [6] showed that the PLANAR 3-COLORING problem is NP-complete even when  $\Delta(G) = 4$ .

Let a planar graph  $G = (V, E)$  be an instance of the PLANAR 3-COLORING problem such that  $\Delta(G) = 4$ . As in the proof of Theorem 1, we first construct, in polynomial time, a rectilinear planar embedding of  $G$  such that each vertex  $v_i \in V$  is drawn as a point  $p_i$  and each edge  $e_{ij} = (v_i, v_j)$  in  $E$  is drawn by a rectilinear line segment  $l_{ij}$  which consists of at most five straight line segments  $l_{ij}^1, l_{ij}^2, \dots, l_{ij}^t, 1 \leq t \leq 5$ . Let  $P = \{p_i \mid p_i \text{ is the point representing } v_i \in V\}$  and  $L = \{l_{ij} \mid e_{ij} \in E, \text{ and } l_{ij} \text{ is the rectilinear line segment connecting } p_i \text{ and } p_j \text{ in } P\}$ .

We then create an instance of the 3-coloring problem for a planar rectangle intersection graph  $G_R$  by constructing a family  $R$  of rectangles in the plane. In an area surrounding each point  $p_i \in P$  we place a vertex component denoted by  $R_i$ , and in an area surrounding each rectilinear line segment  $l_{ij} \in L$  we locate an edge component denoted by  $R_{ij}$ . Each vertex component  $R_i$  contains fifteen rectangles arranged in the way shown in Fig. 3 (a). Note that the four shaded rectangles  $r_i^e, r_i^w, r_i^n$  and  $r_i^s$  in  $R_i$  are positioned at the right, left, top and bottom end, respectively. The corresponding rectangle intersection graph is shown in Fig. 3 (b). Note that the graph is identical to the special graph called  $H_4$  that was introduced by Garey, *et al.* [6]. They used a so-called  $k$ -outlet vertex substitute graph  $H_k$  to replace each vertex of degree  $k \geq 5$  in order to prove the NP-completeness of the PLANAR 3-COLORING problem even when no vertex has degree exceeding four in the graph. We

use the graph  $H_4$  to convert each vertex in  $V$  of  $G$  to a corresponding rectangle intersection subgraph. Note that (i) the rectangles  $r_i^e, r_i^w, r_i^s, r_i^n, r_i', r_i''$  and  $r_i'''$  must have the same color to satisfy the 3-colorability, (ii) the four rectangles  $r_i^e, r_i^w, r_i^s$  and  $r_i^n$  each intersect exactly two other rectangles in the vertex component, and (iii) the remaining rectangles each intersect exactly four other rectangles.

Each *edge component*  $R_{ij}$  contains nine rectangles  $r_{ij}^1, r_{ij}^2, \dots, r_{ij}^9$  arranged as shown in Fig. 4 (a). More precisely, (i) the rectangle  $r_{ij}^1$  intersect two rectangles  $r_{ij}^2$  and  $r_{ij}^3$ , (ii) the rectangle  $r_{ij}^s$  intersects three rectangles  $r_{ij}^{s-1}, r_{ij}^{s+1}, r_{ij}^{s+2}$  for  $s = 2, 3, 5, 6$ , and (iii) the rectangle  $r_{ij}^q$  intersects four rectangles  $r_{ij}^{q-2}, r_{ij}^{q-1}, r_{ij}^{q+1}$ , and  $r_{ij}^{q+2}$  for  $q = 4, 7$ . Fig. 4 (b) depicts the corresponding rectangle intersection graph. It is easy to see that the rectangles  $r_{ij}^1, r_{ij}^4$  and  $r_{ij}^7$  must be assigned the same color and each pair of rectangles,  $\{r_{ij}^2, r_{ij}^3\}$ ,  $\{r_{ij}^5, r_{ij}^6\}$  and  $\{r_{ij}^8, r_{ij}^9\}$  must be assigned the remaining two colors in order to satisfy the 3-colorability. Given a rectilinear line segment  $l_{ij}$  in the embedding, we place an edge component  $R_{ij}$  in its vicinity in such a way that the rectangle  $r_{ij}^1$  in  $R_{ij}$  intersects one of the rectangles  $r_i^e, r_i^w, r_i^n$  and  $r_j^s$  in  $R_i$ , depending on whether  $l_{ij}^1$  is incident upon the point  $p_i$  from the right, left, top and bottom side, respectively, and the rectangles  $r_{ij}^8$  and  $r_{ij}^9$  in  $R_{ij}$  intersect one of  $r_{ij}^e, r_{ij}^w, r_{ij}^n$  and  $r_{ij}^s$  in  $R_j$ , depending on whether  $l_{ij}^t$  is incident upon the point  $p_j$  from the right, left, top and bottom side, respectively. Furthermore, we place rectangle(s)  $r_{ij}^{s+\lceil s/2 \rceil - 1}$  (resp.,  $r_{ij}^{s+\lceil s/2 \rceil - 1}$  and  $r_{ij}^{s+\lceil s/2 \rceil}$ ) in the vicinity of  $l_{ij}^s$  if  $s$  is odd (resp., even) for  $s = 1, 2, \dots, k - 1$ , and the remaining rectangles  $r_{ij}^{t+\lceil t/2 \rceil - 1}, r_{ij}^{t+\lceil t/2 \rceil}, \dots, r_{ij}^9$  in the vicinity of  $l_{ij}^t$ , where  $\lceil x \rceil$  is an integer  $d$  such that  $x \leq d < x + 1$ . Fig. 5 illustrates such a placement for the case of four straight



