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by Xiaobo Tan, John S. Baras

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Optimal Control of Hysteresis in Smart Actuators: A Viscosity Solutions Approach

Xiaobo Tan and John S. Baras

Department of Electrical and Computer Engineering and
Institute for Systems Research
University of Maryland, College Park, MD 20742
{xbtan, baras}@isr.umd.edu

Abstract. Hysteresis in smart materials hinders their wider applicability in actuators. The low dimensional hysteresis models for these materials are hybrid systems with both controlled switching and autonomous switching. In particular, they belong to the class of Duhem hysteresis models and can be formulated as systems with both continuous and switching controls. In this paper, we study the control methodology for smart actuators through the example of controlling a commercially available magnetostrictive actuator. For illustrative purposes, an infinite horizon optimal control problem is considered. We show that the value function satisfies a Hamilton-Jacobi-Bellman equation (HJB) of a hybrid form in the viscosity sense. We further prove uniqueness of the viscosity solution to the (HJB), and provide a numerical scheme to approximate the solution together with a sub-optimal controller synthesis method. Numerical and experimental results based on this approach are presented.

1 Introduction

Materials with the intrinsic characteristics of built-in sensors, actuators, and control mechanism in their microstructures, are called *smart materials*. Smart materials and smart structures have been receiving tremendous interest in the past decade, due to their broad applications in areas of aerospace, manufacturing, defense, and civil infrastructure systems, to name a few. Hysteresis in smart materials, e.g., magnetostrictives, piezoceramics, and shape memory alloys (SMAs), hinders the wider applicability of such materials in actuators. Hysteresis models can be classified into phenomenological models and physics-based models. The most popular phenomenological hysteresis model used in control of smart actuators has been the Preisach model, see e.g., [1–3]. An example of physics-based model is the Jiles-Atherton model for ferromagnetic hysteresis [4], where hysteresis is considered to arise from pinning of domain walls on defect sites.

This paper is aimed at exploring the control methodology for smart actuators exhibiting hysteresis. We illustrate the ideas through the example of controlling a commercially available magnetostrictive actuator. Magnetostriction is the phenomenon of strong coupling between magnetic properties and mechanical properties of some ferromagnetic materials (e.g., Terfenol-D): strains are generated in response to an applied magnetic field, while conversely, mechanical stresses in the materials produce measurable changes in magnetization. This phenomenon can be used for actuation and sensing. Magnetostrictive actuators have applications in micro-positioning, robotics, ultrasonics and vibration control. Figure 1 shows a sectional view of a Terfenol-D actuator manufactured by Etrema Products, Inc. By varying the current in the coil, we vary the magnetic field in the Terfenol-D rod and thus control the motion of the rod head (or the force if the motion is blocked). Figure 2 displays the hysteresis observed in the magnetostrictive actuator.

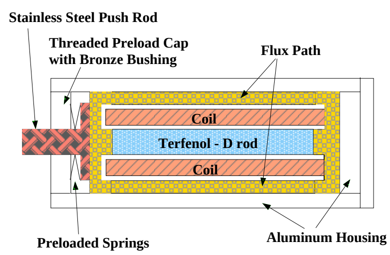

Fig. 1. A Terfenol-D actuator [5](Original source: Etrema Products Inc.)

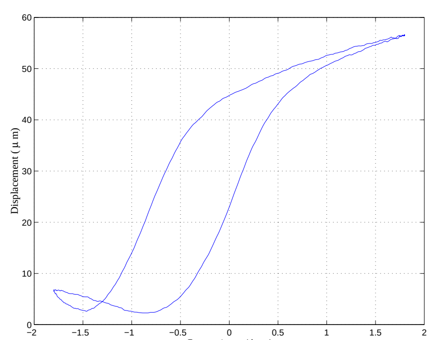

Fig. 2. Hysteresis in the magnetostrictive actuator

When the frequency of the current input I is low, the magnetostriction can be related to the bulk magnetization M along the rod direction through a square law, thus control of the rod head boils down to control of M. We will employ a low dimensional model [6] to represent the ferromagnetic hysteresis between

M and the applied magnetic field H, where H is proportional to I in the low frequency case. The model is a hybrid system with both controlled switching and autonomous switching. It belongs to the class of Duhem hysteresis models and can be rewritten as a system involving both continuous control and switching control. Note that a low dimensional model for ferroelectric hysteresis has been proposed by Smith and Hom [7]. Their model has the same structure as that of the ferromagnetic hysteresis model we use in this paper. Therefore the approach we present in this paper will be fully applicable to control of smart actuators made of ferroelectric materials, e.g., piezoelectrics and electrostrictives.

