ABSTRACT

Title of dissertation: MIXING, FLOCKING AND COOPERATION: ANALYTICAL STUDIES OF TRANSPORT PHENOMENA IN BIOLOGY

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Transport behavior widely takes place in biological phenomena. In this thesis, I study the delicate interactions between transportation and other forces in biology, diffusion, alignment, cooperation included. The thesis focuses on three topics: (i) chemotaxis in moving fluid; (ii) flocking of birds and fish; (iii) multi-species chemotaxis and multi-species flocking dynamics.

We use the Patlak-Keller-Segel equations with additional advection to model the chemotaxis phenomena in the moving fluid. It is well-known that if there is no underlying fluid transport, the total number of cells in the environment determines the long-time behavior of the dynamics. If the number is large enough, cells will concentrate to form clusters and cause break-downs of the model. We discover that external strong fluid flow has the potential to suppress the possible blow-up in the system.

We use the Cucker-Smale model and the Motsch-Tadmor model to describe the flocking behavior of fish or birds. While the well-posedness theory of the PDE in one dimension is well-understood, little is known in dimension two. We give explicit sufficient condition to guarantee the existence of unique global strong solutions and clarify the role played by the stretching and vorticity.

In the last part of the thesis we introduce multi-species concepts into the chemotaxis models and flocking models. We discover new conditions to guarantee the well-posedness of the multi-species Patlak-Keller-Segel systems and the multi-species Cucker-Smale models.

MIXING, FLOCKING AND COOPERATION: ANALYTICAL STUDIES OF TRANSPORT PHENOMENA IN BIOLOGY

by

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Dedicated to my grandparents Yuan Hao, Hu Yun, He Yongke and Xu Fumei.

Dedicated to my parents He Yijun and Yuan Haijing.

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List of Abbreviations

CS	Cucker-Smale equations
MT	Motsch-Tadmor equations
PKS	Patlak-Keller-Segel equations

Chapter 1: Introduction: Transport Phenomena in Biology

The main focus of my research is the Partial Differential Equations (PDE) modeling *collective behavior* in biology. My goal is to give faithful descriptions of these models.

When studying the collective behavior of a large group of agents, say, bacteria, fish or birds, five types of effects play important roles: aggregation, diffusion, advection, alignment and social cooperation. Aggregation describes the tendency for agents to concentrate. Diffusion characterizes the random aspect of the agents' movement. Advection indicates the fluid transport effect in the dynamics. Alignment represents the averaging effect among agents. Social cooperation, on the other hand, reflects interaction between species. Different combinations of these five effects give rise to different models in biology. My work covers three important scenarios: *Chemotaxis in moving fluid* reflects a balance among aggregation, diffusion, and advection; *Flocking* emerges from competition between advection and alignment; *Multi-species dynamics* arises as an interaction of all five effects.

1.1 Chemotaxis in moving fluid

1.1.1 Overview

Bacteria emit chemical signals to attract others. This phenomenon is called the Chemotaxis. It can be viewed as a delicate balance between aggregation and diffusion. Individual agent directs its movement according to the chemical concentration gradient in the environment. This represents the aggregation. On the other hand, the tendency of agents to do random motions always exists. This reflects the diffusion.

The parabolic-elliptic *Patlak-Keller-Segel equation with advection* (PKS) is designed to model Chemotaxis in moving fluid:

$$\begin{cases} \partial_t n = \overbrace{\Delta n}^{\text{diffusion}} - \overbrace{\nabla\cdot(\nabla c n)}^{\text{aggregation}} - \overbrace{Au\cdot\nabla n}^{\text{advection}}, \\ -\Delta c = n; \quad \nabla\cdot u = 0. \end{cases} \tag{1.1.1}$$

The equation governs the evolution of the bacteria density n and the chemical concentration c subject to initial density $n(t=0,x) = n_0(x), x \in \mathbb{R}^2$. The divergence free vector field Au represents the underlying fluid velocity. When $Au \equiv 0$, the system is the classical parabolic-elliptic Patlak-Keller-Segel equation modeling chemotaxis in a static environment; see e.g. [102], [77]. In this case, the first part of (1.1.1) describes the time evolution of the micro-organism density n subject to diffusion and chemo-attractant-triggered aggregation. The second part of (1.1.1) models the time evolution of the chemo-attractant secreted by the micro-organism. The case $A \neq 0$ models the microorganisms suspended in a fluid flow: the elliptic equation

$-\Delta c = n - \bar{n}$ arises as the formal limit as $\epsilon \to 0$ of the advection-diffusion equation

$$\partial_t c + A\partial_x c = \epsilon^{-1}(\Delta c + n),$$

under the assumption that $\epsilon A \ll 1$. In particular, (**??**) requires that the time-scale of equilibriation of c is faster than the transport due to the fluid flow. It is worth mentioning that the model (1.1.1) is one among many attempts to take into account the underlying fluid advection effect, see, e.g. [93], [94], [91], [50], [60].

The static Patlak-Keller-Segel equation ($Au \equiv 0$) and its variations have received considerable mathematical attention over the years, for example, see the review [73] or some of the representative works [22,24–28,39,71,75,99] and the references therein. It is well-known that if the total number of bacteria is small in the sense that the L^1 norm of the initial density is less than 8π ($||n_0||_1 < 8\pi$), the equation (1.1.1) admits global smooth solutions. On the other hand, if the total number of bacteria is large enough, i.e., the L^1 norm of the initial density cross the critical threshold of 8π, aggregation dominates diffusion. As a result, a large number of bacteria concentrate in one point, leading to the blow-up of the solution at a finite time [28]. When $M = 8\pi$, aggregation and diffusion exactly balance each other and solutions with finite second moments form Dirac mass as time approach infinity [27].

I address the problem whether there exist simple fluid flows that suppress the possible blow-up of the equation with supercritical mass. The basic idea is that special fluid flows have either enhanced dissipation or fast splitting effect. These fluid effects restore the balance between aggregation and diffusion and lead to suppression of blow-up.

I mention another result of non-static PKS equation with strong fluid advection preventing the chemotactic blow-up. In [80], A. Kiselev ans X. Xu exploited relaxation enhancing of a vector field u with a large enough amplitude in order to enforce global smooth solution. Here regularity follows due to a *mixing property* of u over $\mathbb{T}^2$ and $\mathbb{T}^3$.

1.1.2 Blow-up and well-posedness theory of the static parabolic-elliptic PKS equation

In this subsection, I review the existence and blow-up theory of the static parabolic-elliptic Patlak-Keller-Segel equation

$$\partial_t n = \Delta n - \nabla \cdot (\nabla c n), \quad -\Delta c = n, \quad n(x,0) = n_0(x). \tag{1.1.2}$$

Since the chemical density solves the Poisson equation on $\mathbb{R}^2$, we could solve it explicitly:

$$c = (-\Delta)^{-1} n = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log(|x-y|) n(y) dy. \tag{1.1.3}$$

If we plug the expression (1.1.3) into the original equation (1.1.2), we end up with one single equation with non-local aggregation nonlinearity:

$$\partial_t n = \Delta n - \nabla \cdot (\nabla (-\Delta)^{-1} n n), \quad n(0,x) = n_0(x). \tag{1.1.4}$$

Since convolution with the fundamental solution of Laplacian is a nonlocal operation, the equation (1.1.1) is a nonlocal nonlinear parabolic PDE.

There are three basic quantities intimately related to the behavior of the equation (1.1.1) - mass, second moment and free energy. First we define the mass as the

L^1 norm of the bacteria density, i.e., $M := ||n||_1$. By the divergence structure of the equation and maximum principle, we have that the density n is positive and the mass is preserved, i.e.,

$$||n(\cdot,t)||_{L^1} = ||n_0(\cdot,t)||_{L^1} = M. \tag{1.1.5}$$

Next we introduce the second moment V:

$$V[n] := \int_{\mathbb{R}^2} n|x|^2 dx. \tag{1.1.6}$$

Through a Virial identity type argument, we can use the second moment to show that all physical solutions with mass greater than 8π blow up in finite time. Finally, the equation (1.1.2) has a naturally dissipative free energy E,

$$E[n(t)] := \int n(t)\log n(t) - \frac{1}{2}\int c(t)n(t) \leq E[n_0], \quad \forall t \geq 0. \tag{1.1.7}$$

It is used to derive the optimal existence results for the system.

With these basic quantities introduced, we present the blow-up result for the equation (1.1.2).

Lemma 1.1.1. *Consider the equation (1.1.2) with C^∞ initial data and finite second moment $V[n_0] < \infty$. Suppose $M = ||n_0||_{L^1} > 8\pi$, then the solution must blow-up in finite time.*

Proof. We prove the Lemma using contradiction. Assume that the solution is smooth for all time. Applying the expression (1.1.3) we can formally calculate the time evolution of the second moment:

$$\begin{aligned}\frac{d}{dt}\int n|x|^2dx =&4\int ndx - \frac{1}{\pi}\iint n(x)\frac{(x-y)\cdot x}{|x-y|^2}n(y)dxdy\\ =&4M-\frac{1}{2\pi}\iint n(x)n(y)\frac{|x-y|^2}{|x-y|^2}dxdy\\ =&4M-\frac{M^2}{2\pi}=4M(1-\frac{M}{8\pi}).\end{aligned}$$

The detailed justification can be made by using smooth compactly supported function to approximate the weight function $|x|^2$. Since $M>8\pi$, we see that the time derivative of the second moment is a strictly negative constant. As a result, there exists a first time $T_\star<\infty$ such that $V[n(T_\star)]=0$. On the other hand, $||n(t)||_{L^1}\equiv M>0$. The only possible density with positive mass and zero second moment is the Dirac mass, which is not a smooth function. As a result, we reach a contradiction. This completes the proof of the lemma. □

Next, we present the global existence theory of free energy solutions of the PKS equation (1.1.2) subject to subcritical mass $M<8\pi$. The free energy solutions are special weak solutions to the PKS equation. To properly define them, we first need the weak formulation of the PKS equation.

Definition 1 (Weak formulation)**.** *The function $n:\mathbb{R}^+\times\mathbb{R}^2\to\mathbb{R}^+$ is said to be the weak solution to (1.1.2) if for $\forall\varphi\in C_c^\infty(\mathbb{R}^2)$, the following equation hold:*

$$\frac{d}{dt}\int_{\mathbb{R}^2}\varphi ndx=\int_{\mathbb{R}^2}\Delta\varphi ndx-\frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{(\nabla\varphi(x)-\nabla\varphi(y))\cdot(x-y)}{|x-y|^2}n(x,t)n(y,t)dxdy. \tag{1.1.8}$$

Second we need to introduce the free energy dissipation relation. The following lemma is essential.

Lemma 1.1.2. *Consider the PKS equation (1.1.2). If the solution is smooth enough, the free energy $E[n(t)]$ is decreasing. Moreover, the following relation is satisfied*

$$E[n_0] \geq E[n(t)] + \int_0^t n(x,s)|\nabla \log n(x,s) - \nabla c(x,s)|^2 dxds, \quad \forall t \in [0,T]. \tag{1.1.9}$$

Proof. The time evolution of the free energy (1.1.7) can be computed,

$$\frac{d}{dt}E[n](t) = -\int n(\nabla \log n - \nabla c)\cdot(\nabla \log n - \nabla c)dx = -\int n|\nabla \log n - \nabla c|^2 dx \leq 0.$$

Time integration yields the lemma. □

Free energy solutions is weak solutions with free energy dissipation constraint (1.1.9).

Definition 2 (Free energy solutions)**.** *For any weak solutions n to the equation (1.1.2) subject to initial data n_0, they are the free energy solutions to (1.1.2) if the following free energy dissipation inequality holds on the time interval $[0,T_\star)$*

$$E[n(t)] + \int_0^t \int_{\mathbb{R}^2} n|\nabla \log n - \nabla c|^2 dxds \leq E[n_0], \quad \forall t \in [0,T_\star). \tag{1.1.10}$$

Now we sketch the proof for the following classical theorem:

Theorem 1. *([28]) Assume that $n_0 \in L^1_+(\mathbb{R}^2,(1+|x|^2)dx)$ and $n_0 \log n_0 \in L^1(\mathbb{R}^2,dx)$. If $M < 8\pi$, then the PKS system (1.1.2) has a global non-negative free energy solution n with initial data n_0 such that*

$$(1+|x|^2+|\log n|)n \in L^\infty_{loc}(\mathbb{R}^+, L^1(\mathbb{R}^2)),$$
$$\int_0^t \int_{\mathbb{R}^2} n|\nabla \log n - \nabla c|^2 dxdt < \infty,$$
$$\int_{\mathbb{R}^2} |x|^2 n(x,t)dx = \int_{\mathbb{R}^2} |x|^2 n_0(x)dx + 4M\left(1-\frac{M}{8\pi}\right)t$$

for $\forall t > 0$. Moreover $n \in L^\infty_{loc}([\delta,\infty), L^p(\mathbb{R}^2))$ for any $p \in (1,\infty)$ and any $\delta > 0$.

We break the proof of Theorem 1 into steps. First we introduce the regularized solutions to the equation and the local existence theorem. Secondly, we present the blow up criterion for the Keller-Segel system (1.1.2). Finally, we show that the blow-up criteria will not be satisfied if the mass is less than 8π. This concludes the proof of the theorem.

The regularized PKS system is defined as follows:

$$\frac{\partial n^\epsilon}{\partial t} + \nabla \cdot (n^\epsilon \nabla c^\epsilon) = \Delta n^\epsilon, \quad c^\epsilon := K^\epsilon * n, \qquad x \in \mathbb{R}^2, t > 0, \tag{1.1.11}$$

with the regularized kernel K^ϵ given by

$$K^\epsilon(z) := K^1\left(\frac{z}{\epsilon}\right) - \frac{1}{2\pi}\log \epsilon, \qquad K^1(z) := \begin{cases} -\frac{1}{2\pi}\log|z|, & \text{if} \quad |z| \geq 4, \\ 0, & \text{if} \quad |z| \leq 1. \end{cases} \tag{1.1.12}$$

Since $||\nabla K^\epsilon||_\infty \leq C_\epsilon < \infty$, it follows from standard convection-diffusion equation theory that the solutions to the equation (1.1.11) exist for all time. We refer the interested reader to the paper [28] for more details.

Now the local existence theorem is expressed as follows:

Proposition 1 (Local existence theorem, [27, 28]). *Suppose $\{n^\epsilon\}_{\epsilon \geq 0}$ are the solutions of the regularized equation (1.1.11) on $[0, T^*)$. If the entropy of the regularized solutions $\{S[n^\epsilon](t) := \int n^\epsilon \log n^\epsilon dx\}_\epsilon$ are bounded from above uniformly in ϵ and in $t \in (0, T^*)$, then the cluster points of $\{n^\epsilon\}_{\epsilon>0}$, in the $L^2_t L^2_x$ strong topology, are non-negative free energy solutions of the PKS system (1.1.2) on $[0, T^*)$.*

We skip the proof of the proposition. This finishes the first step of the proof.

In the second step, we present the blow-up criterion for free energy solutions.

Proposition 2 (Blow-up criterion for free-energy solutions, [27, 28])**.** *Consider the PKS equation* (1.1.2) *subject to the initial condition* $(1+|x|^2)n_0 \in L^1_+(\mathbb{R}^2)$, $n_0 \log n_0 \in L^1(\mathbb{R}^2)$*. Then there exists a maximal existence time* $T^* > 0$ *of a free energy solution to the PKS system* (1.1.2)*. Moreover, if* $T^* < \infty$ *then*

$$\lim_{t \to T^*} \int_{\mathbb{R}^2} n \log n dx = \infty.$$

We observe that if the entropy $S[n](t) = \int n(t) \log n(t)$ is finite for all $t < \infty$, the free energy solution exists for all finite time.

Finally, we use the logarithmic Hardy-Littlewood-Sobolev inequality to get an a priori bound on the entropy $S[n]$.

Theorem 2 (Logarithmic Hardy-Littlewood-Sobolev Inequality, [32])**.** *Let* f *be a nonnegative function in* $L^1(\mathbb{R}^2)$ *such that* $f \log f$ *and* $f \log(1 + |x|^2)$ *belong to* $L^1(\mathbb{R}^2)$*. If* $\int_{\mathbb{R}^2} f dx = M$*, then*

$$\int_{\mathbb{R}^2} f \log f dx + \frac{2}{M} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) \log |x - y| dx dy \geq -C(M) \tag{1.1.13}$$

with $C(M) := M(1 + \log \pi - \log M)$*.*

One can combine the decay of the free energy E (1.1.9) and the logarithmic Hardy-Littlewood-Sobolev inequality to get the A priori bound on the entropy for all finite time. The arguement is as follows. By the free energy dissipation (1.1.9), we obtain that

$$\begin{aligned} E[n_0] \geq E[n(t)] =& \left(1 - \frac{M}{8\pi}\right) \int n \log n dx + \frac{M}{8\pi} \Bigg(\int n \log n dx \\ &+ \frac{2}{M} \iint n(x) n(y) \log |x - y| dx dy \Bigg) \\ \geq& \left(1 - \frac{M}{8\pi}\right) S[n(t)] - \frac{M}{8\pi} C(M). \end{aligned}$$

As a result, we have that

$$S[n(t)] \leq \frac{E[n_0] + \frac{M}{8\pi}C(M)}{1 - \frac{M}{8\pi}} < \infty.$$

In conclusion, we obtain the uniform in time bound on the entropy for any (regularized) solutions with total mass less than 8π. This concludes the proof of Theorem 1.

1.1.3 Hypocoercivity and the enhanced dissipation of shear flows

In chapter 2 and chapter 4 we will use shear flows to suppress the blow-up of PKS equations. Since shear flows have enhanced dissipation effect, it can be used to restore the balance between diffusion and aggregation. To exploit the enhanced dissipation effect, we need to construct specific functional to capture it. We use the Hypocoercivity technique of [8], which builds on the earlier work of [6, 61]. The concept of Hypocoercivity comes from the study of the kinetic equations. As outlined in [115], hypocoercivity techinques are based on finding an energy which extracts the fact that the quadratic quantity $A^{-1}\|\nabla f\|_{L^2}^2 + \|u'\partial_x f\|_{L^2}^2$ is much 'more coercive' than the standard H^1 norm. In this subsection, we focus on the passive scalar equation:

$$\partial_t f + u(y)\partial_x f = \frac{1}{A}\Delta f, \quad (x, y) \in \mathbb{T} \times \mathbb{R}, \mathbb{T}^2. \tag{1.1.14}$$

subject to initial data $f(0,\cdot) = f_0(\cdot)$. Here f is the bacteria density, $\frac{1}{A}$ is the viscosity. We focus on the vanishing viscosity scenario when $\frac{1}{A} \ll 1$.

Since the solution f is periodic in the x variable, it is reasonable to consider the

Fourier transform of the equation (1.1.14) *only in the x variable*:

$$\partial_t \widehat{f}_k + \overbrace{u(y)ik\widehat{f}_k}^{\text{Conservative part}} = \overbrace{\frac{1}{A}\Delta_y \widehat{f}_k - \frac{1}{A}|k|^2\widehat{f}_k}^{\text{Dissipative part}}, \quad k \neq 0. \tag{1.1.15}$$

If we define $P := \partial_y$, $Q := -iu(y)k$, we can rewrite (1.1.15) as

$$\partial_t \widehat{f}_k = P^* P \widehat{f}_k - \frac{|k|^2}{A}\widehat{f}_k - Q\widehat{f}_k. \tag{1.1.16}$$

Standard hypocoercivity functional has the form:

$$\Phi_k[\widehat{f}] = ||\widehat{f}_k||_2^2 + \alpha||P\widehat{f}_k||_2^2 + \beta Re\langle [P,Q]\widehat{f}_k, P\widehat{f}_k\rangle + \gamma||[P,Q]\widehat{f}_k||_2^2, \tag{1.1.17}$$

where $[P,Q]$ is the commutator

$$[P,Q] = PQ - QP = iu'(y)k. \tag{1.1.18}$$

Now following [8], we define the Hypocoercivity functional Φ k-by-k,

$$\Phi_k[f(t)] = ||\widehat{f}_k(t)||_2^2 + ||\sqrt{\alpha}\partial_y \widehat{f}_k(t)||_2^2 + 2kRe\langle i\beta u'\widehat{f}_k(t), \partial_y \widehat{f}_k(t)\rangle + |k|^2||\sqrt{\gamma}u'\widehat{f}_k(t)||_2^2; \tag{1.1.19}$$

Here α, β, and γ are k-dependent constants depending on the property of the shear flow profile $u(y)$. For general shear flows, α, β and γ are k-dependent functions of y. We only focus on two different cases, i.e., nondegenerate shear flows and strictly monotone shear flows[1]. For nondegenerate shear flows, the choices of α, β and γ satisfy

$$\alpha(A,k) = \epsilon_\alpha A^{-1/2}|k|^{-1/2}, \quad \beta(A,k) = \epsilon_\beta |k|^{-1}, \quad \gamma(A,k) = \epsilon_\gamma A^{1/2}|k|^{-3/2}; \tag{1.1.20a}$$

[1]Nondegenerate shear flows are shear flows without degenerate critical points for the function $u(y)$. Strictly Monotone shear flows are shear flows with strictly monotone $u(y)$.

for strictly monotone shear flows, the choices of α, β and γ are as follows:

$$\alpha(A,k) = \epsilon_\alpha A^{-2/3}\,|k|^{-2/3} \quad \beta(A,k) = \epsilon_\beta A^{-1/3}\,|k|^{-4/3} \quad \gamma(A,k) = \epsilon_\gamma\,|k|^{-2}\,, \tag{1.1.21a}$$

where ϵ_α, ϵ_β, and ϵ_γ are small constants depending only on u chosen in [8]. The parameters ϵ_α, ϵ_β, and ϵ_γ are tuned such that,

$$\Phi_k[n] \approx ||\widehat{n}_k||_2^2 + ||\sqrt{\alpha}\partial_y \widehat{n}_k||_2^2 + |k|^2||\sqrt{\gamma}u'\widehat{n}_k||_2^2, \tag{1.1.22}$$

and hence

$$\|\widehat{n}_k\|_2^2 + A^{-1/2}\,|k|^{-1/2}\,\|\partial_y \widehat{n}_k\|_2^2 \lesssim \Phi_k[n] \lesssim \|\widehat{n}_k\|_2^2 + |k|^{1/2}\,A^{1/2}\,\|\widehat{n}_k\|_2^2 + A^{-1/2}\,|k|^{-1/2}\,\|\partial_y \widehat{n}_k\|_2^2\,. \tag{1.1.23}$$

As a result, $\Phi_k(t)$ is equivalent to the H^1 norm of n_k but with constants that depend on A and k. Define the following orthogonal projections:

$$f_0(t,y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t,x,y)dx, \quad f_{\neq}(t,x,y) = f(t,x,y) - f_0(t,y),$$

for "zero frequency" and "non-zero frequency". We can define a functional Φ acting on $f_{\neq}$ as follows:

$$\begin{aligned}\Phi[f_{\neq}(t)] = \sum_{k\neq 0}\Phi_k[f(t)] &= ||f_{\neq}(t)||_2^2 + ||\sqrt{\alpha}\partial_y f_{\neq}(t)||_2^2 + 2\langle \beta u'\partial_x f_{\neq}(t), \partial_y f_{\neq}(t)\rangle \\ &+ ||\sqrt{\gamma}u'|\partial_x| f_{\neq}(t)||_2^2. \end{aligned}\tag{1.1.24}$$

The primary result of [8] is described as follows. Consider the passive scalar equation (1.1.15) subject to nondegenerate or strict monotonic $u(y)$, the norms $\Phi_k[f(t)]$ of the solutions f satisfy the following estimates for some small constants $\tilde{\epsilon}$ independent of k, A (but depending on u):

$$\text{Nondegenerate shear flow: } \frac{d}{dt}\Phi_k[f(t)] \leq -\tilde{\epsilon}\frac{|k|^{1/2}}{A^{1/2}}\Phi_k[f(t)]; \tag{1.1.25}$$

$$\text{Strictly Monotone shear flow: } \frac{d}{dt}\Phi_k[f(t)] \leq -\tilde{\epsilon}\frac{|k|^{2/3}}{A^{1/3}}\Phi_k[f(t)]. \tag{1.1.26}$$

As a result, Φ_k decays with a rate $\approx \frac{|k|^{1/2}}{A^{1/2}}$ (or $\frac{|k|^{2/3}}{A^{1/3}}$), which is much faster than the heat decay rate $\approx \frac{|k|^2}{A}$ if A is large. Since the $\Phi[n_{\neq}]$ dominates the L^2 norm, we have that $||n_{\neq}||_2^2$ is decaying faster than the heat decay. This is the enhanced dissipation effect of the shear flow.

1.1.4 Fast-splitting scenario and the escape from second moment collapse

In the paper [67], we exploit the fast splitting property of the hyperbolic flow to suppress the blow-up of the equation (1.1.1):

$$u(x) := A(-x_1, x_2). \tag{1.1.27}$$

Recall that if there is no ambient fluid flow, the second moment of any solutions with supercritical mass $M > 8\pi$ and finite initial second moment will decrease to zero in finite time, which in turn yields that the solutions must blow up in finite time. In this section, we will show that if the amplitude A of the hyperbolic flow is chosen sufficiently large, the second moment of the solution (1.1.6) will not collapse to zero.

Lemma 1.1.3. *Let $n(x,t)$ be the solution of (3.1.1) with vector $u(x) = A(-x_1, x_2)$, subject to initial data n_0 such that $W_0 := \int_{\mathbb{R}^2} n_0(x)(x_2^2 - x_1^2)dx$ is strictly positive.*

Then, if A is chosen large enough, the second moment of the (classical) solution $V(t) = \int_{\mathbb{R}^2} n(x,t)|x|^2 dx$ increases in time.

Proof. First, the time evolution of $V(t)$ can be calculated as follows:

$$\begin{cases} \frac{d}{dt}V = 4M\left(1 - \frac{M}{8\pi}\right) + 2\int x \cdot u\, n(x,t)dx \\ \qquad = 4M\left(1 - \frac{M}{8\pi}\right) + 2AW, \qquad W(t) = \int_{\mathbb{R}^2} (-x_1^2 + x_2^2) n(x,t)dx. \end{cases} \tag{1.1.28}$$

Next, we compute the time evolution of $W(t)$

$$\begin{aligned} \frac{d}{dt}W &= \int (-x_1^2 + x_2^2)\nabla \cdot (\nabla n - \nabla cn - un)dx \\ &= -\frac{1}{2\pi}\iint n(x,t)(-2x_1, 2x_2) \cdot \frac{x-y}{|x-y|^2} n(y,t)dxdy + 2AV \\ &= -\frac{1}{2\pi}\iint \frac{-(x_1-y_1)^2 + (x_2-y_2)^2}{(x_1-y_1)^2 + (x_2-y_2)^2} n(x,t)n(y,t)dxdy + 2AV, \end{aligned}$$

where the last step follows by symmetrization. Since the first term on the right is bounded from below by $-\frac{1}{2\pi}M^2$, we have

$$\frac{d}{dt}W \geq -\frac{1}{2\pi}M^2 + 2AV. \tag{1.1.29}$$

Finally, notice that since W_0 (and hence V_0) are assumed strictly positive, we can choose A large such that

$$AV_0 - \frac{1}{4\pi}M^2 \geq 0, \quad AW_0 + 2M\left(1 - \frac{M}{8\pi}\right) \geq 0. \tag{1.1.30}$$

Combining (1.1.30), (1.1.28) and (1.1.29) yields that $W(t) > 0,\ V(t) > 0$. ∎

1.1.5 Blow-up and well-posedness theory of the parabolic-parabolic PKS equation

In this subsection, we consider the two-dimensional static parabolic-parabolic Patlak-Keller-Segel equations, which model the chemotaxis phenomena:

$$\partial_t n + \nabla \cdot (n \nabla c) = \Delta n, \tag{1.1.31a}$$

$$\partial_t c = \Delta c + n - c, \tag{1.1.31b}$$

$$n(x,0) = n_0(x), \quad c(x,0) = c_0(x), \quad x \in \mathbb{R}^2. \tag{1.1.31c}$$

It is well known that the Patlak-Keller-Segel equation (1.1.31) is L^1 critical and the L^1 norm of the solution $M := ||n||_1$ is preserved. The existing results for the parabolic-parabolic case can be summarized as follows. In the sub-critical case $M < 8\pi$, the global well-posedness of the free energy solution to (4.1.1) is known [30], [33]. On the other hand, if $M > 8\pi$, it is shown in [106] that there exists finite time blow-up solution on $\mathbb{R}^2$. In higher-dimension, there exist solutions with arbitrary mass which blow up in finite time, [118].

Similar to the parabolic-elliptic PKS equation, the parabolic-parabolic PKS equation has a natural decaying free energy:

$$E[n] = \int_{\mathbb{R}^2} n \log n dx - \int_{\mathbb{R}^2} ncdx + \frac{1}{2}\int_{\mathbb{R}^2} |\nabla c|^2 dx + \frac{1}{2}\int_{\mathbb{R}^2} c^2 dx. \tag{1.1.32}$$

The proof of the global existence of free energy solution with subcritical mass is done in the paper [30]. The theorem is as follows:

Theorem 3 ([30]). *Let (n_0, c_0) be nonnegative initial conditions for the parabolic-parabolic PKS system (1.1.31) such that*

(H1) $n_0 \in L^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2, \log(1+|x|^2)dx)$ *and* $n_0 \log n_0 \in L^1(\mathbb{R}^2)$*;*

(H2) $c_0 \in H^1(\mathbb{R}^2)$*;*

(H3) $n_0 c_0 \in L^1(\mathbb{R}^2)$*. Assume in addition that the mass is sub-critical, i.e.,* $M < 8\pi$*. Then there exists a global weak nonnegative solution* (n, c) *of* (1.1.31) *such that the free energy dissipation relation holds*

$$E[n(t)] + \int_0^t \int_{\mathbb{R}^2} n|\nabla(\log n - c)|^2 dxds + \int_0^t \int_{\mathbb{R}^2} |\partial_t c|^2 dxds \leq E[n_0], \quad a.e.\ t \in [0, \infty). \tag{1.1.33}$$

Moreover, $n \in L^\infty_{loc}((0,\infty); L^p(\mathbb{R}^2))$ *for any* $1 < p < \infty$ *(regularizing effect).*

1.2 Flocking

1.2.1 Overview

Flocking behavior takes place when a large number of agents (fish or birds) move as a group. Alignment effects become apparent here because individual agent tends to adjust its velocity u to a weighted average of its neighbors ($\overline{u}$) in order to avoid running into others. The hydrodynamic flocking equation describes the evolution of the population density ρ and the agent velocity u:

$$\begin{cases} u_t + \overbrace{u \cdot \nabla u}^{\text{advection}} = \overbrace{\alpha\{\overline{u} - u\}}^{\text{alignment}}, \\ \rho_t + \nabla \cdot (\rho u) = 0, \quad (\rho, u)(0) = (\rho_0, u_0). \end{cases} \tag{1.2.1a}$$

The weighted average velocity $\overline{u}$ is dynamically determined through the influence function ϕ:

$$\overline{u} = \frac{\phi * (\rho u)}{\phi * \rho}. \tag{1.2.1b}$$

The Flocking hydrodynamics (1.2.1) reveals the competition between advection and alignment: advection creates overcrowding effect, whereas alignment drives the agents to move in a uniform manner and stabilizes the system.

I study the global well-posedness theory of the equation (1.2.1) in dimension two.

1.2.2 Hydrodynamic flocking model

The classical single-species hydrodynamic flocking model is an Eulerian dynamics of agent density ρ and velocity u subject to nonlocal alignment forcing on $\mathbb{R}^d,\ \ d = 1, 2$:

$$\rho_t + \nabla \cdot (\rho u) = 0, \tag{1.2.2a}$$

$$u_t + u \cdot \nabla u = \alpha(x,t)(\overline{u} - u), \tag{1.2.2b}$$

$$\alpha(x,t) := \begin{cases} \phi * \rho, & \text{Cucker-Smale Model;} \\ 1, & \text{Motsch-Tadmor Model.} \end{cases} \tag{1.2.2c}$$

Here the initial condition $(\rho, u)\big|_{t=0} = (\rho_0, u_0)$ is satisfied. Recall the dynamically weighted average velocity $\overline{u}$ from (1.2.1b). The first equation (1.2.2a) describes the transport of the mass density ρ along the velocity u. The second equation (1.2.2b) governing velocity evolution is a pressure-less compressible Eulerian equation subject to alignment forcing. The model (1.2.2) arises as the hydrodynamic realization of the following agent based dynamics which describes the collective motion of agents each of which adjusts its velocity to a *weighted average velocity* of its neighbors:

$$\dot{x}^i = v^i, \quad x^i \in \mathbb{R}^d, \;\; v^i \in \mathbb{R}^d, \tag{1.2.3a}$$

$$\dot{v}^i = \frac{1}{Deg_i}\sum_{j=1}^{N}\phi(x^j - x^i)(v^j - v^i), \quad i \in \{1, 2, ..., N\}, \tag{1.2.3b}$$

$$Deg_i = \begin{cases} N, \quad \text{Cucker-Smale Model;} \\ \displaystyle\sum_{j=1}^{N}\phi(|x^i - x^j|), \quad \text{Motsch-Tadmor Model.} \end{cases} \tag{1.2.3c}$$

Here x^i, v^i denote the position and velocity of the ith agent subject to initial data $(x^i, v^i)\big|_{t=0} = (x_0^i, v_0^i)$, respectively. The total number of agents is denoted by N. The explicit derivation is carried out in section 1.2.3.

The forces in the equation (1.2.2b) involves nonnegative radially decreasing influence function $\phi(\cdot) \equiv \phi(|\cdot|)$, $\phi(|\cdot|) \in C^1(\mathbb{R}^d)$ and drives the velocity $u(x,t)$ to the meso-stationary state $\overline{u}(x,t) = \dfrac{\phi * (\rho u)}{\phi * \rho}(x,t)$, which neutralize the forcing in the equation (1.2.2b). The $\overline{u}$ is called the *dynamical mean velocity*. Since the agents tend to align their velocity to their neighbors, the velocity u is expected to approach a uniform constant velocity u_∞ as time tends to infinity. This is the *flocking behavior*. We give the explicit definition as follows:

Definition 3. *Consider the solution to the equation* (1.2.2). *We say the solution converges to a flock if the position variation* $D[\rho]$ *stays bounded and velocity variation* $V[u]$ *converges to zero as time approaches infinity, i.e.,*

$$\sup_{t\in[0,\infty)} D[\rho(t)] := \sup_{t\in[0,\infty)} \sup_{x,y\in supp\rho(t)} |x - y| < D_\infty < \infty, \tag{1.2.4a}$$

$$\lim_{t\to\infty} V[u(t)] := \lim_{t\to\infty} \sup_{x,y\in supp\rho(t)} |u(x) - u(y)| \to 0. \tag{1.2.4b}$$

1.2.3 Derivation of the mesoscopic and hydrodynamic models

In this section, we formally derive the equation (1.2.2) from the *microscopic scale* agent-based dynamics (1.2.3). We first derive a mesoscopic Vlasov type equation from the particle dynamics (1.2.3), and then pass to the hydrodynamic limit to obtain the macroscopic equation (1.2.2).

To define the mesoscopic equation, we first define the following empirical probability measure, which represents the probability of finding an agent at position x with velocity v:

$$f(t,x,v) = \frac{1}{N}\sum_{i=1}^{N} \delta_{x^i(t)}\delta_{v^i(t)}, \tag{1.2.5}$$

Here N denotes the number of agents. Next we derive the evolution for the probability density f. To do this, we test $\partial_t f$ against an arbitrary smooth function η and apply equation (1.2.2) to obtain

$$\begin{aligned}\iint \partial_t f(t,x,v)\eta(x,v)dxdv =& \frac{1}{N}\sum_{i=1}^{N} \partial_t \eta(x^i(t), v^i(t)) \\ =& \frac{1}{N}\sum_{i=1}^{N} [\dot{x}^i \cdot \nabla_x \eta(x^i(t), v^i(t)) + \dot{v}^i \cdot \nabla_v(\eta(x^i, v^i))] \\ =& \frac{1}{N}\sum_{i=1}^{N} [v^i \cdot \nabla_x \eta(x^i, v^i) + F^i \cdot \nabla_v \eta(x^i, v^i)], \end{aligned} \tag{1.2.6}$$

where the F^i is defined as follows:

$$F^i := \frac{1}{N}\sum_{j=1}^{N} \phi(|x^j - x^i|)(v^j - v^i) = \underbrace{\iint \phi(|y - x^i|)(w - v^i) f(y,w)dydw}_{=:L(f)(x^i,v^i)}. \tag{1.2.7}$$

Now applying formal integration by part on (1.2.6) yields

$$\begin{aligned}\iint \partial_t f(t,x,v)\eta(x,v)dxdv &= \iint [v\cdot\nabla_x\eta(x,v) + L(f)(x,v)\cdot\nabla_v\eta(x,v)]f(x,v)dxdv \\ &= -\iint \left[v\cdot\nabla_x f(x,v) + \nabla_v\cdot(L(f)f)\right]\eta dxdv.\end{aligned}$$

Since the test function η is arbitrary, the above integral equation yields the following *mesoscopic scale equation*

$$\partial_t f(x,v) + v\cdot\nabla_x f(x,v) + \nabla_v\cdot(L(f)f) = 0. \tag{1.2.8}$$

This completes the derivation from the microscopic agent-based dynamics to the mesoscopic scale dynamics.

From the mesoscopic equation (1.2.8), we could take the hydrodynamic limit. It is formally achieved by calculating the time evolution of the hydrodynamic quantities, e.g., the mass density and the momentum density:

$$\rho(t,x) := \int_{\mathbb{R}^d} f(t,x,v)dv; \tag{1.2.9}$$

$$\rho u(t,x) := \int_{\mathbb{R}^d} vf(t,x,v)dv. \tag{1.2.10}$$

By integrating the mesoscopic equation (1.2.8) in the velocity variable v and applying integration by parts, we derive the *hydrodynamic scale continuity equation* (1.2.2a) for ρ:

$$(\rho)_t + \nabla_x\cdot(\rho u) = 0, \quad \forall\alpha\in\mathcal{I}. \tag{1.2.11}$$

To obtain the evolution equation for u, we multiply the equation (1.2.8) by v and integrate in the v variable and get the following:

$$0 = \int_{\mathbb{R}^d}\left[\partial_t(vf) + v(v\cdot\nabla_x f) + v\nabla_v\cdot(L(f)f)\right]dv =: I + II + III. \tag{1.2.12}$$

The first term I in (1.2.12) can be interpreted using the definition of momentum density ρu (1.2.10)

$$I = \partial_t(\rho u). \tag{1.2.13}$$

For the second term II in (1.2.12), it can be rewritten as

$$II = \nabla_x \cdot (\rho u \otimes u) + \nabla_x \cdot \underbrace{\int_{\mathbb{R}^d} (u - v) \otimes (u - v) f(x, v) dv}_{=:P} \tag{1.2.14}$$

For the third term III in (1.2.12), we use integration by parts to rewrite it as follows

$$\begin{aligned} III = & - \iint \phi(|y - x|)(\rho u)(y) f(x, v) dy dv + \iint \phi(|y - x|) f(y, w)(\rho u)(x) dy dw \\ = & - \rho\phi * (\rho u) + \rho u(\phi * \rho). \end{aligned} \tag{1.2.15}$$

Now combining (1.2.13), (1.2.14), (1.2.15) and (1.2.2a) we obtain the *hydrodynamic momentum equation*

$$\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla_x \cdot P = \sum_{\beta \in \mathcal{I}} \rho \left\{ \phi * (\rho u) - (\phi * \rho) u \right\}, \quad \alpha, \beta \in \mathcal{I}. \tag{1.2.16}$$

Similar to the paper [64] and [76], we choose the mono-kinetic ansatz $f(x, v) = \rho(x)\delta_{u(x)}(v)$ to impose the pressure closure $P \equiv 0$ and end up with the pressureless Eulerian hydrodynamic equation (1.2.2). If we are in the support of the density $\rho(t, \cdot)$, we can divide ρ on both side of the equation (1.2.16) and end up with the macroscopic scale equation (1.2.2b). This completes the derivation of (1.2.2).

1.2.4 Strong solutions must flock

In the paper [112], the authors proved the following theorem:

Theorem 4. *[Strong Solutions must flock] Consider classical solutions* $(\rho(t), u(t)) \in L^\infty \cap L^1 \times W^{1,\infty}$ *to the Hydrodynamic flocking dynamics* (1.2.2) *subject to a compactly supported initial density* $\rho_0 = \rho(0, \cdot) \geq 0$ *and bounded initial velocity* $u_0 = u(0, \cdot) \in W^{1,\infty}$*. Assume that the monotonically decreasing influence function* $\phi \leq \phi(0) = 1$ *is global in the sense that:*

$$V_0 < m_0 \int_{D_0}^{\infty} \phi(r)dr, \quad m_0 := ||\rho_0||_{L^1}, \tag{1.2.17}$$

where D_0 *and* V_0 *are the initial diameters of non-vacuum density and velocity, see, e.g.,* (1.2.4)*. Then* $(\rho(t), u(t))$ *converges to a flock at an exponential rate, namely - the support of* $\rho(\cdot, t)$ *remains within a finite diameter* D_∞ *whose existence follows from assumption* (1.2.17)

$$\sup_{t \geq 0} D(t) \leq D_\infty, \quad \textit{where } m_0 \int_{D_0}^{D_\infty} \phi(s)ds = V_0,$$

and

$$V(t) \leq V_0 e^{-\kappa t} \to 0, \quad \kappa := \begin{cases} m_0 \phi(D_\infty), & \textit{Cucker-Smale Model,} \\ \phi(D_\infty), & \textit{Motsch-Tadmor Model.} \end{cases}$$

Proof. We only prove the result for the Cucker-Smale model. First we pick an arbitrary unit vector $w \in \mathbb{R}^d$. We project the equation (1.2.2) to the vector w and get:

$$(\partial_t + u \cdot \nabla)\langle u(x,t), w\rangle = \int \phi(|x-y|)(\langle u(y,t), w\rangle - \langle u(x,t), w\rangle)\rho(y,t)dy.$$

Now define the quantity $u_+(t) := \max_{x \in supp\{\rho(\cdot,t)\}} \langle u(x,t), w\rangle$, and it satisfies the estimate:

$$\frac{d}{dt}u_+ = \int \phi(|x_+ - y|)(\langle u(y,t), w\rangle - u(x_+,t), w\rangle)\rho(y,t)dy$$
$$\leq \min_{x,y\in supp\{\rho(\cdot,t)\}} \phi(|x-y|)\int(\langle u(y,t), w\rangle - \langle u(x_+,t), w\rangle)\rho(y,t)dy. \tag{1.2.18}$$

Similarly, for the evolution of the quantity $u_-(t) := \min_{x\in supp\{\rho(\cdot,t)\}}\langle u(x,t), w\rangle$, we have the estimate:

$$\frac{d}{dt}u_- \geq \min_{x,y\in supp\{\rho(\cdot,t)\}} \phi(|x-y|)\int(\langle u(y,t), w\rangle - \langle u(x_-,t), w\rangle)\rho(y,t)dy. \tag{1.2.19}$$

The time evolution of the difference between u_+ and u_- can be estimated using the last two inequalities (1.2.18), (1.2.19) as follows:

$$\frac{d}{dt}|u_+(t) - u_-(t)| \leq -\phi(D_\infty)m_0|u_+(t) - u_-(t)|, \quad \phi(D_\infty) = \min_{x,y\in supp\{\rho(t,\cdot)\}}\phi(|x-y|). \tag{1.2.20}$$

Noting that $V(t) = \sup_{|w|=1}|u_+(t) - u_-(t)|$, we have that the $V(t)$ decays exponentially. The conclusion follows. □

1.2.5 Critical Threshold for one dimension

Given the fact that strong solutions subject to global influence function (1.2.17) must flock, the next problem is when there is a global strong solution. This is related to the critical threshold phenomenon, see, e.g., [54], [90], [88].

In this section, we provide the well-posedness theory in 1D. Recall from [35] that the quantity $e := \partial_x u + \phi * \rho$ characterizes the the critical threshold of the equation (1.2.2) in one dimension.

Theorem 5. *[One-dimensional critical threshold, [35]] Consider the Cucker-Smale flocking dynamics (1.2.2) on $\mathbb{R}$ subject to intial data $(\rho_0, u_0) \in (L^1_+(\mathbb{R}), W^{1,\infty}(\mathbb{R};\mathbb{R}))$. If the initial condition satisfies the threshold*

$$\partial_x u_0(x) + \rho * \phi_0(x) \geq 0, \quad \forall x \in \mathbb{R}, \tag{1.2.21}$$

then the flocking dynamics (1.2.2) admits a classical solution for all time. Otherwise there is finite time blow-up.

Proof. Taking the spatial derivative in the second equation (1.2.2b) yields

$$(\partial_t + u\partial_x)(\partial_x u + \rho * \phi) = -\partial_x u \left(\phi * \rho + \partial_x u\right). \tag{1.2.22}$$

One can see that $\partial_x u + \rho * \phi \geq 0$ is an invariant zone. If $\partial_x u_0 + \phi * \rho_0 \geq 0$, then

$$\partial_x u + \rho * \phi \geq 0, \quad \forall t \geq 0. \tag{1.2.23}$$

Since ϕ has upper bound $||\phi||_\infty < \infty$, we get lower bound for $\partial_x u$

$$\partial_x u(x,t) \geq -M||\phi||_\infty, \quad \forall x \in \mathbb{T}, t \in \mathbb{R}_+. \tag{1.2.24}$$

On the other hand we can see directly from the equation (1.2.22) that $\partial_x u$ has an upper bound for all time. Combining this with the lower bound, we have that $||\partial_x u||_\infty \leq C < \infty$ for all time. As a result, we have that the strong solutions exist for all time.

We omit the proof of the blow-up for the sake of brevity. □

1.3 Multi-species dynamics

In all the models above, the agents involved are homogeneous, namely, there is no difference between individuals. In reality, biological agents are more than just identical particles. For example, in a biofilm, i.e., a bacteria colony, bacteria with different functions cooperate with others to generate hard-to-eradicate infections. Chemicals are secreted to enhance communication within the colony. This phenomenon has attracted more and more interest in the theoretical biology and biophysics community. The dynamics of the biofilm cannot be described by single species model. I studied the following multi-species Keller-Segel model:

$$\begin{cases} (n_\alpha)_t + \nabla \cdot (\nabla c_\alpha n_\alpha) = \Delta n_\alpha, \quad \alpha \in \mathcal{I} \\ -\Delta c_\alpha = \sum_{\beta \in \mathcal{I}} b_{\alpha\beta} n, \quad n_\alpha(0) = (n_\alpha)_0. \end{cases} \tag{1.3.1}$$

Here n_α, c_α denote the bacteria and the chemo-attractant densities respectively. The parameters $\alpha, \beta \in \mathcal{I}$ indicate the species of the bacteria and chemo-attractant. The total number of species, which is denoted $|\mathcal{I}|$ throughout the thesis, is assumed to be finite. The first equation in the system (1.3.1) describes the time evolution of the bacteria density n_α subject to chemical density distribution c_α and diffusion. The second equation governs the evolution of the chemical density c_α, which is determined by the collective effect of different species of bacteria n_β. The *chemical generation coefficients* $b_{\alpha\beta}$ represent the relative impact of the bacteria density n_β on the chemical distribution (c_α).

I applied the techniques in the well-posedness theory of the Patlak-Keller-Segel model to study global well-posedness of (1.3.1) and derive simple subcritical mass conditions.

The other related project studies the multi-species hydrodynamic flocking model:

$$\begin{cases} \partial_t \rho_\alpha + \nabla \cdot (u_\alpha \rho_\alpha) = 0 \\ \partial_t(\rho_\alpha u_\alpha) + \nabla \cdot (\rho_\alpha u_\alpha \otimes u_\alpha) = \sum_{\beta \in \mathcal{I}} b_{\alpha\beta} \rho_\alpha \left\{ \phi * (\rho u) - (\phi * \rho) u_\alpha \right\}, \quad \alpha, \beta \in \mathcal{I} \end{cases} \tag{1.3.2}$$

Here ρ_α, u_α denote the density and velocity of the agents in the group α respectively. The parameter α, β which indicate the species of the agents takes value in a finite set $\mathcal{I}$.

I studied the well-posedness of the equation and gave explicit conditions to guarantee the global existence and characterize the long-time behavior of the strong solutions.

The thesis is organized as follows: in chapter 2, 3 and 4, I discuss three results concerning suppression of blow-up in Patlak-Keller-Segel equations through fluid flow; in chapter 5, I discuss the two dimensional well-posedness theory of the hydrodynamic flocking model; in chapter 6, I discuss the multi-species hydrodynamic flocking model; in chapter 7, I discuss the multi-species Patlak-Keller-Segel model; in chapter 8, conclusion is made.

