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Abstract

For a large class of weights $v > 0$, each satisfying a natural concavity condition, the following theorem is proved for $L^1_v(\mathbb{R}) = \{f: \|f\|_v = \int_{-\infty}^{\infty} |f(t)| v(t)dt < \infty\}$: there is a computable constant $c(\varepsilon)$ such that $\|F\|_\infty \le c(\varepsilon) \|f\|_v$ for all $f \in L^1_v(\mathbb{R})$ whose Fourier transform F is supported by $[-\varepsilon, \varepsilon]$ (Theorem 3.5 and Theorem 4.1). A related result demonstrates that Helson sets associated with most Beurling algebras are finite (Theorem 5.2).

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1. Introduction.

We introduce a method which establishes support dependent weighted Fourier transform norm inequalities,

$$(1.1) \qquad \|F\|_{\infty} \leq c(\varepsilon)\|fv\|_1 ,$$

where F is the Fourier transform of f, the support of F denoted by supp F is contained in $[-\varepsilon,\varepsilon]$, the weight v satisfies a concavity condition, and c is an explicit computable function depending on v. Our setting is the real line $\mathbb{R}$ but we anticipate generalizing (1.1) to higher dimensions and formulating it for other L^p-norms.

For motivation we make the following two comments. First, inequalities similar to (1.1) are used to apply the uncertainty principle in signal processing problems; several of the papers in [4] highlight the contributions in this area of Chalk, Fuchs, and Landau, Pollak, and Slepian of Bell Laboratories. Second, related inequalities,

$$(1.2) \qquad \|Fu\|_q \leq c\|fv\|_p , \quad q < \infty,$$

have recently been considered in the context of characterizing weights u and v for which (1.2) is true, e.g., [2].

In Section 2 we establish notation and state some background material for perspective. Section 3 contains our method, Theorem 3.5, and is the major section of the paper. We apply Theorem 3.5 to Beurling algebras $L^1_v(\mathbb{R})$ in Section 4. Finally, Section 5 is devoted to Helson sets in Beurling algebras $L^1_v(\mathbb{R})$. We prove that these v-Helson sets are finite if $v(t) \geq 1+|t|^{\alpha}$.

For eligible weights v, a corollary of our main result from Section 4 is the following: if F_ε is the Fourier transform of $f_\varepsilon \in L^1_v(\mathbb{R})$, and supp $F_\varepsilon \subseteq [-\varepsilon,\varepsilon]$ and $F_\varepsilon(0) = 1$, then $\overline{\lim_{\varepsilon\to 0}} \|f_\varepsilon v\|_1 = \infty$, cf., Remark 5.3c. This fact is clear for certain specific classes $\{F_\varepsilon\}$, such as the de la Vallée-Poussin kernel, but is not apparent for every such class $\{F_\varepsilon\}$. For comparison, if $v(t) = 1$ in which case $L^1_v(\mathbb{R}) = L^1(\mathbb{R})$, we recall that $\{\|f_\varepsilon\|_1\}$ is bounded for the de la Vallée-Poussin kernel $\{F_\varepsilon\}$.

2. Preliminaries

The Fourier transform F of $f \in L^1(\mathbb{R})$, where $L^1(\mathbb{R})$ is the space of Lebesgue integrable functions on $\mathbb{R}$, is $F(\gamma) = \int_{-\infty}^{\infty} f(t)e^{-it\gamma}dt$. F is defined on $\hat{\mathbb{R}}(=\mathbb{R})$; and we write $f \leftrightarrow F$, $\hat{f} = F$, and $F^{\vee} = f$. $L^p(\mathbb{R})$ and $\|\ \|_p$, $1 \le p \le \infty$, are the usual Lebesgue spaces and norms. We shall have occasion to use the weak topology $\sigma(L^p, L^{p'})$, $p > 1$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

If $v \ge 1$ is measurable on $\mathbb{R}$ then $L^1_v(\mathbb{R}) = \{f : \|f\|_v = \|fv\|_1 = \int_{-\infty}^{\infty} |f(t)|v(t)dt < \infty\} \subseteq L^1(\mathbb{R})$ is a Banach space with norm $\|\ \|_v$. $A_v(\hat{\mathbb{R}})$ is the set of Fourier transforms of elements from $L^1_v(\mathbb{R})$; we write $L^1_1(\mathbb{R}) = L^1(\mathbb{R})$ and $A_1(\hat{\mathbb{R}}) = A(\hat{\mathbb{R}})$.