Witsenhausen formulated a class of hybrid-state continuous-time dynamical systems and studied an optimal control problem back in 1966 [8]. The Pontryagin Maximum Principle or its variant was used in optimal control for hybrid systems in [9, 10]. By solving the Bellman inequality, a lower bound on the value function and an approximation to the optimal control law were obtained in [11, 12]. Yong studied the optimal control problem for a system with continuous, switching and impulse controls in [13]. With a unified model for hybrid control, Branicky, Borkar and Mitter proposed a generalized quasi-variational inequalities (GQVI) satisfied by the value function [14]. In this paper, we study optimal control of smart actuators using a viscosity solutions approach.

Dynamic programming is one of the most important approaches in optimal control. When the value function of the control problem is smooth, we can derive the Hamilton-Jacobi-Bellman equation (HJB) from the Dynamic Programming Principle (DPP), and in many cases, solving the (HJB) amounts to solving the optimal control problem. The value function however, in general, is not smooth even for smooth systems, not to mention for a hybrid system, like that in our model. Crandall and Lions [15] introduced the notion of viscosity solutions to Hamilton-Jacobi equations. This turned out to be a very useful concept for optimal control since value functions of many optimal control problems do satisfy the (HJB) in the viscosity sense; and under mild assumptions, uniqueness results for viscosity solutions hold [16]. We will explore this approach for control of smart actuators. This paper provides some flavors of this methodology by considering an infinite horizon control problem.

The paper is organized as follows. In Sect. 2, we present the hysteresis model and examine its properties. In Sect. 3, we formulate an optimal control problem based on the hysteresis model, and show that the value function satisfies a hybrid Hamilton-Jacobi-Bellman equation (HJB) in the viscosity sense. In Sect. 4, We prove that the (HJB) admits a unique solution in the class of functions to which the value function belongs. We describe the numerical scheme to solve the (HJB) in Sect. 5. This establishes the existence of a solution to the (HJB) as well as provides a method for synthesizing a sub-optimal controller. Numerical and experimental results are also reported in Sect. 5. Finally, conclusions and discussions are provided in Sect. 6.

Only key proofs are provided in this paper due to the space limitation. Other proofs can be found in [17].

2 The Bulk Ferromagnetic Hysteresis Model

Jiles and Atherton proposed a low dimensional model for ferromagnetic hysteresis, based upon the quantification of energy losses due to domain wall intersections with inclusions or pinning sites within the material [4]. A modification to the Jiles-Atherton model was made by Venkataraman and Krishnaprasad with rigorous use of the energy balancing principle [6]. The resulting model, named the *bulk ferromagnetic hysteresis model,* has a slightly different form from the Jiles-Atherton model. Also based on the energy balancing principle, they derived a bulk magnetostrictive hysteresis model [18], where high frequency effects are considered. At low frequencies, the magnetostriction can be related to the bulk magnetization through a square law [5], thus control of the bulk magnetization amounts to control of the magnetostriction. In this paper, we will restrict ourselves to the low frequency case to highlight the methodology of hysteresis control. Extension to the high frequency case is straightforward. We now briefly outline the bulk ferromagnetic hysteresis model.

For an external magnetic field H and a bulk magnetization M, we define $H_{\mathrm{e}} = H + \alpha M$ to be the effective field, where α is a mean field parameter representing inter-domain coupling. Through thermodynamic considerations, the *anhysteretic magnetization* M_{an} can be expressed as

$$M_{\mathrm{an}}(H_{\mathrm{e}}) = M_{\mathrm{s}}(\coth(\frac{H_{\mathrm{e}}}{a}) - \frac{a}{H_{\mathrm{e}}}) = M_{\mathrm{s}}\mathcal{L}(z) \ , \tag{1}$$

where $\mathcal{L}(\cdot)$ is the Langevin function, $\mathcal{L}(z) = \coth(z) - \frac{1}{z}$, with $z = \frac{H_{\mathrm{e}}}{a}$, M_{s} is the saturation magnetization of the material and a is a parameter characterizing the shape of M_{an} curve.

