ABSTRACT

Title of dissertation: FUNDAMENTAL DOMAINS FOR PROPER AFFINE ACTIONS OF COXETER GROUPS IN THREE DIMENSIONS

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We study proper actions of groups $G \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ on affine space of three real dimensions. Since G is nonsolvable, work of Fried and Goldman implies that it preserves a Lorentzian metric. A subgroup $\Gamma < G$ of index two acts freely, and $\mathbb{R}^3/\Gamma$ is a Margulis spacetime associated to a hyperbolic surface Σ.

When Σ is convex cocompact, work of Danciger, Guéritaud, and Kassel shows that the action of Γ admits a polyhedral fundamental domain bounded by crooked planes. We consider under what circumstances the action of G also admits a crooked fundamental domain.

We show that it is possible to construct actions of G that fail to admit crooked fundamental domains exactly when the extended mapping class group of Σ fails to act transitively on the top-dimensional simplices of the arc complex of Σ. We also provide explicit descriptions of the moduli space of G actions that admit crooked fundamental domains.

FUNDAMENTAL DOMAINS FOR PROPER AFFINE ACTIONS OF COXETER GROUPS IN THREE DIMENSIONS

by

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Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2016

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Acknowledgments

I am deeply grateful to my advisor William Goldman for his friendship and mentorship, and for many enlightening conversations. I would also like to thank my committee for their interest in my work and for their helpful suggestions.

Many colleagues have contributed to my work. I am especially grateful to Karin Melnick, Virginie Charette, Todd Drumm, and Sean Lawton for their mentorship and guidance.

I would like to thank François Guéritaud for his insight and suggestions about my work, and Yair Minsky for his support and hospitality.

I would like to acknowledge support from U.S. National Science Foundation grants DMS 1107452, 1107263, 1107367 "RNMS: GEometric structures And Representation varieties" (the GEAR Network)."

Finally, I would like to thank my family and friends for their love and support, especially my wife Shannon.

Table of Contents

List of Figures

Chapter 1: Introduction

We consider proper affine isometric actions of a non-solvable group Γ' generated by torsion elements on $\mathbb{R}^3$. Specifically, we are interested in the case where Γ' is isomorphic to the group $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ generated by three non-commuting elements of order two. We classify when such actions admit crooked fundamental domains.

The group $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ contains a free non-abelian subgroup of index two. Denote such a group by $\mathbb{F}_2$. The corresponding subgroup $\Gamma \cong \mathbb{F}_2$ of Γ' acts without fixed points, and the quotient $M := \mathbb{R}^3/\Gamma$ is a geodesically complete flat affine manifold.

By [1], M has a Lorentzian structure. Moreover, M is homotopy equivalent to a complete noncompact hyperbolic surface Σ as follows. The linear part of Γ, which we denote Γ_0, is a discrete group that acts on the hyperbolic plane $\mathbb{H}^2$ by isometries. We define $\Sigma := \mathbb{H}^2/\Gamma_0$. For any affine group G, we let the notation G_0 denote its linear part. We say that G is an *affine deformation* of G_0. Similarly, we say that M is an affine deformation of Σ. We say that Γ is a *proper affine deformation* if it acts properly on $\mathbb{R}^3$.

The linear part Γ_0' of Γ' is a discrete involution group acting on $\mathbb{H}^2$ and containing Γ_0 as an index two subgroup. In Chapter 2, we describe a construction following [2, 3] of moving from representations of $\mathbb{F}_2$ to representations of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. This construction is called a *Coxeter extension*. Γ' is a Coxeter extension of Γ, and Γ'_0 is a Coxeter extension of Γ_0.

We restrict ourselves to the case where Γ_0 is convex cocompact. This allows us to invoke the *Crooked Plane Theorem*, due to Danciger, Guéritaud, and Kassel, which states that the action of Γ necessarily admits a polyhedral fundamental domain bounded by piecewise-linear topological planes called *crooked planes*.

Proper affine actions by torsion groups like Γ', however, need not admit such domains. Charette [4] constructed an example of a group generated by three reflections whose action does not admit a crooked fundamental domain. In her case, the linear holonomy Γ_0 was the holonomy group of a three-holed sphere.

In this thesis, we describe all actions of Γ' that admit crooked fundamental domains. There are four cases to consider, corresponding to the four surfaces Σ with fundamental group isomorphic to $\mathbb{F}_2$. Equivalently, these are the surfaces whose Euler characteristic is -1.

Let $\Sigma_{g,n}$ denote an orientable surface of genus g with n holes. Let $C_{g,n}$ denote a non-orientable surface of genus g with n holes. The surfaces under consideration are:

- The three-holed sphere, $\Sigma_{0,3}$.
- The one-holed torus, $\Sigma_{1,1}$.
- The two-holed projective plane, $C_{1,2}$.

- The one-holed Klein bottle, $C_{2,1}$.

As we have already mentioned, the case of the three-holed sphere originally appears in the work of Charette [4,5]. The case of the two-holed projective plane appears in joint work with Goldman [6] on which part of Chapters 3 and 5 are based.

For each surface, we give an explicit description of the space of affine Coxeter extensions that admit crooked fundamental domains. We also prove a characterization of when an affine deformation of a Coxeter extension fails to admit a crooked fundamental domain in terms of its mapping class group $\mathrm{MCG}^{\pm}(\Sigma)$.

Definition 1.0.1. *The* mapping class group $\mathrm{MCG}^{\pm}(\Sigma)$ *of a surface* Σ *is the group of isotopy classes of diffeomorphisms of* Σ*:*

$$\mathrm{MCG}^{\pm}(\Sigma) := \mathrm{Diff}(\Sigma)/\,\mathrm{Diff}^0(\Sigma)$$

where $\mathrm{Diff}^0(\Sigma)$ *is the group of diffeomorphisms isotopic to the identity.*

In many contexts, for example in [7], the mapping class group is taken to be the group of isotopy classes of *orientation-preserving* diffeomorphisms. We allow orientation-reversing diffeomorphisms since some of the surfaces we are interested in are non-orientable. The group $\mathrm{MCG}^{\pm}(\Sigma)$ is sometimes called the *extended mapping class group*. The notation $\mathrm{MCG}^{\pm}$ emphasizes that mapping classes can reverse orientation. Note that a general element of $\mathrm{MCG}^{\pm}(\Sigma)$ may permute the boundary components of the convex core of Σ.

Theorem 1.0.2. *Every affine deformation* Γ' *of the Coxeter extension* Γ'_0 *admits a crooked fundamental domain if and only if the (extended) mapping class group*

$\mathrm{MCG}^{\pm}(\Sigma)$ *acts transitively on the top-dimensional simplices of the arc complex of* Σ*.*

The top-dimensional simplices of the arc complex correspond to hyperideal triangulations of Σ. Since fundamental domains for the action of Γ_0' on $\mathbb{H}^2$ are built from hyperideal triangles, every such fundamental domain corresponds to a two-simplex, or *tile*, of the arc complex. Theorem 1.0.2 can be interpreted as an assertion that Γ' fails to admit a crooked fundamental domain exactly when the corresponding hyperideal triangulation of Σ fails to be a fundamental domain for Γ_0'.

Theorem 1.0.2 implies that when Γ_0 is the holonomy group of a one-holed torus, Γ' necessarily admits a crooked fundamental domain. For the three remaining surfaces, it is possible to construct affine deformations Γ' that act properly but admit no such domain.

The action of the linear Coxeter extension Γ_0' on $\mathbb{H}^2$ always admits a fundamental polygon τ. We take τ to be an ultraideal triangle, meaning its sides are pairwise ultraparallel. A generator of Γ_0' is a reflection if it reverses the orientation of $\mathbb{H}^2$ or an elliptic element of order two if it preserves orientation. Each reflection fixes a line in $\mathbb{H}^2$ point-wise. Each elliptic element of order two fixes a point p in the interior of $\mathbb{H}^2$ and reverses every line through p. We say that such an element is a *point symmetry* in the point p.

Denote the projection onto the linear factor of $\mathrm{Aff}(\mathbb{R}^3)$ by $\mathbb{L}$. Then $\mathbb{L}(\Gamma) = \Gamma_0$. If $\mathscr{C}$ is a crooked plane, then $\mathbb{L}(\mathscr{C})$ can be interpreted as a geodesic in $\mathbb{H}^2$. A crooked fundamental domain for Γ' linearizes under this operation to a fundamental domain in $\mathbb{H}^2$ for the action of Γ'_0.

Definition 1.0.3. *Let $C = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3\}$ be a triple of crooked planes in $\mathbb{R}^3$. We say that C is a crooked realization of τ if*

- *$\mathscr{C}_1, \mathscr{C}_2$, and $\mathscr{C}_3$ are pairwise disjoint.*
- *$\mathbb{L}(\mathscr{C}_1), \mathbb{L}(\mathscr{C}_2)$, and $\mathbb{L}(\mathscr{C}_3)$ are the sides of τ.*

Theorem 1.0.4. *Let τ be a fundamental domain for the action of Γ'_0. Let $\Sigma = \mathbb{H}^2/\Gamma_0$. Then the space of crooked realizations of τ is the interior of a cone inscribed in the space of all proper affine deformations of Γ_0. This cone is*

- *Six-sided if Σ is a three-holed sphere.*
- *Three-sided if Σ is a one-holed torus.*
- *Five-sided if Σ is a two-holed projective plane.*
- *Four-sided if Σ is a one-holed Klein bottle.*

Each cone has a vertex at the origin, which does not correspond to a proper affine deformation. The space of proper affine deformations of Σ is also the interior of a cone. Quotienting by the action of scalars on the space of affine deformations, the cones become convex sets in $\mathbb{R}P^2$. The projectivized images of the cones in Theorem 1.0.4 are polygons inscribed in the projectivized space of proper affine deformations of Σ, which itself need not be finite-sided.

The polygons can be viewed as blunted two-simplices, where one vertex of the triangle is blunted for each generator of Γ_0' that is a reflection. When Γ_0 is the holonomy group of a one-holed torus, every generator is elliptic and no blunting occurs. When Γ_0 is the holonomy group of a three-holed sphere, then every generator of Γ_0' is a reflection and so all sides are blunted, resulting in a hexagon.

The blunting corresponds to a projective interval's worth of choice of direction in which to move crooked planes apart. The space of directions one can translate a crooked plane to make it disjoint from its two neighbors is called the *stem quadrant*, to be defined in Chapter 2. The blunting can be removed by making a consistent choice of direction in each stem quadrant rather than considering the space of all possible directions simultaneously.

Finally, we describe the space of all affine deformations of Γ_0' that admit a crooked fundamental domain. For an ultraideal triangular fundamental domain τ for the action of Γ_0', let $[\tau]$ be the set of all isotopy classes of τ such that each representative of $[\tau]$ is a fundamental domain for Γ_0'. Let $\mathrm{Crook}([\tau])$ be the set of all crooked realizations of choices of geodesic representatives for any choice of $\tau \in [\tau]$. Theorem 1.0.2 is really a corollary of the following more precise statement.

Theorem 1.0.5. *The space of affine deformations of Γ_0' that admit crooked fundamental domains is the orbit of $\mathrm{Crook}([\tau])$ under the action of $\mathrm{MCG}^{\pm}(\Sigma)$.*

The polygons in the orbit may intersect non-trivially due to blunting. However, one may obtain non-intersecting polygons by consistently choosing directions in the stem quadrants and being careful about the choice of geodesic representatives for the sides of τ.

1.1 Motivation

For our four surfaces, the action of Γ_0 on the Nielsen convex region in $\mathbb{H}^2$ admits a quadrilateral fundamental domain with deleted vertices. For the genus $g \geq 1$ surfaces, these are the familiar quadrilateral domains for $\mathbb{T}^2$, $\mathbb{R}P^2$ and the Klein bottle with the corners removed. This is, of course, a consequence of the classical theorem that every compact surface admits a canonical polygonal fundamental domain that can be turned into a fundamental domain for a noncompact surface by deleting the vertices (see *e.g.* [8] for an excellent historical account).

Since the fundamental group of every Margulis spacetime is isomorphic to the fundamental group of a noncompact surface, one might hope to build crooked fundamental domains based on the classical polygonal domains. Determining whether n crooked planes are disjoint is a non-trivial problem (*e.g.* [9]). Determining when three crooked planes are disjoint is, however, rather tractable. Thus one might hope to build a theory of crooked fundamental domains for Margulis spacetimes by working first with crooked triangular domains for the Coxeter extensions for surfaces of rank two.

This strategy does not work in complete generality, as Charette showed in the case of the three-holed sphere. It does work for the case of the one-holed torus. We show that it fails for the remaining two surfaces and also describe necessary and sufficient conditions for the strategy to succeed in terms of combinatorial properties of systems of arcs on the surfaces.

Note that the sides of the polygons with deleted vertices project to properly embedded geodesic arcs in the quotient. This provides a description of the fundamental domains in the language of the arc complex. The arc complex $\mathscr{A}(\Sigma)$ identifies with the space of proper affine deformations of Σ by [10].