If $v \ge 1$ is not only measurable, but satisfies the arithmetic condition, $v(s+t) \le v(s)v(t)$, then $L^1_v(\mathbb{R})$ is a commutative Banach algebra under convolution. These Banach algebras were introduced by Beurling in 1938; and in this case the function v is a Beurling weight and $L^1_v(\mathbb{R})$ is a Beurling algebra. The main property of Beurling algebras is contained in the following theorem.

Theorem 2.1. Let $L^1_v(\mathbb{R})$ be a Beurling algebra. Spectral analysis holds for $L^1_v(\mathbb{R})$ if and only if i v is continuously differentiable,

ii $v(t)/t^{\alpha}$ is decreasing for some $\alpha \in (0,1)$,

iii v' is locally integrable,

iv v' decreases to 0 .

Example 3.2. The following Beurling weights are α-concave:

i. $v(t) = 1 + |t|^{\alpha}$, $\alpha \in [0,1)$, whose

Fourier transform V is the kernel of the classical potential $U^{\mu}(\gamma) = V*\mu(\gamma)$ where μ is a measure;

ii. $v(t) = 1 + \log(1 + |t|)$.

Lemma 3.3. Let v be an α-concave function and define the integral,

$$\forall\ \gamma \geq 0,\ C_K(\gamma) = \int_0^K \frac{v(t)}{t} \sin \gamma t\ dt,$$

for each $K \in [0,\infty)$. Set $C(\gamma) = C_{\infty}(\gamma)$.

a. $C(\gamma)$ is a positive, continuously differentiable, and locally integrable function on $(0,\infty)$.

b. There is $p > 1$ such that $\lim C_K = C$ in the weak topology $\sigma(L^p, L^{p'})$ on $(0,1)$.

c. $C(\gamma) = \int_0^{\infty} v(\frac{t}{\gamma}) \frac{\sin t}{t}\ dt$ and
$\forall\ \gamma \in (0,\infty)$, $C'(\gamma) = -\frac{1}{\gamma^2} \int_0^{\infty} v'(\frac{t}{\gamma}) \sin t\ dt < 0$.

Proof i. We define

$$a_k(\gamma) = \int_0^{\pi} \frac{v(\frac{t+k\pi}{\gamma})}{t+k\pi} \sin t\ dt > 0,\quad k \geq 0 \text{ and } \gamma > 0 .$$

Note that $a_k(\gamma) > a_{k+1}(\gamma)$ by the strict decrease of $v(t)/t$ on $(0,\infty)$. Further, if $\gamma > 0$ and $k \geq 1$, then the fact that

$v(\frac{t+k\pi}{\gamma})/(\frac{t+k\pi}{\gamma}) \leq v(\frac{\pi}{\gamma})/(\frac{\pi}{\gamma})$ for $t \in [0,\pi]$ and the hypothesis that $v(t)/t$ decreases to 0, as $t \to \infty$, allow us to use the dominated convergence theorem to observe that $\lim_{k\to\infty} \gamma a_k(\gamma) = 0$, and , hence, that $\lim_{k\to\infty} a_k(\gamma) = 0$ for each $\gamma > 0$. Combining these facts we obtain the convergence of the alternating series, $\sum_0^\infty (-1)^k a_k(\gamma) > a_0(\gamma) - a_1(\gamma) > 0$, for each $\gamma > 0$.

Also, we observe that $\sum_0^\infty (-1)^k a_k(\gamma)$ converges uniformly on compacta contained in $(0,\infty)$ because of the estimate

$$\left|\sum_n^\infty (-1)^k a_k(\gamma)\right| \leq a_n(\gamma) \leq \frac{2}{\gamma} \frac{v(\frac{n\pi}{\gamma})}{\frac{n\pi}{\gamma}} \to 0, \quad n \to \infty,$$

where once again we use the decrease of $v(t)/t$.

ii. C_K is infinitely differentiable on $(0,\infty)$.

Also, from part i we see that $C(\gamma)$ exists and equals $\sum_0^\infty (-1) a_k(\gamma)$. In particular, $C(\gamma)$ is a continuous positive function on $(0,\infty)$.

iii. We now show that $C \in L^1(0,1)$.

To this end we first observe that $C_K > 0$ on $(0,\infty)$. In fact, for each $\gamma \in (0,\infty)$, we have

$$C_K(\gamma) = \sum_{k=0}^{N} (a_{2k}(\gamma) - a_{2k+1}(\gamma)) + \frac{1}{\gamma} \int_{2\pi(N+1)}^{K\gamma} \frac{v(u/\gamma)}{u/\gamma} \sin u \, du,$$

where $K\gamma < 2\pi(N+2)$; and the right hand side is positive by the decrease of $v(t)/t$ and the fact that $a_{2k}(\gamma) > a_{2k+1}(\gamma)$ (part i).