Define

$$f_1(H,M) = c\frac{M_{\mathrm{s}}\frac{\partial\mathcal{L}(z)}{\partial z}}{a - \alpha c M_{\mathrm{s}}\frac{\partial\mathcal{L}(z)}{\partial z}} \ ,$$

$$f_2(H,M) = \frac{ckM_{\mathrm{s}}\frac{\partial\mathcal{L}(z)}{\partial z} - \mu_0 a(M_{\mathrm{an}}(H_{\mathrm{e}}) - M)}{k(a - \alpha c M_{\mathrm{s}}\frac{\partial\mathcal{L}(z)}{\partial z}) + \mu_0\alpha a(M_{\mathrm{an}}(H_{\mathrm{e}}) - M)} \ ,$$

$$f_3(H,M) = \frac{ckM_{\mathrm{s}}\frac{\partial\mathcal{L}(z)}{\partial z} + \mu_0 a(M_{\mathrm{an}}(H_{\mathrm{e}}) - M)}{k(a - \alpha c M_{\mathrm{s}}\frac{\partial\mathcal{L}(z)}{\partial z}) - \mu_0\alpha a(M_{\mathrm{an}}(H_{\mathrm{e}}) - M)} \ ,$$

where c is the reversibility constant, μ_0 is the permeability of vacuum, k is a measure for the average energy required to break a pinning site. Note each f_i is smooth in H and M.

The bulk ferromagnetic hysteresis model is as follows [6]:

$$\frac{dM}{dH} = f_i(H,M), \quad \text{where} \quad i = \begin{cases} 1,\, dH < 0,\ M < M_{\mathrm{an}}(H_{\mathrm{e}}) \text{ or} \\ \quad dH \geq 0,\ M \geq M_{\mathrm{an}}(H_e) \\ 2,\, dH < 0,\ M \geq M_{\mathrm{an}}(H_{\mathrm{e}}) \\ 3,\, dH \geq 0,\ M < M_{\mathrm{an}}(H_{\mathrm{e}}) \end{cases} .$$

If we define a control $u = \dot{H}$, the model is rewritten as

$$\begin{pmatrix} \dot{H} \\ \dot{M} \end{pmatrix} = \begin{pmatrix} 1 \\ f_i(H, M) \end{pmatrix} u, \;\text{ where }\; i = \begin{cases} 1,\, u < 0,\; M < M_{\mathrm{an}}(H_{\mathrm{e}}) \text{ or} \\ \quad\; u \geq 0,\; M \geq M_{\mathrm{an}}(H_{\mathrm{e}}) \\ 2,\, u < 0,\; M \geq M_{an}(H_{\mathrm{e}}) \\ 3,\, u \geq 0,\; M < M_{an}(H_{\mathrm{e}}) \end{cases} . \tag{3}$$

Remarks:

- Note that the control u defined above is different from the physical current I we apply to the actuator. The current I is related to the state component H by a constant c_0 (the *coil factor*): $H = c_0 I$. Therefore from the control u, the current we will apply is $I(t) = I(0) + \frac{1}{c_0}\int_0^t u(s)ds$.
- The switching depends on both (the sign of) the continuous control u and the state (H, M), therefore the model (3) is a hybrid system with both controlled switching and autonomous switching [14, 19].

We can represent model (3) in a more compact way. Letting

$$\Omega_1 = \{(H, M) : M < M_{\mathrm{an}}(H_{\mathrm{e}})\}, \;\; \Omega_2 = \{(H, M) : M \geq M_{\mathrm{an}}(H_{\mathrm{e}})\},$$

and $x = (H, M)$, we define

$$f_+(x) = \begin{cases} \begin{pmatrix} 1 \\ f_1(x) \end{pmatrix} & \text{if } x \in \Omega_2 \\ \begin{pmatrix} 1 \\ f_3(x) \end{pmatrix} & \text{if } x \in \Omega_1 \end{cases}, \;\text{ and }\; f_-(x) = \begin{cases} \begin{pmatrix} 1 \\ f_1(x) \end{pmatrix} & \text{if } x \in \Omega_1 \\ \begin{pmatrix} 1 \\ f_2(x) \end{pmatrix} & \text{if } x \in \Omega_2 \end{cases} .$$

Since $f_i, 1 \leq i \leq 3$, coincide on $\{(H, M) : M = M_{\mathrm{an}}(H_{\mathrm{e}})\}$, f_+ and f_- are continuous. We define two continuous control sets

$$U_+ = \{u : u_{\mathrm{c}} \geq u \geq 0\}, \quad U_- = \{u : -u_{\mathrm{c}} \leq u \leq 0\},$$

where $u_{\mathrm{c}} > 0$ represents the operating bandwidth constraint of the actuator (recall $u = c_0 \dot{I}$). To ease the presentation, we make the dependence of switching on u explicit by introducing a discrete control set $D = \{1, 2\}$.

Now the model (3) can be represented as a system with both a continuous control u and a discrete mode (switching) control d:

$$\dot{x} = f(x, u, d) \triangleq \begin{cases} f_+(x)u, & u \in U_+, \text{ if } \; d = 1 \\ f_-(x)u, & u \in U_-, \text{ if } \; d = 2 \end{cases} . \tag{4}$$

The (state-dependent) autonomous switching has now been incorporated into the definitions of f_+, f_-, thanks to the nice structure of the physical model. Note the model (4) belongs to the category of *Duhem* hysteresis model [20].

