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Throughput and Packet Error Probability of Cellular Frequency-Hopped Spread-Spectrum Radio Networks

by

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THROUGHPUT AND PACKET ERROR PROBABILITY OF CELLULAR FREQUENCY-HOPPED SPREAD-SPECTRUM RADIO NETWORKS

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ABSTRACT

We characterize multiple-access interference for cellular mobile networks, in which users are assumed to be Poisson-distributed in the plane and to employ frequency-hopped spread-spectrum signaling with a transmitter-oriented assignment of frequency-hopping patterns. Exact expressions for the bit error probabilities are derived for binary coherently demodulated systems without coding. Approximations for the packet error probability are derived for coherent and noncoherent systems and these approximations are applied when forward-error-control coding is employed. In all cases, the effects of varying interference power are accurately taken into account according to some propagation law.

Numerical results are given in terms of bit error probability for the exact case and throughput for the approximate analyses. Comparisons are made with previously derived bounds and it is shown that these tend to be very pessimistic.

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1 INTRODUCTION

Research in the areas of communication networks and spread-spectrum communications has failed to combine the two; the use of spread-spectrum techniques with networking techniques in order to provide enhanced multi-user capabilities and a higher degree of resistance to interference, whether hostile or benign.

Moreover many of the spread-spectrum network models proposed in the past have lacked precision in their descriptions of the effects of spread-spectrum techniques on network performance. The general practice in papers like [1] and [3] has been to model the effects of spread-spectrum techniques via the use of some threshold (a number of users beyond which communication is not possible) or via some processing gain. Unfortunately, this is not generally considered to be an accurate model, and the results obtained may be optimistic or pessimistic, depending upon assumptions made. In addition, these analyses do not accurately account for data modulation and forward error-control coding.

In [1], a model was presented for a frequency-hopped (FH) spread-spectrum multiple access (SSMA) digital cellular telephone network and an expression was derived for throughput. Notable features of that model were the characterization of user mobility in terms of a two-dimensional Poisson process and the treatment of transmitter power attenuation with distance. This model superceded that of [18].

In [2], the authors described and analyzed a digital cellular mobile network using some of the techniques of [1]. While adding to the model the interference due to users in other cells, the authors left out the spatial Poisson model and did not account explicitly for power attentuation with distance. Spatial Poisson modeling was also used in [3], in which the authors modeled spread-spectrum multihop networks. In [4], a more precise approach was taken to the spread-spectrum multihop network problem, while using some similar network features in the analysis.

In this paper, the modeling techniques described above are combined with a precise characterization of the effects of frequency hopping for the purpose of modeling frequency-hopped digital cellular networks. This analysis also incorporates different data modulation types (coherent and noncoherent) and forward error-control coding. Also, unlike several of the previous analyses, some of the techniques presented here allow for the computation of bit/symbol or packet error probability, as desired. In addition to modeling a cell of a FH/SSMA cellular mobile radio network, this also corresponds

to a more accurate analysis of communications in the presence of secondary multiple-access interference in that it accounts for varying powers for different interferers according to their distances from the receiver (compare, for example, to [11] and [19]).

An overview of the contents of this paper is as follows. In Section II, the network model is presented. Section III contains analyses of the model for coherent and non-coherent modulation/demodulation using approximation techniques. Section IV provides an exact analysis for coherent systems using a characteristic function method. In Section V, numerical results are presented and compared with previously derived results. Section 6 contains the concluding remarks.

2 NETWORK MODEL

The network model used is in this paper similar to the one found in [1]. Users transmit packets to other users via fixed relay (or "base") stations. As is the case with [1], we will focus on the user-to-station channel. Analysis of the station-to-user channel would then proceed in a similar fashion.

Let the number of users per unit area and hop be Poisson distributed with parameter λ. Also, let $M = (x, y)$ represent a position on the plane. Denote the origin by $0 = (0, 0)$; this will be assumed, without loss of generality, to be the position of the fixed station.

Systems like this generally offer voice and data service. In the case of voice service, a steady stream of information must be transmitted and delay can not be tolerated; however, moderate error rates can be tolerated. In the case of data service, traffic is bursty and large delays can be tolerated but small error rates are necessary. In this paper, it is assumed that voice (telephone) service is predominant.

Slotted ALOHA is taken to be the channel access protocol and the system is assumed to utilize (slow) frequency hopping as the means of spreading the spectrum. Transmitter-oriented assignment of hopping patterns is also assumed. Each mobile has its own frequency-hopping pattern for transmission and reception; the base station has a list of all of the patterns and can listen to several of them simultaneously. We restrict our attention to the case of mobile-to-base communication. We consider the situation in which a mobile station has already established communication with the base station at the center of the cell and now communicates in the presence of secondary (or multiple-access) interference from other mobile. It is assumed that only the other terminals in the cell of interest may interfere, which reflects the use of other frequency bands in all adjacent cells, and that frequencies are only reused in cells distant enough so that the effects are negligible.

Synchronization at the packet level is assumed feasible for all users. Thus, uncertainties in the timing between different users can be small relative to packet duration; however, since they might not be small relative to the dwell time of the frequency hopper, we opt for a general asynchronous frequency hopping system.

3 APPROXIMATIONS AND BOUNDS ON PACKET ERROR PROBABILITY AND THROUGHPUT

Our approximation methods are developed below. These can be used for coherently and noncoherently modulated systems with or without forward error-control coding.

3.1 Analysis Based on Gaussian Approximation Techniques

We begin with **noncoherent** signaling. In this case, the system is assumed to utilize M-ary frequency-shift keying (MFSK); binary FSK is a special case of MFSK, and therefore the analysis holds for this, as well. Noncoherent demodulation is used at the receiver.

The signal transmitted by the k-th mobile user is of the form

$$s_k(t) = \sqrt{2P_0}\Psi(t)\ \cos\left\{2\pi\left[f_c + b_k(t)\Delta + f_k(t)\right]t + \theta_k(t)\right\}, \qquad (1)$$

where P_0 is the transmitted signal power and is assumed to be the same for all mobile stations; $b_k(t) = \{2b_n^{(k)}\}$, where $b_n^{(k)} \in \{1, 2, \ldots, M\}$ is the k-th sequence of M-ary information symbols of duration T each; $\Psi(t)$ is a pulse-shaping waveform; $\theta_k(t)$ is the sum of phase shifts due to modulation and frequency hopping generated by the k-th transmitter; f_c is the carrier frequency; 2Δ is the spacing between any two consecutive tones out of the M different tones; and $f_k(t)$ is the k-th hopping sequence.

In our analysis, we consider the attenuation of signal power with distance. Let $P(M)$ denote the received power of a signal originating at the mobile station located at the point M of the plane and received at the origin. We will assume that $P(M)$ is given by

$$P(M) = \begin{cases} P_0 r_0^{-\alpha}, & \text{if } |0M| \le r_0 \\ P_0 |0M|^{-\alpha}, & \text{if } |0M| > r_0, \end{cases} \qquad (2)$$

where $|0M|$ denotes the Euclidean distance between M and the origin 0, r_0 is some small radius within which $P(M)$ is assumed to be approximately constant, P_0 is the transmitted power, and α is a constant that reflects the speed of attenuation. r_0 is necessary because, as M is brought arbitrary close to 0, $|0M|^{-\alpha}$ approaches infinity. A particular signal arriving at the receiver (and originating from M_k) has the form

$$r_k(t) = \sqrt{2P(M_k)}\Psi(t)\ \cos\left\{2\pi\left[f_c + b_k(t)\Delta + f_k(t)\right]t + \tilde{\theta}_k(t)\right\}, \qquad (3)$$

where $\sqrt{2P(M_k)}$ represents the received signal amplitude and $\tilde{\theta}_k(t)$ is the total phase shift. Since the system under consideration is a multiple-access one, the received signal has the form

$$r(t) = \sum_k r_k(t - \tau_k) + n(t), \tag{4}$$

where each $r_k(t)$ is given by (3), τ_k is a random delay, and $n(t)$ is zero-mean additive white Gaussian noise (AWGN) with two-sided spectral density $\frac{N_0}{2}$.

Fig. 1 shows the receiver for the i-th signal in a binary FSK system; the M-ary receiver similar in form but with M branches. The ith signal is the signal of interest. $r(t)$ is dehopped and the dehopper output is given by

$$\begin{aligned} r_d(t) = \sum_k \Bigg\{ & \sqrt{\frac{P(M_k)}{2}} \delta\left[f_k(t - \tau_k), f_i(t)\right] \Psi(t - \tau_k) \\ & \times \cos\left[2\pi\left[f_c + b_k(t - \tau_k)\Delta\right] + \phi_k(t)\right] \Bigg\} + n_d(t). \end{aligned} \tag{5}$$

Here $f_k(\cdot)$ and $f_i(\cdot)$ represent the k-th and ith hopping sequences, respectively; $n_d(t)$ is a zero-mean Gaussian random process with spectral density $\frac{N_0}{8}$; $\delta(u, v) = 1$, if $u = v$, and 0, otherwise; and $\phi_k(t)$ represents the total phase shift of the signal after dehopping. The sufficient statistics $\tilde{R}_m^2$ are computed as shown. Were it not for the multiple-access interference, the computation of the probability of error would be straightforward, as for example in [10]. However, this difficulty can be overcome by means of conditioning the error probability on the potential number of interfering users and the actual number of interfering users causing hits on the desirable received signal; recall that a hit occurs when $f_k(t - \tau_k) = f_i(t)$ (refer to [5] and [6]). To evaluate the conditional symbol-error probability just described, we use the Gaussian approximation technique, as developed in [5] and [6], where the $Z_{c,m}$ and $Z_{s,m}$ were approximated by zero-mean Gaussian random variables with the same second-order moments. Given this, the *symbol error probability* can be derived (see [5] or [6]) and is given by

$$P_{e,i} = \sum_{m=1}^{M-1} \binom{M-1}{m} \frac{(-1)^m}{m+1} \exp\left[-\frac{m}{2(m+1)\tilde{\sigma}_Z^2}\right], \tag{6}$$

in which $\tilde{\sigma}_Z^2$ is the variance of $Z_{c,m}$ and $Z_{s,m}$ and is given by

$$\tilde{\sigma}_Z^2 = \left[2\frac{E_b}{N_0} g(|0M_i|) \log_2 M\right]^{-1} + \frac{1}{g(|0M_i|)} \sum_{k \neq i} g(|0M_k|)\sigma_{k,i}^2. \tag{7}$$

In (7), $E_b = P_0 T$ is the transmitted signal energy; $\sigma^2_{k,i}$ is the variance of the interference with the i-th signal due to the k-th signal; and $g(|0M|) = |0M|^{-\alpha}$, if $|0M| > r_0$, and $r_0^{-\alpha}$, if $|0M| \leq r_0$, which follows from (2).