Since $C_K \geq 0$ we invoke Fatou's lemma and have

$$0 < \int_0^1 C(\gamma)\, d\gamma \leq \varliminf_{K} \int_0^1 C_K(\gamma)\, d\gamma, \tag{3.1}$$

where K ranges over an infinite sequence tending to infinity. To see that the right hand side is finite we note that

(3.2) $$\int_0^1 C_K(\gamma)\, d\gamma = \int_0^K \frac{v(t)}{t}\,\frac{1-\cos t}{t}\, dt \le M,$$

where M is independent of K. This boundedness is a consequence of the α-concavity; in fact, $v(t)(1-\cos t)/t^2 \ge 0$, $\int_0^1 \frac{v(t)(1-\cos t)}{t^2}\, dt < \infty$, and

$$\int_1^K \frac{v(t)(1-\cos t)\, dt}{t^2} \le \int_1^\infty \frac{v(t)(1-\cos t)}{t^\alpha\, t^{2-\alpha}}\, dt < \infty.$$

(3.1) and (3.2) yield the desired integrability.

iv. Part b follows from the classical real variable theorem that

(3.3) $$\sup_K \|C_K\|_p < \infty$$

and $\lim C_K = C$ a.e. on $(0,1)$ imply $\sigma(L^p, L^{p'})$ - convergence on $(0,1)$ when $p \in (1,\infty)$. We already have the pointwise convergence, and so it remains to verify (3.3).

We begin proving (3.3) by choosing $\alpha \in (0,1)$ for which $v(t)/t^\alpha$ is decreasing on $(0,\infty)$, and then taking $p > 1$ for which $p\alpha \in (0,1)$. Because of the monotonicity of $v(t)/t$ we can employ the second mean value theorem for integrals to compute

$$\|C_K\|^p_{L^p(0,1)} \le 2^{p-1} \int_0^1 \left| \int_0^1 \frac{v(t)}{t} \sin \gamma t\, dt\right|^p d\gamma +$$

$$2^{p-1} \int_0^1 \left| v(1) \int_1^{\xi_{K,\gamma}} \frac{\sin \gamma t}{t^{1-\alpha}}\, dt + \frac{v(K)}{K^\alpha} \int_{\xi_{K,\gamma}}^K \frac{\sin \gamma t}{t^{1-\alpha}}\, dt \right|^p d\gamma \le$$

$$\frac{2^{p-1}}{p+1} \left(\int_0^1 v(t) dt\right)^p +$$

$$2^{p-1} \int_0^1 \left| \frac{v(1)}{\gamma^\alpha} \int_\gamma^{\gamma\xi_{K,\gamma}} \frac{\sin u}{u^{1-\alpha}}\, du + \frac{v(K)}{(K\gamma)^\alpha} \int_{\gamma\xi_{K,\gamma}}^{\gamma K} \frac{\sin u}{u^{1-\alpha}}\, du \right|^p d\gamma,$$

where we have used the convexity of t^p for $p > 1$ to effect the first inequality. The "partial sums" $\left|\int_a^b [(\sin u)/u^{1-\alpha}]du\right|$ are uniformly bounded by N and so we obtain

$$\|C_K\|^p_{L^p(0,1)} \leq \frac{2^{p-1}}{p+1}\left(\int_0^1 v(t)\ dt\right)^p + \frac{2^{2p-1}(Nv(1))^p}{1-\alpha p} .$$

Thus, (3.3) is valid and the proof of part b is complete.

v. Define

$$C_n(\gamma) + \sum_0^n (-1)^k a_k(\gamma), \quad \gamma > 0$$

and let $I \subseteq (0,\infty)$ be a closed interval.

We first observe that

$$(3.4) \qquad a_k'(\gamma) = -\frac{1}{\gamma^2}\int_0^\pi v'\left(\frac{t+k\pi}{\gamma}\right) \sin t\ dt, \quad \gamma \in I .$$

To check this, we recall that $a_k(\gamma)$ exists an I, and note that the derivative of its integrand with respect to γ is not only continuous for each $t \in (0,\pi)$ but is bounded by $(1/\xi^2)\ v'(t/\xi)$ which is integrable on $(0,\pi)$ (ξ is the left hand end point of I).

