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An Inverse Eigenvalue Problem with Rotational Symmetry

by

T. I. Seidman

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Thomas I Seidman

Centre for Mathematical Analysis
Australian National University
GPO Box 4
Canberra ACT 2601

Permanent address:

Department of Mathematics
University of Maryland Baltimore County
Catonsville, MD 21228 USA

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ABSTRACT:

We consider convergence of an approximation method for the recovery of a rotationally symmetric potential ψ from the sequence of eigenvalues. In order to permit the consideration of 'rough' potentials ψ (having essentially $H^{-1}(0,1)$ regularity), we first indicate the appropriate interpretation of $-\Delta+\psi$ (with boundary conditions) as a self-adjoint, densely defined operator on $\mathcal{H} := L^2(\Omega)$ and then show a suitable continuous dependence on ψ for the relevant eigenvalues. The approach to the inverse problem is by the method of 'generalized interpolation' and, assuming uniqueness, it is shown that one has convergence to the correct potential ψ (strongly, for an appropriate norm) for a sequence of computationally implementable approximations $(P_{c,N})$.

1 INTRODUCTION

The present paper is intended as an extension of the considerations of [5] to higher dimensional contexts. Our concern will be with operators of the form

(1.1.) $$\underset{\sim}{L} = \underset{\sim}{L}_\psi : u \longrightarrow - \nabla \cdot a \nabla u + \psi u$$

in a context of rotational symmetry in $\mathbb{R}^d$, i.e. assuming that

(1.2.) (i) $a(\cdot), \psi(\cdot)$ depend only on $r := |x|$,

(ii) the domain Ω is the unit ball of $\mathbb{R}^d$ with $d \geq 2$,

(iii) the boundary conditions are radial - of the form[1]

$$au_r = \gamma u \quad \text{on} \quad \partial\Omega \text{ (i.e. at } r = 1) .$$

We assume $a(\cdot)$ is known and bounded with a uniform ellipticity condition:

(1.3.) $$A \geq a(r) \geq \alpha > 0 \quad \text{for} \quad 0 \leq r \leq 1$$

(e.g. $a \equiv 1$ giving $\underset{\sim}{L}_0 = - \Delta$).

Our concern is with the *inverse eigenvalue problem:*

(EVP) Suppose $a(\cdot)$ is known and it is known that $\psi \in \Psi_*$ (some suitable set). If we are given eigenvalues of the self-adjoint operator $\underset{\sim}{A}_\psi$ associated with (1.1) and (1.2)(iii), how can we (computationally) recover the potential ψ ?

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[1]We could equally well consider Dirichlet conditions ($u = 0$ at $r = 1$) which would require minor modification of our presentation - e.g. we would set $\mathcal{V} := H_0^1(\Omega)$ rather than $H^1(\Omega)$ as here, etc.

We do not consider here the deep question of uniqueness: within which sets Ψ_* is ψ uniquely determined by the given eigenvalue information. Rather, this is taken as an *a priori* assumption on the suitability of Ψ_* for (EVP).

On the other hand, as in [5], we are very much concerned with another aspect of the suitability of Ψ_* : For how 'rough' a potential ψ can we construct a workable interpretation of $(-\nabla \cdot a \nabla + \psi)$ as a densely defined, self-adjoint operator $\underset{\sim}{A}_\psi$ on $L^2(\Omega)$ with compact resolvent so that discussion of the "eigenvalues of ... $\underset{\sim}{A}_\psi$" makes sense? The approach, as in [5], is closely related to that of Chapter 3 of [6] with modification to fit the setting under consideration. In [5] it was shown, for the one-dimensional case, that $\underset{\sim}{A}_\psi$ is suitably defined for $\psi \in \mathcal{F}^*$ with $\mathcal{F} := H^1(-1,1)$ and that the eigenvalues then depend continuously[2] on ψ . A principal concern here will be to obtain comparable results for $\psi \in \mathcal{F}^*$ with $\mathcal{F}$ much like $H^1(0,1)$ - viewing $\psi = \psi(r)$ as given for $r \in (0,1)$, rather than on Ω - but now with $\mathcal{F}$ defined through a weighted H^1 - norm, controlling the behaviour near $r = 0$. We are able to get results quite comparable to the one-dimensional case treated in [5] precisely because the radial symmetry permits a treatment through separation of variables which reduces this to one-dimensional considerations.

Once we have developed the setting in which (EVP) is a meaningful problem, our concern is to demonstrate convergence for an approximation method of 'generalized interpolation' type (see, e.g., [4],[6]). We assume in (EVP) that we are given the sequence $(\bar{\lambda}_1, \bar{\lambda}_2, \dots)$ of the eigenvalues of $\underset{\sim}{A}_{\bar{\psi}}$ corresponding to a unique potential $\bar{\psi} \in \Psi_*$ and consider the approximation procedure:

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[2] There appears to be a slight gap in the argument in [5] and it seems necessary to take the strong $\mathcal{V}_0^*$ topology for ψ rather than the sequential weak topology as asserted there. See Theorem 11, below.

(ii) For $a \in L^{\infty}$ and ψ in a certain $L^{\bar{q}}(\Omega)$ (see below) we have a 'weak interpretation' of $(\underset{\sim}{L}_{\psi}+\lambda)$ as a continuous invertible operator : $\mathcal{V} \longrightarrow \mathcal{V}^*$ with $\mathcal{V} = H^1(\Omega)$. We can then take $\mathcal{D}_{\psi}^{(ii)}$ to be the pre-image of $\mathcal{H} \subset \mathcal{V}^*$ for this operator.

(iii) The radial nature of $\underset{\sim}{L}_{\psi}$ induces (e.g., for smooth a, ψ) a canonical decomposition of $\mathcal{H}$ by separation of variables into subspaces of the form $H_{\mu} = \mathfrak{X} \otimes \mathcal{U}_{\mu}$ where each $\mathcal{U}_{\mu}$ is finite-dimensional[3] and $\mathfrak{X}$ is a weighted L^2 - space of functions on (0,1). Associated with $-\nabla \cdot a \nabla \cdot$ and (1.2)(iii) is an ordinary differential operator $\underset{\sim}{M}_{\mu}$ and, following [5], we can interpret $(\underset{\sim}{M}_{\mu}+\psi)$ as a self-adjoint operator on $\mathfrak{X}$ for each relevant μ when ψ is in $\mathcal{P}^*$ where, now, $\mathcal{P}$ is a weighted H^1- space on (0,1). These interpretations $\underset{\sim}{A}_{\mu,\psi} : \mathcal{H}_{\mu} \supset \mathcal{D}_{\mu,\psi} \longrightarrow \mathcal{H}_{\mu}$ can be combined to obtain $\mathcal{D}_{\psi} = \mathcal{D}_{\psi}^{(iii)}$ and the interpretation of (2.1).

We will ultimately use the interpretation (iii) but, of course, wish to know that the interpretations are consistent with each other.

We begin with an abstract construction, following [5]. For the present, we take $\mathfrak{X}$ to be $L^2_{\mu}(\mathcal{S})$ where $\mathcal{S}$ is any set with (positive) bounded measure μ and assume $\mathcal{Y}$ is also a Hilbert space of functions on $\mathcal{S}$ with the pivoting

$$\mathcal{Y} \hookrightarrow \mathfrak{X} \hookrightarrow \mathcal{Y}^*$$

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[3]The elements of $\mathcal{U}_{\mu}$ are just the classical 'angular' functions well-known from analysis of the Laplacian for a ball - i.e., $\{\sin n\theta, \cos n\theta\}$ for d = 2, spherical harmonics for d = 3, etc. The subspaces $\{\mathcal{U}_{\mu}\}$ do not depend on a, ψ or the boundary conditions and give an orthogonal direct sum decomposition of $L^2(\partial\Omega)$.

with dense embeddings. We assume given a linear continuous map $\underset{\sim}{M}: \mathcal{Y} \longrightarrow \mathcal{Y}^*$ for which one has a monotonicity estimate of the form:

$$(2.3) \qquad \langle \underset{\sim}{M}x, x \rangle \geq \underset{\sim}{\alpha} |x|_{\mathcal{Y}}^2 - \beta |x|_{\mathcal{X}}^2 \qquad x \in \mathcal{Y} \subset \mathcal{X}$$

with $\underset{\sim}{\alpha} > 0$. We will also assume symmetry:

$$(2.4) \qquad \langle \underset{\sim}{M}x, y \rangle = \langle x, \underset{\sim}{M}y \rangle \quad \text{for} \quad x,y \in \mathcal{Y} .$$

(The $\mathcal{Y} - \mathcal{Y}^*$ dualities of (2.3), (2.4) are, of course, given by the pivoting through the inner product of $\mathcal{X}$).

We next take $\mathcal{P}$ to be any space containing products xy for $x,y \in \mathcal{Y}$ with a norm such that[4]

$$(2.5) \qquad |xy|_{\mathcal{P}} \leq \bar{C}|x|_{\mathcal{Y}}|y|_{\mathcal{Y}} \quad \text{for} \quad x,y \in \mathcal{Y} .$$

LEMMA 1 *Let* ψ *be a function on* $\mathcal{S}$ *which is in* $\mathcal{P}^*$ *in the sense of the* $\mathcal{P}$–$\mathcal{P}^*$ *duality induced by the* $\mathcal{X}$ *inner product. Then the multiplication operator*

$$\underset{\sim}{\psi}: x \to \psi x: \mathcal{Y} \to \mathcal{Y}^*$$

is well defined and continuous with

$$(2.6) \qquad |\psi x|_{\mathcal{Y}^*} \leq \bar{C}|\psi|_{\mathcal{P}^*}|x|_{\mathcal{Y}} \quad \text{so} \quad \|\underset{\sim}{\psi}\| \leq \bar{C}|\psi|_{\mathcal{P}^*} .$$

Proof: From (2.5) we have

$$|\langle \psi x, y \rangle| = |\langle \psi, xy \rangle| \leq |\psi|_{\mathcal{P}^*}|xy|_{\mathcal{P}} \leq (\bar{C}|\psi|_{\mathcal{P}^*}|x|_{\mathcal{Y}})|y|_{\mathcal{Y}}$$

for arbitrary $x,y \in \mathcal{Y}$. By the definition of the $\mathcal{Y}^*$ norm as $\sup\{|\langle \psi x, y \rangle| : |y|_{\mathcal{Y}} \leq 1\}$, this gives (2.6). □

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[4]E.g. We could take $\mathcal{P}$ to be the completion of $\mathrm{sp}\{xy : x,y \in \mathcal{Y}\}$ with respect to the norm

$$|u|_{\mathcal{P}} := \inf\{\Sigma_j |x_j|_{\mathcal{Y}}|y_j|_{\mathcal{Y}} : \Sigma_j x_j y_j = u\} .$$

Note that for such functions ψ, φ we have $\varphi \geq \psi$ precisely when $\langle(\psi-\varphi)y, y\rangle = \langle\psi-\varphi, y^2\rangle \geq 0$ for $y \in \mathcal{Y}$ and we take this as inducing the order for $\mathcal{F}^*$. We also wish to consider $\varphi \in \mathcal{F}^*$ (so that $\underset{\sim}{\varphi}:\mathcal{Y} \to \mathcal{Y}^*$ is defined) such that:

(2.7) For each $\epsilon > 0$ one has C_ϵ such that

$$|\langle\varphi x, x\rangle| \leq \epsilon|x|_{\mathcal{Y}}^2 + C_\epsilon|x|_{\mathcal{X}}^2 \quad \text{for} \quad x \in \mathcal{Y} \subset \mathcal{X} .$$

LEMMA 2 *Let* $\mathcal{X}, \mathcal{Y}, \mathcal{F}$ *be as above. Let* $\underset{\sim}{M}:\mathcal{Y} \to \mathcal{Y}^*$, *as above, satisfy (2.3),(2.4) and let* $\psi \in \mathcal{F}^*$ *with* $\psi \geq \varphi$ *for* φ *satisfying (2.7). Then* $(\underset{\sim}{M} + \psi)$ *induces a densely defined, self-adjoint operator* $\underset{\sim}{M}_\psi:\mathcal{X} \supset \mathcal{D}_\psi \to \mathcal{X}$. *If the embedding* $\mathcal{Y} \hookrightarrow \mathcal{X}$ *induced by the pivoting is compact, then* $\underset{\sim}{M}_\psi$ *has compact resolvent.*

Proof: For any real λ we have $(\underset{\sim}{A}+\psi+\lambda):\mathcal{Y} \to \mathcal{Y}^*$ continuous with

$$\langle(\underset{\sim}{M}+\psi+\lambda)x, x\rangle = \langle\underset{\sim}{M}x, x\rangle + \langle(\lambda+\varphi)x, x\rangle + \langle\psi-\varphi, x^2\rangle$$

$$\geq [\underset{\sim}{\alpha}|x|_{\mathcal{Y}}^2 - \beta|x|_{\mathcal{X}}^2] + \lambda|x|_{\mathcal{X}}^2 + \langle\varphi x, x\rangle ,$$

using (2.3) and noting $\psi \geq \varphi$. Using an inequality $|x|_{\mathcal{X}} \leq c_o|x|_{\mathcal{Y}}$ and (2.7) with $\epsilon := \underset{\sim}{\alpha}/2c_o$, we obtain the fundamental estimate

$$(2.8) \qquad \langle(\underset{\sim}{M}+\psi+\lambda)x, x\rangle \geq (\underset{\sim}{\alpha}/2)|x|_{\mathcal{Y}}^2 + (\lambda-\beta-c_oC_\epsilon)|x|_{\mathcal{X}}^2 .$$