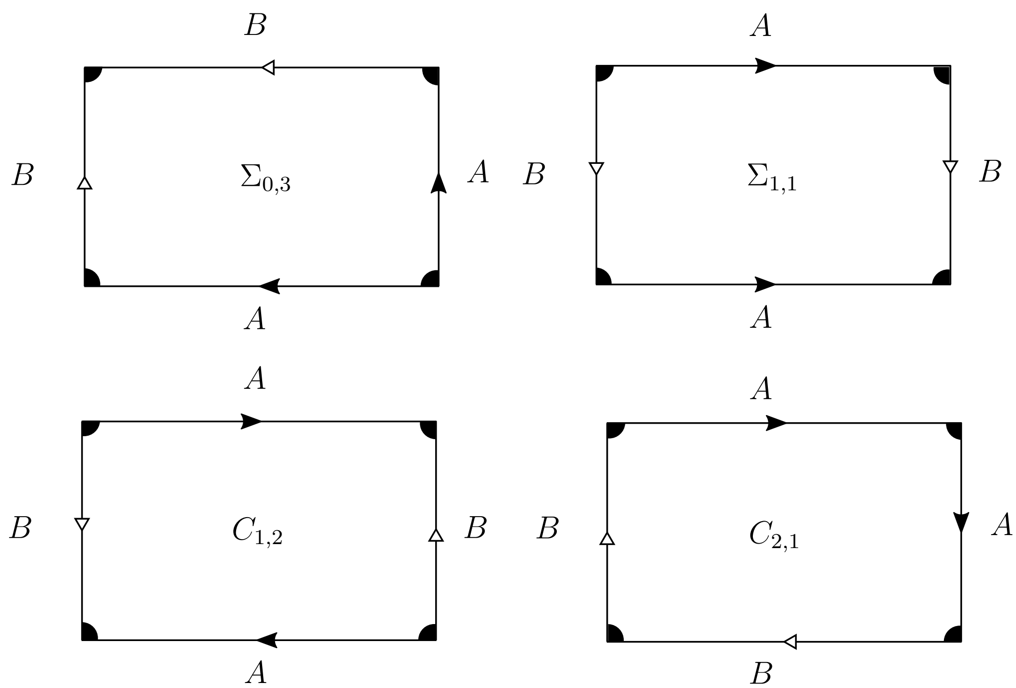

Figure 1.1: Quadrilateral domains for the four surfaces.

When $\pi_1(\Sigma)$ is free of rank two, $\mathscr{A}(\Sigma)$ is dual to the *flip graph* of Σ. The vertices of this graph are hyperideal triangulations, and two vertices share an edge if and only if the triangulations differ by a diagonal flip.

The extended mapping class group $\mathrm{MCG}^{\pm}(\Sigma)$ acts on $\mathscr{A}(\Sigma)$ by automorphisms. We show that the failure of the Coxeter extension strategy can be read directly off of the arc complex by observing that for three of the four surfaces, this action is not transitive on the top-dimensional simplices.

1.2 Plan of Thesis

Chapter Two introduces the necessary tools of affine geometry. A theorem of Fried-Goldman implies that we only need to consider *Minkowski space*, which is a 3-dimensional real vector space with a signature $(2,1)$ inner product. We first review some basic properties of this space, such as the inner and cross products. We consider important groups of affine isometries and discuss the identification of the hyperbolic plane $\mathbb{H}^2$ with translational equivalence classes of timelike geodesics. We also introduce crooked planes, which play a central role in this thesis. We describe a criterion that tells us when triples of crooked planes are disjoint.

Chapter Three begins the study of torsion. We consider the theory of Coxeter extensions of free groups of rank 2. If Σ is a hyperbolic surface with $\chi(\Sigma) = -1$, then $\pi_1(\Sigma) \cong \mathbb{F}_2$. This group can be extended by the addition of an involution that reverses two preferred generators. The geometric version of this construction allows us to extend a representation $\rho_0 : \mathbb{F}_2 \to \mathrm{Isom}(\mathbb{H}^2)$ to a representation

$$\rho' : \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \to \mathrm{Isom}(\mathbb{H}^2).$$

The quotient space $\mathbb{H}^2/\Gamma_0'$ has singular points and so is not a manifold. The algebraic theory allows us to perform the same Coxeter extension construction to free groups of affine transformations. Thus we can study the space of proper affine deformations of the Coxeter group with a fixed covering surface. Because the Coxeter group fixes points in $\mathbb{R}^{2,1}$, the quotient affine space again fails to be a manifold. In particular, it is not a Margulis spacetime and so the Crooked Plane Theorem does not apply. This opens the question of when such actions admit crooked fundamental domains.

Chapter Four introduces some useful tools for the study of proper affine deformations. Danciger, Guéritaud, and Kassel have shown that the projectivized space of proper affine deformations for any convex-cocompact surface is isomorphic to the arc complex of that surface. Surfaces of Euler characteristic -1 have especially nice arc complexes in that they are two-dimensional. For the four surfaces under consideration, the automorphism group of the arc complex identifies with the quotient of the mapping class group by its center. This will be useful for characterizing crooked realizations of fundamental domains.

Chapter Five parametrizes the spaces of crooked fundamental domains for the four Euler-characteristic -1 surfaces. We describe the space of crooked fundamental domains for each type of surface, and prove that there are proper actions of Coxeter groups in the three-holed sphere, two-holed projective plane, and one-holed Klein bottle case that do not admit crooked fundamental domains. Every properly-acting Coxeter extension of the one-holed torus does admit a crooked fundamental domain.

Chapter 2: Minkowski Space, Affine Maps, Crooked Planes

In this chapter, we discuss some basic facts about Margulis spacetimes and related group actions. Many of the facts about affine Lorentzian geometry can be found in [11]. A discussion of Lorentzian vector spaces and their connection with hyperbolic space can be found in [12].

2.1 Affine Representations

Any complete affine manifold M is the quotient of an affine space by a discrete group Γ of affine transformations whose action is proper and free of fixed points. In three dimensions, Fried and Goldman [1] and Mess [13] together imply that either Γ is solvable or it is virtually free.

The case where Γ is solvable was solved in [1]. They show that M admits a finite covering homeomorphic to the total space of a fibration composed of points, circles, annuli and tori. In the case where Γ is virtually free, Fried and Goldman [1] showed that up to conjugation, Γ must preserve an inner product of signature $(2, 1)$.

Let $\mathbb{R}^{2,1}$ denote the vector space $\mathbb{R}^3$ with the inner product given by

$$v \cdot w = v_1 w_1 + v_2 w_2 - v_3 w_3.$$

Let $\mathbb{E}^{2,1}$ denote the affine space modeled on $\mathbb{R}^{2,1}$. We can think of $\mathbb{R}^{2,1}$ as the group of translations acting on $\mathbb{E}^{2,1}$. A choice of a distinguished point $o \in \mathbb{E}^{2,1}$ identifies $\mathbb{E}^{2,1}$ with $\mathbb{R}^{2,1}$ by sending o to the origin. Making a different choice corresponds to conjugating by a translation. We will typically identify $\mathbb{E}^{2,1}$ with $\mathbb{R}^{2,1}$ by implicitly choosing an origin.

Denote the space of affine isometries of $\mathbb{R}^{2,1}$ by $\mathrm{Aff}(2,1)$. This decomposes as the semidirect product $\mathrm{Aff}(2,1) = \mathrm{O}(2,1) \ltimes \mathbb{R}^{2,1}$. Let $\mathbb{L} : \mathrm{Aff}(2,1) \to \mathrm{O}(2,1)$ denote projection onto the first factor. By Let G be any group and $\phi : G \to \mathrm{Aff}(\mathbb{R}^{2,1})$ a representation. The map $\mathbb{L}$ defines a representation of G into $\mathrm{O}(2,1)$:

$$\Phi = \mathbb{L} \circ \phi : G \to \mathrm{O}(2,1).$$

We call Φ the *linear part* of ϕ. The *translational part* of ϕ is denoted $u : G \to \mathbb{R}^{2,1}$ is defined by

$$\phi(g)(x) = \Phi(g)(x) + u(g).$$

We write $\phi = (\Phi, u)$ and say that ϕ is an *affine deformation* of Φ.

The translational part u satisfies a cocycle condition

$$u(g_1 g_2) = u(g_1) + \Phi(g_1) u(g_2).$$

Conjugation by a translation changes ϕ by a coboundary. Translational conjugacy classes identify with the cohomology $H^1(\Gamma, \mathbb{R}^{2,1})$. See [14].

An affine deformation $\phi = (\Phi, u)$ is called *proper* if $\phi(G)$ acts properly discontinuously on $\mathbb{R}^{2,1}$.

2.2 Geometry of $\mathbb{R}^{2,1}$

The following definitions are standard.

Definition 2.2.1. *Let $v \in \mathbb{R}^{2,1} \setminus \{0\}$. We say that v is*

- spacelike *if $v \cdot v > 0$,*

- timelike *if $v \cdot v < 0$,*

- lightlike *or* null *if $v \cdot v = 0$.*

There is a Lorentzian cross product $\boxtimes$, defined as the unique map satisfying

$$(u \boxtimes v) \cdot w = \det(u, v, w).$$

In particular, $u \boxtimes v$ is Lorentz-orthogonal to both u and v. A useful formula is the following, for vectors $u_1, u_2, v_1, v_2 \in \mathbb{R}^{2,1}$.

$$(u_1 \boxtimes v_1) \cdot (u_2 \boxtimes v_2) = -(u_1 \cdot u_2)(v_1 \cdot v_2) + (u_2 \cdot v_2)(v_1 \cdot u_2). \tag{2.1}$$

Denote the set of timelike vectors by $\mathscr{T}$ and the set of null vectors by $\mathscr{N}$. The space $\mathscr{T}$ has two connected components. Select the component containing the vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ to be the *future*. Any other timelike vector in the same connected component will be called *future-pointing*, and vectors in the other component are called *past-pointing*. A choice of future direction is called a *time orientation*.

If $v \in \mathbb{R}^{2,1}$ is spacelike, then $v^{\perp}$ is an indefinite plane. In this case, $\left(v^{\perp} \cap \mathscr{N}\right) \cup \{0\}$ consists of two null lines, which we will call n^- and n^+. Choose future-pointing null vectors v^- and v^+ that span n^- and n^+ respectively. For example, choose v^- and v^+ to have unit length in the standard Euclidean dot product. Then v^- and v^+ (and also n^- and n^+) are uniquely determined by requiring that $\{v^-, v^+, v\}$ is a right-handed basis for $\mathbb{R}^{2,1}$.

The projective or Klein-Beltrami model of the hyperbolic plane classically identifies with $\mathbb{P}(\mathscr{T})$. This can be adapted to the affine setting by identifying a point in $\mathbb{H}^2$ with translational equivalence classes of timelike lines in $\mathbb{E}^{2,1}$. Translational equivalence classes of null lines identify with the ideal boundary $\partial_\infty \mathbb{H}^2$.

Geodesics in $\mathbb{H}^2$ correspond to parallelism classes of linear planes intersecting $\mathscr{T}$. These in turn correspond to translational equivalence classes of spacelike lines in $\mathbb{E}^{2,1}$. If $v\mathbb{R}$ is a spacelike line, its dual geodesic in $\mathbb{H}^2$ is $[v^\perp \cap \mathscr{T}]$. A spacelike vector $s \in \mathbb{R}^{2,1}$ defines an *oriented* geodesic g in $\mathbb{H}^2$ by orienting g from s^- to s^+.

Recall that if g_1 and g_2 are geodesics in $\mathbb{H}^2$, then they are *parallel* if they do not intersect in $\mathbb{H}^2$. They are called *asymptotically parallel* if the geodesics "intersect" at a point in $\partial_\infty \mathbb{H}^2$. Otherwise they are called *ultraparallel.* Two geodesics admit a common perpendicular if and only if they are ultraparallel.

Let s_1 and s_2 be spacelike vectors dual to the hyperbolic geodesics g_1 and g_2. Then g_1 and g_2

- are asymptotically parallel if and only if $s_1 \boxtimes s_2$ is null,
- are ultraparallel if and only if $s_1 \boxtimes s_2$ is spacelike, and
- intersect if and only if $s_1 \boxtimes s_2$ is timelike.

If $s_1 \boxtimes s_2$ is timelike, then the equivalence class $[s_1 \boxtimes s_2] \in \mathbb{H}^2$ is $g_1 \cap g_2$.

2.3 The Isometry Group $\mathrm{Aff}(2,1)$

The group $\mathrm{O}(2,1)$ has four connected components corresponding to whether a matrix $A \in \mathrm{O}(2,1)$ preserves or reverses orientation and preserves or reverses the time orientation. The group $\mathrm{SO}(2,1)$ of orientation-preserving matrices has two connected components. The identity component $\mathrm{SO}(2,1)^0$ consists of elements that preserve both orientation and time orientation. Its complement $\mathrm{SO}(2,1) \setminus \mathrm{SO}(2,1)^0$ preserves orientation but reverses time orientation. Every Margulis spacetime is orientable [15], so if $\rho : G \to \mathrm{Aff}(2,1)$ is the holonomy representation of a Margulis spacetime, then the image of $\mathbb{L} \circ \rho$ is contained in $\mathrm{SO}(2,1)$.

An element of $\mathrm{SO}^0(2,1)$ preserves time orientation, and so preserves each component of $\mathscr{T}$. It acts on $\mathbb{H}^2$ by orientation-preserving isometries. The map $\mathrm{SO}^0(2,1) \to \mathrm{PSL}(2,\mathbb{R})$ is an isomorphism of Lie groups. An element of $\mathrm{SO}(2,1) \setminus \mathrm{SO}(2,1)^0$ reverses the past and future but preserves the set $\mathscr{T}$. Its elements act by orientation-reversing isometries of $\mathbb{H}^2$.

We call an element $A \in \mathrm{SO}^0(2,1)$ *hyperbolic*, *parabolic*, or *elliptic* depending on whether its isomorphic image in $\mathrm{PSL}(2,\mathbb{R})$ is. Similarly, we say an affine transformation $(A,v) \in \mathrm{SO}^0(2,1) \ltimes \mathbb{R}^{2,1}$ is *hyperbolic*, *parabolic*, or *elliptic* depending on whether A is hyperbolic, parabolic or elliptic.

