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On the Computation of the Real Stability Radius*

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Abstract

Recently Qiu et al. obtained a computionally attractive formula for the computation of the real stability radius. This formula involves a global maximization over frequency. Here we show that the frequency range can be limited to a certain finite interval. Numerical experimentation suggests that this interval is often reasonably small.

1 Introduction and Notation

For $k = 1, 2, \ldots$, let $\sigma_k(\cdot)$ denote the kth largest singular value of its matrix argument. The real (structured) stability radius of a real matrix triple $(A, B, C) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times m} \times \mathbf{R}^{p\times n}$, with A Hurwitz stable, is defined by (see [1])

$$r_{\mathbf{R}}(A,B,C) := \min_{\Delta \in \mathbf{R}^{m\times p}} \{\sigma_1(\Delta) : A + B\Delta C \text{ is not Hurwitz stable}\}.$$

Recently Qiu et al. [2] obtained a formula allowing efficient computation of $r_{\mathbf{R}}(A,B,C)$. Specifically they showed that

$$r_{\mathbf{R}}(A,B,C)^{-1} = \max_{\omega \in \mathbf{R}^+} \mu_{\mathbf{R}}\left(C(\mathrm{j}\omega I - A)^{-1}B\right) \tag{1}$$

where $\mathbf{R}^+ = \{\omega \in \mathbf{R} : \omega \geq 0\}$ and where, for any $M \in \mathbf{C}^{m\times p}$,

$$\mu_{\mathbf{R}}(M) := \inf_{\gamma \in (0,1]} \sigma_2\left(\begin{bmatrix} \Re e M & -\gamma \Im m M \\ \gamma^{-1} \Im m M & \Re e M \end{bmatrix}\right). \tag{2}$$

The computation of $\mu_{\mathbf{R}}(M)$ for given M can be carried out at low computational cost as the univariate function to be minimized is unimodal.

In this note, we obtain an upper bound on the magnitude of the global maximizers in (1), computable at a cost negligible compared to that of performing the global maximization.

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Numerical experimentation suggests that this bound is often reasonably small, and in many cases is significantly smaller than a previously obtained bound. Knowledge of such an upper bound simplifies the task of carrying out the numerical maximization.

Given $M \in \mathbf{C}^{m\times p}$, we denote its transpose by M^T and its conjugate transpose by M^H. When $M = M^H$, we denote by $\lambda_k(M)$ its kth largest eigenvalue. For $r \in \mathbf{R}$, $\lfloor r \rfloor$ is the largest integer which is smaller than or equal to r.

2 A Finite Frequency Range

Let $a_0, \ldots, a_n \in \mathbf{R}$ ($a_n = 1$) be the coefficients of the characteristic polynomial of A, i.e.,

$$\det(sI - A) = a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0,$$

and let $R_0, \ldots, R_{n-1} \in R^{n\times n}$ ($R_{n-1} = I$) be the matrix coefficients of $\det(sI - A)(sI - A)^{-1}$, i.e.,

$$(sI - A)^{-1} = \frac{R_{n-1}s^{n-1} + R_{n-2}s^{n-2} + \cdots + R_1 s + R_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}. \tag{3}$$

Also define

$$p_{n-1} := -a_{n-1} + \frac{\sqrt{2}\,\sigma_1(CR_{n-1}B)}{\sigma_1(CA^{-1}B)},$$

$$p_{n-2} := a_{n-2} + \frac{\sqrt{2}\,\sigma_1(CR_{n-2}B)}{\sigma_1(CA^{-1}B)},$$

$$p_{n-3} := a_{n-3} + \frac{\sqrt{2}\,\sigma_1(CR_{n-3}B)}{\sigma_1(CA^{-1}B)},$$

$$\vdots$$

$$p_{n-k} := (-1)^{\lfloor \frac{k+2}{2} \rfloor} a_{n-k} + \frac{\sqrt{2}\,\sigma_1(CR_{n-k}B)}{\sigma_1(CA^{-1}B)},$$

$$\vdots$$

$$p_0 := (-1)^{\lfloor \frac{n+2}{2} \rfloor} a_0 + \frac{\sqrt{2}\,\sigma_1(CR_0 B)}{\sigma_1(CA^{-1}B)}.$$

Our first result provides an outer approximation to a certain level set of $\sigma_1(C(\mathrm{j}\omega I - A)^{-1}B)$.