We can prove that the model enjoys the following properties:

Proposition 1 (Boundedness of f_i). *If the parameters satisfy:*

$$T_1 \triangleq a - \frac{\alpha c M_{\rm s}}{3} > 0 \ , \tag{5}$$

$$T_2 \triangleq k(a - \frac{\alpha c M_{\rm s}}{3}) - 2\mu_0 \alpha a M_{\rm s} > 0 \ , \tag{6}$$

then $0 < f_i \leq C_{\rm f}, i = 1, 2, 3,$ *for some constant* $C_{\rm f} > 0$.

Proof. See [17]. □

Remark : Conditions (5) and (6) are satisfied for typical parameters. For example, taking the parameters identified in [5], $\alpha = 1.9 \times 10^{-4}$, $a = 190$, $k = 48$ Gauss, $c = 0.3$, $M_s = 9.89 \times 10^3$ Gauss and $\mu_0 = 1$, we calculate $T_1 = 189.8$, $T_2 = 8.40 \times 10^3$.

Proposition 2 (Lipshitz Continuity). *Functions $f_+(x)$ and $f_-(x)$ are Lipshitz continuous with some Lipshitz constant L, and $f(x, u, d)$ is Lipshitz continuous with respect to x with Lipshitz constant $L_0 = L u_c$.*

Proof. See [17]. □

3 Optimal Control: The (HJB) and Viscosity Solutions

We first formulate an infinite horizon optimal control problem for the system (4). Define the cost functional with an initial condition x and a control pair $\alpha(\cdot) = \{d(\cdot), u(\cdot)\}$ as

$$J(x, \alpha(\cdot)) = \int_0^\infty l(x(t), u(t)) e^{-\lambda t} dt \ , \tag{7}$$

where the *discount factor* $\lambda \geq 0$. Note the *running cost* $l(\cdot, \cdot)$ is defined to be independent of the switching control d, since this makes sense in the context of smart actuator control. We require $u(\cdot)$ to be measurable. This together with Proposition 2 guarantees that (4) has a unique solution $x(\cdot)$(the dependence of $x(\cdot)$ on x and $\alpha(\cdot)$ is suppressed when no confusion arises).

The optimal control problem is to find the value function

$$V(x) = \inf_{\alpha(\cdot)} J(x, \alpha(\cdot)) \ ,$$

and if $V(x)$ is achievable, find the optimal control $\alpha^*(\cdot)$.

We make the following assumptions about $l(\cdot, \cdot)$:

- (A_1): $l(x, u)$ continuous in x and u, $l(x, u) \geq 0$, $\forall x, u$;
- (A_2): $|l(x_1, u) - l(x_2, u)| \leq C_{\rm l}(1 + |x_1| + |x_2|)|x_1 - x_2|$, $\forall u$, for some $C_{\rm l} > 0$.

Note (A_2) includes the case of quadratic cost.

We can show the value function is locally bounded and locally Lipshitz continuous.

Proposition 3 (Local Boundedness). *Under assumptions (A_1) and (A_2), $\forall \lambda > 0$, $V(x)$ is locally bounded, i.e., $\forall R \geq 0, \exists\, C_{\mathrm{R}} \geq 0$, such that $|V(x)| \leq C_{\mathrm{R}},\ \forall x \in \overline{B}(0,R) \triangleq \{x : |x| \leq R\}$.*

Proof. See [17]. □

Proposition 4 (Local Lipshitz Continuity). *Under assumptions (A_1) and (A_2), $\forall \lambda > 2L_0$ with L_0 as defined in Proposition 2, $V(x)$ is locally Lipshitz, i.e., $\forall R \geq 0, \exists\, L_{\mathrm{R}} \geq 0$, such that $|V(x_1) - V(x_2)| \leq L_{\mathrm{R}}|x_1 - x_2|,\ \forall x_1, x_2 \in \overline{B}(0,R)$. In addition, L_{R} can be chosen to be $C(1+R)$ for some $C > 0$.*

Proof. See [17]. □

Remarks: The proof of Proposition 4 uses the bounds for $|x_1(t) - x_2(t)|$ and $|x_1(t)|$, where $x_1(\cdot), x_2(\cdot)$ are two trajectories starting from x_1 and x_2. The bounds are obtained using Proposition 2 and the Gronwall inequality. One can get a sharper estimate for $|x_1(t)|$ (linear growth) by exploiting Proposition 1. This can be used to weaken the condition $\lambda > 2L_0$ to $\lambda > L_0$ in Proposition 4 and anywhere else it appears.