Because of (3.4) and the decrease of v' we have the estimates, $|a_{k+1}'(\gamma)| \leq |a_k'(\gamma)|$ and

$$|C_n'(\gamma) - C_{m-1}'(\gamma)| = \left|\sum_m^n (-1)^k a'_k(\gamma)\right| \leq |a_m'(\gamma)| \leq$$

$$\frac{1}{\gamma^2}\int_0^\pi v'\left(\frac{m\pi}{\gamma}\right) \sin t\ dt = \frac{2}{\gamma^2}\ v'\left(\frac{m\pi}{\gamma}\right);$$

consequently, the decrease of v' to 0 at ∞ yields the uniform convergence of $\{C_n'\}$ on I. Combining this fact with the already demonstrated (uniform) convergence of $\{C_n\}$ on I we conclude that C is continuously differentiable on $(0,\infty)$ (thus completing part a) and that part c holds.

q.e.d.

Remark 3.4. Because of the elementary claims in the statement of Lemma 3.3, we'd like to point out that their verifications are surprisingly but necessarily arduous.

a. We know that $C, C_K \in L^1(0,1)$, $C, C_K \geq 0$, $\lim C_k = C$ pointwise on $(0,1)$, and $\lim \int_0^1 C_K$ exists. On the other hand, the functions C and C_K do not satisfy the condition, $\lim \int_0^1 |C_K - C| = 0$. To see this, we first observe that pointwise convergence yields convergence in measure since we're on a finite measure space, and so the norm convergence is equivalent to uniform absolute continuity of $\{C_K\}$. A straightforward calculation, even for the trivial weight $v(t) = 1$, shows that this Vitali condition fails.

b. We showed that $C(\gamma)$ is continuously differentiable on $(0,\infty)$ and that the operations of differentiation and integration can be interchanged on the integrand of C (Lemma 3.3c). As such it should be noted that even though

$$C(\gamma) = \int_0^\infty \frac{v(t)}{t} \sin \gamma t \, dt = \int_0^\infty v(\frac{t}{\gamma}) \frac{\sin t}{t} \, dt ,$$

we do not have

$$C'(\gamma) = \int_0^\infty \frac{d}{d\gamma} (\frac{v(t)}{t} \sin \gamma t) \, dt;$$

in fact, this last integral not only does not converge uniformly on compacta in $(0,\infty)$, it does not converge anywhere on $(0,\infty)$.

c. We must use the fundamental theorem of calculus in Theorem 3.5, and for this purpose we only require the absolute continuity of C not the continuity of its derivative. The integrability of v' on $(0,\infty)$ is sufficient for this purpose, but is too strong a condition for the examples we have in mind, e.g., Example 3.2. Thus, we use local integrability to deal with a neighborhood of the origin (we don't have continuity of v' on $[0,\pi]$) and the fact that $C'(\gamma)$ is continuous elsewhere.

Theorem 3.5. Let v be an α-concave function; and let F be an even twice continuously differentiable, non-negative function on $\hat{\mathbb{R}}$ which satisfies the conditions $F(0) = \|F\|_\infty$ and $\operatorname{supp} F \subseteq [-\varepsilon,\varepsilon]$. Then we have $F \in A(\hat{\mathbb{R}})$, $f \leftrightarrow F$, and

$$(3.5) \qquad \|F\|_\infty \le \frac{\pi}{2C(\varepsilon)} \|f\|_v \quad ,$$

where C was defined in Lemma 3.3. If $\int \frac{v(t)}{1+t^2}\, dt < \infty$ then $f \in L^1_v(\mathbb{R})$.

Proof. i. Since F is twice continuously differentiable and has compact support we have $F \in A(\hat{\mathbb{R}})$. A similar computation, for example,

$$\int_{|t|\ge 1} v(t) \; |F^{\vee}(t)|dt = \frac{1}{\pi} \int_{|t|\ge 1} \frac{v(t)}{t^2} \left|\int_0^{\varepsilon} F^{(2)}(\gamma) \cos \gamma t \, d\gamma\right| \, dt \ ,$$

allows us to conclude that $f \in L^1_v(\mathbb{R})$ if $\int \frac{v(t)}{1+t^2}\, dt < \infty$.