Chapter 2: Suppression of blow-up in Patlak-Keller-Segel via shear flows

2.1 Overview

In this section, we consider the parabolic-elliptic Patlak-Keller-Segel model in $\mathbb{T}^d$ with the additional effect of a large shear flow

$$\begin{cases} \partial_t n + Au(y)\partial_x n + \nabla \cdot (n\nabla c) = \Delta n \\ -\Delta c = n - \bar{n} \\ n(t=0,x,y) = n_{\mathrm{in}}(x,y), \end{cases} \tag{2.1.1}$$

where $\bar{n}$ denotes the average of n.

In [80] it was shown that if, instead of a shear flow, one has $A\mathbf{u} \cdot \nabla n$ where $\mathbf{u}$ is *relaxation-enhancing* – a generalization of weakly mixing introduced in [41] – then for each smooth initial datum, one can choose A large enough so that the solution to (2.1.1) does not blow-up in finite time. Such velocity fields are very good mixers, and this ensures that any non-constant density configuration undergoes a large growth of gradients, and hence a large dissipation. The effect at work is then an *enhanced dissipation*. This effect has been studied previously in a variety of contexts, such as [6, 8, 9, 41, 116, 120], in the physics literature [19, 51, 84, 95, 104], and in control theory [4,5]; a closely related effect was also studied in [61].

Mixing due to a shear flow is quite different from that due to a relaxation-enhancing or weakly mixing flow. In particular, data which is independent of x does not mix at all, and so one must separate the evolution of the zero (or low if $x \in \mathbb{R}$) frequencies in x from the non-zero frequencies, which is the decomposition into the nullspace of the transport operator and its orthogonal complement. Enhanced dissipation due to shear flow was shown in [10–12,16,18] to be important for understanding the stability of the Couette flow in the 2D and 3D Navier-Stokes equations at high Reynolds number. For example, [10–12] show that the enhanced dissipation can suppress 3D effects and simplify the dynamics to be essentially 2D. It is intuitive then to expect that a large shear flow can also in some sense suppress one dimension in (2.1.1) and hence make 2D L^1 subcritical and 3D L^1 critical. This is essentially what we prove for $u \in C^3$ with finitely many non-degenerate critical points (the relevance of these hypotheses are discussed after the statements).

Theorem 1. *Let $u \in C^3(\mathbb{T})$ have finitely many, non-degenerate critical points and let $n_{in} \in H^1(\mathbb{T}^2) \cap L^\infty(\mathbb{T}^2)$ be arbitrary. There exists an $A_0 = A_0(u, \|n_{in}\|_{H^1}, \|n_{in}\|_{L^\infty})$ such that if $A > A_0$ then the solution to (2.1.1) is global in time.*

Remark 2.1.1. *Theorem 1 extends to the cylindrical domain $\mathbb{T} \times \mathbb{R}$ provided u' is bounded uniformly away from zero near $y \to \pm\infty$.*

It is clear that Theorem 1 cannot hold in 3D. Indeed, consider any solution to the 3D problem which is constant in the x direction: $n(t, x, y_1, y_2) = n(t, y_1, y_2)$. This solution will solve (2.1.1) on $\mathbb{T}^2$ with $A = 0$ and hence the 8π critical mass will still apply. Our next result shows that for A large the third dimension is suppressed and 8π is indeed the critical mass for (2.1.1) in $\mathbb{T} \times \mathbb{R}^2$ and $\mathbb{T}^3$. As this setting is effectively critical, Theorem 2 is harder to prove than Theorem 1 (which is effectively subcritical, as [80]).

Theorem 2. *(a) Let $u \in C^3(\mathbb{T})$ have finitely many, non-degenerate critical points and let $n_{in} \in H^1(\mathbb{T}^3) \cap L^\infty(\mathbb{T}^3)$ be arbitrary such that $\|n_{in}\|_{L^1} < 8\pi$ and for some $q > 0$, there holds $n_{in}(x) \geq q > 0$ for all $x \in \mathbb{T}^3$. Then there exists an*

$$A_0 = A_0(u, \|n_{in}\|_{H^1}, \|n_{in}\|_{L^\infty}, \|n_{in}\|_{L^1}, q)$$

such that if $A > A_0$ then the solution to (2.1.1) is global in time.

(b) Suppose $u \in C^3(\mathbb{R})$ have finitely many, non-degenerate critical points and u' is bounded uniformly away from zero near infinity. Let $n_{in} \in H^1(\mathbb{T} \times \mathbb{R}^2) \cap L^\infty(\mathbb{T} \times \mathbb{R}^2)$ be arbitrary such that $\|n_{in}\|_{L^1} < 8\pi$ and $I[n_{in}] := \int n_{in}(x, y) |y|^2 \, dxdy < \infty$. Consider the problem

$$\begin{cases} \partial_t n + Au(y_1)\partial_x n + \nabla \cdot (\nabla c n) = \Delta n, \\ -\Delta c = n, \\ n(\cdot, 0) = n_{in}. \end{cases} \tag{2.1.2}$$

Then, there exists an $A_0 = A_0(u, \|n_{in}\|_{L^\infty}, \|n_{in}\|_{H^1}, \|n_{in}\|_{L^1}, I[n_{in}])$, such that if $A > A_0$ then the solution is global in time.

Remark 2.1.2. *It is not clear whether or not one could expect Theorem 2 to hold also in the case $\|n_{in}\|_{L^1} = 8\pi$ as in $\mathbb{R}^2$ [27].*

Let us now briefly discuss the proofs of Theorems 1 and 2. By re-scaling time $t \mapsto A^{-1}t$, the system (2.1.1) is equivalent to

$$\begin{cases} \partial_t n + u\partial_x n + \frac{1}{A}\left(\nabla \cdot (n\nabla c) - \Delta n\right) = 0 \\ -\Delta c = n - \bar{n} \\ n(t=0,x,y) = n_{\text{in}}(x,y), \end{cases} \tag{2.1.3}$$

For our purposes, it is convenient to use the form (2.1.3). In [8], enhanced dissipation was studied for the passive scalar equation

$$\partial_t f + u\partial_x f = \frac{1}{A}\Delta f. \tag{2.1.4}$$

Among other things, it was shown in [8] that for u satisfying the hypotheses of Theorems 1 and 2, there exists some $\tilde{\epsilon} > 0$ such that

$$\left\| f(t) - \frac{1}{2\pi}\int_{\mathbb{T}} f(t,x,\cdot)dx \right\|_{L^2} \lesssim e^{-\frac{\tilde{\epsilon}A^{-1/2}}{1+|\log A|^2}t} \left\| f(0) \right\|_{L^2}.$$

The technique employed in [8] is an energy method known as hypocoercivity, see e.g. the text [115] for an overview or [47, 49, 61, 69, 70] and the references therein. In the proof of Theorem 1 we will couple such hypocoercivity energy estimates to H^1 energy estimates for the zero-in-x frequency as well as to L^p estimates on (2.1.1), similar to the estimates in [28, 29, 75], which do not see the advection term. In the proof of Theorem 2, the x-independent system is now formally L^1 critical, and hence in order to get results for mass up to 8π, we need to employ the free energy in a manner similar to [28]. However, the *two-dimensional* free energy is not a monotonically dissipated quantity for (2.1.1), and hence we need to also couple an estimate on the 2D free energy to the other energy estimates we make and control the errors using the enhanced dissipation. This is particularly tricky if one is interested in the result on $\mathbb{T} \times \mathbb{R}^2$. Enhanced dissipation (or something similar) was studied via hypocoercivity also in [6, 8, 61], however, to the authors' knowledge, this is the first work that uses hypocoercivity to obtain enhanced dissipation estimates for nonlinear problems. We remark that the Fourier analysis methods used in [10, 18] also apply to (2.1.1) in the specific case $u(y) = y$ and $y \in \mathbb{R}$. This approach is much simpler than the hypocoercivity methods we employ, however, the hypocoercivity methods allow us to study a much wider variety of shear flows.

2.1.1 Notations

Miscellaneous

The constants B below are universal constants which have no dependence on any quantities, except perhaps u and M. On the contrary, the dependence of the constants $C_{\dots}$ on various quantities involving n_{in} is more important and will be made a little more explicit. Given quantities X, Y, if there exists a constant B such that $X \leq BY$, we often write $X \lesssim Y$. We will moreover use the notation $\langle x \rangle := (1 + |x|^2)^{1/2}$.

Fourier Analysis

Most of the time, we consider the Fourier transform *only in the x variable*, and denoting it and its inverse as

$$\widehat{f}_k(y) := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x, y) dx, \quad \check{g}(x, y) = \sum_{k=-\infty}^{\infty} g_k(y) e^{ikx}.$$

Define the following orthogonal projections:

$$f_0(t,y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t,x,y)dx,$$

$$f_{\neq}(t,x,y) = f(t,x,y) - f_0(t,y),$$

for "zero frequency" and "non-zero frequency". For any measurable function $m(\xi)$, we define the Fourier multiplier $m(\nabla)f := (m(\xi)\hat{f}(\xi))^{\vee}$.

Functional spaces

The norm for the L^p space is denoted as $||\cdot||_p$ or $||\cdot||_{L^p(\cdot)}$:

$$||f||_p = ||f||_{L^p} = (\int |f|^p dx)^{1/p},$$

with natural adjustment when p is ∞. If we need to emphasize the ambient space, we use the second notation, i.e., $||n_{\neq}||_{L^p(\mathbb{T}\times\mathbb{R}^2)}$. Otherwise, we use the first notation for the sake of simplicity. The Sobolev norm $||\cdot||_{H^s}$ is defined as follow:

$$||f||_{H^s} := ||\langle\nabla\rangle^s f||_{L^2}.$$

For a function of space and time $f = f(t,x)$, we use the following space-time norms:

$$||f||_{L_t^p L_x^q} := || ||f||_{L_x^q} ||_{L_t^p},$$

$$||f||_{L_t^p H_x^s} := || ||f||_{H_x^s} ||_{L_t^p}.$$

2.2 Proof of Theorem 1

2.2.1 Outline of the proof

In this section, we prove Theorem 1. The enhanced dissipation does not act in the nullspace of the advection term, and hence it is reasonable to decompose the solution as follows

$$\begin{aligned}\partial_t n_0+\frac{1}{A}\partial_y(\partial_y c_0 n_0)+\frac{1}{A}(\nabla\cdot(\nabla c_{\neq} n_{\neq}))_0=\frac{1}{A}\partial_{yy}n_0,\\ -\Delta c_0=n_0-\overline{n};\end{aligned}\tag{2.2.1}$$

and,

$$\begin{aligned}\partial_t n_{\neq}+u(y)\partial_x n_{\neq}-\frac{1}{A}(n_0-\overline{n})n_{\neq}-\frac{1}{A}n_0 n_{\neq}+\frac{1}{A}\nabla c_0\cdot\nabla n_{\neq}+\frac{1}{A}\nabla c_{\neq}\cdot\nabla n_0\\ =-\frac{1}{A}(\nabla\cdot(\nabla c_{\neq} n_{\neq}))_{\neq}+\frac{1}{A}\Delta n_{\neq},\\ -\Delta c_{\neq}=n_{\neq}.\end{aligned}\tag{2.2.2}$$

As in [8], it is convenient to consider (2.2.2) after applying the Fourier transform *only in* x. Applying to both sides of (2.2.2) we have,

$$\begin{aligned}\partial_t\widehat{n}_k+NL_k+L_k+u(y)ik\widehat{n}_k=&\frac{1}{A}(\partial_{yy}-|k|^2)\widehat{n_k},\\ -(\partial_{yy}-|k|^2)\widehat{c}_k=&\widehat{n}_k,\quad k\neq 0,\end{aligned}\tag{2.2.3}$$

where L_k, NL_k are defined as follows:

$$NL_k:=-\frac{1}{A}\sum_{\ell\neq 0,k}\widehat{n}_{k-\ell}(y)\widehat{n}_\ell(y)+\frac{1}{A}\sum_{\ell\neq 0,k}\partial_y\widehat{c}_{k-\ell}\partial_y\widehat{n}_\ell-\frac{1}{A}\sum_{\ell\neq 0,k}(k-\ell)\widehat{c}_{k-\ell}\ell\widehat{n}_\ell,\tag{2.2.4a}$$

$$L_k:=-\frac{1}{A}\overline{n}\widehat{n}_k-\frac{2}{A}(n_0-\overline{n})\widehat{n}_k+\frac{1}{A}\partial_y c_0\partial_y\widehat{n}_k+\frac{1}{A}\partial_y\widehat{c}_k\partial_y n_0.\tag{2.2.4b}$$

Here, the L refers to "linear with respect to the nonzero frequencies" and NL refers to "nonlinear with respect to the nonzero frequencies".

For constants $C_{ED}, C_{L^2}, C_{\dot{H}^1}$, and C_∞ determined by the proof, define $T_\star$ to be the end-point of the largest interval $[0, T_\star]$ such that the following hypotheses hold for all $T \le T_\star$:

(1) Nonzero mode $L_t^2 \dot{H}_{x,y}^1$ estimate:

$$\frac{1}{A}\int_0^{T_\star} ||\nabla n_{\neq}||_2^2 dt \le 8||n_{in}||_2^2; \tag{2.2.5a}$$

(2) Nonzero mode enhanced dissipation estimate:

$$||n_{\neq}(t)||_2^2 \le 4C_{ED}||n_{in}||_{H^1}^2 e^{-\frac{ct}{A^{1/2}\log A}}, \tag{2.2.5b}$$

where c is a small constant depending only on u.

(3) Uniform in time estimates on the zero mode:

$$\begin{aligned} ||n_0 - \overline{n}||_{L_t^\infty(0,T_\star;L_y^2)} &\le 4C_{L^2}, \\ ||\partial_y n_0||_{L_t^\infty(0,T_\star;L_y^2)} &\le 4C_{\dot{H}^1}; \end{aligned} \tag{2.2.5c}$$

(4) L^∞ estimate of the whole solution:

$$||n||_{L_t^\infty(0,T_\star;L_{x,y}^\infty)} \le 4C_\infty. \tag{2.2.5d}$$

Moreover, in order to simplify the exposition, we introduce the following constant:

$$C_{2,\infty} := 1 + M + C_{ED}^{1/2}||n_{in}||_{H^1} + C_{L^2} + C_\infty. \tag{2.2.6}$$

Remark 2.2.1. *In the above, C_{ED} is first chosen depending only on u. Then, C_{L^2} is chosen depending only on the initial data n_{in} (and C_{ED}). Then C_∞ is chosen depending only on n_{in}, C_{L^2}, and C_{ED}. Finally, $C_{\dot{H}^1}$ depends on n_{in}, C_{ED}, C_{L^2}, and C_∞. Then, A is chosen large depending on all of these parameters.*

Remark 2.2.2. *Note the small subtlety regarding the inclusion of both the L^2 (2.2.5c) and L^∞ norms (2.2.5d). We need to make a separate estimate on the L^2 norm in (2.2.5c) in a manner which depends directly only on the initial data and C_{ED}, and then we later choose C_∞ (possibly much larger) depending on C_{L^2}. Hence, with the dependencies of the estimates accounted for, in fact (2.2.5d) does not imply (2.2.5c) in a manner which is self-consistent.*

We will refer to the hypotheses (2.2.5a), (2.2.5b), (2.2.5c), and (2.2.5d) together as the *bootstrap hypotheses*, denoted as **(H)**. Notice that by local well-posedness of solutions, see, e.g., [22], [28], the quantities on the left-hand sides of (2.2.5a), (2.2.5b), (2.2.5c), and (2.2.5d) take values continuously in time. Moreover, the inequalities are all satisfied with the 4's replaced by 2's for t sufficiently small. By the standard continuation criteria for (2.1.1), the solution exists and remains smooth on an interval $(0, t_0]$, with $t_0 > T_\star$ such that $t_0 - T_\star$ can be taken to depend only on $\|n(T_\star)\|_{L^2}$. By continuity, the following proposition shows that the solution is global and satisfies the a priori estimates **(H)** for all time.

Proposition 3. *For all n_{in} and u, there exists an $A_0(u, \|n_{in}\|_{H^1}, \|n_{in}\|_{L^\infty})$ such that if $A > A_0$ then the following conclusions hold on the interval $[0, T_\star]$:*

(1)

$$\frac{1}{A}\int_0^{T_\star} ||\nabla_{x,y} n_{\neq}||_2^2 dt \leq 4||n_{in}||_2^2; \tag{2.2.7a}$$

(2) For all $t < T_\star$,

$$||n_{\neq}(t)||_2^2 \leq 2C_{ED}||n_{in}||_{H^1}^2 e^{-\frac{ct}{A^{1/2}\log A}}; \tag{2.2.7b}$$

(3)

$$
\begin{aligned}
||n_0 - \overline{n}||_{L^\infty_t(0,T_\star;L^2_y)} \leq& 2C_{L^2}, \\
||\partial_y n_0||_{L^\infty_t(0,T_\star;L^2_y)} \leq& 2C_{\dot{H}^1};
\end{aligned}
\tag{2.2.7c}
$$

(4)

$$
||n||_{L^\infty(0,T_\star;L^\infty_{x,y})} \leq 2C_\infty. \tag{2.2.7d}
$$

The remainder of the section is dedicated to proving Proposition 3.

We first point out that there is a uniform upper bound on $\|n(t)\|_{L^2}$ over the initial time layer $t \leq \delta A$ for a sufficiently small δ depending only on $\|n_{in}\|_{L^2}$ (as such we can always choose $A > \delta^{-1}$). This is an immediate consequence of the standard local existence theory of (2.1.1) via the time-rescaling used in (2.1.3), however, we include a brief sketch of the a priori estimate for completeness. Proposition 4 and standard higher regularity theory for (2.1.1) (see e.g. [75]) imply that (2.2.7d) holds over the time interval $0 \leq t \leq \delta A$.

Proposition 4. *For all $n_{in} \in L^2(\mathbb{T}^2)$, there exists $\delta = \delta(\|n_{in}\|_{L^2})$ sufficiently small such that for $t \leq \delta A$, the following estimate holds,*

$$
\|n_{\neq}(t)\|_{L^2}^2 + \|n_0(t) - \overline{n}\|_{L^2}^2 = ||n(t) - \overline{n}||_2^2 \leq 2||n_{in} - \overline{n}||_{L^2}^2 \leq 2\,\|n_{in}\|_{L^2}^2\,. \tag{2.2.8}
$$

Proof. The time derivative of the L^2 norm of $n - \overline{n}$ is estimated as follows, using a

Gagliardo-Nirenberg-Sobolev inequality,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||n-\overline{n}||_2^2 =& -\frac{1}{A}||\nabla n||_2^2 - \int \nabla\cdot(\nabla c n)(n-\overline{n})dx \\ =& -\frac{1}{A}||\nabla n||_2^2 + \frac{1}{2A}||n-\overline{n}||_3^3 + \frac{1}{A}\overline{n}||n-\overline{n}||_2^2 \\ \leq& -\frac{1}{A}||\nabla n||_2^2 + \frac{B}{A}||\nabla n||_2||n-\overline{n}||_2^2 + \frac{1}{A}M||n-\overline{n}||_2^2 \\ \lesssim& \frac{1}{A}||n-\overline{n}||_2^4 + \frac{1}{A}M^2.\end{aligned}$$

The desired estimate follows (note that $M \lesssim \|n_{in}\|_{L^2}$). □

2.2.2 A priori estimates

Enhanced dissipation estimate, (2.2.7b)

Proposition 4 implies that (2.2.7b) holds trivially on a time-scale like $t \lesssim A^{1/2}\log A$. In order to deduce the enhanced dissipation effect for longer times, we use the hypocoercivity technique of [8], which builds on the earlier work of [6, 61]. As outlined in [115], hypocoercivity techinques are based on finding an energy which extracts the fact that the quadratic quantity $A^{-1}\|\nabla f\|_{L^2}^2 + \|u'\partial_x f\|_{L^2}^2$ is much 'more coercive' than $A^{-1}\|\nabla f\|_{L^2}^2$. In [8] and here this is done via the following energies, defined k-by-k,

$$\begin{aligned}\Phi_k[n(t)] =& ||\widehat{n}_k(t)||_2^2 + ||\sqrt{\alpha}\partial_y\widehat{n}_k(t)||_2^2 + 2kRe\langle i\beta u'\widehat{n}_k(t), \partial_y\widehat{n}_k(t)\rangle \\ &+ |k|^2||\sqrt{\gamma}u'\widehat{n}_k(t)||_2^2;\end{aligned} \tag{2.2.9}$$

$$\begin{aligned}\Phi[n(t)] =& \sum_{k\neq 0}\Phi_k[n(t)] = ||n_{\neq}(t)||_2^2 + ||\sqrt{\alpha}\partial_y n_{\neq}(t)||_2^2 + 2\langle\beta u'\partial_x n_{\neq}(t), \partial_y n_{\neq}(t)\rangle \\ &+ ||\sqrt{\gamma}u'|\partial_x|n_{\neq}(t)||_2^2.\end{aligned} \tag{2.2.10}$$

Here $\alpha, \beta,$ and γ are k-dependent constants (and hence should be interpreted as Fourier multipliers) satisfying

$$\alpha(A,k) = \epsilon_\alpha A^{-1/2} |k|^{-1/2} \tag{2.2.11a}$$

$$\beta(A,k) = \epsilon_\beta |k|^{-1} \tag{2.2.11b}$$

$$\gamma(A,k) = \epsilon_\gamma A^{1/2} |k|^{-3/2}, \tag{2.2.11c}$$

where ϵ_α, ϵ_β, and ϵ_γ are small constants depending only on u chosen in [8]. Among other things, these are chosen such that $8\beta^2 \leq \alpha\gamma$. Notice that in [8] for treating general situations one must also take $\alpha, \beta,$ and γ to be y-dependent, however, as suggested by [6], this is not necessary to treat shear flows with non-degenerate critical points with $y \in \mathbb{T}$ or $y \in \mathbb{R}$. The parameters ϵ_α, ϵ_β, and ϵ_γ are tuned such that,

$$\Phi_k[n] \approx ||\widehat{n}_k||_2^2 + ||\sqrt{\alpha}\partial_y \widehat{n}_k||_2^2 + |k|^2 ||\sqrt{\gamma} u' \widehat{n}_k||_2^2, \tag{2.2.12}$$

and hence

$$\begin{aligned} &\|\widehat{n}_k\|_2^2 + A^{-1/2} |k|^{-1/2} \|\partial_y \widehat{n}_k\|_2^2 \\ &\quad \lesssim \Phi_k[n] \lesssim \|\widehat{n}_k\|_2^2 + |k|^{1/2} A^{1/2} \|\widehat{n}_k\|_2^2 + A^{-1/2} |k|^{-1/2} \|\partial_y \widehat{n}_k\|_2^2 . \end{aligned} \tag{2.2.13}$$

As a result, $\Phi_k(t)$ is equivalent to the H^1 norm of n_k but with constants that depend on A and k. The primary step in the results of [8] is that for $u(y)$ satisfying the hypotheses in Theorem 1, then for the passive scalar equation on $\mathbb{T}^2$,

$$\partial_t f + u(y)\partial_x f = \frac{1}{A}\Delta f,$$

the norm $\Phi_k[f(t)]$ satisfies the following differential inequality for some small constant $\tilde{\epsilon}$ independent of k, A (but depending on u):

$$\frac{d}{dt}\Phi_k[f(t)] \leq -\tilde{\epsilon} \frac{|k|^{1/2}}{A^{1/2}} \Phi_k[f(t)].$$

The primary step in the proof of (2.2.7b) is the analogous statement (though summed over all k due to the nonlinearity).

Proposition 5. *There exists a small constant $c > 0$ depending only on u such that, under the bootstrap hypotheses and for A sufficiently large, there holds*

$$\frac{d}{dt}\Phi[n(t)] \leq -\frac{c}{A^{1/2}}\Phi[n(t)]. \tag{2.2.14}$$

By (2.2.13), it follows that

$$\|n_{\neq}(t)\|_{L^2}^2 \leq \Phi(0)e^{-cA^{-1/2}t} \lesssim A^{1/2}\,\|n_{in}\|_{H^1}^2\,e^{-cA^{-1/2}t}. \tag{2.2.15}$$

Remark 1. *Propositions 4 and 5 together imply (2.2.7b). Indeed, for A sufficiently large:*

$$\begin{aligned} ||n_{\neq}(t)||_2^2 &\lesssim ||n_{in}||_{H^1}^2 \mathbf{1}_{t \leq \frac{1}{2c}A^{1/2}\log A} + \mathbf{1}_{t \geq \frac{1}{2c}A^{1/2}\log A} A^{1/2}||n_{in}||_{H^1}^2 e^{-\frac{c}{A^{1/2}}t} \\ &\lesssim ||n_{in}||_{H^1}^2 e^{-\frac{c}{2A^{1/2}\log A}t}. \end{aligned}$$

We first compute the time derivative of $\Phi_k[n(t)]$.

Proposition 6. *Let $u \in C^3(\mathbb{T})$ have finitely many, non-degenerate critical points.*

For $\tilde{\epsilon}$ sufficiently small depending only on u, there holds,

$$\begin{aligned}
\frac{d}{dt}\Phi_k[n(t)] \leq & \Bigg\{ -\frac{\tilde{\epsilon}}{2}\frac{|k|^{1/2}}{A^{1/2}}||\widehat{n}_k||_2^2 - \frac{\tilde{\epsilon}}{2}\frac{|k|^{1/2}}{A^{1/2}}||\sqrt{\alpha}\partial_y\widehat{n}_k||_2^2 - \frac{\tilde{\epsilon}}{2}\frac{|k|^{5/2}}{A^{1/2}}||\sqrt{\gamma}u'\widehat{n}_k||_2^2 \\
& - \frac{1}{4A}||\partial_y\widehat{n}_k||_2^2 - \frac{1}{2}|k|^2||\sqrt{\beta}u'\widehat{n}_k||_2^2 - \frac{1}{2A}|k|^2||\widehat{n}_k||_2^2 - \frac{1}{4A}||\sqrt{\alpha}\partial_{yy}\widehat{n}_k||_2^2 \\
& - \frac{1}{4A}|k|^4||\sqrt{\gamma}u'\widehat{n}_k||_2^2 - \frac{1}{4A}|k|^2||\sqrt{\gamma}u'\partial_y\widehat{n}_k||_2^2\Bigg\} \\
& + \Bigg\{ -2Re\langle L_k, \widehat{n}_k\rangle + 2Re\langle \alpha\partial_{yy}\widehat{n}_k, L_k\rangle - 2kRe[\langle i\beta u' L_k, \partial_y\widehat{n}_k\rangle \\
& + \langle i\beta u'\widehat{n}_k, \partial_y L_k\rangle] - 2|k|^2 Re\langle \gamma(u')^2\widehat{n}_k, L_k\rangle\Bigg\} \\
& + \Bigg\{ -2Re\langle NL_k, \widehat{n}_k\rangle + 2Re\langle \alpha\partial_{yy}\widehat{n}_k, NL_k\rangle - 2kRe[\langle i\beta u' NL_k, \partial_y\widehat{n}_k\rangle \\
& + \langle i\beta u'\widehat{n}_k, \partial_y NL_k\rangle] - 2|k|^2 Re\langle \gamma(u')^2\widehat{n}_k, NL_k\rangle\Bigg\} \\
=: & \mathcal{N}_k + \{L_k^1 + L_k^\alpha + L_k^\beta + L_k^\gamma\} + \{NL_k^1 + NL_k^\alpha + NL_k^\beta + NL_k^\gamma\}, \quad (2.2.16)
\end{aligned}$$

where $\mathcal{N}_k$ refers to the negative terms. Recall that L_k, NL_k are defined in (2.2.4b,2.2.4a).

Proof. The estimates from the linear terms (that is, the terms arising from the passive scalar equation (2.1.4)) are made in [8] and are omitted for the sake of brevity. The extra terms from the Keller-Segel nonlinearity in (2.1.3) are immediate. □

The remainder of the section is devoted to controlling L and NL by the negative terms in (2.2.16).

Estimate on the L terms in (2.2.16)

These terms are linear in the k-th mode, and it accordingly makes sense to estimate these terms k-by-k. In this section we prove that for A sufficiently large,

$$L_k^1 + L_k^\alpha + L_k^\beta + L_k^\gamma \leq -\frac{1}{4}\mathcal{N}_k. \quad (2.2.17)$$

We begin by estimating the L_k^1 term in (2.2.16). Integrating by parts and using Lemma 2.4.1, Lemma 2.4.2, and the bootstrap hypotheses, we have, for any fixed constant $B \geq 1$,

$$
\begin{aligned}
|L_k^1| \leq & \frac{2}{A}(2||n_0 - \overline{n}||_\infty + \overline{n})||\widehat{n}_k||_2^2 + \frac{1}{AB}||\partial_y \widehat{n}_k||_2^2 + \frac{B}{A}||\partial_y c_0||_\infty^2 ||\widehat{n}_k||_2^2 \\
& + \left| Re \frac{2}{A}\langle \partial_{yy}\widehat{c}_k \widehat{n}_k + \partial_y \widehat{c}_k \partial_y \widehat{n}_k, n_0 - \overline{n}\rangle \right| \lesssim \frac{BC_{2,\infty}^2}{A}||\widehat{n}_k||_2^2 + \frac{1}{AB}||\partial_y \widehat{n}_k||_2^2.
\end{aligned}
$$

Therefore, we can choose B sufficiently large, and then A sufficiently large, such that the following holds:

$$
\left|L_k^1\right| \lesssim \frac{BC_{2,\infty}^2}{A}\left\|\widehat{n}_k\right\|_2^2 + \frac{1}{AB}\left\|\partial_y \widehat{n}_k\right\|_2^2 \leq -\frac{1}{16}\mathcal{N}_k,
$$

and hence by the definition of $\mathcal{N}_k$, this is consistent with (2.2.17).

Turn next to L_k^α in (2.2.16), which we divide into the following four contributions:

$$
\begin{aligned}
L_k^\alpha &= -2Re\langle \alpha \partial_{yy}\widehat{n}_k, \frac{1}{A}\overline{n}\widehat{n}_k + \frac{2}{A}(n_0 - \overline{n})\widehat{n}_k - \frac{1}{A}\partial_y c_0 \partial_y \widehat{n}_k - \frac{1}{A}\partial_y \widehat{c}_k \partial_y n_0\rangle \\
&=: L_{k,0}^\alpha + L_{k,1}^\alpha + L_{k,2}^\alpha + L_{k,3}^\alpha.
\end{aligned}
\tag{2.2.18}
$$

For the $L_{k,1}^\alpha$ term, we have the following by the bootstrap hypotheses, for any fixed $B \geq 1$:

$$
\begin{aligned}
\left|L_{k,1}^\alpha\right| &\lesssim \frac{1}{AB}||\sqrt{\alpha}\partial_{yy}\widehat{n}_k||_2^2 + \frac{B}{A}||\sqrt{\alpha}(n_0 - \overline{n})\widehat{n}_k||_2^2 \\
&\lesssim \frac{1}{AB}||\sqrt{\alpha}\partial_{yy}\widehat{n}_k||_2^2 + \frac{B}{A^{3/2}}||n_0 - \overline{n}||_\infty^2 ||\widehat{n}_k||_2^2 \\
&\lesssim \frac{1}{AB}||\sqrt{\alpha}\partial_{yy}\widehat{n}_k||_2^2 + \frac{BC_\infty^2}{A^{3/2}}||\widehat{n}_k||_2^2.
\end{aligned}
$$

Recalling the definition $\mathcal{N}_k$ from (2.2.16), it follows that by choosing B, then A, sufficiently large, we can control this term consistent with (2.2.17). The $L_{k,0}^\alpha$ term is treated in the same manner; we omit the details for brevity.

Next, we estimate the second term $L^{\alpha}_{k,2}$ in (2.2.18). Using Lemma 2.4.2, we have the following for any $B \geq 1$:

$$\left|L^{\alpha}_{k,2}\right| \lesssim \frac{1}{BA}||\sqrt{\alpha}\partial_{yy}\widehat{n}_k||_2^2 + \frac{B}{A^{3/2}}||\partial_y c_0||_\infty^2||\partial_y\widehat{n}_k||_2^2 \lesssim \frac{1}{BA}||\sqrt{\alpha}\partial_{yy}\widehat{n}_k||_2^2 + \frac{BC_{2,\infty}^2}{A^{3/2}}||\partial_y\widehat{n}_k||_2^2.$$

Hence, by the bootstrap hypotheses and the definition of $\mathcal{N}_k$, it follows we can choose B large and then A large to control this term consistent with (2.2.17).

Similarly, for $L^{\alpha}_{k,3}$ in (2.2.18), by Lemma 2.4.1

$$\begin{aligned}|L^{\alpha}_{k,3}| \lesssim& \frac{1}{BA}||\sqrt{\alpha}\partial_{yy}\widehat{n}_k||_2^2 + \frac{B}{A^{3/2}}||\partial_y\widehat{c}_k||_\infty^2||\partial_y n_0||_2^2 \\ \lesssim& \frac{1}{BA}||\sqrt{\alpha}\partial_{yy}\widehat{n}_k||_2^2 + \frac{B}{A^{3/2}}||\widehat{n}_k||_2^2||\partial_y n_0||_2^2.\end{aligned}$$

As above, it follows we can choose B large and then A large to control this term consistent with (2.2.17).

Next, turn to the L^{β}_k term in (2.2.16), which we divide into two contributions:

$$\begin{aligned}L^{\beta}_k =& 2kRe\langle i\beta u'\widehat{n}_k, \partial_y(\frac{1}{A}\overline{n}\widehat{n}_k + \frac{2}{A}(n_0-\overline{n})\widehat{n}_k - \frac{1}{A}\partial_y c_0\partial_y\widehat{n}_k - \frac{1}{A}\partial_y\widehat{c}_k\partial_y n_0)\rangle \\ &+ 2kRe\langle i\beta u'(\frac{1}{A}\overline{n}\widehat{n}_k + \frac{2}{A}(n_0-\overline{n})\widehat{n}_k - \frac{1}{A}\partial_y c_0\partial_y\widehat{n}_k - \frac{1}{A}\partial_y\widehat{c}_k\partial_y n_0), \partial_y\widehat{n}_k\rangle \\ =:& L^{\beta}_{k,1} + L^{\beta}_{k,2}. \end{aligned} \tag{2.2.19}$$

By analogy with the α terms, the first term in (2.2.19) is further decomposed via

$$\begin{aligned}L^{\beta}_{k,1} =& 2kRe\langle i\beta u'\widehat{n}_k, \partial_y(\frac{1}{A}\overline{n}\widehat{n}_k + \frac{2}{A}(n_0-\overline{n})\widehat{n}_k - \frac{1}{A}\partial_y c_0\partial_y\widehat{n}_k - \frac{1}{A}\partial_y\widehat{c}_k\partial_y n_0)\rangle \\ =:& L^{\beta}_{k,10} + L^{\beta}_{k,11} + L^{\beta}_{k,12} + L^{\beta}_{k,13}.\end{aligned} \tag{2.2.20}$$

For the $L^{\beta}_{k,11}$ term in (2.2.20), we have the following, (for any fixed $B \geq 1$),

$$\begin{aligned}\left|L^{\beta}_{k,11}\right| =& \left|2kRe\langle i\beta u''\widehat{n}_k + i\beta u'\partial_y\widehat{n}_k, \frac{2}{A}(n_0-\overline{n})\widehat{n}_k\rangle\right| = \left|2kRe\langle i\beta u'\partial_y\widehat{n}_k, \frac{2}{A}(n_0-\overline{n})\widehat{n}_k\rangle\right| \\ \lesssim& \frac{1}{AB}||\partial_y\widehat{n}_k||_2^2 + \frac{B}{A}|k|^2||n_0-\overline{n}||_\infty^2||\sqrt{\beta}u'\widehat{n}_k||_2^2.\end{aligned}$$

By the bootstrap hypotheses and by choosing B, then A, large enough, this term is controlled consistent with (2.2.17). The $L^{\beta}_{k,10}$ term is treated in the same manner; we omit the details for the sake of brevity.

For the $L^{\beta}_{k,12}$ term in (2.2.20), using Lemma 2.4.2, we have that for some fixed $B \geq 1$, the following holds,

$$
\begin{aligned}
|L^{\beta}_{k,12}| \leq & \left| 2kRe\langle i\beta u'\widehat{n}_k, \frac{1}{A}(n_0 - \overline{n})\partial_y \widehat{n}_k\rangle \right| + \left| 2kRe\langle i\beta u'\widehat{n}_k, \frac{1}{A}\partial_y c_0 \partial_{yy}\widehat{n}_k\rangle \right| \\
\lesssim & \frac{1}{AB}||\partial_y \widehat{n}_k||_2^2 + \frac{B|k|^2}{A}||n_0 - \overline{n}||_\infty^2 ||\sqrt{\beta}u'\widehat{n}_k||_2^2 + \frac{1}{AB}||\sqrt{\alpha}\partial_{yy}\widehat{n}_k||_2^2 \\
& + \frac{B|k|^2 M^2}{A^{1/2}}||\sqrt{\beta}u'\widehat{n}_k||_2^2
\end{aligned}
$$

As above, by the bootstrap hypotheses, for B and A sufficiently large, this term is controlled consistent with (2.2.17).

Consider next $L^{\beta}_{k,13}$ in (2.2.20), which we integrate by parts and further subdivide as:

$$
L^{\beta}_{k,13} = 2kRe\langle i\beta u''\widehat{n}_k + i\beta u'\partial_y \widehat{n}_k, \frac{1}{A}\partial_y \widehat{c}_k \partial_y n_0\rangle =: L^{\beta}_{k,131} + L^{\beta}_{k,132}. \tag{2.2.21}
$$

For $L^{\beta}_{k,131}$, by Lemma 2.4.1 and the definition of β, we have the following,

$$
\left| L^{\beta}_{k,131} \right| \leq \frac{|k|^2}{A}||\beta u''\widehat{n}_k||_2^2 ||\partial_y n_0||_2 + \frac{1}{A}||\partial_y \widehat{c}_k||_\infty^2 ||\partial_y n_0||_2 \lesssim \frac{1}{A}||\partial_y n_0||_2 ||\widehat{n}_k||_2^2.
$$

Therefore, by the bootstrap hypotheses, for A large, this term is controlled consistent with (2.2.17). Using Lemma 2.4.1 and the definition of β, the $L^{\beta}_{k,132}$ term in (2.2.21) is handled as follows for a large constant $B \geq 1$:

$$
\left| L^{\beta}_{k,132} \right| \lesssim \frac{1}{AB}||\partial_y \widehat{n}_k||_2^2 + \frac{B}{A}||\widehat{n}_k||_2^2 ||\partial_y n_0||_2^2. \tag{2.2.22}
$$

Therefore, by the bootstrap hypotheses (in particular, (2.2.5c)), for B and A sufficiently large, this is consistent with (2.2.17).

Turn next to $L^{\beta}_{k,2}$, which we sub-divide as follows:

$$\begin{aligned} L^{\beta}_{k,2} =& 2kRe\langle i\beta u'\frac{1}{A}\overline{n}\widehat{n}_k, \partial_y \widehat{n}_k\rangle + 2kRe\langle i\beta u'\frac{2}{A}(n_0-\overline{n})\widehat{n}_k, \partial_y\widehat{n}_k\rangle \\ & - 2kRe\langle i\beta u'\frac{1}{A}\partial_y c_0 \partial_y \widehat{n}_k, \partial_y\widehat{n}_k\rangle - 2kRe\langle i\beta u'\frac{1}{A}\partial_y \widehat{c}_k \partial_y n_0, \partial_y \widehat{n}_k\rangle \\ =:& L^{\beta}_{k,20} + L^{\beta}_{k,21} + L^{\beta}_{k,22} + L^{\beta}_{k,23}. \end{aligned} \tag{2.2.23}$$

Observing that $\langle u'\partial_y c_0\sqrt{\beta}\partial_x\partial_y n, \sqrt{\beta}\partial_y n\rangle = 0$, by anti-symmetry, we have $L^{\beta}_{k,22} = 0$. For the $L^{\beta}_{k,21}$ term, we use the following straightforward estimate for a constant $B \geq 1$:

$$\left|L^{\beta}_{k,21}\right| \lesssim \frac{1}{AB}||\partial_y\widehat{n}_k||_2^2 + \frac{B||n_0-\overline{n}||^2_\infty}{A}|k|^2||\sqrt{\beta}u'\widehat{n}_k||_2^2.$$

This is consistent with (2.2.17) by the bootstrap hypotheses and B,A large. The $L^{\beta}_{k,20}$ is treated similarly, we skip the detail for brevity. The $L^{\beta}_{k,23}$ term can be estimated in the same manner as $L^{\beta}_{k,132}$ above in (2.2.22) and hence is omitted for brevity. This completes the treatment of the L^{β}_{k} term in (2.2.16).

Finally, we estimate L^{γ}_{k} in (2.2.16). We first sub-divide into four parts:

$$\begin{aligned} L^{\gamma}_k &= 2|k|^2Re\langle\gamma(u')^2\widehat{n}_k, \frac{1}{A}\overline{n}\widehat{n}_k + \frac{2}{A}(n_0-\overline{n})\widehat{n}_k - \frac{1}{A}\partial_y c_0\partial_y\widehat{n}_k - \frac{1}{A}\partial_y\widehat{c}_k\partial_y n_0\rangle \\ &=: L^{\gamma}_{k,0} + L^{\gamma}_{k,1} + L^{\gamma}_{k,2} + L^{\gamma}_{k,3}. \end{aligned} \tag{2.2.24}$$

The second term in (2.2.24) is estimated as follows for a fixed constant $B \geq 1$,

$$\begin{aligned} \left|L^{\gamma}_{k,1}\right| &\lesssim \frac{B\gamma}{A\beta}||\sqrt{\beta}u'\widehat{n}_k||_2^2||n_0-\overline{n}||^2_\infty + \frac{|k|^4}{AB}||\sqrt{\gamma}u'\widehat{n}_k||_2^2 \\ &\lesssim \frac{B}{A^{1/2}}||\sqrt{\beta}u'\widehat{n}_k||_2^2||n_0-\overline{n}||^2_\infty + \frac{|k|^4}{AB}||\sqrt{\gamma}u'\widehat{n}_k||_2^2. \end{aligned}$$

As above, this is consistent with (2.2.17) by the bootstrap hypotheses and B,A large. The term $L^{\gamma}_{k,0}$ is treated similarly, hence, we omit the details for the sake of brevity. The term $L^{\gamma}_{k,2}$ in (2.2.24) is similar. Indeed, by Lemma 2.4.2, we have for $B \geq 1$ large,

$$\left|L^{\gamma}_{k,2}\right| \lesssim \frac{B}{A^{1/2}}|k|^2||\sqrt{\beta}u'\widehat{n}_k||_2^2||n_0-\overline{n}||_1^2 + \frac{|k|^2}{AB}||\sqrt{\gamma}u'\partial_y\widehat{n}_k||_2^2.$$

As usual, this is consistent with (2.2.17) by the bootstrap hypotheses and B,A large. The $L^{\gamma}_{k,3}$ term in (2.2.24), is estimated slightly differently; using Lemma 2.4.1, we have for $B \geq 1$ large,

$$\begin{aligned}\left|L^{\gamma}_{k,3}\right| \lesssim & \frac{1}{A^{1/2}B}|k|^{5/2}||\sqrt{\gamma}u'\widehat{n}_k||_2^2 + \frac{B}{A^{3/2}}|k|^{3/2}\gamma||\partial_y\widehat{c}_k||_\infty^2||\partial_y n_0||_2^2 \\ \lesssim & \frac{1}{A^{1/2}B}|k|^{5/2}||\sqrt{\gamma}u'\widehat{n}_k||_2^2 + \frac{B}{A}||\widehat{n}_k||_2^2||\partial_y n_0||_2^2.\end{aligned}$$

This is consistent with (2.2.17) by the bootstrap hypotheses and B,A large. This completes the proof of (2.2.17), and hence, under the bootstrap hypotheses, the contributions of the L terms in (2.2.16) is absorbed by the $\mathcal{N}_k$ terms for A chosen sufficiently large.

Estimate on NL terms

As these terms are nonlinear in non-zero frequencies, it is more natural to consider all of the frequencies at once. For the NL^1_k term in (2.2.16), writing,

$$-\sum_{k\neq 0} 2Re\langle NL_k, \widehat{n}_k\rangle = \frac{2}{A}\langle n_{\neq}\nabla c_{\neq}, \nabla n_{\neq}\rangle \leq \frac{2}{A}\left\|\nabla c_{\neq}\right\|_\infty \left\|\nabla n_{\neq}\right\|_2 \left\|n_{\neq}\right\|_2.$$

By (2.4.3), for some constant $B > 0$,

$$-\sum_{k\neq 0} 2Re\langle NL_k, \widehat{n}_k\rangle \lesssim \frac{1}{AB}\left\|\nabla n_{\neq}\right\|_2^2 + \frac{B}{A}C_{2,\infty}^2\left\|n_{\neq}\right\|_2^2.$$

By first choosing B large relative to the implicit constant, and then choosing A large (relative to constants and B), these terms are absorbed by the negative terms in (2.2.16).

For the NL^{α}_k term in (2.2.16), we use (2.4.3) and the bootstrap hypotheses to deduce (using the definition of α; recall that α is a Fourier multiplier in x),

$$\begin{aligned}
2Re\sum_{k\neq 0}\langle \alpha(ik)\partial_{yy}\widehat{n}_k, NL_k\rangle &= \frac{2}{A}\langle \alpha(\partial_x)\partial_{yy}n_{\neq}, \nabla\cdot(n_{\neq}\nabla c_{\neq})\rangle \\
&\lesssim \frac{1}{A^{5/4}}\left\|\sqrt{\alpha}\partial_{yy}n_{\neq}\right\|_2\left(\|\nabla n_{\neq}\|_2\|\nabla c_{\neq}\|_\infty + \|n_{\neq}\|_2\|n_{\neq}\|_\infty\right) \\
&\lesssim \frac{1}{A^{5/4}}\left\|\sqrt{\alpha}\partial_{yy}n_{\neq}\right\|_2^2 + \frac{C_{2,\infty}^2}{A^{5/4}}\|\nabla n_{\neq}\|_2^2 + \frac{C_\infty^2}{A^{5/4}}\|n_{\neq}\|_2^2,
\end{aligned}$$

and choosing A large, these terms are absorbed by the negative terms in (2.2.16).

There are two terms in NL_k^β in (2.2.16); we estimate the first as follows via integrating by parts in x and distributing the ∇ (using also that $\beta(k)\lesssim |k|^{-1}$ and defines a self-adjoint operator, Lemma 2.4.3, and that u does not depend on x):

$$\begin{aligned}
&-2\sum_{k\neq 0}Re\langle ik\beta(ik)u'NL_k, \partial_y\widehat{n}_k\rangle \\
&\qquad\lesssim \frac{1}{A}\|n_{\neq}\|_2\|n_{\neq}\|_\infty\|\partial_y n_{\neq}\|_2 + \frac{1}{A}\|\beta u'\partial_x\partial_y n_{\neq}\|_2\|\nabla c_{\neq}\|_\infty\|\nabla n_{\neq}\|_2 \\
&\qquad\lesssim \frac{C_\infty^2}{A^{3/4}}\|n_{\neq}\|_2^2 + \frac{1}{A^{5/4}}\|\nabla n_{\neq}\|_2^2 + \frac{C_{2,\infty}^2}{A^{5/4}}\|\sqrt{\gamma}u'\partial_x\partial_y n_{\neq}\|_2^2. \qquad (2.2.25)
\end{aligned}$$

Choosing A large, these terms are absorbed by the negative terms in (2.2.16). For the second term in NL_k^β we use

$$\begin{aligned}
2Re\sum_{k\neq 0}\langle \beta(ik)u'ik\widehat{n}_k, \partial_y NL_k\rangle &= \frac{2}{A}\langle \beta(\partial_x)u'\partial_x n_{\neq}, \partial_y\nabla\cdot(n_{\neq}\nabla c_{\neq})\rangle \\
&= -\frac{2}{A}\langle \beta(\partial_x)u''\partial_x n_{\neq}, \nabla\cdot(n_{\neq}\nabla c_{\neq})\rangle - \frac{2}{A}\langle \beta(\partial_x)u'\partial_x\partial_y n_{\neq}, \nabla\cdot(n_{\neq}\nabla c_{\neq})\rangle \\
&= NL_{k,1}^\beta + NL_{k,2}^\beta.
\end{aligned}$$

Using Lemma 2.4.3, $\beta(\partial_x) = \epsilon_\beta|\partial_x|^{-1}$, and that u does not depend on x, we have,

$$\begin{aligned}
\left|NL_{k,1}^\beta\right| &\lesssim \frac{1}{A}\|\beta u''\partial_x n_{\neq}\|_2\left(\|\nabla n_{\neq}\|_2\|\nabla c_{\neq}\|_\infty + \|n_{\neq}\|_2\|n_{\neq}\|_\infty\right) \\
&\lesssim \frac{1}{BA}\|\nabla n_{\neq}\|_2^2 + \frac{BC_{2,\infty}^2}{A}\|n_{\neq}\|_2^2,
\end{aligned}$$

yielding terms which are absorbed by the negative terms in (2.2.16) for A sufficiently large. The treatment of $NL^{\beta}_{k,2}$ is similar to (2.2.25), hence it is omitted for the sake of brevity.