If a discrete group $\Gamma < \mathrm{O}(2,1) \ltimes \mathbb{R}^{2,1}$ acts properly, then its linear part $\Gamma_0 := \mathbb{L} \circ \Gamma$ acts properly on $\mathbb{H}^2$. The quotient $\Sigma := \mathbb{H}^2 / \Gamma_0$ is a hyperbolic surface. Mess [13] showed that Σ must be non-compact. By the classification of surfaces, $\Gamma_0 \cong \pi_1(\Sigma, *)$ is a non-abelian free group on some number of generators.

A hyperbolic $X \in SO^0(2,1)$ has a unique 1-eigenspace that is spacelike. We choose a unit vector X^0 in this eigenspace by requiring

- $\det(t, X(t), X^0) > 0$ for any timelike vector t, and
- $X^0 \cdot X^0 = 1$.

The action of X on $\mathbb{H}^2$ leaves fixed a geodesic called its *invariant axis*. We denote the axis of X by $\mathrm{Axis}(X)$. $\mathrm{Axis}(X)$ is dual to the spacelike vector X^0. Similarly, if X is a glide reflection then its axis is well-defined and dual to X^0.

For nonzero $u \in \mathbb{R}^{2,1} \setminus \mathscr{N}$ consider the involution $\mathrm{Inv}(u) \in \mathrm{O}(2,1)$ defined by

$$\mathrm{Inv}(u) : v \mapsto -v + 2\frac{v \cdot u}{u \cdot u}u.$$

This maps a spacelike or timelike vector to an involution in the line $\mathbb{R}u$. If u is spacelike, then $\mathrm{Inv}(u)$ is called a (linear) *spine reflection* in [16]. In this case $\mathrm{Inv}(u)$ acts on $\mathbb{H}^2$ as a reflection in the geodesic dual to u. If u is timelike, then $\mathrm{Inv}(u)$ is an elliptic element of order two fixing the point $p := [u] \in \mathbb{H}^2$. We call this a *point symmetry* in the point p.

For $w \in \mathbb{R}^{2,1}$, let τ_w denote translation by w: $\tau_w(v) = v + w$. Define

$$\mathrm{Inv}(u, w) := \tau_w \circ \mathrm{Inv}(u) \circ \tau_{w^{-1}}.$$

Then $\mathrm{Inv}(u, w)$ is an involution in the line $w + \mathbb{R}u$.

2.4 Proper actions and the Margulis Invariant

The existence of proper affine actions of free groups on $\mathbb{R}^3$ was originally demonstrated by Margulis using a Lorentzian generalization of the Riemannian length spectrum. This invariant now bears his name. The Margulis invariant is defined for hyperbolic isometries, although see [17] for a partial generalization to parabolic elements. Given a hyperbolic isometry X and any nonzero point $p \in \mathbb{R}^{2,1}$, the Margulis invariant can be computed as

$$\alpha(X) = (X(p) - p) \cdot X^0.$$

The Margulis invariant $\alpha(X)$ is the signed Lorentzian length of a closed geodesic in $\mathbb{R}^{2,1}/\langle X \rangle$.

Margulis [18, 19] proved the following useful lemma.

Lemma 2.4.1 (Opposite Sign Lemma)**.** *Let γ_1 and γ_2 be hyperbolic elements of* $\mathrm{Aff}(2,1)$*. If $\langle \gamma_1, \gamma_2 \rangle$ acts properly then either $\alpha(\gamma_1)$ and $\alpha(\gamma_2)$ are both positive, or they are both negative.*

As is customary in the literature, we assume that $\alpha(\gamma) > 0$ for all hyperbolic $\gamma \in G$. The case where $\alpha(\gamma) < 0$ follows with a relatively minor adjustments.

We can think of α as a function of Γ_0 and $H^1(\Gamma_0, \mathbb{R}^{2,1})$ by defining for $g \in \Gamma_0$

$$\alpha_{[u]}(g) := \alpha((g, u(g))).$$

For fixed $g \in \Gamma_0 \cong \mathbb{F}_2$, the map $\alpha_g : H^1(\Gamma_0, \mathbb{R}^{2,1}) \to \mathbb{R}$ defined by

$$\alpha_g([u]) := \alpha_{[u]}(g)$$

is linear. Choose a basis A, B for Γ and define $C = B^{-1}A^{-1}$. Then the Margulis invariants of A, B, and C determine an isomorphism of vector spaces:

$$H^1(\Gamma_0, \mathbb{R}^{2,1}) \cong \mathbb{R}^3,$$

given by

$$[u] \mapsto \begin{bmatrix} \alpha_{[u]}(A) \\ \alpha_{[u]}(B) \\ \alpha_{[u]}(C) \end{bmatrix}.$$

This theory is well developed in [3, 15, 20], see also [2] for the description of how the above isomorphism relates to the parametrization of the Fricke space of Σ by traces.

2.5 Crooked Planes and Fundamental Domains

Drumm introduced crooked planes in the early 1990s as a tool for constructing proper affine actions of non-solvable groups based on templates for fundamental domains in $\mathbb{H}^2$. Recall that in the Riemannian setting, convex fundamental domains can be constructing algorithmically using equidistant surfaces. This procedure fails in the pseudo-Riemannian setting, but crooked planes offer a useful partial analog.

Specifically, a crooked plane divides $\mathbb{R}^{2,1}$ into topological halfspaces, called *crooked halfspaces*. By keeping track of the image of these halfspaces, Drumm [21] showed a crooked version of the classical Schottky theorem on fundamental domains. Moreover, Drumm showed that every discrete free subgroup of $SO^0(2,1)$ is the linear part of the holonomy of a Margulis spacetime. The theory of crooked halfspaces is closely studied in [22].

A crooked plane is determined by a spacelike vector, called its *direction vector*, and a point, called its *vertex*. For a point p and a spacelike vector v, define the crooked plane $\mathscr{C}(v,p)$ as the union of two *wings*

$$p + \mathbb{R}_+ v^+ + \mathbb{R}_+ v$$

$$p + \mathbb{R}_+ v^- - \mathbb{R}_+ v$$

and a *stem*

$$p + \left\{ x \in \mathbb{R}^{2,1} \mid v \cdot x = 0, x \cdot x \leq 0 \right\}.$$

Crooked planes have a useful interpretation in the language of hyperbolic geometry, although we will not need it here. The direction vector is dual to a hyperbolic geodesic ℓ. The vector space $\mathbb{R}^{2,1}$ identifies with the Lie algebra of $\mathrm{PSL}(2,\mathbb{R})$, and under this identification a crooked plane is the set of all Killing fields with a non-repelling fixed point on ℓ (see [10]).

For a crooked plane $\mathscr{C} = \mathscr{C}(v,p)$, we define its linearization $\mathbb{L}(\mathscr{C})$ to be its direction vector v:

$$\mathbb{L}(\mathscr{C}(v,p)) := v.$$

We identify the linearization of a crooked plane with an oriented geodesic in $\mathbb{H}^2$.

If u is spacelike, then $\mathrm{Inv}(u)$ fixes the crooked plane $\mathscr{C}(s,p)$ if and only if $s = u$. If u is timelike, then $\mathrm{Inv}(u)$ fixes $\mathscr{C}(s,p)$ if and only if $u \in s^{\perp}$. This happens if and only if the point $[u] \in \mathbb{H}^2$ lies in the $\mathbb{H}^2$ geodesic dual to s. A fundamental domain for $\langle \mathrm{Inv}(u) \rangle$ in either case is given by either crooked half-space bounded by any crooked plane fixed by $\mathrm{Inv}(u)$.

For building fundamental domains, it is important to know when two crooked planes are disjoint. This question has been studied in some detail, for example in [9].

An especially useful criterion is the following characterization in terms of triangles in $\mathbb{H}^2$, due to [20].

Let s_1, s_2, s_3 be the sides of an ideal triangle, with the possibility that s_i may be ultraparallel to s_j. The crooked planes $\{\mathscr{C}(s_i, 0)\}$ directed by the s_i and vertexed at the origin are obviously not disjoint. Not only do they intersect at the origin, but if s_i and s_j are asymptotically parallel, then $\mathscr{C}(s_i)$ and $\mathscr{C}(s_j)$ share a wing.

Translating each crooked plane in a consistent choice of direction of its stem $s^- - s^+$ makes the translated planes disjoint. More generally, we can move them in the direction $as^- - bs^+$, $a, b \in \mathbb{R}_+$. The set of all linear combinations of this form is called the *stem quadrant*. The disjointness criterion formalizes this idea.

Before a formal statement of the criterion, we need a technical definition.

Definition 2.5.1. *Spacelike vectors $v_1, \ldots, v_n \in \mathbb{R}^{2,1}$ are* consistently oriented *if and only if whenever $i \neq j$,*

- $v_i \cdot v_j < 0$
- $v_i \cdot v_j^{\pm} \leq 0.$

Theorem 2.5.2 (Disjointness Criterion)**.** *Let $v_1, v_2 \in \mathbb{R}^{2,1}$ be consistently-oriented spacelike ultraparallel or asymptotically parallel vectors. Let*

$$p_i = a_i v_i^- - b_i v_i^+.$$

Then the crooked planes $\mathscr{C}(v_i, p_i)$ are disjoint if $a_i, b_i > 0$ for $i = 1, 2$.

Chapter 3: Coxeter Extensions

In this chapter, we introduce the necessary formalism for moving between representations of $\mathbb{F}_2$ and representations of $G = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. For this, we need to work with representations into $\mathrm{SL}(2, \mathbb{C})$. The correspondence between the hyperbolic plane and equivalence classes of timelike vectors in $\mathbb{R}^{2,1}$ induces an identification $\mathrm{Isom}(\mathbb{H}^2) \cong \mathrm{SO}(2, 1)$.

Since $\mathbb{F}_2$ is a free group, every representation $\mathbb{F}_2 \to \mathrm{PSL}(2, \mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2)$ lifts to a representation into $\mathrm{SL}(2, \mathbb{C})$. We can represent orientation-reversing isometries of $\mathbb{H}^2$ as purely imaginary matrices of the form iP, $P \in \mathrm{SL}(2, \mathbb{R})$. The matrix iP corresponds under the identification with $\mathrm{SO}(2, 1)$ to a matrix that preserves orientation but reverses time orientation; that is, an element of $\mathrm{SO}(2, 1) \setminus \mathrm{SO}^0(2, 1)$.

3.0.1 Extensions of $\mathbb{F}_2$ Representations

Let $\epsilon \in \mathrm{Aut}(\mathbb{F}_2)$ be the automorphism satisfying $\epsilon(A) = A^{-1}$, $\epsilon(B) = B^{-1}$. The semi-direct product $\mathbb{F}_2' = \mathbb{F}_2 \ltimes_\epsilon \mathbb{Z}_2$ is isomorphic to G:

$$
\begin{aligned}
F_2' &= \langle A, B, \iota_0 \mid \iota_0^2 = 1, \iota_0 A \iota_0 = A^{-1}, \iota_0 B \iota_0 = B^{-1} \rangle \\
&\cong \langle \iota_1, \iota_2, \iota_0 \mid \iota_1^2 = \iota_2^2 = \iota_0^2 = 1 \rangle
\end{aligned}
$$

where the images of A and B under the inclusion $\mathbb{F}_2 \hookrightarrow \mathbb{F}_2'$ are given by

$$A \mapsto \iota_1\iota_0 \tag{3.1}$$

$$B \mapsto \iota_0\iota_2. \tag{3.2}$$

We say that $\mathbb{F}_2'$ is a *Coxeter extension* of $\mathbb{F}_2$. An irreducible representation $\rho_0 : \mathbb{F}_2 \to \mathrm{SL}(2,\mathbb{C})$ admits a unique Coxeter extension. See [2,3].

If Γ_0 is a Fuchsian group free of rank two, then we can always take its Coxeter extension Γ_0'. We can take an affine deformation Γ' of Γ_0', and this is the same as taking the Coxeter extension of an affine deformation Γ of Γ_0. That is, the following diagram commutes:

$$\begin{tikzcd} \Gamma_0 \arrow[r] \arrow[d] & \Gamma \arrow[d] \\ \Gamma_0' \arrow[r] & \Gamma' \end{tikzcd}$$

We call Γ' an *affine Coxeter extension* of Γ_0. If Γ' acts properly we call it a *proper affine Coxeter extension*.

3.0.2 Computations in Coxeter Extensions

For hyperbolic elements $A, B \in \mathrm{SL}(2,\mathbb{C})$, the Lie product $\mathrm{Lie}(A,B) = AB - BA$ is the unique involution whose conjugation action reverses A and B. It follows that the element ι_0 in the Coxeter extension can be computed as

$$\iota_0 = \mathrm{Lie}(A,B)$$

See [2] for additional details.

It is sometimes convenient to also define

$$C := B^{-1}A^{-1} = \iota_2\iota_1$$

and to use the presentation $\mathbb{F}_2 = \langle A, B, C \mid ABC = 1\rangle$. For example, when $\rho : \mathbb{F}_2 \to \mathrm{SO}(2,1)$ is the holonomy of a three-holed sphere, then $\rho(A)$, $\rho(B)$, and $\rho(C)$ are the holonomies of the boundary curves.

The following facts can be verified by simple computation, but will be useful for understanding the geometry of Coxeter extensions of Fuchsian $\mathbb{F}_2$ representations. For the computations in this section, it is convenient to use the notation $\iota_A := \iota_1$, $\iota_B := \iota_2$.

Proposition 3.0.1. *Let $G = \langle \iota_A, \iota_B, \iota_0 \mid \iota_A^2 = \iota_B^2 = \iota_0^2 = 1\rangle$. Let $F = \langle A, B\rangle$ be the index two free group defined by $A = \iota_A\iota_0$, $B = \iota_0\iota_B$. Define $C = B^{-1}A^{-1} = \iota_B\iota_A$.*

Then

- $\iota_0 A \iota_0 = A^{-1}$, $\iota_0 B \iota_0 = B^{-1}$
- $\iota_A A \iota_A = A^{-1}$
- $\iota_B B \iota_B = B^{-1}$
- $\iota_A C \iota_A = \iota_B C \iota_B = C^{-1}$.