Regarding the effects of full and partial hits, their independent consideration may result in a mathematically intractable expression, if we attempt to use (6) and (7) in subsequent expressions. In order to circumvent this problem, we make the assumption that all hits are full hits. Note that this causes the result to be pessimistic, since we assume that the interference caused by any hit is that of a full hit, while the probability of a hit is the sum of the probabilities of full and partial hits. Under this assumption (7) becomes

$$\tilde{\sigma}^2_Z = \left[2\frac{E_b}{N_0} g(|0M_i|) \log_2 M\right]^{-1} + \frac{\sigma_f^2}{g(|0M_i|)} \sum_{k \neq i} g(|0M_k|), \qquad (8)$$

in which σ_f^2 is used to denote the variance of interference under the assumption that any hit that occurs is a full hit (note that this is independent of which of the other signals is interfering). Expressions for σ_f^2 can be found in [5], in which it is also shown that, for random FH patterns, $\sigma_f^2 = m_\psi / M$. For a rectangular shaping waveform $\Psi(t)$, $m_\psi = 1/3$; for a sine shaping waveform, $m_\psi = (15 + \pi^2)/12\pi^2$.

At this point we use a three-term exponential approximation to $P_{c,i}$, the *symbol correctness probability*; an exponential approximation is required to facilitate the remainder of this analysis. Such an approximation can be made using techniques like Prony's method (see [8]), or curve-fitting. A least-squares curve fit was used to obtain the results shown below; as a result, we have

$$\begin{aligned} P_{c,i} &= 1 - P_{e,i}(x) \\ &\approx \sum_{\nu=1}^{3} C_\nu e^{\gamma_\nu x}, \end{aligned} \qquad (9)$$

where we define x to be $\tilde{\sigma}^2_Z$. In using curve-fitting for (9), we considered the range of x to be the interval $[0, 2]$; this was chosen on the basis of the various parameters characterizing the statistics of user population and the varying distances from the base station. If we substitute for x in (9) and

substitute (8) for $\tilde{\sigma}_Z^2$, we get

$$P_{c,i} \approx \sum_{\nu=1}^{3} C_\nu \exp\left[\gamma_\nu \left(\left[2\frac{E_b}{N_0} g(|0M_i|)\log_2 M\right]^{-1} + \frac{\sigma_f^2}{g(|0M_i|)} \sum_{k\neq i} g(|0M_k|)\right)\right]. \tag{10}$$

Let us now write (10) in the form

$$P_{c,i} = \sum_{\nu=1}^{3} C_\nu H(\nu, \tilde{r}) e^{\delta_\nu(\tilde{r}) \sum_{k\neq i} g(|0M_k|)}. \tag{11}$$

In (11), we have introduced the notation

$$H(\nu, \tilde{r}) = \exp\left[\gamma_\nu \left(2\frac{E_b}{N_0} g(\tilde{r})\log_2 M\right)^{-1}\right]$$

and

$$\delta_\nu(\tilde{r}) = \frac{\gamma_\nu \sigma_f^2}{g(\tilde{r})}$$

in order to simplify this and subsequent expressions. Also, we have taken $\tilde{r} = |0M_i|$, the distance of the "terminal of interest."

Let us reiterate that the expression given in (11) represents the conditional probability of symbol correctness, which is conditioned on $\tilde{r}$ (the number of potential interferers present), and on the number of hits that actually occur given the above number of potential interferers. However, since our ultimate goal is to determine *packet* throughput, we require a *packet* error correctness probability.

The development of the probability expression that corresponds to (11) above for the **coherent** case is similar to the one carried out above for the noncoherent case. In this case, the system is assumed to utilize binary phase-shift keying (BPSK) with coherent demodulation. The signal transmitted by the kth mobile station is of the form

$$s_k(t) = \sqrt{2P_0} b_k(t) \Psi(t) \cos\left[2\pi\left[f_c + f_k(t)\right]t + \theta_k(t)\right], \tag{12}$$

in which all quantities are as above, except that $b_n^{(k)} \in \{-1, +1\}$. We now have information bits rather than M-ary symbols. From (12) we obtain the following expression for the k-th received symbol

$$r_k(t) = \sqrt{2P(M_k)} b_k(t) \Psi(t) \cos\left[2\pi\left[f_c + f_k(t)\right]t + \tilde{\theta}_k(t)\right]. \tag{13}$$

The received signal, given by (4), goes through the same dehopper and the dehopper output is given by

$$r_d(t) = \sum_k \left\{ \sqrt{\frac{P(M_k)}{2}} \delta \left[f_k(t-\tau_k), f_i(t) \right] b_k(t-\tau_k) \Psi(t-\tau_k) \right. \\ \left. \cos\left[2\pi f_c t + \phi_k(t)\right] \right\} + n_d(t). \tag{14}$$

This is fed into a coherent demodulator and the sufficient statistic for the reception of the lth bit of the ith transmitted signal is given by

$$Z_i = \int_{lT}^{(l+1)T} r_d(t) \Psi(t) \cos(2\pi f_c t) dt. \tag{15}$$

Assuming that the receiver is synchronized to the ith signal, Z_i can be rewritten as

$$Z_i = \sqrt{\frac{P(M_i)}{8}}\, T\ b_l^{(i)} + \sum_{k \neq i} \sqrt{\frac{P(M_k)}{8}}\, T\ I_{k,i} + \hat{n}, \tag{16}$$

where $I_{k,i}$ represents the normalized interference caused by the kth signal to the reception of the ith signal and $\hat{n}$ is a zero-mean Gaussian random variable with variance $N_0 T/16$.

The decision process is concerned with determining whether the current bit is $+1$ or -1. The test is, therefore, to determine whether $Z_i > 0$ or $Z_i < 0$. This allows some further manipulation of Z_i. $\tilde{Z}_i$, the modified test statistic after we divide by P_0, is of the form

$$\tilde{Z}_i = \sqrt{g(|OM_i|)} b_l^{(i)} + \sum_{k \neq i} \sqrt{g(|OM_k|)} I_{k,i} + \tilde{n}, \tag{17}$$

where $\tilde{n}$ is a Gaussian random variable with variance $N_0/2Eb2E_b$, where $E_b = P_0 T$ is the transmitted bit energy. We utilize the Gaussian approximation technique, namely we approximate $\tilde{Z}_i$ by a zero-mean Gaussian random variable with variance $\sigma^2_{\tilde{Z}_i}$, which denotes the variance of $\tilde{Z}_i$. This is the coherent analog to the Gaussian approximation found in [5] and [6]. Given this approximation, the conditional symbol (bit) error probability is

$$P_{e,i} = \frac{1}{2} \operatorname{erfc} \left[\frac{\sqrt{g(|0M_i|)}}{\sqrt{2\sigma^2_{\tilde{Z}_i}}} \right], \tag{18}$$

in which erfc(·) is the usual complementary error function. ¿From our earlier expressions, it can be shown that

$$\sigma^2_{\tilde{Z}_i} = \frac{1}{2}\left[\frac{E_b}{N_0}\right]^{-1} + \sum_{k\neq i} g(|0M_k|)\sigma^2_{k,i}, \tag{19}$$

where $\sigma^2_{k,i}$ is the variance of $I_{k,i}$. If we once again assume that all hits that occur are full hits, we find that

$$\sigma^2_{\tilde{Z}_i} = \frac{1}{2}\left[\frac{E_b}{N_0}\right] + \sigma^2_f \sum_{k\neq i} g(|0M_k|), \tag{20}$$

where σ^2_f is the variance of the interference terms under the assumption that any hit that occurs is a full hit. It can be shown, using the results of [5], that $\sigma^2_f = m_\psi$, where m_ψ is as discussed above.

At this point we use a three-term exponential approximation to $P_{c,i}$ to obtain

$$\begin{aligned} P_{c,i} &= 1 - P_{e,i} = 1 - \frac{1}{2} \\ &\approx \sum_{\nu=1}^{3} C_\nu e^{\gamma_\nu x}, \end{aligned} \tag{21}$$

where we define x to be $2\sigma^2_{\tilde{Z}_i}/g(|0M_i|)$. Again we use $[0,2]$ as the range of x. If we substitute for x in (21) and substitute (20) for $\sigma^2_{\tilde{Z}_i}$, we get

$$\begin{aligned} P_{c,i} \approx \sum_{\nu=1}^{3} C_\nu \exp\Bigg(\gamma_\nu \Bigg[\left(g(|0M_i|)\frac{E_b}{N_0}\right)^{-1} \\ + \frac{2m_\psi}{g(|0M_i|)} \sum_{k\neq i} g(|0M_k|)\Bigg]\Bigg). \end{aligned} \tag{22}$$

$H(\nu,\tilde{r})$ and $\delta\nu(\tilde{r})$ are given by

$$H(\nu,\tilde{r}) = \exp\left[\gamma_\nu \left(g(\tilde{r})\frac{E_b}{N_0}\right)^{-1}\right]$$

and

$$\delta_\nu(\tilde{r}) = \frac{2\gamma_\nu m_\psi}{g(\tilde{r})}.$$

This completes the development of the conditional coherent bit correctness probability.