The value function satisfies the Dynamic Programming Principle (DPP) :

Proposition 5 (DPP). *Assume (A_1) and (A_2), $\lambda > 0$. We have*

$$V(x) = \inf_{\alpha(\cdot)} \{ \int_0^t e^{-\lambda s} l(x(s), u(s)) ds + e^{-\lambda t} V(x(t)) \},\ \forall t \geq 0,\ \forall x \ . \tag{8}$$

Proof. The argument is standard, see [17]. □

Based on the (DPP), we can show that the value function $V(\cdot)$ satisfies a Hamilton-Jacobi-Bellman equation (HJB) of a hybrid type in the viscosity sense. Viscosity solutions to Hamilton-Jacobi equations were first introduced by Crandall and Lions [15]. Here we use one of the three equivalent definitions [21]:

Definition 1 (Viscosity Solution). *Let W be a continuous function from an open set $O \in \mathbb{R}^n$ into $\mathbb{R}$ and let DW denote the gradient of W (when W is differentiable). We call W a viscosity solution to a nonlinear first order partial differential equation $F(x, W(x), DW(x)) = 0$, provided it is both a viscosity subsolution and viscosity supersolution; and by viscosity sub(super)solution, we mean: $\forall \phi \in C^1(O)$, if $W - \phi$ attains a local maximum (minimum) at $x_0 \in O$, then $F(x_0, W(x_0), D\phi(x_0)) \leq (\geq) 0$.*

Theorem 1 (HJB). *Assume (A_1) and (A_2), $\lambda > 2L_0$. $V(x)$ is a viscosity solution of:*

$$\begin{aligned} \lambda W(x) + \max\{ & \max_{u \in U_+} \{-u f_+(x) \cdot DW(x) - l(x,u)\}, \\ & \max_{u \in U_-} \{-u f_-(x) \cdot DW(x) - l(x,u)\}\} = 0 \ . \end{aligned} \tag{9}$$

Proof. See [17]. □

4 Uniqueness of the Solution to the (HJB)

We would like to characterize the value function V as a unique solution to the (HJB). The uniqueness result basically follows from Theorem 1.5 in [22]. In [22], the author gave only a sketch of proof. For completeness, we will provide the full proof here.

Before stating the theorem, we first identify structural properties of the (HJB). We rewrite (9) as:

$$\lambda W(x) + H(x, DW(x)) = 0 \ , \tag{10}$$

where

$$H(x,p) = \max\{\max_{u\in U_+}\{-uf_+(x)\cdot p - l(x,u)\}, \max_{u\in U_-}\{-uf_-(x)\cdot p - l(x,u)\}\} \ .$$

Proposition 6. *Assume (A_2). $H(x,p)$ satisfies the following:*

$$|H(x_1,p) - H(x_2,p)| \leq C_{\mathrm{R}}(1+|p|)|x_1 - x_2|, \ \forall x_1, x_2 \in \overline{B}(0,R), \ \forall p \ , \tag{11}$$

$$|H(x,p_1) - H(x,p_2)| \leq C_0|p_1 - p_2|, \forall x, \ \forall p_1, p_2 \ , \tag{12}$$

for some $C_{\mathrm{R}} > 0, C_0 > 0$, with C_{R} dependent on R.

Proof. We will only prove (11), since proof of (12) is analogous.

Without loss of generality, suppose $u_1 \in U_-$ attains the maximum in $H(x_1,p)$. Since $H(x_2,p) \geq -u_1 f_-(x_2)\cdot p - l(x_2,u_1)$,

$$\begin{aligned} H(x_1,p) - H(x_2,p) &\leq -u_1 f_-(x_1)\cdot p - l(x_1,u_1) + u_1 f_-(x_2)\cdot p + l(x_2,u_1) \\ &\leq |p|L_0|x_1 - x_2| + C_l(1+|x_1|+|x_2|)|x_1 - x_2| \\ &\leq C_{\mathrm{R}}(1+|p|)|x_1 - x_2| \ , \end{aligned}$$

where C_{R} is a constant dependent on R. By symmetry, we conclude. □

Remarks: As we have seen above, despite the hybrid structure of our physical model, $H(x,p)$ enjoys nice structural properties, which enables us to prove the uniqueness result.

From Proposition 4, we know that the value function $V(\cdot)$ belongs to the class

$$\begin{aligned} \mathcal{P}(\mathbb{R}^2) = \{W(\cdot) : \ & |W(x_1) - W(x_2)| \leq C(1+R)|x_1 - x_2|, \forall x_1, x_2 \in \overline{B}(0,R), \\ & \text{for some } C > 0\}. \end{aligned}$$

The following theorem is adapted from Theorem 1.5 in [22].

