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**Robust H_2 -Type IMC Controller
Design via the Structured Singular
Value**

by

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ROBUST H_2 -TYPE IMC CONTROLLER DESIGN VIA THE STRUCTURED SINGULAR VALUE

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ABSTRACT. The two-step Internal Model Control procedure is used for the synthesis of robust controllers for multivariable open-loop stable or unstable plants. In the first step the controller is designed so that the Integral Squared Error (ISE) is minimized for every external input (setpoint or output disturbance) direction in a set and their linear combinations. In the second step a low-pass filter is designed so that stability and good performance characteristics are maintained in the presence of model-plant mismatch. The problem is formulated as a minimization of the Structured Singular Value (SSV) for robust performance over the filter parameters.

INTRODUCTION

In the past few years, a significant amount of work in the control literature has been directed toward the study of control systems that are robust with respect to model-plant mismatch. The Structured Singular Value theory (Doyle, 1982) has made possible the analysis of the stability and performance of a known control system on the presence of all kinds of structured uncertainties in the plant. In the synthesis aspect of the problem, progress has been made in an H_∞ -type approach (Doyle, 1985). The objective in this case is to minimize the H_∞ -norm of the sensitivity function, that relates the errors to the external system inputs, for the worst case among all possible plants allowed by the available uncertainty description.

In process control however, the use of the H_∞ -type objective is not satisfactory in most cases, because it inherently minimizes the error over types of external inputs (setpoint and/or disturbances) that are never present in process control problems. A more suitable objective is the H_2 -error (Integral Squared Error) for a designer specified set of external inputs, and for the whole set of possible plants. Extension of the H_∞ -approach is not straightforward though. For this reason and in order to incorporate in the controller online tuning parameters, the two-step Internal Model Control design procedure is used in this paper.

1. PRELIMINARIES

1.1 Internal Model Control

The Internal Model Control (IMC) structure (Fig. 1a), introduced by Garcia and Morari (1982), is mathematically equivalent to the classical feedback structure (Fig. 1b). The IMC controller Q and the feedback controller C are related through

$$C = Q(I - \tilde{P}Q)^{-1} \quad (1.1.1)$$

where \tilde{P} is the process model.

When $P = \tilde{P}$

The IMC structure becomes effectively open-loop and the design of Q is simplified. Note that the IMC controller is identical to the parameter of the Q-parametrization (Zames, 1981). Also the addition of a diagonal filter F by writing

$$Q = \tilde{Q}F \quad (1.1.2)$$

introduces parameters (the filter time constants) which can be used for adjusting on-line the speed of response for each process output.

When $P \neq \tilde{P}$, the model-plant mismatch generates a feedback signal in the IMC structure which can cause performance deterioration or even instability. Since the relative modeling error is larger at higher frequencies, the addition of the low-pass filter F adds robustness characteristics into the control system. The overall transfer function connecting the setpoints r and disturbance d to the errors $e = y - r$, where y are the process outputs, is

$$e = y - r = (I - P\tilde{Q}F)(I - (P - \tilde{P})\tilde{Q}F)^{-1}(d - r) \stackrel{\text{def}}{=} E(d - r) \quad (1.1.3)$$

Hence the IMC structure gives rise rather naturally to a two-step design procedure:

Step 1: Design \tilde{Q} , assuming $P = \tilde{P}$.

Step 2: Design F so that the closed-loop characteristics that \tilde{Q} produces in Step 1, are preserved in the presence of model-plant mismatch ($P \neq \tilde{P}$).

1.2 Internal Stability

A linear time invariant control system is internally stable if the transfer functions between any two points of the control system are stable. A more detailed discussion of the concept of internal stability can be found in the literature (e.g., Morari et al., 1987).

Examination of the feedback structure of Fig. 1b results in the requirement that all elements in the matrix $IS1$ in (1.2.1) are stable, where (1.1.1) has been used.

$$IS1 = \begin{pmatrix} Q & PQ & QP & (I - PQ)P \end{pmatrix} \quad (1.2.1)$$

The additional requirements to take care of modeling error are discussed in Section 3.3.

Note that stability of each element in (1.2.1) implies internal stability when the control system is implemented as the feedback structure in Fig. 1b, where C is obtained from the Q used in (1.2.1) through (1.1.1).