Proposition 3.0.2. *Let A and B be hyperbolic elements of $\mathrm{SO}^0(2,1)$. Then*

- $\mathrm{Lie}(A, B)$ *is a reflection if the invariant axes of A and B are ultraparallel.*
- $\mathrm{Lie}(A, B)$ *is elliptic if the invariant axes of A and B intersect.*

Proof. Define $\iota_0 = \mathrm{Lie}(A, B)$. Since ι_0 reverses A and B, it conjugates them to hyperbolic transformations A^{-1}, B^{-1} with the same axes as A and B but in the opposite direction. It follows that ι_0 is an involution that reverses the geodesics $\mathrm{Axis}(A)$ and $\mathrm{Axis}(B)$.

If the geodesics do not intersect, ι_0 must be a reflection through a line intersecting $\mathrm{Axis}(A)$ and $\mathrm{Axis}(B)$ transversely. If the axes intersect in a point, ι_0 must fix that point. Thus ι_0 is an elliptic element of order two, and must be a point symmetry in $\mathrm{Axis}(A) \cap \mathrm{Axis}(B)$.

□

We can extend automorphisms of $\mathbb{F}_2$ to automorphisms of its Coxeter extension:

Proposition 3.0.3. *Let $\phi \in \mathrm{Aut}(\mathbb{F}_2)$. Then*

- *We can extend ϕ to $\mathrm{Aut}(\mathbb{F}_2')$ by defining*

$$\phi' := \phi \circ \epsilon \circ \phi^{-1}.$$

- *If $\rho : \mathbb{F}_2 \to \mathrm{SL}(2, \mathbb{C})$ is an irreducible representation and $\rho' : \mathbb{F}_2' \to \mathrm{SL}(2, \mathbb{C})$ is its Coxeter extension, then*

$$\phi'(\iota_0) = \mathrm{Lie}(\phi(A), \phi(B)).$$

- *Assume ϕ is an outer automorphism induced by a mapping class of $\mathbb{H}_2 / \rho(\mathbb{F}_2)$. Then $\phi'(\iota_0)$ is a reflection (respectively, point symmetry) if and only if ι_0 is a reflection (respectively, point symmetry).*

Proof. We need to check that ϕ' reverses $\phi(A)$ and $\phi(B)$:

$$\begin{aligned}\phi'(\phi(A)) &= (\phi \circ \epsilon \circ \phi^{-1})(\phi(A)) \\ &= (\phi \circ \epsilon)(A) \\ &= \phi(A^{-1}) \\ &= \phi(A)^{-1}\end{aligned}$$

and similarly for $\phi(B)$.

The element $\phi'(\iota_0)$ must be the unique element in $\mathrm{SL}(2, \mathbb{C})$ reversing $\phi(A)$ and $\phi(B)$. Since $\mathrm{Lie}(\phi(A), \phi(B))$ reverses A and B, we have $\phi'(\iota_0) = \mathrm{Lie}(\phi(A), \phi(B))$.

Since ϕ is induced by an element of the mapping class group, it preserves geometric intersection number. Now apply Proposition 3.0.2.

□

The images of ι_A and ι_B under ϕ' can be computed as

$$\begin{aligned}\phi'(\iota_A) &= \phi(A)\phi'(\iota_0) \\ \phi'(\iota_B) &= \phi'(\iota_0)\phi(B).\end{aligned}$$

3.1 Coxeter Extensions of the Four Surfaces

We now describe Coxeter extensions of the four surfaces with fundamental group free of rank 2. These descriptions will be used in Chapter 5 to describe the space of proper affine Coxeter extensions whose actions admit crooked fundamental domains. We start with a fixed representation Γ_0 for each surface as well as a set of generators A, B.

In addition, we parametrize the possibilities for ultraideal triangles τ that bound a fundamental domain for the action of the Coxeter extension Γ_0' subject to the constraint that the generators ι_A, ι_B and ι_0 each pair a side of τ to itself. Since a fundamental domain by definition contains only one representative of each orbit, any point fixed by an element of Γ_0' must lie on the sides of τ.

We will see that the sides of the ultraideal triangles τ intersect the lifts of the boundary curves for Σ. They project to proper arcs in the quotient. We also refer to the sides of τ as arcs except where this would cause confusion.

3.1.1 Three-Holed Sphere

Let Σ be a three-holed sphere with holonomy group Γ_0. We take a presentation $\Gamma_0 = \langle A, B\rangle$ where A and B are hyperbolic isometries with ultraparallel axes. Define $C = (AB)^{-1}$, so that $ABC = 1$. Then A, B, and C correspond to the boundary curves of Σ. We show that Γ_0' is generated by three reflections ι_A, ι_B, ι_0.

By Proposition 3.0.2, ι_0 is a reflection. By Proposition 3.0.1, ι_A reverses the hyperbolic elements A and C whose axes do not intersect, so it is a reflection through a geodesic transverse to $\mathrm{Axis}(A)$ and $\mathrm{Axis}(C)$. Similarly, ι_B is a reflection through a geodesic transverse to $\mathrm{Axis}(B)$ and $\mathrm{Axis}(C)$. In fact, the geodesics fixed by ι_A, ι_B, and ι_0 are the common perpendiculars to the axes of A, B and C.

3.1.1.1 Fundamental Triangle for the Three-Holed Sphere

There is exactly one candidate for τ given by the geodesics fixed by the reflections ι_A, ι_B, and ι_0. Conversely, starting with an ultraideal triangle, the reflections in the sides of the triangle determine a Coxeter group that is the Coxeter extension of the holonomy group of a three-holed sphere. It is well known that the Fricke space for a three-holed sphere is parametrized by the lengths of the three mutual perpendiculars to the lines fixed by the reflections.

3.1.2 One-Holed Torus

Let Σ be a one-holed torus with holonomy group Γ_0. We can take $\Gamma_0 = \langle A, B \rangle$ with A and B hyperbolic isometries with crossing axes. Then Γ_0' is generated by three point symmetries ι_A, ι_B, ι_0.

By Proposition 3.0.2, ι_0 is a point symmetry in $\text{Axis}(A) \cap \text{Axis}(B)$. By an application of Proposition 3.0.1, ι_A must be a point symmetry in $\text{Axis}(A) \cap \text{Axis}(C)$, and ι_B must be a point symmetry in $\text{Axis}(B) \cap \text{Axis}(C)$.

3.1.2.1 Fundamental Triangle for the One-Holed Torus

The generators ι_A, ι_B, ι_0 are point symmetries. Call their fixed points p_A, p_B, p_0 respectively. Choose p_0 to be the origin. The commutator $K = [A, B] = ABA^{-1}B^{-1}$ corresponds to the boundary curve of Σ. The other elements in the conjugacy class of the boundary curve are $\iota_0(K)$, $\iota_A(K)$, and $(\iota_0\iota_A)(K)$. Let $d = \text{Axis}(K)$. A set of representatives for the isotopy class of arcs τ fixed by the generators $\iota_A, \iota_B, \iota_0$ is given by the mutual perpendiculars $\perp(d, \iota_A(d))$, $\perp(\iota_A(d), \iota_0(d))$, and $\perp(d, \iota_0(d))$.

We can get any other choice of geodesic representatives by rotating the above arcs about their respective fixed points p_i while maintaining the condition that they remain disjoint and intersect the boundary. Thinking of τ as a system of arcs on Σ, we can think of this rotation as changing the angle of intersection of the arcs with the boundary of the convex core of Σ.

This gives an open interval's worth of wiggle room for each geodesic representative. The endpoints of each interval depend on the choice of the other two geodesic representatives. To avoid difficulties we may choose them in some fixed order, say in the order ι_A, ι_B, ι_0.

3.1.3 Two-Holed Projective Plane

Let Σ be a two-holed projective plane with holonomy group Γ_0. We can take $\Gamma_0 = \langle A, B \rangle$ with A and B glide reflections whose axes intersect at a point p_0. It is useful to use the redundant presentation $\Gamma_0 = \langle A, B, X, Y \mid X = AB, Y = B^{-1}A \rangle$. Then X and Y are the closed geodesics bounding the two ends of Σ. We show that Γ_0' is generated by two reflections ι_A, ι_B, and a point symmetry ι_0 in p_0.

By Proposition 3.0.2, ι_0 is point symmetry in p_0. Since $\iota_A = A\iota_0$ is the product of an orientation-reversing isometry and an orientation-preserving isometry, it must reverse orientation. It follows that ι_A is a reflection about some line ℓ_A. Similarly ι_B is a reflection about a line ℓ_B.

In Chapter 5, we will need the fact that ℓ_A and ℓ_B intersect both boundary curves X and Y. In fact, ℓ_A is the mutual perpendicular to the axes of X and AYA^{-1}, and ℓ_B is the mutual perpendicular to the axes of X and Y. It is sufficient to show the following:

Proposition 3.1.1. *We have*

- ι_A *reverses* X *and* AYA^{-1}.
- ι_B *reverses* X *and* Y.

Proof. The proof is by a straightforward computation. As an example, we show that ι_A reverses AYA^{-1}. First note that

$$\begin{aligned} AYA^{-1} &= (\iota_A\iota_0)(\iota_B\iota_0\iota_A\iota_0)(\iota_0\iota_A) \\ &= \iota_A\iota_0\iota_B\iota_0 \end{aligned}$$

so its inverse is $\iota_0\iota_B\iota_0\iota_A$.

We compute

$$\begin{aligned} \iota_A AYA^{-1}\iota_A &= \iota_A(\iota_A\iota_0)(\iota_B\iota_0\iota_A\iota_0)(\iota_0\iota_A)\iota_A \\ &= \iota_0\iota_B\iota_0\iota_A. \end{aligned}$$

□

3.1.3.1 Fundamental Triangle for the Two-Holed Projective Plane

Since ι_A and ι_B are reflections, two sides of τ are given by their fixed geodesics ℓ_A and ℓ_B. The point symmetry ι_0 can be chosen to fix the origin, and so it fixes any geodesic representative of the isotopy classes of arcs through the origin that intersect both components of the boundary of Σ's convex core. There are two choices of isotopy classes. One of these separates ℓ_A and ℓ_B, so the triple does not form an ultraideal triangle. Choose ℓ_0 to be the geodesic representative of the remaining choice intersecting the boundary at right angles.

Any other choice of geodesic representative for ℓ_0 is given by an interval's worth of rotation about the origin, requiring that the geodesic intersects the boundary curves and remains disjoint from ℓ_A and ℓ_B.

3.1.4 One-Holed Klein Bottle

Let Σ be a one-holed Klein bottle with holonomy group Γ_0. We consider a presentation $\Gamma_0 = \langle A, B, X \mid X = A^2B^2\rangle$ where A and B are glide reflections whose axes are ultraparallel and where X is the boundary curve. We show that Γ_0' is generated by two point symmetries ι_A and ι_B and a reflection ι_0.

By Proposition 3.0.2, ι_0 is a reflection in a geodesic intersecting $\mathrm{Axis}(A)$ and $\mathrm{Axis}(B)$ transversely. Since A, B, and ι_0 are all orientation-reversing, the involutions ι_A and ι_B are orientation-preserving. Since ι_A reverses both A and C, it is a symmetry in the point $\mathrm{Axis}(A) \cap \mathrm{Axis}(C)$. Similarly, ι_B is a symmetry in the point $\mathrm{Axis}(B) \cap \mathrm{Axis}(C)$.

3.1.4.1 Fundamental Triangle for the One-Holed Klein Bottle

One side of τ must be ℓ_0, the common perpendicular of $\mathrm{Axis}(A)$ and $\mathrm{Axis}(B)$. Take ℓ_0 to go through the origin. As above, there is an interval's worth of choice for the geodesic representatives for the arcs fixed by ι_A and ι_B. One choice of representatives is given by the arcs whose projections to Σ intersect the boundary at right angles.

Chapter 4: The Arc Complex and the Mapping Class Group

In this section, we recall the definition of the arc complex and discuss its relationship with the mapping class group. Let Σ be a noncompact surface.

Definition 4.0.1. *The arc complex $\mathscr{A}(\Sigma)$ is the simplicial complex whose vertices are isotopy classes of essential simple properly embedded arcs in Σ. A set of vertices spans a simplex if and only if the vertices can be represented by pairwise disjoint arcs.*

We can also talk about the arc complex of a compact surface with boundary or marked points. A top-dimensional simplex τ of the arc complex is a maximal collection of isotopy classes of disjoint properly embedded arcs. Choosing a representative in each class gives a hyperideal triangulation of Σ. The interior of $\mathscr{A}(\Sigma)$ is a topological ball, and the boundary corresponds to collections of arcs that may fail to decompose Σ into topological discs [23].

When Σ is a surface of Euler characteristic -1, $\mathscr{A}(\Sigma)$ is a planar simplicial complex. In this case, it is dual to the graph whose vertices are hyperideal triangulations. Two triangulations share an edge if they differ by a diagonal flip. This graph is sometimes called the *flip graph*.

Danciger, Guéritaud, and Kassel [10] showed that the moduli space of Margulis spacetimes with fixed convex cocompact linear holonomy is parametrized by the arc complex.

Theorem 4.0.2 (Danciger-Guéritaud-Kassel)**.** *Let Σ be a convex cocompact hyperbolic surface. The projectivized space of proper affine deformations of Σ is homeomorphic to the interior of the arc complex $\mathscr{A}(\Sigma)$.*

The core idea of the theorem is to construct a homeomorphism by starting with a maximal disjoint collection of arcs, cutting along these arcs and gluing in hyperbolic strips. By an argument of Thurston [24] later proved by Parlier [25] and Papadopoulos and Théret [26], this deformation lengthens all closed geodesics on Σ.