Turn finally to term NL^{γ}_{k} in (2.2.16) associated with γ:

$$\begin{aligned}-2Re\sum_{k\neq 0}\langle |k|^2\gamma(k)u'\widehat{n}_k, u'NL_k\rangle &= -\frac{2}{A}\langle\gamma(\partial_x)u'\partial_x n_{\neq}, u'\partial_x\nabla\cdot(n_{\neq}\nabla c_{\neq})\rangle\\ &= \frac{2}{A}\langle\gamma(\partial_x)u'\partial_x\nabla n_{\neq}, u'\partial_x(n_{\neq}\nabla c_{\neq})\rangle + \frac{4}{A}\langle\gamma(\partial_x)u'u''\partial_x n_{\neq}, \partial_x(n_{\neq}\partial_y c_{\neq})\rangle\\ &=: NL^{\gamma}_{k,1} + NL^{\gamma}_{k,2}. \end{aligned} \tag{2.2.26}$$

Then we use $\gamma(\partial_x) = \epsilon_\gamma A^{1/2}|\partial_x|^{-3/2}$, interpolation, and Lemma 2.4.3 to deduce the following bound for $NL^{\gamma}_{k,1}$:

$$\begin{aligned}&NL^{\gamma}_{k,1}\\ &\lesssim\frac{1}{A}||\sqrt{\gamma}u'\partial_x\nabla n_{\neq}||_2||\sqrt{\gamma}\partial_x(u'n_{\neq}\nabla c_{\neq})||_2 \lesssim \frac{1}{A^{3/4}}\|\sqrt{\gamma}u'\partial_x\nabla n_{\neq}\|_2\left\||\partial_x|^{1/4}(u'n_{\neq}\nabla c_{\neq})\right\|_2\\ &\lesssim\frac{1}{A^{3/4}}\|\sqrt{\gamma}u'\partial_x\nabla n_{\neq}\|_2\|u'n_{\neq}\nabla c_{\neq}\|_2^{3/4}\|\partial_x(u'n_{\neq}\nabla c_{\neq})\|_2^{1/4}\\ &\lesssim\frac{1}{BA}\|\sqrt{\gamma}u'\partial_x\nabla n_{\neq}\|_2^2 + \frac{BC^2_{2,\infty}}{A^{1/2}}\|u'n_{\neq}\|_2^{3/2}\left(\|u'\partial_x n_{\neq}\|_2^{1/2} + \|n_{\neq}\|_2^{1/2}\right).\end{aligned}$$

Hence, for B chosen large, then A chosen large, we may absorb these contributions in the negative terms in (2.2.16).

Next we estimate the $NL^{\gamma}_{k,2}$ term in (2.2.26),

$$NL^{\gamma}_{k,2}\lesssim\frac{1}{A^{3/4}}||\sqrt{\gamma}u'|\partial_x|^{5/4}n_{\neq}||_2||n_{\neq}\nabla c_{\neq}||_2\lesssim\frac{1}{A^{3/4}}||\sqrt{\gamma}u'|\partial_x|^{5/4}n_{\neq}||_2^2 + \frac{C^2_{2,\infty}}{A^{3/4}}||n_{\neq}||_2^2.$$

Hence, for A chosen large, we may absorb these contributions in the negative terms in (2.2.16). This finishes the estimate of the NL terms.

Nonzero mode $L^2_t \dot{H}^1_{x,y}$ estimate (2.2.7a)

The nonzero mode $L^2_t \dot{H}^1_{x,y}$ estimate (2.2.7a) comes from an estimate on the $\frac{d}{dt}||\widehat{n}_{\neq}||_2^2$ and the knowledge that $||\widehat{n}_{\neq}||_2^2$ is bounded by $4C_{ED}||n_{in}||^2_{H^1}$ from Hypothesis (2.2.5b). Indeed, from (2.2.2) and Lemma 2.4.2, there holds for some universal constant B,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||n_{\neq}||_2^2 \leq & -\frac{3}{4A}||\nabla n_{\neq}||_2^2 + \frac{1}{A}||n_{\neq}||_2^2\left(||n_0-\overline{n}||_\infty + \|n_0\|_\infty + \|\nabla c_0\|_\infty^2\right) \\ & + \frac{1}{A}||\partial_y c_{\neq}||_4||\partial_y n_0||_2 \, \|n_{\neq}\|_4 + \frac{1}{2A}\,\|n_{\neq}\|_3^3 \\ \leq & -\frac{1}{2A}||\nabla n_{\neq}||_2^2 + \frac{B(C_{2,\infty}^2 + C_{\dot{H}^1}^2)}{A}||n_{\neq}||_2^2. \end{aligned} \tag{2.2.27}$$

By (2.2.5b), the time integral of $\frac{1}{A}||n_{\neq}||_2^2$ is estimated as

$$\int_0^{T_\star} \frac{1}{A}||n_{\neq}(t)||_2^2 dt \lesssim \frac{\log A}{A^{1/2}}. \tag{2.2.28}$$

Hence, by applying (2.2.5a), integrating (2.2.27), and choosing A large, there holds

$$\frac{1}{A}\int_0^{T_\star} ||\nabla n_{\neq}||_2^2 dt \leq \frac{1}{A^{1/4}} + 2||n_{in}||_2^2 \leq 4||n_{in}||_2^2.$$

As a result, we have proved (2.2.7a).

Zero mode estimate (2.2.7c)

First, by non-negativity, note that $\|n_0\|_{L^1} = \|n\|_{L^1} = M$ is constant in time. We begin by estimating $||n_0-\overline{n}||_2^2$, then go on to estimate $||\partial_y n_0||_2^2$. From (2.2.1) we have, by Minkowski's inequality and (2.4.2),

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||n_0-\overline{n}||_2^2 \\
&= -\frac{1}{A}||\partial_y n_0||_2^2 + \frac{1}{A}\langle \partial_y n_0, \partial_y c_0 (n_0-\overline{n})\rangle + \frac{1}{A}\langle \partial_y n_0, \partial_y c_0 \overline{n}\rangle + \frac{1}{A}\langle \partial_y n_0, (\partial_y c_{\neq} n_{\neq})_0\rangle \\
&\leq -\frac{1}{4A}||\partial_y n_0||_2^2 + \frac{1}{A}||\partial_y c_0||_\infty^2 ||n_0-\overline{n}||_2^2 + \frac{M^2}{A}||\partial_y c_0||_2^2 + \frac{1}{A}||(\partial_y c_{\neq} n_{\neq})_0||_{L^2(\mathbb{T})}^2 \\
&\leq -\frac{1}{4A}||\partial_y n_0||_2^2 + \frac{BM^2}{A}||n_0-\overline{n}||_2^2 + \frac{1}{A}||\partial_y c_{\neq} n_{\neq}||_{L^2(\mathbb{T}^2)}^2.
\end{aligned}
$$

Recall the following Nash inequality on $\mathbb{T}$, under the assumption that $\int_{\mathbb{T}} \rho dx = 0$:

$$
||\rho||_{L^2(\mathbb{T})} \lesssim ||\rho||_{L^1(\mathbb{T})}^{2/3} ||\partial_y \rho||_{L^2(\mathbb{T})}^{1/3}. \tag{2.2.29}
$$

Hence,

$$
-||\partial_y n_0||_2^2 \lesssim -\frac{||n_0-\overline{n}||_2^6}{||n-\overline{n}||_1^4} \lesssim -\frac{||n_0-\overline{n}||_2^6}{M^4}.
$$

Therefore, for a possibly larger universal constant $B>0$, there holds

$$
\begin{aligned}
\frac{d}{dt}||n_0-\overline{n}||_2^2 &\leq -\frac{1}{AB}\frac{||n_0-\overline{n}||_2^6}{M^4} + \frac{BM^2}{A}||n_0-\overline{n}||_2^2 + \frac{B}{A}||\nabla n_{\neq}||_2^{2/3}||n_{\neq}||_2^{10/3} \\
&\leq -\frac{1}{AB}\frac{||n_0-\overline{n}||_2^2}{M^4}(||n_0-\overline{n}||_2^4 - BM^6) + \left\{\frac{1}{AB}||\nabla n_{\neq}||_2^2 + \frac{B}{A}||n_{\neq}||_2^5\right\}.
\end{aligned}
$$

Define the following quantity G to be the time integral of the terms in the $\{\cdot\}$:

$$
G(t) := \int_0^t \frac{1}{AB}||\nabla n_{\neq}||_2^2 + \frac{B}{A}||n_{\neq}||_2^5 d\tau, \quad t \geq 0. \tag{2.2.30}
$$

By the bootstrap hypotheses, there holds $0 \leq G \lesssim ||n_{in}||_2^2 + ||n_{in}||_{H^1}^5 A^{-1/2}\log A$.

Applying this definition,

$$
\begin{aligned}
&\frac{d}{dt}(||n_0-\overline{n}||_2^2 - G(t)) \leq -\frac{1}{AB}\frac{||n_0-\overline{n}||_2^2}{M^4}(||n_0-\overline{n}||_2^4 - BM^6) \\
&\leq -\frac{1}{AB}\frac{||n_0-\overline{n}||_2^2}{M^4}(||n_0-\overline{n}||_2^2 - G(t) - \sqrt{B}M^3)(||n_0-\overline{n}||_2^2 + \sqrt{B}M^3).
\end{aligned}
$$

Choosing A large relative to $||n_{in}||_{H^1}^5$ and universal constants, we have

$$||n_0 - \overline{n}||_2^2 \lesssim \overline{n}^2 + G(t) + M^3 + ||n_{in}||_2^2 \lesssim M^2 + ||n_{in}||_2^2 + M^3 =: C_{L^2}^2(||n_{in}||_2^2, M). \tag{2.2.31}$$

This completes the estimate on $||n_0||_2$, which implies the first estimate in conclusion (2.2.7c).

Next, we use (2.2.31) to estimate $||\partial_y n_0||_2^2$. From (2.2.1) and Minkowski's integral inequality, we have for some $B > 0$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||\partial_y n_0||_2^2 \leq & -\frac{1}{2A}||\partial_{yy} n_0||_2^2 + \frac{B}{A}||\partial_{yy} c_0 n_0||_2^2 + \frac{B}{A}||\partial_y c_0 \partial_y n_0||_2^2 + \frac{B}{A}||(\partial_{yy} c_{\neq} n_{\neq})_0||_{L^2(\mathbb{T})}^2 \\
& + \frac{B}{A}||(\partial_y c_{\neq} \partial_y n_{\neq})_0||_{L^2(\mathbb{T})}^2 \\
\leq & -\frac{1}{2A}||\partial_{yy} n_0||_2^2 + \frac{B}{A}||\partial_{yy} c_0 n_0||_2^2 + \frac{B}{A}||\partial_y c_0 \partial_y n_0||_2^2 \\
& + \frac{B}{A}||n_{\neq}||_{L^2(\mathbb{T}^2)}^2 ||n_{\neq}||_{L^\infty(\mathbb{T}^2)}^2 + \frac{B}{A}||\partial_y c_{\neq}||_{L^\infty(\mathbb{T}^2)}^2 ||\partial_y n_{\neq}||_{L^2(\mathbb{T}^2)}^2.
\end{aligned} \tag{2.2.32}$$

Using (2.4.3) in the above estimate (2.2.32), we have for some B (possibly adjusted from above),

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||\partial_y n_0||_2^2 \leq & -\frac{1}{2A}||\partial_{yy} n_0||_2^2 + \frac{B}{A}||\partial_{yy} c_0 n_0||_2^2 + \frac{B}{A}||\partial_y c_0 \partial_y n_0||_2^2 \\
& + \frac{B}{A}||n_{\neq}||_{L^2(\mathbb{T}^2)}^2 ||n_{\neq}||_{L^\infty(\mathbb{T}^2)}^2 + \frac{BC_{2,\infty}^2}{A}||\partial_y n_{\neq}||_{L^2(\mathbb{T}^2)}^2.
\end{aligned} \tag{2.2.33}$$

Analogously to (2.2.30), we define

$$G(t) := \int_0^t \frac{B}{A}||n_{\neq}||_{L^2(\mathbb{T}^2)}^2 ||n||_{L^\infty(\mathbb{T}^2)}^2 + \frac{BC_{2,\infty}^2}{A}||\partial_y n_{\neq}||_{L^2(\mathbb{T}^2)}^2 d\tau, \quad \forall t \in [0, T_\star]. \tag{2.2.34}$$

By the bootstrap hypothesis (2.2.5a),(2.2.5b) and (2.2.5d) and choosing A large, there holds:

$$G(t) \lesssim \int_0^{T_\star} \frac{C_{2,\infty}^4}{A} e^{-\frac{ct}{A^{1/2}\log A}} + \frac{1}{A}C_{2,\infty}^2 ||\partial_y n_{\neq}||_{L^2(\mathbb{T}^2)}^2 dt \lesssim C_{2,\infty}^4.$$

Therefore, from (2.2.33), we have for some $B > 0$ (using also $\|\partial_y n_0\|_2 \lesssim \|n_0\|_2^{1/2} \|\partial_{yy} n_0\|_2^{1/2}$),

$$\begin{aligned}\frac{d}{dt}(||\partial_y n_0||_2^2 - 2G(t)) \leq & -\frac{||\partial_y n_0||_2^4}{ABC_{L^2}^2} + \frac{B}{A}||n_0||_2^2||\partial_y n_0||_2^2 + \frac{B}{A}||n_0 - \overline{n}||_2^2||\partial_y n_0||_2^2 \\ \leq & -\frac{1}{ABC_{L^2}^2}||\partial_y n_0||_2^2(||\partial_y n_0||_2^2 - 2G(t) - C_{L^2}^4 B^2).\end{aligned}$$

Integrating and applying (2.2.34) implies the following:

$$||\partial_y n_0||_2^2 \leq 2G(t) + C_{L^2}^4 B + \|\partial_y n_{in}\|_2^2 \lesssim C_{2,\infty}^4 + \|\partial_y n_{in}\|_2^2 .$$

Hence, by choosing $C_{\dot{H}^1}^2 \gg C_{2,\infty}^4 + \|\partial_y n_{in}\|_2^2$, we complete the proof of (2.2.7c).

L^∞ uniform control (2.2.7d)

By the bootstrap hypothesis (2.2.5b) and (2.2.5c), it follows that $||n||_2^2 \lesssim ||n_{in}||_{H^1}^2 + C_{L^2}^2(||n_{in}||_2, M) < \infty$ for $t \in [0, T_\star)$. As the L^2 norm is subcritical for 2D Patlak-Keller-Segel, it is standard (see e.g. [29,75,81] and the references therein) that this implies a uniform-in-time L^∞ bound which depends only on $\|n\|_{L^\infty(0,T_\star;L^2)}$. Therefore, by choosing C_∞ appropriately, we have (2.2.7d):

$$||n||_{L^\infty(0,T_\star;L^\infty)} \leq 2C_\infty = 2C_\infty(||n_{in}||_{H^1}).$$

This completes the proof of Proposition 3 and hence Theorem 1.

2.3 Proof of Theorem 2 in the case $\mathbb{T}^3$

Next we turn to the 3D case. Heuristically, we expect the problem to be effectively L^1 critical with critical mass 8π. As in e.g. [28], we will need to use the free energy to obtain such a precise control.

2.3.1 Basic setting and bootstrap

Consider the Patlak-Keller-Segel equation with advection on $\mathbb{T}^3$:

$$\partial_t n + u(y_1)\partial_x n + \frac{1}{A}\nabla\cdot(\nabla c n) = \frac{1}{A}\Delta n, \quad -\Delta c = n - \overline{n}, \quad n(\cdot,0) = n_{in}, \quad (2.3.1)$$

where $(x, y_1, y_2) \in \mathbb{T}^3$. We use the notation

$$(x, y_1, y_2) \in \mathbb{T}\times\mathbb{R}^2, \quad dy = dy_1 dy_2, \quad \nabla_y = (\partial_{y_1}, \partial_{y_2}), \quad \Delta_y = \partial_{y_1y_1} + \partial_{y_2y_2}.$$

As above, the bootstrap argument is applied to prove Theorem 2. For constants $C_{ED}, C_{L^2}, C_{\dot{H}^1}, C_\infty$ determined by the proof, define $T_\star$ to be the end-point of the largest interval $[0, T_\star]$ such that the following hypotheses hold for all $T \le T_\star$:

(1) Nonzero mode $L_t^2\dot{H}_{x,y}^1$ estimates:

$$\frac{1}{A}\int_0^{T_\star} ||\nabla_{x,y} n_{\neq}||_{L^2(\mathbb{T}^3)}^2 dt \le \; 8||n_{in}||_2^2; \quad (2.3.2a)$$

(2) Nonzero mode enhanced dissipation estimate:

$$||n_{\neq}||_{L^2(\mathbb{T}^3)}^2 \le \; 4C_{ED}||n_{in}||_{H^1}^2 e^{-\frac{ct}{A^{1/2}\log A}}, \quad (2.3.2b)$$

where c is a small number independent of A;

(3) Zero mode time independent estimate:

$$||n_0||_{L_t^\infty(0,T_\star;L_y^2)} \le \; 4C_{L^2}, \quad (2.3.2c)$$

$$||\partial_y n_0||_{L_t^\infty(0,T_\star;L_y^2)} \le \; 4C_{\dot{H}^1}; \quad (2.3.2d)$$

(4) $L_t^\infty L_{x,y}^\infty$ estimate of the whole solution:

$$||n||_{L_t^\infty(0,T_\star;L_{x,y}^\infty)} \le 4C_\infty. \quad (2.3.2e)$$

As in the two-dimensional case, we introduce the following constant:

$$C_{2,\infty} := 1 + M + C_{ED}^{1/2}||n_{in}||_{H^1} + C_{L^2} + C_\infty. \tag{2.3.3}$$

Here C_{ED} just depends on the properties of the shear flow u. C_{L^2} just depends on the initial data n_{in}, C_∞ depends on n_{in} and C_{L^2}, and $C_{\dot{H}^1}$ depends on n_{in}, C_{L^2} and C_∞. Recall that we assume that the data is initially bounded strictly away from zero from below:

$$\min_{(x,y_1,y_2)\in\mathbb{T}^3} n_{in}(x, y_1, y_2) \geq q > 0. \tag{2.3.4}$$

As in §2.2, by local well-posedness of mild solutions, the quantities on the left-hand sides of (2.3.2a), (2.3.2b), (2.3.2c), and (2.3.2e) take values continuously in time. Moreover, the inequalities are all satisfied with the 4's replaced by 2's for t sufficiently small. By the standard continuation criteria for (2.1.1), the solution exists and remains smooth on an interval $(0, t_0]$, with $t_0 > T_\star$ such that $t_0 - T_\star$ can be taken to depend only on $\|n(T_\star)\|_{L^2}$. By continuity, the following proposition shows that the solution is global and satisfies the a priori estimates **(H)** for all time.

Proposition 7. *For all n_{in} and u, if the conditions $||n_{in}||_{L^1(\mathbb{T}^3)} < 8\pi$, (2.3.4) and the above bootstrap hypothesis (**H**) are satisfied, there exists an $A_0(\|n_{in}\|_{L^\infty}, \|n_{in}\|_{H^1}, M, q)$ such that if $A > A_0$ then the following conclusions, referred to as (**C**), hold on the* *interval* $[0, T_\star]$:

$$(1) \quad \frac{1}{A}\int_0^{T_\star} ||\nabla_{x,y} n_{\neq}||_2^2 dt \leq 4||n_{in}||_2^2; \tag{2.3.5a}$$

$$(2) \quad ||n_{\neq}||_2^2 \leq 2C_{ED}||n_{in}||_{H^1}^2 e^{-\frac{ct}{A^{1/2}\log A}}; \tag{2.3.5b}$$

$$(3) \quad \begin{cases} ||n_0||_{L_t^\infty(0,T_\star;L_y^2)} \leq 2C_{L^2}, \\ ||\partial_y n_0||_{L_t^\infty(0,T_\star;L_y^2)} \leq 2C_{\dot{H}^1}; \end{cases} \tag{2.3.5c}$$

$$(4) \quad ||n||_{L_t^\infty(0,T_\star;L_{x,y}^\infty)} \leq 2C_\infty. \tag{2.3.5d}$$

The main new difficulty in the 3D case arises in the proof of (2.3.5c): even if the non-zero modes could be neglected entirely, the evolution of n_0 would be given by the L^1 critical parabolic-elliptic Patlak-Keller-Segel. In [28], the free energy, together with the logarithmic Hardy-Littlewood-Sobolev inequality (see e.g. [32]), was applied to prove global existence up to the critical mass. Similarly, here we will estimate the 2D free energy of n_0 (no longer a conserved quantity) and apply the 2D logarithmic Hardy-Littlewood-Sobolev inequality on n_0. We are met with a small difficulty in estimating the effect of non-zero frequencies on the free energy in regions of low density; to help deal with this, we utilize a pointwise lower bound on the solution (See Lemma 2.3.1 below).

2.3.2 A priori estimates

Estimate on the zero mode (2.3.5c)

The idea of the proof is to exploit the fact that the shear flow strongly damps the nonzero frequencies. Hence, even though the equation (2.3.1) is posed on $\mathbb{T}^3$, we can approximate the evolution as the classical Keller-Segel equation in $\mathbb{T}^2$ with a rapidly decaying perturbation $(\nabla \cdot (\nabla c_{\neq} n_{\neq}))_0$ coming from the nonzero modes.

First we derive an exponentially decreasing lower bound for n.

Lemma 2.3.1. *Under the bootstrap hypotheses **(H)** and (2.3.4), there holds the following pointwise lower bound on the solution for all $t \in [0, T^*]$*

$$\left\| \frac{1}{n_0(t)} \right\|_\infty \leq \left\| \frac{1}{n(t)} \right\|_\infty \leq q^{-1} e^{\frac{\overline{n}}{A} t}. \tag{2.3.6}$$

Proof. The equation (2.3.1) implies that at the point $(x_{\min}(t), y_{\min}(t))$ where the minimum in space of the solution is achieved, the following inequality is satisfied:

$$\begin{aligned}(\partial_t n)(x_{\min}, y_{\min}) =& \frac{1}{A}(\Delta n)(x_{\min}, y_{\min}) + \frac{1}{A}(n(x_{\min}, y_{\min}) - \overline{n}) n(x_{\min}, y_{\min}) \\ \geq& -\frac{1}{A}\overline{n} n(x_{\min}, y_{\min}),\end{aligned}$$

which implies that

$$\frac{d}{dt} n_{\min}(t) \geq -\frac{1}{A}\overline{n} n_{\min}(t).$$

Combining this differential inequality with (2.3.4), this yields

$$n_{\min}(t) \geq q e^{-\frac{\overline{n}}{A} t}, \tag{2.3.7}$$

which completes the lemma. □

Next, we study the classical 2D free energy of n_0 on $\mathbb{T}^2$:

$$\mathcal{F}[n_0] = \int_{\mathbb{T}^2} n_0 \log n_0 - \frac{1}{2}(n_0 - \overline{n}) c_0 dy.$$

Lemma 2.3.2. *Under the bootstrap hypotheses **(H)** and (2.3.4), for A sufficiently large, there holds the following uniform bound on $t \in [0, T^*]$,*

$$\mathcal{F}[n_0(t)] \leq 2\mathcal{F}[(n_{in})_0]. \tag{2.3.8}$$

Proof. By applying the hypothesis (2.3.2b,2.3.2e), Minkowski's integral inequality, and (2.3.6), the time derivative of $\mathcal{F}[n_0]$ can be estimated as follows

$$\begin{aligned}\frac{d}{dt}\mathcal{F}[n_0] &= -\frac{1}{A}\int n_0|\nabla_y \log n_0 - \nabla_y c_0|^2 dy - \frac{1}{A}\int (\nabla_y c_{\neq} n_{\neq})_0 \cdot (\nabla_y \log n_0 - \nabla_y c_0) dy \\ &\le -\frac{1}{2A}\int n_0|\nabla_y \log n_0 - \nabla_y c_0|^2 dy + \frac{1}{2A}\int \frac{|(\nabla_y c_{\neq} n_{\neq})_0|^2}{n_0} dy \\ &\le -\frac{1}{2A}\int n_0|\nabla_y \log n_0 - \nabla_y c_0|^2 dy + \frac{1}{2A}\left\|\frac{1}{n_0}\right\|_\infty \|\nabla_y c_{\neq}\|^2_{L^2(\mathbb{T}^3)} \|n_{\neq}\|^2_{L^\infty(\mathbb{T}^3)} \\ &\lesssim -\frac{1}{2A}\int n_0|\nabla_y \log n_0 - \nabla_y c_0|^2 dy + \frac{C^4_{2,\infty}}{2Aq} e^{\left(\frac{\overline{n}}{A} - \frac{c}{A^{1/2}\log A}\right)t}. \end{aligned} \tag{2.3.9}$$

Note that for A sufficiently large yields:

$$\int_0^\infty \frac{C^4_{2,\infty}}{2Aq} e^{\frac{\overline{n}}{A}t - \frac{ct}{A^{1/2}\log A}} dt \le \int_0^\infty \frac{C^4_{2,\infty}}{2Aq} e^{-\frac{ct}{2A^{1/2}\log A}} dt \lesssim \frac{C^4_{2,\infty}}{2Aq} A^{1/2}\log A. \tag{2.3.10}$$

Combining (2.3.9) and (2.3.10) yields the uniform time (2.3.8). □

Next, we use (2.3.8) to get a bound on the entropy:

Lemma 2.3.3. *If $||n_{in}||_{L^1} < 8\pi$ and (2.3.8) hold and A is chosen large enough, there exists a constant $C_{L\log L}(n_{in})$ such that*

$$\int n_0 \log^+ n_0 dy \le C_{L\log L}(n_{in}). \tag{2.3.11}$$

Proof. The following logarithmic Hardy-Littlewood-Sobolev inequality on a compact manifold is needed:

Theorem 6. *[107] Let $\mathcal{M}$ be a two-dimensional, Riemannian, compact manifold. For all $M > 0$, there exists a constant $C(M)$ such that for all non-negative functions $f \in L^1(\mathcal{M})$ such that $f\log f \in L^1$, if $\int_{\mathcal{M}} f dx = M$, then*

$$\int_{\mathcal{M}} f\log f dx + \frac{2}{M}\iint_{\mathcal{M}\times\mathcal{M}} f(x)f(y)\log d(x,y) dxdy \ge -C(M), \tag{2.3.12}$$

where $d(x,y)$ is the distance on the Riemannian manifold.

Let $y \in \mathbb{T}^2$ be fixed. Define the cut-off function $\varphi_y(z) \in C^\infty$ such that

$$\begin{aligned} supp(\varphi_y) =& B(y, 1/4), \\ \varphi_y(z) \equiv& 1, \forall z \in B(y, 1/8), \\ supp(\nabla\varphi_y(z)) \subset& \overline{B}(y, 1/4) \backslash B(y, 1/8). \end{aligned}$$

By extending $n_0(z)$ and $c_0(z)$ periodically to $\mathbb{R}^2$, we can rewrite the equation $-\Delta c_0 = n_0 - \overline{n}$ on $\mathbb{T}^2$ such that it is posed on $\mathbb{R}^2$:

$$-\Delta_z(\varphi_y(z)c_0(z)) = (n_0(z) - \overline{n})\varphi_y(z) - 2\nabla_z\varphi_y(z) \cdot \nabla_z c_0(z) - \Delta_z\varphi_y(z)c_0(z).$$

Using the fundamental solution of the Laplacian on $\mathbb{R}^2$:

$$\begin{aligned} c_0(y) =& c_0(y)\varphi_y(y) \\ =& -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|y-z|\Big((n_0(z) - \overline{n})\varphi_y(z) - 2\nabla_z\varphi_y(z) \cdot \nabla_z c_0(z) - \Delta_z\varphi_y(z)c_0(z)\Big)dz \\ =& -\frac{1}{2\pi}\int_{|y-z|\leq\frac{1}{4}} \log|y-z|(n_0(z) - \overline{n})\varphi_y(z)dz \\ & -\frac{1}{\pi}\int_{|y-z|\leq\frac{1}{4}} \nabla_z \cdot (\log|y-z|\nabla_z\varphi_y(z))c_0(z)dz \\ & +\frac{1}{2\pi}\int_{|y-z|\leq\frac{1}{4}} \log|y-z|\Delta_z\varphi_y(z)c_0(z)dz. \end{aligned}$$

Due to the support of φ_y, we can identify the above with an analogous integral on $\mathbb{T}^2$ with $|y-z|$ replaced by $d(y,z)$. Therefore, we have the following estimate on the interaction energy,

$$
\begin{aligned}
&-\frac{1}{2}\int_{\mathbb{T}^2}(n_0(y)-\overline{n})c(y)dy\\
=&\frac{1}{4\pi}\iint_{\mathbb{T}^2\times\mathbb{T}^2}\log d(y,z)n_0(y)n_0(z)dzdy-\frac{1}{4\pi}\iint_{d(y,z)>\frac{1}{8}}\log d(y,z)n_0(y)n_0(z)dzdy\\
&-\frac{1}{2\pi}\overline{n}\iint_{d(y,z)\leq\frac{1}{8}}\log d(y,z)n_0(y)dzdy+\frac{1}{4\pi}\overline{n}^2\iint_{d(y,z)\leq\frac{1}{8}}\log d(y,z)dzdy\\
&+\frac{1}{4\pi}\iint_{\frac{1}{8}\leq d(y,z)\leq\frac{1}{4}}\log d(y,z)(n_0(y)-\overline{n})(n_0(z)-\overline{n})\varphi_y(z)dzdy\\
&+\frac{1}{2\pi}\iint_{\frac{1}{8}\leq d(y,z)\leq\frac{1}{4}}(n_0(y)-\overline{n})\nabla_z\cdot(\log d(y,z)\nabla_z\varphi_y(z))c_0(z)dzdy\\
&-\frac{1}{4\pi}\iint_{\frac{1}{8}\leq d(y,z)\leq\frac{1}{4}}(n_0(y)-\overline{n})\log d(y,z)\Delta_z\varphi_y(z)c_0(z)dzdy.
\end{aligned}
$$

The 2nd, 3rd, 4th, 5th terms in the last line are bounded below by $-BM^2$ for some constant $B>0$. The 6th and 7th terms are bounded below by $-BM||c_0||_{L^1}$ for some constant $B>0$, using the fact that $\nabla_z\cdot(\log|y-z|\nabla_z\varphi_y(z))$ and $\log|y-z|\Delta_z\varphi_y(z)$ are bounded in the region $\frac{1}{8}\leq|y-z|\leq\frac{1}{4}$. Denoting $K(z)$ to be the fundamental solution of the Laplacian on $\mathbb{T}^2$, by Young's inequality, we have

$$
||c_0||_{L^1(\mathbb{T}^2)}=||K*(n_0-\overline{n})||_{L^1(\mathbb{T}^2)}\leq||K||_{L^1(\mathbb{T}^2)}||n_0-\overline{n}||_{L^1(\mathbb{T}^2)}\lesssim M.
$$

The calculation above hence implies the following for some constant $B>0$,

$$
-\frac{1}{2}\int(n_0-\overline{n})(-\Delta)^{-1}(n_0-\overline{n})dy\geq\frac{1}{4\pi}\iint_{\mathbb{T}^2\times\mathbb{T}^2}\log d(z,y)n_0(z)n_0(y)dzdy-BM^2.
$$

Combining this estimate with (2.3.8) yields

$$
\begin{aligned}
2\mathcal{F}[n_{in}]\geq\left(1-\frac{M}{8\pi}\right)\int_{\mathbb{T}^2}n_0\log n_0dy+\frac{M}{8\pi}\Bigg(\int_{\mathbb{T}^2}n_0\log n_0dy\\
+\frac{2}{M}\iint_{\mathbb{T}^2\times\mathbb{T}^2}n_0(z)\log d(z,y)n_0(y)dzdy\Bigg)-BM^2.
\end{aligned}
$$

Applying (2.3.12) in the above estimate, we obtain

$$2\mathcal{F}[n_{in}] \geq \left(1 - \frac{M}{8\pi}\right) \int_{\mathbb{T}^2} n_0 \log n_0 dy - C(M) - BM^2,$$

which results in

$$\int_{\mathbb{T}^2} n_0 \log n_0 dy \leq \frac{2\mathcal{F}[n_{in}] + C(M) + BM^2}{1 - \frac{M}{8\pi}}.$$

As $x \log x$ is bounded below, this implies the following for a suitable constant $C_{L\log L}$ depending only on the initial data due to $y \in \mathbb{T}^2$:

$$\int_{\mathbb{T}^2} n_0 \log^+ n_0 dy \leq C_{L\log L}(n_{in}) < \infty.$$

This completes the proof of the lemma. $\square$

Lemma 2.3.4. *The bound on the entropy (2.3.11) yields a uniform in time L^2 bound of n_0, that is,*

$$||n_0||_{L^2} \leq C_{L^2}(n_{in}). \tag{2.3.13}$$

Proof. The proof is a small variation of classical Patlak-Keller-Segel techniques (see e.g. [28, 75]).

The following Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{T}^d$ is needed in the proof:

Theorem 3. *(Lemma 9.2 in [80]) Suppose $v \in C^\infty(\mathbb{T}^d), d \geq 2$, and the set where v vanishes is nonempty. Assume that $q, r > 0, \infty > q > r$, and $\frac{1}{d} - \frac{1}{2} + \frac{1}{r} > 0$. Then*

$$||v||_{L^q} \leq C(d,q)||\nabla v||_{L^2}^a ||v||_{L^r}^{1-a}, \quad a = \frac{\frac{1}{r} - \frac{1}{q}}{\frac{1}{d} - \frac{1}{2} + \frac{1}{r}}. \tag{2.3.14}$$

For a fixed d, the constant $C(d,q)$ is bounded uniformly when q varies in any compact set in $(0,\infty)$.

Let $K > \max\{1, \overline{n}\}$ be a constant, to be chosen later. Observe that (2.3.11) implies the following:

$$\int (n_0 - K)_+ dy \leq \int_{n_0 > K} n_0 dx \leq \frac{1}{\log(K)} \int_{n_0 > K} n_0 \log^+(n_0) dy \leq \frac{C_{L \log L}}{\log(K)}. \quad (2.3.15)$$

Next, via (2.2.1), there holds

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int (n_0 - K)_+^2 dy \\
&= -\frac{1}{A}\int |\nabla((n_0 - K)_+)|^2 dy + \frac{1}{2A}\int (n_0 - K)_+^3 dy + \frac{3K - \overline{n}}{2A}\int (n_0 - K)_+^2 dy \\
&\quad + \frac{K^2 - K\overline{n}}{A}\int (n - K)_+ dy + \frac{1}{A}\int \nabla(n_0 - K)_+ \cdot (\nabla_y c_{\neq} n_{\neq})_0 dy \\
&\leq -\frac{7}{8A}\int |\nabla((n_0 - K)_+)|^2 dy + \frac{1}{2A}\int (n_0 - K)_+^3 dy + \frac{3K - \overline{n}}{2A}\int (n_0 - K)_+^2 dy \\
&\quad + \frac{(K^2 - K\overline{n})M}{A} + \frac{B}{A}||(\nabla_y c_{\neq} n_{\neq})_0||_{L^2(\mathbb{T}^2)}^2. \qquad (2.3.16)
\end{aligned}$$

We start with the second term in (2.3.16). As long as $K \geq \overline{n}$, the function $(n - K)_+$ must vanish somewhere on $\mathbb{T}^2$, and hence the Gagliardo-Nirenberg-Sobolev inequality (2.3.14) is applied to deduce:

$$\int |(n_0 - K)_+|^3 dy \lesssim \int |\nabla(n_0 - K)_+|^2 dy \int (n_0 - K)_+ dy. \quad (2.3.17)$$

From (2.3.15), we choose K depending only on $C_{L \log L}$ such that

$$-\frac{7}{8A}\int |\nabla((n_0 - K)_+)|^2 dy + \frac{1}{2A}\int (n_0 - K)_+^3 dy \leq -\frac{1}{2A}\int |\nabla((n_0 - K)_+)|^2 dy. \quad (2.3.18)$$

Next, we apply Minkowski's inequality, the elliptic estimate (2.4.5), and the hypothesis (2.3.2e) to control the non-zero mode contribution $||(\nabla_y c_{\neq} n_{\neq})_0||_{L^2(\mathbb{R}^2)}^2$ in (2.3.16):

$$\frac{1}{A}||(\nabla_y c_{\neq} n_{\neq})_0||_{L^2(\mathbb{T}^2)}^2 \lesssim \frac{1}{A}||\nabla_y c_{\neq} n_{\neq}||_{L^2(\mathbb{T}^3)}^2 \lesssim \frac{1}{A}||\nabla_y c_{\neq}||_\infty^2 ||n_{\neq}||_2^2 \lesssim \frac{1}{A}C_{2,\infty}^2 ||n_{\neq}||_2^2. \quad (2.3.19)$$

Plugging (2.3.18) and (2.3.19) into (2.3.16) yields the following for some universal $B > 0$,

$$\frac{1}{2}\frac{d}{dt}\int (n_0 - K)_+^2 dy \leq -\frac{1}{2A}\int |\nabla((n_0 - K)_+)|^2 dy + \frac{KB}{A}\int (n_0 - K)_+^2 dy + \frac{BK^2M}{A} + \frac{BC_{2,\infty}^2}{A}||n_{\neq}||_2^2. \tag{2.3.20}$$

Recalling the Gagliardo-Nirenberg-Sobolev inequality (2.3.14), for a function v which vanishes in a nonempty set of $\mathbb{T}^2$, the following Nash inequality holds

$$||v||_{L^2(\mathbb{T}^2)}^2 \lesssim ||\nabla v||_{L^2(\mathbb{T}^2)}||v||_{L^1(\mathbb{T}^2)}.$$

Applying the Nash inequality in the estimate (2.3.20) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||(n_0 - K)_+||_2^2 \\ &\leq -\frac{1}{2AB}\frac{||(n_0 - K)_+||_2^4}{M^2} + \frac{3KB}{2A}||(n_0 - K)_+||_2^2 + \frac{BK^2M}{A} + \frac{B}{A}C_{2,\infty}^2\left\|n_{\neq}\right\|_2^2.\end{aligned} \tag{2.3.21}$$

Applying an argument similar to the one used in Section 2.2.2 to deduce (2.2.31), choosing A sufficiently large implies $\int (n - K)_+^2 dy \leq C(n_{in})$. Recall the following classical inequality (see e.g. [29, 75])

$$\|n_0\|_{L^2} \lesssim \|(n_0 - K)_+\|_{L^2} + K^{1/2}M^{1/2}, \tag{2.3.22}$$

where the implicit constant is independent of K and M. The inequality (2.3.13) hence follows. □

Next, we prove the higher regularity estimate (2.3.2c) using (2.3.13).

Lemma 2.3.5. *For A sufficiently large, provided (2.3.13) holds, the following improvement to (2.3.2c) holds on $[0, T_\star]$ for a suitable choice of $C_{\dot{H}^1}$:*

$$||\nabla_y n_0||_{L^2(\mathbb{T}^2)} \leq 2C_{\dot{H}^1}. \tag{2.3.23}$$

Proof. We employ the following standard multi-index notation:

$$\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2, \quad \partial_y^\alpha = \partial_{y_1}^{\alpha_1}\partial_{y_2}^{\alpha_2}, \quad ||\partial_y^s n_0||_2^2 = \sum_{|\alpha|=s} ||\partial_y^\alpha n_0||_2^2.$$

Let α be such that $|\alpha| = 1$. Computing the time derivative of $||\partial_y^\alpha n_0||_2^2$ and applying ϵ-Young's inequality:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\partial_y^\alpha n_0||_2^2 \\
&= -\frac{1}{A}\int |\partial_y^\alpha \nabla n_0|^2 dy + \frac{1}{A}\int \partial_y^\alpha \nabla n_0 \cdot (\partial_y^\alpha \nabla c_0 n_0) dy \\
&\quad + \frac{1}{A}\int \partial_y^\alpha \nabla n_0 \cdot \nabla c_0 \partial_y^\alpha n_0 dy + \frac{1}{A}\int \partial_y^\alpha \nabla n_0 \cdot \partial_y^\alpha (\nabla c_{\neq} n_{\neq})_0 dy \\
&\leq -\frac{7}{8A}\int |\partial_y^\alpha \nabla n_0|^2 dy + \frac{B}{A}||\partial_y^\alpha \nabla c_0 n_0||_2^2 + \frac{B}{A}||\nabla c_0 \partial_y^\alpha n_0||_2^2 + \frac{B}{A}||\partial_y^\alpha (\nabla c_{\neq} n_{\neq})_0||_2^2 \\
&=: -\frac{7}{8A}\int |\partial_y^\alpha \nabla n_0|^2 dy + T_1 + T_2 + NZ.
\end{aligned} \tag{2.3.24}$$

We first estimate the term T_1 in (2.3.24). Combining the bound (2.3.13), the Gagliardo-Nirenberg-Sobolev inequality (2.3.14), and the L^4 boundedness of the Riesz transform on $\mathbb{T}^d$ yields

$$\begin{aligned}
T_1 \leq & \frac{B}{A}\left\|\partial_y^\alpha \nabla c_0\right\|_4^2 ||n_0||_4^2 \lesssim \frac{B}{A}||n_0||_4^4 \lesssim \frac{B}{A}||n_0 - \overline{n}||_4^4 + \frac{BM^4}{A} \\
\lesssim & \frac{B}{A}||n_0 - \overline{n}||_2^2 ||\nabla_y n_0||_2^2 + \frac{BM^4}{A} \lesssim \frac{B}{A}C_{L^2}^2 ||\nabla_y n_0||_2^2 + \frac{BM^4}{A}.
\end{aligned} \tag{2.3.25}$$

Next for the second term T_2 in (2.3.24), combining the elliptic estimate (2.4.5) and the hypothesis (2.3.2e) yields

$$T_2 \leq \frac{B}{A}||\nabla_y c_0||_\infty^2 ||\partial_y^\alpha n_0||_2^2 \lesssim \frac{B}{A}C_{2,\infty}^2 ||\nabla_y n_0||_2^2. \tag{2.3.26}$$

Similar to the two dimensional case, the NZ term in (2.3.24) is estimated using the elliptic estimate (2.4.5), (2.3.2b) and (2.3.2e) as follows:

$$NZ \leq \frac{B}{A}||n_{\neq}||_2^2||n_{\neq}||_\infty^2 + \frac{B}{A}||\nabla_y c_{\neq}||_\infty^2||\nabla_y n_{\neq}||_2^2 \lesssim \frac{BC_{2,\infty}^2}{A}||n_{\neq}||_2^2 + \frac{BC_{2,\infty}^2}{A}||\nabla_y n_{\neq}||_2^2. \tag{2.3.27}$$

Combining the the above estimates (2.3.24),(2.3.25),(2.3.26),(2.3.27) and summing over $\alpha = 1, 2$ yield

$$\frac{1}{2}\frac{d}{dt}||\nabla_y n_0||_2^2 \leq -\frac{1}{2A}||\partial_y^2 n_0||_2^2 + \frac{B}{A}C_{2,\infty}^2||\nabla_y n_0||_2^2 + \frac{BM^4}{A} + G'(t), \tag{2.3.28}$$

where $G(t)$ is defined as

$$G(t) := \int_0^t \frac{BC_{2,\infty}^2}{A}||n_{\neq}||_2^2 + \frac{BC_{2,\infty}^2}{A}||\nabla_y n_{\neq}||_2^2 d\tau, \quad \forall t \in [0, T_\star].$$

Applying an argument similar to the one used in Section 2.2.2 to prove (2.2.31), choosing A sufficiently large implies that

$$||\nabla_y n_0||_2^2 \lesssim C_{2,\infty}^4 \tag{2.3.29}$$

which is independent of A and $C_{\dot{H}^1}$. Note that we still have the freedom to pick our $C_{\dot{H}^1}$, and we choose it such that $C_{\dot{H}^1}^2$ is much bigger than the right hand side of (2.3.29). This finishes the proof of Lemma 2.3.5 and the conclusion (2.3.5c) follows.

□

Enhanced dissipation estimate, (2.3.5b)

The estimates in this section is similar to §2.2.2. For the sake of brevity, we skip the proofs and refer the interested readers to [13] for further details.

Nonzero mode $L_t^2\dot{H}^1_{x,y}$ estimate (2.3.5a)

Computing $\frac{d}{dt}||n_{\neq}||_2^2$ and applying (2.4.5),

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||n_{\neq}||_2^2 =&\langle n_{\neq}, \frac{1}{A}\Delta n_{\neq} + \frac{1}{A}(n_0-\overline{n})n_{\neq} + \frac{1}{A}n_{\neq}n_0 - \frac{1}{A}\nabla c_0\cdot\nabla n_{\neq} \\
&- \frac{1}{A}\nabla c_{\neq}\cdot\nabla n_0 - u(y)\partial_x n_{\neq} - \frac{1}{A}(\nabla\cdot(\nabla c_{\neq}n_{\neq}))_{\neq}\rangle \\
\lesssim& -\frac{1}{2A}||\nabla n_{\neq}||_2^2 + \frac{B}{A}||n_{\neq}||_2^2||\nabla c_0||_\infty^2 + \frac{B}{A}||\nabla c_{\neq}||_\infty \left\|n_{\neq}\right\|_2 ||\nabla_y n_0||_2 \\
&+ \frac{1}{A}||n_{\neq}||_2^2||n_0-\overline{n}||_\infty + \frac{1}{A}||n_{\neq}||_2^2 + \frac{1}{A}||\nabla n_{\neq}||_2||\nabla c_{\neq}||_4||n_{\neq}||_4 \\
\lesssim& -\frac{1}{2A}||\nabla n_{\neq}||_2^2 + \frac{C_{2,\infty}^2}{A}||n_{\neq}||_2^2 + \frac{C_{2,\infty}C_{\dot{H}^1}}{A}\left\|n_{\neq}\right\|_2 \\
&+ \frac{1}{A}||\nabla n_{\neq}||_2||\nabla c_{\neq}||_4||n_{\neq}||_4. \qquad (2.3.30)
\end{aligned}$$

Note that, due to the bootstrap hypothesis (2.3.2b), there holds

$$\begin{aligned}
&\int_0^{T_\star}\frac{C_{2,\infty}^2}{A}||n_{\neq}||_2^2 + \frac{C_{2,\infty}C_{\dot{H}^1}}{A}\left\|n_{\neq}\right\|_2 dt \\
&\quad\lesssim \frac{\log A}{A^{1/2}}C_{ED}^2(1+\|n_{in}\|_{H^1}^2)\left(C_{2,\infty}^2 + C_{2,\infty}C_{\dot{H}^1}\right), \qquad (2.3.31)
\end{aligned}$$

which can be made arbitrarily small by choosing A large. The latter term is treated via the Gagliardo-Nirenberg-Sobolev inequality, s

$$\begin{aligned}
\frac{1}{A}||\nabla n_{\neq}||_2||\nabla c_{\neq}||_4||n_{\neq}||_4 \lesssim& \frac{1}{A}||\nabla n_{\neq}||_2||\nabla c_{\neq}||_2^{1/4}||\nabla^2 c_{\neq}||_2^{3/4}||n_{\neq}||_2^{1/4}||\nabla n_{\neq}||_2^{3/4} \\
\lesssim& \frac{1}{AB}||\nabla n_{\neq}||_2^2 + \frac{B}{A}||n_{\neq}||_2^{10} \qquad (2.3.32) \\
\lesssim& \frac{1}{AB}||\nabla n_{\neq}||_2^2 + \frac{BC_{2,\infty}^8}{A}||n_{\neq}||_2^2.
\end{aligned}$$

Hence, by choosing B then A sufficiently large, we have following $L_t^2\dot{H}^2_{x,y}$ estimate:

$$\frac{1}{A}\int_0^{T\star}||\nabla n_{\neq}||_2^2 dt \le \frac{1}{A^{1/4}} + 2||n_{in}||_2^2 \le 4||n_{in}||_2^2.$$

As a result, we have proven (2.3.5a).

Remainder of the proof of Theorem 2 in the case $\mathbb{T}^3$

The remaining steps in the proof of Proposition 7 is the proof of (2.3.5d). Since L^2 is subcritical in 3D, the proof of (2.3.5d) follows as in §2.2.2 by standard methods. This completes the proof of Proposition 7 and hence also Theorem 2 in the $\mathbb{T}^3$ case.