In order for the control system to be stable when implemented in the IMC structure of Fig. 1a, internal stability arguments (Morari et al., 1987) lead to the requirement that P be stable.

Hence if the process P is open-loop unstable, the control system has to be implemented in the feedback structure of Fig. 1b. Still, the two step IMC design procedure can be used for the design of Q , as described in the following sections. C can then be obtained from (1.1.1) and the structure in Fig. 1b implemented. However special care has to be taken in the construction of C so that all the

common closed RHP zeros of Q and $(I - PQ)$ are cancelled in (1.1.1). Minimal or balanced realization software can be used to accomplish that.

2. Step 1: Design of \tilde{Q} .

Throughout this section the assumption is made that $P = \tilde{P}$.

2.1 Performance Objectives

The performance objective adopted in this paper is to minimize the Integral Squared Error (ISE) for the error signal e given by (1.1.3).

For a specified external system input v ($v = d$ for $r = 0$; $v = -r$ for $d = 0$), the ISE is given by the square of the H_2 -norm of e :

$$\Phi(v) \stackrel{\text{def}}{=} \|e\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^*(i\omega)e(i\omega) d\omega \quad (2.1.1)$$

where the superscript $*$ denotes complex conjugate transpose. Hence one objective could be

$$\min_{\tilde{Q}} \Phi(v) \quad (O1)$$

for a particular input v .

What is desirable, however, is to find a \tilde{Q} , that minimizes $\Phi(v)$ for every single v in a set of external inputs v of interest for the particular process. For an $n \times n$ P this set can be defined as

$$\mathcal{V} = \{v^i(s); i = 1, \dots, n\} \quad (2.1.2)$$

where $v^1(s), \dots, v^n(s)$ are vectors that describe the expected directions and frequency content of the external system inputs, e.g., steps, ramps or other types of inputs. The objective can then be written as

$$\min_{\tilde{Q}} \Phi(v) \quad \forall v \in \mathcal{V} \quad (O2)$$

It should be noted however that a linear time invariant \tilde{Q} that solves (O2) does not necessarily exist. In Section 2.3, it will be shown that this is the case for some \mathcal{V} 's. An alternative objective in such a case would be:

$$\min_{\tilde{Q}} [\Phi(v^1) + \Phi(v^2) + \dots + \Phi(v^n)] \quad (O3)$$

In this case the objective is to minimize the sum of the ISE's that each of the inputs v^i would cause when applied to the system separately.

2.2 Parametrization of All Stabilizing \tilde{Q} 's.

The following assumption simplifies the solution of the optimization problem:

Assumption A.1. If π is a pole of the model \tilde{P} in the open RHP, then the order of π is equal to 1. Also \tilde{P} has no zeros at $s = \pi$.

Assumption A.1.a is not overly restrictive in practice. Assumption A.1 is not made for poles at $s = 0$. The following assumption true for all practical process control problems is made:

Assumption A.2. Any poles of \tilde{P} or P on the imaginary axis are at $s = 0$. Also \tilde{P} has no finite zeros on the imaginary axis.

Theorem 2.2.1. (Zafriou, 1987). Assume that $Q_0(s)$ satisfies the internal stability requirements of section 1.2, i.e., it produces a matrix $IS1$ with stable elements. Then all Q 's that make $IS1$ stable are given by

$$Q(s) = Q_0(s) + Q_1(s) \quad (2.2.1)$$

where Q_1 is any stable transfer matrix such that PQ_1P has no poles in the closed RHP.

2.3 Objective (O1)

For every external input v that will be considered in this paper, the following assumptions can be made.

Assumption A.3. Every nonzero element of v (or \hat{v}) includes all the open RHP poles of \tilde{P} , each of them with degree 1.

Assumption A.4. Let l_i be the maximum number of poles at $s = 0$ that an element of the i^{th} row of P has. Then $v_i(s)$ has at least l_i poles at $s = 0$. Also v has no other poles on the imaginary axis and its elements have no finite zeros on the imaginary axis.

The above assumptions are not restrictive in the case where v is an output disturbance d , because in a practical situation we want to design for an output disturbance produced by a disturbance that has passed through the process. The assumptions may be restrictive in the case of setpoints, though. However, for setpoint tracking the use of the Two-Degree-of-Freedom structure (Vidyasagar, 1985) allows us to disregard the existence of any unstable poles of P and therefore this assumption need not be made for setpoints.