Theorem 2. *If (10) has a viscosity solution in $\mathcal{P}(\mathbb{R}^2)$, it is unique.*

Proof. Without loss of generality, we take $\lambda = 1$. Let $W(\cdot), V(\cdot) \in \mathcal{P}(\mathbb{R}^2)$ be viscosity solutions to (10). For $\epsilon > 0, \alpha > 0, m > 2$, define

$$\Phi(x,y) = W(x) - V(y) - \frac{|x-y|^2}{\epsilon} - \alpha(< x >^m + < y >^m) \ ,$$

with $< x > \triangleq \sqrt{1+|x|^2}$. Since $W(\cdot), V(\cdot) \in \mathcal{P}(\mathbb{R}^2)$, $\lim_{|x|+|y|\to\infty} \Phi(x,y) = -\infty$. By continuity of $\Phi(\cdot,\cdot)$, there exists (x_0, y_0) where Φ attains the global maximum. First we want to obtain bounds for $|x_0|, |y_0|$ and $|x_0 - y_0|$.

From $\Phi(0,0) \le \Phi(x_0,y_0)$, and $W(\cdot), V(\cdot) \in \mathcal{P}(\mathbb{R}^2)$, we can get

$$< x_0 >^m + < y_0 >^m \le C_\alpha(1 + < x_0 >^2 + < y_0 >^2) \ ,$$

where C_α is a constant independent of ϵ (but dependent on α). Since $m > 2$, there exists $R_\alpha > 0$ (independent of ϵ), such that $|x_0| \le R_\alpha$, $|y_0| \le R_\alpha$.

From $\Phi(x_0,x_0) + \Phi(y_0,y_0) \le 2\,\Phi(x_0,y_0)$, we can derive

$$|x_0 - y_0| \le \epsilon C'_\alpha \ , \tag{13}$$

with C'_α depending on α only.

Define

$$\phi(x) = V(y_0) + \frac{1}{\epsilon}|x - y_0|^2 + \alpha(< x >^m + < y_0 >^m) \ ,$$
$$\psi(y) = W(x_0) - \frac{1}{\epsilon}|x_0 - y|^2 - \alpha(< x_0 >^m + < y >^m) \ .$$

Since $W - \phi$ achieves maximum at x_0, and $V - \psi$ achieves minimum at y_0,

$$W(x_0) + H(x_0, D\phi(x_0)) \le 0 \ , \tag{14}$$
$$V(y_0) + H(y_0, D\psi(y_0)) \ge 0 \ . \tag{15}$$

Subtracting (15) from (14) and using Proposition 6, we have

$$\begin{aligned} W(x_0) - V(y_0) \le\ & C_{R_\alpha}(1 + \frac{2}{\epsilon}|x_0 - y_0|)|x_0 - y_0| \\ & + \alpha C_0 m(< x_0 >^{m-1} + < y_0 >^{m-1}) \ . \end{aligned}$$

Now fix α, construct a sequence $\{\epsilon_k\}$ with $\lim_{k\to\infty} \epsilon_k = 0$. We denote the corresponding maximizers of Φ as (x_{0k}, y_{0k}). Since $\forall k, (x_{0k}, y_{0k}) \in \overline{B}(0, R_\alpha)$, by extracting a subsequence if necessary, we get

$$\lim_{k\to\infty} (x_{0k}, y_{0k}) \to (x_\alpha, y_\alpha) \in \overline{B}(0, R_\alpha) \ . \tag{16}$$

From (13), we have $x_\alpha = y_\alpha$. For each ϵ_k, from $\Phi(x,x) \le \Phi(x_{0k}, y_{0k})$, we can get

$$\begin{aligned} W(x) - V(x) - 2\alpha < x >^m \le & C_{R_\alpha}(1 + \frac{2}{\epsilon_k}|x_{0k} - y_{0k}|)|x_{0k} - y_{0k}| \\ & + \alpha C_0 m(< x_{0k} >^{m-1} + < y_{0k} >^{m-1}) - \alpha(< x_{0k} >^m + < y_{0k} >^m) \ , \end{aligned}$$

and letting $k \to \infty$,

$$W(x) - V(x) \le 2\alpha(C_0 m < x_\alpha >^{m-1} - < x_\alpha >^m) + 2\alpha < x >^m \quad .$$

Since $C_0 m < x_\alpha >^{m-1} - < x_\alpha >^m \le C''$ for some $C'' > 0$,

$$W(x) - V(x) \le 2\alpha(C'' + < x >^m) \ .$$

Letting $\alpha \to 0$, we get $W(x) - V(x) \le 0$, $\forall x$. We conclude by noting W and V are symmetric. □

From Theorem 2, if we can solve for a solution to (10) in $\mathcal{P}(\mathbb{R}^2)$, it must be the value function. One way to solve it is by discrete-time approximation.