Proposition 1: The polynomial

$$P(\omega) := \omega^n - p_{n-1}\omega^{n-1} - \cdots - p_0$$

has at least one zero in $\mathbf{R}^+$. Furthermore, any $\hat{\omega} \geq 0$ such that

$$\sigma_1\left(C(\mathrm{j}\hat{\omega} I - A)^{-1}B\right) \geq \sigma_1(CA^{-1}B), \tag{4}$$

satisfies

$$\hat{\omega} \leq \rho_P := \max\{\omega \in \mathbf{R}^+ : P(\omega) = 0\}.$$

□

Thus the level set $\{\omega \geq 0 : \ \sigma_1(C(\mathrm{j}\omega I - A)^{-1}B) \geq \sigma_1(CA^{-1}B)\}$ (which is nonempty since it contains the origin) is contained in the interval $[0, \rho_P]$. An immediate consequence of this is that the complex (structured) stability radius $r_{\mathbf{C}}(A, B, C)$ (see [1]), whose inverse is given by

$$r_{\mathbf{C}}(A, B, C)^{-1} = \max_{\omega \in \mathbf{R}^+} \sigma_1(C(\mathrm{j}\omega I - A)^{-1}B), \tag{5}$$

can be obtained based instead on the formula

$$r_{\mathbf{C}}(A,B,C)^{-1} = \max_{\omega \in [0,\rho_P]} \sigma_1(C(j\omega I - A)^{-1}B).$$

This however is of little value as efficient schemes exist for solving (5) [3–5]. Of more interest is the following result concerning the computation of $r_{\mathbf{R}}(A,B,C)$. Here dependence of ρ_P on the triple (A,B,C) is made explicit.

Theorem 1:

$$r_{\mathbf{R}}(A,B,C)^{-1} = \max_{\omega \in [0,\rho_P(A,B,C)]} \mu_{\mathbf{R}}(C(j\omega I - A)^{-1}B). \tag{6}$$

Moreover, for fixed $(\hat{A},\hat{B},\hat{C})$, with $\hat{A}$ Hurwitz stable, the mapping

$$(A,B,C) \mapsto \max_{\omega \in [0,\rho_P(A,B,C)]} \mu_{\mathbf{R}}(\hat{C}(j\omega I - \hat{A})^{-1}\hat{B})$$

is continuous at $(A,B,C) = (\hat{A},\hat{B},\hat{C})$. □

While ρ_P may not be continuous as a function of (A,B,C) (largest real zero of a polynomial), the second statement in Theroem 1 validates computation of $r_{\mathbf{R}}$ by means of (6) whenever $r_{\mathbf{R}}$ is continuous (if $r_{\mathbf{R}}$ is discontinuous, there is no reliable way to compute it in the presence of numerical errors). Also, note that the computational cost of evaluting ρ_P is negligible compared to that of carring out the maximization (1). In particular, the a_i's and R_i's can be computed efficiently using the Souriau-Frame-Fadeev Algorithm (see, e.g., [6, Theorem 5.3.10]).