The limit of the process of gluing in increasingly thinner strips is an infinitesimal deformation that lengthens all closed geodesics on the surface. Goldman, Labourie, and Margulis [27] had previously shown that the space of proper affine deformations of a convex cocompact surface is an open cone corresponding to infinitesimal deformations of Σ that uniformly lengthen or shorten every closed curve. The key technical result of [10] is to show that—up to making some choices about how to glue in the strips—every Margulis spacetime is uniquely determined by a single point in the arc complex.

A cocycle in the arc complex gives a recipe for moving crooked planes to be disjoint. Danciger-Guéritaud-Kassel used this to prove the Crooked Plane Theorem by constructing a crooked fundamental domain for each point in the arc complex. The proof of the Crooked Plane Theorem uses crucially that the fundamental group of Σ is torsion-free. In particular, it does not apply to Coxeter extensions of rank-two free groups, as was already known from the work of Charette [4, 16].

Nevertheless, the arc complex *does* parametrize the space of proper affine deformations of Γ_0'. This is because Γ' acts properly if and only if Γ does, and the space of proper affine deformations of Γ_0 is given by the arc complex. In general, the space of affine deformations of Γ_0' that admit crooked fundamental domains is a proper subset of the arc complex.

We will see in Chapter 5 that the space of crooked realizations of a fixed fundamental domain for the action of Γ_0' on $\mathbb{H}^2$ corresponds to a two-simplex of $\mathscr{A}(\Sigma)$. The correspondence is not in general a simplicial map, however. We call a two-simplex of $\mathscr{A}(\Sigma)$ a *tile*.

To find the space of all proper affine deformations of Γ_0', we will need to understand the orbit of one of these tiles under the automorphism group of $\mathscr{A}(\Sigma)$.

The mapping class group $\mathrm{MCG}^{\pm}(\Sigma)$ acts on $\mathscr{A}(\Sigma)$ by simplicial automorphisms. It turns out that the representation $\mathrm{MCG}^{\pm}(\Sigma) \to \mathrm{Aut}\,(\mathscr{A}(\Sigma))$ is surjective.

Theorem 4.0.3 ([28, 29]). *Let Σ be a surface of Euler characteristic -1. Let Z be the center of the mapping class group $\mathrm{MCG}^{\pm}(\Sigma)$. Then*

$$\mathrm{Aut}(\mathscr{A}(\Sigma)) \cong \mathrm{MCG}^{\pm}(\Sigma)/Z.$$

When Σ is orientable, Irmak and McCarthy [28] actually prove that $\mathrm{Aut}(\mathscr{A}(\Sigma)) \cong \mathrm{MCG}^{\pm}(\Sigma)$ whenever Σ is not a disc, annulus, pair of pants, or one-holed torus. Thus the orientable surfaces with Euler characteristic -1 are exceptional. For a general non-orientable surface, $\mathrm{Aut}(\mathscr{A}(\Sigma)) \cong \mathrm{MCG}^{\pm}(\Sigma)/Z$.

Theorem 4.0.3 is proved in the context of compact surfaces with boundary, but where the mapping class group is not required to fix the boundary components. Under these assumptions, for a noncompact Σ, the inclusion of the mapping class group of the convex core into the mapping class group of Σ is an isomorphism, which provides Theorem 4.0.3 for our purposes.

To apply these theorems, we need to know the mapping class groups of the four surfaces under consideration. It is easy to see that the one-holed torus and the one-holed Klein bottle both have infinite mapping class groups. Indeed, both contain a non-separating curve. A Dehn twist about this curve generates an infinite subgroup of the mapping class group. The three-holed sphere and the two-holed projective plane both have finite mapping class groups.

It is well known that the mapping class group of the one-holed torus is $\mathrm{GL}(2,\mathbb{Z})$. See, for example, [3, 7, 28]. Its center is $\{\pm I\}$. Thus

$$\mathrm{Aut}\,(\mathscr{A}(\Sigma_{1,1})) \cong \mathrm{PGL}(2,\mathbb{Z}).$$

The mapping class group of the three-holed sphere is the group $S_3 \ltimes \mathbb{Z}/2\mathbb{Z}$ where S_3 is symmetric group on three boundary curves of Σ. Its center is the cyclic group $\mathbb{Z}/2\mathbb{Z}$ generated by an orientation-reversing mapping class that preserves each boundary curve.

$$\mathrm{Aut}\,(\mathscr{A}(\Sigma_{0,3})) = S_3.$$

See [2] for a geometric exposition of this fact which relates closely to the theory of Coxeter extensions.

An important mapping class for non-orientable surfaces of genus ≥ 1 is given by sliding a Möbius band around a closed 1-sided curve. This is called a *y-homeomorphism*

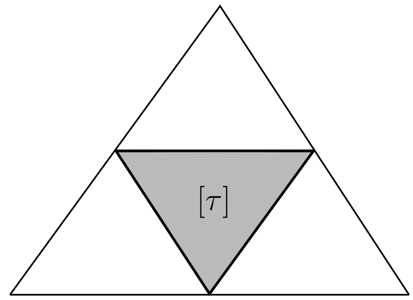

Figure 4.1: The arc complex of $\Sigma_{0,3}$.

in [30] and a *crosscap slide* in [31].

Proposition 4.0.4 ([31]). *The mapping class group of the two-holed projective plane is isomorphic to the Dihedral group of order* 8*. Its center is generated by the product of two crosscap slides. The mapping class group modulo its center is isomorphic to* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$*.*

Proposition 4.0.5 ([32]). *The mapping class group of the one-holed Klein bottle is isomorphic to* $(\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z}$*. The mapping class group modulo its center is isomorphic to* $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$*.*

Scharlemann [33] described the arc complexes of both the two-holed projective plane and the one-holed Klein bottle. The arc complex of the two-holed projective plane is depicted below.

The arc complex of a one-holed Klein bottle is as follows. Let ψ be the Dehn twist about the two-sided curve a. Then $\psi(b) = ab$. Let a', b', and c' be the arcs indicated in Figure 4. Then the vertices of the arc complex are $\{a', \psi^n(b'), \psi^n(c')\}$ for an integer n. The triples $\{b', c', \psi^{-1}(b')\}$ and $\{b', \psi^{-1}(b'), a'\}$ can be realized disjointly,

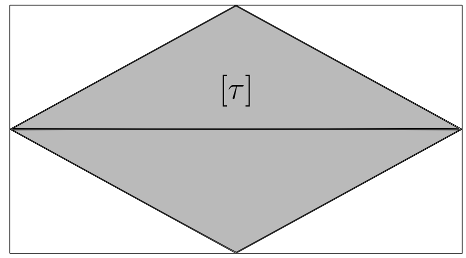

Figure 4.2: The arc complex of $C_{1,2}$.

so they correspond to the top-dimensional simplices of $\mathscr{A}(\Sigma)$. See Figure 4.

4.0.1 Transitivity of the action of $\mathrm{MCG}^{\pm}(\Sigma)$ on the tiles of $\mathscr{A}(\Sigma)$

The following proposition is now straightforward:

Proposition 4.0.6. *The action of the mapping class group is transitive on the top-dimensional simplices of $\mathscr{A}(\Sigma)$ when Σ is a one-holed torus, but not when Σ is a three-holed sphere, two-holed projective plane, or one-holed Klein bottle.*

Proof. The transitivity of the action in the case of a one-holed torus is a consequence of the classical fact that $\mathrm{Aut}\left(\mathscr{A}(\Sigma)\right) \cong \mathrm{PGL}(2, \mathbb{Z})$ acts transitively on the vertices of the Farey graph.

The three-holed sphere and two-holed projective plane have finite arc complexes, and it is clear that the automorphism groups of these complexes do not act transitively on the two-simplices. For example, a tile with three neighbors (*e.g.* one of the gray tiles in Figures 4.1 and 4.2) will never be taken to a tile with only one neighbor (the white tiles).

The action in the case of a one-holed Klein bottle cannot be transitive because

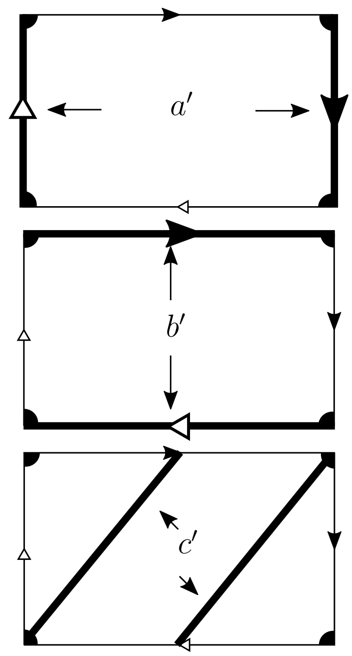

Figure 4.3: Arcs on $C_{2,1}$ after Scharlemann.

c' is a separating arc while a' and b' are non-separating. Since mapping classes preserve the property of being separating, c' is never the image of a' or b' under a mapping class. In particular, the orbits of $\{b', c', \psi^{-1}(b')\}$ and $\{b', \psi^{-1}(b'), a'\}$ are distinct. These are, in fact, the only two orbits. $\square$

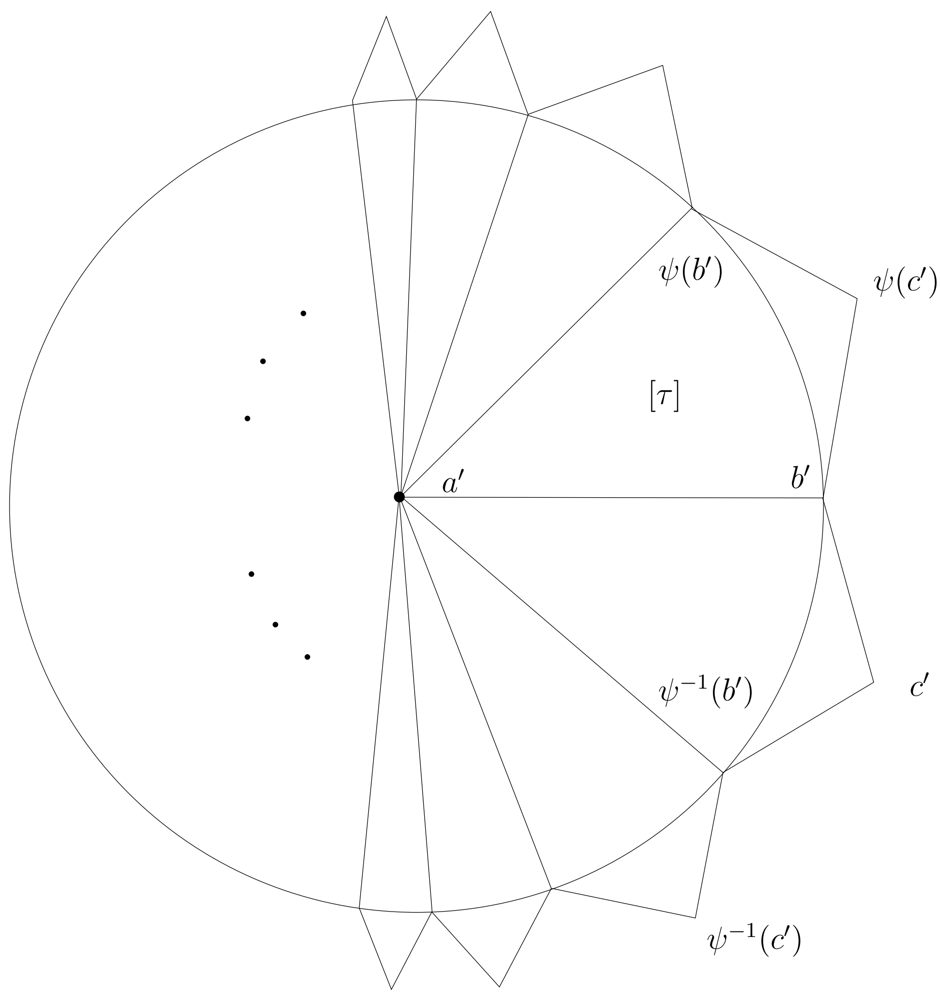

Figure 4.4: The arc complex of $C_{2,1}$ after Scharlemann.

Chapter 5: The Deformation Spaces

In this section, we describe the space of crooked realizations of ultraideal fundamental domains for the actions of Γ_0' on $\mathbb{H}^2$ and prove Theorems 1.0.2, 1.0.4, and 1.0.5.

As in Chapter 3, let τ denote an ultraideal triangle bounding a fundamental domain for the action of Γ_0'. As noted before, τ corresponds to a tile $[\tau]$ of $\mathscr{A}(\Sigma)$. We view $[\tau]$ as consisting of affine deformations $[u] \in \mathbb{H}^1(\Gamma_0, \mathbb{R}^{2,1})$ arising by inserting infinitesimal hyperbolic strips along the sides of τ.

Let $\mathrm{Crook}(\tau)$ be the projectivized space of crooked realizations of τ that bound a fundamental domain for the action of Γ'. $\mathrm{Crook}(\tau)$ is in general *not* a two-simplex. We show that $\mathrm{Crook}(\tau)$ arises from a two-simplex by *blunting*; that is, some vertices are replaced by intervals. These projective intervals correspond to the freedom in choosing a direction in the stem quadrant for each reflection generator of Γ_0'. A choice of direction in each stem quadrant does determine a two-simplex inscribed in $\mathrm{Crook}(\tau)$. This is an easy consequence of the description of the map M below.