Considering $\lambda \geq \beta + c_oC_\epsilon =: \bar{\lambda}_\varphi$, this makes $(\underset{\sim}{M}+\psi+\lambda): \mathcal{Y} \to \mathcal{Y}^*$ strictly monotone, hence invertible. We set

$$(2.9) \qquad \mathcal{D}_\psi := \mathcal{R}((\underset{\sim}{M}+\psi+\lambda)^{-1}|_{\mathcal{X}})$$

$$= \{x \in \mathcal{Y}:(\underset{\sim}{M}+\psi+\lambda)x =: z \in \mathcal{X}\} \subset \mathcal{X} .$$

$$\underset{\sim}{M}_\psi x := z-\lambda x \quad \text{for} \quad x \in \mathcal{D}_\psi \quad \text{with} \quad z := (\underset{\sim}{M}+\psi+\lambda)x \in \mathcal{X}$$

The continuity of $(\underset{\sim}{A}+\psi+\lambda)^{-1}: \mathcal{X} \hookrightarrow \mathcal{Y}^* \to \mathcal{Y}$ ensures that $\underset{\sim}{A}_\psi$ is a closed operator. Clearly, this definition is independent of the particular choice of (large enough) λ. Note that $\bar{\lambda}_\varphi$ depends on ψ only through the lower bound φ but (2.9) does not depend on the particular choice of φ. Fixing φ and taking $\lambda \geq \bar{\lambda}$, (2.8) gives

$$|x|_{\mathcal{Y}} \leq M|(\underset{\sim}{M}_\psi+\lambda)x|_{\mathcal{X}} \quad \text{for} \quad x \in \mathcal{D}_\psi \tag{2.10}$$

uniformly on $\{\psi \in \mathcal{P}^*:\psi \geq \varphi\}$ with $M = 2/\alpha\bar{C}$ where $\bar{C}:=$ [norm of embedding: $\mathcal{Y} \to \mathcal{X}$] . Note that if $\mathcal{Y} \to \mathcal{X}$ is compact then (2.10) makes $\{(\underset{\sim}{M}_\psi+\lambda)^{-1} : \lambda \geq \bar{\lambda}_\varphi , \psi \geq \varphi , \psi \in \mathcal{P}^*\}$ collectively compact for any φ satisfying (2.7).

To see that $\mathcal{D}_\psi$ is dense in $\mathcal{X}$, i.e. that $(\underset{\sim}{M}+\psi+\lambda)^{-1}: \mathcal{X} \to \mathcal{X}$ has dense range, we proceed by contradiction. Were $\mathcal{D}_\psi$ not dense there would exist $\bar{x} \in \mathcal{X}$ orthogonal to $\mathcal{D}_\psi$ with $\bar{x} \neq 0$. We can find $\bar{z} \in \mathcal{Y}$ with $(\underset{\sim}{M}+\psi+\lambda)\bar{z} = \bar{x}$ so, using (2.4),

$$\begin{aligned}\langle \bar{z}, z\rangle &= \langle(\underset{\sim}{M}+\psi+\lambda)\bar{z}, (\underset{\sim}{M}+\psi+\lambda)^{-1}z\rangle \\ &= \langle\bar{x}, (\underset{\sim}{M}+\psi+\lambda)^{-1}z\rangle = 0 \quad \text{since} \quad (\underset{\sim}{M}+\psi+\lambda)^{-1}z \in \mathcal{D}_\psi\end{aligned}$$

for any $z \in \mathcal{X}$. Hence $\bar{z} = 0$ so $\bar{x} = 0$ – a contradiction.

Finally, the assumed symmetry of $\underset{\sim}{M}$ makes $\underset{\sim}{M}_\psi$ formally self-adjoint but we must verify that the domain of $(\underset{\sim}{M}_\psi)^*$ is precisely $\mathcal{D}_\psi$, i.e. that $\mathcal{X}$-continuity (on the dense set $\mathcal{D}_\psi$) of the functional: $y \to \langle\underset{\sim}{M}_\psi y,x\rangle$ implies $x \in \mathcal{D}_\psi$ (noting that the inverse implication is clear). Note that this continuity implies existence of $z \in \mathcal{X}$ such that $\langle\underset{\sim}{M}_\psi y,x\rangle = \langle y,z\rangle$ for each $y \in \mathcal{D}_\psi$; set

$$\tilde{x} := (\underset{\sim}{M}+\psi+\lambda)^{-1}[z+\lambda x] \in \mathcal{D}_\psi .$$

We then have, by the symmetry,

$$\langle(\underset{\sim}{M}+\psi+\lambda)y,\tilde{x}\rangle = \langle\underset{\sim}{M}y,\tilde{x}\rangle + \langle(\psi+\lambda)y,\tilde{x}\rangle$$

$$= \langle y,\underset{\sim}{M}\tilde{x}\rangle + \langle y,(\psi+\lambda)\tilde{x}\rangle$$

$$= \langle y,(\underset{\sim}{M}+\psi+\lambda)\tilde{x}\rangle$$

$$= \langle y,z+\lambda x\rangle = \langle\underset{\sim}{M}_{\psi}y,x\rangle + \lambda\langle y,x\rangle$$

$$= \langle(\underset{\sim}{M}+\psi+\lambda)y,x\rangle \ .$$

Since $(\underset{\sim}{M}+\psi+\lambda)y$ ranges over $\mathscr{X}$ as y ranges over $\mathscr{D}_{\psi}$, this shows $\tilde{x} = x$ so $x \in \mathscr{D}_{\psi}$. □

For the interpretation (2.2) (ii) of $\underset{\sim}{L}_{\psi}$, we take[5] $\mathscr{Y} = \mathscr{V} := H^1(\Omega)$ and $\mathscr{X} = \mathscr{H} := L^2(\Omega)$. The standard weak formulation of $\underset{\sim}{A} := -\nabla\cdot a\nabla\cdot$ with (1.2) (iii) is given by

$$\text{(2.11)} \qquad \langle\underset{\sim}{A}u,v\rangle = \int_{\Omega}(-\nabla\cdot a\nabla u)v$$

$$= \int_{\Omega} a\nabla u\cdot\nabla v - \gamma\int_{\partial\Omega} uv \ .$$

Since [1] the Dirichlet trace is compact : $\mathscr{Y} \to L^2(\partial\Omega)$, we have

$$\text{(2.12)} \qquad \left|\int_{\partial\Omega} uv\right| \le \epsilon|u|_{\mathscr{V}}|v|_{\mathscr{V}} + C_{\epsilon}|u|_{\mathscr{H}}|v|_{\mathscr{H}} \quad \text{for} \quad u,v \in \mathscr{V}$$

by a standard functional analysis result.[6] Hence, taking

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[5]As noted earlier, we have $\mathscr{V} := H^1_0(\Omega)$ in the case of Dirichlet boundary conditions.

[6]Given Banach spaces $\mathscr{U},\mathscr{V},\mathscr{W}$ and linear maps $\underset{\sim}{A} : \mathscr{U} \to \mathscr{V}$ and $\underset{\sim}{B} : \mathscr{U} \to \mathscr{W}$ with $\underset{\sim}{A}$ compact and $\mathscr{N}(\underset{\sim}{B}) \subset \mathscr{N}(\underset{\sim}{A})$, one has an inequality:

$$|\underset{\sim}{A}u|_{\mathscr{V}} \le \epsilon|u|_{\mathscr{U}} + C_{\epsilon}|\underset{\sim}{B}u|_{\mathscr{W}} \quad \text{for} \quad u \in \mathscr{U}$$

for arbitrary $\epsilon > 0$ and some C_{ϵ}.

(2.13) $$|u|_{\mathscr{V}} := \left[\int_\Omega |\nabla u|^2 + |u|^2\right]^{1/2} ,$$

we obtain (2.3) for $\underset{\sim}{M} = \underset{\sim}{A}$ with $\underline{\alpha}$ arbitrarily close to the α in (1.3) and a correspondingly determined β. We now take[7] $\bar{q} := 2d/(d+2) > 1$ so that (with $1/\bar{q} + 1/\bar{p} = 1$) standard results [1] give continuous embedding : $\mathscr{V} \hookrightarrow L^{2\bar{p}}(\Omega)$ whence

$$|x^2|_{\bar{p}} = |x|^2_{2\bar{p}} \le C|x|^2_{\mathscr{V}} .$$

This means that we can take $\mathscr{P} = L^{\bar{p}}(\Omega)$ and $\psi \in \mathscr{P}^* = L^{\bar{q}}(\Omega)$. For $q > \bar{q}$ (or $q = \bar{q}$ when $d = 2$) the embedding : $\mathscr{V} \to L^{2p}(\Omega)$ (with $1/q + 1/p = 1$) is compact so we have (2.7), for $\varphi \in L^q(\Omega)$. We have thus shown:

(2.14) For any $\psi \in L^{\bar{q}}(\Omega)$, i.e., $\int r^{d-1}|\psi(r)|^{\bar{q}}dr < \infty$, bounded below by $\varphi \in L^q(\Omega)$, the construction (2.2) (ii) via (2.9) defines $\underset{\sim}{A}_\psi$, corresponding to $\underset{\sim}{L}_\psi$ in (1.1), as a densely defined, self-adjoint operator on $L^2(\Omega)$ with compact resolvent. □

When a,ψ are smooth it is a standard regularity result that $u \in H^2(\Omega)$ for $\underset{\sim}{L}_\psi u \in L^2(\Omega)$ so the definitions (2.2) (i), (ii) are then equivalent.

We proceed now to develop (2.2) (iii). A formal calculation[8], imposing the *ansatz*

(2.15) $$u(x) = R(r)U(\omega) \quad \text{for} \quad x = r\omega \in \Omega$$

with $r \in (0,1)$ and $\omega \in S^{d-1} = \partial\Omega$, gives

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[7]This is for $d > 2$. For $d = 1$ we could take $\mathscr{P} = \mathscr{Y}$ as in [5] while for $d = 2$ we take any $\bar{\varphi} > 1$ and continue.

[8]This is valid pointwise for a,R,U smooth.

$$(2.16) \qquad -\nabla\cdot a\nabla u = (\underset{\sim}{M}_0 R)U + \frac{a}{r^2}R(\underset{\sim}{S}U)$$

where $\underset{\sim}{M}_0$ is the ordinary differential operator given formally on (0,1) by

$$(2.17) \qquad \underset{\sim}{M}_0 : f \to -(af')' - \frac{d-1}{r}af'$$

and $\underset{\sim}{S}$ is a second order elliptic operator, acting as a densely defined, self-adjoint, semi-definite operator on $\mathcal{U} := L^2(S^{d-1})$ with compact resolvent.

A significant observation is that the *spherical operator* $\underset{\sim}{S}$ does not depend on $\underset{\sim}{A}$, i.e. on the particular choice of $a(\cdot)$, γ in (1.1) and (1.2) (iii). We write $\{\mu_j : j = 0,1, \ldots\}$ for the *distinct* eigenvalues of $\underset{\sim}{M}_0$ so

$$0 = \mu_0 < \mu_1 < \ldots \to \infty$$

and, for each $\mu = \mu_j$, we let

$$(2.18) \qquad \mathcal{U}_\mu := \{U : \underset{\sim}{S}U = \mu U\} \subset \mathcal{U} := L^2(S^{d-1})$$

be the corresponding eigenspace. Note that each $\mathcal{U}_\mu$ is finite-dimensional and that the elements of $\mathcal{U}_\mu$ (eigenfunctions of $\underset{\sim}{S}$) are just the classical 'angular' functions. The subspaces $\{\mathcal{U}_\mu\}$ are othogonal, giving a direct sum decomposition:

$$(2.19) \qquad \mathcal{U} = \{\mathcal{U}_\mu : \mu = 0, \mu_1, \ldots\} .$$

Now let $\mathcal{X}$ be a weighted $L^2(0,1)$, i.e., with the inner product

$$(2.20) \qquad \langle x,y\rangle_{\mathcal{X}} := \int_0^1 r^{d-1}\, x(r)y(r)dr$$

and corresponding norm. For $\mu = 0, \mu_1, \ldots$ we set

$$(2.21) \qquad \mathcal{H}_\mu := \mathcal{X} \otimes \mathcal{U}_\mu \quad \text{(tensor product)}$$

$$:= \overline{\mathrm{sp}}\ \{u = R(r)U(w) : R \in \mathcal{X}\ ,\ U \in \mathcal{U}_\mu\}$$

$$= \left\{\sum_{j=1}^{J(\mu)} R_j(r)\ \bar{U}_j(\omega) : R_j \in \mathcal{X}\right\} \subset \mathcal{H} := L^2(\Omega)$$

where $\{\bar{U}_j = \bar{U}_{\mu,j} : j=1,\ldots,J(\mu)\}$ is an othonormal basis for $\mathcal{U}_\mu$. Corresponding to (2.19) we then have an orthogonal direct sum decomposition:

$$(2.22) \qquad \mathcal{H} = \oplus\{\mathcal{H}_\mu : \mu = 0,\mu_1,\ldots\}\ .$$

Note that the norm $|\cdot|_{\mathcal{X}}$ corresponding to (2.20) gives $|RU|_{\mathcal{H}} = |R|_{\mathcal{X}}|U|_{\mathcal{U}}$ for (2.15) and if, corresponding to (2.21), (2.22), we consider $u,v \in \mathcal{H}$ expanded as

$$u = \sum_\mu \sum_{j=1}^{J(\mu)} R_{\mu,j}\bar{U}_{\mu,j}\ , v = \sum_\mu \sum_{j=1}^{J(\mu)} \hat{R}_{\mu,j}\bar{U}_{\mu,j}$$

with $R_{\mu,j}$, $\hat{R}_{\mu,j} \in \mathcal{X}$, then

$$\langle u,v\rangle_{\mathcal{H}} = \sum_\mu \sum_{j=1}^{J(\mu)} \langle R_{\mu,j}, \hat{R}_{\mu,j}\rangle_{\mathcal{X}}$$

in view of the orthonormality of $\{\bar{U}_{\mu,j} : j = 1,\ldots,J(\mu);\mu = 0,\mu_1,\ldots\}$. Note that for u as in (2.15) with $U \in \mathcal{U}_\mu$ (e.g. $U = \bar{U}_{\mu,j}$) we have

$$\underset{\sim}{L}_\psi u = [(\underset{\sim}{M}_0+\mu a/r^2+\psi)R]U$$

and we are led to analyze $\underset{\sim}{M}_\mu := \underset{\sim}{M}_0+\mu a/r^2$ $(\mu = \mu_0,\mu_1,\ldots)$.