As mentioned earlier, in order to compute packet throughput we need a probability of packet correctness or error. We would like to get an expected packet error probability with respect to $\tilde{r}$ and the number of users. We do this by taking a known error probability expression and substituting it in our result. Such probability expressions can be found in or adapted from expressions found in many sources (e.g. [9]). Expectations are then taken with respect to $\tilde{r}$, the number of users, and the number of hits. This method is demonstrated below for the cases of uncoded packets and RS(n,k)-coded (encoded with a Reed-Solomon code of length n and rate k/n) packets; the development is the same for other coding schemes.

For a general *uncoded packet* of length L we have

$$P_{e,pkt} = 1 - P_c^L.$$

Since $P_c(\tilde{r}, K)$ in (11) is dependent on $\tilde{r}$, on the number of potential interferers, and on the number of hits given K, we must take expectations over all of these. Thus,

$$\overline{P}_{e,pkt} = 1 - E_{\tilde{r}} E_K E_h \{P_c^L(\tilde{r}, K)\}. \tag{23}$$

The inner expectation in (23) requires some explanation. The probability of having k hits, when there are in addition to the user i, K other users present, is given by $\binom{K}{k} P_h^k (1-P_h)^{K-k}$, where P_h is the probability of a full hit. As discussed in [7], for stationary Markov random hopping patterns P_h is given by

$$P_h = \frac{1}{q}\left(1 + \frac{1}{N_s}\right), \tag{24}$$

where q is the number of different frequencies and N_s the number of symbols per hop. As discussed in [7], for moderately large values of q, (24) either bounds or approximates the hit probabilities for memoryless random hopping patterns and Reed-Solomon periodic hopping patterns. Thus (24) is a good general expression for most purposes.

Suppose now that the summation in the exponent in (11) can be decomposed into collections of interferers having equal $g(|0M|)$. We would then have

$$\sum_{k \neq i} g(|0M_k|) = \sum_{j=1}^{n} k_j g_j, \tag{25}$$

where k_j is the number of interferers with $g(|0M|) = g_j$.

Let us consider some planar region A and a partition of A given by $\{A_j\}_{j=1}^n$ such that for all mobile users located in A_j, $g(|0M|) = g_j$. Let $N(A_j)$ denote the number of such mobile users. We now have a total number of users with a given attenuation and a probability that, out of a given population of K potential interferers, there are k hits. Then one may take $K_j = N(A_j)$ to be the population in that probability expression.

¿From (11) we get by expanding the trinomial

$$\begin{aligned} P_c^L(\tilde{r}, K) &= \sum_{m=0}^{L}\sum_{i=0}^{m}\binom{L}{m}\binom{m}{i}[C_1H(1,\tilde{r})]^m[C_2H(2,\tilde{r})]^i \\ &\quad\times[C_3H(3,\tilde{r})]^{L-m-i}\exp\Big[[m\delta_1(\tilde{r})+i\delta_2(\tilde{r}) \\ &\quad+(L-m-i)\delta_3(\tilde{r})]\sum_{k\neq i}g(|0M_k|)\Big]. \end{aligned} \tag{26}$$

Considering this, the discussion of $E_h\{\cdot\}$, and the fact that $\tilde{C}_i(\tilde{r}) = C_iH(i,\tilde{r})$, we remit (23) as

$$\begin{aligned} \overline{P}_{e,pkt} &= 1-\sum_{m=0}^{L}\sum_{i=0}^{m}\binom{L}{m}\binom{m}{i} \\ &\quad\times E_{\tilde{r}}E_KE_h\Big\{\tilde{C}_1^m(\tilde{r})\tilde{C}_2^i(\tilde{r})\tilde{C}_3^{L-m-i}(\tilde{r}) \\ &\quad\times\exp\left([m\delta_1(\tilde{r})+i\delta_2(\tilde{r})+(L-m-i)\delta_3(\tilde{r})\right] \\ &\quad\times\sum_{j=1}^{n}g_jk_j\Big)\Big\} \\ &= 1-\sum_{m=0}^{L}\sum_{i=0}^{m}\binom{L}{m}\binom{m}{i} \\ &\quad\times E_{\tilde{r}}\Big\{\tilde{C}_1^m(\tilde{r})\tilde{C}_2^i(\tilde{r})\tilde{C}_3^{L-m-i}(\tilde{r}) \\ &\quad\times E_K\Big\{\prod_{j=1}^{n}\sum_{k_j=0}^{K_j}\binom{K_j}{k_j}P_h^{k_j}(1-P_h)^{K_j-k_j} \\ &\quad\times\exp\left([m\delta_1(\tilde{r})+i\delta_2(\tilde{r})+(L-m-i)\delta_3(\tilde{r})\right] \\ &\quad\times g_jk_j)\}\Big\} \\ &= 1-\sum_{m=0}^{L}\sum_{i=0}^{m}\binom{L}{m}\binom{m}{i} \end{aligned}$$

$$
\begin{aligned}
&\times E_{\tilde{r}}\left\{\tilde{C}_1^m(\tilde{r})\tilde{C}_2^i(\tilde{r})\tilde{C}_3^{L-m-i}(\tilde{r})\right. \\
&\quad \times E_K\left\{\prod_{j=1}^{n}\left[P_h e^{[m\delta_1(\tilde{r})+i\delta_2(\tilde{r})+(L-m-i)\delta_3(\tilde{r})]g_j}\right.\right. \\
&\quad\quad \left.\left.+1-P_h\right]^{K_j}\right\}\Bigg\} \\
= \;& 1-\sum_{m=0}^{L}\sum_{i=0}^{m}\binom{L}{m}\binom{m}{i} \\
&\times E_{\tilde{r}}\left\{\tilde{C}_1^m(\tilde{r})\tilde{C}_2^i(\tilde{r})\tilde{C}_3^{L-m-i}(\tilde{r})\right. \\
&\quad \times E_K\left\{\sum_{j=1}^{n} K_j \ln\left[P_h \exp\left([m\delta_1(\tilde{r})+i\delta_2(\tilde{r})\right.\right.\right. \\
&\quad\quad \left.\left.\left.+(L-m-i)\delta_3(\tilde{r})\right]g_j\right)+1-P_h\right]\right\}\Bigg\}. \qquad (27)
\end{aligned}
$$

$E_K\{\cdot\}$ is taken by following a technique used in [1]. For any partition of A, say $\{A_j\}_{j=1}^n$, and constants $\{a_j\}_{j=1}^n$,

$$
E\left\{\exp\left[\sum_{j=1}^{n} N(A_j)a_j\right]\right\} = \exp\left[\lambda\sum_{j=1}^{n} S(A_j)\left(e^{a_j}-1\right)\right], \qquad (28)
$$

where $S(A_j)$ denotes the area of A_j. Because $K_j = N(A_j)$, (27) and (28) can be combined to yield

$$
\begin{aligned}
\overline{P}_{e,pkt} = \;& 1-\sum_{m=0}^{L}\sum_{i=0}^{m}\binom{L}{m}\binom{m}{i} \\
&\times E_{\tilde{r}}\left\{\tilde{C}_1^m(\tilde{r})\tilde{C}_2^i(\tilde{r})\tilde{C}_3^{L-m-i}(\tilde{r})\right. \\
&\quad \times \exp\left[\lambda\sum_{j=1}^{n} S(A_j)\left(P_h\left[\exp\left([m\delta_1(\tilde{r})+i\delta_2(\tilde{r})\right.\right.\right.\right. \\
&\quad\quad \left.\left.\left.\left.+(L-m-i)\delta_3(\tilde{r})\right]g_j\right)-1\right]\right)\right]\Bigg\}. \qquad (29)
\end{aligned}
$$

Generally, $g(|0M|)$ can be considered to be a limit of step functions, which allows $\overline{P}_{e,pkt}$ to be written as

$$
\begin{aligned}
\overline{P}_{e,pkt} = \;& 1-\sum_{m=0}^{L}\sum_{i=0}^{m}\binom{L}{m}\binom{m}{i} \\
&\times E_{\tilde{r}}\left\{\tilde{C}_1^m(\tilde{r})\tilde{C}_2^i(\tilde{r})\tilde{C}_3^{L-m-i}(\tilde{r})\right.
\end{aligned}
$$

$$\times \exp\left[\lambda \int_A \left(P_h\left[\exp\left([m\delta_1(\tilde{r}) + i\delta_2(\tilde{r}) + (L-m-i)\delta_3(\tilde{r})\right] g(|0M|)\right) - 1\right]\right)\right] \quad (30)$$

where $\int_A(\cdot)d^2M$ denotes the integral over the surface A. This completes the analysis because $E_{\tilde{r}}\{\cdot\}$ is simply an expectation over the entire area A under consideration.

We now consider RS(n,k)-coded packets with error-only correction decoding. The applicable error probability expression is given by

$$P_{e,RS} = 1 - \sum_{\ell=0}^{t} \binom{n}{\ell} P_c^{n-\ell}(1-P_c)^{\ell}, \quad (31)$$

where $t = \lfloor n - k/2 \rfloor$ is the error correction capability of the code. This can also be expressed as

$$P_{e,RS} = \sum_{\ell=0}^{t} \sum_{m=0}^{\ell} \binom{n}{\ell}\binom{\ell}{m}(-1)^m P_c^{n-\ell+m}. \quad (32)$$

With respect to the required expectations, this can be treated exactly like the uncoded finally packet error probability expression. Therefore, we obtain

$$\begin{aligned}\overline{P}_{e,RS} = 1 - \sum_{\ell=0}^{t}\sum_{m=0}^{\ell}\sum_{i=0}^{n-\ell+m}\sum_{j=0}^{i}\binom{n}{\ell}\binom{\ell}{m}\binom{n-\ell+m}{i} \\ \times \binom{i}{j}(-1)^m E_{\tilde{r}}\left\{\tilde{C}_1^i(\tilde{r})\tilde{C}_2^j(\tilde{r})\tilde{C}_3^{n-\ell+m-i-j}(\tilde{r})\right. \\ \times \exp\left[\lambda \int_A \left(P_h\left[\exp\left([i\delta_1(\tilde{r}) + j\delta_2(\tilde{r}) + (n-\ell+m-i-j)\delta_3(\tilde{r})\right] g(|0M|)\right)\right.\right. \\ \left.\left.\left. -1\right]\right) d^2M\right]\Big\} \end{aligned} \quad (33)$$

It should be noted that in the case of coherent BPSK, the three-term exponential approximation must approximate $P_{c,i}^m$, rather than $P_{c,i}$ when considering Reed-Solomon codes. The m noted here refers to the number of bits in BPSK per Reed-Solomon symbol; for example, if RS(32,16) code is used, $m = 5$. It should also be noted, when discussing Reed-Solomon codes, that the signal-to-noise ratio is normalized by the code rate.