Let $v_0(s)$ be the scalar allpass that includes the common RHP zeros of the elements of v . Factor v as follows:

$$v(s) = v_0(s) (\hat{v}_1(s) \dots \hat{v}_n(s))^T \stackrel{\text{def}}{=} v_0(s)\hat{v}(s) \quad (2.3.1)$$

The plant P can be factored into a stable allpass portion P_A and a minimum phase (MP) portion P_M such that

$$P = P_A P_M \quad (2.3.2)$$

Hence P_A is stable and such that $P_A^*(i\omega)P_A(i\omega) = I$. Also P_M^{-1} is stable. This "inner-outer" factorization can be accomplished through the spectral factorization of $P(-s)^T P(s)$. Details on these problems can be found in the literature (Anderson, 1967; Chu, 1985; Doyle, 1984).

Theorem 2.3.1 (Zafriou, 1987). The set of controllers \tilde{Q} that solve (O1) satisfy

$$\tilde{Q}\hat{v} = P_M^{-1}\{P_A^{-1}\hat{v}\}_* \quad (2.3.3)$$

where the operator $\{\cdot\}_*$ denotes that after a partial fraction expansion of the operand all terms involving the poles of P_A^{-1} are omitted. Furthermore, for $n \geq 2$ the number of stabilizing controllers that satisfy (2.3.3) is infinite. Guidelines for the construction of such a controller are given in the proof.

2.4 Objectives (O2) and (O3)

We shall consider objective (O3) first. Define

$$V(s) \stackrel{\text{def}}{=} (v^1(s) \ v^2(s) \ \dots \ v^n(s)) \quad (2.4.1)$$

where v^1, \dots, v^n satisfy assumptions A.3, A.4. An additional assumption on V is needed:

Assumption A.5. V has no zeros at the location of its unstable poles or on the imaginary axis and V^{-1} cancels the closed RHP poles of \tilde{P} in $V^{-1}\tilde{P}$.

Note that satisfaction of assumptions A.3. and A.4 for each column of V does not necessarily imply satisfaction of A.5. However such a V can be easily constructed. A simple way is to use a diagonal V , in which case satisfaction of A.3 and A.4 by every column of V implies satisfaction of A.5 by V . This situation is discussed further in Corollary 2.4.1.

Factor V similarly to P (use $V(s)V^T(-s)$ if spectral factorization theory is used):

$$V = V_M V_A \quad (2.4.2)$$

Theorem 2.4.1 (Zafriou, 1987). The controller

$$\tilde{Q} = P_M^{-1} \{ P_A^{-1} V_M \} \cdot V_M^{-1} \quad (2.4.3)$$

is the unique solution to (O3).

Let us now consider the more meaningful objective (O2). Clearly, a \tilde{Q} that solves (O2) will also solve (O3). Hence from Theorem 2.4.1, it follows that if a solution to (O2) exists, it is given by (2.4.3). Factor each of the v^i in the way used in (2.3.1) and define

$$\hat{V} \stackrel{\text{def}}{=} (\hat{v}^1 \ \hat{v}^2 \ \dots \ \hat{v}^n) \quad (2.4.4)$$

Theorem 2.4.2 (Zafriou, 1987).

- i) If $\hat{V}(z)$ is non-minimum phase, then there exists no solution to (O2).
- ii) If $\hat{V}(z)$ is minimum phase, then use of \hat{V} instead of V_M in (2.4.3) yields exactly the same \tilde{Q} , which also solves (O2) and it minimizes $\Phi(v)$ for any v that is a linear combination of v^i 's that have the same v_0^i 's.

Corollary 2.4.1. Let

$$V = \text{diag}(v_1, v_2, \dots, v_n) \quad (2.4.5)$$

where $v_1(s), \dots, v_n(s)$ are scalars. Then use of \hat{V} instead of V_M in (2.4.3) yields exactly the same \tilde{Q} , which minimizes $\Phi(v)$ for the following n vectors:

$$v = (v_1 \dots 0)^T, \dots, (0 \dots v_n)^T \quad (2.4.6)$$

and their multiples, as well as for the linear combinations of those directions that correspond to v_i 's with the same open RHP zeros in the same degree and the same time delays.

3. MODEL UNCERTAINTY

3.1 Structured Singular Value.

The Structured Singular Value (SSV), introduced by Doyle (1982), takes into account the structure of the model uncertainty and it allows the non-conservative quantification of the concept of robust performance.