Finally, another upper bound for the frequency range to which the maximization in (1) may be restricted can be obtained from a simple extension of a result of J.M. Martin [7]. Specifically (4) also holds whenever $\omega \in [0,\rho_M]$, where

$$\rho_M := \sigma_1(A) + \frac{\sigma_1(C)\sigma_1(B)}{\sigma_1(CA^{-1}B)},$$

and an argument identical to that used in the proof of the first claim of Theorem 1 shows that the maximization in (1) can be limited to $[0,\rho_M]$ (and ρ_M is continuous in (A,B,C)). It follows that

$$r_{\mathbf{R}}(A,B,C)^{-1} = \max_{\omega \in [0,\rho^*]} \mu_{\mathbf{R}}(C(j\omega I - A)^{-1}B)$$

where $\rho^* := \min\{\rho_P, \rho_M\}$.

3 Examples

In the first 3 examples, borrowed from [8], $m = p = n$ and $C = B = I$.

Example 1:

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -0.01 & 0 & 0 \\ 0 & -2 & 0 & 0 & -0.01 & 0 \\ 0 & 0 & -10 & 0 & 0 & -0.01 \end{pmatrix}.$$

The global maximizer in (1) is 1.4142; $\rho_P = 6.2301$, $\rho_M = 10.995$. □

Example 2:

$$A = \begin{pmatrix} -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}.$$

The global maximizer in (1) is 1; $\rho_P = 3.2075$, $\rho_M = 3.0000$. □

Example 3:

$$A = \begin{pmatrix} -1 & 1000 & 0.001 \\ -1 & -1 & 0 \\ 1 & 1 & -100 \end{pmatrix}.$$

The global maximizer in (1) is 31.391 (it is erroneously printed in [8] as 3.1624); $\rho_P = 49.7810$, $\rho_M = 1001.0000$. □

Our last example is taken from [2].

Example 4:

$$A = \begin{pmatrix} 79 & 20 & -30 & -20 \\ -41 & -12 & 17 & 13 \\ 167 & 40 & -60 & -38 \\ 33.5 & 9 & -14.5 & -11 \end{pmatrix},$$

$$B = \begin{pmatrix} 0.2190 & 0.9347 \\ 0.047 & 0.3835 \\ 0.6789 & 0.5194 \\ 0.6793 & 0.8310 \end{pmatrix}, \quad C = \begin{pmatrix} 0.0346 & 0.5297 & 0.0077 & 0.0668 \\ 0.0533 & 0.6711 & 0.3834 & 0.4175 \end{pmatrix}.$$

The global maximizer in (1) is 1.3; $\rho_P = 13.9073$, $\rho_M = 216.8366$. □

4 Proofs

Our proof of Proposition 1 makes use of the following result.

Lemma 1: For any $\omega \in \mathbf{R}$,

$$|\det(j\omega I - A)| \geq \frac{\sqrt{2}}{2}\left(\omega^n + a_{n-1}\omega^{n-1} + \cdots + (-1)^{\lfloor \frac{k}{2} \rfloor} a_{n-k}\omega^{n-k} + \cdots + (-1)^{\lfloor \frac{n}{2} \rfloor} a_0\right).$$

Proof:

$$\begin{aligned} |\det(j\omega I - A)|^2 &= (\Re e(\det(j\omega I - A)))^2 + (\Im m(\det(j\omega I - A)))^2 \\ &= \left(a_0 - a_2\omega^2 + \cdots + (-1)^{\lfloor \frac{n}{2} \rfloor} a_{2\lfloor \frac{n}{2} \rfloor}\omega^{2\lfloor \frac{n}{2} \rfloor}\right)^2 + \\ &+ \left(a_1\omega - a_3\omega^3 + \cdots + (-1)^{\lfloor \frac{n-1}{2} \rfloor} a_{2\lfloor \frac{n-1}{2} \rfloor + 1}\omega^{2\lfloor \frac{n-1}{2} \rfloor + 1}\right)^2. \end{aligned}$$