line segments. A simple example is given in Fig. 6 to illustrate the entire transformation.

Since the sizes of a vertex component and an edge component are fifteen and nine, respectively, the size of the family  $R$  of rectangles constructed by the above procedure is  $15|V| + 9|E|$  and hence the transformation can be done in polynomial time. It is easy to see that the rectangle intersection graph  $G_R$  for  $R$  is planar and  $\Delta(G_R) = 4$ .

Consider an edge component  $R_{ij}$  which is placed in the vicinity of the rectilinear line segment  $l_{ij}$ . Without loss of generality, assume that the rectangle  $r_{ij}^1$  in  $R_{ij}$  intersects the rectangle  $r_i^e$  of the vertex component  $R_i$  and the rectangles  $r_{ij}^8$  and  $r_{ij}^9$  in  $R_{ij}$  intersect the rectangle  $r_j^w$  of the vertex component  $R_j$ . As noted above, in any 3-coloring  $r_{ij}^1 \in R_{ij}$  is assigned one color and  $r_{ij}^8$  and  $r_{ij}^9$  in  $R_{ij}$  are assigned the remaining two colors. Therefore, the color of  $r_{ij}^1 \in R_{ij}$  must be the same as that of  $r_j^w \in R_j$  and hence  $r_i^e \in R_i$  and  $r_j^w \in R_j$  must have different colors in order to satisfy the 3-colorability. Since the rectangles  $r_i^e, r_i^w, r_i^n$  and  $r_i^s$  in any vertex component  $R_i$  must have the same color, it is clear that  $G$  is 3-colorable if and only if the rectangle intersection graph  $G_R$  for the family  $R$  of rectangles constructed above is 3-colorable.  $\square$

The rectangle overlap graph for the family  $R$  of rectangles constructed in the above proof is the same as the rectangle intersection graph for  $R$  since no rectangle in  $R$  contains any other rectangle. Therefore, we immediately obtain the following theorem.

**Theorem 6.** The 3-coloring problem is NP-complete for a planar rectangle overlap graph  $G_R$  even when  $\Delta(G_R) = 4$ .  $\square$

## 4. The Maximum Clique Problem

Unlike the maximum independent set and minimum coloring problems, the maximum clique problem is solvable in polynomial time for a rectangle intersection graph and a rectangle overlap graph. Lee [11] developed an  $O(n \log n)$  time algorithm for finding a maximum clique of a rectangle intersection graph, where  $n$  is the number of rectangles. We show, in this section, that a maximum clique of a rectangle overlap graph can also be found in polynomial time. We describe an  $O(n^{14/3})$  time algorithm which uses a 0–1 maximum flow algorithm [3].