For a constant complex matrix M the definition of the SSV $\mu_\Delta(M)$ depends also on a certain set Δ . Each element Δ of Δ is a block diagonal complex matrix with a specified dimension for each block, i.e.,

$$\Delta = \{ \text{diag}(\Delta_1, \Delta_2, \dots, \Delta_n) \mid \Delta_j \in \mathbb{C}^{m_j \times m_j} \} \quad (3.1.1)$$

Then

$$\frac{1}{\mu_\Delta(M)} = \min_{\Delta \in \Delta} \{ \delta(\Delta) \mid \det(I - M\Delta) = 0 \} \quad (3.1.2)$$

and $\mu_\Delta(M) = 0$ if $\det(I - M\Delta) \neq 0 \quad \forall \Delta \in \Delta$. Note that δ is the maximum singular value of the corresponding matrix.

3.2 Block Structure.

In order to effectively use the SSV for designing F , some rearrangement of the block structure is necessary. The IMC structure of Fig. 1a can be written as that of Fig. 2 where Δ is a block diagonal matrix representing the model uncertainty, which has the additional property that

$$\delta(\Delta) \leq 1 \quad \forall \omega \quad (3.2.1)$$

G^F is a function of F and it also depends on the specific uncertainty description available for the model \hat{P} . We shall not demonstrate in detail here, how G^F can be obtained. For some common cases of model uncertainty, the expressions can be found in Zafriou (1987).

3.3 Robust Stability.

We now require that the matrix ISI as given by (1.2.1) is stable for all possible plants P . The design of \tilde{Q} according to Section 2 resulted in a stable ISI for $P = \hat{P}$. The SSV can be used to determine if any crossings of the imaginary axis occur for any P . The system is stable for any of the plants in the set defined from the bounds on the model uncertainty and which have the same number of RHP poles as the model, if and only if (Doyle, 1985)

$$\mu_\Delta(G_{11}^F) < 1 \quad \forall \omega \quad (3.3.1)$$

3.4 Robust Performance.

In the first step of the IMC design procedure a controller \tilde{Q} is obtained, which produces satisfactory disturbance rejection and/or setpoint tracking. This response is described by the "sensitivity" function \tilde{E} defined in (1.1.3) for $P + \hat{P}$. Since \tilde{E} connects the external inputs to the error e , a well-designed control system produces a relatively "small" \tilde{E} . A measure of the magnitude of the known \tilde{E} is its maximum singular value. Let $b(\omega)$ be a frequency function such that

$$\delta(\tilde{E}(i\omega)) < b(\omega) \quad \forall \omega \quad (3.4.1)$$

In order for the performance of the control system to remain robust with respect to model-plant mismatch we have to keep E small in spite of the modeling error. Hence we require that

$$\sup_{\omega} \delta(b(\omega)^{-1} E(i\omega)) < 1 \quad \forall \Delta \in \Delta \quad (3.4.2)$$

We can now use the properties of the SSV (Doyle, 1985) to obtain

$$\begin{aligned} \sup_{\omega} \delta(b(\omega)^{-1} E(i\omega)) < 1 \quad \forall \Delta \in \Delta \\ \iff \sup_{\omega} \mu_{\Delta^0}(G^b) < 1 \end{aligned} \quad (3.4.3)$$

where

$$G^b = \begin{pmatrix} I & 0 \\ 0 & b^{-1} \end{pmatrix} G^F \quad (3.4.4)$$

$$\Delta^0 = \{ \text{diag}(\Delta, \Delta^0) \mid \Delta \in \Delta, \Delta^0 \in \mathbb{C}^{m \times n} \} \quad (3.4.5)$$

4. STEP 2: DESIGN OF F

4.1 Filter Structure.

The filter parameters can now be computed so that the robustness conditions that were discussed in Section 3 are satisfied. To do so, some structure will have to be assumed for F , which can be of any general type that the designer wishes. However in order to keep the number of variables in the optimization problem small, a rather simple structure like a diagonal F with first or second order terms would be recommended.

i) Open-loop unstable plants.

The IMC filter $F(s)$ is chosen to be a diagonal rational function that satisfies the following requirements.