A general two-simplex $[\kappa]$ may not correspond to $\mathrm{Crook}(\kappa)$ for any ultraideal triangle κ. That is, $[\kappa]$ may correspond to proper affine deformations of Γ_0' that admit no crooked fundamental domain.

We now describe Crook(τ) for a fixed τ for each of the four surfaces. Theorem 1.0.4 is a consequence of the following lemma.

Lemma 5.0.1. Crook(τ) *is*

1. *A hexagon if* Σ *is a three-holed sphere.*

2. *A triangle if* Σ *is a one-holed torus.*

3. *A pentagon if* Σ *is a two-holed projective plane.*

4. *A quadrilateral if* Σ *is a one-holed Klein bottle.*

The three-holed sphere case was proved by Charette [16] using different methods. The two-holed projective plane case initially appears in Goldman and Laun [6].

5.1 Preparation for Proof of Lemma 5.0.1

We prove Lemma 5.0.1 by cases in Section 5.1.1, 5.1.2, 5.1.3 and 5.1.4. In this section we developed the necessary tools and notation.

By assumption, the sides of τ are geodesics fixed by the generators ι_A, ι_B, ι_0 of Γ_0'. To facilitate indexing, relabel:

$$\iota_1 := \iota_A$$

$$\iota_2 := \iota_B.$$

Let ℓ_i denote the fixed geodesic of ι_i. For each ℓ_i, let w_i be its spacelike dual.

The crooked planes $\{\mathscr{C}(w_i, 0)\}$ linearize to τ, but they are not disjoint. To make them disjoint, we need to choose a triple of points q_1, q_2, q_0 in the stem quadrants. This corresponds to choosing pairs of positive real numbers (u_i^-, u_i^+).

Define

$$q_1 = u_1^- w_1^- - u_1^+ w_1^+$$

$$q_2 = u_2^- w_2^- - u_2^+ w_2^+$$

$$q_0 = u_0^- w_0^- - u_0^+ w_0^+.$$

If ι_i is a reflection, define an affine spine reflection $\tilde{\iota}_i$ by

$$\tilde{\iota}_i := \mathrm{Inv}(w_i, q_i).$$

If ι_i is an involution with fixed point $[t_i] \in \mathbb{H}^2$, let t_i be the future-pointing timelike vector in the translational equivalence class of $[t_i]$ going through 0. Then define

$$\tilde{\iota}_i := \mathrm{Inv}(t_i, q_i).$$

The group Γ' generated by the $\tilde{\iota}_i$ is an affine deformation of Γ_0' acting properly with fundamental domain $\mathscr{C}(w_i, q_i)$. Conversely, any affine deformation of Γ_0' admitting a crooked fundamental domain is of this form, by linearizing.

Taking the index two subgroup, define

$$\widetilde{A} := \widetilde{\iota}_1 \widetilde{\iota}_0$$

$$\widetilde{B} := \widetilde{\iota}_0 \widetilde{\iota}_2$$

$$\widetilde{C} := \widetilde{\iota}_2 \widetilde{\iota}_1.$$

Let

$$V = V(w_1) \oplus V(w_2) \oplus V(w_0)$$

be the direct sum of the stem quadrants, thought of as the positive orthant in $\mathbb{R}^6$.

Each $v \in V$ determines a cocycle $[u] \in H^1(\Gamma_0, \mathbb{R}^{2,1})$. The map $M : \mathbb{R}^6 \to \mathbb{R}^3$ by

$$Mv := \begin{pmatrix} \alpha_{[u]}(A) \\ \alpha_{[u]}(B) \\ \alpha_{[u]}(C) \end{pmatrix}$$

is a linear surjection and the projectivized image of V is the space $\mathrm{Crook}(\tau)$ of crooked realizations of τ.

We identify the linear map M with its matrix. Decompose M as the sum of three 3×2 matrices M_i:

$$M = M_1 \begin{pmatrix} u_1^- \\ u_1^+ \end{pmatrix} + M_2 \begin{pmatrix} u_2^- \\ u_2^+ \end{pmatrix} + M_3 \begin{pmatrix} u_3^- \\ u_3^+ \end{pmatrix}.$$

Lemma 5.0.1 can now be proved by considering the ranks of the matrices M_i. The image of M is a polygon inscribed in the projectivized space of proper affine deformations of Γ_0, thought of as the arc complex of Σ. The number of vertices in the polygon depends on the rank of the M_i.

The following lemma was proved in the one-holed torus case in [3].

Lemma 5.1.1. *We can compute the Margulis invariants as*

$$\alpha(\widetilde{A}) = 2(q_1 - q_0) \cdot A^0$$

$$\alpha(\widetilde{B}) = 2(q_0 - q_2) \cdot B^0$$

$$\alpha(\widetilde{C}) = 2(q_1 - q_2) \cdot C^0.$$

Proof. We compute $\alpha(\widetilde{A})$. The others are similar. Recall that $\alpha(\widetilde{A})$ can be computed as $(\widetilde{A}x-x)\cdot A^0$ for any $x \in \mathbb{R}^{2,1}$. Let $x = q_0$. By definition, $\widetilde{A} = \widetilde{\iota}_1\widetilde{\iota}_0$. By construction, $\widetilde{\iota}_0 q_0 = q_0$. We compute $\widetilde{A}q_0 = \widetilde{\iota}_1 q_0$. Let s_1 be w_1 or t_1 depending as ι_1 is a reflection or point symmetry.

$$\begin{aligned}
\widetilde{\iota}_1 q_0 &= \mathrm{Inv}(s_1, q_1)(q_0) \\
&= (\tau_{q_1} \circ \mathrm{Inv}(s_1) \circ \tau_{q_1}^{-1})(q_0) \\
&= (\tau_{q_1} \circ \mathrm{Inv}(s_1))(q_0 - q_1) \\
&= q_1 + \mathrm{Inv}(s_1)(q_0 - q_1) \\
&= q_1 + (-q_0 + q_1) \mod s_1 \\
&= 2q_1 - q_0 \mod s_1.
\end{aligned}$$

Since $A^0 \cdot s_1 = 0$, we have

$$\begin{aligned}
(\widetilde{A}q_0 - q_0) \cdot A^0 &= (2q_1 - 2q_0 \mod s_1) \cdot A^0 \\
&= 2(q_1 - q_0) \cdot A^0
\end{aligned}$$

as required. $\square$

We now compute the matrices M_1, M_2, and M_3.

If $u_2^{\pm} = 0$ and $u_3^{\pm} = 0$, then $Mv = M_1 \begin{pmatrix} u_1^- \\ u_1^+ \end{pmatrix}$. Let e_1, e_2 be the standard basis vectors of $\mathbb{R}^2$. Then

$$\mathrm{M}_1 e_1 = \begin{pmatrix} \alpha(A) \\ \alpha(B) \\ \alpha(C) \end{pmatrix} = \begin{pmatrix} 2q_1 \cdot A^0 \\ 0 \\ 2q_1 \cdot C^0 \end{pmatrix}$$
$$= \begin{pmatrix} 2w_1^- \cdot A^0 \\ 0 \\ 2w_1^- \cdot C^0 \end{pmatrix}.$$

Similarly, $M_1 e_2 = \begin{pmatrix} -2w_1^+ \cdot A^0 \\ 0 \\ -2w_1^+ \cdot C^0 \end{pmatrix}.$

We find M_1

$$M_1 = 2 \begin{pmatrix} w_1^- \cdot A^0 & -w_1^+ \cdot A^0 \\ 0 & 0 \\ w_1^- \cdot C^0 & -w_1^+ \cdot C^0 \end{pmatrix}.$$

The remaining matrices M_2 and M_3 are analogous. Explicitly:

$$M_2 = 2 \begin{pmatrix} 0 & 0 \\ -W_2^- \cdot B^0 & w_2^+ \cdot B^0 \\ -w_2^- \cdot C^0 & w_2^+ \cdot C^0 \end{pmatrix}$$

$$M_3 = 2 \begin{pmatrix} -w_3^- \cdot A^0 & w_3^+ \cdot A^0 \\ w_3^- \cdot B^0 & -w_3^+ \cdot B^0 \\ 0 & 0 \end{pmatrix}.$$

Proposition 5.1.2. *For $i = 0, 1, 2$, define M_i' to be the 2×2 submatrix of M_i consisting of nonzero entries. Define $d_i := \det M_i'$. Then,*

$$d_1 = c_1(A^0 \boxtimes C^0) \cdot w_1$$

$$d_2 = c_2(B^0 \boxtimes C^0) \cdot w_2$$

$$d_3 = c_0(A^0 \boxtimes B^0) \cdot w_0$$

where the c_i are nonzero real numbers.

Proof. We prove the proposition for M_1. The remaining cases are similar.

Recall the definition of M_1 as

$$M_1 = 2 \begin{pmatrix} w_1^- \cdot A^0 & -w_1^+ \cdot A^0 \\ 0 & 0 \\ w_1^- \cdot C^0 & -w_1^+ \cdot C^0 \end{pmatrix}.$$

Let M_1' be the 2×2 submatrix consisting of the nonzero entries in M_1. Then we can write it as the product

$$M_1' = 2 \begin{pmatrix} A^0 \\ C^0 \end{pmatrix} \cdot \begin{pmatrix} w_1^- & -w_1^+ \end{pmatrix}.$$

The determinant is

$$\begin{aligned} 4(w_1^- \cdot A^0)(w_1^+ \cdot C^0) - (w_1^- \cdot C^0)(w_1^+ \cdot A^0) &= -4(A^0 \boxtimes C^0) \cdot (w_1^- \boxtimes w_1^+) \\ &= -4(A^0 \boxtimes C^0) \cdot w_1 \end{aligned}$$

by 2.1. $\square$

Proposition 5.1.3. *Let $d = c(X^0 \boxtimes Y^0) \cdot w$ for $X, Y \in \mathrm{SO}(2,1)$ and w a spacelike vector. Define $k := X^0 \boxtimes Y^0$. Then*

- *If k is timelike and $[k]$ lies on the $\mathbb{H}^2$ geodesic dual to w, then $d = 0$.*

- *If k is spacelike and if the geodesic dual to w is the common perpendicular to the axes of X and Y, then $d \neq 0$.*

Proof. Recall that X^0 and Y^0 are dual to the axes of X and Y respectively. If k is timelike, then these axes intersect in $\mathbb{H}^2$, and $[k]$ is their intersection. If $[k]$ lies on the geodesic dual to w, then $\mathbb{R}k \in w^\perp$ so by the definition of perpendicularity, $k \cdot w = 0$.

If the axes of X and Y are ultraparallel, then the spacelike vector $X^0 \boxtimes Y^0$ is dual to their mutual perpendicular. If w is also dual to the mutual perpendicular, then $w = \lambda X^0 \boxtimes Y^0$ for some nonzero constant λ. Then $k \cdot w = \lambda \|w\|^2 \neq 0$ since w is not null. $\square$

Proposition 5.1.3 gives a geometric interpretation to the determinants in Proposition 5.1.2. The ranks of the M_i can be read directly off of a picture of the axes of the generators of Γ_0.

In the pictures below, the axis of A is colored red, that of B is colored blue, and that of C is colored green.

5.1.1 The Three-Holed Sphere

The projectivized space of proper affine deformations of Σ is a triangle bounded by the vanishing lines of $\alpha(\widetilde{A})$, $\alpha(\widetilde{B})$, and $\alpha(\widetilde{C})$ [20]. This triangle identifies with the arc complex $\mathscr{A}(\Sigma)$. We saw in Chapter 3 that the arcs ℓ_i are the mutual perpendiculars of the axes. Since these axes don't intersect, the determinants d_i are

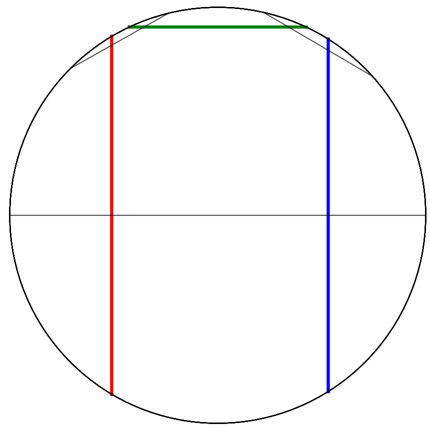

Figure 5.1: Axes and arcs for $\Sigma_{0,3}$.

all nonzero. The images of the M_i are all two-dimensional, giving a six-sided cone in $H^1(\Gamma_0, \mathbb{R}^{2,1})$, which projectivizes to a hexagon.

It is clear from the formulas for the M_i that the image of M_1 lies in $\ker \alpha(\widetilde{B})$, that the image of M_2 lies in $\ker \alpha(\widetilde{A})$ and that M_3 lies in $\ker \alpha(\widetilde{C})$. Thus the hexagon $\mathbb{P}(M(V))$ is inscribed in the space of proper affine deformations.

In Figure 5.2, the black lines are vanishing lines for the Margulis invariants of hyperbolic words in Γ_0. By [27], the space of proper affine deformations of Γ_0 is the intersection of the half planes bounded by these vanishing lines.

5.1.2 The Two-Holed Projective Plane

When Σ is a two-holed projective plane, Charette, Drumm, and Goldman [15] showed that the projectivized space of proper affine deformations of Σ is a quadrilateral $\boldsymbol{Q}$ bounded by the vanishing of the Margulis invariants for the generators $\widetilde{A}$, $\widetilde{B}$ and the boundary curves $\widetilde{X} = \widetilde{A}\widetilde{B}$, $\widetilde{Y} = \widetilde{B}^{-1}\widetilde{A}$. The quadrilateral $\boldsymbol{Q}$ identifies

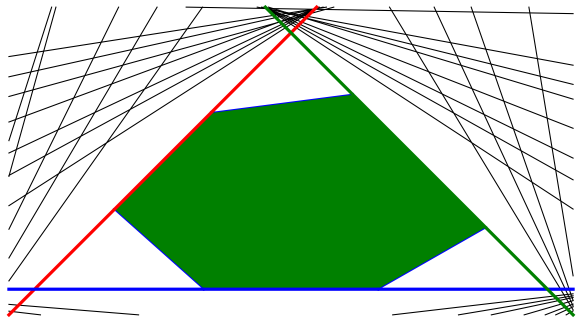

Figure 5.2: Crook(τ) for $\Sigma_{0,3}$.

with the arc complex of Σ.