5 The Discrete Approximation Scheme

The approximation will be accomplished in two steps. First we approximate the continuous time optimal control problem by a discrete time problem, derive the hybrid discrete Bellman equation (DBE), and show the value function of the discrete time problem converges to that of the continuous time problem locally uniformly. Following [16], we call this step "semi-discrete" approximation. Then we indicate how to further discretize (DBE) in the spatial variable, which is called "fully-discrete" approximation. The approaches we take here follow closely those in [16](Chapter VI and Appendix A).

Consider a discrete time problem obtained by discretizing the original continuous time one with time step $h \in (0, \frac{1}{\lambda})$. The dynamics is given by

$$x[n] = x[n-1] + hf(x[n-1], u[n-1], d[n-1]),\ x[0] = x \ , \tag{17}$$

and the cost is given by

$$J_h(x, \alpha[\cdot]) = \sum_{n=0}^{\infty} hl(x[n], u[n])(1 - \lambda h)^n \ , \tag{18}$$

where $\alpha[\cdot] = \{d[\cdot], u[\cdot]\}$ is the control. The value function is defined to be

$$V_h(x) = \inf_{\alpha[\cdot]} J_h(x, \alpha[\cdot]) \ . \tag{19}$$

It's not hard to show:

Proposition 7. *Assume A_1 and A_2, $\lambda > 2L_0$. Then $V_h(\cdot) \in \mathcal{P}(\mathbb{R}^2)$, and the coefficient C in defining $\mathcal{P}(\mathbb{R}^2)$ can be made independent of h.*

Following standard arguments, one can show:

Proposition 8 (DBE). *$V_h(\cdot)$ satisfies:*

$$\begin{aligned} V_h(x) = \ & \min\{\min_{u \in U_+}\{(1 - \lambda h)V_h(x + huf_+(x)) + hl(x, u)\}, \\ & \min_{u \in U_-}\{(1 - \lambda h)V_h(x + huf_-(x)) + hl(x, u)\}\} \ . \end{aligned} \tag{20}$$

It's of interest to know whether (20) characterizes the value function $V_h(x)$. Unlike in [16](Chapter VI), where a bounded value function was considered, we have V_h unbounded. But it turns out that with a little bit additional assumption, (20) has a unique solution.

Proposition 9. *There exists a unique solution in $\mathcal{P}(\mathbb{R}^2)$ to (20), if*

$$\frac{(1-\lambda h)(\sqrt{C_0^2+4}+C_0)}{\sqrt{C_0^2+4}-C_0} < 1 \ , \tag{21}$$

where $C_0 = hu_{\mathrm{c}} \left| \begin{matrix} 1 \\ C_{\mathrm{f}} \end{matrix} \right|$ and C_{f} is as defined in Proposition 1.

Proof. Let $\widetilde{V}_h(x) = V_h(x) < x >^{-m}, m > 2$, where $< x > \stackrel{\triangle}{=} \sqrt{1+|x|^2}$. Since $V_h \in \mathcal{P}(\mathbb{R}^2)$, $\widetilde{V}_h$ is bounded. In terms of $\widetilde{V}_h$, (20) is rewritten as

$$\begin{aligned} &\widetilde{V}_h(x) = (\mathcal{G}(\widetilde{V}_h))(x) \stackrel{\triangle}{=} \min\{ \\ &\min_{u\in U_+}\{(1-\lambda h)\widetilde{V}_h(x+huf_+(x))\frac{< x+huf_+(x) >^m}{< x >^m} + hl(x,u) < x >^{-m}\}, \\ &\min_{u\in U_-}\{(1-\lambda h))\widetilde{V}_h(x+huf_-(x))\frac{< x+huf_-(x) >^m}{< x >^m} + hl(x,u) < x >^{-m}\}\} \ . \end{aligned} \tag{22}$$

It suffices to show (22) has a unique solution. It's clear that the operator $\mathcal{G}(\cdot)$ maps any $\widetilde{W} \in BC(\mathbb{R}^2)$ into $BC(\mathbb{R}^2)$, where $BC(\mathbb{R}^2)$ denotes the set of bounded continuous functions. When (21) is satisfied, one can show that $\mathcal{G}(\cdot)$ is a contraction mapping. Hence we conclude using the Contraction Mapping Principle. □

The following theorem asserts that $V_h(\cdot)$ converges to $V(\cdot)$ as $h \to 0$. The proof can be found in [16](Chapter VI)(with minor modification).