Integrating by parts and using the boundary conditions (1.2) (iii), we obtain the weak formulation of $\underset{\sim}{M}_\mu$:

$$(2.23) \qquad \langle \underset{\sim}{M}_\mu f,g\rangle_{\mathcal{X}} = \int_0^1 r^{d-1}a(r)[f'g'+\frac{\mu}{r^2}fg]dr - \gamma f(1)g(1)\ .$$

To proceed it is necessary to distinguish the two cases: $\mu = \mu_0 = 0$ and $\mu = \mu_1, \mu_2, \ldots > 0$. In each case we take $\mathcal{Y}$ to be a weighted H^1 space, but use slightly different norms. We take $\mathcal{Y}_0, \mathcal{Y}_+$ to be the Hilbert spaces of functions f on $(0,1)$ induced[9] respectively, by the norms

$$\|f\|_0 := \left(\int_0^1 r^{d-1}[|f'|^2+|f|^2]dr\right)^{1/2}, \tag{2.24}$$

$$\|f\|_+ := \left(\int_0^1 [r^{d-1}|f'|^2+\mu_1 r^{d-3}|f|^2]dr\right)^{1/2}.$$

Observe that $\|\cdot\|_+$ dominates $\|\cdot\|_0$ since $r^{-2}>1$ and $\mu_1 > 0$ so $\mathcal{Y}_+ \subset \mathcal{Y}$ with, clearly, a dense embedding. We complete the weak formulation of

$$\underset{\sim}{M}_\mu : \mathcal{Y} \to \mathcal{Y}^* \quad (\mathcal{Y} := \mathcal{Y}_0 \text{ for } \mu = 0;\ \mathcal{Y} = \mathcal{Y}_+ \text{ for } \mu > 0)$$

by specifying that (2.23) is to hold for $f, g \in \mathcal{Y}$, as appropriate.

From standard embedding results [1], one easily sees that $\mathcal{Y}_0$ (*a fortiori* $\mathcal{Y}_+$) embeds in $C^{1/2}[\bar{r},1]$ for any $\bar{r} > 0$ so one has an estimate

$$|f(1)| \le C_1\|f\|_0 \quad \text{for } f \in \mathcal{Y}_0 .$$

Also, $\mathcal{Y}$ embeds compactly in $C[\bar{r},1]$ for $\bar{r} > 0$ from which it follows, as earlier for (2.12), that

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[9]We take the closure (with respect to $\|\cdot\|_0$ or $\|\cdot\|_+$) of the set of smooth functions compactly supported in $(0,1]$ and satisfying the boundary conditions. Hence, the specification of $\mathcal{Y}$ includes the requirement that $f(1) = 0$ in the case of Dirichlet conditions. Note that for $\mu = 0$ we have $\mathcal{U}_0 = \{\text{constants on } S^{d-1}\}$ so $\mathcal{H}_0$ is just the subspace of radial functions in $L^2(\Omega)$ while (2.24) makes $\mathcal{Y}_0 \tilde{=} \mathcal{Y}_0 \otimes \mathcal{U}_0$ the subspace of $\mathcal{V} := H^1(\Omega)$ (or $H_0^1(\Omega)$) consisting of radial functions with a norm isometry.

$$(2.25)\qquad |f(1)|^2 \le \epsilon\, \|f\|_0^2 + C_\epsilon |f|_{\mathfrak{X}}^2 \quad \text{for} \quad f \in \mathcal{Y}_0$$

for any $\epsilon > 0$. We will need more precise information about the behaviour of $f \in \mathcal{Y}_0$ as $r \to 0+$. For $0 < r < 1$ we have

$$|f(1) - f(r)| = \left|\int_r^1 s^{-(d-1)/2}[s^{(d-1)/2}f'(s)]ds\right|$$

$$\le \left(\int_r^1 s^{1-d}ds\right)^{1/2} \|f\|_0$$

whence, as $|f(r)| \le |f(1)| + |f(1) - f(r)|$, we have

$$(2.26)\qquad |f(r)| \le \begin{cases} [C_1 + \ell n^{1/2} r]\|f\|_0 & d = 2 \\ C_* \, r^{1-d/2}\|f\|_0 & d = 3,4,\ldots \end{cases}$$

with C_1, C_* depending only on d. If we define $\mathfrak{X}_\nu$ as the space of functions continuous on $(0,1]$ for which the norm

$$(2.27)\qquad |f|_{[\nu]} := \sup\{r^{\nu/2}|f(r)| \,:\, 0 < r \le 1\}$$

is finite, then (2.26) shows that $\mathcal{Y}_0$ embeds (continuously, by the Closed Graph Theorem) in $\mathfrak{X}_\nu$ for $\nu \ge \bar{\nu} := d - 2$ ($\nu > \bar{\nu} = 0$ for $d = 2$).

LEMMA 3 *$\mathcal{Y}_0$ embeds compactly in $\mathfrak{X}_\nu$ for $\nu > \bar{\nu} := d - 2$ ($d \ge 2$).*

Proof: Suppose $d > 2$ and $\{f_k\}$ is bounded in $\mathcal{Y}_0$. We can extract a subsequence (again denoted by $\{f_k\}$) converging $\mathcal{Y}_0$ - weakly, say to $\bar{f}$, and we will show $f_k \to \bar{f}$ in $\mathfrak{X}_\nu$. Note that $f_k \to \bar{f}$ uniformly on $[\bar{r},1]$ for each $\bar{r} > 0$ since the embedding : $\mathcal{Y}_0 \to C[\bar{r},1]$ is compact (as $\|\cdot\|_0$ dominates the $H^1(\bar{r},1)$ - norm); cf. e.g. [1]. Since $\nu > \bar{\nu}$ we have, by (2.26).

$$(2.28)\qquad r^{\nu/2}|f_k(r) - \bar{f}(r)| \le C_* r^{\nu'} \|f_k - \bar{f}\|_0 \le \bar{C} r^{\nu'}$$

with $2\nu' = \nu - \bar{\nu} > 0$ and with $\bar{C}$ fixed for the sequence. Given any $\epsilon > 0$ we can choose $\bar{r}$ so the right side of (2.28) is less than ϵ on $(0,\bar{r})$. Then, noting the uniform convergence $f_k \to \bar{f}$ on $[\bar{r},1]$, we can choose $K = K(\epsilon)$ large enough that the left side of (2.28) is less than ϵ on $[\bar{r}, 1]$ for each $k \ge K$, giving $|f_k - f|_{[\nu]} \le \epsilon$.

For $d = 2$ we maytake any $\tilde{\nu} \in (0,\nu)$ and note that (2.26) gives

$$|f(r)| \le C_* r^{-\tilde{\nu}/2} \|f\|_0$$

(C_* now depending on the choice of $\tilde{\nu}$), giving (2.28) with $2\nu' = \nu - \tilde{\nu}$. The proof concludes as before. □

Next, we consider the weighted $H^1(0,1)$ spaces $\hat{\mathcal{Y}}_\nu$ induced by the norms

$$(2.29)\qquad \|f\|_{[\nu]} := \left(\int_0^1 r^\nu [|f'|^2 + |f|^2] dr\right)^{1/2}$$

for $\nu \ge 0$; observe that $\mathcal{Y}_0 = \hat{\mathcal{Y}}_{d-1}$. We will set $\mathcal{P} := \hat{\mathcal{Y}}_{\bar{\nu}}$ with $\bar{\nu} := 2d-3$ for $d > 2$ (any $\bar{\nu} > 1$ for $d = 2$).

LEMMA 4 *Let $\varphi \in (\hat{\mathcal{Y}}_\nu)^*$ for some $\nu > 2d-3$ $(d \ge 2)$. Then (2.7) holds with $\mathcal{Y} = \mathcal{Y}_0$ (a fortiori with $\mathcal{Y} = \mathcal{Y}_+$).*

Proof: Set $\tilde{\nu} := \nu - (d-1)$ so $\nu > 2d-3$, as assumed, gives $\tilde{\nu} > d-2$ whence, by Lemma 3, the embedding : $\mathcal{Y}_0 \hookrightarrow \mathcal{X}_{\tilde{\nu}}$ is compact. Again this gives an estimate

$$(2.30)\qquad r^{\tilde{\nu}}|f(r)|^2 \le \hat{\epsilon}\,\|f\|_0^2 + \hat{C}\,|f|_{\mathcal{X}}^2 \quad \text{for } f \in \mathcal{Y}_0 \subset \mathcal{X}$$

for arbitrary $\hat{\epsilon} > 0$ and with $\hat{C}$ depending on $\hat{\epsilon}, \nu, d$. We have $(f^2)' = 2ff'$ so

$$\int r^{\nu}|(f^2)'|^2dr \le 4\ \sup\{r^{\tilde{\nu}}|f(r)|^2\} \int r^{d-1}|f'(r)|^2dr$$

$$\le 4\ (\hat{\varepsilon}\|f\|_0^2+\hat{C}|f|_{\mathfrak{X}}^2)|f'|_{\mathfrak{X}}^2$$

$$\int r^{\nu}|f^2(r)|^2dr \le (\hat{\varepsilon}\|f\|_0^2+\hat{C}|f|_{\mathfrak{X}}^2)|f|_{\mathfrak{X}}^2 .$$

$$\|f^2\|_{[\nu]}^2 \le 4(\hat{\varepsilon}\|f\|_0^2+\hat{C}|f|_{\mathfrak{X}}^2)\|f\|_0^2 .$$

Setting $4\hat{\varepsilon} := \epsilon^2$ (also fixing $\hat{C}$) , we then have

$$\|f^2\|_{[\nu]} \le \epsilon\ \|f\|_0^2 + C|f|_{\mathfrak{X}}^2 \tag{2.31}$$

for any C large enough that $2\epsilon C \ge \hat{\varepsilon} + 4\hat{C}$ and also $C^2 \ge \hat{C}$.

For $\varphi \in (\hat{\mathcal{Y}}_{\nu})^*$ let $M := [(\hat{\mathcal{Y}}_{\nu})^*$- norm of $\varphi] < \infty$ and note that (2.31) gives

$$|\langle\varphi x,x\rangle| = |\langle\varphi,x^2\rangle| \le M\|x^2\|_{[\nu]} \le (\epsilon M)|x|_{\mathcal{Y}}^2 + (CM)|x|_{\mathfrak{X}}^2$$

for $x \in \mathcal{Y} = \mathcal{Y}_0$ (or $\mathcal{Y}_+ \subset \mathcal{Y}_0$) and any $\epsilon M > 0$ and correspondingly determined CM . This is just (2.7). □

LEMMA 5 *With* $\mathcal{F} := \hat{\mathcal{Y}}_{\bar{\nu}}$, *as above, we have* (2.5) *with* $\mathcal{Y} = \mathcal{Y}_0$ *(a fortiori, with* $\mathcal{Y} = \mathcal{Y}_+$*).*

Proof: Note that $\tilde{\nu} := \bar{\nu} - (d-1) > 0$ for any $d \ge 2$. We no longer have compactness but (2.26) gives

$$r^{\tilde{\nu}}|f(r)|^2 \le C_*|f|_{\mathcal{Y}}^2 \quad \text{for} \quad f \in \mathcal{Y}$$

corresponding to (2.30) so we obtain, as for (2.31), the estimate

$$|x^2|_{\mathcal{F}} := \|x^2\|_{[\bar{\nu}]} \le C^*|x|_{\mathcal{Y}}^2 \quad \text{for} \quad x \in \mathcal{Y} ,$$

absorbing $|x|_{\mathfrak{X}}$ terms in $|x|_{\mathcal{Y}}$. From the identity

$$xy = [(cx+y/c)^2 - (cx-y/c)^2]/4$$

we then obtain

$$|xy| \le (C^*/4)[|cx+y/c|_{\mathcal{Y}}^2 + |cx-y/c|_{\mathcal{Y}}^2]$$
$$\le (C^*/2)[c|x|_{\mathcal{Y}} + |y|_{\mathcal{Y}}/c]^2$$

which is just (2.5) with $\bar{C} := 2C^*$ on setting $c^2 := |y|_{\mathcal{Y}}/|x|_{\mathcal{Y}}$. □

With these lemmas in hand we are ready to proceed to the construction (2.2) (iii).