Once one has obtained an expression for packet error probability, the expected throughput can be obtained by subtracting it from one and multiplying by the offered traffic load, which is given by $G_{tot} = \lambda\pi S(A)$,

$S(A)$ being the area of the region of concern. In the cases presented here, a bandwidth expansion factor must also be introduced.

$$\overline{S} = G_{tot}(1-\overline{P}_e)B. \tag{34}$$

B, the bandwidth expansion factor, is given by

$$B = \frac{1}{q}R_c, \tag{35}$$

where q is the number of hop frequencies and R_c the code rate R_c is 1, for uncoded packets, and k/n, for $RS(n,k)$ coded packets.

3.2 Bounds

Error probabilities have been upper bounded giving lower bounds on system throughput. These will be presented here for the purpose of comparison. The probability of a symbol error at the receiver $p_s(K)$, when K users transmit, can be upper-bounded as

$$p_s(K) \leq 1-(1-P_0)(1-P_h)^{K-1}, \tag{36}$$

where P_0 is the error probability for a single user system in the presence of AWGN with or without fading, and P_h is the probability of a hit from a single other user; expressions for P_h can be found in [7]. For random frequency-hopping patterns, we can use $P_h = (1+\log_2 M/N_b)\frac{1}{q}$, where N_b denotes the number of bits per dwell time and q is the number of frequencies.

Let us discuss now the calculation of throughput when the total number of users (packets) transmitting in a slot (a packet slot, which is L symbols long in the uncoded case) is K. Let $s(K)$ and $P_e(K)$ denote the average number of packets received successfully and the average probability of failure (erroneous reception) of a typical packet, respectively, when K packets are transmitted. This gives use $s(K) = K - KP_e(K)$.

Let K be Poisson-distributed on an area A with average intensity λ (packets per unit area). Let $\overline{s}_e(G)$ and $\overline{P}_e(G)$ denote the averages of the quantities $s(K)$ and $P_e(K)$ defined in the previous paragraph with respect to the Poisson distribution $p(K) = e^{-G}G^K/K!$, where $E\{K\} = \lambda A = G$ (the offered load). Then $\overline{P}_e(G) = 1-\overline{s}_e(G)/G = E\{KP_e(K)\}/G$ and the normalized throughput is given by $\eta(G) = G/q\left[1-\overline{P}_e(G)\right]$.

We can actually show that $\overline{P}_e(G) = E\{P_e(K+1)\}$. Since the packet error probability $P_e(K)$ is, through the symbol error probability $P_s(K)$, a

function of $K-1$ (the number of interfering packets), $\overline{P}_e(G)$ is a function of K and not of $K-1$. This implies that in the calculation of the throughput, instead of working with the total number of transmitted packets, we can equivalently work with the number of interfering packets and treat that as being Poisson-distributed with the same parameter G. This argument also justifies the assumption that the number of interfering users in Poisson-distributed with parameter G.

We can now write

$$\begin{aligned}\overline{P}_c(G) &= 1-\overline{P}_e(G) = E\left\{[1-p_s(K+1)]\right\}^L \\ &\leq E\left\{\left[(1-P_0)(1-P_h)^K\right]^L\right\} \\ &= (1-P_0)^L e^{-G\left[1-(1-P_h)^L\right]}, \end{aligned} \tag{37}$$

where L is the number of M-ary symbols or bits transmitted in a packet $L = 1$ can be used to bound the bit error probability. The throughput can be then lower-bounded as

$$\eta(G) \geq \frac{G}{q}(1-P_0)^L e^{-G\left[1-(1-P_h)^L\right]}. \tag{38}$$

If the system employs error-control coding, we can follow the analysis in [11]. The analysis will be shown for a Reed-Solomon (RS) errors-only correction scheme; other schemes, such as RS with side information, have been analzyed similarly.

The probability of a symbol error $p_s(K)$ is upper-bounded in a way similar to that in REF(36), where P_h is the probability of a hit for an RS symbol, consisting of m_c m-ary symbols; thus, we must use $m_c \log_2 M$ instead of $\log_2 M$ in the expression that provides the probability of a hit. When the $RS(n,k)$ code errors-only correction is employed and one codeword per packet is transmitted, the probability of correct reception of a codeword is given by

$$P_c(K) \geq \sum_{j=1}^{t} \binom{n}{j} [p_s(K)]^j [1-p_s(K)]^{n-j}, \tag{39}$$

where $t = \lfloor n-k/2 \rfloor$ is the error-correction capability of the code and $P_s(K) \leq 1-(1-P_0)^{K-1}$. Finally, we can get an expression for the average probability of correct packet reception $\overline{P}_c(G)$ that allows us to write the following expression for the normalized throughput (per frequency slot)

$\eta(G)$:

$$\begin{aligned}\eta(G) &\geq \frac{G}{q}\frac{k}{n}\sum_{K=0}^{\infty}\frac{e^{-G}G^K}{K!}\sum_{j=0}^{t}(-j^n)\left[1-(1-P_0)^{m_c}(1-P_h)^K\right]^j \\ &\quad\times\left[(1-P_0)^{m_c}(1-P_h)^K\right]^{n-j} \\ &= \frac{G}{q}\frac{k}{n}\sum_{j=0}^{t}\sum_{i=0}^{j}\binom{n}{j}\binom{j}{i}(-1)^i(1-P_0)^{m_c(n+i-j)_e-G[1-(1-P_h)^{n+i-j}]}\end{aligned} \tag{40}$$

4 ACCURATE EVALUATION OF BIT ERROR PROBABILITY

If **coherent** modulation is employed, the resulting system can be analyzed by means of a characteristic function method, as will be demonstrated below.

4.1 Analysis Based on Integrating the Characteristic Function of the Interference

In this case, the system will be assumed to utilize binary phase-shift keying (BPSK) with coherent demodulation. The signal transmitted by the kth mobile station is of the form given in (12). We assume here that Rayleigh fading may or may not be present. We modify (13) to get

$$r_k(t) = F_k\sqrt{2P(M_k)}b_k(t)\Psi(t)\cos[2\pi[f_c + f_k(t)]t + \tilde{\theta}_k(t)], \qquad (41)$$

where F_k is a Rayleigh fading factor, if fading is assumed to be present and all other quantities are as previously described.

The dehopper output, $r_d(t)$ which is used in (15) is now given by

$$r_d(t) = \sum_k \left\{ F_k\sqrt{\frac{P(M_k)}{2}}\delta\left[f_k(t-\tau_k), f_i(t)\right] b_k(t-\tau_k)\Psi(t-\tau_k) \right.$$
$$\left. \times \cos\left[2\pi f_c t + \phi_k(t)\right]\right\} + n_d(t). \qquad (42)$$

We proceed until we again have a modified test statistic $\tilde{Z}_i$ of the form given in (17). Our goal is to derive expressions for the probability of error and the expected throughput η. Since a coherent receiver is being used here, the characteristic function method, described in [13] and simplified to the case where the signature sequences are identically 1, will be used to derive the desired probability expression.

The computation of the error probability involves the characteristic function of the noise $\tilde{n}$, which, in our case, is $\Phi_{\tilde{n}}(u) = \exp\left(-u^2/2 \cdot N_0/2E_{b,0}\right)$, and the characteristic function of the interference process, which is the middle term in (17). This will be derived here for the synchronous case and for the asynchronous case, both with and without Rayleigh fading.

In the **synchronous frequency-hopping case**, we have $\tau_k = 0$ for all k. The k-th signal interferes with the i-th signal if and only if the two signals are exactly superimposed upon each other; this suggest that user k interferes with user i if there is a full hit, when two signals are sent in the

same frequency "bin" at the same time. Thus, if q denotes the number of different frequency bins, the probability of a hit is $1/q$.

A result used in [1] can now be adapted to give an expression for $\Phi_I(u)$, the characteristic function of the interference process. The result used in [1] is that, if the interference process can be written as $I = \sum_k X_k h(M_k)$ for a random process $\{X_k\}$ with pdf $p_X(\cdot)$ and, if the number of users is Poisson distributed on the area A with parameter (mean intensity) λ, then

$$\Phi_I(u) = \exp\left[\lambda \int_A \left(\left[\int p_X(x) e^{iuxh(M)} dx\right] - 1\right) d^2M\right], \qquad (43)$$

in which $\int_A f(M)\, d^2M$ denotes a surface integral and A represents the area of the cell (area of concern). This, however, is not the exact form that must be used here, since the interference process in our problem has the form of the middle term of (17) and the probability of hits must also be taken into account. As a result, the characteristic function of the interference process takes the form

$$\Phi_I(u) = \exp\left\{\frac{\lambda}{q} \int_A \left[\Phi\left(u\sqrt{g(|OM|)}\right) - 1\right] d^2M\right\}, \qquad (44)$$

where

$$\Phi(u) = \frac{1}{2\pi} \int_0^{2\pi} \exp(iu\cos\theta) d\theta = J_0(u). \qquad (45)$$

In (45), the $\cos\theta$ term was obtained by considering (43) and (17), letting $\tau_k = 0$ and assuming uniformly distributed random phase processes (θ is uniform on the interval $[0, 2\pi)$); $J_0(\cdot)$ is the Bessel function of the first kind and zeroth order.

If we consider our region of concern to be a disk of radius R, then the integral in (44) can be easily computed. Let us denote by r the radial distance $|OM|$ and consider the annulus $(r, r+dr)$. Then we can write that $\int_A \left[J_0\left(u\sqrt{g(|OM|)}\right) - 1\right] d^2M = \int_0^R 2\pi r \left[J_0\left(u\sqrt{g(r)}\right) - 1\right] dr$, where $g(\cdot)$ is as in (17). This expression can be evaluated using formulas found in [14] to yield series solution.