- a. Pole-zero excess. The controller $Q = \tilde{Q}F$ must be proper. Assume that the designer has specified a pole-zero excess of κ for the filter $F(s)$.
- b. Internal stability. ISI in (1.2.2) must be stable.
- c. Asymptotic setpoint tracking and/or disturbance rejection. $(I - \hat{P}\tilde{Q}F)v$ must be stable.

Write

$$F(s) = \text{diag}(f_1(s), \dots, f_n(s)) \quad (4.1.1)$$

Under assumptions A.1,2,3,4,5, (b),(c) are equivalent to the following conditions. Let π_i ($i = 1, k$) be an open RHP pole of \tilde{P} and $\pi_0 = 0$ and m_{0i} the largest multiplicity of such a pole in any element of the i^{th} row of V . From assumptions A.1, 2, 3, 4, 5 and the fact that \tilde{Q} makes $(I - P\tilde{Q})V$ stable, it follows that the i^{th} element, f_i of the filter F must satisfy:

$$f_i(\pi_i) = 1, \quad i = 0, 1, \dots, k \quad (4.1.2)$$

$$\frac{d^j}{ds^j} f_i(s)|_{s=\pi_0} = 0, \quad j = 1, \dots, m_{0i} - 1 \quad (4.1.3)$$

(4.1.2) clearly shows the limitation that RHP poles place on the robustness properties of a control system designed for an open-loop unstable plant. Since because of (4.1.2) one cannot reduce the nominal ($P = \tilde{P}$) closed-loop bandwidth of the system at frequencies corresponding to the RHP poles of the plant, one can only tolerate a relatively small model error at those frequencies.

Experience has shown that the following structure for a filter element $f_i(s)$ is reasonable:

$$f_i(s) = \frac{a_{\nu_i-1}s^{\nu_i-1} + \dots + a_{1,i}s + a_{0,i}}{(\lambda s + 1)^{\nu_i-1}} \quad (4.1.4)$$

where

$$\nu_i = m_{0i} + k \quad (4.1.5)$$

and then compute the numerator coefficients for a specific tuning parameter λ from (4.1.2), (4.1.3). This involves solving a system of ν_i linear equations with ν_i unknowns. ii) Ill-conditioned plants.

The problems arise because the optimal controller \tilde{Q} designed for \tilde{P} tends to be an approximate inverse of \tilde{P} and as a result \tilde{Q} is ill-conditioned as well. A way to address this problem is to try to use a filter that acts directly on the singular values of \tilde{Q} , at the frequency where the condition number of \tilde{Q} is highest, say ω^* . Let

$$\tilde{Q}(i\omega^*) = U_Q \Sigma_Q V_Q^* \quad (4.1.6)$$

be the SVD of \tilde{Q} at ω^* and let R_u, R_v , be real matrices that solve the pseudo-diagonalization problems:

$$U_Q^* R_u \approx I \quad (4.1.7)$$

$$V_Q^* R_v \approx I \quad (4.1.8)$$

Then for the IMC controller Q that includes the filter, use the expression

$$Q(s) = R_u F_1(s) R_v^{-1} \tilde{Q}(s) F_2(s) \quad (4.1.9)$$

or

$$Q(s) = \tilde{Q}(s) R_v F_1(s) R_u^{-1} F_2(s) \quad (4.1.10)$$

where $F_1(s), F_2(s)$ are diagonal filters, such that $F_1(0) = F_2(0) = I$ if integral action is desired. Note for F_1, m_{0i} should be used in (4.1.3), (4.1.5), for all i , instead of m_{0i} , where $m_{0i} = \max_i m_{0i}$.

Note that one can put this control structure in the form of Fig. 2 (Zafriou, 1987) where

$$F(s) = \text{diag}(F_1(s), F_2(s)) \quad (4.1.11)$$

4.2 Objective

We can write

$$F \stackrel{\text{def}}{=} F(s; \Lambda) \quad (4.2.1)$$

where Λ is an array with the filter parameters. The problem can now be formulated as a minimization problem over the elements of the array Λ . Our goal is to satisfy (3.4.3). The filter parameters can be obtained by solving

$$\min_{\Lambda} \sup_{\omega} \mu_{\Delta^{\circ}}(G^b) \quad (O4)$$

One should note that in (O4), the objective function is not convex. Hence a local minimum could be the result of solving (O4). Use of a number of different initial values for Λ can help circumvent this problem. Also, good initial guesses can usually be obtained for the filter parameters (elements of Λ) by matching them with the frequencies where the peaks of $\mu_{\Delta^{\circ}}(G^b)$ appear for $F = I$. The computational issues are discussed by Zafriou & Morari (1986).