THEOREM 6 *Let* $a(\cdot)$ *satisfy* (1.3) *with the boundary conditions* (1.2) (iii). *With* $\mathcal{P} := \hat{\mathcal{Y}}_{\bar{v}}$ *as above, assume* $\psi \in \mathcal{P}^*$ *with* $\psi \ge \varphi$ *for some* φ *as in Lemma 4. Then* $\underset{\sim}{L}_\psi = \underset{\sim}{A} + \underset{\sim}{\psi}$ *induces a closed, densely defined, self-adjoint operator* $\underset{\sim}{A}_\psi$ *on* $\mathcal{H}$, *as in* (2.1).

Proof: We begin by considering the ordinary differential operator $(\underset{\sim}{M}_\mu + \underset{\sim}{\psi})$, first for $\mu > 0$ so we are taking $\mathcal{Y} = \mathcal{Y}_+$ and $|\cdot|_{\mathcal{Y}} = \|\cdot\|_+$. From (2.23), using (1.3) and (2.25), we have

$$\langle \underset{\sim}{M}_\mu \; x, x\rangle \ge \alpha \int_0^1 [r^{d-1}|x'|^2 + \mu\, r^{d-3}|x|^2]dr - \gamma_+|x(1)|^2$$
$$\ge \alpha|x|_{\mathcal{Y}}^2 + (\mu-\mu_1)\alpha|x|_{\mathcal{X}}^2 - \gamma_+[\epsilon|x|_{\mathcal{Y}}^2 + C_\epsilon|x|_{\mathcal{X}}^2]$$

where $\gamma_+ := \max\{\gamma, 0\}$. Choose $\epsilon < \alpha/\gamma_+$ (thus also fixing C_ϵ) and one obtains (for $\mu \ge 1$) the monotonicity estimate[10]

(2.32) $\quad \langle \underset{\sim}{M}_\mu\, x,\, x\rangle \ge \underline{\alpha}\; |x|_{\mathcal{Y}}^2 + (\alpha\mu - \beta_0)|x|_{\mathcal{X}}^2 \;$ for $\; x \in \mathcal{Y} = \mathcal{Y}_+$

where $\underline{\alpha} := \alpha - \gamma_+^\epsilon$ and $\beta_0 := \alpha\mu_1 + \gamma_+ C_\epsilon$, i.e. (2.3) holds. The symmetry condition (2.4) is clear from (2.23) and Lemmas 3 and 4 ensure

[10] We only need (2.3) immediately but emphasise that $\underline{\alpha}$ and β_0 are independent of μ for $\mu = \mu_1, \mu_2, \ldots$.

the hypotheses on ψ, φ for applicability of Theorem 2. Thus we know that for each $\mu = \mu_1, \mu_2, \ldots$ there is a well-defined self-adjoint operator

$$\underset{\sim}{M}_{\mu,\psi} : \mathfrak{X} \supset \mathfrak{D}_{\mu,\psi} \to \mathfrak{X} \tag{2.33}$$

which maps : $x \to (\underset{\sim}{M}x + \psi x) \in \mathfrak{X}$ whenever $x \in \mathfrak{D}_{\mu,\psi} \subset \mathcal{Y}_+ \subset \mathfrak{X}$. For $\mu = \mu_0 = 0$, taking $\mathcal{Y} = \mathcal{Y}_0$ and $|\cdot|_{\mathcal{Y}} = \|\cdot\|_0$, one similarly obtains (2.3) with $\underline{\alpha}$ and $\beta = \beta_0$ exactly as in (2.32). Thus, we have (2.33) for *every* $\mu = \mu_0, \mu_1, \ldots$.

At this point we recall (2.21), (2.22) and note that each $u \in \mathcal{H}$ has the orthogonal expansion

$$u(x) = \sum_{\mu} \sum_{j=1}^{J(\mu)} R_{\mu,j}(r)\, \bar{U}_{\mu,j}(\omega) \qquad (x = r\omega) \tag{2.34}$$

where the outer sum is over $\mu = \mu_0, \mu_1, \ldots$ and each $R_{\mu,j}$ is in $\mathfrak{X}$. The orthonormality of $\{\bar{U}_{\mu,j}\}$ gives the norm identity

$$|u|^2_{\mathcal{H}} = \sum_{\mu} \sum_{j=1}^{J(\mu)} |R_{\mu,j}|^2_{\mathfrak{X}} \tag{2.35}$$

with a corresponding formula for the $\mathcal{H}$ inner product. We now define, in terms of (2.33),

$$\underset{\sim}{A}_{\psi}\, u := \sum_{\mu} \sum_{j=1}^{J(\mu)} [\underset{\sim}{M}_{\mu,\psi} R_{\mu,j}] \bar{U}_{\mu,j} \tag{2.36}$$

for u, given by (2.34), in

$$\mathfrak{D}_{\psi} := \{u \in \mathcal{H}: \text{ each } R_{\mu,j} \text{ of (2.34) is in } \mathfrak{D}_{\mu,\psi} \text{ with}$$
$$\sum_{\mu} \sum_{j=1}^{J(\mu)} |\underset{\sim}{M}_{\mu,\psi} R_{\mu,j}|^2_{\mathfrak{X}} < \infty\} .$$

It is easy to verify that this definition of $\underset{\sim}{A}_{\psi}$, $\mathfrak{D}_{\psi}$ is independent of the particular choices of orthonormal bases $\{\bar{U}_{\mu,j}: j = 1, \ldots, J(\mu)\}$ made for each $\mathcal{U}_{\mu}$.

Since each $\mathcal{D}_{\mu,\psi}$ is dense in $\mathfrak{X}$ we have the set {*finite* sums (2.34) with each $R_{\mu,j} \in \mathcal{D}_{\mu,\psi}$} dense in $\mathcal{H}$ so $\underset{\sim}{A}_\psi$ is densely defined. If $u^k \to \tilde{u}$ in $\mathcal{H}$ (with each $u^k \in \mathcal{D}_\psi$) and also $\underset{\sim}{A}_\psi u^k \to w$ in $\mathcal{H}$, then each $R^k_{\mu,j} \to \tilde{R}_{\mu,j}$ in $\mathfrak{X}$ and $\underset{\sim}{M}_{\mu,\psi} R^k_{\mu,j}$ converges in $\mathfrak{X}$, necessarily to $\underset{\sim}{M}_{\mu,\psi} \tilde{R}_{\mu,j}$, with

$$\sum_\mu \sum_j |\underset{\sim}{M}_{\mu,j} \tilde{R}_{\mu,j}|^2 = |w|^2_{\mathcal{H}} < \infty$$

so $\tilde{u} \in \mathcal{D}_\psi$ with $\underset{\sim}{A}_\psi \tilde{u} = w := \lim_k \underset{\sim}{A}_\psi u^k$. I.e., $\underset{\sim}{A}_\psi$ is closed. Next, suppose $\tilde{u} \in \mathcal{D}(\underset{\sim}{A}^*_\psi)$, meaning $\mathcal{H}$-continuity (on $\mathcal{D}_\psi$) of the functional : $u \to \langle \underset{\sim}{A}_\psi u, \tilde{u}\rangle$ so we have an identity $\langle \underset{\sim}{A}_\psi u, \tilde{u}\rangle = \langle u, \hat{u}\rangle$ for some $\hat{u} \in \mathcal{H}$. With the obvious notation, taking $u := R_{\mu,j} \bar{U}_{\mu,j}$ gives

$$\langle \underset{\sim}{M}_{\mu,\psi} R_{\mu,j}, \tilde{R}_{\mu,j}\rangle_{\mathfrak{X}} = \langle R_{\mu,j}, \hat{R}_{\mu,j}\rangle \quad \text{for} \quad R_{\mu,j} \in \mathcal{D}_{\mu,\psi}$$

whence, as $\underset{\sim}{M}_{\mu,\psi}$ is self-adjoint, we have $\tilde{R}_{\mu,j} \in \mathcal{D}_{\mu,\psi}$ and $\hat{R}_{\mu,j} = \underset{\sim}{M}_{\mu,\psi} \tilde{R}_{\mu,j}$. As in (2.35), this (for each μ, j) gives

$$\sum_\mu \sum_j |\underset{\sim}{M}_{\mu,j} \tilde{R}_{\mu,j}|^2 = |\hat{u}|_{\mathcal{H}} < \infty$$

whence $\hat{u} \in \mathcal{D}_\psi$. Thus $\mathcal{D}(\underset{\sim}{A}^*_\psi) \subset \mathcal{D}_\psi =: \mathcal{D}(\underset{\sim}{A}_\psi)$. Since one obviously has the reverse inclusion, it follows that $\underset{\sim}{A}_\psi$, is self-adjoint. □

We remark that $\underset{\sim}{A}_\psi$, as defined in (2.36), has compact resolvent but it is convenient to defer proof of this until the discussion of spectral analysis of $\underset{\sim}{A}_\psi$ in the next section. The final task of this section is verification of the consistency of (2.2) (ii) and (iii).

LEMMA 7 *Suppose* $\underset{\sim}{A}_\psi, \mathcal{D}_\psi$ *are defined as in* (2.14), *directly by application of Theorem* 2, *and also as in* (2.36). *Then these definitions are equivalent*

Since each $\mathfrak{D}_{\mu,\psi}$ is dense in $\mathfrak{X}$ we have the set {*finite* sums (2.34) with each $R_{\mu,j} \in \mathfrak{D}_{\mu,\psi}$} dense in $\mathcal{H}$ so $\underset{\sim}{A}_\psi$ is densely defined. If $u^k \to \tilde{u}$ in $\mathcal{H}$ (with each $u^k \in \mathfrak{D}_\psi$) and also $\underset{\sim}{A}_\psi u^k \to w$ in $\mathcal{H}$, then each $R^k_{\mu,j} \to \tilde{R}_{\mu,j}$ in $\mathfrak{X}$ and $\underset{\sim}{M}_{\mu,\psi} R^k_{\mu,j}$ converges in $\mathfrak{X}$, necessarily to $\underset{\sim}{M}_{\mu,\psi} \tilde{R}_{\mu,j}$, with

$$\sum_\mu \sum_j |\underset{\sim}{M}_{\mu,j} \tilde{R}_{\mu,j}|^2 = |w|^2_{\mathcal{H}} < \infty$$

so $\tilde{u} \in \mathfrak{D}_\psi$ with $\underset{\sim}{A}_\psi \tilde{u} = w := \lim_k \underset{\sim}{A}_\psi u^k$. I.e., $\underset{\sim}{A}_\psi$ is closed. Next, suppose $\tilde{u} \in \mathfrak{D}(\underset{\sim}{A}^*_\psi)$, meaning $\mathcal{H}$-continuity (on $\mathfrak{D}_\psi$) of the functional : $u \to \langle \underset{\sim}{A}_\psi u, \tilde{u}\rangle$ so we have an identity $\langle \underset{\sim}{A}_\psi u, \tilde{u}\rangle = \langle u, \hat{u}\rangle$ for some $\hat{u} \in \mathcal{H}$. With the obvious notation, taking $u := R_{\mu,j} \bar{U}_{\mu,j}$ gives

$$\langle \underset{\sim}{M}_{\mu,\psi} R_{\mu,j}, \tilde{R}_{\mu,j}\rangle_{\mathfrak{X}} = \langle R_{\mu,j}, \hat{R}_{\mu,j}\rangle \quad \text{for } R_{\mu,j} \in \mathfrak{D}_{\mu,\psi}$$

whence, as $\underset{\sim}{M}_{\mu,\psi}$ is self-adjoint, we have $\tilde{R}_{\mu,j} \in \mathfrak{D}_{\mu,\psi}$ and $\hat{R}_{\mu,j} = \underset{\sim}{M}_{\mu,\psi} \tilde{R}_{\mu,j}$. As in (2.35), this (for each μ,j) gives

$$\sum_\mu \sum_j |\underset{\sim}{M}_{\mu,j} \tilde{R}_{\mu,j}|^2 = |\hat{u}|_{\mathcal{H}} < \infty$$

whence $\hat{u} \in \mathfrak{D}_\psi$. Thus $\mathfrak{D}(\underset{\sim}{A}^*_\psi) \subset \mathfrak{D}_\psi =: \mathfrak{D}(\underset{\sim}{A}_\psi)$. Since one obviously has the reverse inclusion, it follows that $\underset{\sim}{A}_\psi$. is self-adjoint. □

We remark that $\underset{\sim}{A}_\psi$, as defined in (2.36), has compact resolvent but it is convenient to defer proof of this until the discussion of spectral analysis of $\underset{\sim}{A}_\psi$ in the next section. The final task of this section is verification of the consistency of (2.2) (ii) and (iii).