ii. Using Parseval's theorem we make the estimate

$$\|f\|_v \ge \int_{\frac{1}{K}\le |t| \le K} v(t) \; |f(t)| \; dt \ge \left| \int_{\frac{1}{K}\le |t| \le K} f(t) t \, \frac{v(t)}{|t|} \operatorname{sgn} t \, dt \right| =$$

$$\left|\frac{1}{2\pi} \int_{-\infty}^{\infty} i \; F'(-\gamma) \int_{\frac{1}{K}\le |t| \le K} \frac{v(t)}{|t|} (\operatorname{sgn} t) e^{-it\gamma} dt \; d\gamma\right| =$$

$$\left|\frac{1}{\pi} \int_{-\varepsilon}^{\varepsilon} F'(\gamma) \int_{1/K}^{K} \frac{v(t)}{t} \sin \gamma t \; dt \; d\gamma\right| \ ,$$

where $K > 1$. Consequently, we've established the inequality,

$$(3.6) \qquad \forall K > 1, \; \frac{\pi}{2}\|f\|_v \ge \left|\int_0^{\varepsilon} F'(\gamma) \int_{1/K}^{K} \frac{v(t)}{t} \sin \gamma t \; dt \; d\gamma\right| .$$

Since

$$\left|\int_0^{\varepsilon} F'(\gamma) \int_0^{1/K} \frac{v(t)}{t} \sin \gamma t \, dt \, d\gamma\right| \le \int_0^{\varepsilon} |\gamma F'(\gamma)| d\gamma \int_0^{1/K} v(t) dt = \varepsilon_{K,F} ,$$

we can replace (3.6) by

$$(3.7) \quad \forall\, K > 1, \quad \frac{\pi}{2} \| f \|_v > \left|\int_0^{\varepsilon} F'(\gamma) \int_0^{K} \frac{v(t)}{t} \sin \gamma t \, dt \, d\gamma\right| - \varepsilon_{K,F} ,$$

where $\lim_{K\to\infty} \varepsilon_{K,F} = 0$. Applying Lemma 3.3b to (3.7) we obtain the inequality,

$$(3.8) \qquad \frac{\pi}{2} \| f \|_v \ge \left|\int_0^{\varepsilon} F'(\gamma) \int_0^{\infty} \frac{v(t)}{t} \sin \gamma t \, dt \, d\gamma\right| .$$

iii. We now choose some necessary constants and parameters. First note that

$$(3.9) \qquad \forall\, \gamma \in (0,1], \quad 0 \le \gamma\, C(\gamma) \le a_0(1) .$$

Since $C(\gamma) \ge 0$ and $C \in L^1(0,1)$ (Lemma 3.3a) we know that $\int_0^{\gamma} C(\lambda)\, d\lambda$ is increasing on $[0,1]$. We are given $\varepsilon > 0$ and we take $\nu \in (0,1]$. By the continuity of F' and the hypothesis that $F'(0) = 0$ we can choose $\eta = \eta(\varepsilon,\nu) \in (0, \min(\varepsilon,1))$ such that

$$(3.10) \qquad \sup_{\gamma\in[0,\eta]} |F'(\gamma)| \le \frac{\nu}{2\int_0^1 C(\lambda) d\lambda}$$

and

$$(3.11) \qquad \sup_{\gamma\in[0,\eta]} |F'(\gamma)| \le \frac{\nu}{2\, a_0(1)} .$$

Using (3.10) we make the estimate,

$$\left|\int_0^{\eta} F'(\gamma)\, C(\gamma) d\gamma\right| \le \frac{\nu}{2} ,$$

and, thus, we obtain

$$(3.12) \qquad \frac{\pi}{2} \| f \|_v \ge \left|\int_{\eta}^{\varepsilon} F'(\gamma)\, C(\gamma)\, d\gamma\right| - \frac{\nu}{2}$$

from (3.8), where $\nu > 0$ is arbitrary and η depends on ε and ν.

iv. Formally, we compute

$$\left|\int_\eta^\varepsilon F'(\gamma)C(\gamma)d\gamma\right| = \left|- F(\eta)\ C(\eta) - \int_\eta^\varepsilon F(\gamma) \int_0^\infty \frac{d}{d\gamma} v(\frac{u}{\gamma}) \frac{\sin u}{u} du\ d\gamma\right| =$$

$$= \left|F(\eta)\ C(\eta) - \int_\eta^\varepsilon \frac{F(\gamma)}{\gamma^2} \int_0^\infty v'(\frac{u}{\gamma}) \sin u\ du\ d\gamma\right| \geq$$