2.4 Appendix

2.4.1 Chemical gradient ∇c estimates

We have applied various estimates on $\nabla c_0, \nabla c_{\neq}$; while all are standard, we list them here for the readers' convenience. The proofs are omitted for the sake of brevity, we refer the interested readers to the paper [13] for further details.

Lemma 2.4.1. *In the two-dimensional case, the following estimate holds for uniformly for all $k \in \mathbb{Z} \setminus \{0\}$ and $(k^2 - \partial_{yy})\widehat{c}_k = \widehat{n}_k$:*

$$|k|^{1/2}||\partial_y \widehat{c}_k||_{L^\infty(\mathbb{T})} \lesssim ||\widehat{n}_k||_{L^2(\mathbb{T})}. \tag{2.4.1}$$

Lemma 2.4.2. *In the two-dimensional case, the following estimate on ∇c_0 holds:*

$$||\partial_y c_0||_{L^\infty(\mathbb{T})} \lesssim ||n_0 - \overline{n}||_{L^1(\mathbb{T})}. \tag{2.4.2}$$

Lemma 2.4.3. *In the two-dimensional case, the following elliptic estimate holds:*

$$||\nabla(c_{\neq})||_{L^\infty(\mathbb{T}^2)} \lesssim \|n_{\neq}\|_{L^3(\mathbb{T}^2)}. \tag{2.4.3}$$

In the 3-dimensional case, we need the following lemmas.

Lemma 2.4.4. *In the 3-dimensional case, the following mode by mode estimates are true:*

$$\begin{aligned} ||\nabla_y \widehat{c}_k||_{L^\infty(\mathbb{R}^2)} &\lesssim ||\widehat{n}_k||_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} ||\nabla_y \widehat{n}_k||_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \\ ||\nabla_y \widehat{c}_k||_{L^\infty(\mathbb{T}^2)} &\lesssim ||\widehat{n}_k||_{L^2(\mathbb{T}^2)}^{\frac{1}{2}} ||\nabla_y \widehat{n}_k||_{L^2(\mathbb{T}^2)}^{\frac{1}{2}}. \end{aligned} \tag{2.4.4}$$

Other than the lemma above, we need the following 3D elliptic estimates.

Lemma 2.4.5. *In the three-dimensional case, the following elliptic estimates are true:*

$$\begin{aligned} ||\nabla c_{\neq}||_{L^\infty(\mathbb{T}\times\mathbb{R}^2)} &\lesssim ||n_{\neq}||_{L^4(\mathbb{T}\times\mathbb{R}^2)}, \\ ||\nabla c_{\neq}||_{L^\infty(\mathbb{T}^3)} &\lesssim ||n_{\neq}||_{L^4(\mathbb{T}\times\mathbb{R}^2)}, \\ ||\nabla_y c_0||_{L^\infty(\mathbb{R}^2)} &\lesssim ||n_0||_{L^1(\mathbb{R}^2)}^{1/4} ||n_0||_{L^3(\mathbb{R}^2)}^{3/4}, \\ ||\nabla_y c_0||_{L^\infty(\mathbb{T}^2)} &\lesssim \|n_0 - \overline{n}\|_{L^3(\mathbb{T}^2)}. \end{aligned} \tag{2.4.5}$$

2.5 Conclusions

In this paper we consider the parabolic-elliptic Patlak-Keller-Segel models in $\mathbb{T}^d$ with $d = 2, 3$ with the additional effect of advection by a large shear flow. Without the shear flow, the model is L^1 critical in two dimensions with critical mass 8π; solutions with mass less than 8π are global and solutions with mass larger than 8π with finite second moment, all blow up in finite time. In three dimensions, the model is $L^{3/2}$ critical and L^1 supercritical; there exist solutions with arbitrarily small mass which blow up in finite time arbitrarily fast. We show that the additional shear flow, if it is chosen sufficiently large, suppresses one dimension of the dynamics and hence can suppress blow-up. In two dimensions, the problem becomes effectively L^1 subcritical and so all solutions are global in time (if the shear flow is chosen large). In three dimensions, the problem is effectively L^1 critical, and solutions with mass less than 8π are global in time (and for all mass larger than 8π, there exists solutions which blow up in finite time).

Chapter 3: Suppressing chemotactic blow-up through a fast splitting scenario on the plane

3.1 Overview

Consider the PKS equation on $\mathbb{R}^2$

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \nabla c) + \mathbf{b} \cdot \nabla n = \Delta n, \quad x = (x_1, x_2) \in \mathbb{R}^2, \tag{3.1.1}$$

subject to prescribed initial conditions $n(x, 0) = n_0(x)$. Here the divergence free vector field $\mathbf{b}(\cdot)$ represents the environment of an background fluid transported with velocity $\mathbf{b}(x,t) := \nabla H(x,t)$.

We find that already the simplest case of linear stationary vector field, $\mathbf{b} = A(-x_1, x_2)$, corresponding to $H(x) = \frac{1}{2}A(x_2^2 - x_1^2)$, prevents chemotactic blow-up for $M < 16\pi$. As we shall see, the presence of such an ambient fluid transport creates what we call a 'fast splitting scenario' which competes with the focusing effect of aggregation so that 'enough mass' is able to escape a finite time blow-up, at least for $M < 16\pi$. This scenario is likely to be enhanced even further when larger amount of mass can be transported by a more pronounced ambient field $\mathbf{b}(x,t) = \nabla H(\cdot, t) \gg |x|^q$ at $|x| \gg 1$.

3.1.1 A fast splitting scenario

Here, we exploit a mechanism that suppress the possible chemotactic blow up of the equation (3.1.1), where the underlying fluid flow splits the population of bacteria with density n exponentially fast, resulting in several isolated subgroups with mass smaller than the critical 8π. In this manner, an initial total mass greater than 8π is able to escape the finite-time concentration of Dirac mass. This provides a first no blow-up scenario over $\mathbb{R}^2$, at least for M up to 16π.

We now fix the vector field driving a hyperbolic flow as the strain flow in [78]):

$$\mathbf{b}(x) := A(-x_1, x_2). \tag{3.1.2}$$

Our aim is to show that a large enough amplitude, $A \gg 1$, guarantees the global existence of solution of PKS (3.1.1) subject to initial mass $M < 16\pi$. Observe that an initial center of mass at the origin is an invariant of the flow. Intuitively, the large enough amplitude $A \gg 1$ is required so that the ambient field $A(-x_1, x_2)$ *'pushes away' highly concentrated mass near the x_1-axis*, namely, $\displaystyle\int_{|x_2|\leq\epsilon} n_0(x)dx \gg 1$. With this we can state the main theorem of the paper.

Theorem 7. *Consider the PKS equation (3.1.1),(3.1.2) subject to initial data, $n_0 \in H^s(s \geq 2)$ with total mass, $M := |n_0|_1 < 16\pi$, such that $(1+|x|^2)n_0 \in L^1(\mathbb{R}^2)$ and $n_0 \log n_0 \in L^1(\mathbb{R}^2)$. Assume n_0 is symmetric about the x_1-axis, and that the "y-component" of its center of mass in the upper half plane*

$$y_+(t) := \frac{1}{M_+}\int_{x_2\geq 0} n(x,t)x_2 dx, \quad M_+ := \int_{x_2\geq 0} n(x,t)dx \equiv \frac{M}{2},$$

is not too close to the x_1-axis in the sense that

$$y_+^2(0) > \frac{2}{M_+}V_+(0), \qquad V_+(t) := \int_{x_2 \geq 0} n(x,t)|x_2 - y_+|^2 dx. \tag{3.1.3}$$

Then there exists a large enough amplitude, $A = A(M, y_+(0), V_+(0))$, such that the weak solution of (3.1.1),(3.1.2) exists for all time and the free energy

$$E[n](t) := \int \big(n \log n - \frac{1}{2}cn - H(x)n(x,t)\big)dx, \qquad H(x) = \frac{A}{2}(x_2^2 - x_1^2), \tag{3.1.4}$$

satisfies the dissipation relation

$$E[n](t) + \int_0^t \int_{\mathbb{R}^2} n|\nabla \log n - \nabla c - \mathbf{b}|^2 dx ds \leq E[n_0]. \tag{3.1.5}$$

We conclude the introduction with three remarks.

Remark 2 (Why large enough stationary field prevents blow-up)**.** *Our main theorem extends the amount of critical mass, so that global regularity of (3.1.1),(3.1.3) prevails for $M < 16\pi$, provided A is large enough. To realize how large the amplitude A should be and thus clarifying the reason behind this doubling the initial mass threshold for global regularity, we express (3.1.3) as*

$$R^2 := M_+ \frac{y_+^2(0)}{2V_+(0)} > 1.$$

Then we can choose $A = M_+\delta^{-2}$ with small enough δ so that $\delta \leq (R-1)\sqrt{\frac{2V_+(0)}{M_+}}$.

Remark 3 (On the free energy)**.** *We note that when $\mathbf{b} = 0$, $E[n]$ becomes the classical dissipative free energy*

$$\mathcal{F} = \int_{\mathbb{R}^2} n \log n dx - \frac{1}{2}\int_{\mathbb{R}^2} ncdx. \tag{3.1.6}$$

Due to the importance of the property (3.1.5)*, a weak solution of* (3.1.1) *satisfying* (3.1.5) *will be called a* free energy solution*. One of the important properties of the PKS equation* (3.1.1) *with background flow velocity* (3.1.2) *is the dissipation of its free energy* $E[n]$*. The formal computation, indicating the energy dissipation in non-static smooth solutions, is the content of our last lemma in this section.*

Lemma 3.1.1. *Consider the PKS equation* (3.1.1) *with background fluid velocity* (3.1.2)*. If the solution is smooth enough, the free energy* $E[n](t)$ *is decreasing.*

Proof. The time evolution of the free energy (3.1.4) can be computed in terms of the potential $H = \frac{1}{2}A(x_2^2 - x_1^2)$,

$$\begin{aligned}
\frac{d}{dt}E[n](t) &= \int n_t(\log n - c - H)dx \\
&= -\int n(\nabla \log n - \nabla c - \mathbf{b}) \cdot (\nabla \log n - \nabla c - \nabla H)dx \\
&= -\int n|\nabla \log n - \nabla c - \mathbf{b}|^2 dx \leq 0.
\end{aligned}$$

This completes the proof of the lemma. □

Remark 4. *Arguing along the lines [52], one should be able to prove that the free energy solution is smooth for all positive time,* $n \in C_c^\infty((0,T]; C_x^\infty)$ *for all* $T < \infty$ *and thus our global weak solution is in fact a global strong solution.*

Our paper is organized as follows. In section 2, we introduce the regularized problem to (3.1.1) which leads to the local existence results. In section 3, we prove the main theorem, and in the appendix, we give detailed proofs of the results stated in section 2.

3.2 Local existence

3.2.1 Weak formulation

It is standard to understand the Keller-Segel equation (3.1.1) with background fluid velocity (3.1.2) in the following weak formulation.

Definition 4 (weak formulation)**.** *n is said to be the weak solution to (3.1.1) if for $\forall \varphi \in C_c^\infty(\mathbb{R}_+^2)$, the following equation hold:*

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2}\varphi n dx = \int_{\mathbb{R}^2}\Delta\varphi n dx - \frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{(\nabla\varphi(x)-\nabla\varphi(y))\cdot(x-y)}{|x-y|^2}n(x,t)n(y,t)dxdy \\ + \int_{\mathbb{R}^2}\nabla\varphi\cdot\mathbf{b}n dx.\end{aligned} \tag{3.2.1}$$

Taking advantage of the assumed symmetry across the x_1-axis, one can further simplify the notation of a weak formulation adapted to the upper half plane, $\mathbb{R}_+^2 = \{(x_1, x_2)\,|\,x_2 \geq 0\}$.

Theorem 8. *If n is a weak solution to the equation (3.1.1), then for $\forall \varphi \in C_c^\infty(\mathbb{R}_+^2)$, the following holds:*

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}_+^2}\varphi n_+ dx \\ &= \int_{\mathbb{R}_+^2}\Delta\varphi n_+ dx - \frac{1}{4\pi}\int_{\mathbb{R}_+^2\times\mathbb{R}_+^2}\frac{(\nabla\varphi(x)-\nabla\varphi(y))\cdot(x-y)}{|x-y|^2}n_+(x,t)n_+(y,t)dxdy \\ &\quad + \int_{\mathbb{R}_+^2}\nabla c_-\cdot\nabla\varphi n_+ dx + \int_{\mathbb{R}_+^2}\nabla\varphi\cdot\mathbf{b}n_+ dx.\end{aligned} \tag{3.2.2}$$

Here $\left\{\begin{array}{lcl} n_+ & := & n\mathbf{1}_{x_2\geq 0} \\ n_- & := & n\mathbf{1}_{x_2\leq 0}\end{array}\right\}$, *and* $\nabla c_-(x) := -\int_{y\in\mathbb{R}_-^2}\frac{x-y}{2\pi|x-y|^2}n_-(y)dy,\ .$

Proof. Rewrite (3.2.1) as follows:

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2_+}\varphi n_+ dx\\
&=\int_{\mathbb{R}^2_+}\Delta\varphi n_+ dx-\frac{1}{4\pi}\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+}\frac{(\nabla\varphi(x)-\nabla\varphi(y))\cdot(x-y)}{|x-y|^2}n_+(x,t)n_+(y,t)dxdy\\
&\quad-\frac{1}{2\pi}\int_{\mathbb{R}^2_+\times\mathbb{R}^2_-}\frac{\nabla\varphi(x)\cdot(x-y)}{|x-y|^2}n_+(x,t)n_-(y,t)dxdy+\int_{\mathbb{R}^2_+}\nabla\varphi\cdot\mathbf{b}n_+dx.
\end{aligned}$$

The third term can be rewritten as $\int_{\mathbb{R}^2_+}\nabla c_-\cdot\nabla\varphi\, n_+(x)dx$, and we get (3.2.2). □

3.2.2 Regularized equation and local existence theorems

In this section we introduce the local existence theorem and the blow up criterion for the Keller-Segel equation with advection. The theorems are standard, so the proofs are postponed to the appendix. The interested reader are referred to the papers [27], [28] for further details.

In order to prove the local existence theorem and the blow up criterion for the Keller-Segel system with advection (3.1.1), we regularize the system as follows:

$$\frac{\partial n^\epsilon}{\partial t}+\nabla\cdot(n^\epsilon\nabla c^\epsilon)+\mathbf{b}\cdot\nabla n^\epsilon=\Delta n^\epsilon,\quad c^\epsilon:=K^\epsilon * n,\qquad x\in\mathbb{R}^2,t>0,\tag{3.2.3}$$

with the regularized kernel, K^ϵ, given by

$$K^\epsilon(z):=K^1\left(\frac{z}{\epsilon}\right)-\frac{1}{2\pi}\log\epsilon,\qquad K^1(z):=\begin{cases}-\frac{1}{2\pi}\log|z|, & \text{if}\quad |z|\geq 4,\\ 0, & \text{if}\quad |z|\leq 1.\end{cases}\tag{3.2.4}$$

Noting that $|\nabla K^\epsilon(z)|\leq C_\epsilon$ for all $z\in\mathbb{R}^2$, it follows that the solutions to the equation (3.2.3) exist for all time. The proof is similar to the corresponding proof in the classical case. We refer the interested reader to the paper [28] for more details.

Before stating the local existence theorems, we introduce the entropy of the solution

$$S[n] := \int_{\mathbb{R}^2} n \log n dx. \tag{3.2.5}$$

Now the local existence theorems are expressed as follows:

Proposition 8. (*Local Existence Criterion*). *Assume that* $|\mathbf{b}|(x) \leq C|x|, \forall x \in \mathbb{R}^2$. *Suppose* $\{n^\epsilon\}_{\epsilon \geq 0}$ *are the solutions of the regularized equation* (3.2.3) *on* $[0, T^*)$. *If* $\{S[n^\epsilon](t)\}_\epsilon$ *is bounded from above uniformly in* ϵ *and in* $t \in [0, T^*)$, *then the cluster points of* $\{n^\epsilon\}_{\epsilon \to 0}$, *in a suitable topology, are non-negative weak solutions of the PKS system with advection* (3.1.1) *on* $[0, T^*)$ *and satisfies the relation* (3.1.5).

Proposition 9. (*Maximal Free-energy Solutions*). *Assume the boundedness of the vector field* $|\mathbf{b}|(x) \lesssim |x|$ *and the integrability of initial data*

$$(1 + |x|^2)n_0 \in L^1_+(\mathbb{R}^2), \;\; n_0 \log n_0 \in L^1(\mathbb{R}^2).$$

Then there exists a maximal existence time $T^* > 0$ *of a free energy solution to the PKS system with advection* (3.1.1),(3.1.5). *Moreover, if* $T^* < \infty$ *then*

$$\lim_{t \to T^*} \int_{\mathbb{R}^2} n \log n dx = \infty.$$

For the sake of brevity, we skip the proofs of these two propositions and refer the interested readers to the paper [67] for further details.

We conclude that if the entropy $S[n](t) = \int n \log n$ is bounded, then the free energy solution of (3.1.1) exists locally. Moreover, if $S[n](t) < \infty$ for all $t < \infty$, the solution exists for all time.

3.3 Proof of the main results

3.3.1 The three-step 'battle-plan'

We proceed in three steps. The *first step* carried in section 3.3.2 below, is to control cell density distribution. From the last section, we see that an entropy bound is essential for derivation of local existence theorems for the PKS equation (3.1.1),(3.1.2). To this end, information about the distribution of cell density is crucial. The following lemma is the key to the proof of the main results. It shows that mass cannot concentrate along the the x_1-axis, since we can find a thin enough strip along the x_1-axis with controlled amount of mass.

Lemma 3.3.1. *Suppose a sufficiently smooth n_0 is symmetric about the x_1 axis and assume that*

$$R^2 := M_+ \frac{y_+^2(0)}{2V_+(0)} > 1. \tag{3.3.1}$$

Fix a small enough $0 < \eta \ll 1$. Then there exists $\delta = \delta(y_+(0), V_+(0), M, \eta)$ such that if we choose $A > \frac{M_+}{\delta^2}$, the smooth solutions to the regularized $(3.2.3)_\epsilon$ satisfy, uniformly for small enough ϵ,

$$\int_{|x_2|\leq 2\delta} n^\epsilon(x,t)dx \leq \frac{(1+\eta)^2}{2R^2} M. \tag{3.3.2}$$

Condition (3.3.2) implies, at least for $M < 16\pi$, that the mass inside that δ-strip is less than 8π. On the other hand, it indicates the reason for the limitation $R > 1$: for if $R < 1$, then the bound (3.3.2) would allow a concentration of mass $\frac{M}{2R^2} \geq 8\pi$ inside the strip $|x_2| \leq 2\delta$, which in turn could lead to a finite-time blow-up.

The proof of lemma 3.3.1 is based on the following simple observation. Given f with $\mathbb{R}^2_+$-center of mass at $(\cdot, y_f)$ and variation $V_f = \int |x_2 - y_f|^2 f(x)dx$, we find that its total mass *outside* the strip $\mathcal{S}[y_f, r] := \{(x_1, x_2)||x_2 - y_f| \leq r\}$ with radius $r = R\sqrt{2V_f/M_f}$, does not exceed

$$\int_{|x_2-y_f|>r} f(x)dx = \int_{|x_2-y_f|>r} f(x)\frac{|x_2-y_f|^2}{|x_2-y_f|^2}dx \leq \frac{M_f}{2R^2V_f}\int f(x)|x_2-y_f|^2dx = \frac{M_f}{2R^2}$$

If we can find the δ such that our target strip $\mathcal{S}_\delta := \{|x_2| \leq 2\delta\}$ is lying below and *outside* the strip $\mathcal{S}[y_f, r]$, then the total mass in the strip $\mathcal{S}_\delta$ would be smaller than $\frac{1}{2R^2}M_f$. When $n^\epsilon(x,t)$ takes the role of $f(x)$ with $(y_f, V_f) \mapsto (y_+(t), V_+(t))$, the aim is to bound the strip $\mathcal{S}[y_+(t), r(t)]$ with radius $r(t) = R\sqrt{2V_+(t)/M_+}$ away from a fixed strip $\mathcal{S}_\delta$. To this end we collect the necessary estimates on $y_+(t), V_+(t)$ and complete the proof of the lemma in section 3.3.2.

The *second step*, carried in section 3.3.3, is to prove the main theorem with moderate mass constraint. Equipped with lemma 3.3.1 we can control the entropy and prove a weaker form of our main theorem for moderate size mass M (which is still larger than the 8π barrier):

Theorem 9. *Consider the PKS equation (3.1.1) with background fluid velocity (3.1.2) subject to $H^s(s \geq 2)$ initial data with mass $M = |n_0|_1$, such that $(1+|x|^2)n_0 \in L^1(\mathbb{R}^2)$, $n_0 \log n_0 \in L^1(\mathbb{R}^2)$. Furthermore, assume n_0 is symmetric about the x_1-axis, that (3.1.3) holds. If the total mass does not exceed*

$$M < \frac{1}{1+(1+\eta)^2/R^2}16\pi, \tag{3.3.3}$$

then there exists an $A = A(M, y_+(0), V_+(0), \eta)$ large such that the free energy solution to PKS (3.1.1),(3.1.2) exists for all time.

Thus, as the ratio increases over the range $1 < R < \infty$, (9) yields global existence with an increasing amount of mass $8\pi < M < 16\pi$. Although theorem 9 is not as sharp as the main theorem, its proof is more illuminating and can be extended easily to prove the main theorem for the 'limiting case' of any $M < 16\pi$. We therefore include its proof in section 3.3.3.

Finally, the *third step* carried in section 3.3.4 presents the proof of the main theorem 7.

We turn to a detailed discussion of the three steps.

3.3.2 Step 1— control of the cell density distribution

As pointed out before, the proof involves the calculation of $y_+(t)$ and $V_+(t)$, summarized in the following two lemmas. Here and below, we let $A \lesssim B$ denote the relation $A \le CB$ with a constant C which is independent of δ.

Lemma 3.3.2. *Consider the regularized PKS equation (3.2.3) with background fluid velocity (3.1.2). Assume that the initial center of mass $y_+(0)$ is separated from the x_1-axes in the sense that (3.3.1) holds. Then , there exists a constant such that the time evolution of $y_+(t)$ remains bounded from below*

$$y_+(t) \ge [y_+(0) - C\delta]\, e^{At}, \quad . \tag{3.3.4}$$

Lemma 3.3.3. *Consider the regularized PKS equation (3.2.3) with background fluid velocity (3.1.2). Assume that the initial variation around the center of mass $V_+(0)$ is not too large in the sense that (3.3.1) holds. Then , there exists a constant $C =$*

$C(V_+(0))$ *such that the variation* $V_+(t)$ *remains bounded from above,*

$$V_+(t) \leq [CM_+\delta + V_+(0)]\, e^{2At}, \quad . \tag{3.3.5}$$

We note that all the calculations made below should be carried out at the level of the regularized equation (3.2.3), but for the sake of simplicity, we proceed at the formal level using the weak formulation (3.2.2). We explicitly point when there is a technical subtlety in the derivation due to a difference between the regularized and weak formations.

We begin with the proof of Lemma 3.3.2. First of all, as $x_2 \notin C_c^\infty(\mathbb{R}_+^2)$, we introduce an approximate test function φ to x_2:

$$\varphi := \begin{cases} x_2 & x_2 \in (2\delta, \infty), \\ 0 & x_2 \in (-\infty, \delta), \\ smooth & x_2 \in (\delta, 2\delta). \end{cases}$$

Note that there exists a constant C_φ, independent of δ, such that $|\varphi| \leq 2\delta, \quad \forall x_2 \leq 2\delta$ and $|\nabla\varphi| + \delta|\nabla^2\varphi| \leq C_\varphi$. Here and below, we use C_φ to denote φ-dependent constants . Here note that φ is still not in $C_c^\infty(\mathbb{R}_+^2)$, but we can truncate φ at sufficiently large scale to overcome this technical deficiency. For the sake of simplicity, we still use φ in the argument.

By replacing $\int_{\mathbb{R}_+^2} x_2 n dx$ with $\int_{\mathbb{R}_+^2} \varphi n dx$, we lose information on the strip

$$\{(x_1, x_2) | |x_2| \leq 2\delta\}.$$

However, the contribution of this part is small in the sense that:

$$\left| \int_{0 \leq x_2 \leq 2\delta} \varphi n_+ dx - \int_{0 \leq x_2 \leq 2\delta} x_2 n_+ dx \right| \leq 4M_+\delta. \tag{3.3.6}$$

Next, one can use φ and the weak formulation (3.2.2) to extract information about y_+:

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2_+}\varphi n_+ dx\\
=&\int_{\mathbb{R}^2_+}\Delta\varphi n_+ dx-\frac{1}{4\pi}\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+}\frac{(\nabla\varphi(x)-\nabla\varphi(y))\cdot(x-y)}{|x-y|^2}n_+(x,t)n_+(y,t)dxdy\\
&+\int_{\mathbb{R}^2_+}n_+\nabla c_-\cdot\nabla\varphi dx+\int_{\mathbb{R}^2_+}\nabla\varphi\cdot\mathbf{b}n_+dx\\
=&I+II+III+IV.
\end{aligned}\tag{3.3.7}
$$

Now we estimate the right hand side of (3.3.7) term by term. The first and second terms are relatively easy to control from above as follows:

$$
I=\left|\int_{\mathbb{R}^2_+}\Delta\varphi n_+dx\right|\le\frac{C_\varphi M_+}{\delta},\tag{3.3.8}
$$

$$
II\le\frac{1}{4\pi}|\nabla^2\varphi|_\infty\int\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+}n_+(x)n_+(y)dxdy\le\frac{1}{4\pi}\frac{C_\varphi}{\delta}M_+^2.\tag{3.3.9}
$$

In order to estimate the third term in (3.3.7), we need a pointwise estimate on the $\partial_{x_2}c_-$:

$$
\begin{aligned}
|\partial_{x_2}c_-(x)|=&\frac{1}{2\pi}\left|\int_{\mathbb{R}^2_+}\frac{(x-y)_2}{|x-y|^2}n_-(y)dy\right|\\
&\le\frac{1}{2\pi}\frac{1}{|x_2|}\int_{\mathbb{R}^2_+}\frac{|x_2|}{|x_2|+|y_2|}n_-(y)dy\le\frac{1}{2\pi}\frac{M_-}{|x_2|}=\frac{1}{2\pi}\frac{M_+}{|x_2|}.
\end{aligned}\tag{3.3.10}
$$

Now we can use the above estimate to control the third term in (3.3.7):

$$
III=\left|\int_{\mathbb{R}^2_+}\partial_{x_2}c_-n_+\partial_{x_2}\varphi dx\right|\le\frac{C_\varphi}{2\pi}\frac{M_+}{\delta}\int_{\mathbb{R}^2_+}n_+dx\le\frac{C_\varphi}{2\pi}\frac{M_+^2}{\delta}.\tag{3.3.11}
$$

Here we have used the fact that $supp(\varphi)$ is δ away from the x_1 axis. As a result, $\frac{1}{|x_2|}\le\frac{1}{\delta}$.

Remark 5. *The only difference in estimating the regularized solutions (3.2.3) vs. the formal calculation we have done above is in terms II and III. In the calculation for the (3.2.3), we will need the estimate*

$$|\nabla K^\epsilon(z)| \le \frac{1}{2\pi|z|}, \quad \forall z \in \mathbb{R}^2.$$

Here we show how to get a similar estimate for term II in (3.3.7) for the regularized equation (3.2.3):

$$\begin{aligned} II =& \left| \iint_{\mathbb{R}^2_+ \times \mathbb{R}^2_+} \nabla\varphi(x)\nabla_x[K^\epsilon(|x-y|)]n^\epsilon(y)n^\epsilon(x)dxdy \right| \\ =& \frac{1}{2}\left| \iint_{\mathbb{R}^2_+ \times \mathbb{R}^2_+} \frac{(\nabla\varphi(x)-\nabla\varphi(y))\cdot(x-y)}{|x-y|}\nabla K^\epsilon(|x-y|)n^\epsilon(x)n^\epsilon(y)dxdy \right| \\ \le& \frac{1}{4\pi}\left| \iint_{\mathbb{R}^2_+ \times \mathbb{R}^2_+} \frac{|\nabla\varphi(x)-\nabla\varphi(y)|}{|x-y|}n^\epsilon(x)n^\epsilon(y)dxdy \right| \le \frac{1}{4\pi}|\nabla^2\varphi|_\infty M_+^2 \le \frac{1}{4\pi}\frac{C_\varphi}{\delta}M_+^2. \end{aligned}$$

The treatment of term III is similar to the one we gave above.

Finally, we need to address additional transport term IV in (3.3.7) to compete with the focusing effect. Recall that $\mathbf{b} = A(-x_1, x_2)$ with $A = \frac{M_+}{\delta^2}$. First we write IV down explicitly,

$$IV = \int \mathbf{b}\cdot n_+\nabla\varphi dx = \frac{M_+}{\delta^2}\int_{\mathbb{R}^2_+} x_2\partial_{x_2}\varphi n_+ dx.$$

Next we replace the right hand side by $\int \varphi n_+ dx$. Due to the fact that $x_2\partial_{x_2}\varphi = x_2 = \varphi$ for $x_2 > 2\delta$, the error introduced in this process originates from the thin 2δ-strip:

$$\left|\int_{\delta<x_2<2\delta}(x_2\partial_{x_2}\varphi - \varphi)n_+dx\right| \le \int_{\delta<x_2<2\delta}|x_2\partial_{x_2}\varphi-\varphi|n_+dx \le C_\varphi\delta M_+,$$

and we conclude that

$$IV \ge \frac{M_+}{\delta^2}\left(\int_{\mathbb{R}^2_+}\varphi n_+dx - C_\varphi\delta M_+\right) \ge \frac{M_+}{\delta^2}\int_{\mathbb{R}^2_+}\varphi n_+dx - \frac{C_\varphi M_+^2}{\delta}. \qquad (3.3.12)$$

Combining the equation (3.3.7) and estimates (3.3.8), (3.3.9), (3.3.11) and (3.3.12) yields the following

$$\frac{d}{dt}\int_{\mathbb{R}^2_+}\varphi n_+ dx \geq -C_\varphi \frac{M_+^2}{\delta} + A\int_{\mathbb{R}^2_+}\varphi n_+ dx,$$

which implies that

$$\int_{\mathbb{R}^2_+}\varphi n_+ dx \geq \left(\int_{\mathbb{R}^2_+}\varphi n_0 dx - C_\varphi M_+\delta\right)e^{At}. \tag{3.3.13}$$

Finally, we calculate the center of mass of the upper half plane using the lower bound (3.3.13) and the error control (**??**):

$$\begin{aligned}
y_+(t) =& \frac{1}{M_+}\left(\int_{0\leq x_2\leq 2\delta} x_2 n_+ dx - \int_{0\leq x_2\leq 2\delta}\varphi n_+ dx + \int_{\mathbb{R}^2_+}\varphi n_+ dx\right)\\
\geq& -\frac{1}{M_+}\left|\int_{0\leq x_2\leq 2\delta} x_2 n_+ dx - \int_{0\leq x_2\leq 2\delta}\varphi n_+ dx\right| + \frac{1}{M_+}\int_{\mathbb{R}^2_+}\varphi n_+ dx\\
\geq& -4\delta + \frac{1}{M_+}\left(\int_{\mathbb{R}^2_+}\varphi n_0 dx - C_\varphi M_+\delta\right)e^{At}\\
\geq& \left(y_+(0) - C_\varphi\delta\right)e^{At}.
\end{aligned}$$

This completes the proof of lemma 3.3.2.□

Next we address the proof of Lemma 3.3.3. The main goal is to calculate time evolution of the variation

$$V_+(t) := \int_{\mathbb{R}^2_+}|x_2 - y_+(t)|^2 n(x,t)dx.$$

We again use C to denote constants which may change from line to line but are independent of δ.

The first obstacle is that we cannot choose $|x_2 - y_+|^2$ as a test function due to the fact that $y_+(t)$ depends on the solution. However, by the definition of y_+ we can expand the V_+-integrand, ending up with the usual

$$V_+(t) = \int_{\mathbb{R}^2_+} |x_2|^2 n_+(x,t)dx - M_+ y_+^2(t). \tag{3.3.14}$$

Since we already know y_+, it is enough to calculate the $\int_{\mathbb{R}^2_+} |x_2|^2 n(x,t)dx$. For simplicity, we plug $|x_2|^2$ inside the weak formulation (3.2.2) and (3.3.14) to get the time evolution of V_+. Of course, what one really does is to use a test function to approximate the $|x_2|^2$. Furthermore, when we use the weak formulation, we formally integrated by part twice, but since the value and the first derivative of the function $|x_2|^2$ are zero on the boundary, we will not create extra dangerous boundary term.

First combining (3.2.2) and (3.3.14) yields

$$\begin{aligned}
\frac{d}{dt}V_+ =& \frac{d}{dt}\int_{\mathbb{R}^2_+} |x_2|^2 n_+(x,t)dx - M_+\frac{d}{dt}y_+^2(t) \\
=& \int_{\mathbb{R}^2_+} \Delta\varphi n_+ dx - \frac{1}{4\pi}\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \frac{(\nabla\varphi(x)-\nabla\varphi(y))\cdot(x-y)}{|x-y|^2} n_+(x,t)n_+(y,t)dxdy \\
&+ \int_{\mathbb{R}^2_+} \nabla c_- n_+ \nabla\varphi dx + \int_{\mathbb{R}^2_+} \nabla\varphi\cdot\mathbf{b}n_+ dx - \frac{d}{dt}\big(M_+(y_+)^2\big) \\
=& I + II + III + IV - M_+\frac{d}{dt}y_+^2(t)
\end{aligned} \tag{3.3.15}$$

Next we estimate every term on the right hand side of (3.3.15). The first two terms are estimated as follows:

$$|I| = \left|\int_{\mathbb{R}^2_+} \Delta(x_2^2) n_+ dx\right| = 2M_+, \tag{3.3.16}$$

and

$$\begin{aligned}
|II| =& \left|-\frac{1}{4\pi}\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \frac{(\nabla(x_2^2)-\nabla(y_2^2))\cdot(x-y)}{|x-y|^2} n_+(x,t)n_+(y,t)dxdy\right| \\
\leq& \frac{1}{4\pi}\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} 2n_+(x)n_+(y)dxdy \leq \frac{1}{2\pi}M_+^2.
\end{aligned} \tag{3.3.17}$$

Now for the third term III in (3.3.15), applying (3.3.10) yields

$$|III| = \left| \int_{\mathbb{R}_+^2} \partial_{x_2} c_- n_+ 2x_2 dx \right| \leq \frac{1}{2\pi} \int_{\mathbb{R}_+^2} 2|x_2| \frac{M_+}{|x_2|} n_+ dx \leq \frac{1}{\pi} M_+^2. \tag{3.3.18}$$

Note that for the term II and III, we only estimate them formally above, one can prove the estimates explicitly using the same techniques as the one in Remark 5. For the IV term in (3.3.15), we use the (3.3.14) again to obtain

$$|IV| = \left| \int_{\mathbb{R}_+^2} \nabla x_2^2 \cdot \mathbf{b} n_+ dx \right| = 2A \left| \int_{\mathbb{R}_+^2} x_2^2 n_+ dx \right| = 2A(V_+ + M_+ y_+^2) \tag{3.3.19}$$

Collecting equation (3.3.15) and all the estimates (3.3.16), (3.3.17), (3.3.18) and (3.3.19) above, we have the following differential inequality,

$$\frac{d}{dt} \left(\frac{1}{M_+} V_+(t) + y_+^2(t) \right) \leq C(1 + M_+) + 2A \left(\frac{1}{M_+} V_+(t) + y_+^2(t) \right),$$

which yields

$$\frac{1}{M_+} V_+(t) + y_+^2(t) \leq C \left(1 + \frac{1}{M_+} \right) \delta^2 e^{2At} + \frac{1}{M_+} V_+(t) e^{2At} + y_+^2(0) e^{2At}. \tag{3.3.20}$$

Combining (3.3.4) with (3.3.20) yields

$$\begin{aligned} \frac{1}{M_+} V_+(t) + \left[\left(y_+(0) - C\delta \right) e^{At} \right]^2 &\leq \frac{1}{M_+} V_+(t) + y_+^2(t) \\ &\leq C \left(1 + \frac{1}{M_+} \right) \delta^2 e^{2At} + \frac{1}{M_+} V_+(0) e^{2At} + y_+^2(0) e^{2At}. \end{aligned}$$

By collecting similar terms, we finally have

$$\frac{1}{M_+} V_+(t) \leq \left[2C\delta y_+(0) + C \left(1 + \frac{1}{M_+} \right) \delta^2 + \frac{1}{M_+} V_+(0) \right] e^{2At}. \tag{3.3.21}$$

which completes the proof of Lemma 3.3.3. □

Equipped with the estimate on $V_+(t)$, we can now conclude the proof of Lemma 3.3.1.

Proof. (Lemma 3.3.1) Once $0 < \eta \ll 1$ was fixed, we can clearly choose a small enough δ such that by $(3.3.5)_\delta$, there holds

$$V_+(t) \le (1+\eta)V_+(0)e^{2At}. \tag{3.3.22}$$

Now recalling that $R = y_+(0)\sqrt{\frac{M_+}{2V_+(0)}} > 1$, then we can use (3.3.4), (3.3.22) and further choose δ small enough to get:

$$\begin{aligned}
y_+(t) - \frac{R}{1+\eta}\sqrt{\frac{2V_+(t)}{M_+}} \ge& \left[y_+(0) - C\delta - \frac{R}{\sqrt{1+\eta}}\sqrt{\frac{2V_+(0)}{M_+}}\right]e^{At} \\
=& \left[\left(1 - \frac{1}{\sqrt{1+\eta}}\right)y_+(0) - C\delta\right]e^{At} \\
\ge& 2\delta e^{At} \ge 2\delta.
\end{aligned}$$

Thus, the 'thin' δ-strip along the x_1-axis, $\mathcal{S}_\delta := \{(x_1,x_2)|0 \le x_2 \le 2\delta\}$, lies *outside* the strip centered around $y_+(t)$, uniformly in time,

$$\mathcal{S}_\delta \subset \{(x_1,x_2)||x_2 - y_+(t)| > R_\eta(t)\}, \qquad R_\eta(t) := \frac{R}{1+\eta}\sqrt{\frac{2V_+(t)}{M_+}}.$$

It follows that thanks to our choice of δ, the mass inside the δ-strip $\mathcal{S}_\delta$ does not exceed

$$\begin{aligned}
\int_{\mathcal{S}_\delta} n_+(x,t)dx \le& \int_{\mathbb{R}^2_+\cap\{|x_2-y_+|>R_\eta\}} n_+(x,t)dx \le \int_{\mathbb{R}^2_+\cap\{|x_2-y_+|>R_\eta\}} n_+(x,t)\frac{|x_2-y_+|^2}{|x_2-y_+|^2}dx \\
\le& \frac{(1+\eta)^2}{R^2 2V_+/M_+}\int_{\mathbb{R}^2_+} n_+(x,t)|x_2-y_+|^2dx = \frac{(1+\eta)^2}{R^2 2V_+/M_+}V_+ \\
\le& \frac{(1+\eta)^2}{2R^2}M_+.
\end{aligned}$$

By symmetry, the mass inside the symmetric δ-strip, $\{(x_1,x_2)||x_2| \le 2\delta\}$ is smaller than $\frac{(1+\eta)^2}{2R^2}2M_+ = \frac{(1+\eta)^2}{2R^2}M$, uniformly in time, which completes the proof of Lemma 3.3.1. □

Remark 6. *One can do a similar computation to get the evolution for the higher moment estimates* $\int_{\mathbb{R}^2_+} n(x,t)|x|^{2k}dx$*, and derive similar results to Lemma 3.3.1.*

3.3.3 Step 2 — proof of the main theorem with moderate mass constraint

With the Lemma 3.3.1 at our disposal, we can now turn to the proof of theorem 9 along the lines of [27]. Note that the actual calculation are to be carried out with the regularized solutions n^ϵ of (3.2.3), though for the sake of simplicity, we only do the formal calculation on $n(x) = n(\cdot, t)$.

The key is to use the logarithmic Hardy-Littlewood-Sobolev inequality (1.1.13) to get a bound on the entropy $S[n]$.

Remark 7. *It is pointed out in [27] that by multiplying f by indicator functions, one can prove that the inequality* (1.1.13) *remains true with $\mathbb{R}^2$ replaced by any bounded domains $\mathcal{D} \subset \mathbb{R}^2$.*

The idea of the proof goes as follows. By observing that the mass in the upper half plane and lower half plane are subcritical ($||n_\pm||_1 < 8\pi$), we plan to use the logarithmic Hardy-Littlewood-Sobolev inequality on these sub-domains to get uniform bound on the entropy. However, without extra information concerning the cell density distribution, naive application of logarithmic Hardy-Littlewood-Sobolev inequality fails. For this approach to work, the density distribution constraint required is that the cells in the upper and lower half plane are well-separated by a 'cell clear strip' in which the total number of cells is sufficiently small. The strip is constructed through applying Lemma 3.3.1. Combining the logarithmic Hardy-Littlewood-Sobolev inequality and the cell seperation constraint, we can use a 'total entropy reconstruction' trick introduced in [27] to obtain the entropy bound.

Now let's start the whole proof.

Proof. First we contruct the 'cell clear strip'. Define the following three regions:

$$\Gamma_1 = \{x_2 \,|\, x_2 > 2\delta\}, \quad \Gamma_2 = \{x_2 \,|\, x_2 < -2\delta\}, \quad \Gamma_3 = \mathbb{R}^2 \backslash (\Gamma_1 \cup \Gamma_2). \tag{3.3.23}$$

Here region Γ_1 contains points in the upper half plane which are 2δ away from the x_1 axis, whereas region Γ_2 contains points in the lower half plane with the same property. Region Γ_3 is a closed neighborhood of the x_1 axis. The δ neighborhood of the Γ_1, Γ_2 region is denoted as follows:

$$\Gamma_1^{(\delta)} = \{x_2 \mid x_2 > \delta\}, \quad \Gamma_2^{(\delta)} = \{x_2 \,|\, x_2 < -\delta\}. \tag{3.3.24}$$

We further decompose Γ_3 into subdomains:

$$S_1 = \{x_2 | \delta < x_2 \leq 2\delta\}, \quad S_2 = \{x_2 | -\delta > x_2 \geq -2\delta\}, \quad S_3 = \{x_2 \mid |x_2| \leq \delta\}. \tag{3.3.25}$$

Applying Lemma 3.3.1 yields that the total mass inside Γ_3 is small, i.e.

$$\int_{\Gamma_3} n dx = \int_{\{x_2 || x_2| \leq 2\delta\}} n dx \leq \frac{(1+\eta)^2}{2R^2} M. \tag{3.3.26}$$

Therefore, the Γ_3 strip is the 'cell clear strip'.

Next, we estimate the entropy. First recall that the free energy $E[n](t)$ (3.1.4) is decreasing, i.e,

$$\begin{aligned} E[n_0] \geq E[n] = & \left(1 - \frac{K}{8\pi}\right) \int n \log n dx + \frac{1}{8\pi} \bigg(K \int n \log n dx \\ & + 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} n(x) n(y) \log |x-y| dx dy \bigg) - \int G n dx \\ =: & \left(1 - \frac{K}{8\pi}\right) S[n] + T_1 - T_2. \end{aligned} \tag{3.3.27}$$

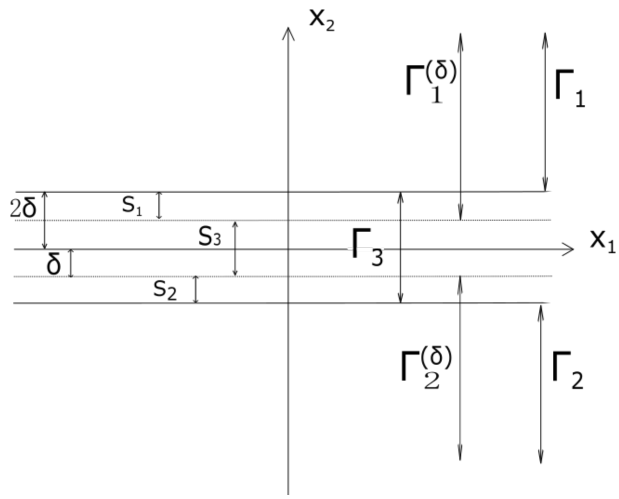

Figure 3.1: Regions $\Gamma_1, \Gamma_2, \Gamma_3$

To obtain the entropy bound, we need to estimate T_1 from below for some $K < 8\pi$ and estimate T_2 from above. We start by estimating T_1. Similar to [27], we apply the Logarithmic Hardy-Littlewood-Sobolev inequality in the three regions $\Gamma_2^{(\delta)}, \Gamma_2^{(\delta)}, \Gamma_3$ and obtain:

$$\int_{\Gamma_1^{(\delta)}} n(x)dx \int_{\Gamma_1^{(\delta)}} n \log n dx + 2 \iint_{\Gamma_1^{(\delta)} \times \Gamma_1^{(\delta)}} n(x)n(y) \log |x-y| dxdy \geq C,$$
$$\int_{\Gamma_2^{(\delta)}} n(x)dx \int_{\Gamma_2^{(\delta)}} n \log n dx + 2 \iint_{\Gamma_2^{(\delta)} \times \Gamma_2^{(\delta)}} n(x)n(y) \log |x-y| dxdy \geq C,$$
$$\int_{\Gamma_3} n(x)dx \int_{\Gamma_3} n \log n dx + 2 \iint_{\Gamma_3 \times \Gamma_3} n(x)n(y) \log |x-y| dxdy \geq C.$$

Combining the above inequalities yields

$$\begin{aligned}
-C \leq & K\int_{\mathbb{R}^2} n\log^+ n dx + 2\iint_{\mathbb{R}^2\times\mathbb{R}^2} n(x)n(y)\log|x-y|dxdy \\
& - 4\iint_{((\Gamma_1^{(\delta)})^c\times\Gamma_1)\cup(\Gamma_2\times(S_1\cup S_3))} n(x)n(y)\log|x-y|dxdy \\
& + 2\iint_{(S_1\times S_1)\cup(S_2\times S_2)} n(x)n(y)\log|x-y|dxdy \\
=: & I_1 + I_2 - I_3 + I_4. \qquad (3.3.28)
\end{aligned}$$

Here the number K is defined as

$$K := \max\left\{\int_{\Gamma_1^{(\delta)}} ndx + \int_{\Gamma_3} ndx, \int_{\Gamma_2^{(\delta)}} ndx + \int_{\Gamma_3} ndx\right\}. \qquad (3.3.29)$$

Combining the definition (3.3.29) and (3.3.26) yields

$$K \leq \left(\frac{1}{2} + \frac{(1+\eta)^2}{2R^2}\right)M.$$

By the moderate mass constraint (3.3.3), we have $K < 8\pi$. Next applying the fact that $|x-y| \geq \delta$ for all (x,y) in the integral domain of I_3, we estimate the I_3 and I_4 terms in (3.3.28) as follows

$$\begin{aligned}
I_3 \geq & 4M^2\log\delta, \\
I_4 \leq & 2\iint_{(S_1\times S_1)\cup(S_2\times S_2)} n(x)n(y)\log^+|x-y|dxdy \\
\leq & C\iint_{(S_1\times S_1)\cup(S_2\times S_2)} n(x)n(y)(1+|x|^2+|y|^2)dxdy \leq C(M^2 + 2M\int|x|^2 n(x)dx).
\end{aligned} \qquad (3.3.30)$$

Combining (3.3.30) with (3.3.28) yields

$$\begin{aligned}
K\int_{\mathbb{R}^2} n\log^+ n dx + 2\iint_{\mathbb{R}^2\times\mathbb{R}^2} n(x)n(y)\log|x-y|dxdy \\
\geq 4M^2\log\delta - C(1+M^2+M\int n|x|^2dx).
\end{aligned} \qquad (3.3.31)$$

Recall the well-known upper bound on the negative part of the entropy:

Lemma 3.3.4. *([27]) For f positive function, the following estimate holds*

$$\int_{\mathbb{R}^2} f \log^- f dx \leq \frac{1}{2}\int_{\mathbb{R}^2} |x|^2 f dx + \log(2\pi)\int_{\mathbb{R}^2} f dx + \frac{1}{e} \leq C(1 + M + \int |x|^2 ndx). \tag{3.3.32}$$

Proof. For the proof of this lemma, we refer the interested readers to the papers [28], [27]. □

Combining (3.3.32) and (3.3.31) yields the lower bound of T_1 in (3.3.27)

$$T_1 \geq -C(M+1)\left(1 + M + \int n|x|^2 dx\right) + 4M^2 \log \delta \tag{3.3.33}$$

for $0 < K < 8\pi$.