Let  $R$  be a family of iso-oriented rectangles in the plane. A *horizontal* (resp., *vertical*) *extension line* is a horizontal (resp., vertical) line of infinite length such that its  $y(x)$ -coordinate is the same as that of a horizontal (resp., vertical) line segment which forms the boundary of any rectangle in  $R$ . A *cell* is a rectangular region in the plane which is bounded by two adjacent horizontal and two adjacent vertical extension lines. It is easy to see that there are at most  $(2n - 1)^2$  such cells.

For any clique of the rectangle overlap graph  $G_R$  for  $R$ , the rectangles corresponding to the vertices of the clique share a common intersection region which contains at least one such cell. Therefore, a maximum clique can be found by the following procedure:

1. For each cell  $c$ , find a subfamily  $R(c)$  of  $R$  of rectangles that contain  $c$ . Then, find a maximum clique of the rectangle overlap graph  $G_{R(c)}$  for  $R(c)$ .
2. Select a maximum cardinality one among the cliques found in Step 1.

In the remaining part of this section, we describe how to find a maximum clique of  $G_{R(c)}$ .

Let  $\vec{G} = (V, \vec{E})$  be a directed graph. We call  $\vec{G}$  a *transitive graph* if the existence of the directed edges  $(u, v)$  and  $(v, w)$  in  $\vec{E}$  implies the existence of the directed edge  $(u, w)$  in  $\vec{E}$ . It is known that a maximum independent set (of the undirected version) of a transitive graph  $\vec{G} = (V, \vec{E})$  can be found in  $O(|V|^{8/3})$  time by using a 0–1 maximum flow algorithm [8].

Given a family  $R$  of iso-oriented rectangles in the plane, we construct a directed graph  $\vec{G} = (V, \vec{E})$  as follows. For each rectangle  $r_i$  in  $R$ , we create a vertex  $v_i \in V$ , and for any two rectangles  $r_i$  and  $r_j$  in  $R$  such that  $r_i$  contains  $r_j$ , we create a directed edge  $(v_i, v_j) \in \vec{E}$ . It is easy to see that, for any triple of rectangles  $r_i, r_j$  and  $r_k$  in  $R$ , if  $r_i$  contains  $r_j$  and  $r_j$  contains  $r_k$ , then  $r_i$  contains  $r_k$ . Therefore, the graph  $\vec{G} = (V, \vec{E})$  is a transitive graph.

Let  $R(c)$  be a family of rectangles that contain a cell  $c$ . Since either any two rectangles in  $R(c)$  overlap each other or one rectangle contains the other, there is an edge between two vertices in the rectangle overlap graph  $G_{R(c)}$  for  $R(c)$  if and only if there is no edge between the corresponding vertices in the transitive graph derived from  $R(c)$  in the way mentioned above. Therefore, a maximum clique of  $G_{R(c)}$  corresponds to a maximum independent set of the corresponding transitive graph, and hence we can find a maximum clique of the rectangle overlap graph  $G_{R(c)}$  for  $R(c)$  in  $O(|R(c)|^{8/3})$  time.

Since the number of cells in the plane is  $O(n^2)$  and  $|R(c)| \leq n$ , we obtain the following theorem.

**Theorem 7.** A maximum clique of a rectangle overlap graph can be found in  $O(n^{14/3})$  time, assuming that the graph is given in the form of a family of  $n$  iso-oriented rectangles in the plane.  $\square$

## 5. Conclusion

We have shown that the maximum independent set problem is NP-hard for a cubic planar rectangle intersection or overlap graph. We have also shown that the 3-coloring problem is NP-complete for a planar rectangle intersection or overlap graph  $G_R$  even when  $\Delta(G_R) = 4$ . These results are obtained for the tightest degree constraint cases. Finally, we have described how to find, in  $O(n^{14/3})$  time, a maximum clique of a rectangle overlap graph with  $n$  vertices. It is interesting to develop a faster maximum clique algorithm for a rectangle overlap graph.