(3.13)
$$F(\eta)\ C(\eta) - \left|\int_\eta^\varepsilon \frac{F(\gamma)}{\gamma^2} \int_0^\infty v'(\frac{u}{\gamma}) \sin u\ du\ d\gamma\right| \geq$$

$$F(\eta)\ C(\eta) - \|F\|_\infty \int_\eta^\varepsilon \frac{1}{\gamma^2} \int_0^\infty v'(\frac{u}{\gamma}) \sin u\ du\ d\gamma =$$

$$F(\eta)\ C(\eta) + \|F\|_\infty \int_\eta^\varepsilon \frac{d}{d\gamma} C(\gamma)\ d\gamma =$$

$$C(\varepsilon)\|F\|_\infty - C(\eta)(\|F\|_\infty - F(\eta)) .$$

To justify (3.13) we first note that $\int_0^\infty v'(\frac{u}{\gamma}) \sin u\ du$ exists and is non-negative since v' decreases to 0 on $(0,\infty)$ and because the integral is an alternating series. The second inequality of (3.13) follows.

Next, Lemma 3.3b yields the first and penultimate equalities in (3.13). The remaining inequality and equalities in (3.13) are clear.

v. To complete the proof we observe that

$$C(\eta)(\|F\|_\infty - F(\eta)) = |C(\eta)(\|F\|_\infty - F(\eta))| \leq$$

(3.14)

$$\eta\ C(\eta) \sup_{\gamma\in[0,\eta]} |F'(\gamma)| \leq \frac{\eta\nu\ C(\eta)}{2\ a_0(1)} \leq \frac{\nu}{2}$$

by (3.9) and (3.11); and, hence, by combining (3.12), (3.13), and (3.14) we obtain $(\pi/2)\|f\|_V \geq C(\varepsilon)\ \|F\|_\infty - \nu$ which yields (3.5) since ν is arbitrary.

q.e.d.

Example 3.6a. By the definition of $C(\varepsilon)$, we can replace (3.5) by

$$\|F\|_\infty \le \frac{\pi}{2(a_0(\varepsilon)-a_1(\varepsilon))}\|f\|_v$$

where $a_k(\varepsilon) = \int_0^\pi \frac{v(t/\varepsilon)}{t} \sin t \, dt$.

b. If $v(t) = 1 + |t|^\alpha$, $\alpha \in (0,1)$, then

$$C(\varepsilon) = \frac{\pi}{2} + \frac{1}{\varepsilon^\alpha}\int_0^\infty \frac{\sin t}{t^{1-\alpha}}\,dt = \frac{\pi}{2} + \frac{1}{\varepsilon^\alpha}\frac{\Gamma(\alpha)}{\csc(\frac{\pi\alpha}{2})} .$$

4. Fourier transform norm inequalities for Beurling algebras

We emphasize that the constant in the following result is support dependent.

Theorem 4.1. Let v be an α-concave Beurling weight. For each $f \in L^1_v(\mathbb{R})$, whose Fourier transform F is supported by $[-\varepsilon,\varepsilon]$, we have

(4.1) $$\|F\|_\infty \le \left(\frac{2\pi}{C(2\varepsilon)}\right)^{1/4} \|f\|_v ,$$

where C was defined in Lemma 3.3.

Proof. i. Suppose F is an even, twice continuously differentiable, non-negative function on $\hat{\mathbb{R}}$ which satisfies the condition, supp $F \subseteq [-\varepsilon,\varepsilon]$. Let $\|F\|_\infty = F(\gamma)$, $\gamma \in [0,\varepsilon)$, and define $G(\lambda) = F(\lambda-\gamma)F(-\lambda-\gamma)$. Clearly, we have $\|G\|_\infty = G(0) = \|F\|^2_\infty$ and supp $G \subseteq [-2\varepsilon, 2\varepsilon]$. G satisfies the conditions of Theorem 3.5, and so

$$\|F\|^2_\infty = \|G\|_\infty \le \frac{\pi}{2C(2\varepsilon)}\|g\|_v \le \frac{\pi}{2C(2\varepsilon)}\|f\|^2_v ,$$

where $\hat{g} = G$ and where the second inequality follows since v is an (even) Beurling weight. Consequently, we've shown the inequality

(4.2) $$\|F\|_\infty \le \left(\frac{\pi}{2C(2\varepsilon)}\right)^{1/2} \|f\|_v ,$$

for such $f \leftrightarrow F$.

ii. Next, suppose F is twice continuously differentiable on $\hat{\mathbb{R}}$ and that $\operatorname{supp} F \subseteq [-\varepsilon,\varepsilon]$. Define the functions $G(\lambda) = F(\lambda)\overline{F(\lambda)}$ and $H(\lambda) = G(\lambda) + G(-\lambda)$.