LEMMA 7 *Suppose* $\underset{\sim}{A}_\psi, \mathfrak{D}_\psi$ *are defined as in* (2.14), *directly by application of Theorem* 2, *and also as in* (2.36). *Then these definitions are equivalent*

Proof: For u of the form (2.15) with $U \in \mathcal{U}_\mu$ we have $u \in \mathcal{D}_\psi^{(ii)}$ if and only if $R \in \mathcal{D}_{\mu,\psi} \subset \mathcal{Y}_0$. Since elements of $\mathcal{U}_\mu$ are smooth, this gives $u \in \mathcal{V}$ and, from (2.16), etc., we have

$$(\underset{\sim}{A} + \psi + \lambda)[RU] = [(\underset{\sim}{M}_\mu + \psi + \lambda)R]U =: \hat{R}U + \lambda u$$

so $\underset{\sim}{M}_{\mu,\psi} R := \hat{R} \in \mathcal{X}$ implies $RU \in \mathcal{D}_\psi^{(ii)}$ and $\underset{\sim}{A}_\psi^{(ii)}u = \hat{R}U = \underset{\sim}{A}_\psi^{(iii)}u$. Conversely, $RU \in \mathcal{D}_\psi^{(ii)}$ means $(\hat{R}+\lambda)U \in \mathcal{H}$ whence $\hat{R} \in \mathcal{X}$ so $R \in \mathcal{D}_{\mu,\psi}$. For either of the definitions one obtains a closed operator and the span of such u = RU is dense in each graph. Thus, the definitions of $\underset{\sim}{A}_\psi$, $\mathcal{D}_\psi$ coincide. □

3 SPECTRAL THEORY: CONTINUITY

We will be considering the eigenvalues of $\underset{\sim}{A}_\psi$ (defined as in Theorem 6 for $\psi \in \mathcal{F}^*$ with suitable lower bound φ), taken in increasing order with multiplicities:

$$(3.1) \qquad \lambda_1(\psi) \leq \lambda_2(\psi) \leq \ldots \to \infty ,$$

as a sequence of nonlinear functionals of ψ. The principal result of this section, after verifying (3.1), is that each functional: $\psi \mapsto \lambda_k(\psi)$ is continuous, topologizing ψ in $\mathcal{F}^*$ (with a suitable one-sided estimate: $\psi \geq \varphi$).

From the proof of Theorem 6 (and under those hypotheses) we know that each of the operators $\underset{\sim}{M}_{\mu,\psi}$ $(\mu = 0, \mu_1, \ldots)$ is self-adjoint with compact resolvent and so we have eigenpairs[11] $\{[\sigma_{\mu,k}, y_{\mu,k}] : k = 1,2,\ldots\}$ such that, for each $\mu = 0, \mu_1, \ldots$,

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[11]That is, $y_{\mu,k} \in \mathcal{D}_{\mu,\psi} \subset \mathcal{X}$ with $\underset{\sim}{M}_{\mu,\psi}\, y_{\mu,k} = \sigma_{\mu,k}\, y_{\mu,k}$. (We are using $\sigma_{,,}$ to denote the eigenvalues of $\underset{\sim}{M}_{\mu,\psi}$ to avoid confusion with the eigenvalues $\{\mu_0, \ldots\}$ of $\underset{\sim}{S}$ or $\{\lambda_1, \ldots\}$ of $\underset{\sim}{A}_\psi$. We remark at this point that in standard Sturm-Liouville theory one shows, using properties of the *initial value problem* for $[\underset{\sim}{M}_0 + \mu a/r^2 + \psi]y = 0$, that these eigenvalues are simple (strict inequalities in (3.2) (ii) with certain nodal properties for the eigenfunctions. For ψ as rough as here it is not clear that this remains valid.

(3.2) (i) $\{y_{\mu,k}: k = 1,\ldots\}$ is an orthonormal basis for $\mathcal{X}$;

(ii) $\sigma_{\mu,1} \leq \sigma_{\mu,2} \leq \cdots \to \infty$.

LEMMA 8 *Each of the functions* $y_{\mu,k}(r)\overline{U}_{\mu,j}$ *(for* $\mu = 0,\mu_1,\ldots;$ $k = 1,2,\ldots;$ $j = 1,\ldots,J(\mu)$) *is an eigenfunction of* $\underset{\sim}{A}_\psi$ *with corresponding eigenvalue* $\sigma_{\mu,k}$. *This set of functions is an orthonormal basis for* $\mathcal{H}$.

Proof: For $y,\hat{y} \in \mathcal{X}$ and $U,\hat{U} \in \mathcal{U}$ we have

$$\langle yU, \hat{y}\hat{U}\rangle_{\mathcal{H}} = \langle y,\hat{y}\rangle_{\mathcal{X}} \langle U,\hat{U}\rangle_{\mathcal{U}}$$

so, since $\{\overline{U}_{\mu,j}: \mu = 0,\mu_1,\ldots;\ j = 1,\ldots,J(\mu)\}$ is an orthonormal basis for $\mathcal{U}$ and each $\{y_{\mu,k}: k = 1,\ldots\}$ is an orthonormal basis for $\mathcal{X}$, it follows that $\{y_{\mu,k}\ \overline{U}_{\mu,j}\}$ is an orthonormal basis for $\mathcal{H} = \mathcal{X}\otimes\mathcal{U}$. Our construction of $\underset{\sim}{A}_\psi$ gives

$$\underset{\sim}{A}_\psi(yU) := (\underset{\sim}{M}_{\mu,\psi}\, y)U \quad \text{for} \quad U \in \mathcal{U}_\mu\ , y \in \mathcal{D}_{\mu,\psi} ;$$

$$\underset{\sim}{A}_\psi(y_{\mu,k}\ \overline{U}_{\mu,j}) = (M_{\mu,\psi}\ y_{\mu,k})\overline{U}_{\mu,j} = (\sigma_{\mu,k}\ y_{\mu,k})\overline{U}_{\mu,j}$$

so each $(y_{\mu,k}\ \overline{U}_{\mu,j})$ is an eigenfunction of $\underset{\sim}{A}_\psi$. □

LEMMA 9 *For any* $\hat{\lambda} \in \mathbb{R}$ *there are only finitely many* $\mu \in \sigma(\underset{\sim}{L})$ *for which* $\sigma(\underset{\sim}{M}_{\mu,\psi}) \cap (-\infty,\hat{\lambda}]$ *is nonempty so (counting multiplicities in* $\sigma(\underset{\sim}{A}_\psi)$) *the set* $\{\sigma_{\mu,k} : \sigma_{\mu,k} \leq \hat{\lambda}\}$ *is finite.*

Proof: Suppose $\sigma \lesssim \hat{\lambda}$ is an eigenvalue of $\underset{\sim}{M}_{\mu,\psi}$ with corresponding eigenfunction $x \in \mathcal{D}_{\mu,\psi} \subset \mathfrak{X}$, normalized so $|x|_{\mathfrak{X}} = 1$. Then

$$\sigma = \langle \underset{\sim}{M}_{\mu,\psi} x,x\rangle$$

$$= \langle \underset{\sim}{M}_{\mu} x,x\rangle + \langle \varphi x,x\rangle + \langle \psi-\varphi,x^2\rangle$$

$$\geq \langle \underset{\sim}{M}_{\mu} x,x\rangle - \|\varphi\| \, \|x^2\|_{[\upsilon]}$$

where we take $\|\varphi\|$ in $(\hat{\mathcal{Y}}_{\upsilon})^*$. From (2.32) and (2.31),

$$\sigma \geq \underline{\alpha} \; |x|^2_{\mathcal{Y}} + (\alpha\mu - \beta_0) - \|\varphi\|(\epsilon |x|^2_{\mathcal{Y}} + C_\epsilon) \; .$$

Choosing $\epsilon := \underline{\alpha} / \|\varphi\|$, this gives[12]

(3.3) $$\mu \lesssim [\sigma + \beta_0 + \|\varphi\| \, C_\epsilon]/\alpha \; .$$

Since $\alpha, \underline{\alpha}, \beta_0, \|\varphi\|, C_\epsilon$ are independent of σ,μ, we see that a bound $\hat{\lambda}$ on σ bounds μ. Since we only consider $\mu \in \sigma(\underset{\sim}{L}) = \{0,\mu_1, \ldots\}$, this restricts us to a finite set.

In particular, given σ, the set $\mathcal{M}(\sigma) := \{\mu \in \sigma(S)\colon \sigma \in \sigma(\underset{\sim}{M}_{\mu,\psi})\}$ is finite. Further, σ occurs with finite multiplicity K_μ for each $\mu \in \mathcal{M}(\sigma)$. For each occurrence of σ in $[\sigma(\underset{\sim}{M}_{\mu,\psi})$ with multiplicities], say with eigenfunction y, it occurs $J(\mu)$ times in $[\sigma(\underset{\sim}{A}_{\psi})$ with multiplicities] - with corresponding eigenfunctions $\{y\bar{U}_{\mu,j}\colon j = 1,\ldots,J(\mu)\}$. Thus, the set $\{y_{\mu,k}\bar{U}_{\mu,j}\}$ contains exactly

[12]Note that C_ϵ here, coming from (2.31) in Lemma 5, depends only on $\underline{\alpha}$ and $\|\varphi\|$ whereas the C_ϵ appearing in the definition of β_0 for (2.32) comes from (2.25), depending only on the relation of $\underline{\alpha}$ to the α in (1.3).

$$(3.4) \qquad \sum\{J(\mu)K_{\mu}: \mu \in \mathcal{M}(\sigma)\} =: m_{\sigma}$$

eigenfunctions associated with σ as an eigenvalue of $\underset{\sim}{A}_{\psi}$.

From the above it follows that $\{\sigma_{\mu,k}\}$ is a discrete set which, when sorted in increasing order (with multiplicities), we can relabel as $(\lambda_1,\lambda_2,\ldots)$ with $\lambda_k \to \infty$ as in (3.1) with an associated orthonormal basis for $\mathcal{H}$ consisting of eigenfunctions[13] of $\underset{\sim}{A}_{\psi}$ which we relabel as $\{w_k = w_k(\psi): k = 1,2,\ldots\}$ so $\underset{\sim}{A}_{\psi}w_k = \lambda_k w_k$. It follows that $[\sigma(\underset{\sim}{A}_{\psi})$ with multiplicities] is precisely $\{\lambda_1,\lambda_2,\ldots\}$ with the multiplicities given by (3.4). □

An immediate corollary to the spectral expansion given by the eigenpairs $\{[\lambda_k,w_k]\}$:

$$(3.5) \qquad u = \sum_k \upsilon_k w_k \quad (|u|_{\mathcal{H}}^2 = \sum_k |\upsilon_k|^2);$$

$$\underset{\sim}{A}_{\psi}u = \sum_k(\lambda_k\upsilon_k)w_k \quad \text{for} \quad u \in \mathcal{D}_{\psi} \quad \text{where}$$

$$u \in \mathcal{D}_{\psi} \Leftrightarrow \sum_k |\lambda_k\upsilon_k|^2 < \infty$$

is that $\underset{\sim}{A}_{\psi}$ has compact resolvent (since $(\lambda-\lambda_k)^{-1} \to 0$ for $\lambda \notin \sigma(\underset{\sim}{A}_{\psi}) = \{\lambda_1,\ldots\})$. From (3.5) we easily obtain

$$(3.6) \qquad \langle \underset{\sim}{A}_{\psi}\, u,u\rangle = \sum_k \lambda_k \upsilon_k^2 \quad \text{for} \quad u \in \mathcal{D}_{\psi} \quad \text{as in (3.5)}$$

and note, from this and (3.1), that $\langle \underset{\sim}{A}_{\psi}\, u,u\rangle$ attains its minimum on

$$(3.7) \qquad \{u \in \mathcal{D}_{\psi}: \ |u|_{\mathcal{H}} = 1 \ \text{with} \ u \perp w_k \ \text{for} \ k < K\}$$

at, e.g. $u = w_K$ with the minimum value λ_K . It will be convenient to obtain a slightly different recursive variational characterization of λ_K . For $\mu \in \sigma(\underset{\sim}{L})$ we set

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[13] We canonically take these to have the form $w = y\bar{U}_{\mu,j}$ (once we have fixed the orthonormal bases $\{\bar{U}_{\mu,j}\}$ for each $\mathcal{U}_{\mu}$) with y an eigenfunction of $\underset{\sim}{M}_{\mu,\psi}$.

(3.8) $\mathcal{S}_{\mu,K} := \{u = y\bar{U}:\ y \in \mathcal{Y}\ \text{with}\ |y|_{\mathcal{X}} = 1;\ \bar{U} \in \{\bar{U}_{\mu,j}\};$
$u \perp w_k\ \text{for}\ k < K\}$

with $\mathcal{Y} := \mathcal{Y}_0$ for $\mu = 0$ and $\mathcal{Y}_+$ for $\mu = \mu_1, \ldots$.

LEMMA 10 *For the problem*

(3.9) *minimize* $\langle(\underset{\sim}{A}+\psi)u,u\rangle$ *subject to:* $u \in \bigcup_{\mu}\mathcal{S}_{\mu,K}$

the minimum is attained with the minimum value λ_K . *The minimizer* $\bar{u}$ *is an eigenfunction of* $\underset{\sim}{A}_{\psi}$ *which can be taken to be* w_K .