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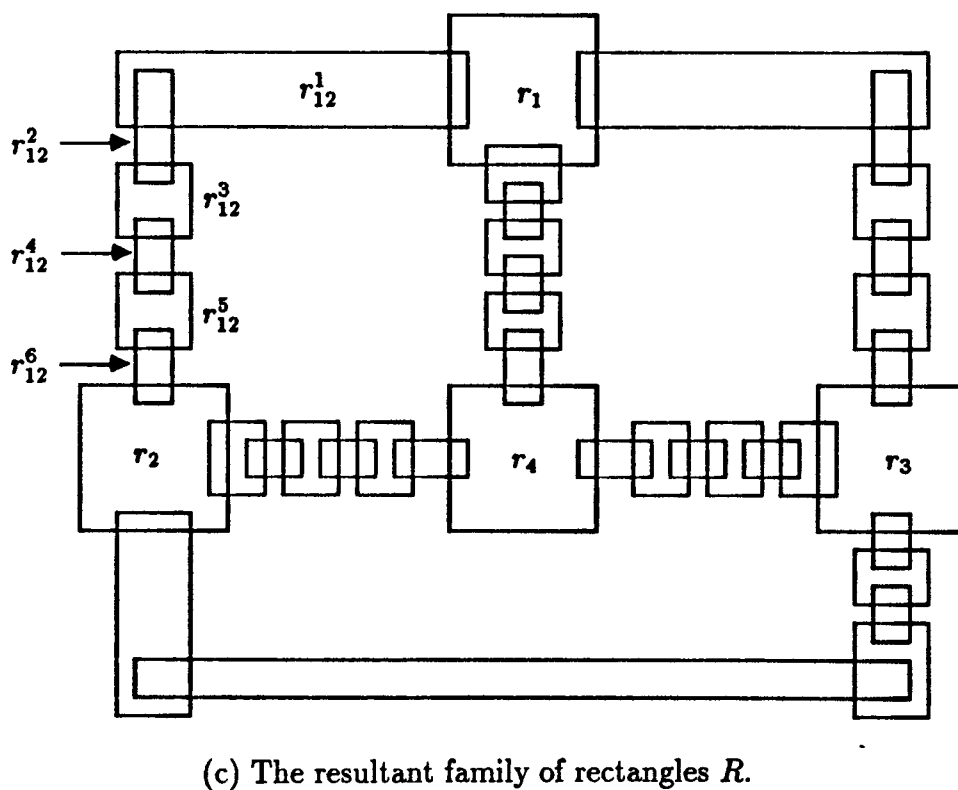
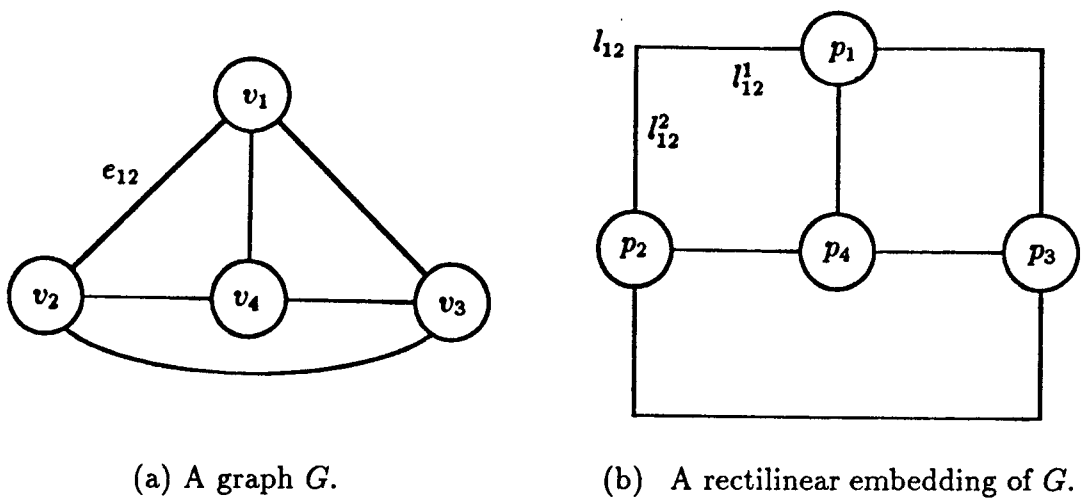


Fig. 1. An illustration of the transformation for the proof of Theorem 1.