$\overline{F} \in A_v(\hat{\mathbb{R}})$ since $f \in L^1_v(\mathbb{R})$ and v is even. Thus, $G \geq 0$ is an element of $A_v(\hat{\mathbb{R}})$ by the submultiplicativity of v. Further, G is twice continuously differentiable and $\operatorname{supp} G \subseteq [-\varepsilon,\varepsilon]$. Finally, $\|G\|_\infty = \|F\|^2_\infty$ since $|F(\gamma)|^2 = F_1(\gamma)^2 + F_2(\gamma)^2 = G(\gamma)$ where $F = F_1 + i\,F_2$ and $\|F\|_\infty = |F(\gamma)|$.

By definition, H is an even, twice continuously differentiable, non-negative function on $\hat{\mathbb{R}}$ which satisfies the condition, $\operatorname{supp} H \subseteq [-\varepsilon,\varepsilon]$. We have $\|H\|_\infty \geq \|G\|_\infty = \|F\|^2_\infty$ because $G \geq 0$; and we apply (4.2) to H obtaining the inequality,

(4.3) $$\|F\|^2_\infty \leq \|H\|_\infty \leq \left(\frac{\pi}{2C(2\varepsilon)}\right)^{1/2}\|h\|_v \leq \left(\frac{2\pi}{C(2\varepsilon)}\right)^{1/2}\|g\|_v \leq \left(\frac{2\pi}{C(2\varepsilon)}\right)^{1/2}\|f\|_v ,$$

where $\hat{g} = G$ and $\hat{h} = H$ and where we've used the evenness and submultiplicativity of v to obtain the last two inequalities.

iii. For the general case, take $f \in L^1_v(\mathbb{R})$ where $\operatorname{supp} F \subseteq [-\varepsilon,\varepsilon]$.

Let ρ_η be a non-negative, even, twice continuously differentiable function which satisfies the conditions, $\frac{1}{2\pi}\int \rho_\eta(\lambda)\, d\lambda = 1$ and $\operatorname{supp} \rho_\eta \subseteq [-\eta,\eta]$.

Set $G_\eta = F*\rho_\eta$ so that G_η is twice continuously differentiable and $\operatorname{supp} G_\eta \subseteq [-(\varepsilon+\eta),\ \varepsilon+\eta]$. Since $|\rho_\eta^\vee(t)| \leq 1$ we have $\|g_\eta\|_v \leq \|f\|_v$ where $\hat{g}_\eta = G_\eta$; in particular, $g_\eta \in L^1_v(\mathbb{R})$. We can apply (4.3) to G_η, and we obtain

$$(4.4) \qquad \|G_\eta\|_\infty^2 \le \left(\frac{2\pi}{C(2(\varepsilon+\eta))}\right)^{1/2} \|g_\eta\|_v^2 \le \left(\frac{2\pi}{C(2(\varepsilon+\eta))}\right)^{1/2} \|f\|_v^2 \quad .$$

For each $\nu > 0$, choose $\eta_\nu > 0$ such that $|\|G_\eta\|_\infty - \|F\|_\infty| < \nu$ for $\eta \in (0,\eta_\nu]$. Thus, (4.4) yields the inequality,

$$\|F\|_\infty \le \left(\frac{2\pi}{C(2(\varepsilon+\eta))}\right)^{1/4} \|f\|_v + \nu \ ,$$

from which we deduce (4.1) by the continuity of C.

q.e.d.

5. Helson sets in Beurling algebras

Definition 5.1. Let $E \subseteq \hat{\mathbb{R}}$ be compact, let $L^1_v(\mathbb{R})$ be a regular Beurling algebra, let $A_v(E)$ be the restriction algebra (to E of elements from $A_v(\hat{\mathbb{R}})$, and let $C(E)$ be the algebra of continuous functions on E. E is a v-Helson set if $A_v(E) = C(E)$.

If $v(t) = 1$ then v-Helson sets are the classical Helson sets used to study the structure, including the arithmetic structure, of group algebras, e.g., [1]. Clearly, v-Helson sets are Helson sets, and the following result describes their trivial nature for most Beurling weights.

Theorem 5.2. Let $E \subseteq \hat{\mathbb{R}}$ be compact, let $L^1_v(\mathbb{R})$ be a regular Beurling algebra, and suppose that $v(t) \ge w(t) = 1 + |t|^\alpha$ on $\mathbb{R}$ for some $\alpha \in (0,1)$. If E is a v-Helson set then E is finite.