5. ILLUSTRATION

The design of a robust IMC controller will be demonstrated for a 2×2 high purity distillation column. Skogestad and Morari (1986) used the following model:

$$\tilde{P}(s) = \frac{1}{75s + 1} \begin{pmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{pmatrix} \quad (5.0.1)$$

The problems arise from the fact that high purity distillation columns are ill-conditioned at steady-state. For the case in (5.0.1) the condition number is equal to 142. We have

$$\tilde{Q}(s) = \tilde{P}(s)^{-1} \quad (5.0.2)$$

The IMC filter has now to be designed for robustness. Input uncertainty will be assumed in this case. Skogestad and Morari (1986) proposed the following uncertainty bound l , and performance bound b (used in (3.4.2)):

$$\sigma(\tilde{P}^{-1}(P - \tilde{P})) \leq l_i(s) = 0.2 \frac{5s + 1}{0.5s + 1}, \quad s = j\omega \quad (5.0.3)$$

$$b(s) = \frac{20s}{10s + 1} \quad (5.0.4)$$

First contrary to Section 4.1.ii, a simple diagonal filter will be used to demonstrate the problem. A gradient search procedure based on the analytic gradient expressions given by Zafriou and Morari (1986), is used to solve (O4') for a one parameter Λ , i.e., a scalar times identity filter is employed. The result is

$$F(s) = \frac{1}{7.28s + 1} I \quad (5.0.5)$$

Plots of μ for robust stability (given by (3.3.1)) and robust performance (given by (3.4.3)) are shown in Fig. 3. Clearly, the performance is expected to deteriorate. This is confirmed by the simulations shown in Fig. 7 for a step set-point change in output 1. For the nominal case ($P = \tilde{P}$), the outputs are decoupled and the performance is acceptable (Fig. 4a). However in the case where

$$P(s) = \tilde{P}(s) \begin{pmatrix} 1.2 & 0 \\ 0 & 0.8 \end{pmatrix} \quad (5.0.6)$$

the performance deteriorates to the point where it is totally unacceptable (Fig. 4b). Filters with different elements and higher orders and zeros were also used in the optimization but the results showed that these filters produce no worthwhile difference for the one-filter controller. The reason is that one diagonal filter cannot in general significantly affect the condition number of \tilde{Q} .

We shall now proceed and use the filter structure suggested for ill-conditioned systems in Section 4.1.ii. In this case $\omega^* = 0$ and therefore the diagonalizations (4.1.7) and (4.1.8) are exact. Hence (4.1.9) and (4.1.10) produce the same Q . Objective (O4') was solved with a gradient search method. Different filter orders were used and a few different initial guesses were tried to avoid local minima. The final result for filters with two parameters in each element, was:

$$F_1(s) = \begin{pmatrix} \frac{0.244s+1}{(0.184s+1)^2} & 0 \\ 0 & \frac{0.00284s+1}{(8.72s+1)^2} \end{pmatrix} \quad (5.0.8)$$

$$F_2(s) = \begin{pmatrix} \frac{0.164s+1}{(0.446s+1)^2} & 0 \\ 0 & \frac{0.213s+1}{(0.476s+1)^2} \end{pmatrix} \quad (5.0.9)$$

The values of μ for robust stability and performance are shown in Fig. 3b. The clear improvement over the diagonal filter is verified by the simulations in Fig. 5. It is interesting to note that the responses in Figs. 4 and 5 are similar for the nominal case. Hence, as expected, the increase in robustness was not the result of additional detuning, which is something that if it was sufficient it would have been accomplished by the minimization problem solved for the diagonal filter. The reason for the improvement is that the two-filter structure acted directly on the singular values of \hat{Q} and, in addition to appropriate detuning, it also reduced its condition number at the critical frequency range.

Finally, a last comparison will be made between the performance obtained by the two-filter IMC controller and the "true" μ -optimum controller, defined as the result of minimizing $\sup_{\Delta} \mu_{\Delta}(G^b)$ over not a specified filter structure but over any stabilizing controller Q or C . However, the iterative approach suggested by Doyle (1985) for solving this problem is not guaranteed to converge and indeed, it has often failed to converge. For this particular example though, Skogestad and Morari (1986) obtained this μ -optimal controller. The values of μ for robust performance and stability are shown in Fig. 3 for the same bounds as in (5.0.3) and (5.0.4). Clearly the difference is not significant and this is verified by the simulations shown in Fig. 6.