Let us consider the same model with Rayleigh fading. This is accounted for by means of F_k in the expressions above. The F_k are assumed to be independent and identically distributed with a Rayleigh parameter b^2. As a result of the addition of fading, (45) becomes

$$\Phi(u) = \int_0^\infty P_F(y) \frac{1}{2\pi} \int_0^{2\pi} \exp(iuy\cos\theta) d\theta dy$$

$$= \exp(-\frac{b^2u^2}{2}) \tag{46}$$

In (46), $P_F(y) = y^2/b^2 \exp(-y^2/2b^2)$ is the Rayleigh density function with parameter b^2. Since the product of a Rayleigh random variable and the sine or cosine of a uniform on $[0, 2\pi]$ random variable yields a Gaussian random variable, $G = F\cos\theta$ has pdf $p_G(\eta) = 1/\sqrt{2\pi}b \exp(-\eta^2/2b^2)$ and the right-hand equality is derived by performing the integration with respect to η.

If the circular region model mentioned above is used, (44) becomes

$$\begin{aligned}\Phi_I(u) &= \exp\left\{\frac{\lambda}{q}\int_0^R \left(\exp\left[-g(r)\frac{b^2u^2}{2}\right]-1\right)2\pi r dr\right\} \\ &= \exp\left\{\frac{\lambda}{q}\left(\left[e^{-B(u)r_0^{-\alpha}}-1\right]\pi r_0^2\right.\right. \\ &\quad \left.\left.+\int_{r_0}^R \left[e^{-B(u)r^{-\alpha}}-1\right]2\pi r dr\right)\right\}\end{aligned} \tag{47}$$

where $B(u) = b^2u^2/2$. A closed-form result can be obtained using a formula from [16], if α is known. If the typical value $\alpha = 3.5$ is used, (47) becomes

$$\begin{aligned}\Phi_I(u) &= \exp\left\{\frac{\lambda}{q}\left(\pi r_0^2[e^{-B(u)r_0^{-3.5}}-1] - \pi[R^2-r_0^2]\right.\right. \\ &\quad + \left[7\pi e^{-B(u)x}\left[-\frac{x^6}{B(u)}\right.\right. \\ &\quad \left.\left.\left.+\sum_{k=1}^{6}(-1)^k\frac{6\cdot 5\cdots(6-k+1)}{[-B(u)]^{k+1}}x^{6-k}\right]\right]_{x=r_0^{-3.5}}^{x=R^{-3.5}}\right)\Bigg\}.\end{aligned} \tag{48}$$

We now focus on the **asynchronous frequency-hopping case.** It will be assumed that the τ_k in (42) are independent and identically uniformly distributed on the interval $[0, T_h)$. As a result of these delays, there are three possible interference scenarios for a signal transmitted in a given frequency bin: *no hit, partial hit,* or *full hit.* These must all be accounted for in the computation of the characteristic function of the interference process.

Since the interference term in (17) is a sum of independent random variables, the overall characteristic function can be seen as a product of the

individual characteristic functions, that is,

$$\begin{aligned}\Phi_I(u) &= E\left\{\prod_{k\neq i}\Phi^{(k,i)}(u)\right\} \\ &= E\left\{\prod_{k\neq i}\left[1-P+P_f\Phi_f^{(k,i)}(u)+P_p\Phi_p^{(k,i)}(u)\right]\right\} \qquad (49)\end{aligned}$$

In (49), E denotes expectation with respect to the number of interfering users, P_f is the probability of a full hit, P_p is the probability of a partial hit, $1-P=1-P_f-P_p$ is the probability of no hit occurring, and $\Phi_f^{(k,i)}(u)$ and $\Phi_p^{(k,i)}(u)$ are the characteristic functions of $I_{k,i}$, given that a full hit or a partial hit has occurred. Note that for the case of no hit, $I_{k,i}=0$, which results in a characteristic function equal to 1. The validity of the second equality in (49) was proven in [13].

The quantity inside the expectation in (49) can be written in the form $\exp\left\{\sum_{k\neq i}\ln\left[1-P+P_f\Phi_f^{(k,i)}(u)+P_p\Phi_p^{(k,i)}(u)\right]\right\}$. Let us consider a step function $g(|OM|)=\sum_{j=1}^{n} g_j\, I_{A_j}(M)$, where $\{A_j\}_{j=1}^{n}$ is a partition of the region A; $I_B(M)$ is 1, if $M\in B$, and 0 otherwise; and $g(|OM|)=g_j$ for all $M\in A_j$. If we denote by $N(A_j)$ the number of users in the area A_j, we obtain that

$$\Phi_I(u) = E\left\{\exp\left(\sum_{j=1}^{n} N(A_j)\ln\left[1-P+P_f\Phi_f(u\sqrt{g_j})+P_p\Phi_p(u\sqrt{g_j})\right]\right)\right\}, \qquad (50)$$

in which $\Phi_f(u\sqrt{g_j})$ and $\Phi_p(u\sqrt{g_j})$ denote the characteristic functions of the interference due to any particular user located in A_j for the cases of full hits and partial hits, respectively. Then the expectation can be written as

$$\begin{aligned}\Phi_I(u) = \exp\Bigg\{&\lambda P_f\sum_{j=1}^{n} S(A_j)\left[\Phi_f(u\sqrt{g_j})-1\right] \\ &+\lambda P_p\sum_{j=1}^{n} S(A_j)\left[\Phi_p(u\sqrt{g_j})-1\right]\Bigg\}. \qquad (51)\end{aligned}$$

In the general case, $g(|OM|)$ is a limit of step functions, implying that $\Phi_f(u\sqrt{g(|OM|)})$ and $\Phi_p(u\sqrt{g(|OM|)})$ are such limits, too, and the overall

characteristic function becomes

$$\Phi_I(u) = \exp\left\{\lambda P_f \int_A \left[\Phi_f\left(u\sqrt{g(|OM|)}\right) - 1\right] d^2M + \lambda P_p \int_A \left[\Phi_p\left(u\sqrt{g(|OM|)}\right) - 1\right] d^2M\right\}. \quad (52)$$

Expressions for P_f and P_p were given in [13] for first-order Markov random hopping patterns, for memoryless random hopping patterns, and for Reed-Solomon periodic hopping patterns. Following [13] for the case in which the signature sequence is identically 1, we obtain that

$$\Phi_f(u) = \frac{1}{T}\int_0^T J_0\left(\frac{u}{T}[R_\psi(\tau) + \hat{R}_\psi(\tau)]\right) d\tau$$
$$\Phi_p(u) = \frac{1}{2T}\int_0^T \left[J_0\left(\frac{u}{T}R_\psi(\tau)\right) + J_0\left(\frac{u}{T}\hat{R}_\psi(\tau)\right)\right] d\tau, \quad (53)$$

in the absence of fading, and

$$\Phi_f(u) = \frac{1}{T}\int_0^T \exp\left\{-\frac{b^2u^2}{2}\left[R_\Psi(\tau) + \hat{R}_\Psi(\tau)\right]^2\right\} d\tau$$
$$\Phi_p(u) = \frac{1}{2T}\int_0^T \left\{\exp\left[-\frac{b^2u^2}{2}R_\Psi^2(\tau)\right] + \exp\left[-\frac{b^2u^2}{2}\hat{R}_\Psi^2(\tau)\right]\right\} d\tau \quad (54)$$

in the presence of Rayleigh fading. In (53) and (54), the functions R_ψ and $\hat{R}_\psi$ are the partial autocorrelation functions of the shaping waveform $\psi(t)$ and are defined by $\hat{R}_\psi(s) = \int_s^T \psi(t)\psi(t-s)dt$ and $R_\psi(s) = \hat{R}_\psi(T-s)$.

The relationship in [13], which expresses the bit error probability of the receiver as an integral involving the characteristic functions of the noise and the interference process (see [6]), can be modified to depend on the position of the "transmitter of concern" in the following way

$$P_b(|0M|) = Q\left(\sqrt{2E_{b,0}/N_0}\right) + \frac{1}{\pi}\int_0^\infty \frac{\sin[g(|0M|)u]}{u}\,\Phi_{\tilde{n}}(u)\,[1 - \Phi_I(u)]\,du, \quad (55)$$

where $|0M|$ represents the distance between the receiving (base) station and the "transmitter of concern" (the ith user in the analysis above), $Q(x)$ is the

complementary error function given by $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp(-y^2/2)\, dy$, and $\Phi_{\tilde{n}}(u)$ was defined earlier. $\Phi_I(u)$ is given by (44) for synchronous systems, by (45) for no fading, and (46) for Rayleigh fading. For asynchronous systems, it is given by (52) and (53) for no fading and by (54) for Rayleigh fading.

The result in (55) is useful because it gives us a measure of the correctness of each individual bit. Unfortunately, to evaluate the throughput one needs the packet error probability $P_e(|OM|)$, which can not be expressed as an explicit function of $P_b(|OM|)$. The error probability of each one of the L bits in a packet depends on the number of interfering users and their received power at the base station. Therefore, one should first derive the bit error probability given the number of interfering users; from that one should derive the packet error probability, given again the number of interfering users, and then finally average the latter with respect to the Poisson distributed number of interfering users. As a result, the best that can be done using this method for packets (coded or uncoded ones) is an approximation.

5 NUMERICAL RESULTS

In discussing our numerical results, we must note several things. First, we take A to be a circular region of radius R and then we set $R = 1$. This can be viewed as a normalization and the results can be generalized to regions of larger or smaller size; this is especially evident if one considers that the expected number of users in A is given by $N = \lambda \pi R^2$ and that our results are indexed by N (and therefore by λ).