Proof: Let $\hat{\lambda} := \inf\ \{\langle(\underset{\sim}{A}+\psi)\,u,u\rangle:\ u \in \bigcup_{\mu}\mathcal{S}_{\mu,K}\}$. We already know from (3.5),(3.6) that λ_K is attained at $u = w_K \in \bigcup_{\mu}\mathcal{S}_{\mu,K}$ so in (3.9) we need only consider u for which $\langle(\underset{\sim}{A}+\psi)u,u\rangle \leq \lambda_K$. As in the argument for (3.3) we then have, for $u = y\bar{U} \in \bigcup_{\mu}\mathcal{S}_{\mu,K}$,

$$\lambda_K \geq \langle(\underset{\sim}{A}+\psi)u,u\rangle = \langle(\underset{\sim}{M}_{\mu}+\psi)y,y\rangle_{\mathcal{X}}$$

$$\geq \underline{\alpha}|y|^2_{\mathcal{Y}} + (\alpha\mu - \beta_0) - \|\varphi\|(\epsilon|y|^2_{\mathcal{Y}} + C_\epsilon) \ ,$$

$$(\underline{\alpha}/2\alpha)\,|y|^2_{\mathcal{Y}} + \mu \leq (\lambda_K+\beta_0+\|\varphi\|C_\epsilon)/\alpha$$

where we have here taken $\epsilon = \underline{\alpha}/2\|\varphi\|$ for use in (2.31).

If we consider a minimizing sequence for (3.9), we see that only finitely many $\mu \in \sigma(\underset{\sim}{S})$ need be considered. As there are then only finitely many relevant $\{\bar{U}_{\mu,j}\}$,we may extract a (minimising) subsequence of the form $y_k\bar{U}$ with $\bar{U} = \bar{U}_{\mu,j}$ (μ, j fixed), $y_k \in \mathcal{Y}$, and

$$\langle(\underset{\sim}{A}+\psi)u_k,u_k\rangle = \langle(\underset{\sim}{M}_{\mu}+\psi)y_k,y_k\rangle_{\mathcal{X}} \to \hat{\lambda} \ .$$

Further, we have $\{|y_k|_{\mathcal{Y}}\}$ bounded so we may also assume $y_k \rightharpoonup \bar{y}$ (weak convergence in $\mathcal{Y}$) .

Choosing λ large enough that $(\underset{\sim}{M}_\mu+\psi+\lambda)\colon \mathcal{Y} \to \mathcal{Y}^*$ is strictly monotone, the quadratic form $[y \to \langle(\underset{\sim}{M}_\mu+\psi+\lambda)y,y\rangle_{\mathcal{X}}]$ is convex and so lower semicontinuous with respect to weak convergence in $\mathcal{Y}$. Thus

$$\begin{aligned}\langle(\underset{\sim}{M}_\mu+\psi)\bar{y},\bar{y}\rangle &= \langle(\underset{\sim}{M}_\mu+\psi+\lambda)y,y\rangle - \lambda \\ &\leq \liminf \langle(\underset{\sim}{M}_\mu+\psi+\lambda)y_k,y_k\rangle - \lambda \\ &= \liminf \langle(\underset{\sim}{M}_\mu+\psi)y_k,y_k\rangle = \hat{\lambda}\end{aligned}$$

and the minimization (3.9) is attained at $\bar{u} = \bar{y}\bar{U}$; set $\bar{z} := (\underset{\sim}{M}_\mu+\psi)\bar{y} \in \mathcal{Y}^*$.

Let $\{\hat{y}_1,\ldots,\hat{y}_{K'}\} = \{y \in \mathcal{X}\colon y\bar{U} \in \{w_1,\ldots,w_{K-1}\}\}$ with $\bar{U} := \bar{U}_{\mu,j}$ for the fixed (μ,j). Then $y\bar{U} \perp w_k(K < K)$ in $\mathcal{H}$ precisely when $y \perp \hat{y}_k(k=1,\ldots,K')$ in $\mathcal{X}$. We set

$$\mathcal{S}_* := \{y \in \mathcal{Y}\colon |y|_{\mathcal{X}} = 1 \text{ and } y \perp y_k(k=1,\ldots,K')\}\ ,$$

$$\begin{aligned}\rho(t) = \rho(t;y) &:= \langle(\underset{\sim}{M}+\psi+\lambda)(\bar{y}+ty),\bar{y}+ty\rangle_{\mathcal{X}} \quad \text{for } y \in \mathcal{S}_* \\ &= (\hat{\lambda}+\lambda) + 2t[\langle\bar{z},y\rangle_{\mathcal{X}} + \lambda\langle\bar{y},y\rangle_{\mathcal{X}}] + t^2\langle(\underset{\sim}{M}_\mu+\psi+\lambda)y,y\rangle_{\mathcal{X}}\end{aligned}$$

with λ as above. The minimization property of $\bar{y}$ ensures that $\rho(t)$ is minimized at 0 for any $y \in \mathcal{S}_*$ with $y \perp \bar{y}$. Thus, $\langle\bar{z},y\rangle_{\mathcal{X}} = 0$ for such y. Since $\bar{z}$ is orthogonal (in the sense of the $\mathcal{Y}$–$\mathcal{Y}^*$ duality corresponding to the $\mathcal{X}$ inner product) to everything in $\mathcal{S}_*$ which is orthogonal to $\bar{y}$, i.e.,

$$\bar{z} \perp \{y \in \mathcal{Y}\colon y \perp \mathrm{sp}\ \{\hat{y}_1,\ldots,\hat{y}_{K'},\bar{y}\}\}\ .$$

we must have $\bar{z} \in \mathrm{sp}\{\hat{y}_1,\ldots,\hat{y}_{K'},\bar{y}\}$. Hence $\bar{z} \in \mathcal{Y} \subset \mathcal{X}$ so $\bar{y} \in \mathcal{D}_{\mu,\psi}$ and $\bar{z} = \underset{\sim}{M}_{\mu,\psi}\,\bar{y}$. Also, for $k = 1,\ldots,K'$ we have

$$\langle \hat{y}_k, \bar{z} \rangle_{\mathfrak{X}} = \langle \hat{y}_k, \underset{\sim}{M}_{\mu,\psi}\, \bar{y} \rangle_{\mathfrak{X}}$$

$$= \langle \underset{\sim}{M}_{\mu,\psi}\, \hat{y}_k, \bar{y} \rangle_{\mathfrak{X}} = \lambda_{k'} \langle \hat{y}_k, \bar{y} \rangle_{\mathfrak{X}} = 0$$

where $\lambda_{k'}(k'<K)$ is the eigenvalue corresponding to $w_{k'} = \hat{y}_k \bar{U}$. Hence, $\bar{z} \in sp\{\bar{y}\}$, i.e. $\bar{z} = \tilde{\lambda}\, \bar{y}$ for some $\tilde{\lambda}$. Clearly $\tilde{\lambda} = \hat{\lambda} \leq \lambda_K$ and $\bar{u} := \bar{y}\bar{U}$ is an eigenfunction of $\underset{\sim}{A}_\psi$ with $\underset{\sim}{A}_\psi \bar{u} = (\underset{\sim}{M}_{\mu,\psi}\, \bar{y})\, \bar{U} = \bar{z}\bar{U} = \tilde{\lambda}\, \bar{u}$. The ordering (3.1), i.e. the definition of λ_K, then ensures $\tilde{\lambda} \geq \lambda_K$ so $\underset{\sim}{A}_\psi \bar{u} = \lambda_K \bar{u}$. To within the arbitrariness in the specification of the eigenfunctions we can take $w_K = \bar{u}$. □

This argument is essentially the Courant Minimax Theorem (cf., e.g. [6]), adapted to the present definition of $\underset{\sim}{A}_\psi$. The characterization by (3.9) permits us, as in [5], to show the continuous dependence of each eigenvalue $\lambda_k = \lambda_k(\psi)$ (and of each corresponding eigenfunction w_k, to within the arbitrariness inherent in specificaiton of the eigenfunctions) on ψ, topologized by the $\mathcal{P}^*$ norm.

THEOREM 11 *Let* $\psi = \psi_i \to \bar{\psi}$ *strongly in* $\mathcal{P}^*$ *and assume there is a bounded sequence* $\{\varphi = \varphi_i\}$ *in* $(\hat{\mathcal{Y}}_\upsilon)^*$ *such that* $\psi_i \geq \varphi_i$. *Then as* $i \to \infty$ *one has*

$$\lambda_k = \lambda_{k,i} := \lambda_k(\psi_i) \to \bar{\lambda}_k := \lambda_k(\bar{\psi}) \tag{3.10}$$

for each $k = 1,2,\ldots$. *Correspondingly, we have*

$$w_k = w_{k,i} := w_k(\psi_i) \to \bar{w}_k := w_k(\bar{\psi}) \quad in \quad \mathcal{V} = \mathcal{H}'(\Omega) \tag{3.11}$$

to within the arbitrariness inherently associated with our specification of the eigenfunctions.

Proof: The argument is essentially the same as the corresponding argument in [5], inductively using the variational characterization:

$$(3.12)\qquad \lambda_K(\psi) = \min\{\langle(\underset{\sim}{A}+\psi)u,u\rangle : u \in \cup_\mu \mathcal{S}_{\mu,K}(\psi)\};$$

$$\mathcal{S}_{\mu,K}(\psi) := \{u = y\bar{U} : y \in \mathcal{Y} \text{ with } |y|_{\mathfrak{X}} = 1;\ \bar{U} = \bar{U}_{\mu,j};$$

$$u \perp w_k(\psi) \text{ for } k < K\};$$

$$w_k(\psi) = \arg\min\{\langle(\underset{\sim}{A}+\psi)w,w\rangle : w \in \cup_\mu \mathcal{S}_{\mu,k}(\psi)\}$$

given by Lemma 10. The inductive hypothesis is to assume the result (3.10), (3.11) known for $k < K$ and fix K. We now write $\lambda = \lambda_i$, $w = w_i$, $\bar{\lambda}$, and $\bar{w}$ for $\lambda_K = \lambda_{K,i}$, $w_K = w_{K,i}$, $\bar{\lambda}_K$, and $\bar{w}_K$.

We first wish to show that $\limsup \lambda_i \le \bar{\lambda}$. To this end, obtain $\tilde{u} = \tilde{u}_i$ by applying the Gram-Schmidt procedure to $\{w_{1,i},\dots,w_{K-1,i},\bar{w}\}$ so

$$(3.13)\qquad \tilde{u} = N(\bar{w} - \textstyle\sum_{k<K} C_k\, w_k)$$

where $N = N_i$ is a normalizing constant and, noting the orthonormality of $\{w_k = w_{k,i} : k < K\}$, we have $C_k = C_{k,i} = N\langle w_k,\bar{w}\rangle$. Since, by the inductive hypothesis, $w_k \to \bar{w}_k$ and $w_k \perp \bar{w}$ for $k < K$, we have

$$(3.14)\qquad N = N_i \to 1, \qquad C_k = C_{k,i} \to 0 \quad \text{for } k < K .$$

From (3.13), (3.14) it follows that $\tilde{u} \to \bar{w}$ in $\mathcal{V}$. Actually, we know that $\bar{w}$ has the form $\bar{y}\bar{U}$ with $\bar{U} = \bar{U}_{\mu,j}$ (some fixed μ,j) and $\bar{y} \in \mathcal{Y}$ with $|\bar{y}|_{\mathfrak{X}} = 1$ and $\underset{\sim}{M}_{\mu,\bar{\psi}}\,\bar{y} = \bar{\lambda}\,\bar{y}$. In (3.13) we have $C_k = 0$ for any w_k not corresponding to the same (μ,j) so we can set

$$\tilde{y} := N\bar{y} - \textstyle\sum_{k<K} C_k\, y_k$$

where $w_k = y_k\,\bar{U}_{\mu,j}$ (any μ,j) and have $\tilde{u} = \tilde{y}\,\bar{U}$ (same $\bar{U}$ as for $\bar{w}$). We have $\tilde{y} = \tilde{y}_i \to \bar{y}$ in $\mathcal{Y}$ whence also $\{\tilde{y}_i^{\,2}\}$ is convergent in $\mathcal{P}$.

Now Lemma 10 gives

$$\lambda - \lambda_{K,i} \leq \langle (\underset{\sim}{A}+\psi)\tilde{u}, \tilde{u} \rangle$$

$$= \langle (\underset{\sim}{M}_\mu+\bar{\psi})\tilde{y},\tilde{y}\rangle_{\mathcal{X}} + \langle \psi-\bar{\psi},\tilde{y}^2\rangle_{\mathcal{X}}$$

We have

$$\langle (\underset{\sim}{M}_\mu+\bar{\psi})\tilde{y},\tilde{y}\rangle \to \langle (\underset{\sim}{M}_\mu+\bar{\psi})\bar{y},\bar{y}\rangle = \bar{\lambda}$$

as $\tilde{y} \to \bar{y}$ in $\mathcal{Y}$ and

$$\langle \psi - \bar{\psi} , \tilde{y}^2 \rangle \to 0$$

as $\tilde{y}^2 \to \bar{y}^2$ in $\mathcal{P}$ and $\psi \to \bar{\psi}$ in $\mathcal{P}^*$ (even weak convergence would suffice). Thus, lim sup $\lambda \lesssim \bar{\lambda}$.