Proof. i. Since $A_v(\hat{\mathbb{R}}) \subseteq A_w(\hat{\mathbb{R}})$, it is sufficient to prove that E is finite when E is a w-Helson set. We denote $A_w(\hat{\mathbb{R}})$ by $A_\alpha(\hat{\mathbb{R}})$ and $A_w(E)$ by $A_\alpha(E)$.

ii. Define the quantity,

$$\|F\|_\alpha = \sup_{\substack{\gamma \\ \lambda\neq 0}} \frac{|F(\gamma+\lambda) - F(\gamma)|}{|\lambda|^\alpha} .$$

$\Lambda_\alpha(\hat{\mathbb{R}})$ is the space of continuous functions F on $\hat{\mathbb{R}}$ for which $\|F\|_\alpha < \infty$. $\Lambda_\alpha(E)$ is the space of restrictions of $\Lambda_\alpha(\hat{\mathbb{R}})$ to E.

iii. In this subsection we'll verify the classical fact,

$$A_\beta(\hat{\mathbb{R}}) \subseteq \Lambda_\alpha(\hat{\mathbb{R}}),\ \beta \geq \alpha ; \tag{5.1}$$

and, consequently, by the compactness of E, we obtain

$$A_\alpha(E) \subseteq \Lambda_\alpha(E) . \tag{5.2}$$

As a preliminary calculation, note that $|e^{iu} - 1|^2/|u|^2 \leq 2$, $u \neq 0$; and so, for any $K > 0$, we find that

$$\frac{|e^{it\lambda} - 1|}{|\lambda|^\alpha} \leq \begin{cases} \sqrt{2}\ |\lambda|^{1-\alpha}\ |t| \leq \sqrt{2}\ K^{1-\alpha}\ |t|, & \text{if } |\lambda| \leq K, \\ 2/K^\alpha, & \text{if } |\lambda| > K . \end{cases}$$

Setting these two bounds equal implies $K = \sqrt{2}/|t|$. Consequently, we have $|e^{it\lambda} - 1|/|\lambda|^\alpha \leq 2^{(1-\alpha/2)}|t|^\alpha$.

Using this calculation and taking $\hat{f} = F \in A_\beta(\hat{\mathbb{R}})$ we make the estimate,

$$\|F\|_\alpha \leq \sup_{\lambda\neq 0} \int_{-\infty}^{\infty} |f(t)|\frac{|e^{it\lambda} - 1|}{|\lambda|^\alpha}\, dt \leq$$

$$2^{(1-\alpha/2)} \int_{-\infty}^{\infty} |f(t)|(1 + |t|^\alpha)\, dt \leq 2^{(1-\alpha/2)}\ \|f\|_w \quad ;$$

and, therefore, (5.1) and (5.2) are obtained.

iv. It is clear that $\Lambda_\alpha(E) \subseteq C(E)$; and this inclusion, combined with (5.2) and the hypothesis that E is v-Helson, imply that $\Lambda_\alpha(E) = C(E)$. It is easy to see that this equality forces E to be a finite set.

q.e.d.

Remark 5.3. a. Our original argument for Theorem 5.2, which we give in Remark 5.3c, was intimately related to Theorem 4.1. This argument did not involve Λ_α and was valid for Beurling algebras larger than $L^1_w(\mathbb{R})$, but not for $L^1(\mathbb{R})$.

b. To outline the argument alluded to in Remark 5.3a we first state the following adaptation of a technique formulated and implemented by Varopoulos and Drury, e.g., [1]. Let E be a v-Helson set, with Helson constant K, and a v-spectral synthesis set; E satisfies the condition that for each closed disjoint pair $E_1, E_2 \subseteq E$, there exists $\hat{f} = F$ with the properties that $F = 1$ on a neighborhood of E_1, $F = 0$ on a neighborhood of E_2, and $\|f\|_v \leq K$.

c. It is easy to see that w-Helson sets are totally disconnected and that closed countable subsets of w-Helson sets are w-Helson. Also, using the fact that points are w-strong Ditkin sets, we can check by conventional methods that closed countable sets are sets of w-spectral synthesis. Thus, given an infinite w-Helson and w-spectral synthesis set, we can choose a countably infinite w-Helson and w-spectral synthesis subset E. We see that the condition from the result in Remark 5.3b fails because of Theorem 4.1; and so, since the w-spectral synthesis of countable sets is a fact, we are forced to conclude that w-Helson sets are finite.