A brief discussion of the issue of normalizing the radius R of the cell is in order here. We assume that in the mobile-to-base mode of operation, all mobiles transmit with the same signal-to-noise ratio E_b/N_0, which is such that the signal-to-noise ratio received at the base from a mobile at a distance r, given by $E_b'/N_0 = (E_b/N_0)\, g(r)$, is at least as large as $\overline{E}_b/N_0 = (E_b/N_0)\, g(R)$, the received energy from a mobile at the periphery of the cell. Therefore, if $\overline{E}_b/N_0$ is set at the minimum acceptable level for reliable communications, the transmitted signal-to-noise ratio for all mobiles is given by $E_b/N_0 = \left(\overline{E}_b/N_0\right)/g(R)$; if $R = 1$, $E_b/N_0 = \overline{E}_b/N_0$, while if $R > 1$, $g(R)$ can be computed and used to compute E_b/N_0 from the above formula, while using $R = 1$ for all other calculations.

It is also important to consider the computational requirements of our results. Looking at (30) or (33), one can see that there are multiple levels

of summations and integrals involved in the computation of each value of the packet error probability (note that $E_{\tilde{r}}\{\cdot\}$ is an integral). In computing the results shown here, we sought to avoid this last expectation over $\tilde{r}$ and explored two alternative methods.

The first method was to set $\tilde{r}$ equal to some value such that a given probability of acceptable communication is maintained. As discussed in [1], it can be shown that in this case, $\tilde{r} = \sqrt{P}R$, where P is the probability of acceptable communication to be maintained. If one considers values of P close to one–as would be the range of values one might like to examine–one can see that the results obtained will be overly pessimistic.

As a result of the pessimistic nature of the first method, we have opted to use another method, namely to set $\tilde{r}$ equal to a value such that an average value of $g(\tilde{r})$ is obtained; then $\tilde{r} = [E_r\{g(r)\}]^{-1/\alpha}$, where α and $g(\cdot)$ are as in (2). We find that the expectation is given by

$$E_r\{g(r)\} = \frac{r_0^{2-\alpha}}{R^2} + \frac{2}{(\alpha-2)R^2}(r_0^{2-\alpha} - R^{2-\alpha}) = g(\tilde{r}). \qquad (56)$$

The value of this method is due to the fact that the probability of symbol or packet correctness is usually a convex $\cap$ function of $g(r)$ in part of the range of r. In this range, we can use Jensen's inequality

$$E_r\{P_c(g(r))\} \leq P_c\{E_r(g(r))\} = P_c(g(\tilde{r})) \qquad (57)$$

to obtain the upper bound on the right-hand side which depends only on $g(\tilde{r})$. In the range of r in which the function P_c is not convex $\cap$ in r, we can still use $P_c(g(\tilde{r})$ as an approximation.

Fig. 2 compares the results using the Gaussian approximation technique, described in Section III-A, with the bound presented in Section III-B for the case of coherent signaling and RS(32, 16) coding with error-only correction decoding. This demonstrates that the bound tends to be extremely pessimistic. On the other hand, the Gaussian approximation tends to be moderately optimistic for most of the range of N (the average number of neighbors). This last argument is justified as follows: Gaussian approximations tend to be usually optimistic when applied to multiple-access problems (refer to [5] and [6]). However, the assumption that all interfering users cause full hits is pessimistic. Furthermore, the use of $g(\tilde{r})$ results in pessimistic values of the probability of correct reception for part of the range of r. Thus overall the Gaussian approximation used here is expected to be moderately optimistic. This can be actually verified through a comparison to the exact expressions in Fig. 6 and 7 below.

Fig. 3 deals with the coherent case and demonstrates the decrease in throughput with decreasing Reed-Solomon code rate. As one might expect, throughput is proportional to code rate due to the normalization by the code rate. However, the RS(32,8) code provides a bit more than half of the RS(32,16) throughput when $N = 150$, which is explained by an improvement in the expected probability of error. In both Fig. 2 and 3, the values of the throughput for large N (larger than 150) are optimistic and should not be considered fully dependable.

We now move on to the noncoherent case (32-ary FSK with noncoherent demodulation). Fig. 4 corresponds to Fig. 2. Once again it is demonstrated that the bound is very pessimistic and the Gaussian approximation overly optimistic for large N.

Fig. 5 presents results using RS(32, 16) and RS(32, 8) coding. The same observations that were made regarding Fig. 3 should be made here.

Fig. 6 demonstrates the relationships among the exact expression developed in this section and the Gaussian approximation and bound presented in Section III. The bound tends to be very pessimistic. The Gaussian approximation, on the other hand, tends to be somewhat pessimistic for low to moderate values of N (the average number of neighbors), while it provides a closer approximation and even becomes slightly optimistic for moderate to large values of N. This reflects on the previous throughput results: for small N, the approximation is closer to the exact value than the bound but is pessimistic; for moderate N, it is close to the exact value; for large N, the results exceed what they should be and this effect is magnified by the RS coding.

Fig. 7 shows results for synchronous and asynchronous systems. These results are so close, that the appear as a single curve in the figure. The closeness of the results for the synchronous and asynchronous cases is due to the fact that the probability of a hit is $P_h = 1/q$ for synchronous systems and $P_h = (1 + 1/N_b)/q$ for asynchronous systems, whereas $m_\psi = 1/3$ for asynchronous systems and $m_\psi = 1/2$ for synchronous systems, thus balancing out the effect of other-user interference of the two systems. Also shown is a curve that represents an asynchronous system with Rayleigh fading (parameter $b^2 = 0.01$). As expected, this system does significantly worse than its counterpart without fading.

6 CONCLUDING REMARKS

We presented methods by which error probability and/or throughput can be computed for FH cellular mobile packet redio networks. These methods consist of a Gaussian approximation applicable to all systems and an exact method that we could only apply to coherent systems. It was shown that the previously derived lower bound on system throughput (upper bound on error probability) is a loose, pessimistic bound in comparison to the approximation and the exact results. The approximation was shown to have satisfactory accuracy for the lower-to-moderate values of the average number of neighbors, but to be overly optimistic for large values of the average number of neighbors.

One major problem encountered lay in the computational complexity of the expressions. In general, each curve in the figures presented required a considerable amount of CPU time to compute. In the case of some of the exact expressions (especially the asynchronous case without fading), this computation was extremely long. We used basically a VAX 780 computer and a few hours of supercomputer CPU time. What this indicates is that this work may very well represent a practical limit as to how precisely one can characterize these systems and still be able to compute analytically the various performance measures of interest.

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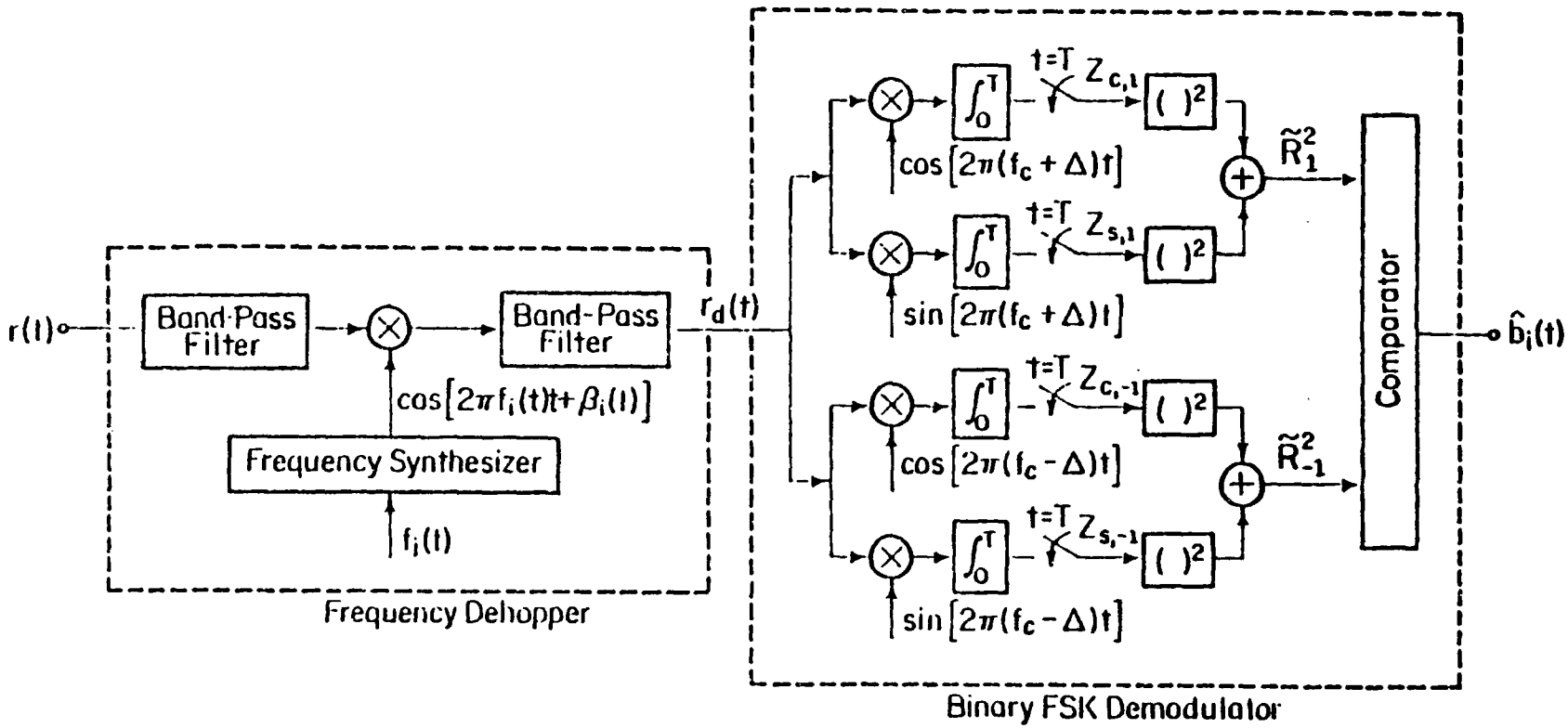

Figure 1. BFSK SFH/SSMA receiver model.