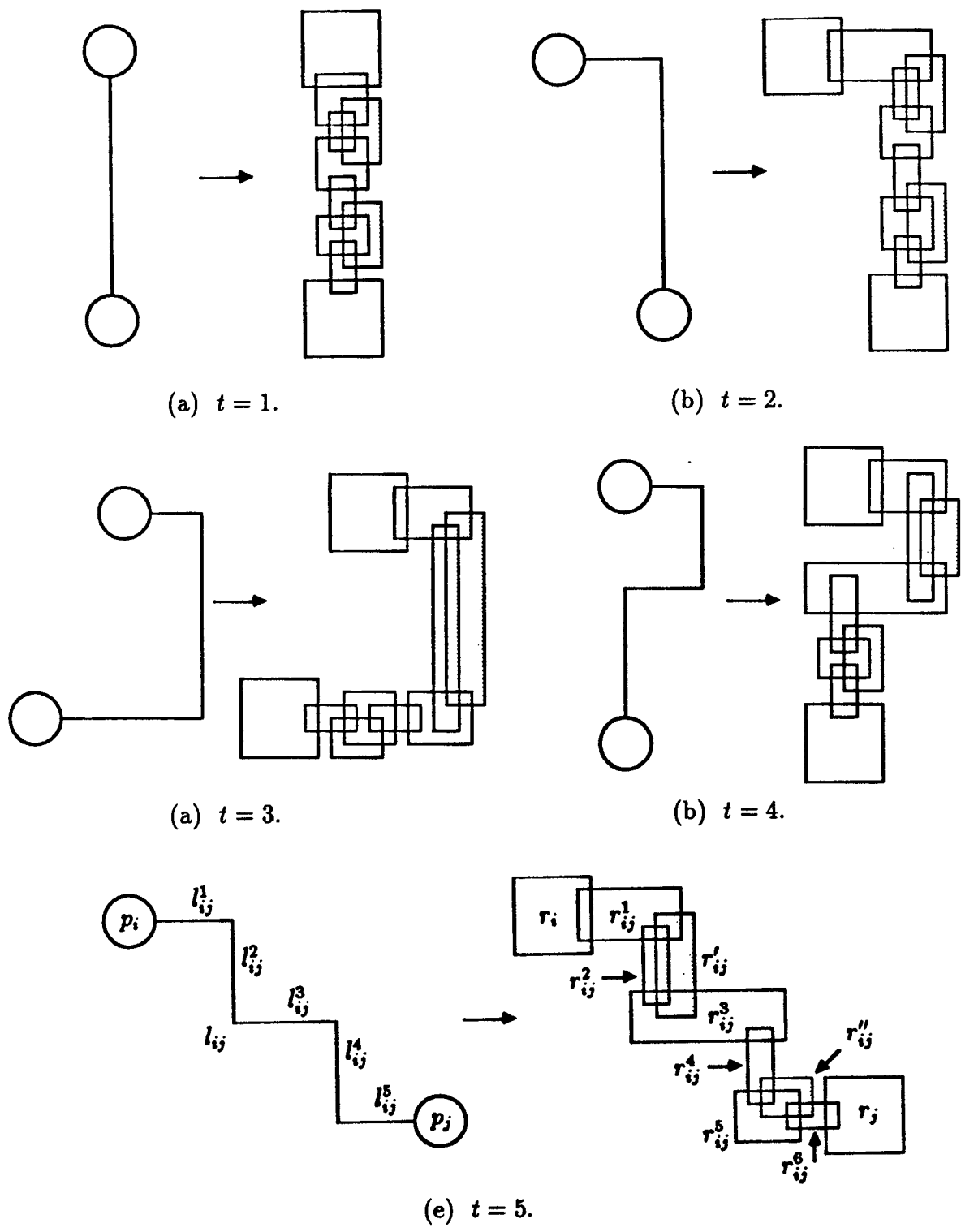
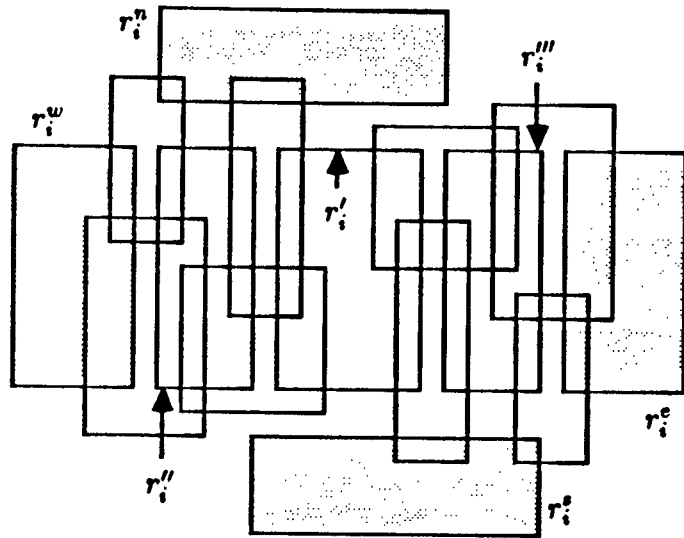
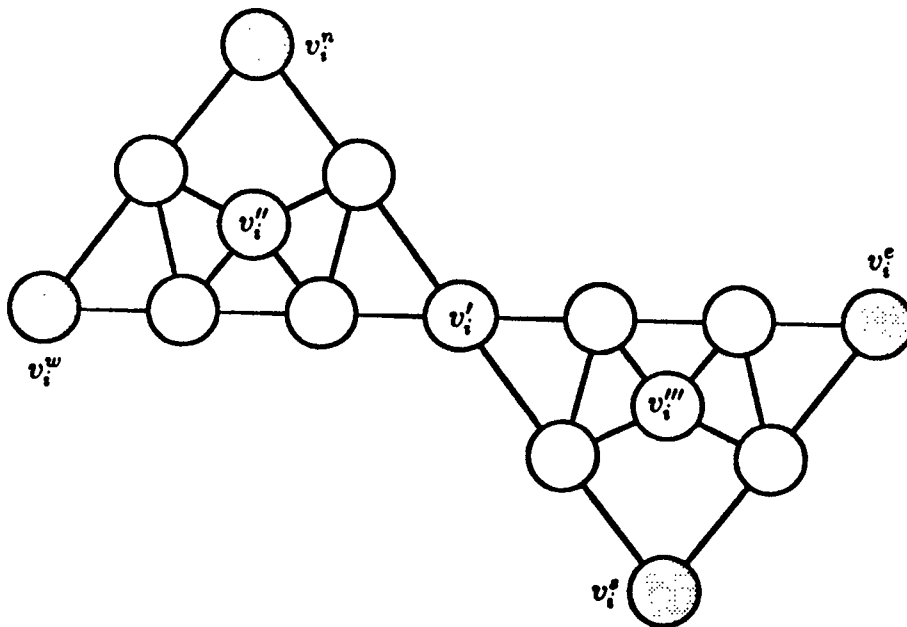