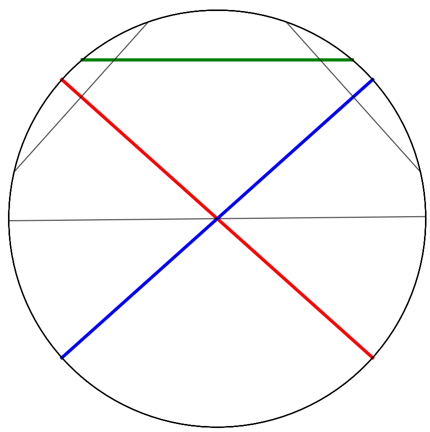

Figure 5.3: Axes and arcs for $C_{1,1}$.

The axes of A and B intersect at a point p_0 on the line ℓ_0. The determinant of M_3 is 0, and M_3 is a rank-one matrix. Both M_1 and M_2 are rank-two matrices since the axis of C is disjoint from the axes of A and B. The image of V under M projectivizes to a pentagon $\boldsymbol{P}_1$. As in the three-holed sphere case, $\boldsymbol{P}_1$ is inscribed in the arc complex $\boldsymbol{Q}$.

At this point, we can say a word of motivation for Theorem 1.0.5, which is

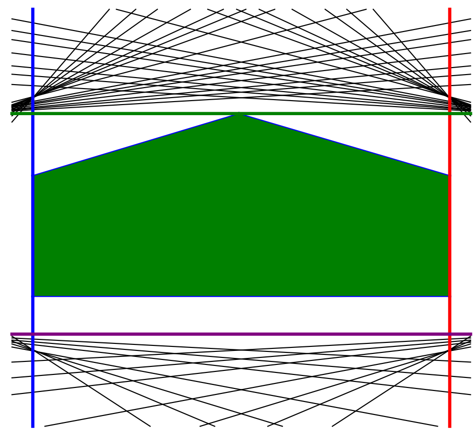

Figure 5.4: Crook(τ) for $C_{1,2}$.

proved below. The image of M_3 is a point on the projective image of $\ker \alpha(\widetilde{X})$ at the top of the pentagon $\boldsymbol{P}_1$ in the diagram. There is an asymmetry between $\alpha(\widetilde{Y})$ and $\alpha(\widetilde{X})$. Specifically, $\alpha(\widetilde{X})$ can vanish while $\alpha(\widetilde{A})$, and $\alpha(\widetilde{B})$ are nonzero. However $\alpha(\widetilde{Y})$ vanishes if and only if the Margulis invariants of all the generators vanish. This is an artifact of working with the fundamental domain τ adapted to the generators $\iota_0, \iota_1, \iota_2$. This fundamental domain is asymmetric with respect to X and Y: it contains a self-loop at Y but not one at X.

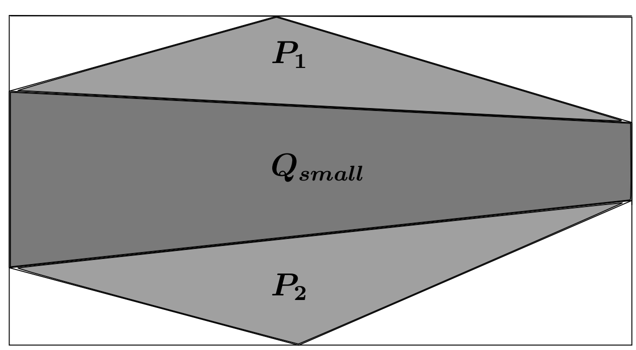

Figure 5.5: The hexagon $\boldsymbol{H}$ as the union of pentagons $\boldsymbol{P_1}$, $\boldsymbol{P_2}$.

One can recover the symmetry of the problem as follows. There is a mapping class of Σ that interchanges the boundary components. The induced automorphism ϕ of $\pi_1(\Sigma)$ is defined by $\phi(A) = A$, $\phi(B) = B^{-1}$. By Proposition 3.0.3, ϕ extends to an automorphism ϕ' of the Coxeter extension. Then $\phi(\tau)$ is an ultraideal triangle fundamental domain for Γ_0' with side pairings given by the generating set ι_1, $\iota_0\iota_2\iota_0$, ι_0. The space Crook$(\phi(\tau))$ is a pentagon $\boldsymbol{P_2}$ with a vertex on the line $\alpha(\widetilde{Y}) = 0$ such that $\boldsymbol{P_1} \cap \boldsymbol{P_2}$ is a quadrilateral $\boldsymbol{Q_{small}}$ inscribed in the larger quadrilateral $\boldsymbol{Q}$ defining the space of all proper affine deformations of Γ_0.

Applying ϕ corresponds to considering the image of V under the matrix

$$M^\phi v := \begin{pmatrix} \alpha_{[u]}\phi(A) \\ \alpha_{[u]}\phi(B) \\ \alpha_{[u]}\phi(X) \end{pmatrix} = \begin{pmatrix} \alpha_{[u]}(A) \\ \alpha_{[u]}(B) \\ \alpha_{[u]}(Y) \end{pmatrix}.$$

The equality on B is due to the fact that the Margulis invariant satisfies $\alpha_{[u]}(B^{-1}) = \alpha_{[u]}(B)$ for any B and $[u]$.

Using M^ϕ in place of M swaps the roles of X and Y in Lemma 5.1.1. Following the same argument for $\boldsymbol{P_1}$, we get maps M_1^ϕ, M_2^ϕ and M_3^ϕ. For any vectors v, w $M_1^\phi v + M_2^\phi w = M_1 v + M_2 w$. This image is the quadrilateral $\boldsymbol{Q_{small}}$. Like M_3, M_3^ϕ has a one-dimensional image. However, its image corresponds to a point on the line $\alpha_{[u]}(\widetilde{Y}) = 0$, giving another pentagon $\boldsymbol{P_2}$ with a vertex on the line in $\mathbb{R}P^2$ corresponding to the image of $\ker \alpha_{[u]}(\widetilde{Y})$ and such that $\boldsymbol{P_1} \cap \boldsymbol{P_2} = \boldsymbol{Q_{small}}$.

5.1.3 One-holed Torus

Let Σ be a one-holed torus. Its arc complex identifies with the interior of a convex region in $\mathbb{R}P^2$ cut out by the vanishing lines of the Margulis invariants of infinitely-many elements of Γ_0 [3].

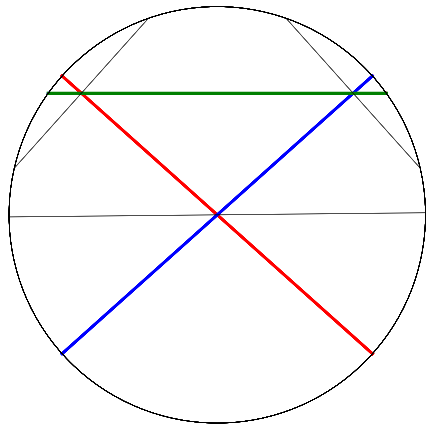

Figure 5.6: Axes and arcs for $\Sigma_{1,1}$.

Each arc intersects two axes. Each matrix M_i has rank 1 and the corresponding polygon is a triangle.

A careful choice of geodesic representatives for τ allows for the tiles to be disjoint. This, plus the transitivity of the action of $\mathrm{MCG}^{\pm}(\Sigma)$ on the arc complex allows for the tiling of the space of proper affine deformations of Σ by tiles corresponding to Coxeter deformations.

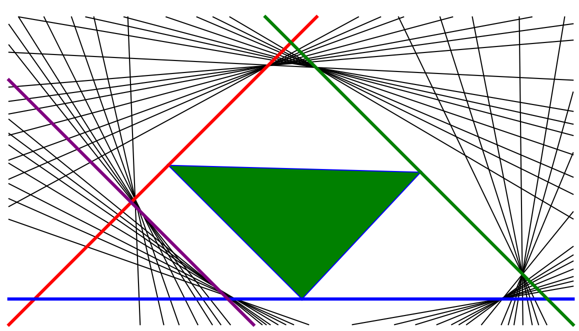

Figure 5.7: Crook(τ) for $\Sigma_{1,1}$.

This is achieved in [3] in different language. In that paper, for each basis A, B of $\mathbb{F}_2$, the triple $(A, B, B^{-1}A^{-1})$ is called a *super basis*. On the one-holed torus, a superbasis corresponds to a triple of curves with pairwise geometric intersection number 1. This gives a graph structure that is sometimes called the *pants graph*. [1] A superbasis also describes an ideal triangulation of the one-holed torus, and so a vertex of the flip graph. Thus, one can also interpret the main result of [3] as a parametrization of the space of proper affine deformations of Σ by the arc complex.

5.1.4 One-holed Klein Bottle

Let Σ be a one-holed Klein bottle. Its arc complex identifies with the interior of a convex region in $\mathbb{R}P^2$ cut out by the vanishing lines of the Margulis invariants of infinitely-many elements of Γ_0.

The axes of A and C intersect on the line ℓ_1 and the axes of B and C intersect on the line ℓ_2. The matrices M_1 and M_2 have rank 1 while matrix M_3 has rank 2. The corresponding polygon is a quadrilateral.

5.2 Dependency of τ on Parameters

As noted in Chapter 3, the choice of an ultraideal triangle τ bounding a fundamental domain for Γ_0' depends on a choice of parameter for every elliptic generator.

[1] Typically, the pants graph is taken to be collections of curves with pairwise intersection number 0. However, this complex is a single point for the one-holed torus since all closed curves intersect. For this reason, one often uses minimal geometric intersection number to get an interesting combinatorial structure.

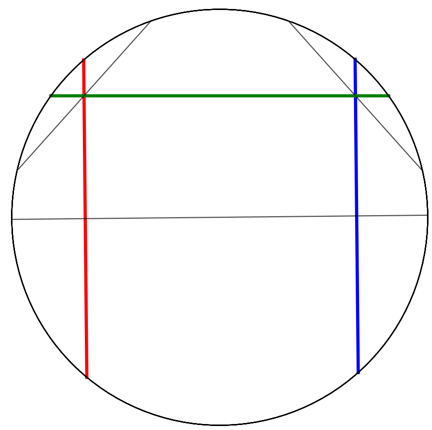

Figure 5.8: Axes and arcs for $C_{2,1}$.

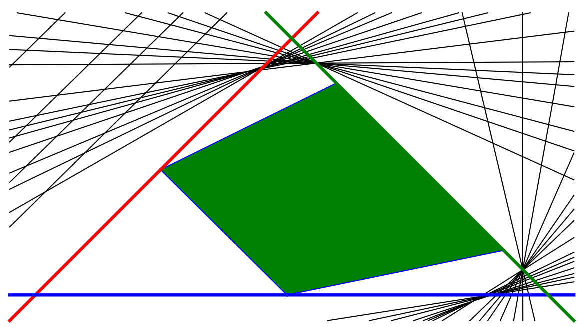

Figure 5.9: Crook(τ) for $C_{2,1}$.

The parameters can be taken to be the angle that the geodesic representatives of the arcs in $[\tau]$ make with the boundary. If ι_i is an elliptic generator of Γ_0', call the corresponding parameter θ_i. Let $\boldsymbol{\theta}$ be the set of θ_is for all the elliptic generators. Let $\tau(\boldsymbol{\theta})$ denote the unique τ depending on these parameters. Then Crook($\tau(\boldsymbol{\theta})$) depends on $\boldsymbol{\theta}$.

The union $\bigcup_{\boldsymbol{\theta}}$ Crook($\tau(\boldsymbol{\theta})$) is a hexagon. By the proof of Lemma 5.0.1, for each $\boldsymbol{\theta}$, Crook($\tau(\boldsymbol{\theta})$) is a blunted triangle with one side blunted for each reflection generator. The union $\bigcup_{\boldsymbol{\theta}} \mathrm{Crook}(\tau(\boldsymbol{\theta}))$ adds blunting for each elliptic generator as follows.

If ι_i is an elliptic generator, then the spacelike vector w_i depends on θ_i: $w_i = w_i(\theta_i)$. For any pair $(u_i^-, u_i^+) \in \mathbb{R}_+^2$, we get a point $q_i(\theta_i)$ in the stem quadrant. The matrix $M = M(\boldsymbol{\theta})$ now depends on θ_i for each elliptic generator ι_i. Consider the matrices M_i as above. If M_i is rank 2 for some choice of $\boldsymbol{\theta}$, then it is rank two for all choices of $\boldsymbol{\theta}$. The corresponding vertices of the polygon do not depend on $\boldsymbol{\theta}$. However, if M_i was rank 1, then varying $\boldsymbol{\theta}$ gives a continuous family of rank-one matrices whose images are all contained on a line bounding the space of proper affine deformations of Γ_0. The union of their images is thus an interval on this line, resulting in blunting of the corresponding vertex of the tile $[\tau]$.

Note that $[\tau]$ does not depend on $\boldsymbol{\theta}$. In what follows, define

$$\mathrm{Crook}([\tau]) = \bigcup_{\boldsymbol{\theta}} \mathrm{Crook}(\tau(\boldsymbol{\theta}))$$

In the case where Σ is a two-holed projective plane, the orbit of $\mathrm{Crook}([\tau])$, which is an octagon, is shown in Figure 5.10.