Since for any two real numbers a and b, $a^2 + b^2 \geq \frac{1}{2}(|a| + |b|)^2$, it follows that

$$\begin{aligned} &|\det(j\omega I - A)| \\ &\geq \tfrac{\sqrt{2}}{2}\left(|a_0 + \cdots + (-1)^{\lfloor \frac{n}{2} \rfloor} a_{2\lfloor \frac{n}{2} \rfloor}\omega^{2\lfloor \frac{n}{2} \rfloor}| + |a_1\omega + \cdots + (-1)^{\lfloor \frac{n-1}{2} \rfloor} a_{2\lfloor \frac{n-1}{2} \rfloor + 1}\omega^{2\lfloor \frac{n-1}{2} \rfloor + 1}|\right) \\ &= \tfrac{\sqrt{2}}{2}\left(|(-1)^{\lfloor \frac{n}{2} \rfloor}(a_0 + \cdots + (-1)^{\lfloor \frac{n}{2} \rfloor} a_{2\lfloor \frac{n}{2} \rfloor}\omega^{2\lfloor \frac{n}{2} \rfloor})| + |(-1)^{\lfloor \frac{n-1}{2} \rfloor}(a_1\omega + \cdots + (-1)^{\lfloor \frac{n-1}{2} \rfloor} a_{2\lfloor \frac{n-1}{2} \rfloor + 1}\omega^{2\lfloor \frac{n-1}{2} \rfloor + 1})|\right) \end{aligned}$$

and the claim follows from the triangle inequality in $\mathbf{C}$. □

Proof of Proposition 1: Let $\hat{\omega}$ be such that (4) holds. It follows from (3) and the triangle inequality ($\sigma_1(\cdot)$ is a norm) that

$$\sigma_1(CA^{-1}B) \leq \frac{\sigma_1(CR_{n-1}B)\hat{\omega}^{n-1} + \cdots + \sigma_1(CR_1B)\hat{\omega} + \sigma_1(CR_0B)}{|\det(\mathrm{j}\hat{\omega}I - A)|}, \tag{7}$$

or equivalently

$$|\det(\mathrm{j}\hat{\omega}I - A)| \leq \frac{\sigma_1(CR_{n-1}B)\hat{\omega}^{n-1} + \cdots + \sigma_1(CR_1B)\hat{\omega} + \sigma_1(CR_0B)}{\sigma_1(CA^{-1}B)}. \tag{8}$$

In view of Lemma 1 and the definition of the p_i's, this implies that

$$P(\hat{\omega}) \leq 0. \tag{9}$$

Since $P(\omega)$ goes to infinity as ω tends to infinity, the claims follow. □

Proof of the Theorem 1: First note that, for any $X, Y \in \mathbf{R}^{n\times n}$,

$$\sigma_1(X + \mathrm{j}Y) = \sigma_2\left(\begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}\right). \tag{10}$$

Indeed $X + \mathrm{j}Y$ and $X - \mathrm{j}Y$ have the same singular values, so that

$$\sigma_1\left(\begin{bmatrix} X - \mathrm{j}Y & 0 \\ 0 & X + \mathrm{j}Y \end{bmatrix}\right) = \sigma_2\left(\begin{bmatrix} X - \mathrm{j}Y & 0 \\ 0 & X + \mathrm{j}Y \end{bmatrix}\right) = \sigma_1(X + \mathrm{j}Y),$$

and

$$\begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} = U \begin{bmatrix} X - \mathrm{j}Y & 0 \\ 0 & X + \mathrm{j}Y \end{bmatrix} V^H$$

where U and V are unitary matrices given by

$$U = V = \frac{\sqrt{2}}{2}\begin{bmatrix} I & \mathrm{j}I \\ \mathrm{j}I & I \end{bmatrix},$$