Fig. 2. Placements of additional rectangles  $r'_{ij}$  and  $r''_{ij}$  for rectilinear line segments with  $t$  straight line segments.



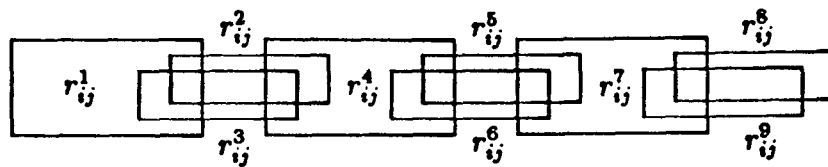


(a) A vertex component  $R_i$ .

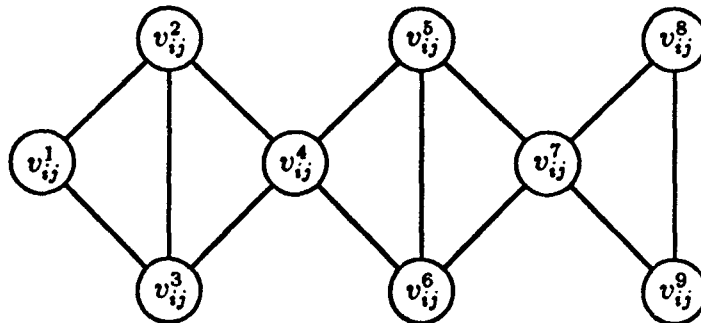


(b) The rectangle intersection graph  $G_{R_i}$  for  $R_i$ .