We now wish to show, conversely, that lim inf $\lambda_{K,i} \geq \bar{\lambda}$, giving (3.10), and that (3.11) holds. Each $w = w_{K,i}$ has the form $y\bar{U}$ by our specifications and, as for Lemma 10, the upper bound on λ which we have just obtained restricts attention to $\bar{U} = \bar{U}_{\mu,j}$ for a finite set of relevant (μ,j) . Thus, possibly subdividing $\{w_{K,i}\}$ into alternative[14] subsequences, we may assume a fixed $\bar{U}$ and that this $\bar{U}$ is to be used in specifying $\bar{w}$. For each $w = w_{K,i}$, then, we have $w = y\bar{U}$ with $y \in \mathcal{Y}$, $|y|_{\mathcal{X}} = 1$, and $\underset{\sim}{M}_{\mu,\psi}\, y = \lambda y$. The same estimate as in Lemma 10 (recalling the assumed boundedness of $\{\varphi = \varphi_i\}$) bounds $\{y = y_i\}$ in $\mathcal{Y}$ so we may assume (again possibly taking a subsequence) that $\{y_i\}$ converges (weakly in $\mathcal{Y}$ so strongly in $\mathcal{X}$) to some $\bar{y} \in \mathcal{Y}$ with $|\bar{y}|_{\mathcal{X}} = 1$. For $k < K$ we need consider only $\{\hat{y}_1,\ldots,\hat{y}_{K'}\} = \{y{:}w_k{=}y\bar{U}$ with $k < K\}$ as earlier, except that $\hat{y}_j = \hat{y}_{j,i}$ now (but we are considering the fixed $\bar{U}$ as for $w,\bar{w}$) and, similarly, $\bar{y}_j := \hat{y}_j(\bar{\psi})$; note that the corresponding indices are independent of i by our inductive assumption. Then

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[14]These alternatives would correspond to equally valid ways of specifying $\bar{w}_K$, as is shown by the subsequent argument.

$$\langle y,\bar{y}_j\rangle_{\mathfrak{X}} = \langle y,\bar{y}_j - \hat{y}_j\rangle_{\mathfrak{X}} \to 0$$

since $\langle y,\hat{y}_j\rangle_{\mathfrak{X}} = 0$ and, inductively, (3.11) giving $w_k = \hat{y}_j \to \bar{w}_k = \bar{y}_j$ in $\mathcal{H}$ corresponds to $|\bar{y}_j - \hat{y}_j|_{\mathfrak{X}} \to 0$. Hence, in the limit $\langle \bar{y}, \bar{y}_j\rangle_{\mathfrak{X}} = 0$ and $\langle \bar{y}\bar{U},w_k\rangle_{\mathcal{H}} = 0$ for each $k < K$. We also have

$$\langle \psi - \bar{\psi},y^2\rangle \to 0$$

since we have assumed $\psi \to \bar{\psi}$ strongly[15] in $\mathcal{G}^*$ and $\{y^2\}$ is bounded in $\mathcal{G}$ by Lemma 5. Now choose $\tilde{\lambda}$ large enough that $(\underset{\sim}{M}_\mu + \bar{\psi} + \tilde{\lambda})\colon \mathcal{Y} \to \mathcal{Y}^*$ is (strictly) monotone so the functional: $y \mapsto \langle(\underset{\sim}{M}_\mu + \bar{\psi} + \tilde{\lambda})y,y\rangle$ is (strictly) convex on $\mathcal{Y}$ and so lower semicontinuous with respect to the weak topology of $\mathcal{Y}$. We have

$$\lambda = \langle(\underset{\sim}{M}_\mu + \bar{\psi} + \tilde{\lambda})y,y\rangle_{\mathfrak{X}} - \tilde{\lambda} + \langle\psi - \bar{\psi},\ y^2\rangle_{\mathfrak{X}}$$

so, as $y \rightharpoonup \bar{y}$ in $\mathcal{Y}$, we have

$$\begin{aligned} \liminf\ \lambda &\geq \langle(M_\mu + \bar{\psi} + \tilde{\lambda})\bar{y},\bar{y}\rangle - \tilde{\lambda} \\ &= \langle(\underset{\sim}{M}_\mu + \bar{\psi})\bar{y},\bar{y}\rangle_{\mathfrak{X}} = \langle(\underset{\sim}{A} + \psi)(\bar{y}\bar{U}),\bar{y}\bar{U}\rangle_{\mathcal{H}} \\ &\geq \min\ \{\langle(\underset{\sim}{A} + \bar{\psi})u,u\rangle\colon u \in \cup_\mu \mathcal{S}_{\mu,K}\} = \bar{\lambda}_K . \end{aligned}$$

This shows that $\lambda = \lambda_K(\psi_i) \to \bar{\lambda}_K$ along subsequences for which $w_{K,i} = y_i\bar{U}$ with $\bar{U}$ fixed and $\{y_i\}$ weakly convergent in $\mathcal{Y}$. The uniqueness

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[15]This corrects a minor error in [5] where, at the corresponding point, only weak convergence $\psi_i \rightharpoonup \bar{\psi}$ in $\mathcal{G}^*$ was assumed – which seems inadequate if one has only weak convergence: $y_i \rightharpoonup \bar{y}$ in $\mathcal{Y}$. Note that we need only $\limsup \langle\psi_i - \bar{\psi},y^2\rangle \leq 0$ so weak convergence would be adequate if supplemented by a one-sided bound: $\psi \leq \bar{\psi} + \theta_i$ with strong converence: $\theta_i \to 0$ in $\mathcal{G}^*$. It remains open as to whether weak convergence could suffice in general.

This completes the inductive step and, since the inductive hypothesis is vacuous for $K = 1$, the proof of the Theorem is complete by induction on K. □

4 THE APPROXIMATION SCHEME

The *method of generalized interpolation* [3], [4], is a quite general approach to the approximate solution of ill-posed problems. Typically, one must first observe the equivalence of the problem to specification of the values for a sequence of functionals $\{\lambda_k(\cdot)\}$ - but here, as in [5], the nature of the problem already presents it in this form.

The simplest version of the method is the procedure (P_N) described in the Introduction. The relevant hypotheses[18] are:

(4.1) The norm $\|\cdot\|_*$ (determining a reflexive Banach space $\mathcal{P}_*$) topologizing the relevant potentials is such that if $\psi_\nu \rightharpoonup \bar{\psi}$ weakly in $\mathcal{P}_*$ with $\|\psi_\nu\|_* \to \|\bar{\psi}\|_*$, then $\psi_\nu \to \bar{\psi}$ strongly in $\mathcal{P}_*$.

(4.2) Weak convergence $\psi_\nu \rightharpoonup \bar{\psi}$ in $\mathcal{X}_*$ implies $\lambda_k(\psi_\nu) \to \lambda_k(\bar{\psi})$ for each $k = 1,2,\ldots$.

(4.3) The constraint set $\Psi_* \subset \mathcal{P}_*$ is such that the problem:

$$\lambda_k(\psi) = \bar{\lambda}_k \quad \text{for} \quad k = 1,2,\ldots \text{ with } \psi \in \Psi_*$$

has at most one (minimum norm) solution $\bar{\psi}$.

Under these hypotheses (4.1)-(4.3) it is a general result [3] that

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[18]The property (4.1) is referred to as the "Efimov-Stečkin property". We refer to (4.2),(4.3) briefly as "(*weak*) *continuity*" and "*uniqueness*", respectively.

(4.4) $\psi_N \to \bar{\psi}$ (strongly in $\mathcal{P}_*$) as $N \to \infty$

where we assume the data $\{\bar{\lambda}_k\}$ is consistent (i.e. a solution exists in (4.3)) and, as for (P_N), each ψ_N $(N = 1,2,\ldots)$ is defined as the minimum norm element of Ψ_* subject to matching the given values $\bar{\lambda}_k$ of $\lambda_k(\cdot)$ for $k = 1, \ldots, N$.

Rather than prove the result in this form, we turn instead to consideration of a more general version which permits the use of (implementable) approximate procedures for the 'N-th stage' computations. Before doing this we comment on the hypotheses.

It is known (cf. e.g. [2]) that (4.1) holds for any uniformly convex Banach space, in particular for Hilbert spaces. Our major effort, to this point, has been to show that one obtains continuity of the eigenvalues, viewed as nonlinear functionals on the (radial) potential ψ , using anorm convergence in the specific space $\mathcal{P}^*$ and subject to a lower bound condition. Our first observation is that (4.1), (4.2) need only hold on the constraint set Ψ_* . We will assume[19]:

(4.5) The constraint set Ψ_* is in $\mathcal{P}^*$ and for each $\mathcal{P}_*$-bounded subset $\Psi_0 \subset \Psi_*$, there exists a suitable υ and a number m such that each $\psi \in \Psi_0$ has a lower bound $\varphi \in (\hat{\mathcal{Y}}_\upsilon)^*$ with $\psi \geq \varphi$, $\|\varphi\| \leq m$.

and obtain the (restricted form of the) condition (4.2) by requiring *compact embedding* : $\mathcal{P}_* \to \mathcal{P}^*$. We would like to permit consideration of potentials ψ involving (radial) measures and note that our efforts in working with such a weak space as $\mathcal{P}^*$ do, indeed, have the value of

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[19]The simplest form of this, of course, would be to have $\psi \geq 0$ for $\psi \in \Phi_*$ or, slightly more generally, a one-sided condition that $\psi \geq \underline{C}$ for a constant $\underline{C}$ depending only on the $\mathcal{P}^*$ norm of ψ .

permitting this, even after the norm is strengthened (defining $\mathcal{P}_*$) to have this compact embedding. Note that we are *not* assuming that Ψ_* itself is compact[20] in $\mathcal{P}^*$ but only a relative pre-compactness in Ψ_* of sets bounded with respect to $\mathcal{P}_*$-norm without having to specify any particular such $\mathcal{P}_*$ bound in specifying Ψ_* .

The uniqueness property (4.3) is, at present, *terra incognita* for (EVP), even for the case of radial potentials. In the one-dimensional case $(\Omega := (-1,1) \subset \mathbb{R}^1)$ radiality just means that the potential is known to be symmetric on the interval and that is known to ensure uniqueness [3]. This suggests the possibility that (4.3) may hold[21] for quite general $\Psi_* \subset \mathcal{P}^*$, satisfying (4.5), but this remains entirely conjectural at present. Here we take the uniqueness condition (4.3) as an *a priori* hypothesis without investigating specific settings (i.e. more concrete conditions) permitting its direct verification.

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[20] This assumption (corresponding, e.g. to an assumed *a priori* bound on Ψ_* in a space as $\mathcal{P}_*$) would permit a simpler approach. The map

$$\Lambda : \psi \mapsto [\lambda_1(\psi), \ldots]: \Psi_* \to \mathbb{R}^\infty$$

(taking $\mathbb{R}^\infty$ with the product topology) would be a continuous injective map from a compact Hausdorff space. By a standard result of point-set topology, Λ would then have compact range and a uniformly continuous inverse. The (uniform) continuity of the inverse would mean that, in specifying Ψ , we have assumed away the ill-posedness of the inverse problem (EVP). The difficulty lies in justification of any *a priori* bound on the potential $\bar{\psi}$.

[21] An interesting stronger conjecture is that one might be able to recover ψ from knowledge only of thoseeigenvalues of $\underset{\sim}{A}_\psi$ associated with purely radial eigenfunctions, i.e. from $\sigma(\underset{\sim}{M}_{0,\psi})$. This seems unlikely, however, as in the one-dimensional case it would correspond to knowing ψ symmetric but only giving alternate eigenvalues – those with even eigenfunctions.

We turn now to the more general approximation procedure, relaxing (P_N) somewhat. For this we assume that we are given $\delta_N > 0$ and positive functions $\epsilon_{N,k}(\psi) > 0$ for $k \in N$ and $\psi \in \Phi_*$. The procedure (at this 'N-th stage') is then:

$(P_{a,N})$ Let $\Phi_N := \{\psi \in \Phi_*: |\lambda_k(\psi) - \bar{\lambda}_k| \leq \epsilon_{N,k}(\psi)$ for $k = 1,\dots,N\}$ and select $\psi_N \in \Phi_N$ such that $\|\psi_N\|_* \leq \inf \{\|\psi\|_*: \psi \in \Phi_N\} + \delta_N$.

We note that $(P_{a,N})$ does not determine ψ_N uniquely.

THEOREM 12 *Let $\mathcal{F}^*$ be as in Theorem 11 and let $\mathcal{F}_*$ be a reflexive Banach space (with norm $\|\cdot\|_*$) embedding compactly in $\mathcal{F}^*$ and satisfying (4.1). Let Ψ_* be a closed convex*[22] *subset of $\mathcal{F}_* \subset \mathcal{F}^*$ satisfying (4.5). Assume $0 < \delta_N \to 0$ and $0 < \epsilon_{N,k}(\cdot) \to 0$ for each fixed k, uniformly on $\mathcal{F}_*$ - bounded subsets of Ψ_*. The operator $\underset{\sim}{A}_\psi$ is defined (as in Section 2) for radial potentials $\psi \in \Psi_*$ and the spectrum $[\lambda_k(\psi): k = 1,2,\dots] = \sigma(\underset{\sim}{A}_\psi)$ is as in (3.1), in increasing order with multiplicities. Suppose, for $k = 1,2,\dots,$ we have $\lambda_k = \lambda_k(\bar{\psi})$ for some unique*[23] *$\bar{\psi} \in \Psi_*$. Then (4.4) holds for any sequence $\{\psi_N\}$ in Ψ_* obtained by the procedure $(P_{a,N})$ for $N = 1,2,\dots$.*

Proof: The first observation is that $\bar{\psi} \in \Psi_N$ so

(4.6) $$\|\psi_N\|_* \leq \|\bar{\psi}\|_* + \delta_N \; ; \; \limsup \|\psi_N\|_* \leq \|\bar{\psi}\|_* .$$

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[22] It is sufficient that Ψ_* be closed in the weak topology of $\mathcal{F}_*$.