From (10) and from the definition (2) of $\mu_{\mathbf{R}}$ it follows that

$$\mu_{\mathbf{R}}(CA^{-1}B) = \sigma_1(CA^{-1}B)$$

and, for any $\omega \geq 0$,

$$\begin{aligned} \mu_{\mathbf{R}}(C(\mathrm{j}\omega I - A)^{-1}B) &\leq \sigma_2\left(\begin{bmatrix} \Re e(C(\mathrm{j}\omega I - A)^{-1}B) & -\Im m(C(\mathrm{j}\omega I - A)^{-1}B) \\ \Im m(C(\mathrm{j}\omega I - A)^{-1}B) & \Re e(C(\mathrm{j}\omega I - A)^{-1}B) \end{bmatrix}\right) & (11) \\ &= \sigma_1(C(\mathrm{j}\omega I - A)^{-1}B). & (12) \end{aligned}$$

In view of Proposition 1, this implies that any $\hat{\omega} \geq 0$ such that

$$\mu_{\mathbf{R}}(C(\mathrm{j}\hat{\omega}I - A)^{-1}B) \geq \mu_{\mathbf{R}}(CA^{-1}B)$$

must satisfy

$$\hat{\omega} \leq \rho_P.$$

Thus the level set $\{\omega \geq 0 : \mu_{\mathbf{R}}(C(j\omega I - A)^{-1}B) \geq \mu_{\mathbf{R}}(CA^{-1}B)\}$ (which is nonempty since it contains the origin) is contained in the interval $[0, \rho_P]$. The first claim is a direct consequence of this fact. Now let $\hat{\rho} := \rho_P(\hat{A}, \hat{B}, \hat{C})$. Uniform continuity of $\sigma_1(C(j\omega I - A)^{-1}B)$ over compact sets preserving Hurwitz stability of A implies that, given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|\sigma_1(C(j\omega I - A)^{-1}B) - \sigma_1(\hat{C}(j\omega I - \hat{A})^{-1}\hat{B})| < \epsilon/2 \quad \forall \omega \in [0, \hat{\rho}] \tag{13}$$

whenever $||(A, B, C) - (\hat{A}, \hat{B}, \hat{C})|| < \delta$, where $||\cdot||$ denotes an arbitrary norm. Now let (A, B, C) be such that $||(A, B, C) - (\hat{A}, \hat{B}, \hat{C})|| < \delta$, and let $\rho := \rho_P(A, B, C)$. We show that

$$\max_{\omega \in [0,\rho]} \mu_{\mathbf{R}}(\hat{C}(j\omega I - \hat{A})^{-1}\hat{B}) \geq r_{\mathbf{R}}(\hat{A}, \hat{B}, \hat{C})^{-1} - \epsilon \tag{14}$$

thus proving the second claim. If $\rho \geq \hat{\rho}$, the claim follows trivially. Thus suppose $\rho < \hat{\rho}$. From (13) and Proposition 1 (at (A, B, C)), it follows that

$$\sigma_1(\hat{C}(j\omega I - \hat{A})^{-1}\hat{B}) \leq \sigma_1(\hat{C}\hat{A}^{-1}\hat{B}) + \epsilon \quad \forall \omega \in (\rho, \hat{\rho}]$$

which, in view of (12) implies that

$$\mu_{\mathbf{R}}(\hat{C}(j\omega I - \hat{A})^{-1}\hat{B}) \leq \sigma_1(\hat{C}\hat{A}^{-1}\hat{B}) + \epsilon \quad \forall \omega \in (\rho, \hat{\rho}].$$

It follows that

$$r_{\mathbf{R}}(\hat{A}, \hat{B}, \hat{C})^{-1} \leq \max\{\max_{\omega \in [0,\rho]} \mu_{\mathbf{R}}(\hat{C}(j\omega I - \hat{A})^{-1}\hat{B}), \sigma_1(\hat{C}\hat{A}^{-1}\hat{B}) + \epsilon\}.$$

Since

$$\max_{\omega \in [0,\rho]} \mu_{\mathbf{R}}(\hat{C}(j\omega I - \hat{A})^{-1}\hat{B} \geq \mu_{\mathbf{R}}(\hat{C}\hat{A}^{-1}\hat{B}) = \sigma_1(\hat{C}\hat{A}^{-1}\hat{B}),$$

(14) follows. □

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