Fig. 3. A vertex component and its corresponding rectangle intersection graph.

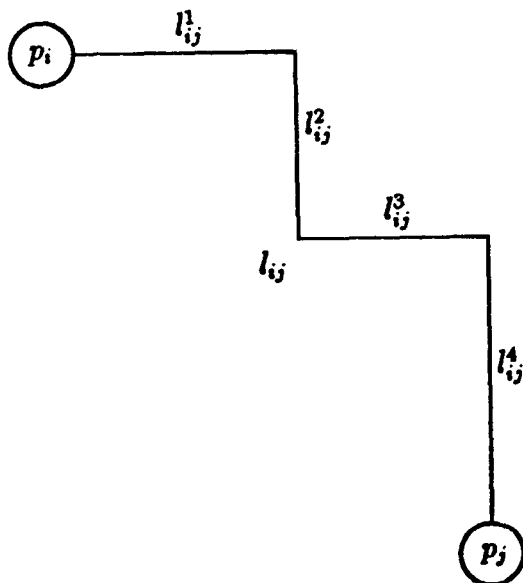


(a) An edge component  $R_{ij}$ .

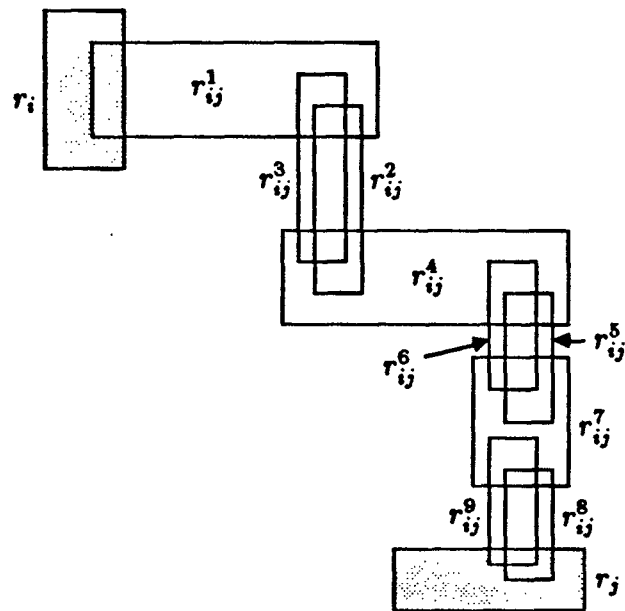


(b) The rectangle intersection graph  $G_{R_{ij}}$  for  $R_{ij}$ .

Fig. 4. An edge component and its corresponding rectangle intersection graph.

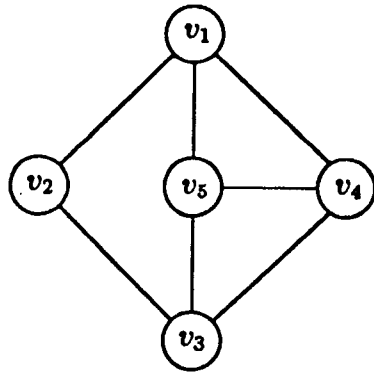


(a) A rectilinear line segment  $l_{ij}$  with four straight line segments.

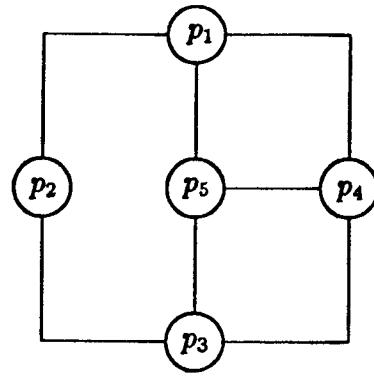


(b) The corresponding family of rectangles  $R_{ij}$ .