[23] It is sufficient that $\bar{\psi}$ be unique among minimum norm solutions. The general condition (4.3) asserts that for *any* eigenvalue sequence $(\bar{\lambda}_k)$ which can arise (consistency) the minimum norm solution in Φ_* is unique as asserted.

We may (extracting a subsequence if necessary) then assume $\psi_N \rightharpoonup \tilde{\psi}$: weak convergence in $\mathcal{P}_*$ for some $\tilde{\psi}$. As we assumed Ψ_* closed and convex, it follows that $\tilde{\psi} \in \Psi_*$ and we will show that $\tilde{\psi}$ is a (minimum norm) solution of

$$\text{(4.7)} \qquad \lambda_k(\psi) = \bar{\lambda}_k \quad \text{for} \quad k = 1,2,\ldots$$

whence $\tilde{\psi} = \bar{\psi}$ by the assumed uniqueness. The uniqeness of the limit shows that the possible extraction of a subsequence above was nugatory: one has $\psi_N \rightharpoonup \bar{\psi}$ for the full sequence $\{\psi_N\}$.

To see (4.7) for $\tilde{\psi}$, note that boundedness of $\{\psi_N\}$ in $\mathcal{P}_*$ gives $\epsilon_{N,k} := \epsilon_{N,k}(\psi_N) \to 0$ so $\lambda_k(\psi_N) \to \bar{\lambda}_k$ by $(P_{a,N})$ - considering only $N \geq k$, of course for each $k = 1,2,\ldots$. On the other hand, the assumed compactness of the embedding: $\mathcal{P}_* \to \mathcal{P}^*$ means that *weak* convergence: $\psi_N \rightharpoonup \tilde{\psi}$ in $\mathcal{P}_*$ implies *strong* convergence: $\psi_N \to \tilde{\psi}$ in $\mathcal{P}^*$. The condition (4.5), with boundedness in $\mathcal{P}_*$ of $\{\psi_N\}$ also gives the "lower bound condition" $(\psi_N \geq \varphi_N)$ of Theorem 11, so Theorem 11 applies to give[24] $\lambda_k(\psi_N) \to \lambda_k(\tilde{\psi})$ for each k whence $\lambda_k(\tilde{\psi}) = \bar{\lambda}_k$.

At this point we have weak convergence $\psi_N \rightharpoonup \tilde{\psi}$ in $\mathcal{P}_*$ (along the subsequence). The convexity of the norm gives lower semicontinuity with respect to the weak topology so $\psi_N \rightharpoonup \tilde{\psi}$ implies

$$\text{(4.8)} \qquad \|\tilde{\psi}\|_* \leq \liminf \|\psi_N\|_* \leq \|\bar{\psi}\|_* .$$

Since $\bar{\psi}$ is a *minimum norm* solution of (4.7) by assumption, the solution $\bar{\psi}$ cannot have smaller norm; hence $\|\tilde{\psi}\|_* = \|\bar{\psi}\|_*$ from (4.8) and $\tilde{\psi}$ is also a minimum norm solution. The assumed uniqueness of $\bar{\psi}$ then implies $\tilde{\psi} = \bar{\psi}$. As noted above, this gives weak convergence $\psi_N \rightharpoonup \bar{\psi}$ in $\mathcal{P}_*$ (along the full sequence) without yet using (4.1).

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[24]For present purposes (3.10) suffices; (3.11) is relevant only as part of the inductive argument for Theorem 11.

Now if we combine (4.6) with (4.8) we see that $\|\psi_N\|_* \to \|\bar{\psi}\|_*$. This, with the weak convergence, gives (4.4) subject to the assumption (4.1). □

For implementation we note that one does not attempt to construct Ψ_N and need not even construct $\psi_N \in \Psi_N$ directly as in $(P_{a,N})$. If one could produce any element $\tilde{\psi}_N \in \mathcal{P}_*$ for which one would have an estimate (for *some* ψ_N as in $(P_{a,N})$, $N = 1,2,\ldots$):

$$\|\tilde{\psi}_N - \psi_N\|_* \le \delta_N'(\psi_N) \; , \tag{4.9}$$

with $\delta_N'(\cdot) \to 0$ uniformly on $\mathcal{P}_*$ - bounded sets in Ψ_*, then as an immediate corollary of Theorem 12, one also has $\tilde{\psi}_N \to \bar{\psi}$ in $\mathcal{P}_*$ as $N \to \infty$. We will not, however, attempt to reduce the proof of convergence of our computational implementation to Theorem 12, but instead will use an essentially similar argument to prove convergence directly.

Parametrized by $h > 0$, we will need a family of computational approximations $\Psi_*(h)$ to Ψ_* and an algorithm[25] which takes h, N, and (a representation of) $\psi \in \Psi_*(h)$ as inputs and returns

$$\Lambda_N(\psi;h) = [\hat{\lambda}_1(\psi;h),\ldots,\hat{\lambda}_N(\psi;h)] \; ,$$

approximating $[\lambda,(\psi),\ldots,\psi_N(\psi)]$. Reasonable properties of such a computational procedure would be:

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[25]One could attempt a finite element discretization of $(\underset{\sim}{A}+\psi)$ from (2.11), using a finite element subspace corresponding to a mesh parameter h, if ψ were moderately smooth (or first approximate ψ by a smoother $\tilde{\psi}$). In view of the analysis above, one might more plausibly use such finite element discretizations to $(\underset{\sim}{M}_{\mu} + \psi)$ for relevant μ, assuming $\sigma(\underset{\sim}{S})$ accurately known. This effectively produces (sparse) $n(h) \times n(h)$ symmetric matrices whose 'first' N eigenvalues could be computed and taken as giving $\Lambda_N(\psi;h)$. In general, such a procedure would give $\hat{\lambda}_k(\psi;h) \to \lambda_k(\psi)$ as $h \to 0$.

(4.10) Given any $\psi \in \Psi_*$ there exists $\tilde{\psi} \in \Psi_*(h)$ with $\|\psi-\tilde{\psi}\|_* \le \delta(\psi;h)$ and given any $\tilde{\psi} \in \Psi_*(h)$ there exists $\psi \in \Psi_*$ with $\|\psi-\tilde{\psi}\|_* \le \delta(\tilde{\psi};h)$ there exists $\psi \in \Psi_*$ with $\|\psi-\tilde{\psi}\|_* \le \delta(\tilde{\psi}:h)$ where $\delta(\cdot;h) \to 0$ as $h \to 0$ uniformly on $\mathcal{P}_*$ -bounded sets; we may also assume (4.5) for $\Psi_*(h)$, uniformly in h for small h ;

(4.11) $|\hat{\lambda}_k(\psi;h) - \lambda_k(\psi)| \to 0$ as $h \to 0$ for each fixed $k = 1,2,\ldots$, uniformly on $\mathcal{P}_*$ -bounded sets.

Without further concern for the details of possible construction of such algorithms, we indicate how the availability of a computational implementation satisfying (4.10), (4.11) could be used to obtain a computable sequence $\{\tilde{\psi}_N\}$ converging in $\mathcal{P}_*$ -norm to $\bar{\psi}$.

We wish to replace the approximation procedure $(P_{a,N})$ by a more explicitly implementable computational procedure:

$(P_{c,N})$ Choose $h = h_N$ small enough that $\Psi_N(h) := \{\psi \in \Psi_*(h) : |\hat{\lambda}_k(\psi;h) - \bar{\lambda}_k| \le \tilde{\epsilon}_{N,k}$ for $k \le N\}$ is nonempty and select $\tilde{\psi}_N \in \tilde{\Psi}_N := \Psi_N(h_N)$ such that $\|\tilde{\psi}_N\|_* \le \tilde{\upsilon}_N + \tilde{\delta}_N$ where $\tilde{\upsilon}_N := \inf \{\|\tilde{\psi}\|_* : \tilde{\psi} \in \tilde{\Psi}_N\}$.

The actual computation involved in $(P_{c,N})$ would be the use of some (standard) algorithm for nonlinear constrained optimization to minimize $\|\tilde{\psi}\|_*$ (using a stopping criterion giving approximate minimization to within $\tilde{\delta}_N > 0$ of the infimum $\tilde{\upsilon}_N$) subject to the constraint : $\tilde{\psi} \in \tilde{\Psi}_N$. The computational difficulty of this will depend on the nature of $\|\cdot\|_*$, on the sizes of $\epsilon_{N,k}$, on the size of h_N and the computational difficulty in implementing $\Lambda_N(\cdot;h_N)$, etc.

THEOREM 13 *Let* $\underset{\sim}{A}$ $,\mathcal{P}^{*}$ $,\mathcal{P}_{*}$ $,\Psi_{*}$ $,\{\bar{\lambda}_k\}$, *be as for Theorem 12 and assume implementable computational approximations* $\Psi_{*}(h)$, $\Lambda_N(\cdot;h)$ *are available (for small* $h > 0$) *satisfying* (4.10), (4.11) . *Assume* $0 < \tilde{\delta}_N \to 0$ *and* $\tilde{\epsilon}_{N,k} \to 0$ *for* $k = 1,2,\ldots$ *as* $N \to \infty$. *Then, for each* $N = 1,2,\ldots,$ *one can choose* $h = h_N$ *so* $\tilde{\Psi}_N := \Psi_N(h_N)$ *is nonempty (further requiring that* $h_N \to 0$) *and select* $\tilde{\psi}_N \in \tilde{\Psi}_N$ *as in* $(P_{c,N})$. *For any such computed sequence* $\{\tilde{\psi}_N\}$ *we have*

$$(4.12) \qquad \tilde{\psi}_N \to \bar{\psi} \ \text{ in } \ \mathcal{P}_{*}\text{-norm as } \ N \to \infty$$

where $\bar{\psi}$ *is given by* (4.3) .

Proof: Since $\bar{\psi} \in \Psi_{*}$ one has, by (4.10), existence of $\bar{\psi}(h) \in \Psi_{*}(h)$ with $\|\bar{\psi} - \bar{\psi}(h)\|_{*} \leq \tilde{\delta}(\bar{\psi};h) \to 0$. This makes $\{\bar{\psi}(h)\colon 0 < h \leq h_0\}$ bounded so (4.11) gives

$$|\hat{\lambda}_k(\bar{\psi}(h);h) - \lambda_k(\bar{\psi}(h))| \leq \epsilon_{N,k}/2 \quad \text{for} \quad k = 1,\ldots,N$$

for small enough h . On the other hand, $\bar{\psi}(h) \to \bar{\psi}$ in $\mathcal{P}_{*}$ (*a fortiori*, in $\mathcal{P}^{*}$) and we have assumed (4.5) for $\Psi_{*}(h)$ so Theorem 11 applies to give

$$|\lambda_k(\bar{\psi}(h)) - \bar{\lambda}_k| \leq \epsilon_{N,k}/2 \quad \text{for} \quad k = 1,\ldots,N .$$

Combining these gives $\bar{\psi}(h) \in \Psi_N(h)$ for h small enough so then $\Psi_N(h) \neq \emptyset$. Also requiring $h_N \leq h_N'$ for any given sequence: $0 < h_N' \to 0$ lets us fix[26] h_N . This fixes $\tilde{\Psi}_N := \Psi_N(h_N)$ and we can find $\tilde{\psi}_N \in \tilde{\Psi}_N$, approximately minimizing the norm, as in $(P_{c,N})$.

To show (4.12), we proceed as in the proof of Theorem 12 . Since $\bar{\psi}(h_N)$, as above, is in $\tilde{\Psi}_N$ for each N and $\bar{\psi}(h_N) \to \bar{\psi}$, we see that

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[26]Computationally, one might start with a trial $h_n = h_N'$ and then, say, successively halve h_N until one can obtain/compute some $\tilde{\psi} \in \Psi_N(h_N)$.

$$\tilde{\upsilon}_N \leq \|\bar{\psi}(h_N)\|_* \to \|\bar{\psi}\|_*$$

so $\{\tilde{\upsilon}_N\}$ is bounded and, as in (4.6), we have

$$\limsup \|\tilde{\psi}_N\|_* \leq \|\bar{\psi}\|_* \tag{4.13}$$

so (extracting a subsequence if necessary) we may assume weak convergence: $\tilde{\psi}_N \rightharpoonup \tilde{\psi}$ in $\mathcal{H}_*$. By (4.11) and the definitiion of $\tilde{\Psi}_N$ (noting that $\tilde{\epsilon}_{N,k} \to 0$) we have $\lambda_k(\tilde{\psi}_N) \to \bar{\lambda}_k$ while Theorem 11 gives $\lambda_k(\tilde{\psi}_N) \to \lambda_k(\tilde{\psi})$. It follows that $\tilde{\psi}$ is a solution of (4.7) with $\|\tilde{\psi}\|_* \leq \liminf \|\tilde{\psi}_N\| \leq \|\bar{\psi}\|_*$; the uniqueness property (4.3) then gives $\tilde{\psi} = \bar{\psi}$ and weak convergence $\tilde{\psi}_N \rightharpoonup \bar{\psi}$ for the full sequence. From (4.13) we also have $\|\tilde{\psi}_N\|_* \to \|\bar{\psi}\|_*$ so, by (4.1), we have (4.12) as asserted. □

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