For the T_2 term in (3.3.27), it is bounded above by $\left|\int Hndx\right| \leq A\int |x|^2 ndx$. Therefore it is enough to show that the second moment is bounded for any finite time. The time evolution of the second moment can be estimated as follows:

$$\frac{d}{dt}\int n|x|^2 dx \leq 4AM + 4A\int Hn(x)dx \leq 4AM + \frac{A^2}{2}\int |x|^2 ndx.$$

Gronwall inequality yields that the second moment is bounded for all finite time:

$$\int n|x|^2 dx \leq C(A,T) < \infty, \quad \forall T < \infty. \tag{3.3.34}$$

Therefore $T_2 \leq C(A,T)$. Combining this with (3.3.27) and (3.3.33), and recalling that $K < 8\pi$ yield

$$S[n](T) \leq \frac{1}{(1-\frac{K}{8\pi})}\left(E[n_0] + C(M,A,T) - \frac{1}{2\pi}M^2 \log \delta\right), \quad \forall T < \infty. \tag{3.3.35}$$

As a result, we see from (3.3.35) that the entropy $S[n^\epsilon]$ is uniformly bounded independent of ϵ for any finite time interval $[0,T]$, $\quad T < \infty$. Now by the Proposition 8, 9, we have that the free energy solution exists on any time interval $[0,T]$, $\quad \forall T < \infty$. □

3.3.4 Step 3 — proof of the main theorem

In the proof of Theorem 9, we see that the cell population is separated by a 'cell clear zone' near the x_1 axis. Since total mass in the "cell clear zone" is small, we can heuristically treat the total cell population as a union of two subgroups with subcritical mass ($< 8\pi$). However, since we lack sufficiently good control over the total number of cells near the x_1 axis, we cannot use this idea to prove the optimal result as stated in Theorem 7. The idea of proving Theorem 7 is that instead of considering the total cell population as the union of two subgroups separated by one fixed 'cell clear zone', we treat it as the union of three subgroups with subcritical mass, namely, the cells in the upper half plane, the lower half plane and the neighborhood of the x_1 axis, respectively. These three subgroups of cells are separated by two 'cell clear zones' varying in time.

The main difficulty in the proof is setting up the three new regions such that:

1. mass inside each region is smaller than 8π;

2. the total mass of cells near their boundaries is well-controlled.

Once the construction is completed, the remaining steps will be similar to step 2.

Proof of Theorem 7. We start by constructing the three regions. First we note that the Lemma 3.3.1 implies that there exists $\delta > 0$ such that the following estimate is satisfied for a fixed $R > 1$ and η chosen small enough:

$$\int_{|x_2|\le 2\delta} n dx \le \frac{(1+\eta)^2}{R^2} \int_{\mathbb{R}^2} n dx \le \frac{1}{2}M, \quad \forall t > 0. \tag{3.3.36}$$

Now the region $L = \{(x_1, x_2)||x_2| \leq 2\delta\}$ have total mass less than $\frac{1}{2}M = M_+ < 8\pi$ for all time.

Secondly, we subdivide the region L into J pieces:

$$
\begin{aligned}
L =& \cup_1^J L^i, \\
L^i :=& \{(x_1, x_2)|\frac{2\delta}{J}(i) > |x_2| \geq \frac{2\delta}{J}(i-1)\}.
\end{aligned}
$$

Here $J = J(M) \geq 10$, to be determined later, depends on M. By the pigeon hole principle, there is at least three strips L^i such that

$$
\int_{L^i} n(x)dx \leq \frac{2}{J}M_+.
$$

Suppose there are only two strips with mass smaller than $\frac{2}{J}M_+$, then total mass in L will be bigger than $(J-2)\frac{2}{J}M_+ > M_+$, a contradiction. Now we pick from these three strips the one which is neither L^1 nor L^J. As a result, this strip L^i does not touch the x_1 axis nor the boundary of L. We denote this i by i^*. The L^{i^*} is the 'cell clear zone'. Notice that here $i^* = i^*(n, t)$ depends on time.

Finally, we use this i^* to define the regions. First we define the three regions, each of which has total mass smaller than 8π:

$$
\Gamma_1 = \left\{x_2 \geq \frac{2\delta}{J}i^*\right\}, \quad \Gamma_2 = \left\{x_2 \leq -\frac{2\delta}{J}i^*\right\}, \quad \Gamma_3 = \left\{|x_2| \leq \frac{2\delta}{J}(i^*-1)\right\}. \tag{3.3.37}
$$

Next we set

$$
\rho = \frac{2\delta}{3J}, \tag{3.3.38}
$$

and define the ρ neighborhood of the above three regions:

$$
\Gamma_1^{(\rho)} = \{x_2 > \frac{2\delta}{J}(i^* - \frac{1}{3})\} \quad \Gamma_2^{(\rho)} = \left\{x_2 < -\frac{2\delta}{J}(i^* - \frac{1}{3})\right\}, \quad \Gamma_3^{(\rho)} = \{|x_2| < \frac{2\delta}{J}(i^* - \frac{2}{3})\}. \tag{3.3.39}
$$

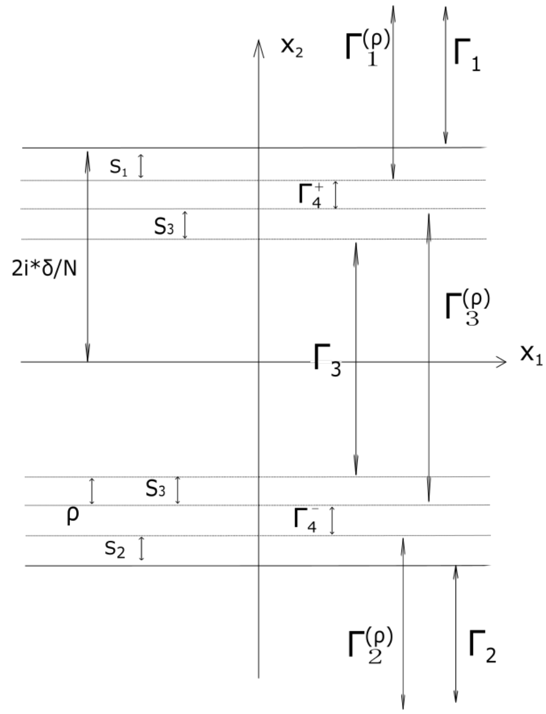

Figure 3.2: Regions $\Gamma_1, \Gamma_2, \Gamma_3$ in the proof of the main theorem

Now we define the complement Γ_4 of the above three regions $\Gamma_i^{(\rho)}, i = 1, 2, 3$:

$$\Gamma_4 = \left\{ \frac{2\delta}{J}\left(i^* - \frac{2}{3}\right) \le |x_2| \le \frac{2\delta}{J}\left(i^* - \frac{1}{3}\right) \right\} = \Gamma_{4+} \cup \Gamma_{4-}, \tag{3.3.40}$$

$$\Gamma_{4\pm} = \Gamma_4 \cap \mathbb{R}^2_\pm. \tag{3.3.41}$$

Now we define the complement $\Gamma_4^{(\rho)}$ of $\cup_{i=1}^3 \Gamma_i$ and decompose it into subdomains:

$$\Gamma_4^{(\rho)} = \left(\cup_{i=1}^3 \Gamma_i\right)^c = \left(\Gamma_{4+}^{(\rho)}\right) \cup \left(\Gamma_{4-}^{(\rho)}\right), \quad \left(\Gamma_{4\pm}^{(\rho)}\right) = \Gamma_4^{(\rho)} \cap \mathbb{R}^2_\pm, \tag{3.3.42}$$

$$\Gamma_4^{(\rho)} = \Gamma_4 \cup S_1 \cup S_2 \cup S_3, \tag{3.3.43}$$

$$S_1 = \Gamma_1^{(\rho)} \backslash \Gamma_1, \quad S_2 = \Gamma_2^{(\rho)} \backslash \Gamma_2, \quad S_3 = \Gamma_3^{(\rho)} \backslash \Gamma_3. \tag{3.3.44}$$

Remark 8. *It is important to notice that the regions we are constructing are changing with respect to the given time t. Therefore, by doing the argument below, we can only* *show that the entropy is bounded at time t, but since t is an arbitrary finite time, we have the bound on entropy for $\forall t \in [0,T], \forall T < \infty$.*

We start estimating the entropy. By the free energy dissipation, we obtain

$$\begin{aligned}
E[n_0] \geq & \left(1-\frac{K}{8\pi}\right)\int n\log n dx + \frac{1}{8\pi}\left(K\int n\log n dx \right.\\
& \left. + 2\iint_{\mathbb{R}^2\times\mathbb{R}^2} n(x)n(y)\log|x-y|dxdy\right) - \int Hndx \\
= & \left(1-\frac{K}{8\pi}\right) S[n(T)] + T_1 - T_2. \qquad (3.3.45)
\end{aligned}$$

To derive entropy bound, we need the estimate T_1 form below for $K < 8\pi$ and estimate T_2 from above. We start by estimating T_1. Combining the definition of L^{i^*} and (3.3.36) yields

$$\int_{\Gamma_1^{(\rho)}} ndx \leq M_+ = \frac{1}{2}M < 8\pi, \qquad (3.3.46)$$

$$\int_{\Gamma_2^{(\rho)}} ndx \leq M_+ = \frac{1}{2}M < 8\pi, \qquad (3.3.47)$$

$$\int_{\Gamma_3^{(\rho)}} ndx \leq M_+ = \frac{1}{2}M < 8\pi, \qquad (3.3.48)$$

$$\int_{\Gamma_4^{(\rho)}} ndx \leq \int_{L^{i^*}} ndx \leq \frac{2}{J}(M_+). \qquad (3.3.49)$$

Now by the log-Hardy-Littlewood-Sobolev inequality (1.1.13), we have that

$$\int_{\Gamma_i^{(\rho)}} n(x)dx \int_{\Gamma_i^{(\rho)}} n(x)\log n(x)dx + 2\iint_{\Gamma_i^{(\rho)}\times\Gamma_i^{(\rho)}} n(x)n(y)\log|x-y|dxdy \geq -C, \quad i=1,2,3, \qquad (3.3.50)$$

$$\int_{\Gamma_{4\pm}^{(\rho)}} n(x)dx \int_{\Gamma_{4\pm}^{(\rho)}} n(x)\log n(x)dx + 2\iint_{\Gamma_{4\pm}^{(\rho)}\times\Gamma_{4\pm}^{(\rho)}} n(x)n(y)\log|x-y|dxdy \geq -C. \qquad (3.3.51)$$

Same as in subsection 3.3, we use these estimates to reconstruct the entropy and the potential on the whole $\mathbb{R}^2$ as follows:

$$\begin{aligned}
-C \leq & K \int_{\mathbb{R}^2} n(x) \log^+ n(x) dx + 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} n(x)n(y) \log|x-y| dxdy \\
& - 2 \iint_R n(x)n(y) \log|x-y| dxdy \\
& + 2 \iint_{(S_1 \times S_1) \cup (S_2 \times S_2) \cup (S_3^+ \times S_3^+) \cup (S_3^- \times S_3^-)} n(x)n(y) \log|x-y| dxdy \\
=: & I_1 + I_2 - I_3 + I_4. \qquad (3.3.52)
\end{aligned}$$

The region R[1] and the integral domain of I_4 is indicated in Figure 3. The K in (3.3.52) can be estimated using (3.3.49) as follows

$$K := M_+ + \int_{\Gamma_4^{(\rho)}} n dx \leq \left(1 + \frac{2}{J}\right) M_+. \qquad (3.3.53)$$

By the assumption $M_+ < 8\pi$, we can make J big such that $K < 8\pi$. This is where we choose the $J = J(M)$. Applying the fact that $|x - y| \geq \frac{2\delta}{3J}$, $\forall (x, y) \in R$, the I_3 and I_4 terms in (3.3.52) can be estimated as follows:

$$\begin{aligned}
I_3 \geq & CM^2 \log \frac{2\delta}{3J}, \\
I_4 \leq & 2 \iint_{(S_1 \times S_1) \cup (S_2 \times S_2) \cup (S_3^+ \times S_3^+) \cup (S_3^- \times S_3^-)} n(x)n(y) \log^+ |x-y| dxdy \\
\leq & C \iint_{(S_1 \times S_1) \cup (S_2 \times S_2) \cup (S_3^+ \times S_3^+) \cup (S_3^- \times S_3^-)} n(x)n(y)(1 + |x|^2 + |y|^2) dxdy \\
\leq & C(M^2 + M \int |x|^2 n dx). \qquad (3.3.54)
\end{aligned}$$

[1] Region R is the union of the following nine regions:

1)$\Gamma_1 \times (\Gamma_1^{(\rho)})^c$, 2)$S_1 \times (\Gamma_1 \cup (\Gamma_4^{(\rho)})^+)^c$, 3)$\Gamma_4^+ \times ((\Gamma_4^{(\rho)})^+)^c$, 4)$S_3^+ \times (\Gamma_3^{(\rho)} \cup (\Gamma_4^{(\rho)})^+)^c$, 5)$\Gamma_3 \times (\Gamma_3^{(\rho)})^c$,

6)$S_3^- \times (\Gamma_3^{(\rho)} \cup (\Gamma_4^{(\rho)})^-)^c$, 7)$\Gamma_4^- \times ((\Gamma_4^{(\rho)})^-)^c$, 8)$S_2 \times (\Gamma_2^{(\rho)} \cup (\Gamma_4^{(\rho)})^-)^c$, 9)$\Gamma_2 \times (\Gamma_2^{(\rho)})^c$.

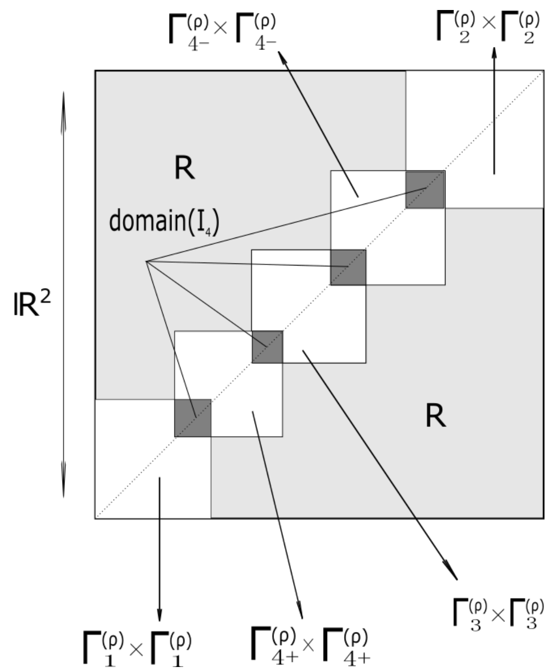

Figure 3.3: Region R in $\mathbb{R}^2 \times \mathbb{R}^2$

Combining (3.3.52), (3.3.53) and (3.3.54) yields

$$
\begin{aligned}
K \int n(x) \log^+ n(x) dx + 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} & n(x) n(y) \log |x - y| dx dy \\
\geq & C M^2 \log \frac{2\delta}{3J} - C(1 + M^2 + M \int |x|^2 n dx).
\end{aligned}
$$

Moreover, applying (3.3.32) yields

$$
T_1 \geq -C(M+1)(1 + M + \int |x|^2 n dx) + C M^2 \log \frac{2\delta}{3J}. \tag{3.3.55}
$$

Combining (3.3.55), (3.3.34), (3.3.45) yields

$$
S[n](T) \leq \frac{1}{(1 - \frac{K}{8\pi})} \left(E[n_0] + C(M, A, T) - C M^2 \log \frac{2\delta}{3J} \right) < \infty, \forall T < \infty.
$$

Once the entropy is bounded for any finite time, the existence is guaranteed by Proposition 8 and Proposition 9. $\square$

3.4 Conclusion

We revisit the question of global regularity for the Patlak-Keller-Segel (PKS) chemotaxis model. The classical 2D hyperbolic-elliptic model blows up for initial mass $M > 8\pi$. We consider more realistic scenario which takes into account the flow of the ambient environment induced by *harmonic* potentials, and thus retain the identical elliptic structure as in the original PKS. Surprisingly, we find that already the simplest case of *linear* stationary vector field, $Ax^\top$, with large enough amplitude A, prevents chemotactic blow-up. Specifically, the presence of such an ambient fluid transport creates what we call a 'fast splitting scenario', which competes with the focusing effect of aggregation so that 'enough mass' is pushed away from concentration along the x_1-axis, thus avoiding a finite time blow-up, at least for $M < 16\pi$. Thus, the enhanced ambient flow doubles the amount of allowable mass which evolve to global smooth solutions.

Chapter 4: Suppression of blow up in parabolic-parabolic Patlak-Keller-Segel systems via strictly monotone shear flows

4.1 Overview

In this paper, we consider the two-dimensional parabolic-parabolic Patlak-Keller-Segel equations with additional effect of advection by a shear flow, which model the chemotaxis phenomena in a moving fluid:

$$\partial_t n + \nabla \cdot (n\nabla c) + Au(y)\partial_x n = \Delta n, \tag{4.1.1a}$$

$$\partial_t c + Au(y)\partial_x c = \Delta c + n - c, \tag{4.1.1b}$$

$$n(x, y, 0) = n_{in}(x, y), \quad c(x, y, 0) = c_{in}(x, y), \quad (x, y) \in \mathbb{T} \times \mathbb{R}. \tag{4.1.1c}$$

In contrast to the parabolic-elliptic PKS equation with external large shear flow (Chapter 2), the mixing of the shear flow has both stabilizing and destabilizing effect on the system (4.1.1). On the one hand, same as in the parabolic-elliptic case, mixing enhances the dissipation in the micro-organism evolution equation (4.1.1a) and hence stabilizes the dynamics. On the other hand, the extra shear flow advection term $Au(y)\partial_x c$ in the chemo-attractant evolution (4.1.1b) creates large gradient in the chemical density c. To better understand this destabilizing effect, we take a look at the passive scalar equation on a Torus $\mathbb{T}^2$:

$$\partial_t \rho + Au(y)\partial_x \rho = \Delta \rho, \quad \rho(t=0,\cdot) = \rho_0(\cdot),$$

where ρ have average zero. We calculate the time evolution of $||\nabla \rho||_2^2$ as follows

$$\frac{d}{dt}||\nabla \rho||_2^2 \leq - \overbrace{||\nabla \rho||_2^2}^{\text{Dissipation}} + \overbrace{A||u'(y)\partial_x \rho||_2 ||\nabla \rho||_2}^{\text{Shear flow contribution}}.$$

One observe that the dissipation start to take effect on the time scale $O(1)$ but the shear flow effect takes effect on the time scale $O(1/A)$. Therefore in the time scale between $O(1/A)$ and $O(1)$, shear flow effect dominates the dissipation and creates large growth in the gradient. This is called the *destabilization effect of the shear flow*. The large growth in the chemical gradient destabilizes the dynamics through the aggregation nonlinearity $\nabla \cdot (\nabla c n)$ in the micro-organism evolution (4.1.1a). It is worth noting that this destabilizing effect of shear flow does not exist in the parabolic-elliptic regime due to the fast relaxation of chemical density to equilibrium. As a result, it is reasonable to expect that an extra smallness assumption is needed to control the mixing destabilizing effect. In this paper, it is assumed that the x-dependent part of the initial chemical gradient is small. Since only the x-dependent part of $\partial_{y_1} c$ is strongly forced by the shear flow, this smallness restriction is sufficient to control the growth of the chemo-attractant gradient and hence keep the aggregation nonlinearity in (4.1.1a) bounded independent of A. Now the situation is similar to the parabolic-elliptic case, hence one can show suppression of chemotactic blow-up through shear flow.

Denote the following projections for function $g(x,y)$:

$$g_0(y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x,y)dx, \quad g_{\neq}(x,y) = g(x,y) - g_0(y).$$

The main theorem of this paper is as follows.

Theorem 4. *Let the shear flow profile $u \in C^3(\mathbb{R})$ be a strictly monotone function whose derivative approaches nonzero numbers at $\pm\infty$ and $||u'||_{W^{2,\infty}} < \infty$. Consider the equation (4.1.1) subject to initial condition $n_{in} \in H^1 \cap W^{1,\infty}(\mathbb{T}\times\mathbb{R})$, $c_{in} \in H^2 \cap W^{2,\infty}(\mathbb{T}\times\mathbb{R})$. Then the solution to (4.1.1) is global in time if the amplitude A takes values in the interval $(A_0, ||\nabla(c_{in})_{\neq}||^{q_\star}_{H^1\cap W^{1,\infty}}]$, where $q_\star \in (-2,0)$ and $A_0 = A_0(u, \|n_{in}\|_{H^1\cap W^{1,\infty}}, ||\nabla c_{in}||_{H^1\cap W^{1,\infty}})$ is independent of $||\nabla(c_{in})_{\neq}||_{H^1\cap W^{1,\infty}}$.*

We make several remarks concerning the main theorems.

Remark 9. *For the interval $(A_0, ||\nabla(c_{in})_{\neq}||_{H^1\cap W^{1,\infty}}]$ to be nonempty, we implicitly assume that $||\nabla(c_{in})_{\neq}||_{H^1\cap W^{1,\infty}}$ is small compared to $A_0^{1/q_\star}$, $q_\star \in (-2,0)$. As explained before, this smallness is applied to control the destabilizing effect of the strong shear flow. Note that if $c_{in} \equiv 0$, then the interval is always nonempty. This corresponds to the situation that at the initial time of the chemotaxis experiment, no chemo-attractant exists in the environment.*

Remark 10. *The difficulty is twofold. First we need to construct a hypocercivity functional adapted to the parabolic-parabolic PKS equation, which is significantly more subtle than the one in the parabolic-elliptic case [13]. Secondly, one needs to control $||\nabla c_{\neq}||_\infty$ uniformly independent of A for all time. This is delicate due to destabilizing effect of the strong shear flow.*

4.1.1 Notations

Miscellaneous

Given quantities X, Y, if there exists a constant B such that $X \leq BY$, we often write $X \lesssim Y$. We will moreover use the notation $\langle x \rangle := (1 + |x|^2)^{1/2}$.

Fourier Analysis

For $f(x, y)$ we define the Fourier transform $\widehat{f}(k, y)$ only in terms of variable x, and the inverse Fourier transform as follows:

$$\widehat{f}(k, y) = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-ikx} f(x, y) dx, \quad \check{g}(x, y) = \sum_{k=-\infty}^{\infty} g(k, y) e^{ikx}.$$

Define the following orthogonal projections:

$$f_0(t, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t, x, y) dx,$$

$$f_{\neq}(t, x, y) = f(t, x, y) - f_0(t, y).$$

Here '0' and '$\neq$' stand for "zero frequency" and "non-zero frequencies". For any measurable function $m(k)$, we define the Fourier multiplier $m(\partial_x) f := (m(k) \hat{f}(k, y))^{\vee}$.

Functional spaces

The norm for the L^p space is denoted as $|| \cdot ||_p$ or $|| \cdot ||_{L^p(\cdot)}$:

$$||f||_p = ||f||_{L^p} = \left(\int |f|^p dx \right)^{1/p},$$

with natural adjustment when p is ∞. If we need to emphasize the ambient space, we use the second notation, i.e., $||n_{\neq}||_{L^p(\mathbb{T}\times\mathbb{R})}$. Otherwise, we use the first notation for the sake of simplicity. The Sobolev norm $||\cdot||_{H^s}$ is defined as follow:

$$||f||_{H^s} := ||\langle\nabla\rangle^s f||_{L^2}.$$

For a function of space and time $f = f(t,x)$, we use the following space-time norms:

$$||f||_{L_t^p L_x^q} :=||||f||_{L_x^q}||_{L_t^p},$$

$$||f||_{L_t^p H_x^s} :=||||f||_{H_x^s}||_{L_t^p}.$$

The paper is organized as follows: in section 2, we set up the bootstrap argument; in section 3, we prove the enhanced dissipation of the x-depending part of the solution; in section 4, we prove the $L_t^2 \dot{H}_{x,y}^1$ estimate of x-dependent part the micro-organism density; in section 5, we estimate the x independent part of the solution; in section 6, we prove the uniform in time L^∞ estimate of the solution.

4.2 Preliminaries and Bootstrap

4.2.1 Reformulation of Theorem 4

In this paper, we will prove the following theorem, which implies the Theorem 4.

Theorem 5. *Let the shear flow profile u satisfy the conditions in Theorem 4. Consider the equation (4.1.1) subject to initial conditions $n_{in} \in H^1 \cap W^{1,\infty}$, $c_{in} \in H^2 \cap W^{2,\infty}$ and $||\nabla(c_{in})_{\neq}||_{H^1\cap W^{1,\infty}} \le C_{in}A^{-q}$, $q > 1/2$ for any constant C_{in} indepen-*

dent of A. Then there exists an $A_0 = A_0(u, \|n_{in}\|_{H^1\cap W^{1,\infty}}, ||\nabla c_{in}||_{H^1\cap W^{1,\infty}})$ such that if $A > A_0$, the solution to (4.1.1) is global in time.

Proof of Theorem 4. Choosing $C_{in} = 1$ in Theorem 5, we have the following relation

$$||\nabla (c_{in})_{\neq}||_{H^1\cap W^{1,\infty}}^{-1/q} \geq A. \tag{4.2.1}$$

Combining it with the relation $A \geq A_0$, we end up with $A \in (A_0, ||\nabla (c_{in})_{\neq}||_{H^1\cap W^{1,\infty}}^{-1/q})$, $q \in (1/2, \infty)$. Define $q_\star = -\frac{1}{q}$, we end up with the condition specified in Theorem 4. □

4.2.2 Bootstrap argument

Same as in the paper [13], we rescale in time and decompose the solution into x-independent part n_0, c_0 and x-dependent part $n_{\neq}$, $c_{\neq}$

$$\partial_t n_0 + \frac{1}{A}\partial_y(\partial_y c_0 n_0) + \frac{1}{A}(\nabla \cdot (\nabla c_{\neq} n_{\neq}))_0 = \frac{1}{A}\partial_{yy} n_0, \tag{4.2.2a}$$

$$\partial_t c_0 = \frac{1}{A}\Delta c_0 + \frac{1}{A} n_0 - \frac{1}{A} c_0; \tag{4.2.2b}$$

and,

$$\partial_t n_{\neq} + u(y)\partial_x n_{\neq} + \frac{1}{A}\nabla \cdot (\nabla c_{\neq} n_0) + \frac{1}{A}\partial_{y_1}(\partial_{y_1} c_0 n_{\neq}) + \frac{1}{A}(\nabla \cdot (\nabla c_{\neq} n_{\neq}))_{\neq} = \frac{1}{A}\Delta n_{\neq}, \tag{4.2.3a}$$

$$\partial_t c_{\neq} + u(y)\partial_x c_{\neq} = \frac{1}{A}\Delta c_{\neq} + \frac{1}{A} n_{\neq} - \frac{1}{A} c_{\neq}. \tag{4.2.3b}$$

To apply the machinery of the paper [8], we apply the Fourier transform *only in the x variable* to both sides of (4.2.3a,4.2.3b) to obtain

$$\partial_t \widehat{n}_k + NL_k + L_k + u(y)ik\widehat{n}_k = \frac{1}{A}(\partial_{yy} - |k|^2)\widehat{n}_k, \tag{4.2.4a}$$

$$\partial_t \widehat{c}_k + u(y)ik\widehat{c}_k = \frac{1}{A}(\partial_{yy} - |k|^2)\widehat{c}_k + \frac{1}{A}\widehat{n}_k - \frac{1}{A}\widehat{c}_k, \quad k \neq 0, \tag{4.2.4b}$$

where L_k, NL_k are defined as follows:

$$NL_k := \frac{1}{A}\sum_{\ell\neq 0,k} \partial_y(\partial_y \widehat{c}_{k-\ell}\widehat{n}_\ell) - \frac{1}{A}k\sum_{\ell\neq 0,k}(k-\ell)\widehat{c}_{k-\ell}\widehat{n}_\ell, \tag{4.2.5}$$

$$L_k := \frac{1}{A}\partial_y(\partial_y c_0 \widehat{n}_k) + \frac{1}{A}\nabla\cdot(\nabla\widehat{c}_k n_0) = \frac{1}{A}\partial_y(\partial_y c_0\widehat{n}_k) - \frac{1}{A}k^2\widehat{c}_k n_0 + \frac{1}{A}\partial_y(\partial_y\widehat{c}_k n_0). \tag{4.2.6}$$

Here, the L refers to "linear with respect to the nonzero frequencies" and NL refers to "nonlinear with respect to the nonzero frequencies".

As is standard in the study of nonlinear mixing, we use a bootstrap argument to prove the main theorem. For constants C_{ED}, C_{n_0,L^2}, $C_{n_0,\dot{H}^1}$, $C_{n,\infty}$, $C_{\nabla c_{\neq},\infty}$ and A_0 determined by the proof, define $T_\star$ to be the end-point of the largest interval $[0,T_\star]$ such that the following hypotheses hold for all $t \leq T_\star$:

(1) Nonzero mode $L^2_t\dot{H}^1_{x,y}$ estimate:

$$\frac{1}{A}\int_0^{T_\star} ||\nabla_{x,y} n_{\neq}||_2^2 dt \leq 8||n_{in}||_2^2; \tag{4.2.7a}$$

(2) Nonzero mode enhanced dissipation estimate:

$$||n_{\neq}(t)||_2^2 + ||\nabla c_{\neq}(t)||_2^2 \leq 4C_{ED}(||n_{in}||_{H^1}^2 + 1)e^{-\frac{\eta t}{A^{1/3}}}, \quad \forall t < T_\star, \tag{4.2.7b}$$

where η is a small constant depending only on u.

(3) Uniform in time estimates on the zero mode:

$$\begin{aligned} ||\partial_y c_0||_{L^\infty(0,T_\star;L_y^\infty)} + ||n_0||_{L_t^\infty(0,T_\star;L_y^2)} \leq & 4C_{n_0,L^2}, \\ ||\partial_y n_0||_{L_t^\infty(0,T_\star;L_y^2)} \leq & 4C_{n_0,\dot{H}^1}; \end{aligned} \tag{4.2.7c}$$

(4) L^∞ estimate of the solution n:

$$||n||_{L_t^\infty(0,T_\star;L_{x,y}^\infty)} \leq 4C_{n,\infty}; \tag{4.2.7d}$$

(5) L^∞ estimate of the x-dependent part of the chemical gradient $\nabla c_{\neq}$:

$$||\nabla c_{\neq}||_{L_t^\infty(0,T_\star;L_{x,y}^\infty)} \leq 4C_{\nabla c_{\neq},\infty}. \tag{4.2.7e}$$

Furthermore, we define the following constant to simplify the notation:

$$C_{2,\infty} := 1 + M + C_{ED}^{1/2}||n_{in}||_{H^1} + C_{n_0,L^2} + C_{n,\infty} + C_{\nabla c_{\neq},\infty} + ||\nabla(c_{in})_0||_{H^1\cap W^{1,\infty}}. \tag{4.2.7f}$$

Note that $C_{n_0,\dot{H}^1}$ is not included in $C_{2,\infty}$.

The goal is to prove the following improvement to the above hypotheses:

Proposition 10. *For all n_{in}, c_{in} and u satisfying the assumption of Theorem 5, there exists an $A_0 = A_0(u, \|n_{in}\|_{H^1\cap L^\infty}, \|\nabla c_{in}\|_{H^1\cap W^{1,\infty}})$ such that if $A > A_0$ then the following conclusions hold on the interval $[0, T_\star]$:*

$$\frac{1}{A}\int_0^{T_\star} ||\nabla_{x,y} n_{\neq}||_2^2 dt \leq 4||n_{in}||_2^2; \tag{4.2.8a}$$

$$||n_{\neq}(t)||_2^2 + ||\nabla c_{\neq}(t)||_2^2 \leq 2C_{ED}(||n_{in}||_{H^1}^2 + 1)e^{-\frac{\eta t}{A^{1/3}}}, \quad \forall t < T_\star; \tag{4.2.8b}$$

$$\begin{aligned} ||\partial_{y_1} c_0||_{L_t^\infty(0,T_\star;L_y^\infty)} + ||n_0||_{L_t^\infty(0,T_\star;L_y^2)} \leq & 2C_{n_0,L^2}, \\ ||\partial_y n_0||_{L_t^\infty(0,T_\star;L_y^2)} \leq & 2C_{n_0,\dot{H}^1}; \end{aligned} \tag{4.2.8c}$$

$$||n||_{L^\infty(0,T_\star;L^\infty_{x,y})} \leq 2C_{n,\infty}; \tag{4.2.8d}$$

$$||\nabla c_{\neq}||_{L^\infty_t(0,T_\star;L^\infty_{x,y})} \leq 2C_{\nabla c_{\neq},\infty}. \tag{4.2.8e}$$

Proposition 10 together with the local wellposedness of the equation (4.1.1) implies that the time interval $[0,T_\star]$ on which the estimates (4.2.7) hold is both open and closed on $\mathbb{R}_+$. Since the estimates are trivially satisfied at the initial time, we obtain that $[0,T_\star]$ is nonempty and hence $T_\star$ must be infinity, which in term implies Theorem 5.

Remark 11. *For the sake of completeness, we prove the blow-up criterion for the system* (4.1.1) *in the appendix. The criteria implies that as long as $||n||_\infty$ is bounded uniformly in time, all initial bounds on higher H^s norms of the solution can be propagated.*

Remark 12. *The constants in the proof are determined in the following order*

$$C_{ED} \Rightarrow C_{n_0,L^2} \Rightarrow C_{n,\infty}, C_{\nabla c_{\neq},\infty} \Rightarrow C_{n_0,\dot{H}^1} \Rightarrow A_0. \tag{4.2.9}$$

The magnitude of the flow A_0 will be chosen large depending on the constants in the hypotheses and the intermediate constants in the proof.

Remark 13. *We need to control the destabilizing effect of the shear flow in the proof of* (4.2.8b), (4.2.8d) *and* (4.2.8e).

4.2.3 Chemical gradient $\partial_y c_0$ estimate

The following estimate of the chemical gradient $\partial_y c_0$ is applied in the latter sections.

Lemma 4.2.1. *Consider the solution to* (4.2.2a) *subject to initial data* $(c_{in})_0$*. For* $\forall s \in \mathbb{N}$ *and any* (p, q) *pair such that either* $2 \le p < \infty, 1 \le q \le p$ *or* $p = \infty, 1 < q \le p$ *is satisfied, the following estimates hold for the solution* c_0

$$\begin{aligned} ||\partial_{y_1} c_0(t)||_p &\lesssim_{p,q} \sup_{0 \le \tau \le t} ||n_0(\tau)||_q + ||(\partial_{y_1} c_{in})_0||_p; \\ ||\partial_{y_1}^{s+1} c_0(t)||_p &\lesssim_{p,q} \sup_{0 \le \tau \le t} ||\partial_{y_1}^s n_0(\tau)||_q + ||(\partial_{y_1}^{s+1} c_{in})_0||_p, \quad 2 \le p \le \infty. \end{aligned} \tag{4.2.10}$$

Proof. For the sake of brevity, we skip the proof. □

4.3 Enhanced dissipation estimate (4.2.8b)

Enhanced dissipation functional $\mathcal{F}$

In this subsection, we construct the functional $\mathcal{F}$ to exploit the enhanced dissipation in the equation (4.1.1).

We start by introducing the basic ideas of Hypocoercivity ([8], [115]). Consider the following passive scalar equation on $\mathbb{T} \times \mathbb{R}$,

$$\partial_t f + u(y)\partial_x f = \frac{1}{A}\Delta f, \quad f(t = 0, \cdot) = f_{in}(\cdot). \tag{4.3.1}$$

By applying Fourier transform in the x variable, we obtain the following equation:

$$\partial_t \widehat{f}_k + u(y)ik\widehat{f}_k = \frac{1}{A}(\partial_y)^2 \widehat{f}_k - \frac{|k|^2}{A}\widehat{f}_k.$$

The term $u(y)ik\widehat{f}_k$ is called the conservative part of the equation (4.3) because it generates a unitary semigroup which preserves the L^2 norm. The terms $-\frac{1}{A}(\partial_y)^2\widehat{f}_k - \frac{|k|^2}{A}\widehat{f}_k$ are the dissipative part of the dynamics because they cause decay in the L^2 norm $||\widehat{f}_k||_2^2$. The idea of Hypocoercivity is to construct a functional Φ, which is more "coercive" than the H^1 norm, to exploit the commutator structure between the conservative part and the dissipative part of the dynamics. The functional is defined as:

$$\Phi_k[f] := ||\widehat{f}_k||_2 + \alpha||\partial_y \widehat{f}_k||_2^2 + \beta Re\langle[\partial_y, u(y)ik]\widehat{f}_k, \partial_y \widehat{f}_k\rangle_{L^2} + \gamma||[\partial_y, u(y)ik]\widehat{f}_k||_2^2,$$

where $[\cdot,\cdot]$ denotes the commutator of operators and α, β and γ are constants chosen properly. By noting that

$$[\partial_y, u(y)ik]\widehat{f}_k = \partial_y(u(y)ik\widehat{f}_k) - u(y)ik\partial_y\widehat{f}_k = u'(y)ik\widehat{f}_k,$$

the functional can be represented as follows

$$\begin{aligned}\Phi_k[f(t)] =& ||\widehat{f}_k(t)||_2^2 + ||\sqrt{\alpha}\partial_y\widehat{f}_k(t)||_2^2 + 2kRe\langle i\beta u'\widehat{f}_k(t), \partial_y\widehat{f}_k(t)\rangle \\ &+ |k|^2||\sqrt{\gamma}u'\widehat{f}_k(t)||_2^2; \end{aligned}\tag{4.3.2}$$

$$\begin{aligned}\Phi[f(t)] =& \sum_{k\neq 0}\Phi_k[f(t)] = ||f_{\neq}(t)||_2^2 + ||\sqrt{\alpha}\partial_y f_{\neq}(t)||_2^2 + 2\langle\beta u'\partial_x f_{\neq}(t), \partial_y f_{\neq}(t)\rangle \\ &+ ||\sqrt{\gamma}u'|\partial_x|f_{\neq}(t)||_2^2. \end{aligned}\tag{4.3.3}$$

Here α, β, and γ are A, k-dependent constants

$$\alpha(A,k) = \epsilon_\alpha A^{-2/3}\,|k|^{-2/3} \tag{4.3.4a}$$

$$\beta(A,k) = \epsilon_\beta A^{-1/3}\,|k|^{-4/3} \tag{4.3.4b}$$

$$\gamma(A,k) = \epsilon_\gamma\,|k|^{-2}\,, \tag{4.3.4c}$$

where ϵ_α, ϵ_β, and ϵ_γ are small constants depending only on u[1]. Since we are concerned with strictly monotone shear flows instead of nondegenerate shear flows, we employ

[1]The constants ϵ_α, ϵ_β and ϵ_γ are chosen so that all the potentially positive terms in the time derivative of Φ_k are absorbed by the negative terms in the $\frac{d}{dt}\Phi_k$. Since the explicit form is too complicated, we refer the interested to the paper [8].

slightly different multipliers α, β, γ from the ones in the paper [13]. Notice that in [8] for treating general situations one must also take α, β, and γ to be y-dependent, however, as suggested by [6], this is not necessary to treat strictly monotone shear flows with $y \in \mathbb{R}$. The parameters ϵ_α, ϵ_β, and ϵ_γ are tuned such that,

$$\Phi_k[f] \approx \left\|\widehat{f}_k\right\|_2^2 + \left\|\sqrt{\alpha}\partial_y \widehat{f}_k\right\|_2^2 + |k|^2 \left\|\sqrt{\gamma}u' \widehat{f}_k\right\|_2^2, \tag{4.3.5}$$

and hence

$$\Phi_k[f] \approx \left\|\widehat{f}_k\right\|_2^2 + |k|^{-2/3} A^{-2/3} \left\|\partial_y \widehat{f}_k\right\|_2^2. \tag{4.3.6}$$

As a result, $\Phi_k[f(t)]$ is equivalent to the H^1 norm of f_k but with constants that depend on A and k. The primary step in the results of [8] is that for $u(y)$ satisfying the hypotheses in Theorem 5, then for the passive scalar equation (4.3.1), the norm $\Phi_k[f(t)]$ satisfies the following differential inequality for some small constant $\tilde{\epsilon}$ independent of k, A (but depending on u):

$$\frac{d}{dt}\Phi_k[f(t)] \leq -\tilde{\epsilon}\frac{|k|^{2/3}}{A^{1/3}}\Phi_k[f(t)].$$

Note that the decay rate of the functional $\Phi_k[f]$ $\left(= \frac{\tilde{\epsilon}|k|^{2/3}}{A^{1/3}}\right)$ is much larger than the classical heat decay rate $\left(= \frac{1}{A}\right)$ for the passive scalar equation (4.3.1) when A is chosen big. This is the enhanced dissipation effect of the shear flow.

Recall the estimate of the time evolution of $\Phi_k[f(t)]$ in [8].

Proposition 11. *([8]) Consider the solution to the passive scalar equation* (4.3.1)*.*

For $\tilde{\epsilon}$ sufficiently small depending only on u, there holds,

$$
\begin{aligned}
\frac{d}{dt}\Phi_k[f(t)] \leq &-\frac{\tilde{\epsilon}}{2}\frac{|k|^{2/3}}{A^{1/3}}||\widehat{f}_k||_2^2-\frac{\tilde{\epsilon}}{2}\frac{|k|^{2/3}}{A^{1/3}}||\sqrt{\alpha}\partial_y\widehat{f}_k||_2^2-\frac{\tilde{\epsilon}}{2}\frac{|k|^{8/3}}{A^{1/3}}||\sqrt{\gamma}u'\widehat{f}_k||_2^2-\frac{1}{4A}||\partial_y\widehat{f}_k||_2^2\\
&-\frac{1}{2}|k|^2||\sqrt{\beta}u'\widehat{f}_k||_2^2-\frac{1}{2A}|k|^2||\widehat{f}_k||_2^2-\frac{1}{4A}||\sqrt{\alpha}\partial_{yy}\widehat{f}_k||_2^2\\
&-\frac{1}{4A}|k|^4||\sqrt{\gamma}u'\widehat{f}_k||_2^2-\frac{1}{4A}|k|^2||\sqrt{\gamma}u'\partial_y\widehat{f}_k||_2^2\\
=:&\mathcal{N}_k[f]. \qquad (4.3.7)
\end{aligned}
$$

Remark 14. *The notation "$\mathcal{N}$" stands for "negative terms".*

Remark 15. *In Theorem 2.1 of the paper [8], it is proved that*

$$\frac{d}{dt}\Phi_k[f(t)] \leq -\tilde{\epsilon}\widetilde{\lambda}_{A^{-1},k}\Phi_k[f(t)], \qquad (4.3.8)$$

where $\widetilde{\lambda}_{A^{-1},k} = |k|^{2/3}A^{-1/3}$ for strictly monotone shear flows. By the equivalence relation (4.3.6), we obtain the first three negative terms in the time evolution estimate (4.3.7). The other negative terms are the remnant of the negative terms in the time derivative of $\Phi_k[f]$. We refer the interested reader to the Lemma 2.2 in the paper [8] for further calculation details.

The functional we construct to exploit the enhanced dissipation effect in the equation (4.1.1) is the following:

Definition 5. *Define the functional $\mathcal{F}$ as*

$$\mathcal{F}_k :=\Phi_k[n_{\neq}]+\Phi_k[\partial_y c_{\neq}]+\Phi_k[\partial_x c_{\neq}]+A|k|\Phi_k[c_{\neq}]; \qquad (4.3.9)$$

$$\mathcal{F} :=\sum_{k\neq 0}\Phi_k[n_{\neq}]+\sum_{k\neq 0}\Phi_k[\partial_y c_{\neq}]+\sum_{k\neq 0}\Phi_k[\partial_x c_{\neq}]+\sum_{k\neq 0}A|k|\Phi_k[c_{\neq}]=\sum_{k\neq 0}\mathcal{F}_k. \qquad (4.3.10)$$

The goal in this subsection is to show that

Theorem 6. *Assume the hypothesis of Proposition 10. There exists a constant $\eta > 0$ depending only on u such that the following time decay estimate holds if A is chosen large enough*

$$\frac{d}{dt}\mathcal{F} \leq -\frac{\eta}{A^{1/3}}\mathcal{F}. \tag{4.3.11}$$

The theorem implies the conclusion (4.2.8b).

Proof of the conclusion (4.2.8b). Combining Theorem 6 and the equivalence (4.3.6) yields the conclusion (4.2.8b). First by solving the differential inequality (4.3.11), we have that

$$\mathcal{F}(t) \leq \mathcal{F}(0)e^{-\frac{\eta t}{A^{1/3}}}. \tag{4.3.12}$$

Thanks to the assumption on the initial chemical gradient

$$||\nabla(c_{in})_{\neq}||_{H^1} \leq C_{in}A^{-q}, \quad q > 1/2,$$

the initial value $F(0)$ is bounded

$$\begin{aligned}\mathcal{F}(0) &\leq C(\epsilon_\alpha, \epsilon_\beta, \epsilon_\gamma, u, C_{in})\bigg(||(n_{in})_{\neq}||_{H^1}^2 + ||(\nabla c_{in})_{\neq}||_{H^1}^2(1+A)\bigg)\\ &\leq C(\epsilon_\alpha, \epsilon_\beta, \epsilon_\gamma, u, C_{in})(||(n_{in})_{\neq}||_{H^1}^2 + 1).\end{aligned}$$

Here we can choose the C_{ED} in (4.2.7b) to be much larger than the constant appeared in the estimate and obtain

$$\mathcal{F}(0) \leq 2C_{ED}(\epsilon_\alpha, \epsilon_\beta, \epsilon_\gamma, u, C_{in})(||(n_{in})_{\neq}||_{H^1}^2 + 1). \tag{4.3.13}$$

The equivalence relation (4.3.6) yields

$$||n_{\neq}(t)||_2^2 + ||\nabla c_{\neq}(t)||_2^2 \leq \mathcal{F}(t).$$

Combining this with the estimates (4.3.12) and (4.3.13), we obtain (4.2.8b). $\square$

In order to show the idea behind the construction of the functional $\mathcal{F}$, we first list all the related equations here:

$$\partial_t \widehat{n}_k = \frac{1}{A}\partial_{yy}\widehat{n}_k - \frac{|k|^2}{A}\widehat{n}_k - u(y)ik\widehat{n}_k - L_k - NL_k; \tag{4.3.14}$$

$$\partial_t\partial_y \widehat{c}_k = \frac{1}{A}\partial_{yy}\partial_y\widehat{c}_k - \frac{|k|^2}{A}\partial_y\widehat{c}_k - u(y)ik\partial_y\widehat{c}_k - u'(y)ik\widehat{c}_k + \frac{1}{A}\partial_y\widehat{n}_k - \frac{1}{A}\partial_{y_1}\widehat{c}_k; \tag{4.3.15}$$

$$\partial_t \widehat{c}_k = \frac{1}{A}\partial_{yy}\widehat{c}_k - \frac{|k|^2}{A}\widehat{c}_k - u(y)ik\widehat{c}_k + \frac{1}{A}\widehat{n}_k - \frac{1}{A}\widehat{c}_k, \tag{4.3.16}$$

where L_k, NL_k are defined as in (4.2.6) and (4.2.5). Our primary goal is to obtain the L^2 enhanced dissipation estimate of $n_{\neq}$. However, we are not able to close the estimate on $d\Phi_k[n_{\neq}]/dt$ without further information about the chemical gradient $\partial_{y_1}c_{\neq}$. Specifically speaking, the terms in L_k, NL_k involving $\partial_y(\partial_y\widehat{c}_{\neq}\widehat{n}_{0,\neq})$ cannot be absorbed by the negative terms in $d\Phi_k[n_{\neq}]/dt$. Therefore, in the first step, we add $\Phi_k[\nabla c_{\neq}]$ in the functional $\mathcal{F}$ to make use of the extra negative terms in $d\Phi[\nabla c_{\neq}]/dt$. The drawback is that it introduces destabilizing effect of the strong shear flow into the functional since problematic terms involving $-u'(y)ik\widehat{c}_k$ are created. These terms will typically involve large powers of A and $|k|$. In the second step, we add the term $A|k|\Phi_k[c_{\neq}]$ in $\mathcal{F}$ to compensate for this destabilizing effect of shear flow. Finally, we show that the negative terms in $d\Phi_k[n_{\neq}]/dt$ absorb all terms involving $n_{\neq}$ in $A|k|\Phi_k[c_{\neq}]$. By completing this loop, we have shown that all the terms are absorbed by the negative terms in the time derivative of $\mathcal{F}$ and the exponential decay (4.3.11) follows.