Theorem 3. *Assume A_1 and A_2, $\lambda > 2L_0$, and (21). Then*

$$\sup_{x\in\mathcal{K}} |V_h(x) - V(x)| \to 0 \ \ as \ \ h \to 0, \tag{23}$$

for every compact $\mathcal{K} \subset \mathbb{R}^2$.

It was also shown in [16] that one can obtain a sub-optimal control for the continuous time problem when solving the (DBE). Theoretically the solution to (20) can be obtained by successive approximation. A practical approximation scheme for solving the (DBE) is described in [16](Appendix A, by Falcone), which we have followed in the numerical simulation. It was shown there that when space discretization gets finer and finer, the solution obtained via solving a finite system of equations converges to $V_h(\cdot)$.

Computation can only be done in a bounded domain. The domain we used in simulation is of the form $\overline{\Omega} = \{(H, M) : H_{\min} \le H \le H_{\max}, |M| \le M_{\mathrm{s}}\}$. The constraint $|M| \le M_{\mathrm{s}}$ arises from the physics, while the constraint on H is due to limitation on the range of current input. The value function of an optimal control problem with state-space constraints is a *constrained* viscosity solution to the (HJB) in $\overline{\Omega}$ [23], namely, a solution in the interior of $\overline{\Omega}$ and a supersolution in $\overline{\Omega}$. Theoretical results for the constrained state-space case are omitted in this paper.

The values we used for the model parameters are as those in the remarks following Proposition 1. The running cost was defined as $l(H, M, u)=100(H - H_0)^2 + 0.1M^2 + 0.01u^2$, where H_0 corresponds to some desired steady current input. Other parameters: $H_{\min} = 19.8$, $H_{\max} = 407.8$, $H_0 = 213.8$, $\lambda = 1.58 \times 10^3$, $h = 5 \times 10^{-4}$, $u_c = 1.22 \times 10^3$. Each of U_+ and U_- is discretized into 20 levels, while each dimension of the state space is discretized into 40 levels.

Figure 3 shows the value function and the optimal feedback control map. Optimal trajectories obtained through simulation and experiments from three different initial conditions (A, B, C) are shown in Fig. 4, where the arrows indicate directions of evolution as well as stationary points of the closed-loop systems. Figure 5 shows the experimental setup.

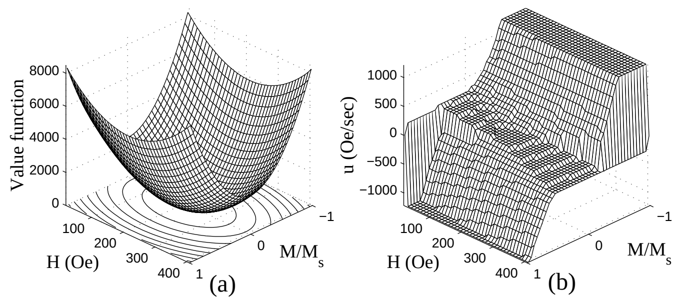

Fig. 3. (a) Value function and its level curves (b) Optimal feedback control map

6 Conclusions and Discussions

In this paper, we have studied the viscosity solutions approach for optimal control of a smart actuator based on the low dimensional hysteresis model. We took the infinite horizon optimal control problem as an example and characterized the value function as the (unique) solution of a Hamilton-Jacobi-Bellman equation of a hybrid form. We pointed out how to solve the (HJB) and obtain a sub-optimal control by discrete time approximation.

Future work includes extension of this approach to other control problems of practical interest, including the finite horizon control problem, the time-optimal control problem and the H_∞ control problem. Also the stability issues associated with the closed-loop systems need to be investigated.

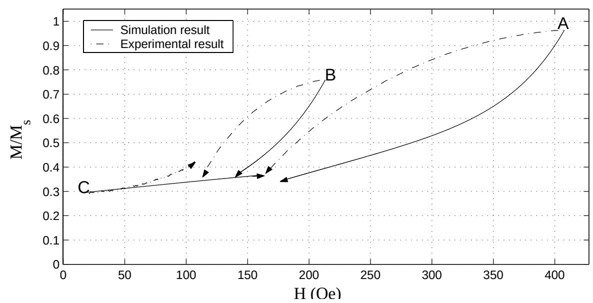

Fig. 4. Optimal trajectories

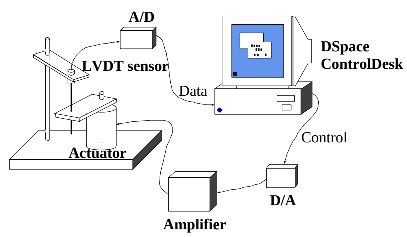

Fig. 5. Experimental setup

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