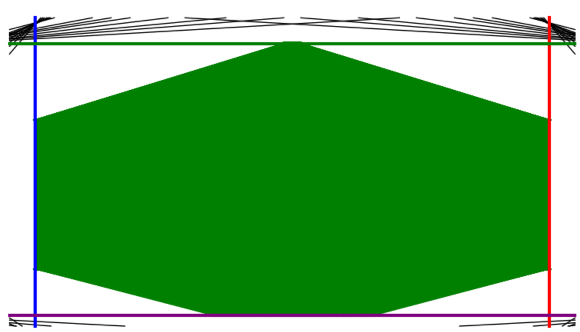

Figure 5.10: The orbit of $\mathrm{Crook}([\tau])$ for a two-holed projective plane.

5.3 Proof of the Main Theorems

In this section, we show that the full space of proper affine deformations of Γ_0' that admit crooked fundamental domains is the orbit of $\mathrm{Crook}([\tau])$ under $\mathrm{MCG}^{\pm}(\Sigma)$.

Fix a tiling of $\mathscr{A}(\Sigma)$ as the space of proper affine deformations of Σ. Given a tile $[\kappa] \in \mathscr{A}(\Sigma)$ and a cocycle $[u] \in [\kappa]$, one can construct a quadrilateral fundamental domain for the action of $\Gamma_{[u]}$. If the action of the Coxeter extension $\Gamma_{[u]}'$ admits a crooked fundamental domain, then in general $[\kappa] \subset \mathrm{Crook}([\kappa])$. Indeed $[\kappa]$ can be recovered from $\mathrm{Crook}([\kappa])$ by choosing a direction in each stem quadrant for a fixed set of geodesic representatives for $[\kappa]$. The larger set $\mathrm{Crook}([\kappa])$ also includes cocycles arising from other choices of direction vector and other choices of geodesic representative.

Theorem 5.3.1. *Let $[\kappa] = \phi([\tau])$ for some mapping class $\phi \in \mathrm{MCG}^{\pm}(\Sigma)$. Then any choice of geodesic representatives κ of $[\kappa]$ is a fundamental domain for the action of Γ_0'. Moreover, the tile $[\kappa]$ consists of affine deformations of Γ_0' that admit crooked fundamental domains that linearize to $\kappa(\boldsymbol{\theta})$ for some choice of $\boldsymbol{\theta}$. That is, $[\kappa] \subset \mathrm{Crook}([\kappa])$.*

Proof. By abuse of notation, let ϕ also denote the automorphism of $\mathbb{F}_2$ induced by ϕ. Extend ϕ an automorphism ϕ' of the Coxeter extension, as in Chapter 3.

Once the parameters $\boldsymbol{\theta}$ are fixed, the construction of τ in Chapter 3 depended only on properties of simple closed curves on Σ that are preserved by mapping classes. In fact, the construction depended only on the intersection properties of

Axis(A), Axis(B), and Axis(C). Thus we can repeat this construction for the generators $\phi(A)$, $\phi(B)$, and $\phi(C)$ to build a fundamental domain for the action of Γ_0' with generators $\phi'(\iota_i)$. By possibly changing the angle of intersection with the boundary curves, this fundamental domain is bounded by the geodesics in κ.

To find crooked realizations of the fundamental domain based on κ, apply the disjointness criterion 5.3 just as we did with τ.

□

Naturally, Theorem 1.0.5 implies that if the action is transitive then every proper affine deformation of Γ_0' admits a crooked fundamental domain. For in this case *every* κ is the image of τ under some mapping class.

Theorem 5.3.2. *Let κ be any choice of geodesic representatives for a tile $[\kappa] \in \mathscr{A}(\Sigma)$ in the complement of the orbit of $[\tau]$. Then κ is not a fundamental domain for the action of Γ_0'. In addition, there is no crooked fundamental domain for Γ' that linearizes to κ.*

Proof. The proof is by analysis of cases.

If Σ is homeomorphic to a three-holed sphere, then the fundamental domain adapted to the generators of Γ_0' is canonical because the reflections must fix specific geodesics in $\mathbb{H}^2$. These geodesics project to arcs in Σ, and τ is necessarily the union of these arcs. It was shown in Chapter 3 that each of the three arcs connects two distinct boundary curves of Σ. This is true of the image of τ under any mapping class. [2] Any other tile $[\kappa]$ in $\mathscr{A}(\Sigma)$ contains an arc connecting some boundary

[2] In fact, this orbit has exactly one point; namely, τ.

component with itself. The lift of such an arc to $\mathbb{H}^2$ is not preserved by any of the reflections, so κ cannot bound a fundamental domain for the action of the Coxeter extension.

The case of the two-holed projective plane is similar. The two reflections preserve arcs connecting two distinct boundary components. The tiles $[\kappa]$ in the complement of the orbit of $[\tau]$ contain two self-arcs. There is no choice κ of geodesic representatives such that the lifts of the arcs in κ are preserved by both reflections. Thus κ is not a fundamental domain for the action of Γ_0'.

If Σ is a one-holed Klein bottle, then the orbit of any ultraideal triangle κ containing an image of the separating arc c' fails to be a fundamental domain for the action of Γ_0'. For suppose κ is such an ultraideal triangle containing the separating arc $\phi(c')$ for some mapping class ϕ. Then the line fixed by the reflection $\phi(\iota_0)$ intersects the interior of κ and so the triangle bounded by κ contains fixed points for the action of Γ_0' and cannot be a fundamental domain.

In each case, $\mathrm{Crook}([\kappa]) = \emptyset$, for otherwise linearizing would produce a fundamental domain for some set of geodesic representatives κ, which is impossible. $\square$

The argument about the fixed lines of the reflections in the proof also works for the three-holed sphere and two-holed projective plane cases as well.

Theorem 5.3.2 allows us to deduce the existence of proper affine Coxeter extensions of the three-holed sphere, two-holed projective plane, and one-holed Klein bottle whose action does not admit a crooked fundamental domain:

Theorem 5.3.3. *Let* $\Sigma = \mathbb{H}^2/\Gamma_0$ *be a rank-two surface such that* $\mathrm{MCG}^{\pm}(\Sigma)$ *does* not *act transitively on the top-dimensional simplices of* $\mathscr{A}(\Sigma)$. *Then some affine deformation* $\Gamma' = \Gamma'_{[u]}$ *does not admit a crooked fundamental domain.*

Proof. Choose a tile $[\kappa]$ that is outside the orbit of $[\tau]$ but which shares an edge with $[\tau]$. By examining the arc complexes of the three-holed sphere, the two-holed projective plane, and the one-holed Klein bottle, we see that $[\kappa]$ is not adjacent to any other tile in the orbit of $[\tau]$. Since $\mathscr{A}(\Sigma)$ is a simplicial complex, $[\tau]$ does not intersect the interior of $[\kappa]$. However, due to blunting $\mathrm{Crook}([\tau])$ may intersect the interior of $[\kappa]$.

For any of the four surfaces, $\mathscr{A}(\Sigma)$ is compact in some affine patch. Choose a metric for this patch. Let v be the unique vertex of $[\kappa]$ that is disjoint from the boundary of $[\tau]$. Let $\mathscr{B}(v,\epsilon)$ be a small metric ball around v.

Claim 5.3.4. *For sufficiently small* ϵ, $\mathscr{B}(v,\epsilon)$ *is disjoint from* $\mathrm{Crook}([\tau])$.

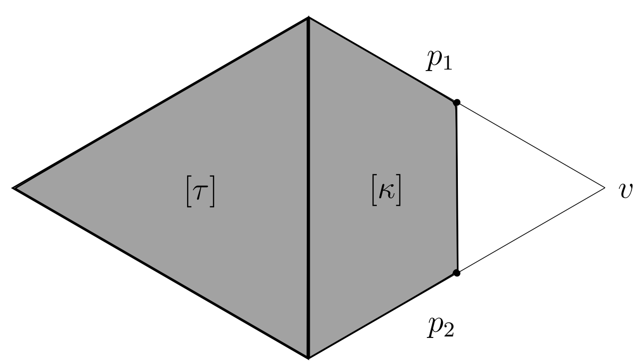

Figure 5.11: $\mathrm{Crook}([\tau])$ intersecting $[\kappa]$. Only two simplices of $\mathscr{A}(\Sigma)$ are shown.

Proof. The tile $[\kappa]$ shares one side with $[\tau]$. The intersection $\mathrm{Crook}([\tau]) \cap [\kappa]$ is depicted in Figure 5.11. The points p_1 and p_2 are the projectivized images under

M of two standard basis vectors in $\mathbb{R}^6$. If $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are the standard basis vectors in $\mathbb{R}^2$, then the decomposition of M above implies that $p_i = \mathbb{P}(M_j e_k)$ where $j \in \{1, 2, 3\}$ and $i, k \in \{1, 2\}$.

We want to show that the p_i are bounded some positive distance away from v. If M_j is rank two, then its image does not depend on $\boldsymbol{\theta}$, and the corresponding p_i is bounded away from v by the matrix formulas as follows.

The sides of $[\kappa]$ containing the points p_i are the vanishing lines for the Margulis invariants of two hyperbolic elements of Γ. Call them γ_1 and γ_2. It is easy to verify that $\gamma_1, \gamma_2 \in \{A, B, C\}$ for each of the surfaces under consideration. The explicit formulas for the M_i imply that as long as one of the $u_i^{\pm}$ is greater than 0, then two of the Margulis invariants are positive. This implies that the image of the positive orthant—which includes the point p_i—is bounded away from the intersection of the vanishing lines for the Margulis invariants for any two elements of $\{A, B, C\}$. In particular, p_i is bounded away from v.

If M_1 or M_2 is rank one, then its image depends on $\boldsymbol{\theta}$. For each fixed choice of $\boldsymbol{\theta}$, the arcs ℓ_i remain ultraparallel and the matrix formulas again imply that the image of $M_i(\boldsymbol{\theta})$ is bounded away from the vertex v. As the parameters θ_i approach the endpoints of their intervals, the distance between some arcs in τ decreases to 0. In the limit, the arcs become asymptotically parallel. The disjointness criterion also applies in the case of asymptotically parallel vectors, and the matrices M_i have the same formulas. Thus the limit p_i' of p_i is itself bounded away from v and we are done.

$\square$

Fix ϵ to be sufficiently small, and let $[u] \in \mathscr{B}(v, \epsilon) \setminus v$. By construction, $[u]$ gives an affine deformation of Γ_0' that admits no crooked fundamental domain. $\square$

We may need to choose ϵ to be quite small, as this example in the case of the three-holed sphere shows. This completes the proof of Theorems 1.0.2, 1.0.4,

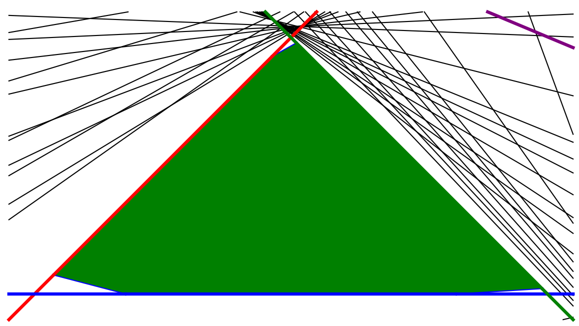

Figure 5.12: $\Sigma_{0,3}$ with considerable blunting.

and 1.0.5.

It is worth emphasizing that Crook($[\tau]$) is generally not disjoint from its image under a mapping class. For example, in the diagram of the two-holed projective plane, the pentagons $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ intersect in a quadrilateral. The blunting in the case of reflections is due to the fact that the image of the stem quadrant under the rank-two matrices is two-dimensional. This blunting can be removed by choosing a direction in each stem quadrant rather than computing the image of the entire stem quadrant. The blunting corresponding to elliptic generators can be removed by choosing a set of parameters $\boldsymbol{\theta}$; that is, choosing geodesic representatives of the arcs.

The parametrization of the proper affine deformation space by the arc complex in [10] makes such choices. In that paper, what they call a choice of *waist* forces a choice of direction in the stem quadrant. They also choose a length of the vector in the stem quadrant, giving a parameter they call the *width*.

The tiling of the space of proper affine deformations given in the case of the one-holed torus in [3] makes a choice of geodesic representative. Since every generator of the Coxeter group is elliptic in this case, this removes all blunting. A similar tiling of the proper affine deformation space of the one-holed Klein bottle is achieved in [34] by choosing geodesic representatives and directions in the stem quadrant.

However, there are two important points to contrast in the case of Coxeter extensions for our four surfaces. First, it is impossible in general to tile the space of proper affine deformations of Γ_0' with cocycles corresponding to affine deformations that admit crooked fundamental domains. By making choices, one can get triangular regions rather than hexagonal regions, but the regions are only a proper convex subset of the space of proper affine deformations.

Second, the tilings given for the one-holed torus in [3] and the one-holed Klein bottle in [34] use ideal triangles rather than ultraideal triangles. This allows the theory to apply to the case where Γ_0 contains parabolic elements at the cost that the edges of the tiles no longer form crooked fundamental domains for the action of the Coxeter group. See the section "Living on the Edge" in [3]. The edges correspond to moving only two of the crooked planes in the direction of their stem quadrants. When the crooked planes linearize to an ideal triangle, the crooked planes intersect pairwise if only two are moved. However, if they linearize to an ultraideal triangle, moving two crooked planes is sufficient to ensure disjointness. For this reason, the edges of the polygons $\mathrm{Crook}(\tau(\boldsymbol{\theta}))$ and $\mathrm{Crook}([\tau])$ in this paper consist of cocycles whose actions *do* admit crooked fundamental domains.

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