Proposition 12. *For $\tilde{\epsilon}$ sufficiently small depending only on u, there holds,*

$$\begin{aligned}
\frac{d}{dt}\Phi_k[n_{\neq}(t)] \leq & \mathcal{N}_k[n_{\neq}] + \Big\{ 2Re\langle -L_k, \widehat{n}_k\rangle - 2Re\langle \alpha\partial_{yy}\widehat{n}_k, -L_k\rangle - 2kRe[\langle i\beta u' L_k, \partial_y \widehat{n}_k\rangle \\
& + \langle i\beta u'\widehat{n}_k, \partial_y L_k\rangle] + 2|k|^2 Re\langle \gamma(u')^2\widehat{n}_k, -L_k\rangle \Big\} \\
& + \Big\{ -2Re\langle NL_k, \widehat{n}_k\rangle + 2Re\langle \alpha\partial_{yy}\widehat{n}_k, NL_k\rangle - 2kRe[\langle i\beta u' NL_k, \partial_y \widehat{n}_k\rangle \\
& + \langle i\beta u'\widehat{n}_k, \partial_y NL_k\rangle] - 2|k|^2 Re\langle \gamma(u')^2\widehat{n}_k, NL_k\rangle \Big\} \\
=: & \mathcal{N}_{n,k} + \{L_k^1 + L_k^\alpha + L_k^\beta + L_k^\gamma\} + \{NL_k^1 + NL_k^\alpha + NL_k^\beta + NL_k^\gamma\}.
\end{aligned} \tag{4.3.17}$$

Recall that $\mathcal{N}_k$ is defined in (4.3.7) and L_k, NL_k are defined in (4.2.5,4.2.6). The time derivative of $\Phi_k[\partial_y c_{\neq}], \Phi_k[\partial_x c_{\neq}]$ are bounded,

$$\begin{aligned}
& \frac{d}{dt}\Phi_k[\partial_y c_{\neq}(t)] \\
\leq & \mathcal{N}_k[\partial_y c_{\neq}] + \Big\{ 2Re\left\langle \frac{\partial_y \widehat{n}_k}{A}, \partial_y \widehat{c}_k \right\rangle - 2Re\left\langle \alpha\partial_{yy}\partial_y \widehat{c}_k, \frac{\partial_y \widehat{n}_k}{A}\right\rangle + 2kRe\Big[\left\langle i\beta u' \frac{\partial_y \widehat{n}_k}{A}, \partial_y\partial_y \widehat{c}_k\right\rangle \\
& + \left\langle i\beta u'\partial_y \widehat{c}_k, \frac{\partial_y\partial_y \widehat{n}_k}{A}\right\rangle\Big] + 2|k|^2 Re\left\langle \gamma(u')^2\partial_y \widehat{c}_k, \frac{\partial_y \widehat{n}_k}{A}\right\rangle \Big\} \\
& + \Big\{ -2Re\,\langle u' ik\widehat{c}_k, \partial_y \widehat{c}_k\rangle + 2Re\,\langle \alpha\partial_{yy}\partial_y \widehat{c}_k, u' ik\widehat{c}_k\rangle - 2kRe\Big[\langle i\beta(u')^2 ik\widehat{c}_k, \partial_y\partial_y \widehat{c}_k\rangle \\
& + \langle i\beta u'\partial_y \widehat{c}_k, \partial_y(u' ik\widehat{c}_k)\rangle\Big] - 2|k|^2 Re\langle \gamma(u')^2\partial_y \widehat{c}_k, u' ik\widehat{c}_k\rangle \Big\} - 4kRe\left\langle i\beta u' \frac{\partial_{y_1}\widehat{c}_k}{A}, \partial_{yy}\widehat{c}_k\right\rangle \\
=: & \mathcal{N}_{\partial_y c,k} + \{T^1_{\partial_y c,1;k} + T^\alpha_{\partial_y c,1;k} + T^\beta_{\partial_y c,1;k} + T^\gamma_{\partial_y c,1;k}\} \\
& + \{T^1_{\partial_y c,2;k} + T^\alpha_{\partial_y c,2;k} + T^\beta_{\partial_y c,2;k} + T^\gamma_{\partial_y c,2;k}\} + T^\beta_{\partial_{y_1} c,3;k},
\end{aligned} \tag{4.3.18}$$

$$
\begin{aligned}
&\frac{d}{dt}\Phi_k[\partial_x c_{\neq}(t)] \\
\leq&\mathcal{N}_k[\partial_x c_{\neq}] + \left\{2Re\left\langle \frac{ik\widehat{n}_k}{A}, ik\widehat{c}_k\right\rangle - 2Re\left\langle \alpha\partial_{yy}ik\widehat{c}_k, \frac{ik\widehat{n}_k}{A}\right\rangle + 2kRe\left[\left\langle i\beta u'\frac{ik\widehat{n}_k}{A}, \partial_y ik\widehat{c}_k\right\rangle\right.\right. \\
&\left.\left.+\left\langle i\beta u' ik\widehat{c}_k, \frac{ik\partial_y\widehat{n}_k}{A}\right\rangle\right] + 2|k|^2 Re\left\langle \gamma(u')^2 ik\widehat{c}_k, \frac{ik\widehat{n}_k}{A}\right\rangle\right\} - 4kRe\left\langle i\beta u'\frac{ik\widehat{c}_k}{A}, ik\partial_{y_1}\widehat{c}_k\right\rangle \\
=:&\mathcal{N}_{\partial_x c,k} + \{T^1_{\partial_x c,1;k} + T^\alpha_{\partial_x c,1;k} + T^\beta_{\partial_x c,1;k} + T^\gamma_{\partial_x c,1;k}\} + T^\beta_{\partial_x c,2;k}.
\end{aligned}
\tag{4.3.19}
$$

The time derivative of $A|k|\Phi_k[c_{\neq}]$ is bounded,

$$
\begin{aligned}
&\frac{d}{dt}A|k|\Phi_k[c_{\neq}(t)] \\
\leq&A|k|\mathcal{N}_k[c_{\neq}] + A|k|\left\{2Re\left\langle \frac{\widehat{n}_k}{A}, \widehat{c}_k\right\rangle - 2Re\left\langle \alpha\partial_{yy}\widehat{c}_k, \frac{\widehat{n}_k}{A}\right\rangle + 2kRe\left[\left\langle i\beta u'\frac{\widehat{n}_k}{A}, \partial_y\widehat{c}_k\right\rangle\right.\right. \\
&\left.\left.+\left\langle i\beta u'\widehat{c}_k, \frac{\partial_y\widehat{n}_k}{A}\right\rangle\right] + 2|k|^2 Re\left\langle \gamma(u')^2\widehat{c}_k, \frac{\widehat{n}_k}{A}\right\rangle\right\} - A|k|4kRe\left\langle i\beta u'\frac{\widehat{c}_k}{A}, \partial_{y_1}\widehat{c}_k\right\rangle \\
=:&A|k|\mathcal{N}_{c,k} + A|k|\{T^1_{c,1;k} + T^\alpha_{c,1;k} + T^\beta_{c,1;k} + T^\gamma_{c,1;k}\} + A|k|T^\beta_{c,2;k}.
\end{aligned}
\tag{4.3.20}
$$

Proof. Applying the equations (4.3.7), (4.3.14), (4.3.15), (4.3.16) and integration by parts, the estimates follow. □

Sketch of the proof of Theorem 6. The main idea of the proof of Theorem 6 is to estimate all the terms $L_k^{(\cdot)}$, $NL_k^{(\cdot)}$, $T^{(\cdot)}_{\partial_y c,\cdot;k}$, $T^{(\cdot)}_{\partial_x c,\cdot;k}$ and $A|k|T^{(\cdot)}_{c,\cdot;k}$ in (4.3.17), (4.3.18), (4.3.19) and (4.3.20), and show that the sum of all these terms are smaller than $-\frac{1}{4}\big(\mathcal{N}_{n,k} + \mathcal{N}_{\partial_y c,k} + \mathcal{N}_{\partial_x c,k} + A|k|\mathcal{N}_{c,k}\big)$. Once we show this estimate, we end up with

$$
\frac{d}{dt}\mathcal{F} \leq \frac{3}{4}\sum_{k\neq 0}\big(\mathcal{N}_{n,k} + \mathcal{N}_{\partial_y c,k} + \mathcal{N}_{\partial_x c,k} + A|k|\mathcal{N}_{c,k}\big) \leq -\frac{3}{4}\frac{\widetilde{\epsilon}}{A^{1/3}}\mathcal{F}. \tag{4.3.21}
$$

This is the same as (4.3.11). □

The remaining part of this section is organized as follows: in section 3.2, we estimate all the terms in (4.3.20); in section 3.3, we estimate (4.3.18) and (4.3.19); in section 3.4, we estimate (4.3.17).

Time evolution estimates: $A|k|\frac{d}{dt}\Phi_k[c_{\neq}]$

In this subsection, we estimate terms in (4.3.20). First the $A|k|T^1_{c,1;k}$ term in (4.3.20) can be estimated using Hölder inequality and Young's inequality:

$$T^1_{c,1;k} = 2Re\left\langle \frac{\widehat{n}_k}{A}, \widehat{c}_k \right\rangle \leq \frac{B/\tilde{\epsilon}}{A^{5/3}|k|^{2/3}}||\widehat{n}_k||_2^2 + \frac{|k|^{2/3}}{A^{1/3}B}\tilde{\epsilon}||\widehat{c}_k||_2^2. \tag{4.3.22}$$

We show that $A|k|T^1_{c,1;k}$ is consistent with (4.3.11) given that B, then A, are chosen large. For the second term in (4.3.22), it can be absorbed by the negative term $A|k|\mathcal{N}_k[c_{\neq}]$ in (4.3.20) given B chosen large enough. For the first term, we can use the negative term $-\frac{\tilde{\epsilon}}{2}\frac{|k|^{2/3}}{A^{1/3}}||\widehat{n}_k||_2^2$ in (4.3.17) to absorb it given A chosen large enough compared to B and $1/\tilde{\epsilon}$, i.e.

$$A|k|\frac{B/\tilde{\epsilon}}{A^{5/3}|k|^{2/3}}||\widehat{n}_k||_2^2 - \frac{1}{2}\tilde{\epsilon}\frac{|k|^{2/3}}{A^{1/3}}||\widehat{n}_k||_2^2 \leq -\frac{7}{16}\tilde{\epsilon}\frac{|k|^{2/3}}{A^{1/3}}||\widehat{n}_k||_2^2.$$

The second term $A|k|T^{\alpha}_{c,1;k}$ in (4.3.20) is estimated using Hölder inequality, Young's inequality and the definition of α (4.3.4):

$$T^{\alpha}_{c,1;k} \leq \frac{1}{AB}||\sqrt{\alpha}\partial_{yy}\widehat{c}_k||_2^2 + \frac{B}{A^{5/3}|k|^{2/3}}||\widehat{n}_k||_2^2,$$

which by (4.3.7), (4.3.17) and (4.3.20) is consistent with (4.3.11) given A large. For the $A|k|T^{\beta}_{c,1;k}$ term in (4.3.20), we estimate it using the fact that $||u''||_\infty \leq C$, the definition of β (4.3.4), Hölder inequality and Young's inequality as follows

$$\begin{aligned} T^{\beta}_{c,1;k} =& 4kRe\left\langle i\beta u'\frac{\widehat{n}_k}{A}, \partial_y\widehat{c}_k\right\rangle - 2kRe\left\langle i\beta u''\widehat{c}_k, \frac{\widehat{n}_k}{A}\right\rangle \\ \lesssim& \frac{B|k|^{2/3}}{A^{4/3}}||\sqrt{\beta}u'\widehat{n}_k||_2^2 + \frac{||\partial_y\widehat{c}_k||_2^2}{AB} + \frac{||\widehat{c}_k||_2^2}{BA^{1/3}} + \frac{B||\widehat{n}_k||_2^2}{A^{7/3}|k|^{2/3}}, \end{aligned}$$

which by (4.3.7), (4.3.17) and (4.3.20) is consistent with (4.3.11) given B, then A large. Similarly, the $A|k|T^{\gamma}_{c,1;k}$ term in (4.3.20) can be estimated using Hölder inequality and Young's inequality

$$T^{\gamma}_{c,1;k} = 2|k|^2Re\left\langle \gamma(u')^2\widehat{c}_k, \frac{\widehat{n}_k}{A}\right\rangle \leq \frac{|k|^{8/3}}{A^{1/3}B}||\sqrt{\gamma}u'\widehat{c}_k||_2^2 + \frac{B|k|^{4/3}}{A^{5/3}}||\sqrt{\gamma}u'\widehat{n}_k||_2^2,$$

which is consistent with (4.3.11) given that B, then A, are chosen large enough thanks to (4.3.7), (4.3.17) and (4.3.20). The $A|k|T^{\beta}_{c,2;k}$ term in (4.3.20) can be estimated using Hölder inequality, Young's inequality and the definition of β (4.3.4) as follows

$$A|k|T^{\beta}_{c,2;k} \leq A|k|\left(\frac{8|k|^2||\sqrt{\beta}u'\widehat{c}_k||_2^2}{A} + \frac{||\partial_y\widehat{c}_k||_2^2}{2A^{4/3}}\right), \tag{4.3.23}$$

which can be absorbed by $A|k|\mathcal{N}_k[c_{\neq}]$ in (4.3.20) given that A is chosen large enough. This completes the estimation of all the terms in (4.3.20).

Time evolution estimates: $\frac{d}{dt}\Phi[\nabla c_{\neq}]$

In this subsection, we estimate the time evolution of $\Phi[\nabla c_{\neq}]$ (4.3.18) and (4.3.19). We start by estimating the terms in $\frac{d}{dt}\Phi[\partial_y c_{\neq}]$ since they involve destabilizing effect of strong shear flow. First we estimate the term $T^1_{\partial_y c,2;k}$ in (4.3.18) using the definition of β (4.3.4), Hölder inequality and Young's inequality as follows:

$$T^1_{\partial_y c,2;k} \lesssim \frac{|k|^{2/3}||\partial_y\widehat{c}_k||_2^2}{BA^{1/3}} + B\left(A^{2/3}|k|^{2/3}\right)|k|^2||\sqrt{\beta}u'\widehat{c}_k||_2^2.$$

Now we see that the first term is absorbed by the negative terms in (4.3.18) given B chosen large enough, and the second term can be absorbed by the term $-A|k|^3||\sqrt{\beta}u'\widehat{c}_k||_2^2$ in (4.3.20) given A chosen large enough. Now we see that this term is consistent with (4.3.11). Next, combining the definition of α, β (4.3.4), Hölder inequality and Young's inequality, the α term in $T^{\alpha}_{\partial_{y_1}c,2;k}$ can be estimated as follows:

$$T^{\alpha}_{\partial_{y_1}c,2;k} \lesssim \frac{1}{AB}||\sqrt{\alpha}\partial_{yyy}\widehat{c}_k||_2^2 + B\left(A^{2/3}|k|^{2/3}\right)|k|^2||\sqrt{\beta}u'\widehat{c}_k||_2^2,$$

which is consistent with (4.3.11) given that B, then A, are chosen large. For the first β term in $T^{\beta}_{\nabla c,2;k}$, combining the definition of β (4.3.4), the fact that $||u'||_{W^{1,\infty}} \leq C$, integration by parts, Hölder inequality and Young's inequality yields

$$\begin{aligned}2kRe\langle i\beta u'(-u'ik\widehat{c}_k), \partial_{yy}\widehat{c}_k\rangle &= -2|k|^2Re\langle\beta 2u'u''\widehat{c}_k, \partial_y\widehat{c}_k\rangle - 2|k|^2Re\langle\beta u'^2\partial_y\widehat{c}_k, \partial_y\widehat{c}_k\rangle\\ &\leq 2|k|^2||\sqrt{\beta}u''\widehat{c}_k||_2^2 + 2|k|^2||\sqrt{\beta}u'\partial_y\widehat{c}_k||_2^2 - 2|k|^2||\sqrt{\beta}u'\partial_y\widehat{c}_k||_2^2 \lesssim \frac{|k|^{2/3}}{A^{1/3}}||\widehat{c}_k||_2^2,\end{aligned}$$

which can be absorbed by the negative term $A|k|\mathcal{N}_k[c_{\neq}]$ in (4.3.20) given A large enough. By applying integration by parts, we see that the second β term in $T^{\beta}_{\partial_{y_1}c,2;k}$ is equivalent to the first one up to the following term, which can be estimated using the definition of β (4.3.4), $||u''||_\infty \leq C$, Hölder inequality and Young's inequality

$$2kRe\langle i\beta u''\partial_y\widehat{c}_k, u'ik\widehat{c}_k\rangle \lesssim \frac{|k|^{2/3}||\partial_y\widehat{c}_k||_2^2}{A^{1/3}B} + B|k|^2||\sqrt{\beta}u'\widehat{c}_k||_2^2.$$

Since the first terms can be absorbed by $\mathcal{N}_k[\partial_{y_1}c_{\neq}]$ and the second term can be absorbed by $A|k|\mathcal{N}_k[c_{\neq}]$, this is consistent with (4.3.11) given that B, then A, are chosen large. The $T^{\gamma}_{\partial_{y_1}c,2;k}$ term in (4.3.18) can be estimated using $||u'||_\infty \leq C$, Hölder inequality and Young's inequality as follows:

$$T^{\gamma}_{\partial_{y_1}c,2;k} \lesssim \frac{|k|^{8/3}}{A^{1/3}B}||\sqrt{\gamma}u'\partial_y\widehat{c}_k||_2^2 + B\left(A^{2/3}|k|^{2/3}\right)\frac{||\sqrt{\gamma}u'\widehat{c}_k||_2^2|k|^{8/3}}{A^{1/3}}.$$

Now we see that the first term is absorbed by the negative term $\mathcal{N}_k[\partial_{y_1} c_{\neq}]$ in (4.3.18) if B is chosen large, and the second term is absorbed by $A|k|\mathcal{N}_k[c_{\neq}]$ in (4.3.20) given that A is chosen large. This finishes the estimation of the terms $T^{(\cdot)}_{\partial_{y_1} c,2;k}$ in (4.3.18).

For the terms of the form $T^{(\cdot)}_{\partial_{y_1} c,1;k}$ in (4.3.18), we will use the negative terms in (4.3.17) and (4.3.18) to absorb them. For the $T^{1}_{\partial_{y_1} c,1;k}$ in (4.3.18), we have that by Hölder inequality and Young's inequality,

$$T^{1}_{\partial_{y_1} c,1;k} \leq \frac{1}{A^{5/4}}||\partial_y \widehat{n}_k||_2^2 + \frac{1}{A^{3/4}}||\partial_y \widehat{c}_k||_2^2.$$

By choosing A large, these two terms can be absorbed by the negative terms in (4.3.17) and (4.3.18). Combining the definition of α (4.3.4), Hölder inequality and Young's inequality, the $T^{\alpha}_{\partial_{y_1} c,1;k}$ term in (4.3.18) can be estimated as follows,

$$T^{\alpha}_{\partial_{y_1} c,1;k} \leq \frac{1}{A^{4/3}|k|^{1/3}}||\sqrt{\alpha}\partial_{yyy} \widehat{c}_k||_2^2 + \frac{1}{A^{4/3}|k|^{1/3}}||\partial_{y_1} \widehat{n}_k||_2^2,$$

which is consistent with (4.3.11) for A large enough. For the first β term in $T^{\beta}_{\partial_{y_1} c,1;k}$, we can estimate it using the definition of β (4.3.4), the fact that $||u'||_\infty \leq C$, Hölder inequality and Young's inequality as follows

$$2kRe\left\langle i\beta u' \frac{\partial_y \widehat{n}_k}{A}, \partial_{yy} \widehat{c}_k \right\rangle \lesssim \frac{1}{A^{4/3}}||\partial_y \widehat{n}_k||_2^2 + \frac{1}{A^{4/3}}||\partial_{yy} \widehat{c}_k||_2^2.$$

This term is consistent with (4.3.11) given A chosen large. The second term in $T^{\beta}_{\partial_{y_1} c,1;k}$ is the same as the first one through integration by part up to a controllable term, which can be estimated using the definition of β (4.3.4), the fact that $||u''||_\infty \leq C$, Hölder inequality and Young's inequality as follows

$$-2kRe\left\langle i\beta u'' \partial_y \widehat{c}_k, \frac{\partial_y \widehat{n}_k}{A} \right\rangle \lesssim \frac{1}{A^{4/3}}||\partial_y \widehat{c}_k||_2^2 + \frac{1}{A^{4/3}}||\partial_y \widehat{n}_k||_2^2.$$

As long as A is large enough, these two terms can be absorbed by the negative terms in (4.3.17) and (4.3.18). Finally, for the γ term $T^{\gamma}_{\partial_{y_1}c,1;k}$, we estimate it using the definition of γ (4.3.4), $||u'||_{W^{1,\infty}} \leq C$, Hölder inequality and Young's inequality as follows

$$T^{\gamma}_{\partial_{y_1}c,1;k} \lesssim \frac{||\partial_{y_1}\widehat{c}_k||_2^2}{A^{2/3}} + \frac{||\partial_{y_1}\widehat{n}_k||_2^2}{A^{4/3}}.$$

This is consistent with (4.3.11) given that A is chosen large enough. The treatment of the term $T^{\beta}_{\partial_{y_1}c,3;k}$ in (4.3.18) is similar to the treatment of (4.3.23), so we omit the estimate for the sake of brevity. This concludes the estimate of the time evolution $\frac{d}{dt}\Phi_k[\partial_y c_{\neq}]$.

The estimate of the time derivative $\frac{d}{dt}\Phi_k[\partial_x c_{\neq}]$ is similar to the estimates of the terms $T^{(\cdot)}_{\partial_{y_1}c,1;k}$ and $T^{\beta}_{\partial_{y_1}c,3;k}$ in (4.3.18), hence we omit it for the sake of brevity.

Time evolution estimates: $\frac{d}{dt}\Phi[n_{\neq}]$

Estimate on the L terms in (4.3.17)

The treatment of the L terms has similar flavour to the corresponding treatment of the L terms in the parabolic-elliptic case. Unfortunately, we need to omit it for the sake of brevity.

Estimate on NL terms

The treatment of the NL terms has similar flavour to the corresponding treatment of the NL terms in the parabolic-elliptic case. Unfortunately, we need to omit it for the sake of brevity.

4.4 Nonzero mode $L_t^2\dot{H}^1_{x,y}$ estimate (4.2.8a)

The estimates in this section has similar flavour to the corresponding proof in the parabolic-elliptic case. For the sake of brevity, we omit the proof.

4.5 Zero mode estimate (4.2.8c)

The proof of the zero mode estimate has similar flavour to the corresponding proof in the parabolic-elliptic case. For the sake of brevity, we omit the proof and refer the interested readers to the paper [65].

4.6 Uniform L^∞ control (4.2.8d) and (4.2.8e)

In this section we prove the uniform L^∞ control (4.2.8d) and (4.2.8e). We separate the proof into two different time regimes, namely, the initial time $t \leq A^{1/3+\epsilon}$ and the long time $t \geq A^{1/3+\epsilon}$. Here $\epsilon > 0$ is a small constant determined by the proof. For the sake of clarity, we use $C^{in}_{n,\infty}$, $C^{in}_{\nabla c_{\neq},\infty}$ to denote bounds in the initial time and $C^{long}_{n,\infty}$, $C^{long}_{\nabla c_{\neq},\infty}$ to denote bounds in the long time. At the end of the proof, we will take the $C_{n,\infty}$ to be large compared to $C^{in}_{n,\infty}$ and $C^{long}_{n,\infty}$ and take the $C_{\nabla c_{\neq},\infty}$ large compared to $C^{in}_{\nabla c_{\neq},\infty}$ and $C^{long}_{\nabla c_{\neq},\infty}$.

Initial Time Layer Estimate

In this subsection, we would like to prove the following lemma:

Lemma 4.6.1. *Under the assumptions of Proposition 10, there exist a constant* $0 <$

$\epsilon < \frac{1}{12}$ *independent of the solution and constants* $C_{n,\infty}, C_{\nabla c_{\neq},\infty}, C_{\partial_x n,\infty}$ *depending on* C_{ED}, n_{in}, M *such that the following estimates hold on the time interval* $0 \le t \le A^{1/3+\epsilon}$ *when* A *is chosen large enough:*

$$||n(t)||_\infty \le C^{in}_{n,\infty}(n_{in}, C_{ED}, M); \tag{4.6.1a}$$

$$||\nabla c_{\neq}(t)||_\infty \le C^{in}_{\nabla c_{\neq},\infty}(n_{in}, C_{ED}, M); \tag{4.6.1b}$$

$$||\partial_x n(t)||_\infty \le C_{\partial_x n,\infty}(||n_{in}||_{H^1}), \quad \forall t \in [0, A^{1/3+\epsilon}]. \tag{4.6.1c}$$

Remark 16. *In the proof of the lemma, the destabilizing effect of shear flow has to be treated carefully because the enhanced dissipation effect of the shear flow is too weak at the initial time. We will propagate the estimates* (4.6.1) *till* $t = A^{1/3+\epsilon}$. *After this time threshold, the enhanced dissipation kicks in to stabilize the dynamics.*

Proof. We use a bootstrap argument to prove the lemma. Assume that for constants $C^{in}_{n,\infty}$, $C^{in}_{\nabla c_{\neq},\infty}$, $C_{\partial_x n,\infty}$ depending on the proof, $T_{\star\star} \in [0, A^{1/3+\epsilon}]$ is the maximal time on which the following hypothesis is satisfied:

$$||n(t)||_\infty \le 2C^{in}_{n,\infty}; \tag{4.6.2a}$$

$$||\nabla c_{\neq}(t)||_\infty \le 2C^{in}_{\nabla c_{\neq},\infty}; \tag{4.6.2b}$$

$$||\partial_x n(t)||_\infty \le 2C_{\partial_x n,\infty}, \quad \forall t \in [0, T_{\star\star}), T_{\star\star} \le \min\{A^{1/3+\epsilon}, T_\star\}. \tag{4.6.2c}$$

We will show that all the estimates (4.6.2) hold on the same time interval $[0, T_\star]$ with '1' instead of '2' if we choose A_0 large. These improvements combined with the local well-posedness of the equation (4.1.1) yield (4.6.1).

We split the proof into three steps. In the first step, we obtain the improvement to (4.6.2a) together with a suboptimal estimate of $||\nabla c_{\neq}||_p, \forall p < \infty$. Here the estimate in $||\nabla c_{\neq}||_p, \forall p < \infty$ is suboptimal in the sense that on the interval $[0, T_{\star\star})$, the estimate loses a small power of A, i.e., $||\nabla c_{\neq}||_p \lesssim A^{\delta}, \delta > 0$. In order to compensate for the loss in powers of A, we need information about the higher regularity of $n_{\neq}$. This is why we propagate another estimate (4.6.1c) in the initial time layer $[0, T_{\star\star})$. In the second step, we complete the proof of (4.6.1c). In the last step, we use the extra regularity information to get the optimal L^∞ bound of $\nabla c_{\neq}$.

First step: We prove the improvement to (4.6.2a) on $[0, T_{\star\star})$. We start with the estimate on $||\partial_x c_{\neq}||_4$. Direct energy estimate yields

$$\frac{d}{dt}||\partial_x c_{\neq}||_4^4 \leq -\frac{3}{2A}||\nabla(\partial_x c_{\neq})^2||_2^2 + \frac{6}{A}||\partial_x c_{\neq}||_4^2||n_{\neq}||_4^2 - \frac{4}{A}||\partial_x c_{\neq}||_4^4. \tag{4.6.3}$$

Integration in time yields

$$||\partial_x c_{\neq}(t)||_4 \leq \sqrt{3}\frac{\sqrt{t}}{A^{1/2}} \sup_{0\leq s\leq t} ||n_{\neq}(s)||_4 + ||\partial_x (c_{in})_{\neq}||_4. \tag{4.6.4}$$

With the equation (4.2.3b), we estimate the time evolution of the L^4 norm of $\partial_y c_{\neq}$:

$$\frac{d}{dt}||\partial_y c_{\neq}||_4^4 \leq -\frac{3}{2A}||\nabla(\partial_y c_{\neq})^2||_2^2 + \frac{6||n_{\neq}||_4^2||\partial_y c_{\neq}||_4^2}{A} + 4||\partial_y c_{\neq}||_4^3||u'\partial_x c_{\neq}||_4 - \frac{4}{A}||\partial_y c_{\neq}||_4^4.$$

As in the $\partial_x c_{\neq}$ case, we drop the negative term at the moment, and end up with the following inequality

$$\frac{d}{dt}||\partial_y c_{\neq}||_4^4 \leq \frac{6||n_{\neq}||_4^2||\partial_y c_{\neq}||_4^2}{A} + 4||\partial_y c_{\neq}||_4^3||u'\partial_x c_{\neq}||_4. \tag{4.6.5}$$

Recall from the statement of the main theorem 5 that $||\nabla(c_{in})_{\neq}||_{H^1\cap W^{1,\infty}} \leq C_{in}A^{-q}, q > 1/2$. Now the idea is to compare $||\partial_y c_{\neq}||_4$ with the solution to the following differential equation,

$$\frac{d}{dt}f^4 = 10f^3\left(||u'\partial_x c_{\neq}||_4 + \frac{||n_{\neq}||_4^2}{A} + \frac{1}{A}\right), \tag{4.6.6}$$

$$f(0) = 1 > C_{in}A^{-q} \geq ||\partial_y (c_{in})_{\neq}||_4. \tag{4.6.7}$$

and show that $||\partial_y c_{\neq}(t)||_4 \leq f(t)$ for $t \leq T_{\star\star}$. The function f is estimated using (4.6.4) and the fact $q > 1/2$ as follows:

$$\begin{aligned} f(t) \lesssim & 1 + \frac{1}{A^{1/2}} + \int_0^t ||u'\partial_x c_{\neq}(s)||_4 + \frac{||n_{\neq}(s)||_4^2}{A} ds \\ \lesssim & 1 + A^{1/2-q} + \frac{t^{3/2}}{A^{1/2}} \sup_{0\leq s\leq t} ||n_{\neq}(s)||_4 + \frac{1}{A^{1/2}} \sup_{0\leq s\leq t} ||n_{\neq}(s)||_4^2, \quad \forall t \leq A^{1/3+\epsilon}. \end{aligned} \tag{4.6.8}$$

Next we show that $||\partial_y c_{\neq}||_4 \leq f$ for $\forall t \in [0, T_{\star\star})$. Since f is strictly increasing in time, $f \geq 1$. Assume that there exists a first time $t^\star \leq T_{\star\star}$ such that $||\partial_y c_{\neq}(t^\star)||_4^4$ is equal to the function $f^4(t^\star)$. At time $t^\star$, we have $||\partial_y c_{\neq}(t^\star)||_4 = f(t^\star) \geq 1$, which yields the following relation

$$||\partial_y c_{\neq}(t^\star)||_4^3 \geq ||\partial_y c_{\neq}(t^\star)||_4^2. \tag{4.6.9}$$

Combining this with (4.6.5), (4.6.6) yields that at time $t^\star$,

$$\frac{d}{dt}||\partial_y c_{\neq}||_4^4\bigg|_{t=t^\star} \leq \left(\frac{6||n_{\neq}||_4^2}{A} + 4||u'\partial_x c_{\neq}||_4\right)||\partial_y c_{\neq}||_4^3\bigg|_{t=t^\star} < \frac{d}{dt}f^4\bigg|_{t=t^\star}. \tag{4.6.10}$$

On the other hand, $\frac{d}{dt}||\partial_y c_{\neq}||_4^4\Big|_{t=t^\star} \geq \frac{d}{dt}f^4\Big|_{t=t^\star}$ at the first break-through time $t^\star$, which is a contradiction. As a result, we have that $||\partial_y c_{\neq}(t)||_4 \leq f(t)$, $\forall t \leq T_{\star\star}$, which together with (4.6.8) yields the following estimate

$$||\partial_y c_{\neq}(t)||_4 \lesssim 1 + \frac{t^{3/2}}{A^{1/2}} \sup_{0\leq s\leq t} ||n_{\neq}(s)||_4 + \frac{1}{A^{1/2}} \sup_{0\leq s\leq t} ||n_{\neq}(s)||_4^2, \quad \forall t \leq T_{\star\star}. \tag{4.6.11}$$

Recall the Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{T}\times\mathbb{R}$,

$$||f||_4 \lesssim ||\nabla f||_2^{1/2}||f||_2^{1/2} + ||f||_2.$$

Combining this with Lemma 4.2.10, $||\nabla c_{\neq}||_4$ estimates (4.6.4)and (4.6.11), we estimate the time evolution of $||n||_4^4$ as follows

$$\begin{aligned}\frac{d}{dt}||n||_4^4 \lesssim& -\frac{3}{2A}||\nabla(n^2)||_2^2 + \frac{(||\nabla(n^2)||_2^{3/2}||n^2||_2^{1/2} + ||\nabla(n^2)||_2||n^2||_2)(||\nabla c_{\neq}||_4 + ||\partial_{y_1}c_0||_4)}{A}\\ \lesssim& \frac{||n||_4^4}{A}\bigg(1 + M^4 + ||\nabla(c_{in})_0||_4^4 + \frac{t^2}{A^2}\sup_{0\le s\le t}||n_{\neq}(s)||_4^4 + \frac{t^6}{A^2}\sup_{0\le s\le t}||n_{\neq}(s)||_4^4\\ &+ \frac{1}{A^2}\sup_{0\le s\le t}||n_{\neq}(s)||_4^8\bigg).\end{aligned}$$

Thanks to the hypothesis (4.2.7d), conservation of mass and Hölder inequality, we can take A large enough such that the above estimate can be simplified as follows:

$$\frac{d}{dt}||n||_4^4 \le \frac{C||n||_4^4}{A}(M^4 + 1 + ||\partial_{y_1}(c_{in})_0||_4^4 + A^{6\epsilon}\sup_{0\le s\le t}||n(s)||_4^4),$$

where the constant C is the implicit constant in the estimate above. Now we can compare the $||n||_4^4$ to the solution to the following differential equation:

$$\frac{d}{dt}f = \frac{2Cf}{A}(M^4 + ||\partial_{y_1}(c_{in})_0||_4^4 + 1 + A^{6\epsilon}f), \quad f(0) > \max\{1, ||n_{in}||_4^4\}.$$

The strictly increasing solution f is bounded $f \le C(n_{in})$ on the interval $[0, A^{1/3+\epsilon}]$ if ϵ is chosen small enough and A is chosen large enough compared to M, $||\partial_{y_1}(c_{in})_0||_4$ and C. Assume that there exists a first time $0 < t_\star \le A^{1/3+\epsilon}$ such that $||n(t_\star)||_4^4$ is equal to the function $f(t_\star)$. Since f is strictly increasing, at the first break-through time $t_\star$, we have $||n(t_\star)||_4 = \sup_{0\le s\le t_\star}||n(s)||_4$, which yields the following relation

$$\frac{d}{dt}||n||_4^4\bigg|_{t=t_\star} \le \frac{C||n||_4^4}{A}(M^4 + ||\partial_{y_1}(c_{in})_0||_4^4 + 1 + A^{6\epsilon}||n||_4^4)\bigg|_{t=t_\star} < \frac{d}{dt}f\bigg|_{t=t_\star}. \quad (4.6.12)$$

On the other hand, $\frac{d}{dt}||n||_4^4\Big|_{t=t_\star} \geq \frac{d}{dt}f\Big|_{t=t_\star}$ at the first break-through time $t_\star > 0$, which is a contradiction. As a result, we have that

$$||n(t)||_4 \leq C^{in}_{n,L^4}(n_{in}), \quad \forall t \in [0, T_{\star\star}). \tag{4.6.13}$$

Next we start the iteration process. Assume that $||n||_p$ is bounded, we estimate the $||n||_{2p}$ in terms of $||n||_p$. We start with estimating the $||\partial_x c_{\neq}||_{2p}^{2p}$. By calculating the time derivative, we see that

$$\begin{aligned}
&\frac{1}{2p}\frac{d}{dt}||\partial_x c_{\neq}||_{2p}^{2p} \\
&= -\frac{2p-1}{Ap^2}||\nabla(\partial_x c_{\neq})^p||_2^2 + \frac{2p-1}{Ap}||\nabla(\partial_x c_{\neq})^p||_2||(\partial_x c_{\neq})^{p-1}n_{\neq}||_2 - \frac{1}{A}||\partial_x c_{\neq}||_{2p}^{2p} \\
&\leq -\frac{2p-1}{2Ap^2}||\nabla(\partial_x c_{\neq})^p||_2^2 + \frac{p}{A}||\partial_x c_{\neq}||_{2p}^{2p-2}||n_{\neq}||_{2p}^2 - \frac{1}{A}||\partial_x c_{\neq}||_{2p}^{2p}.
\end{aligned}$$

As a result, we have that

$$\frac{d}{dt}||\partial_x c_{\neq}||_{2p}^2 \leq \frac{2p}{A}||n_{\neq}||_{2p}^2, \tag{4.6.14}$$

which yields

$$||\partial_x c_{\neq}(t)||_{2p} \lesssim \sqrt{p}\sup_{0\leq s\leq t}||n_{\neq}(s)||_{2p}\frac{t^{1/2}}{A^{1/2}} + ||\partial_x(c_{in})_{\neq}||_{2p}, \quad \forall t \in [0, A^{1/3+\epsilon}]. \tag{4.6.15}$$

Next we estimate the time evolution of $||\partial_y c_{\neq}||_{2p}^{2p}$,

$$\frac{d}{dt}||\partial_y c_{\neq}||_{2p}^{2p} \leq 2p||u'\partial_x c_{\neq}||_{2p}||\partial_y c_{\neq}||_{2p}^{2p-1} + \frac{2p^2}{A}||\partial_y c_{\neq}||_{2p}^{2p-2}||n_{\neq}||_{2p}^2 - \frac{2p}{A}||\partial_y c_{\neq}||_{2p}^{2p}. \tag{4.6.16}$$

By comparing the solution with the following strictly increasing function f

$$\frac{d}{dt}f^{2p} = 4pf^{2p-1}\left(||u'\partial_x c_{\neq}||_{2p} + p\frac{||n_{\neq}||_{2p}^2}{A} + \frac{1}{A}\right), \quad f(0) = 1 > C_{in}A^{-q} \geq ||\partial_y(c_{in})_{\neq}||_{2p}, \tag{4.6.17}$$

and applying a similar argument to prove (4.6.13), we have that

$$||\partial_y c_{\neq}(t)||_{2p} \leq f(t) \lesssim 1 + \frac{t^{3/2}}{A^{1/2}}\sqrt{p} \sup_{0\leq s\leq t}||n_{\neq}(s)||_{2p} \\ + A^{-q+1/3+\epsilon} + p\frac{\sup_{0\leq s\leq t}||n_{\neq}(s)||_{2p}^2}{A^{1/2}}, \quad \forall t\in[0,T_{\star\star}]. \tag{4.6.18}$$

Next we estimate the time evolution of $||n||_{2p}^{2p}$. Applying the hypothesis, $||\nabla c_{\neq}||_{2p}$ estimates (4.6.18), (4.6.15), Lemma 4.2.10 and the Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{T}\times\mathbb{R}$

$$||f||_2 \lesssim ||\nabla f||_2^{1/2}||f||_1^{1/2} + ||f||_1,$$

we have the following estimate by picking A large

$$\begin{aligned}
&\frac{1}{2p}\frac{d}{dt}||n||_{2p}^{2p} \\
&= -\frac{2p-1}{Ap^2}||\nabla(n^p)||_2^2 + \frac{2p-1}{Ap}||\nabla(n^p)||_2||n^p\nabla c||_2 \\
&\leq -\frac{2p-1}{Ap^2}||\nabla(n^p)||_2^2 + \frac{2p-1}{Ap}||\nabla(n^p)||_2||n^p||_2^{1-1/p}||n||_\infty||\nabla c||_{2p} \\
&\leq -\frac{2p-1}{2Ap^2}||\nabla(n^p)||_2^2 + \frac{Cp^3}{A}||n^p||_1^{\frac{2p-2}{p+1}}||n||_\infty^{\frac{4p}{p+1}}||\nabla c||_{2p}^{\frac{4p}{p+1}} + \frac{Cp^2}{A}||n^p||_1^{\frac{2p-2}{p}}||n||_\infty^2||\nabla c||_{2p}^2 \\
&\leq \frac{Cp^7}{A}(||n^p||_1^{\frac{2p-2}{p+1}} + ||n^p||_1^{2-\frac{2}{p}})C_{2,\infty}^4\left(C(M,C_{n_0,L^2},\partial_y(c_{in})_0) + A^{3\epsilon/2}\sup_{0\leq s\leq t}||n_{\neq}(s)||_{2p}\right)^4,
\end{aligned}$$

where the constant C is a universal constant depending on the constant in the Gagliardo-Nirenberg-Sobolev inequality. Time integrating on both side of the estimate and applying the hypothesis (4.2.7d), conservation of mass and Hölder inequality, we have

$$\begin{aligned}
&\sup_{0\leq s\leq T_{\star\star}}||n(s)||_{2p}^{2p} \\
&\leq \frac{p^8}{A^{2/3-7\epsilon}}C(C_{2,\infty},\partial_y(c_{in})_0)\left(\sup_{0\leq s\leq T_{\star\star}}||n(s)||_p^{2p(\frac{p-1}{p+1})} + \sup_{0\leq s\leq T_{\star\star}}||n(s)||_p^{2(p-1)}\right) + ||n_{in}||_{2p}^{2p}.
\end{aligned} \tag{4.6.19}$$

Finally, we use the (4.6.19) together with (4.6.13) to prove the $||n||_{L^\infty(0,T_{\star\star};L^\infty)} \leq C^{in}_{n,\infty}$. Note that if for $\forall j \in \mathbb{N}$, $\sup_{0\leq s<T_{\star\star}} ||n(s)||_{2^j} \leq 1$, we have that $\sup_{0\leq s<T_{\star\star}} ||n||_\infty \leq 1$, and the result follows. Therefore, we define $4 < p_\star = 2^{j_\star} \in 2^{\mathbb{Z}}$ to be the first integer such that $\sup_{0\leq s<T_{\star\star}} ||n||_{p_\star} \geq 1$. Note that for $p = p_\star/2$,

$$||n||_{L^\infty_t(0,T_{\star\star};L^{p_\star/2}_{x,y})} \leq \max\{C^{in}_{n,L^4}, 1\}. \tag{4.6.20}$$

In the following argument, we will only care about $p > p_\star$ since we want to find the limit of $||n||_{L^\infty_t(0,T_{\star\star};L^p_{x,y})}$ as $p \to \infty$.

By the Hölder's inequality,

$$1 \leq \sup_{0\leq s\leq T_{\star\star}} ||n(s)||_{p_*} \leq ||n(t)||_1^\theta \sup_{0\leq s\leq T_{\star\star}} ||n(s)||_p^{1-\theta}, \quad \forall p > p_\star.$$

Combining this estimate with the conservation of mass, we can get a lower bound for $\sup_{0\leq s\leq T_{\star\star}} ||n(s)||_p$

$$\sup_{0\leq s\leq T_{\star\star}} ||n(s)||_p \geq (1+M)^{-\frac{\theta}{1-\theta}} \geq (1+M)^{-2/5}, \quad \forall p = 2^j \in 2^{\mathbb{N}}, j > j_\star > 2. \tag{4.6.21}$$

Combining this with (4.6.19), we have that

$$\sup_{0\leq s\leq T_{\star\star}} ||n(s)||_{2p}^{2p} \leq \frac{p^8}{A^{2/3-7\epsilon}} C(C_{2,\infty}) \sup_{0\leq s\leq T_{\star\star}} ||n(s)||_p^{2p} + ||n_{in}||_{2p}^{2p}. \tag{4.6.22}$$

Now we can pick the A big such that

$$\sup_{0\leq s\leq T_{\star\star}} ||n(s)||_{2p}^{2p} \leq p^8 \sup_{0\leq s\leq T_{\star\star}} ||n(s)||_p^{2p} + ||n_{in}||_{2p}^{2p}, \quad \forall p = 2^j \geq p_\star, j \in \mathbb{N}. \tag{4.6.23}$$

Now by the L^4 bound of n (4.6.13), the $L^{p_\star/2}$ bound of n (4.6.20) and the standard Moser-Alikakos iteration ([1]),we have that

$$\sup_{0\leq s\leq T_{\star\star}} ||n(s)||_\infty \leq C^{in}_{n,\infty}(n_{in}). \tag{4.6.24}$$

Second step: We prove the improvement to (4.6.2c). First we estimate the time evolution of $||\partial_{xx}c_{\neq}||_{2p}^{2p}$:

$$\frac{1}{2p}\frac{d}{dt}||\partial_{xx}c_{\neq}||_{2p}^{2p} \leq -\frac{2p-1}{2Ap^2}||\nabla(\partial_{xx}c_{\neq})^p||_2^2 + \frac{p}{A}||\partial_x n||_{2p}^2||\partial_{xx}c_{\neq}||_{2p}^{2p-2} - \frac{1}{A}||\partial_{xx}c_{\neq}||_{2p}^{2p}.$$

Here we use the fact that $\partial_x n = \partial_x n_{\neq}$. As a result, we see that

$$||\partial_{xx}c_{\neq}(t)||_{2p} \lesssim \sqrt{\frac{2pt}{A}}\sup_{0\leq s\leq t}||\partial_x n(s)||_{2p} + ||\partial_x^2(c_{in})_{\neq}||_{2p}, \quad \forall t\in[0,T_{\star\star}). \tag{4.6.25}$$

By a similar argument as in the estimate of the term $||\partial_y c_{\neq}||_{2p}$ in (4.6.18), we have that

$$||\partial_{xy}c_{\neq}(t)||_{2p} \lesssim 1 + \frac{\sqrt{p}t^{3/2}}{A^{1/2}}\sup_{0\leq s\leq t}||\partial_x n(s)||_{2p} + p\frac{\sup_{0\leq s\leq t}||\partial_x n(s)||_{2p}^2}{A^{1/2}}, \quad t\in[0,T_{\star\star}). \tag{4.6.26}$$

Now we can calculate the time evolution of $||\partial_x n||_{2p}^{2p}$:

$$\begin{aligned}\frac{1}{2p}\frac{d}{dt}||\partial_x n||_{2p}^{2p} \leq& -\frac{2p-1}{Ap^2}||\nabla(\partial_x n)^p||_2^2 + \frac{2p-1}{Ap}\bigg(||\nabla(\partial_x n)^p||_2||\partial_x n||_{2p}^{p-1}||\partial_x\nabla c_{\neq}||_{2p}||n||_\infty \\ &+||\nabla(\partial_x n)^p||_2||(\partial_x n)^p\nabla c||_2\bigg) \\ =:& -\frac{2p-1}{Ap^2}||\nabla(\partial_x n)^p||_2^2 + T_1 + T_2. \end{aligned} \tag{4.6.27}$$

In the first line, we have used the fact that $\partial_x\nabla c = \partial_x\nabla c_{\neq}$. Now we need to separate the estimate into two cases, $p=1$ and $p\neq 1$. First we discuss the $p=1$ case. The T_1 term in (4.6.27) can be estimated using the $||\nabla\partial_x c_{\neq}||_{2p}$ estimates (4.6.25) and (4.6.26) as follows:

$$\begin{aligned}T_1 \lesssim& \frac{1}{4A}||\nabla(\partial_x n)||_2^2 + \frac{1}{A}||\partial_x\nabla c_{\neq}||_2^2||n||_\infty^2 \\ \lesssim& \frac{1}{4A}||\nabla(\partial_x n)||_2^2 + \frac{1}{A}\bigg(\frac{2}{A^{2/3-\epsilon}}\sup_{0\leq s\leq t}||\partial_x n(s)||_2^2 + 1 + A^{3\epsilon}\sup_{0\leq s\leq t}||\partial_x n(s)||_2^2 \\ &+\frac{1}{A}\sup_{0\leq s\leq t}||\partial_x n(s)||_2^4\bigg)||n||_\infty^2 \end{aligned} \tag{4.6.28}$$

The T_2 in (4.6.27) can be estimated using $\nabla c_{\neq}$ L^4 estimates (4.6.4), (4.6.11), Lemma 4.2.10, Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{T}\times\mathbb{R}$ and Hölder inequality as follows:

$$\begin{aligned}
T_2 \lesssim & \frac{1}{4A}||\nabla(\partial_x n)||_2^2 + \frac{1}{A}||\partial_x n||_2^2(||\nabla c||_4^4 + ||\nabla c||_4^2) \\
\lesssim & \frac{1}{4A}||\nabla(\partial_x n)||_2^2 + \frac{B}{A}||\partial_x n||_2^2\bigg(1 + M^4 + ||\partial_{y_1}(c_{in})_0||_4^4 + \frac{t^2}{A^2}\sup_{0\le s\le t}||n(s)||_4^4 \\
& + \frac{t^6}{A^2}\sup_{0\le s\le t}||n(s)||_4^4 + \frac{1}{A^2}\sup_{0\le s\le t}||n(s)||_4^8\bigg). \qquad (4.6.29)
\end{aligned}$$

Now combining (4.6.24), (4.6.27), (4.6.28), (4.6.29), Lemma 4.2.1, and, we obtain that

$$\frac{d}{dt}||\partial_x n||_2^2 \le \frac{C(C_{2,\infty},\partial_y(c_{in})_0)}{A^{1-6\epsilon}}\bigg(1 + \sup_{0\le s\le t}||\partial_x n(s)||_2^4\bigg).$$

Now use a comparison argument similar to the one used to prove (4.6.13), we end up with the following estimate given A chosen large enough

$$||\partial_x n(t)||_2 \le C_{\partial_x n, L^2}(n_{in}), \quad \forall t\in[0,T_{\star\star}). \qquad (4.6.30)$$

This finishes the treatment of the case $p=1$.