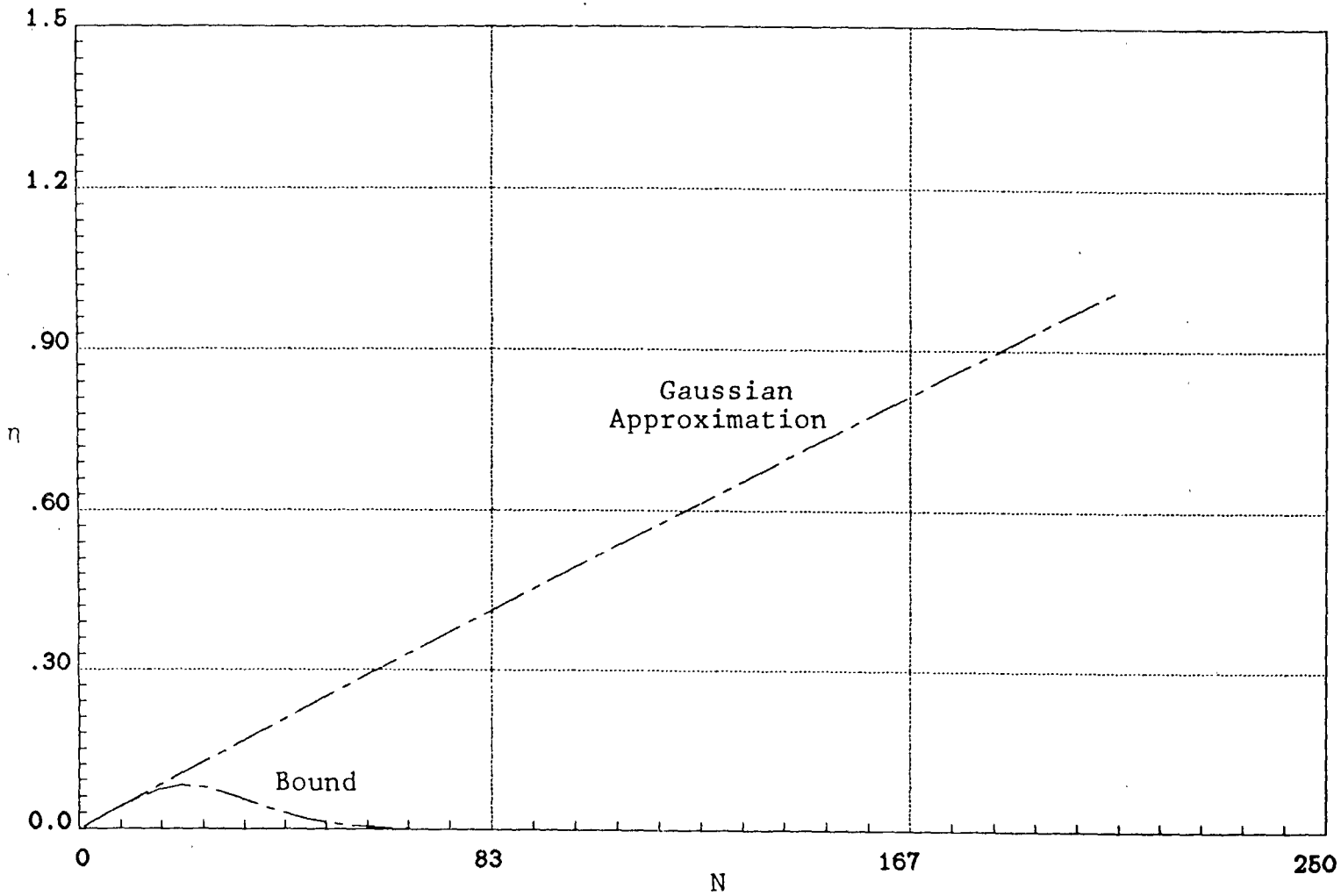

Figure 2. Throughput versus Average Number of Neighbors: Gaussian Approximation versus Bound (RS(32,16) coding with error-only correction decoding, coherent BPSK, SNR=15dB, q=100, N_b=10)

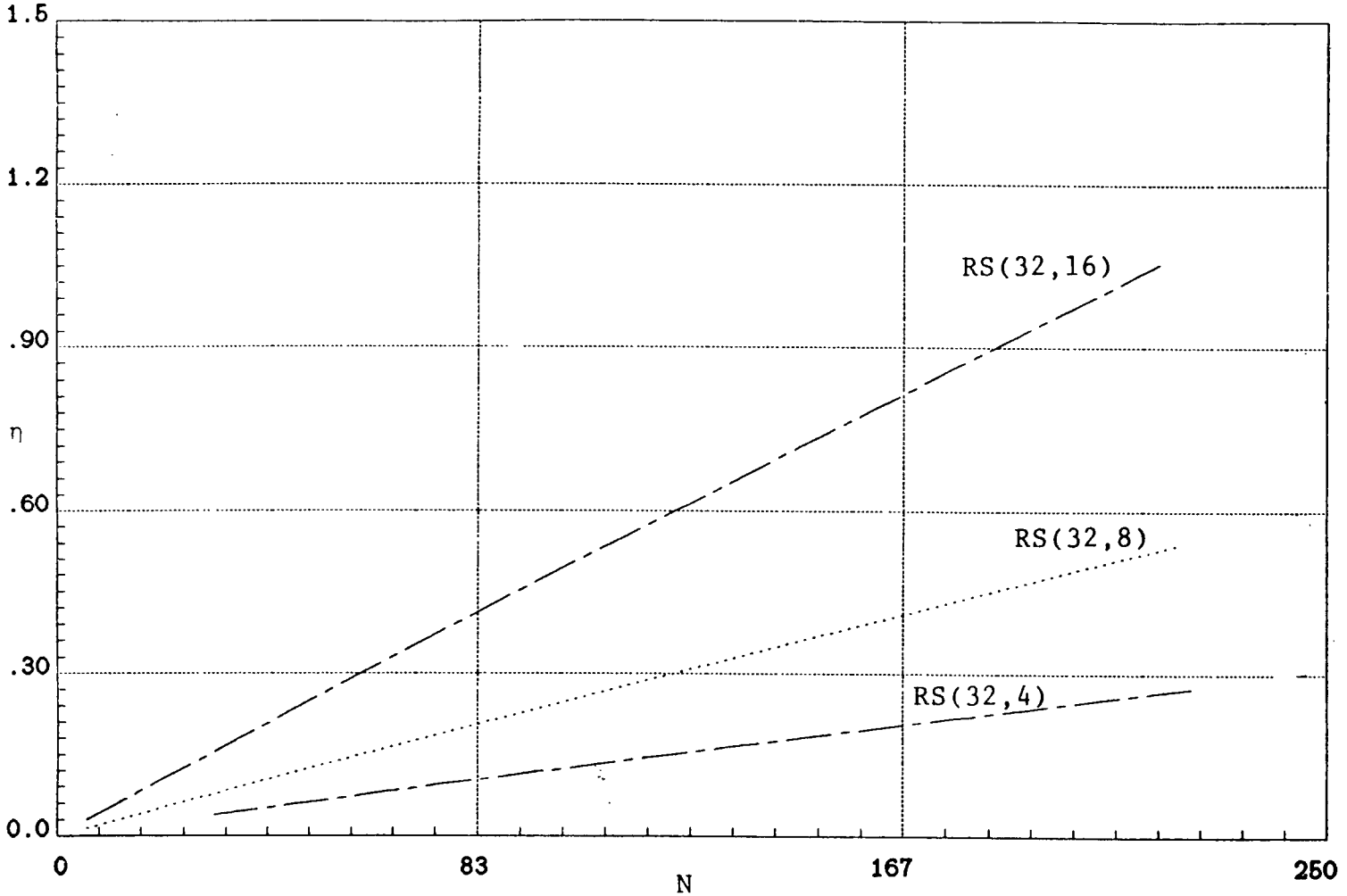

Figure 3. Throughput versus Average Number of Neighbors for Varying Code Rate (Reed-Solomon coding with error-only correction decoding, coherent BPSK, SNR=15dB, q=100, N_b=10)