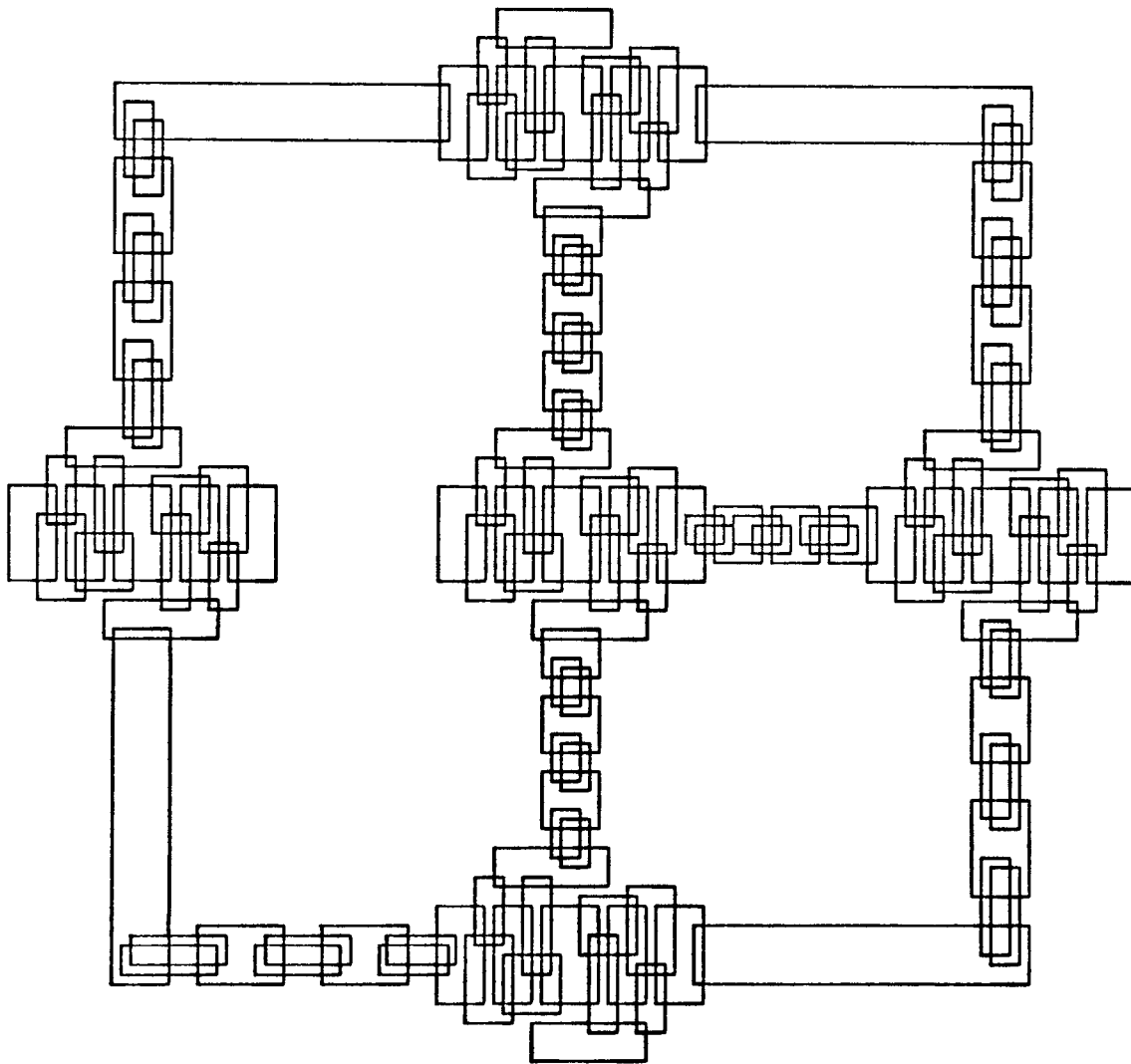
Fig. 5. A rectilinear line segment and the placement of rectangles of its edge component.



(a) A graph  $G$ .



(b) A rectilinear embedding of  $G$ .



(c) The resultant family of rectangles  $R$ .

Fig. 6. An illustration of the transformation for the proof of Theorem 5.