For the $p\neq 1$ case, there exists a large B such that the T_1 term in (4.6.27) can be estimated as follows:

$$\begin{aligned}
T_1 \le & \frac{2p-1}{BAp^2}||\nabla(\partial_x n)^p||_2^2 + \frac{BCp^3}{A}||(\partial_x n)^p||_1^{2(1-\frac{2}{p+1})}||\nabla\partial_x c_{\neq}||_{2p}^{\frac{4p}{p+1}}||n||_\infty^{\frac{4p}{p+1}} \\
& + \frac{BCp^2}{A}||(\partial_x n)^p||_1^{2(1-\frac{1}{p})}||\nabla\partial_x c_{\neq}||_{2p}^2||n||_\infty^2,
\end{aligned}$$

which combined with $\nabla\partial_x c_{\neq}$ L^{2p} estimates (4.6.25), (4.6.26), hypothesis (4.6.2c) and

L^2 estimate of $\partial_x n$ in the initial time layer (4.6.30) yields

$$\begin{aligned} T_1 \le & \frac{2p-1}{BAp^2}||\nabla(\partial_x n)^p||_2^2 \\ & + \frac{Bp^7}{A^{1-6\epsilon}}\left(||(\partial_x n)^p||_1^{2(1-\frac{2}{p+1})} + ||(\partial_x n)^p||_1^{2(1-\frac{1}{p})}\right) C(C_{2,\infty}, C_{\partial_x n,\infty}, n_{in}). \end{aligned} \tag{4.6.31}$$

For the T_2 in (4.6.27), we can estimate it using Lemma (4.2.10), L^∞ estimate of n (4.6.24), $\nabla c_{\neq}$ L^{2p} estimates (4.6.15), (4.6.18) and the Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{T}\times\mathbb{R}$ as follows:

$$\begin{aligned} T_2 \le & \frac{2p-1}{Ap}||\nabla(\partial_x n)^p||_2||(\partial_x n)^p||_{16/7}||\nabla c||_{16} \\ \le & \frac{2p-1}{BAp^2}||\nabla(\partial_x n)^p||_2^2 + \frac{BCp^4}{A}||(\partial_x n)^p||_1^2\left(||\nabla c||_{16}^{32/7} + ||\nabla c||_{16}^2\right) \\ \le & \frac{2p-1}{BAp^2}||\nabla(\partial_x n)^p||_2^2 + \frac{Bp^4}{A^{1-7\epsilon}}||\partial_x n||_p^{2p}C(C_{2,\infty}). \end{aligned} \tag{4.6.32}$$

Combining (4.6.27), (4.6.31) and (4.6.32) and integrating in time, we have that

$$\begin{aligned} & \frac{1}{2p}||\partial_x n(t)||_{2p}^{2p} \\ \le & \frac{1}{2p}||\partial_x n_{in}||_{2p}^{2p} + \frac{Bp^7 t}{A^{1-6\epsilon}}\sup_{0\le s\le t}\left(||\partial_x n(s)||_p^{2p(1-\frac{2}{p+1})} + ||\partial_x n||_p^{2(p-1)}\right)C(C_{2,\infty}, C_{\partial_x n,\infty}, n_{in}) \\ & + \frac{Bp^4 t}{A^{1-7\epsilon}}\sup_{0\le s\le t}||\partial_x n(s)||_p^{2p}C(C_{2,\infty}), \quad \forall t\in[0,T_{\star\star}]. \end{aligned} \tag{4.6.33}$$

Finally, we use the (4.6.33) together with (4.6.30) to get the $||\partial_x n||_{L_t^\infty(0,T_{\star\star};L_{x,y}^\infty)} \le 2C_{\partial_x n,\infty}$. Note that if for $\forall j\in\mathbb{N}$,

$$\sup_{0\le s<T_{\star\star}}||\partial_x n(s)||_{2^j} \le 1,$$

we have that $\sup_{0\le s<T_{\star\star}}||\partial_x n(s)||_\infty \le 1$, and the result follows. Therefore, we assume that there exists $4 \le p_\star = 2^{j_\star} \in 2^{\mathbb{N}}$ such that it is the first integer that $\sup_{0\le s<T_{\star\star}}||\partial_x n||_{p_\star} \ge 1$. For $p = p_\star/2$,

$$||\partial_x n||_{L_t^\infty(0,T_{\star\star};L_{x,y}^{p_\star/2})} \le \max\{C_{\partial_x n,L^2}, 1\}. \tag{4.6.34}$$

We will only care about $p = 2^j > p_\star$, $j \in \mathbb{N}$. By the Hölder's inequality,

$$1 \leq \sup_{0\leq s\leq T_{\star\star}} ||\partial_x n(s)||_{p_*} \leq \sup_{0\leq s\leq T_{\star\star}} ||\partial_x n(s)||_2^\theta \sup_{0\leq s\leq T_{\star\star}} ||\partial_x n(s)||_p^{1-\theta}, \quad p > p_\star, p \in 2^{\mathbb{N}}.$$

Now combining this with (4.6.30), we have a lower bound for $\sup_{0\leq s\leq T_{\star\star}} ||\partial_x n(s)||_p$:

$$\sup_{0\leq s\leq T_{\star\star}} ||\partial_x n(s)||_p \geq (1 + C_{\partial_x n, L^2})^{-3}, \quad \forall p \geq p_\star, \ \ p \in 2^{\mathbb{N}}. \tag{4.6.35}$$

Now combining this with (4.6.33), we have that

$$\sup_{0\leq s\leq T_{\star\star}} ||\partial_x n(s)||_{2p}^{2p} \leq ||\partial_x n_{in}||_{2p}^{2p} + \frac{Bp^8}{A^{2/3-8\epsilon}} \sup_{0\leq s\leq T_{\star\star}} ||\partial_x n(s)||_p^{2p} C(C_{2,\infty}, C_{\partial_x n,\infty}, n_{in}). \tag{4.6.36}$$

Now we can take the A large such that

$$\sup_{0\leq s\leq T_{\star\star}} ||\partial_x n(s)||_{2p}^{2p} \leq ||\partial_x n_{in}||_{2p}^{2p} + p^8 \sup_{0\leq s\leq T_{\star\star}} ||\partial_x n(s)||_p^{2p}. \tag{4.6.37}$$

Combining L^2 estimate of $\partial_x n$ (4.6.30), $L^{p_\star/2}$ estimate of $\partial_x n$ (4.6.34) and the standard Moser-Alikakos iteration yields

$$\sup_{0\leq s\leq T_{\star\star}} ||\partial_x n(s)||_\infty \leq C(n_{in}). \tag{4.6.38}$$

Now by picking $2C_{\partial_x n,\infty} \geq C(n_{in})$, we finishes the proof of the improvement to (4.6.2c).

Third step: We prove the (4.6.1b). First we calculate the time evolution of $||\partial_x c_{\neq}||_{2p}$ using (4.6.30) and (4.6.38):

$$\frac{1}{2p}\frac{d}{dt}||\partial_x c_{\neq}||_{2p}^{2p} \leq \frac{1}{A}||\partial_x n||_{2p}||\partial_x c_{\neq}||_{2p}^{2p-1} \leq \frac{1}{A}(C_{\partial_x n,2}(n_{in}) + C_{\partial_x n,\infty}(n_{in}))||\partial_x c_{\neq}||_{2p}^{2p-1}.$$

This implies that

$$||\partial_x c_{\neq}(t)||_{2p} \leq \frac{t}{A}(C_{\partial_x n,2} + C_{\partial_x n,\infty}) + ||\partial_x (c_{in})_{\neq}||_{2p}, \quad \forall p \in [2, \infty). \tag{4.6.39}$$

Therefore, by the assumption that $||\nabla(c_{in})_{\neq}||_{H^1\cap W^{1,\infty}} \leq CA^{-q},\ q > 1/2$, we have that

$$||\partial_x c_{\neq}(t)||_\infty \leq \frac{t}{A}(C_{\partial_x n,2} + C_{\partial_x n,\infty}) + CA^{-q}. \quad (4.6.40)$$

For $t \leq T_{\star\star} \leq A^{1/3+\epsilon}$, we have the following estimate for A chosen large enough

$$||\partial_x c_{\neq}(t)||_\infty \leq 1, \quad \forall t \in [0, T_{\star\star}). \quad (4.6.41)$$

In order to estimate the norm $||\partial_y c_{\neq}||_{2p}$, we need to introduce a time weighted norm. To define it, we first consider the following simpler equation only taking into account the destabilizing effect of strong shear flow

$$\frac{d}{dt}f = -u'(y)\partial_x c_{\neq} - u(y)\partial_x f, \quad f_{in} = \partial_{y_1}(c_{in})_{\neq}.$$

We can estimate the time evolution of the L^{2p} norm of the solution using (4.6.39) as follows:

$$\frac{1}{2p}\frac{d}{dt}||f||_{2p}^{2p} \leq ||u'\partial_x c_{\neq}||_{2p}||f||_{2p}^{2p-1} \leq \left(\frac{t}{A}(C_{\partial_x n,2} + C_{\partial_x n,\infty}) + CA^{-q}\right)||u'||_\infty||f||_{2p}^{2p-1}. \quad (4.6.42)$$

Time integration yields

$$||f(t)||_{2p} \leq \frac{t^2}{A}||u'||_\infty(C_{\partial_x n,2}+C_{\partial_x n,\infty})+C||u'||_\infty A^{-q}t+CA^{-q} =: G_\infty(t), 0 \leq t < T_{\star\star}, \forall p \geq 2. \quad (4.6.43)$$

Note the following relation:

$$G'_\infty(t) = 2\frac{t}{A}(C_{\partial_x n,2} + C_{\partial_x n,\infty})||u'||_\infty + C||u'||_\infty A^{-q} \geq ||u'\partial_x c_{\neq}||_{2p}. \quad (4.6.44)$$

Next we consider the following time weighted norm:

$$\mathcal{F}_p^{1/p}(t) := \frac{||\partial_y c_{\neq}(t)||_p}{e^{G_\infty(t)}}. \quad (4.6.45)$$

Since G_∞ is bounded by a universal constant if we choose A large enough, the norm $\mathcal{F}_p^{1/p}$ is equivalent to the L^p norm. However, the quantity $\mathcal{F}_p$ has better property than the usual L^p norm. When we take the time derivative of $\mathcal{F}_p$, the weight $\frac{1}{e^{pG_\infty(t)}}$ will contribute extra negative term to compensate for the destabilizing effect of strong shear flow.

The time derivative of the $\mathcal{F}_{2p}$ can be estimated with the L^∞ bound of n in the initial time layer (4.6.24) and Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{T}\times\mathbb{R}$ as follows

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_{2p} \leq \frac{2p}{e^{G_\infty 2p}}\bigg(&-\frac{2p-1}{Cp^2A}\frac{||\partial_y c_{\neq}||_{2p}^{4p}}{||\partial_y c_{\neq}||_p^{2p}}+\frac{2p-1}{p^2A}||\partial_y c_{\neq}||_p^{2p}+\frac{2p-1}{A}||\partial_y c_{\neq}||_{2p}^{2p-2}(M+C_{n,\infty}^{in})^2\\&+||\partial_y c_{\neq}||_{2p}^{2p-1}||u'\partial_x c_{\neq}||_{2p}-G_\infty'(t)||\partial_y c_{\neq}||_{2p}^{2p}\bigg).\end{aligned}$$

If $\sup_{0\leq s\leq T_{\star\star}}\frac{||\partial_y c_{\neq}(s)||_{2p}}{e^{G_\infty(s)}}\leq 1$, we have

$$\mathcal{F}_{2p}(t)\leq 1,\quad \forall t\in[0,T_{\star\star}]. \tag{4.6.46}$$

Otherwise if $\sup_{0\leq s\leq T_{\star\star}}\frac{||\partial_y c_{\neq}(s)||_{2p}}{e^{G_\infty(s)}}\geq 1$, we have that at the maximum point $t_\star$ of $\mathcal{F}_{2p}$, $||\partial_y c_{\neq}(t_\star)||_{2p}\geq 1$. Combining this fact, Hölder inequality and the hypothesis e(4.2.7b), we obtain the following:

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_{2p}\bigg|_{t=t_\star}\leq& \frac{2p}{e^{G_\infty(t_\star)2p}}||(\partial_y c_{\neq})^p(t_\star)||_2^2\times\\ &\times\bigg(-\frac{2p-1}{CAp^2}\frac{||\partial_y c_{\neq}(t_\star)||_{2p}^{2p}/e^{G_\infty(t_\star)2p}}{(||\partial_y c_{\neq}(t_\star)||_p^p/e^{G_\infty(t_\star)p})^2}+\frac{32}{pA}(C_{ED}^2||n_{in}||_{H^1}^4+1)+\frac{2p-1}{A}(M+C_{n,\infty}^{in})^2\bigg).\end{aligned}$$

Now we have that

$$\sup_{0\leq s\leq T_{\star\star}}F_{2p}(s)\leq C(M,C_{n,\infty}^{in},C_{ED},||n_{in}||_{H^1})p^2\sup_{0\leq s\leq T_{\star\star}}F_p(s)^2+\frac{||\partial_{y_1}(c_{in})_{\neq}||_{2p}^{2p}}{e^{2pCA^{-q}}}. \tag{4.6.47}$$

Combining (4.6.46) and (4.6.47), and noting that $C_{n,\infty}^{in}$ only depends on n_{in} (4.6.24), we have

$$\sup_{0\le s\le T_{\star\star}} F_{2p}(s) \le \max\{C(M, C_{ED}, n_{in})p^2 \sup_{0\le s\le T_{\star\star}} F_p^2(s), 1\} \tag{4.6.48}$$

for A large enough. Combining this with the fact that $||\partial_y c_{\neq}||_2 \le \sqrt{C_{ED}}(||n_{in}||_{H^1} + 1) < \infty$ from the hypothesis (4.2.7b) and using similar Moser-Alikakos iteration argument as before, we end up with

$$\sup_{0\le s\le T_{\star\star}} ||\partial_y c_{\neq}(s)||_\infty \le C(M, C_{ED}, n_{in}). \tag{4.6.49}$$

Combining this with (4.6.41), we have proven that

$$||\nabla c_{\neq}(t)||_\infty \le C_{\nabla c_{\neq},\infty}^{in}(M, C_{ED}, n_{in}), \quad \forall t \in [0, T_{\star\star}). \tag{4.6.50}$$

Now since we have proven the bootstrap conclusion (4.6.38), $T_{\star\star}$ can be extended all the way to $A^{1/3+\epsilon}, \epsilon < \frac{1}{12}$. Therefore all the estimates we got above can be extended to $[0, A^{1/3+\epsilon}]$. This completes the proof of the lemma. $\square$

Remark 17. *One might slightly improve the value of q in Theorem 5. However, if q is too small, we are not able to prove Theorem 5. For example, we could not prove the theorem with $||\nabla(c_{in})_{\neq}||_{H^1\cap W^{1,\infty}} \approx A^{-1/4}$. The main obstacle is the estimate of the chemical gradient near the initial time. From the $||\partial_x c_{\neq}||_4$ estimate (4.6.4), we can only obtain that*

$$||\partial_x c_{\neq}(t)||_4 \le \sqrt{3}\frac{\sqrt{t}}{A^{1/2}} \sup_{0\le s\le t} ||n_{\neq}||_4 + A^{-1/4}, \quad t \in [0, T_{\star\star}].$$

Combining this and the estimate (4.6.8) and the fact that $||\partial_y c_{\neq}(t)||_4 \le f(t)$, we obtain that

$$||\partial_y c_{\neq}(t)||_4 \lesssim A^{1/12+\epsilon} + \frac{t^{3/2}}{A^{1/2}} \sup_{0\le s\le t} ||n_{\neq}(s)||_4 + \frac{1}{A^{1/2}} \sup_{0\le s\le t} ||n_{\neq}(s)||_4^2, \quad \forall t \le T_{\star\star},$$

which depends badly on A. Since we do not have a bound which is independent of A, we could not continue the proof.

Long time estimate

In this subsection, we prove (4.2.8d) and (4.2.8e) in the time interval $[A^{1/3+\epsilon}, T_\star)$. For the sake of brevity, we omit the proof and refer the interested readers to the paper [65] for further details.

4.7 Conclusion

In this paper we consider the parabolic-parabolic Patlak-Keller-Segel models in $\mathbb{T}\times\mathbb{R}$ with advection by a large strictly monotone shear flow. Without the shear flow, there exist solutions with mass larger than 8π which blow up in finite time [106]. We show that the additional shear flow, if it is chosen sufficiently large, suppresses one dimension of the dynamics and hence can suppress blow-up.

Chapter 5: Global Regularity of Two-Dimensional Flocking Hydrodynamics

5.1 Overview

We consider the system of Eulerian dynamics where the density $\rho(x,t)$ and velocity field $\mathbf{u}(x,t) = (u_1, \dots u_n) : \mathbb{R}^n \times \mathbb{R}_+ \mapsto \mathbb{R}^n$ are driven by nonlocal alignment forcing,

$$\left\{\begin{aligned} & \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ & \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = \int a(x,y,t)(\mathbf{u}(y,t) - \mathbf{u}(x,t))\rho(y,t)\mathrm{d}y \end{aligned}\right\} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+. \tag{5.1.1}$$

A solution $(\rho, \mathbf{u})$ is sought subject to the compactly supported initial density $\rho(x,0) = \rho_0(x) \in L^1_+(\mathbb{R}^n)$ and uniformly bounded initial velocity $\mathbf{u}(x,0) = \mathbf{u}_0(x) \in W^{1,\infty}(\mathbb{R}^n)$. The alignment forcing on the right hand side of (5.1.1) involves the non-negative interaction kernel $a(x,y,t)$. Different models involve different interaction kernels. We focus on two cases. The Cucker-Smale (CS) model [46] is subject to a *symmetric* interaction kernel $a(x,y) = \phi(|x-y|)$. The Motsch-Tadmor (MT) model [97] utilizes a more realistic interaction kernel $a(x,y,t) = \dfrac{\phi(|x-y|)}{(\phi * \rho)(x,t)}$. The kernel is non-symmetric but normalized such that $\int a(x,y,t)\rho(y,t)\mathrm{d}y = 1$.

It is shown in Chapter 1 that taking hydrodynamic limit on the particle system yields the following conservative equation

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0 \\ (\rho \mathbf{u})_t + \nabla(\rho \mathbf{u} \otimes \mathbf{u}) = \dfrac{\alpha(x,t)}{(\phi * \rho)(x,t)} \displaystyle\int \phi(|x-y|)(\mathbf{u}(y,t) - \mathbf{u}(x,t))\rho(x,t)\rho(y,t)\mathrm{d}y. \end{cases} \tag{5.1.2}$$

Here $\alpha(x,t)$ is the amplitude of alignment, $\alpha(x,t) = (\phi * \rho)(x,t)$ in the case of CS model, and $\alpha(x,t) \equiv 1$ in MT model. When classical solutions of these equations are restricted to the support of $\rho(\cdot,t)$, one ends with the equivalent system (5.1.1) with $a(x,y,t) = \alpha(x,t)\phi(|x-y|)/(\phi * \rho)(x,t)$, namely

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = \dfrac{\alpha(x,t)}{(\phi * \rho)(x,t)} \displaystyle\int \phi(|x-y|)(\mathbf{u}(y,t) - \mathbf{u}(x,t))\rho(y,t)\mathrm{d}y. \end{cases} \tag{5.1.3}$$

One aspect of the long time behavior of (5.1.1) is the emergence of flocking phenomena, which can be characterized in terms of the diameters

$$D(t) := \sup_{x,y \in \mathrm{supp}\{\rho(\cdot,t)\}} |x-y|, \qquad V(t) := \sup_{x,y \in \mathrm{supp}\{\rho(\cdot,t)\}} |\mathbf{u}(x,t) - \mathbf{u}(y,t)|.$$

The system (5.1.1) converges to a flock if there exists a finite D_∞ such that

$$\sup_{t \geq 0} D(t) \leq D_\infty \quad \text{and} \quad V(t) \overset{t \to \infty}{\longrightarrow} 0. \tag{5.1.4}$$

5.1.1 Strong solutions must flock

For the reader's convenience, we recall the following strong solution must flock result from Chapter one.

Theorem 10 (Strong solutions must flock [112]). *Let* $(\rho(\cdot,t),\mathbf{u}(\cdot,t)) \in (L^\infty \cap L^1) \times W^{1,\infty}$ *be a global strong solution of the system (5.1.1) subject to a compactly supported initial density* $\rho_0 = \rho(\cdot,0) \geq 0$ *and bounded initial velocity* $\mathbf{u}_0 = \mathbf{u}(\cdot,0) \in W^{1,\infty}$. *Assume that a monotonically decreasing influence function* $\phi \leq \phi(0) = 1$ *is global in the sense that*[1]

$$V_0 < m_0 \int_{D_0}^{\infty} \phi(r)dr, \qquad m_0 := |\rho_0|_1, \tag{5.1.5}$$

where D_0 *and* V_0 *are the initial diameters of non-vacuum density and velocity. Then* $(\rho,\mathbf{u})$ *converges to a flock at exponential rate, namely — the support of* $\rho(\cdot,t)$ *remains within a finite diameter* D_∞ *whose existence follows from assumption* (5.1.5)

$$\sup_{t\geq 0} D(t) \leq D_\infty \quad \textit{where} \quad m_0 \int_{D_0}^{D_\infty} \phi(s)ds = V_0, \tag{5.1.6a}$$

and

$$V(t) \leq V_0 e^{-\kappa t} \longrightarrow 0, \qquad \kappa := \begin{cases} m_0\phi_\infty, & \text{CS} \quad \textit{model}, \\ \phi_\infty, & \text{MT} \ \textit{model}, \end{cases} \qquad \phi_\infty := \phi(D_\infty). \tag{5.1.6b}$$

In particular, if $|\phi|_1 = \infty$ *then there is an* unconditional flocking *in the sense that* (5.1.6) *holds for all finite* V_0.

5.1.2 Critical thresholds

Theorem 10 raises the problem whether solutions of the hydrodynamic model (5.1.1) remain smooth for all time. This question was addressed in [35, 112], proving that if the compactly supported initial data stay below certain critical threshold

[1] We let $|\cdot|_p$ denote the usual L^p norm.

in configuration space then initial smoothness propagates and, as a result, the corresponding strong solutions will flock. Recall the finite-time blow-up of compactly supported density in the presence of *local* pressure [92, 110] and even in the presence of global Poisson forcing [96]. In both cases, a positive lower-bound on the (potential of) the forcing — the pressure, the Poisson forcing, etc., over the *finite* $\mathrm{supp}\{\rho(\cdot,t)\}$ leads to finite time blow up. In contrast, here the non-local character of the influence function ϕ guarantees global regularity, at least for sub-critical initial data. This type of conditional regularity for Eulerian dynamics depending on a *critical threshold* in configuration space, was advocated in a series of papers [54, 85, 87, 89, 90, 113]. Here, we pursue this approach to derive sharp critical thresholds for propagation of regularity of the *two-dimensional* flocking hydrodynamics.

5.1.3 Vacuum and the finite horizon alignment

According to (5.1.5), if the influence function is global in the sense that $\displaystyle\int^{\infty}\phi(r)dr = \infty$, then the alignment dynamics (5.1.1) admits *unconditional* flocking in the sense that (5.1.6) holds for all V_0's. This holds for both the symmetric CS model and non-symmetric MT model [98, proposition 2.9]. In this case, alignment in (5.1.3) is active *throughout* $\mathbb{R}^n$, inside and outside $\mathrm{supp}\{\rho(\cdot,t)\}$. Indeed, one has a global lower-bound on the action of alignment for all $x \in \mathbb{R}^n$, [112, proposition 6.1]

$$(\phi * \rho)(x,t) \geq m_0\phi(d(x,t) + D_\infty) > 0, \qquad d(x,t) = \mathrm{dist}\{x, \mathrm{supp}\{\rho(\cdot,t)\}\}$$

The flocking behavior of such a global approach was pursued in [112].

Another possible approach to study (5.1.3) is to focus on a specific initial configuration with finite velocity variation $V_0 < \infty$. Then, since $\text{supp}\{\rho(\cdot,t)\}$ cannot grow beyond a maximal diameter of size D_∞ dictated by (5.1.6a), it follows that the alignment term on the right of the underlying conservative formulation (5.1.2),

$$\phi(|x-y|)(\mathbf{u}(y,t)-\mathbf{u}(x,t))\rho(x,t)\rho(y,t) \equiv 0, \qquad |x-y| > D_\infty,$$

independently of the values of $\{\phi(r),\ r > D_\infty\}$. Alternatively, we can fix a compactly support influence function ϕ and view (5.1.6a) as a restriction on initial velocities whose variation is "not too large", so that they lead to flocking. With either one of these two points of view, the values of $\phi(r)$ for $r > D_\infty$ play no role in the dynamics. We therefore may set $\phi(r)_{|r>D_\infty} \equiv 0$ which in turn sets a *finite horizon* on the action of alignment. Namely, the alignment in (5.1.3) is still active in the vacuous annulus *outside* $\text{supp}\{\rho(\cdot,t)\}$,

$$A(t) := \{x \mid 0 < \text{dist}\{x, \text{supp}\{\rho(\cdot,t)\}\} < D_\infty\},$$

and (5.1.3) applies in $\text{supp}\{\rho(\cdot,t)\} \cup A(t)$,

$$\left\{\begin{array}{l} \rho_t + \nabla\cdot(\rho\mathbf{u}) = 0, \\ \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \dfrac{\alpha(x,t)}{\phi*\rho}\displaystyle\int \phi(|x-y|)(\mathbf{u}(y)-\mathbf{u}(x))\rho(y)\mathrm{d}y \end{array}\right\} \ \text{dist}\{x, \text{supp}\{\rho(\cdot,t)\}\} < D_\infty. \tag{5.1.7a}$$

However, since $\phi(|x-y|)\rho(y)$ is supported for y's in the intersection $y \in Y_x(t) := \text{supp}\{\rho(\cdot,t)\} \cap B_{D_\infty}(x)$, it implies the alignment bound

$$\left|\int \phi(|x-y|)(\mathbf{u}(y,t)-\mathbf{u}(x,t))\rho(y,t)dy\right| \leq V(t)\cdot|\rho(\cdot,t)|_\infty \times \int_{y\in Y_x(t)} \phi(|x-y|)dy.$$

It follows that the alignment on the right of (5.1.7a) approaches zero, as $x \in A(t)$ approaches the "horizon" boundary $\text{dist}\{x, \text{supp}\{\rho(\cdot,t)\}\} = D_\infty$ and $\text{vol}(Y_x(t)) \to$

0. In particular, $(\phi * \rho)(x,t) \equiv 0$ beyond the horizon $\mathrm{dist}\{x, \mathrm{supp}\{\rho(\cdot,t)\}\} > D_\infty$, where the momentum equation is reduced to inviscid pressureless equations, $\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = 0$. Accordingly, (5.1.7a) can be complemented with *constant* far-field boundary conditions, in agreement with [112, Remarks 2.8 & 6.6],

$$\mathbf{u}(x,t) \equiv \mathbf{u}_\infty, \quad \text{for} \;\; \mathrm{dist}\{x, \mathrm{supp}\{\rho(\cdot,t)\}\} > D_\infty. \tag{5.1.7b}$$

5.2 Cucker-Smale hydrodynamics: global regularity and fast alignment

5.2.1 Global regularity

We begin by recalling the one-dimensional Cucker-Smale model for $(\rho, u) : (\mathbb{R}, \mathbb{R}_+) \mapsto (\mathbb{R}_+, \mathbb{R})$,

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ u_t + u u_x = \displaystyle\int_{\mathbb{R}} \phi(|x-y|)(u(y,t) - u(x,t))\rho(y,t)\mathrm{d}y \end{cases} \quad (x,t) \in (\mathbb{R}, \mathbb{R}_+). \tag{5.2.1}$$

In [35] it was proved that (5.2.1) has a global classical solution if and only if the initial data satisfies

$$\partial_x u_0(x) \geq -(\phi * \rho_0)(x), \quad \text{for all } x \quad \in \mathbb{R}. \tag{5.2.2}$$

Condition (5.2.2) separates the space of initial configurations into two distinct regimes: a sub-critical regime of initial data satisfying $\partial_x u_0(x) \geq -\phi * \rho_0(x), \forall x \in supp(\rho_0)$, which guarantee *global* smooth solutions; and a supercritical regime of initial conditions such that $\partial_x u_0(x_0) \leq -\phi * \rho_0(x_0)$ for some $x_0 \in \mathbb{R}$, which leads to a finite time blowup. This is a typical one-dimensional example for the critical threshold behavior.

Condition (5.2.2) provides a sharp improvements to the earlier critical threshold results in [86, 105, 112]. Recent results in [48, 109] prove the global regularity of (5.2.1) for singular kernels $\phi(|x|) = |x|^{-(1+\alpha)}$ for $\alpha \in (0,2)$ *independent* of any finite critical threshold. Singularity helps!.

A first attempt to extend the study of critical threshold to the *two-dimensional* CS model was derived in [112]. Here, we improve this result with a simplified derivation of a sharper critical threshold condition, leading to alignment decay of order $e^{-\kappa t}$. We recall (5.1.6b) which set $\kappa = m_0\phi_\infty$ in the present case of CS model.

Theorem 11 (Critical threshold for 2D Cucker-Smale hydrodynamics)**.** *Consider the two-dimensional CS model*

$$\left\{\begin{aligned} &\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0, \\ &\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \int \phi(|x-y|)(\mathbf{u}(y,t) - \mathbf{u}(x,t))\rho(y,t)\,dy \end{aligned}\right\} \qquad x\in\mathbb{R}^2, t\geq 0, \tag{5.2.3}$$

subject to initial conditions, $(\rho_0, \mathbf{u}_0) \in (L^1_+(\mathbb{R}^2), W^{1,\infty}(\mathbb{R}^2))$, with compactly supported density, $D_0 < \infty$, and such that the variation of the initial velocity satisfies the strengthened bound

$$V_0 \leq m_0\cdot\min\left\{|\phi|_1, \frac{\phi_\infty^2}{4|\phi'|_\infty}\right\}, \qquad V_0 = \max_{x,y\in supp(\rho_0)} |\mathbf{u}_0(x) - \mathbf{u}_0(y))|, \quad \phi_\infty = \phi(D_\infty). \tag{5.2.4}$$

Assume that the following critical threshold condition holds.

(i) The initial velocity divergence satisfies

$$\operatorname{div}\mathbf{u}_0(x) \geq -\phi * \rho_0(x) \quad \textit{for all} \quad x\in\mathbb{R}^2. \tag{5.2.5}$$

(ii) Let $S = \frac{1}{2}\{(\partial_j u_i + \partial_i u_j)\}$ denote the symmetric part of the velocity gradient with eigenvalues $\mu_i = \mu_i(S)$. Then the initial spectral gap *$\eta_{S_0} := \mu_2(S_0) - \mu_1(S_0)$ is*

bounded

$$\max_x \left|\eta_{S_0}(x)\right| \le \frac{1}{2} m_0 \phi_\infty, \qquad \eta_S = \mu_2(S(x,t)) - \mu_1(S(x,t)). \tag{5.2.6}$$

Then the class of such sub-critical initial conditions (5.2.5),(5.2.6) *admit a classical solution*

$(\rho(\cdot,t), \mathbf{u}(\cdot,t)) \in C(\mathbb{R}^+; L^\infty \cap L^1(\mathbb{R}^2)) \times C(\mathbb{R}^+; \dot{W}^{1,\infty}(\mathbb{R}^2))$ *with large time hydrodynamics flocking behavior* (5.1.6b), $\max_{x,y \in supp(\rho(\cdot,t))} |\mathbf{u}(x,t) - \mathbf{u}(y,t)| \lesssim e^{-\kappa t}$.

Before turning to the proof of theorem 11, we comment on its assumptions.

Remark 18 (on the critical threshold (5.2.5),(5.2.6))**.** *Theorem 11 recovers the one-dimensional critical threshold* (5.2.2)*. It amplifies the same theme of critical threshold required for global regularity of other* two-dimensional *Eulerian dynamics found in restricted Euler-Poisson equations [89], rotational Euler equations [90],etc., namely — if the initial divergence is "not too negative", as in* (5.2.5)*, and the initial spectral gap is "not too large", as in* (5.2.6)*, then global regularity persists for all time. In particular, since* $\eta_S = \sqrt{(\partial_1 u_1 - \partial_2 u_2)^2 + (\partial_1 u_2 + \partial_2 u_1)^2}$ *we find that both* (5.2.5),(5.2.6) *hold if*

$$|\partial_j u_i(x,0)| \le \frac{1}{4\sqrt{2}} m_0 \phi_\infty.$$

Remark 19 (on the finite variation (5.2.4))**.** *Observe that* (5.2.4) *places a restriction on the size of* V_0 *even in the case of unconditional flocking,* $|\phi|_1 = \infty$*. Specifically, recall that* V_0 *dictates the maximal diameter of the flock in* (5.1.6a) *and thus,* (5.2.4) *amounts to*

$$\int_{D_0}^{D_\infty} \phi(s)ds \le \frac{\phi^2(D_\infty)}{4\max_{s \le D_\infty} |\phi'(s)|}. \tag{5.2.7}$$

Since the term on the left is increasing while the term on the right is decreasing as functions of D_∞, it follows that (5.2.7) *is satisfied for diameters D_∞ up to some maximal finite size, that is — the condition made in* (5.2.4) *is met for finite $V_0 = m_0 \int^{D_\infty} \phi(s)ds$ depending on the influence function ϕ. This finite restriction on V_0 can probably be improved, but unlike the one-dimensional case it cannot be completely removed. In fact, since $V_0 \leq (\mu_2(S_0) + \omega_0)D_\infty$, the bound sought in* (5.2.4) *places a purely two-dimensional restriction on the size of* initial vorticity*.*

Remark 20 (on the finite horizon)**.** *Observe that in the case of alignment with a finite horizon, the critical threshold* (5.2.5) *requires that* $\operatorname{div} \mathbf{u}_0(x) \geq 0$ *for dist$\{x, supp\{\rho_0\}\} > D_\infty$. This is precisely the critical threshold condition which rules out finite time blow-up in the pressure-less equations [111], which is satisfied when prescribing far-field constant velocity* (5.1.7b)*. In this case, the critical threshold* (5.2.5) *needs to be verfied within the finite horizon dist$\{x, supp\{\rho_0\}\} < D_\infty$.*

Proof. Our purpose is to show that the derivative $\{\partial_j u_i\}$ are uniformly bounded. We proceed in four steps.

Step #1 — the dynamics of $\operatorname{div} \mathbf{u} + \phi * \rho$. Differentiation of (5.1.1) implies that the 2×2 velocity gradient matrix, $M_{ij} := \partial_j u_i$, satisfies

$$M_t + \mathbf{u} \cdot \nabla M + M^2 = -(\phi * \rho)M + R, \qquad R_{ij} := \partial_j \phi * (\rho u_i) - u_i \partial_j \phi * \rho. \quad (5.2.8)$$

The entries of the residual matrix $\{R_{ij}\}$ can be bounded by the commutator estimate

[112, proposition 4.1] in terms of $V(t) = \sup_{\text{supp}(\rho)} |u_i(x,t) - u_i(y,t)| \leq V_0 e^{-\kappa t}$,

$$|R_{ij}| = \left| \int_{\mathbb{R}^n} \partial_j \phi(|x-y|)(u_i(y,t) - u_i(x,t))\rho(y,t)\mathrm{d}y \right| \leq |\phi'|_\infty m_0 V_0 e^{-\kappa t}, \qquad \kappa = m_0 \phi_\infty.$$

The first step is to bound the divergence: taking the trace of (5.2.8) we find that $\mathsf{d} := \nabla \cdot \mathbf{u}$ satisfies

$$\mathsf{d}_t + \mathbf{u} \cdot \nabla \mathsf{d} + \operatorname{Tr} M^2 = -(\phi * \rho)\mathsf{d} + \operatorname{Tr} R.$$

Expressed in terms of the material derivative along particle path, $X' := (\partial_t + \mathbf{u} \cdot \nabla)X$, we have $\mathsf{d}' + \operatorname{Tr} M^2 = -(\phi * \rho)\mathsf{d} + \operatorname{Tr} R$. We now make a key observation that $\operatorname{Tr} R$ is in fact an exact derivative along particle path. Indeed, as in [35] we invoke the mass equation,

$$\operatorname{Tr} R = \phi * \nabla \cdot (\rho \mathbf{u}) - \mathbf{u} \cdot \nabla \phi * \rho = -(\phi * \rho)_t - \mathbf{u} \cdot \nabla \phi * \rho = -(\phi * \rho)',$$

and we end up with

$$(\mathsf{d} + \phi * \rho)' + \operatorname{Tr} M^2 = -(\phi * \rho)\mathsf{d}. \tag{5.2.9}$$

To proceed, we express $\operatorname{Tr} M^2 \equiv \dfrac{\mathsf{d}^2 + \eta_{_M}^2}{2}$ in terms of the *spectral gap*, $\eta_{_M} := \lambda_2(M) - \lambda_1(M)$, associated with the eigenvalues of M,

$$(\mathsf{d} + \phi * \rho)' = -\frac{1}{2}\eta_{_M}^2 - \frac{1}{2}\mathsf{d}(\mathsf{d} + 2\phi * \rho). \tag{5.2.10}$$

We need to follow the dynamics of the spectral gap η_M. To this end, one may try to use the *spectral dynamics* associated with M, [87]: by (5.2.8) the λ_i's satisfy

$$\lambda_i' + \lambda_i^2 = -(\phi * \rho)\lambda_i + \langle \boldsymbol{\ell}_i, R\mathbf{r}_i \rangle, \qquad i = 1, 2,$$

where $\{\boldsymbol{\ell}_i, \mathbf{r}_i\}$ are the left and right eigenvectors associated with λ_i, normalized such that $\langle \boldsymbol{\ell}_i, \mathbf{r}_i \rangle = 1$. Taking the difference of these two equations shows that the spectral gap $\eta_{_M} = \lambda_2 - \lambda_1$, satisfies the transport equation

$$\eta_{_M}' + (\mathsf{d} + \phi * \rho)\eta_{_M} = \langle \boldsymbol{\ell}_2, R\mathbf{r}_2 \rangle - \langle \boldsymbol{\ell}_1, R\mathbf{r}_1 \rangle.$$

Here one faces the difficulty which arises with the term on the right, namely — even with the control of the entries $\{R_{ij}\}$, we may still encounter an ill-conditioned M with $|\boldsymbol{\ell}_i| \cdot |\mathbf{r}_i| \gg 1$ so that the magnitude of this term is left unchecked. To circumvent this difficulty, we proceed along the lines argued in [111]: we split M into its symmetric and antisymmetric parts $M = S + \Omega$ and use the identity[2]

$$\eta_{_M}^2 \equiv \eta_{_S}^2 - 4\omega^2, \qquad M = S + \Omega, \quad \Omega := \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}, \tag{5.2.11}$$

where ω is the *scaled* vorticity[3] $\omega = \frac{1}{2}(\partial_1 u_2 - \partial_2 u_1)$. Expressed in terms of $\eta_{_S}$, the trace dynamics (5.2.10) now reads

$$(\mathsf{d} + \phi * \rho)' = \frac{1}{2}(4\omega^2 - \eta_{_S}^2) - \frac{1}{2}\mathsf{d}(\mathsf{d} + 2\phi * \rho).$$

This calls for the introduction of the new "natural" variable $\mathsf{e} = \mathsf{d} + \phi * \rho$, satisfying

$$\mathsf{e}' = \frac{1}{2}\left((\phi * \rho)^2 + 4\omega^2 - \eta_{\mathsf{S}}^2 - \mathsf{e}^2\right). \tag{5.2.12}$$

Our purpose is to show that $\{x \mid \mathsf{e}(\mathsf{x}, \mathsf{t}) \geq 0\}$ is invariant region of the dynamics (5.2.12).

[2]Equating the trace of M^2 with that of $S^2 + \Omega^2 + S\Omega + \Omega S$ we find $\operatorname{Tr} M^2 = \operatorname{Tr} S^2 - 2\omega^2$. Using $\operatorname{Tr} X^2 = \frac{1}{2}(\mathsf{d}^2 + \eta_{_X}^2)$ with $X = M$ on the left and $X = S$ on the right implies (5.2.11).

[3]The use of such scaling simplifies the computation below.

Step #2 — bounding the spectral gap η_S. Consider the dynamics of the symmetric part of (5.2.8)

$$S' + S^2 = \omega^2 \mathbb{I}_{2\times 2} - (\phi * \rho)S + R_{\text{sym}}, \qquad R_{\text{sym}} = \frac{1}{2}(R + R^\top),$$

The spectral dynamics of its eigenvalues, $\mu_2(S) \geq \mu_1(S)$, is governed by

$$\mu_i' + \mu_i^2 = \omega^2 - (\phi * \rho)\mu_i + \big\langle \mathbf{s}_i, R_{\text{sym}} \mathbf{s}_i \big\rangle \tag{5.2.13}$$

driven by the *orthonormal* eigenpair $\{\mathbf{s}_1, \mathbf{s}_2\}$ of the symmetric S. Taking the difference, we find that $\eta_S := \mu_2(S) - \mu_1(S) \geq 0$ satisfies,

$$\eta_S' + \mathsf{e}\eta_{\mathsf{s}} = \mathsf{q}, \qquad \mathsf{e} = \mathsf{d} + \phi * \rho. \tag{5.2.14}$$

This is the *same* dynamics found with η_M except that the different residual on the right of (5.2.14) given by

$$q := \big\langle \mathbf{s}_2, R_{\text{sym}} \mathbf{s}_2 \big\rangle - \big\langle \mathbf{s}_1, R_{\text{sym}} \mathbf{s}_1 \big\rangle,$$

is now controlled by the size of $\{R_{ij}\}$: since $\mathbf{s}_i$ are normalized,

$$|q(\cdot, t)| \leq 2 \max_{ij} |R_{ij}(\cdot, t)| \leq 2|\phi'|_\infty m_0 V_0 e^{-\kappa t}, \qquad \kappa = m_0 \phi_\infty. \tag{5.2.15}$$

Hence, as long as $\mathsf{e}(\cdot, \mathsf{t})$ remains positive then η_S remain uniformly bounded

$$|\eta_S(x,t)| \leq \max_x |\eta_S(x,0)| + 2\frac{|\phi'|_\infty}{\phi_\infty} V_0 < \max_x |\eta_S(x,0)| + \frac{1}{2} m_0 \phi_\infty < m_0 \phi_\infty \tag{5.2.16}$$

The first inequality on the right follows from integration of (5.2.14)-(5.2.15); the second follows from the V_0-bound in (5.2.4) and the third from the assumed bound on η_{S_0} in (5.2.6).

Step #3 — the invariance of $\mathsf{e}(\cdot,\mathsf{t}) \geq 0$. We return to (5.2.12): expressed in terms of $c(x,t) := \sqrt{(\phi * \rho)^2 - \eta_S^2}$ we have

$$\mathsf{e}' \geq \frac{1}{2}\left(\mathsf{c}^2(\mathsf{x},\mathsf{t}) - \mathsf{e}^2\right), \qquad \mathsf{c}(\mathsf{x},\mathsf{t}) = \sqrt{(\phi * \rho)^2 - \eta_{\mathsf{S}}^2}. \tag{5.2.17}$$

Observe that $c(\cdot)$ is well-defined in $\mathbb{R}$: the upper-bound (5.2.16) and the lower-bound $\phi * \rho \geq m_0\phi_\infty$ imply that as long as $\mathsf{e} \geq 0$, the right term on the right of (5.2.17) remains boundedly positive

$$c(x,t) \geq \sqrt{m_0^2\phi_\infty^2 - \max_x \eta_S^2(x,t)} \geq c_{\min} > 0.$$

Since $\mathsf{e}' \geq \frac{1}{2}(\mathsf{c}_{\min}^2 - \mathsf{e}^2) = \frac{1}{2}(\mathsf{c}_{\min} - \mathsf{e})(\mathsf{c}_{\min} + \mathsf{e})$, it follows that e is increasing whenever $\mathsf{e} \in (-\mathsf{c}_{\min}, \mathsf{c}_{\min})$ and in particular, if $\mathsf{e}_0 \geq 0$ then $\mathsf{e}(\mathsf{x},\mathsf{t})$ remains positive at later times. Thus, if the initial data are *sub-critical* in the sense that (5.2.5) holds

$$\mathsf{e}_0 = \operatorname{div} \mathbf{u}_0(\mathsf{x}) + \phi * \rho_0(\mathsf{x}) \geq 0,$$

then $\mathsf{e}(\cdot,\mathsf{t}) \geq 0$ and $\eta_S(\cdot,t)$ remains bounded.

Step #4 — an upper-bound of $\mathsf{e}(\cdot,\mathsf{t})$. The lower-bound $\mathsf{e} \geq 0$ implies that the vorticity is bounded. Indeed, the anti-symmetric part of (5.2.8) yields that the vorticity $\omega = \frac{1}{2}\operatorname{Tr} JM$ satisfies

$$\omega' + \mathsf{e}\omega = \frac{1}{2}\operatorname{Tr} \mathsf{J}\mathsf{R}, \qquad \mathsf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{5.2.18}$$

hence

$$|\omega|' \leq -\mathsf{e}|\omega| + \frac{1}{2}|\mathsf{q}|, \qquad |\mathsf{q}(\cdot,\mathsf{t})| \leq 2|\phi'|_\infty \mathsf{m}_0\mathsf{V}_0\mathsf{e}^{-\kappa\mathsf{t}}, \quad \kappa = \mathsf{m}_0\phi_\infty, \tag{5.2.19}$$

and we end up with same upper-bound on ω as with η_S,

$$|\omega(x,t)| \leq \omega_{\max}, \qquad \omega_{\max} := \max_x |\omega_0| + \frac{1}{2}m_0\phi_\infty. \tag{5.2.20}$$

Returning to (5.2.12) we have (recall $\phi \leq 1$)

$$\mathsf{e}' \leq \frac{1}{2}\Big((\phi * \rho)^2 + 4\omega^2 - \mathsf{e}^2\Big) \leq \frac{1}{2}\Big(\mathsf{m}_0^2 + 4\omega_{\max}^2 - \mathsf{e}^2\Big),$$

which implies that $\mathsf{e}(\mathsf{x}, \mathsf{t}) \leq \mathsf{e}_{\max} < \infty$. The uniform bound on e implies that $\operatorname{div} u$ is uniformly bounded, $|\operatorname{div} \mathbf{u}| \leq |\mathsf{e}|_\infty + |\phi * \rho|_\infty \leq \mathsf{e}_{\max} + \mathsf{m}_0$, and together with the bound on the spectral gap (5.2.16), it follows that the symmetric part $\{S_{ij}\}$ is bounded. Finally, together with the vorticity bound (5.2.20) it follows that $\{\partial_j u_i\}$ are uniformly bounded which completes the proof. □

Remark 21. *Observe that the region of sub-critical configuration leading global regularity becomes* larger *for* $|\omega_0| \gg 1$ *in agreement with the statements made in [38, 90] that rotation prevents or at least delays finite-time blow-up. Specifically, if* $|\omega_0(\cdot)| \geq \omega_{min} > 0$ *then one can set a larger lower barrier* $c = \sqrt{(\phi * \rho)^2 + 4\omega_{min}^2 - \eta_S^2}$ *in* (5.2.17) *leading to the improved threshold* $\operatorname{div} \mathbf{u}_0 > -\phi * \rho_0 - \omega_{min}$. *In particular, if* ω *is large enough so that* $4\omega^2 - \eta_S^2 > 0$, *that is — if* M *has complex-valued eigenvalues, then the invariance of the positivity of* e *follows at once from the fact that* (5.2.12) *is dominated equation by* $\mathsf{e}' \geq \frac{1}{2}\left((\phi * \rho)^2 - \mathsf{e}^2\right)$. *As in the 2D restricted Euler-Poisson equations [89], the difficulty lies with the case of real eigenvalues.*

Remark 22. *The proof of theorem 11 reveals two main aspects. First, the* commutator structure *of the alignment term on the right of* $(5.2.3)_2$, *expressed as* $[\phi *, u](\rho)$, *leads to the 'residual terms'* R_{ij} *with exponentially decaying bound. The role of commutator structure was highlighted in our recent work [109]. Second, the use of spectral dynamics, [85, 87, 89], to trace the propagation of regularity for the remaining, non-residual terms in* (5.2.8).