6. CONCLUDING REMARKS

The work presented in this paper extends the two-step IMC design procedure to open-loop unstable systems. It proposes a meaningful way for the selection of the "weight" in the H_2 -minimization problem solved in the first step of the synthesis procedure, through Theorem 2.4.2 and Corollary 2.4.1. The method quantifies the problem of the design of the IMC filter via the use of the Structured Singular Value and provides analytic expressions for the gradients of the objective functions. The special filter structures needed for open-loop unstable plants and ill-conditioned plants are given. The promise of the approach as a practical way for the design of robust multivariable controllers is demonstrated by the high purity distillation column example examined in the paper.

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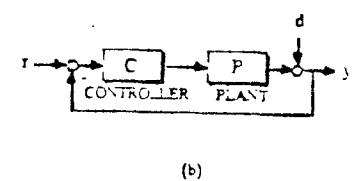
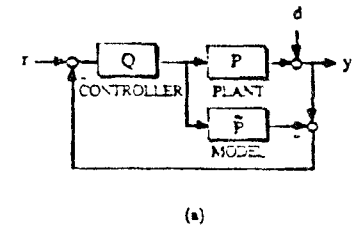


Figure 1.

- (a) Internal Model Control structure.
- (b) Feedback Control structure.

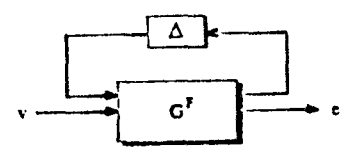


Figure 2

SSV block diagram.

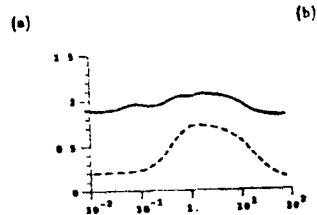
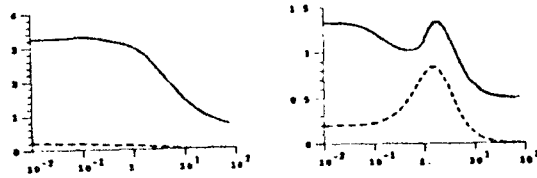


Figure 3.

Plots of $\mu(\omega)$. Solid line: Robust Performance. Dashed line: Robust stability.
 (a) One-filter IMC controller.
 (b) Two-filter IMC controller.
 (c) μ -optimal controller.

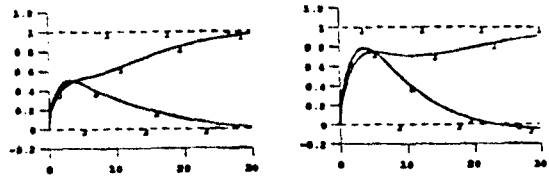


Figure 6.

Time response for the μ -optimal controller.
 - Dashed lines: Setpoints; Solid lines: Outputs.
 (a) $P = \tilde{P}$.
 (b) $P \neq \tilde{P}$.

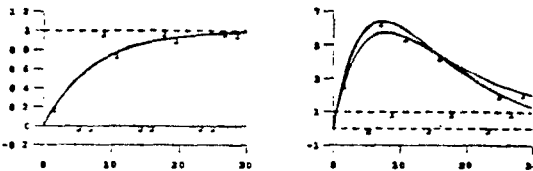


Figure 4.

Time response for the one-filter IMC controller.
 - Dashed lines: Setpoints; Solid lines: Outputs.
 (a) $P = \tilde{P}$.
 (b) $P \neq \tilde{P}$.

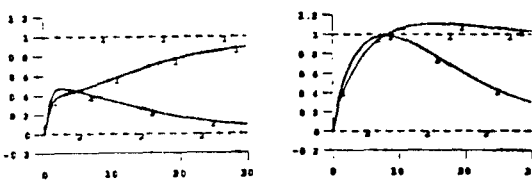


Figure 5.

Time response for the two-filter IMC controller.
 - Dashed lines: Setpoints; Solid lines: Outputs.
 (a) $P = \tilde{P}$.
 (b) $P \neq \tilde{P}$.