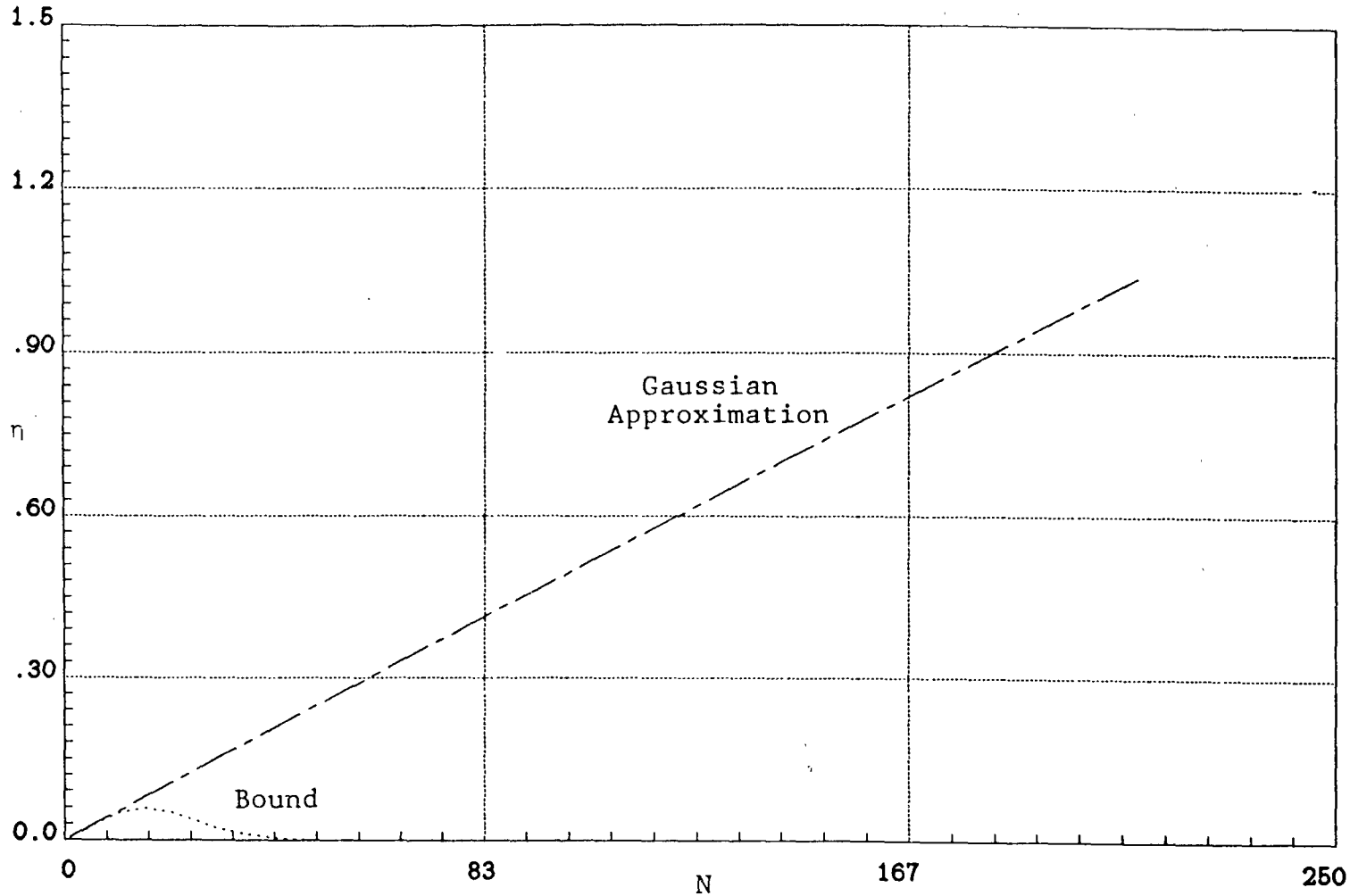

Figure 4. Throughput versus Average Number of Neighbors: Gaussian Approximation versus Bound (RS(32,16) coding with error-only correction decoding, noncoherent 32-ary FSK, SNR=15dB, q=100, N_b=10)

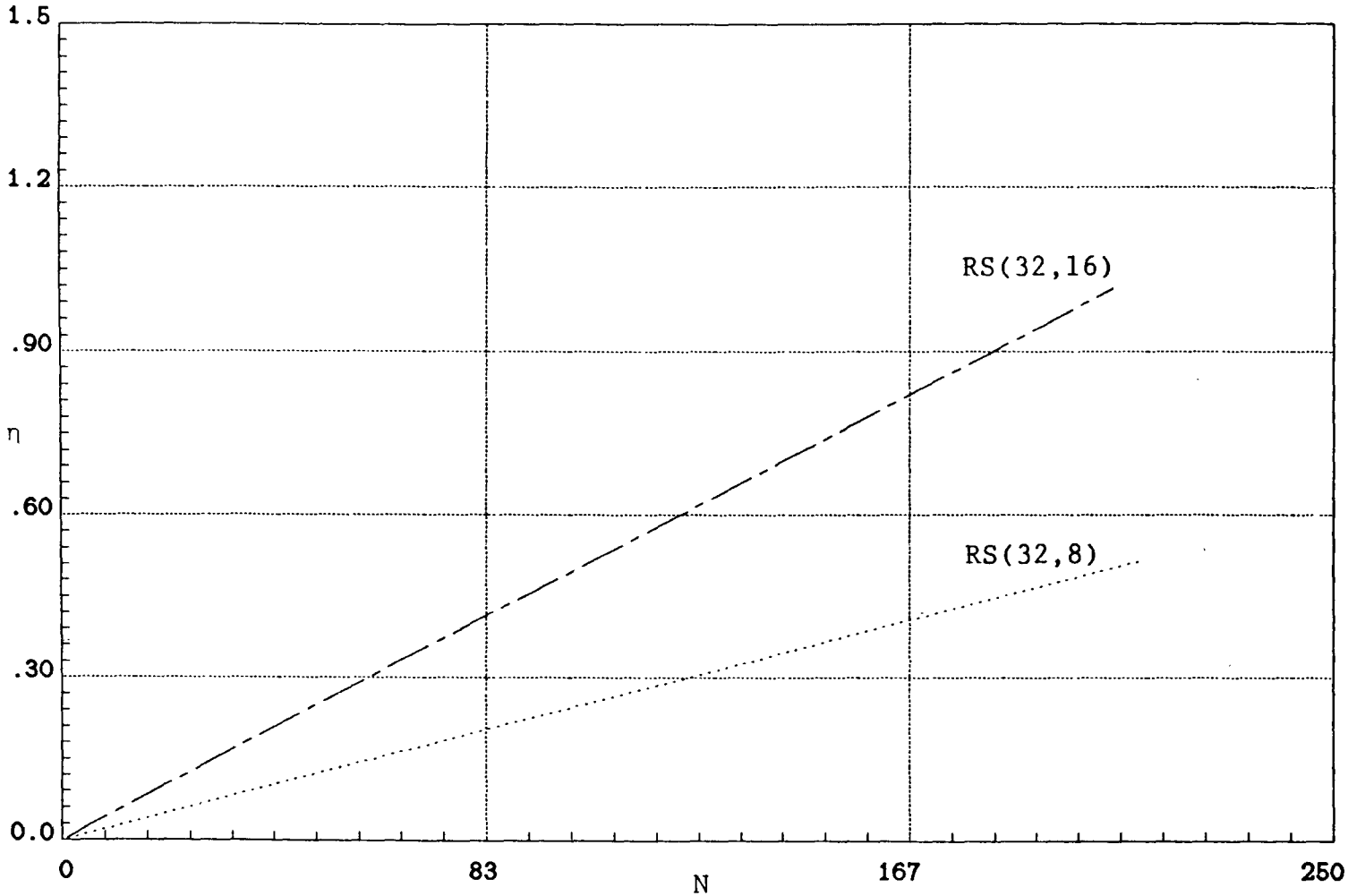

Figure 5. Throughput versus Average Number of Neighbors for Varying Code Rate (Reed-Solomon coding with error-only correction decoding, noncoherent 32-ary FSK, SNR=15dB, q=100, N_b=10)

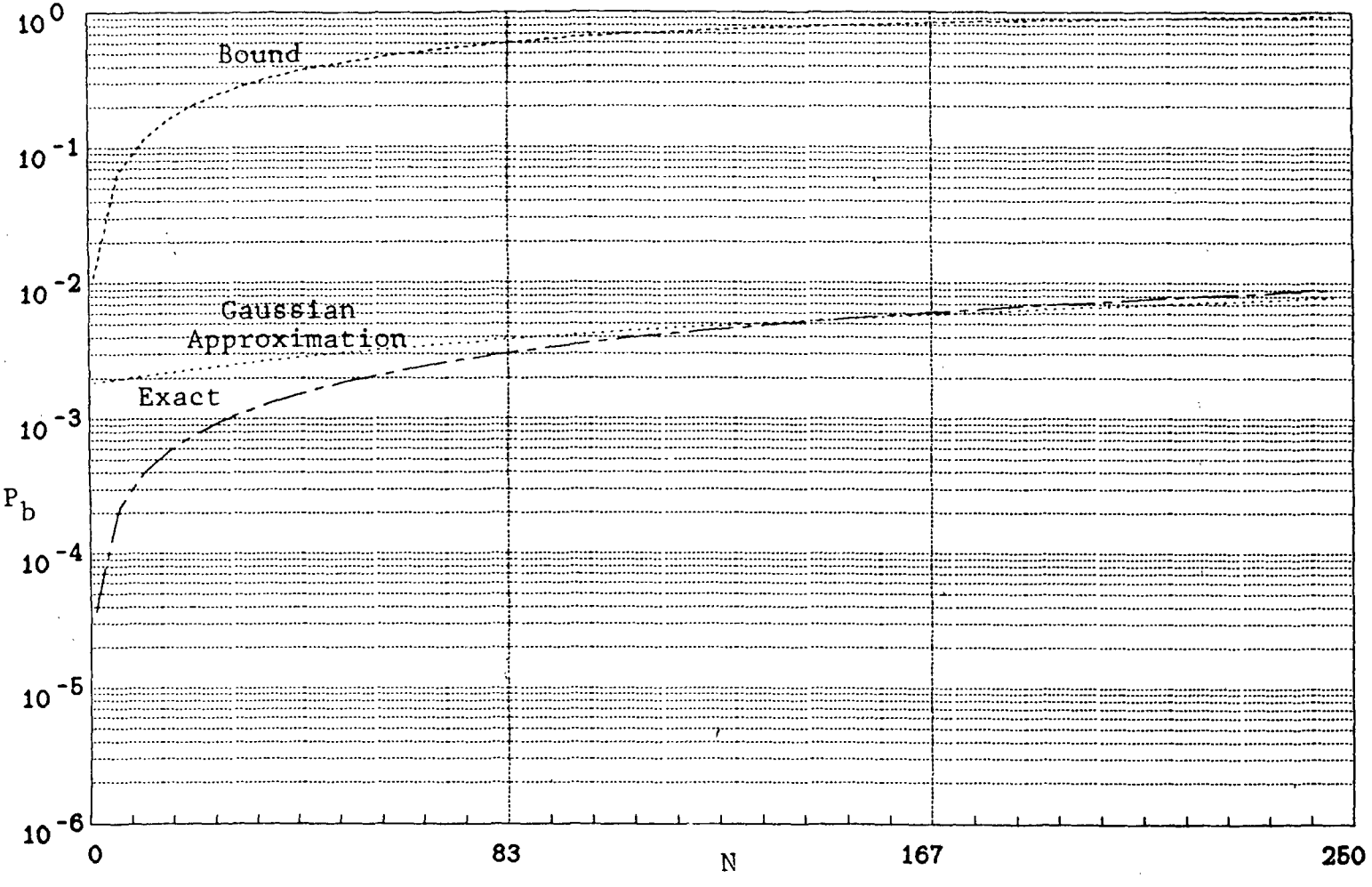

Figure 6. Bit Error Probability versus Average Number of Neighbors: Exact Method, Gaussian Approximation, and Bound (coherent BPSK, asynchronous, SNR=10dB, q=100, N_b=10)