5.2.2 Fast alignment

We extend the one-dimensional arguments of [109] that show an exponentially rapid convergence towards a *flocking state*, consisting of a constant 2-vector velocity $\bar{\mathbf{u}} \in \mathbb{R}^2$ and a traveling density profile $\bar{\rho}(x,t) = \rho_\infty(x - t\bar{\mathbf{u}})$. We only indicate the main aspects in the passage to the present system. We start by noting that the positivity of e implies more than the mere boundedness of the spectral gap η_S and the vorticity ω. Indeed, (5.2.14) and (5.2.19) imply that these quantities follow the exponential decay of q in (5.2.15)

$$|\eta_S(\cdot,t)|_\infty + |\omega(\cdot,t)|_\infty \lesssim e^{-\kappa t}.$$

This shows that modulo rapidly decaying error terms $E(t)$ of order $E(t) \lesssim e^{-\kappa t}$, equation (5.2.12) which governs e takes the form

$$\mathsf{e_t} + \mathbf{u} \cdot \nabla \mathsf{e} = \frac{1}{2}\left(\mathsf{h}^2 - \mathsf{e}^2\right) + \mathsf{E(t)}, \qquad \mathsf{h} := \phi * \rho$$

Moreover, convolving the mass equation with ϕ we find

$$\mathsf{h_t} + \mathbf{u} \cdot \nabla \mathsf{h} = \int \nabla\phi(|\mathsf{x} - \mathsf{y}|) \cdot (\mathbf{u}(\mathsf{x},\mathsf{t}) - \mathbf{u}(\mathsf{y},\mathsf{t}))\rho(\mathsf{y},\mathsf{t})\mathsf{dy}. \tag{5.2.21}$$

Observe that the quantity on the right of rapidly decaying, being upper-bounded by $\lesssim |\phi'|_\infty V(t) \lesssim e^{-\kappa t}$. Hence, the difference $\mathsf{d} = \mathsf{e} - \mathsf{h}$ satisfies

$$\mathsf{d}_t + \mathbf{u} \cdot \nabla \mathsf{d} = -\frac{1}{2}(\mathsf{e} + \mathsf{h})\mathsf{d} + \mathsf{E(t)}.$$

The positivity of $\mathsf{e}+\mathsf{h}$ then implies the rapid decay of the divergence, $|\operatorname{div}\mathbf{u}(\cdot,t)|_\infty \lesssim e^{-\kappa t}$. The exponential decay of the divergence, the vorticity and the spectral gap imply that $|\partial_j u_i(\cdot,t)|_\infty \lesssim e^{-\kappa t}$. Let $\bar{\mathbf{u}}$ be a large-time limiting value of $\mathbf{u}(\cdot,t)$. The mass equation reads

$$\rho_t + \bar{\mathbf{u}} \cdot \nabla\rho = -\mathsf{d}\rho + (\bar{\mathbf{u}} - \mathbf{u}) \cdot \nabla\rho.$$

The term on the right is rapidly decaying because d and $(\bar{\mathbf{u}}-\mathbf{u})$ are, and one concludes along the lines of [108], that there exists a traveling density profile such that $\rho(x,t) - \rho_\infty(x - t\bar{\mathbf{u}}) \to 0$.

5.3 Motsch-Tadmor hydrodynamics

In this section, we study the flocking hydrodynamics which arises from MT model (**??**) with $\kappa = \phi_\infty$. We begin by recalling the one-dimensional case

$$\begin{aligned} &\rho_t + (\rho u)_x = 0, \qquad (x,t) \in (\mathbb{R}, \mathbb{R}_+) \\ &u_t + uu_x = \int \frac{\phi(|x-y|)}{(\phi * \rho)(x,t)}(u(y,t) - u(x,t))\rho(y,t)\mathrm{d}y. \end{aligned} \tag{5.3.1}$$

System (5.3.1) was recently studied in [21], as the hydrodynamic description for agent-based model of "emotional contagion", and in [62] in the context of stable swarming. In [35] it was proved that (5.3.1) has a global classical solution for sub-critical initial data such that

$$\partial_x u_0(x) \geq -\sigma_+(V_0) \quad \text{for all} \quad x \in \mathbb{R}, \tag{5.3.2}$$

for a certain critical curve $\sigma_+ \geq 0$. We now make a precise statement of the critical threshold for both the one - and two-dimensional MT model.

Theorem 12 (Critical threshold for 2D Motsch-Tadmor hydrodynamics)**.** *Consider* *the two-dimensional MT model in* $(x,t) \in (\mathbb{R}^2, \mathbb{R}_+)$,

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = \displaystyle\int a(x,y,t)(\mathbf{u}(y,t) - \mathbf{u}(x,t))\rho(y,t)dy, \qquad a(x,y,t) := \frac{\phi(|x-y|)}{(\phi * \rho)(x,t)}, \end{cases} \tag{5.3.3}$$

subject to initial conditions $(\rho_0, \mathbf{u}_0) \in (L^1, W^{1,\infty}(\mathbb{R}^2))$, *with compactly supported density,* $D_0 < \infty$ *and initial velocity of finite variation*

$$V_0 \le m_0 \cdot \min\left\{ |\phi|_1, \frac{\phi_\infty^2}{4|\phi'|_\infty(1+2\phi_\infty)} \right\}, \qquad \phi_\infty = \phi(D_\infty). \tag{5.3.4}$$

Assume that the following critical threshold condition holds.

(i) The initial velocity divergence satisfies

$$\operatorname{div} \mathbf{u}_0(x) \ge -1 \quad \textit{for all} \quad x \in \mathbb{R}^2. \tag{5.3.5}$$

(ii) Then the initial spectral gap $\eta_{S_0} := \mu_2(S_0) - \mu_1(S_0)$ *is bounded*

$$\max_x \left|\eta_{S_0}(x)\right| \le \frac{1}{2}, \qquad \eta_S = \mu_2(S(x,t)) - \mu_1(S(x,t)). \tag{5.3.6}$$

Then the class of such sub-critical initial conditions (5.3.5),(5.3.6) *give rise to a classical solution* $(\rho(t), \mathbf{u}(t) \in C(\mathbb{R}^+; L^\infty(\mathbb{R}^2)) \times C(\mathbb{R}^+; \dot{W}^{1,\infty}(\mathbb{R}^2))$ *with large time hydrodynamics flocking behavior* (5.1.6b) $\max_{x \in supp(\rho)} |\mathbf{u}(x,t) - \mathbf{u}(y,t)| \lesssim e^{-\kappa t}$.

Remark 23. *In the case of finite horizon alignment encoded in* (5.1.7) *with* $\alpha = \phi * \rho$, *the critical thresholds* (5.3.5),(5.3.6) *can be restricted to the finite set dist*$\{x, supp\{\rho_0\}\}$.

For the sake of brevity, we skip the proof and refer the interested readers to the paper [68] for further details.

5.4 Conclusion

We study the systems of Euler equations which arise from agent-based dynamics driven by velocity *alignment*. It is known that smooth solutions of such systems must flock, namely — the large time behavior of the velocity field approaches a limiting "flocking" velocity. To address the question of global regularity, we derive sharp critical thresholds in the phase space of initial configuration which characterize the global regularity and hence flocking behavior of such *two-dimensional* systems. Specifically, we prove for that a large class of *sub-critical* initial conditions such that the initial divergence is "not too negative" and the initial spectral gap is "not too large", global regularity persists for all time.

Chapter 6: Collective behavior of Multi-species

6.1 Overview

6.1.1 Multi-species hydrodynamic flocking model

The classical single-species hydrodynamic flocking model is an Eulerian dynamics of agent density ρ and velocity u subject to nonlocal alignment forcing on $\mathbb{T}^d$, $d = 1, 2$:

$$\rho_t + \nabla \cdot (\rho u) = 0, \tag{6.1.1a}$$

$$u_t + u \cdot \nabla u = \phi * (\rho u) - \phi * \rho u. \tag{6.1.1b}$$

Here the initial condition $(\rho, u)\big|_{t=0} = (\rho_0, u_0)$ is satisfied. The first equation (6.1.1a) describes the transportation of the mass density ρ along the velocity u. The second equation (6.1.1b) governing velocity evolution is a pressure-less compressible Eulerian equation subject to alignment forcing. The forces in the equation (6.1.1b) involve a nonnegative radially decreasing influence function $\phi(\cdot) \equiv \phi(|\cdot|)$, $\phi(|\cdot|) \in C^1(\mathbb{T}^d)$ and drives the velocity $u(x,t)$ to the dynamic mean velocity $\overline{u}(x,t) = \dfrac{\phi * (\rho u)}{\phi * \rho}(x,t)$, which neutralize the forcing in the equation (6.1.1b). The $\overline{u}$ is called the *dynamical mean velocity*. Since the agents tend to align their velocity to their neighbors, the velocity u is expected to approach a uniform constant velocity $u_\infty = \dfrac{\int \rho u dx}{\int \rho dx}$ as time tends to infinity. This is the *flocking behavior*.

In this paper, we extend the model (6.1.1) to the Eulerian multi-species hydrodynamic flocking model on $\mathbb{T}^d, d = 1, 2$:

$$\partial_t \rho_\alpha + \nabla \cdot (u_\alpha \rho_\alpha) = 0; \tag{6.1.2a}$$

$$\partial_t(\rho_\alpha u_\alpha) + \nabla \cdot (\rho_\alpha u_\alpha \otimes u_\alpha) = \sum_{\beta\in\mathcal{I}} b_{\alpha\beta}\rho_\alpha \left\{\phi * (\rho_\beta u_\beta) - (\phi * \rho_\beta)u_\alpha\right\}, \quad \alpha, \beta \in \mathcal{I}, \tag{6.1.2b}$$

subject to initial condition $(\rho_\alpha, u_\alpha)\big|_{t=0} = ((\rho_\alpha)_0, (u_\alpha)_0) \in L^1(\mathbb{T}^d)\times W^{1,\infty}(\mathbb{T}^d; \mathbb{R}^d), \ \ \forall \alpha \in \mathcal{I}$. Here ρ_α, u_α denote the density and velocity of the species α, respectively. The parameters $\alpha, \beta \in \mathcal{I}$ indicate the species of the agents. The total number of species $|\mathcal{I}|$ is assumed to be finite. The alignment forces in (6.1.2b) now also involve interations between species determined by the *coupling coefficients* $b_{\alpha\beta}$. For the most part of the paper, the coupling coefficients $b_{\alpha\beta} \geq 0$ are assume to be symmetric and stochastic, i.e.

$$b_{\alpha\beta} = b_{\beta\alpha}, \quad \sum_{\beta\in\mathcal{I}} b_{\alpha\beta} = 1, \quad \forall \alpha, \beta \in \mathcal{I}. \tag{6.1.3}$$

Clearly, a row stochastic matrix $B = (b_{\alpha\beta})_{\alpha,\beta\in\mathcal{I}}$ has an eigenvalue 1 with the corresponding eigenvector $w_1 = (1, 1, 1, ..., 1)^T$. We say that the symmetric row stochastic matrix B is *essentially negative* if all its eigenvalues are negative except for the simple eigenvalue $\lambda_1 = 1$. These matrices have the general form:

$$B = 1w_1w_1^T - \lambda_2 w_2 w_2^T - \lambda_3 w_3 w_3^T - ... - \lambda_{|\mathcal{I}|} w_{|\mathcal{I}|} w_{|\mathcal{I}|}^T, \quad \lambda_1 = 1, \quad \lambda_2, \lambda_3, ...\lambda_{|\mathcal{I}|} < 0, \tag{6.1.4}$$

where $\{w_i\}_{i=1}^{|\mathcal{I}|}$ is an orthonormal basis of $\mathbb{R}^{|\mathcal{I}|}$. To ensure positivity of each entry of B, it suffices to choose $0 \geq \sum_{i=2}^{|\mathcal{I}|} \lambda_i \geq -1/|\mathcal{I}|$.

Remark 24. *If the coupling coefficient matrix is the identity $b_{\alpha\beta} = \delta_{\alpha\beta}$, the matrix B fails to be essentially negative. In this case, there will be no interaction between different groups. On the other hand, if the matrix $(b_{\alpha\beta})$ is one of the following matrices,*

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}, \cdots \tag{6.1.5}$$

the essential negativity condition is satisfied. In this case, every group couples with all the other groups.

The model (6.1.2) arises as the hydrodynamic realization of the following agent based dynamics which describes the collective motion of agents each of which adjusts its velocity to a *weighted average velocity* of its neighbors from different species:

$$\dot{x}^i_\alpha = v^i_\alpha, \quad x^i_\alpha \in \mathbb{T}^d, \;\; v^i_\alpha \in \mathbb{R}^d, \tag{6.1.6a}$$

$$\dot{v}^i_\alpha = \sum_{\beta \in \mathcal{I}} \frac{1}{N_\beta} \sum_{j=1}^{N_\beta} b_{\alpha\beta} \phi(x^j_\beta - x^i_\alpha)(v^j_\beta - v^i_\alpha), \quad i \in \{1, 2, ..., N_\alpha\}. \tag{6.1.6b}$$

Here x^i_α, v^i_α denote the position and velocity of the ith agent subject to initial data $(x^i_\alpha, v^i_\alpha)\big|_{t=0} = ((x^i_\alpha)_0, (v^i_\alpha)_0)$ in species α, respectively. The total number of agents in each species α is denoted by N_α. The explicit derivation is carried out in section 2.2.

For the single-species model (6.1.1), the long time behavior has been studied by several authors. It is shown in [112] that if the influence function decays slow enough, the strong solution exhibits *flocking* behavior, i.e., the velocity $u(x, t)$ aligns to a limiting velocity u_∞ as $t \to \infty$. This fact raises the problem whether the solutions to the models (6.1.2) stay smooth for all time. In one-dimension, the well-posedness theory is complete [35]. It is related to the critical threshold phenomenon, see, e.g., [54], [90], [88]. In dimension two, the critical threshold is partially characterized in the papers [112], [68].

In this paper, we extend the single species results to multi-species equation (6.1.2). Before stating the main theorems, we need to introduce some terminologies and assumptions. First, as a multispecies counterpart of flocking, we say that the system (6.1.2) converges to a flock on the Torus $\mathbb{T}^d$ if the velocity variation approaches zero, i.e.,

$$\lim_{t\to\infty} \delta V(t) = 0, \quad \delta V(t) := \sup_{x,y\in\cup_{\alpha\in\mathcal{I}} supp\rho_\alpha(t)} |u_\alpha(x,t) - u_\beta(y,t)|, \quad \forall \alpha, \beta \in \mathcal{I}, \quad (6.1.7)$$

which is also equivalent to saying that there exists a constant velocity u_∞ such that all velocities $u_\alpha(x)$ approach u_∞ as time tends to infinity. Next, we assume that all the initial densities $(\rho_\alpha)_0$ are bounded away from vacuum, i.e.,

$$\min_{\forall \alpha\in\mathcal{I}, x\in\mathbb{T}^d} (\rho_\alpha)_0(x) \geq q > 0. \quad (6.1.8)$$

The main theorems are as follows:

Theorem 13. *[Strong Solutions must flock] Consider classical solutions to the multi-species flocking dynamics* (6.1.2) *subject to global influence function ϕ on a torus $\mathbb{T}^d$, i.e.,*

$$\min_{x,y\in\mathbb{T}^d} \phi(x-y) \geq \phi_{min} > 0.$$

Assume that the coupling matrix $(b_{\alpha\beta})_{\alpha,\beta\in\mathcal{I}}$ is essentially negative (6.1.4)*. Then the classical solutions exhibit flocking behavior in the sense of* (6.1.7) *with limiting velocity*

$$u_\infty = \frac{\sum_{\alpha\in\mathcal{I}} \int (\rho_\alpha u_\alpha)_0 dx}{\sum_{\alpha\in\mathcal{I}} \int (\rho_\alpha)_0 dx}. \tag{6.1.9}$$

Remark 25. *The conservative form of the equation (6.1.2) implies that the mass $||\rho_\alpha||_1 =: M_\alpha, \ \ \forall \alpha \in \mathcal{I}$ and the total momentum $\sum_{\alpha\in\mathcal{I}} \int \rho_\alpha u_\alpha dx$ is conserved in time. As a result, the only possible u_∞ in the flocking (6.1.7) is u_∞ defined in (6.1.9).*

Similar to the case of one species, we developed well-posed-ness theorems for the multi-species flocking dynamics (6.1.2). We discuss one dimensional case and two-dimensional case separately.

One-dimensional Flocking. Recall from [35] that the quantity $e := \partial_x u + \phi * \rho$ characterizes the the critical threshold of the equation (6.1.2) in one dimension. If $e(x) \geq 0, \ \ \forall x \in \mathbb{R}$, then the solution exists globally. Otherwise there is finite time blow-up. In the multi-species case, we find the corresponding quantity

$$e_\alpha = \partial_x u_\alpha + \sum_{\beta\in\mathcal{I}} b_{\alpha\beta} \phi * \rho_\beta, \quad \forall \alpha \in \mathcal{I}.$$

The one-dimensional flocking result is summarized in the following theorem.

Theorem 14. *[One-dimensional Flocking] Consider the multi-species flocking dynamics (6.1.2) on $\mathbb{T}$ subject to intial data $\{((\rho_\alpha)_0, (u_\alpha)_0)\}_{\alpha\in\mathcal{I}} \in (L^1_+(\mathbb{T}), W^{1,\infty}(\mathbb{T};\mathbb{R}))^{|\mathcal{I}|}$. If the initial condition satisfies the threshold*

$$\partial_x u_\alpha(x, t=0) + \sum_{\beta\in\mathcal{I}} b_{\alpha\beta} \rho_\beta * \phi(x, t=0) \geq 0, \quad \forall x \in \mathbb{T}, \alpha \in \mathcal{I}, \tag{6.1.10}$$

then the multi-species flocking dynamics (6.1.2) admits classical solution for all time.

Two-dimensional Flocking. In order to state two-dimensional result, we introduce some terminology. First we define the energy deviation as follows:

$$\delta E := \sum_{\alpha,\beta\in\mathcal{I}} \iint \rho_\alpha(x)\rho_\beta(y)|u_\alpha(x)-u_\beta(y)|^2 dxdy. \tag{6.1.11}$$

Next we define the velocity gradient matrix as $(\nabla u_\alpha)^i_j = (\partial_j u^i_\alpha),\ \ i,j\in\{1,2\}$. Let $S_\alpha := \frac{1}{2}(\nabla u_\alpha + (\nabla u_\alpha)^T)$ denote the symmetric part of the matrix ∇u_α with eigenvalues $\lambda^\ell_\alpha(S_\alpha)$. Finally, similar to the paper [35], [68], we need to define a shifted eigenvalue

$$\Lambda^\ell_\alpha = \lambda^\ell_\alpha(S_\alpha) + \frac{1}{2}\sum_{\beta\in\mathcal{I}} b_{\alpha\beta}(\phi * \rho_\beta). \tag{6.1.12}$$

With these concepts and assumptions, we can state the well-posedness theorem on $\mathbb{T}^2$.

Theorem 15. *[Two-dimensional Flocking] Consider the multi-species flocking dynamics (6.1.2) on $\mathbb{T}^2$ subject to initial condition*

$$\{((\rho_\alpha)_0,(u_\alpha)_0)\}_{\alpha\in\mathcal{I}} \in (L^1_+(\mathbb{T}^2), W^{1,\infty}(\mathbb{T}^2,\mathbb{R}^2))^{|\mathcal{I}|}$$

and (6.1.8). Assume that the influence function is smooth $\phi(|\cdot|)\in C^1(\mathbb{T}^2)$ and the coupling matrix $(b_{\alpha\beta})_{\alpha,\beta\in\mathcal{I}}$ is essentially negative. There exists a constant $C_0 = C_0(\phi_{min}, ||\phi||_{C^1}, b_{\alpha\beta}, M_\alpha)$ such that if the initial velocity variation and energy deviation is bounded by C_0, i.e.,

$$\delta V(0) + \delta E(0) \le C_0, \tag{6.1.13}$$

and the shifted eigenvalues Λ^ℓ_α are bounded below

$$(\Lambda^\ell_\alpha)_0 \ge -\frac{1}{3}\sum_{\beta\in\mathcal{I}} b_{\alpha\beta}\phi_{min}M_\beta, \quad \forall\alpha\in\mathcal{I}, \ell\in\{1,2\},$$

then the multi-species flocking dynamics (6.1.2) *admits a classical solution* $(\rho_\alpha, u_\alpha) \in C(\mathbb{R}^+; L^\infty \cap L^1(\mathbb{T}^2)) \times C(\mathbb{R}^+; \dot{W}^{1,\infty}(\mathbb{T}^2; \mathbb{R}^2))$, $\forall \alpha \in \mathcal{I}$ *with large time hydrodynamics flocking behavior* $\delta V(t) \lesssim e^{-\kappa t}$, $\kappa := \frac{1}{2} \min_{\alpha\in\mathcal{I}} \sum_{\beta\in\mathcal{I}} b_{\alpha\beta} M_\beta \phi_{min}$.

6.1.2 Multi-species aggregation model

The next model we analyse is the multi-species aggregation equation modelling opinion dynamics. Before giving the full equation, we recall the single-species aggregation equation describing evolution of agent density ρ subject to strictly positive radially symmetric influence function ϕ:

$$\partial_t \rho + \nabla \cdot ((\phi x) * \rho \rho) = 0, \quad \rho(t = 0, x) = \rho_0(x), \quad x \in \mathbb{R}^d. \tag{6.1.14}$$

The equation (6.1.14) can be viewed as the macroscopic realization of the agent-based dynamics of N agents each of which has position x^i:

$$\dot{x}^i = \frac{1}{N} \sum_{i=1}^{N} \phi(x^i - x^j)(x^j - x^i). \tag{6.1.15}$$

In this paper, we extend the model (6.1.14) to the multi-species setting

$$(\rho_\alpha)_t + \sum_{\beta\in\mathcal{I}} b_{\alpha\beta} \nabla \cdot ((\phi x) * \rho_\beta)\rho_\alpha) = 0, \quad \alpha, \beta \in \mathcal{I} \tag{6.1.16}$$

$$\rho_\alpha(t = 0) = (\rho_\alpha)_0, \quad x \in \mathbb{R}^d. \tag{6.1.17}$$

Here ρ_α denotes the agent density in the group α. The parameters α, β take value in a finite index set $\mathcal{I}$. The strictly positive influence function ϕ is a radially symmetric decreasing function. Since it is radially symmetric, we slightly abuse the notation and write $\phi(\cdot) = \phi(|\cdot|)$. The positive coupling coefficients $b_{\alpha\beta} \geq 0$ are assumed to be symmetric:

$$b_{\alpha\beta} = b_{\beta\alpha}. \tag{6.1.18}$$

The single-species aggregation equation (6.1.14) is well-understood. We refer the interested readers to the following representative papers, [20], [74], [58], [36], [63], [103]. We summarize the results in our framework. If the influence function satisfies the following Osgood condition

$$\int_0^1 \frac{1}{\phi(s)s} ds = \infty, \tag{6.1.19}$$

then the solution exists for all time and the support of the solution shrink to zero as time approach infinity. Otherwise, if the integral is smaller than infinity, the solution blows up in finite time. One observe that, for a strictly positive C^2 radially symmetric influence function ϕ, the condition (6.1.19) is always satisfied.

There is also an increasing interest in two species aggregation model. We refer the interested readers to the papers [63], [59], [53], and [56]. In the paper [59], the authors used the optimal transport and Wasserstein gradient flow techniques to show existence and uniqueness of measure solutions to the two-species system. In the paper [53], the authors studied measure-valued solutions - the solutions which capture the dynamics after the blow-ups - to the one dimensional hyperbolic aggregation equation with two species and gave interesting numerical results. In [56], the authors categorize the possible steady states of the two-species system.

In this paper, we extends the results to the multi-species setting and give explicit sufficient condition to guarantee consensus under the assumption that the influence function is sufficiently smooth $\phi \in C^2$. Our main theorem is as follows:

Theorem 16. *Consider the equation (6.1.16) subject to strictly positive radially symmetric influence function $\phi(\cdot) = \phi(|\cdot|) \in C^2(\mathbb{R}^d)$ and compactly supported initial data $(\rho_\alpha)_0$. Further assume that the influence function ϕ is decreasing and $|\phi'|(s) \lesssim \frac{1}{s}$. If the coupling coefficient matrix $(b_{\alpha\beta})_{\alpha,\beta\in\mathcal{I}}$ satisfies the essential negativity condition (6.1.4), then the solutions $(\rho_\alpha)_{\alpha\in\mathcal{I}}$ exist globally and converge to single Dirac mass as time approaches infinity.*

The paper is organized as follows: In section 2, we will derive the macroscopic model (6.1.2) from the microscopic model (6.1.6); in section 3, we prove theorem 13; in section 4, we prove theorem 14; in section 5, we prove theorem 15; in section 6, we prove the Theorem 16.

6.2 Derivation of the mesoscopic and hydrodynamic models

Since the derivation is similar to the one carried out in chapter one, we omit the details here.

6.3 Hydrodynamic flocking

In this section, we prove Theorem 13. Applying the continuity equation (6.1.2a) in the momentum equation (6.1.2b), we could write the equation in the following form as long as $\rho_\alpha > 0$:

$$\partial_t \rho_\alpha + \nabla \cdot (u_\alpha \rho_\alpha) = 0; \tag{6.3.1a}$$

$$\partial_t u_\alpha + u_\alpha \cdot \nabla u_\alpha = \sum_{\beta\in\mathcal{I}} b_{\alpha\beta} \Big\{ \phi * (\rho_\beta u_\beta) - (\phi * \rho_\beta) u_\alpha \Big\}. \tag{6.3.1b}$$

subject to initial data $(\rho_\alpha, u_\alpha)(t=0,x) = ((\rho_\alpha)_0, (u_\alpha)_0), \;\; \alpha \in \mathcal{I}$. The equation (6.3.1) will be our main object of study later on.

First of all, since the coupling matrix $(b_{\alpha\beta})_{\alpha,\beta\in\mathcal{I}}$ is symmetric, the total momentum of the agents are preserved, i.e.,

$$\frac{d}{dt}\sum_{\alpha\in\mathcal{I}}\int_{\mathbb{T}^d}\rho_\alpha u_\alpha dx = 0. \tag{6.3.2}$$

Up to change of coordinates in the velocity space, it is reasonable to assume that the initial total momentum is zero, i.e.,

$$\sum_{\alpha\in\mathcal{I}}\int_{\mathbb{T}^d}(\rho_\alpha u_\alpha)_0 dx = 0. \tag{6.3.3}$$

A natural consequence of this assumption is that the final flocking velocity is zero, i.e., $u_\infty = 0$. Under this assumption, we define the initial total energy as follows:

$$E := \sum_{\alpha\in\mathcal{I}}\int \rho_\alpha |u_\alpha|^2 dx. \tag{6.3.4}$$

Note that the condition (6.1.13) implies the following estimate up to adjusting the constant C_0:

$$\delta V(0) + E(0) \le C_0(||\phi(|\cdot|)||_{C^1}, \phi_{min}, b_{\alpha\beta}, M_\alpha). \tag{6.3.5}$$

Before proving Theorem 13, we need some preparation.

Lemma 6.3.1. *The essential negativity of the matrix $(b_{\alpha\beta})_{\alpha,\beta\in\mathcal{I}}$ (6.1.4) is equivalent to the following condition,*

the matrix $J^T(b_{\alpha\beta}\mathbb{I}_d)J$ is negative definite, where J^T is the $d|\mathcal{I}| \times d(|\mathcal{I}|+1)$ matrix

$$\begin{pmatrix} \mathbb{I}_d & 0 & 0 & 0 & -\mathbb{I}_d \\ 0 & \mathbb{I}_d & 0 & 0 & -\mathbb{I}_d \\ 0 & 0 & ... & 0 & ... \\ 0 & 0 & 0 & \mathbb{I}_d & -\mathbb{I}_d \end{pmatrix}. \tag{6.3.6}$$

Proof. First we prove the result for the one dimension case, i.e. $d = 1$. To prove the essential negativity condition is necessary, note that the image of J is exactly in $S = \mathbf{1}_{|\mathcal{I}|}^{\perp}$. If $(b_{\alpha\beta})$ has positive eigenvalue which has $w \in \mathbf{1}_{|\mathcal{I}|}^{\perp}$, $w^T(b_{\alpha\beta})w \geq 0$, then condition (6.3.6) does not hold.

Next we prove that the essential negativity implies (6.3.6). If all eigenvalue of $(b_{\alpha\beta})_{\alpha,\beta\in\mathcal{I}}$ is negative, then the result is true. If not, first note that

$$(b_{\alpha\beta}) = O^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -c_2 & 0 & 0 \\ 0 & 0 & -... & 0 \\ 0 & 0 & 0 & -c_{|\mathcal{I}|} \end{pmatrix} O, \quad O = \begin{bmatrix} \frac{1}{\sqrt{|\mathcal{I}|}} & \frac{1}{\sqrt{|\mathcal{I}|}} & \cdots & \frac{1}{\sqrt{|\mathcal{I}|}} \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}.$$

Note that if $w \in \mathbf{1}_{|\mathcal{I}|}^{\perp}$, $Ow = (0, a_2, a_3, ..., a_{|\mathcal{I}|})$ and

$$\forall w \in \mathbf{1}^{\perp}, (Ow)^T diag(1, -c_2, -c_3, ..., -c_{|\mathcal{I}|})Ow = \sum_{j=2}^{|\mathcal{I}|}(-c_j)a_j^2 \leq 0.$$

For dimension d higher than one, the proof is as follows. The image of J contains the subset

$$(\underbrace{a_1, ..., a_1}_{d}, \underbrace{a_2, ..., a_2}_{d}, ..., \underbrace{a_{|\mathcal{I}|}, ..., a_{|\mathcal{I}|}}_{d}), \quad s.t. \quad \sum_{i=1}^{|\mathcal{I}|} a_i = 0.$$

If the matrix $(b_{\alpha\beta})$ has eigenfunction $w = (w_1, w_2, ..., w_{|\mathcal{I}|})$ corresponding to positive eigenvalue which lies in $\mathbf{1}_{|\mathcal{I}|}^{\perp}$, we can choose the vector

$$W = (\underbrace{w_1, ..., w_1}_{d}, \underbrace{w_2, ..., w_2}_{d}, ..., \underbrace{w_{|\mathcal{I}|}, ..., w_{|\mathcal{I}|}}_{d})) \in Image(J$$

so that $W^T(b_{\alpha\beta}\mathbf{1}_d)_{\alpha,\beta\in\mathcal{I}}W > 0$, which is a contradiction.

To prove the essential negativity condition implies (6.3.6), we could decompose the matrix $(b_{\alpha\beta}\mathbb{I}_d)_{\alpha,\beta\in\mathcal{I}}$ as follows:

$$(b_{\alpha\beta}\mathbb{I}_d) = O^T \begin{bmatrix} \mathbb{I}_d & 0 & 0 & 0 \\ 0 & -c_2\mathbb{I}_d & 0 & 0 \\ 0 & 0 & -... & 0 \\ 0 & 0 & 0 & -c_{|\mathcal{I}|}\mathbb{I}_d \end{bmatrix} O, \quad O = \begin{bmatrix} \frac{\mathbb{I}_d}{\sqrt{|\mathcal{I}|}} & \frac{\mathbb{I}_d}{\sqrt{|\mathcal{I}|}} & \cdots & \frac{\mathbb{I}_d}{\sqrt{|\mathcal{I}|}} \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}.$$

Note that if $W \in Image(J)$, $OW = (\vec{0}, \vec{a}_2, \vec{a}_3, ..., \vec{a}_{|\mathcal{I}|})$, $(OW)^T(b_{\alpha\beta}\mathbb{I}_d)OW = -\sum_2^{|\mathcal{I}|} c_j\vec{a}_j^2 \leq 0$. The result follows. □

We use a sequel of lemmas to prove Theorem 13. First we estimate the total energy of the solution.

Lemma 6.3.2. *Consider the strong solution to the equation (6.1.2). Assume that the initial total first momentum is zero, i.e.,*

$$\sum_{\alpha\in\mathcal{I}} \int (\rho_\alpha u_\alpha)_0(x)dx = 0. \tag{6.3.7}$$

If the matrix $(b_{\alpha\beta})_{\alpha,\beta\in\mathcal{I}}$ is essentially negative (6.1.4), then the following energy-enstrophy relation holds:

$$\frac{d}{dt}\bigg(\sum_{\alpha\in\mathcal{I}} \int \rho_\alpha|u_\alpha|^2dx\bigg) \leq -2\phi_{min}\eta\bigg(\sum_{\alpha\in\mathcal{I}} \int \rho_\alpha|u_\alpha|^2\bigg), \quad \eta := \min_{\alpha\in\mathcal{I}} \underbrace{\bigg(\sum_{\beta\in\mathcal{I}} b_{\alpha\beta}M_\beta\bigg)}_{=:\eta_\alpha} > 0. \tag{6.3.8}$$

Specifically, we have the following estimate

$$\sum_{\alpha\in\mathcal{I}}\int \rho_\alpha|u_\alpha|^2dx \le e^{-2\phi_{min}\eta t}\sum_{\alpha\in\mathcal{I}}\int(\rho_\alpha)_0|(u_\alpha)_0|^2dx. \tag{6.3.9}$$

Proof. First we estimate the time derivative of the total energy E using the equations (6.1.2)

$$\begin{aligned}
&\frac{d}{dt}\left(\sum_{\alpha\in\mathcal{I}}\int \rho_\alpha|u_\alpha|^2dx\right)\\
&\quad=2\sum_{\alpha,\beta\in\mathcal{I}} b_{\alpha\beta}\iint \rho_\alpha(x)u_\alpha(x)\cdot u_\beta(y)\phi(|x-y|)\rho_\beta(y)-\rho_\alpha(x)u_\alpha^2(x)\phi(|x-y|)\rho_\beta(y)dxdy\\
&\quad=-\sum_{\alpha,\beta\in\mathcal{I}} b_{\alpha\beta}\iint \phi(|x-y|)\rho_\alpha(x)\rho_\beta(y)(u_\alpha(x)-u_\beta(y))^2dxdy\\
&\quad\le-\phi_{min}\sum_{\alpha,\beta\in\mathcal{I}} b_{\alpha\beta}\iint |u_\alpha(x)-u_\beta(y)|^2\rho_\alpha(x)\rho_\beta(y)dxdy\\
&\quad\le-\phi_{min}\sum_{\alpha,\beta\in\mathcal{I}} b_{\alpha\beta}\left(M_\beta\int|u_\alpha(x)|^2\rho_\alpha(x)dx+M_\alpha\int|u_\beta(x)|^2\rho_\beta(x)dx\right)\\
&\qquad+2\phi_{min}\sum_{\alpha,\beta\in\mathcal{I}} b_{\alpha\beta}\int(u_\alpha\rho_\alpha)(x)dx\cdot\int(u_\beta\rho_\beta)(y)dy. \qquad (6.3.10)
\end{aligned}$$

If we can prove that the last term in (6.3.10) is negative, the result of the lemma follows. Note that we can rewrite the last term in (6.3.10) as follows:

$$2\phi_{min}\sum_{\alpha,\beta\in\mathcal{I}} b_{\alpha\beta}M_\alpha\overline{u}_\alpha\cdot M_\beta\overline{u}_\beta.$$

Since the total momentum is zero (6.3.7), we have the following relation:

$$M_{|\mathcal{I}|}\overline{u}_{|\mathcal{I}|}=-\sum_{\alpha\ne|\mathcal{I}|}M_\alpha\overline{u}_\alpha.$$

Now we see the following relation

$$
\begin{aligned}
&(M_1\overline{u}_1{}^T, M_2\overline{u}_2{}^T, ..., M_{|\mathcal{I}|}\overline{u}_{|\mathcal{I}|}{}^T) \\
&\qquad = (M_1\overline{u}_1{}^T, M_2\overline{u}_2{}^T, ..., M_{|\mathcal{I}|-1}\overline{u}_{|\mathcal{I}|-1}{}^T)\begin{pmatrix} \mathbb{I}_d & 0 & 0 & 0 & -\mathbb{I}_d \\ 0 & \mathbb{I}_d & 0 & 0 & -\mathbb{I}_d \\ 0 & 0 & ... & 0 & -\mathbb{I}_d \\ 0 & 0 & 0 & \mathbb{I}_d & -\mathbb{I}_d \end{pmatrix} \\
&\qquad = (M_1\overline{u}_1{}^T, M_2\overline{u}_2{}^T, ..., M_{|\mathcal{I}|-1}\overline{u}_{|\mathcal{I}|-1}{}^T)J^T.
\end{aligned} \tag{6.3.11}
$$

Therefore,

$$
\begin{aligned}
&\sum_{\alpha,\beta\in\mathcal{I}} b_{\alpha\beta} M_\alpha \overline{u}_\alpha \cdot M_\beta \overline{u}_\beta \\
&\qquad = (M_1\overline{u}_1{}^T, M_2\overline{u}_2{}^T, ..., M_{|\mathcal{I}|-1}\overline{u}_{|\mathcal{I}|-1}{}^T)J^T(b_{\alpha\beta}\mathbb{I}_d)_{\alpha,\beta}J\begin{pmatrix} M_1\overline{u}_1 \\ M_2\overline{u}_2 \\ ... \\ M_{|\mathcal{I}|-1}\overline{u}_{|\mathcal{I}|-1} \end{pmatrix}.
\end{aligned}
$$

By Lemma 6.3.1 and the assumption that $(b_{\alpha\beta})_{\alpha,\beta\in\mathcal{I}}$ is essentially negative, $J^T(b_{\alpha\beta}\mathbb{I}_d)J$ is negative definite. As a result, we have that the last term in (6.3.10) is negative.

□

Not only the L^2-based velocity variation is decreasing, the L^∞-based velocity variation is also decreasing. This is the main content of the following lemma.

Lemma 6.3.3. *Consider the strong solution to* (6.3.1) *subject to the condition* (6.3.6) *and non-vacuum condition* (6.1.8)*. Then the total variation* δV (6.1.7) *decays to zero*

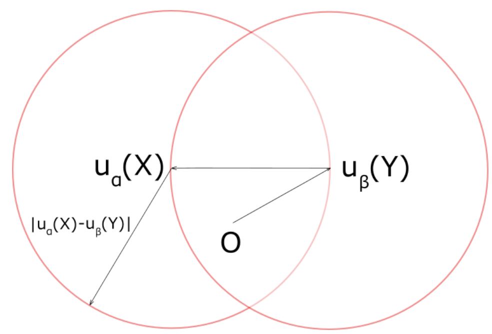

Figure 6.1: The geometric relation

as times goes to infinity. Moreover, the time evolution of δV *is bounded:*

$$\frac{d}{dt}\delta V^2 \leq -\phi_{min}\eta\delta V^2 + 2\sqrt{2}\delta V||\phi||_\infty \max_{\alpha\in\mathcal{I}} \eta_\alpha^{1/2}E^{1/2}, \tag{6.3.12}$$

where we recall η, η_α *from* (6.3.8) *and the total energy* E *from* (6.3.4)*.*

Proof. First we need to show that given condition (6.1.8), there is no vacuum formation for all finite time, i.e., $\min_{\alpha,x}(\rho_\alpha)_0(t,x) > 0,\ \ t \in [0,\infty)$. Consider the following trajectories:

$$\frac{dX}{dt} = u_\alpha(X(t),t);$$
$$\frac{dY}{dt} = u_\beta(Y(t),t).$$

We could calculate the time evolution of the density ρ_α along the partical trajectory:

$$(\partial_t + u_\alpha \cdot \nabla)\rho_\alpha = -\nabla \cdot u_\alpha\rho_\alpha. \tag{6.3.13}$$

Since the solution is assumed to be strong for all time, we have that $|\nabla \cdot u_\alpha| < \infty$. As a result, $\rho_\alpha > 0$ if initially the condition (6.1.8) is satisfied. Therefore, all points (t,x) are connected to $(0,x_0)$ through some trajectory $X(t)$ corresponds to $u_\alpha,\ \ \forall\alpha \in \mathcal{I}(?)$.

Applying the facts that $0 \leq b_{\alpha\beta} \leq 1$ and $|u_\alpha(X) - u_\beta(Y)| \leq \delta V$, we estimate the time evolution of $|u_\alpha(X) - u_\beta(Y)|^2$ as follows:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u_\alpha(X) - u_\beta(Y)|^2 \\
=&\langle u_\alpha(X) - u_\beta(Y), \sum_{\gamma\in\mathcal{I}} b_{\alpha\gamma}\{\phi * (\rho_\gamma u_\gamma)(X) - u_\alpha(X)(\phi * \rho_\gamma)(X)\} \\
&- \sum_{\gamma\in\mathcal{I}} b_{\beta\gamma}\{\phi * (\rho_\gamma u_\gamma)(Y) - u_\beta(Y)(\phi * \rho_\gamma(Y)\}\rangle_{\mathbb{T}^2} \\
=&\langle u_\alpha(X) - u_\beta(Y), \sum_{\gamma\in\mathcal{I}} b_{\alpha\gamma}\phi * (\rho_\gamma u_\gamma)(X) - \sum_{\gamma\in\mathcal{I}} b_{\beta\gamma}\phi * (\rho_\gamma u_\gamma)(Y)\rangle_{\mathbb{T}^2} \\
&+ \left\langle u_\alpha(X) - u_\beta(Y), -u_\alpha(X)\left(\sum_{\gamma\in\mathcal{I}} b_{\alpha\gamma}(\phi * \rho_\gamma)(X) - \eta\right)\right. \\
&\left. + u_\beta(Y)\left(\sum_{\gamma\in\mathcal{I}} b_{\beta\gamma}(\phi * \rho_\gamma(Y) - \eta\right)\right\rangle_{\mathbb{T}^2} - \eta\phi_{\min}|u_\alpha(X) - u_\beta(Y)|^2 \\
\leq& 2\delta V||\phi||_\infty \max_{\alpha,\beta\in\mathcal{I}}\left(\sum_{\gamma\in\mathcal{I}}(b_{\alpha\gamma} + b_{\beta\gamma})M_\gamma\right)^{1/2}\left(\sum_{\gamma\in\mathcal{I}}\int \rho_\gamma u_\gamma^2 dx\right)^{1/2} \\
&+ \left\langle u_\alpha(X) - u_\beta(Y), -u_\alpha(X)\underbrace{\left(\sum_{\gamma} b_{\alpha\gamma}(\phi * \rho_\gamma)(X) - \eta\right)}_{=:c}\right. \\
&\left. + u_\beta(Y)\underbrace{\left(\sum_{\gamma\in\mathcal{I}} b_{\beta\gamma}(\phi * \rho_\gamma)(Y) - \eta\right)}_{=:d}\right\rangle_{\mathbb{T}^2} \\
&- \eta\phi_{\min}|u_\alpha(X) - u_\beta(Y)|^2.
\end{aligned}
\tag{6.3.14}
$$

If the second line in (6.3.14) is negative, the result of Theorem 6.3.3 follows. To prove this, we first need to derive a geometric constraint. Since $\mathbb{T}^2$ is compact, we can assume that the δV is realized by $|u_\alpha(X) - u_\beta(Y)|$. First look at Figure 6.3. Since $|u_\alpha(X) - u_\beta(Y)|$ saturates the variation δV, all the possible velocities of the agents lie in the region

$$\Gamma := B(u_\alpha(X), |u_\alpha(X) - u_\beta(Y)|) \cap B(u_\beta(Y), |u_\alpha(X) - u_\beta(Y)|).$$

Since the total momentum is assumed to be zero (6.3.3), the total average velocity $u_\infty = \frac{\sum_{\alpha\in\mathcal{I}} \int \rho_\alpha u_\alpha dx}{\sum_{\alpha\in\mathcal{I}} \int \rho_\alpha dx}$ is zero and hence the origin must lie inside Γ. Observe from Figure 6.3 that

$$\langle u_\alpha(X) - u_\beta(Y)), u_\beta(Y)\rangle_{\mathbb{T}^2} \leq 0. \tag{6.3.15}$$

This is the geometric constraint we are after. Next we recall the definition of η (6.3.8), and one observe that $\eta \leq \sum_{\beta\in\mathcal{I}} b_{\alpha\beta}(\phi * \rho_\beta)$, which in turn implies $c, d \geq 0$ in (6.3.14). Without loss of generality, we assume that $0 \leq c \leq d$. Combining it with (6.3.15), we estimate the second line of (6.3.14)

$$\begin{aligned}
&\Big\langle u_\alpha(X) - u_\beta(Y), -cu_\alpha(X) + cu_\beta(Y) + (d-c)u_\beta(Y)\Big\rangle_{\mathbb{T}^2} \\
&= -c|u_\alpha(X) - u_\beta(Y)|^2 + (d-c)\langle u_\alpha(X) - u_\beta(Y)), u_\beta(Y)\rangle_{\mathbb{T}^2} \leq 0.
\end{aligned}$$

Therefore, the second line in (6.3.14) is negative. This completes the proof of Lemma 6.3.3. □

Proof of Theorem 13. Combining Lemma 6.3.1, Lemma 6.3.2, Lemma 6.3.3, the result follows. □

6.4 Global well-posedness of the multi-species system

6.4.1 Critical threshold in one-dimensional flocking dynamics

Proof of Theorem 14. Taking the spatial derivative in the second equation (6.1.2b) yields

$$(\partial_t + u_\alpha \partial_x)(\partial_x u_\alpha + \sum_{\beta \in \mathcal{I}} b_{\alpha\beta} \rho_\beta * \phi) = -\partial_x u_\alpha \left(\sum_{\beta \in \mathcal{I}} b_{\alpha\beta} \phi * \rho_\beta + \partial_x u_\alpha \right), \quad \forall \alpha \in \mathcal{I}. \tag{6.4.1}$$

One can see that $\partial_x u_\alpha + \sum_\beta b_{\alpha\beta} \rho_\beta * \phi \geq 0$ is invariant zone. If

$$\left(\partial_x u_\alpha + \sum_\beta b_{\alpha\beta} \rho_\beta * \phi \right) \Big|_{t=0} \geq 0,$$

then

$$\partial_x u_\alpha + \sum_{\beta \in \mathcal{I}} b_{\alpha\beta} \rho_\beta * \phi \geq 0, \quad \forall t \geq 0. \tag{6.4.2}$$

Since ϕ has upper bound $||\phi||_\infty < \infty$, we get lower bound for $\partial_x u_\alpha$

$$\partial_x u_\alpha(x,t) \geq -\sum_{\beta \in \mathcal{I}} b_{\alpha\beta} M_\beta ||\phi||_\infty, \quad \forall x \in \mathbb{T}, t \in \mathbb{R}_+, \alpha \in \mathcal{I}. \tag{6.4.3}$$

On the other hand we can see directly from the equation (6.4.1) that $\partial_x u_\alpha$ has an upper bound for all time. Combining this with the lower bound, we have that $||\partial_x u_\alpha||_\infty \leq C < \infty$ for all time. As a result, we have that the strong solutions exist for all time. $\square$

6.4.2 Critical threshold in two-dimensional flocking dynamics

Proof of Theorem 15. Here the key is to prove that the actual solution is strong given some condition. From (6.3.9) and (6.3.12), we have that

$$\delta V(t) \leq \delta V(0) e^{-\frac{1}{2}\phi_{min}\eta t} + 2\sqrt{2}\frac{||\phi||_\infty}{\phi_{min}\eta}(e^{-\frac{1}{2}\phi_{min}\eta t})C(b_{\alpha\beta}, M_\alpha)E^{1/2}(0). \tag{6.4.4}$$

By choosing $\delta V(0) + E(0)$ small enough (6.3.5), we can make the $\delta V(t)$ small for all time:

$$\delta V(t) \leq \widetilde{\epsilon}, \quad \forall t \in [0, \infty]. \tag{6.4.5}$$

The $\widetilde{\epsilon}$ can be made arbitrarily small. We will determine $\widetilde{\epsilon}$ later in the proof. The smallness of $\delta V(t)$ guarantees that the 2-dimensional effects from the alignment forcing are small and do not have drastic effect on our spectral analysis.

Following the single species case, we calculate the time evolution of the velocity gradient matrix $\nabla u_\alpha := (\partial_j u^i_\alpha)^2_{i,j=1}$,

$$\begin{aligned}(\nabla u_\alpha)_t + u_\alpha \cdot \nabla(\nabla u_\alpha) + (\nabla u_\alpha)^2 = -\sum_{\beta\in\mathcal{I}} b_{\alpha\beta}\phi * \rho_\beta \nabla u_\alpha + R_\alpha,\\ R^i_{\alpha,j} := \sum_{\beta\in\mathcal{I}}\int b_{\alpha\beta}\nabla\phi(x-y)(u^i_\beta(y) - u^i_\alpha(x))\rho_\beta(y)dy.\end{aligned} \tag{6.4.6}$$

Combining the smallness of δV (6.4.4), one could see that the R_α is small for all time:

$$|R^i_{\alpha,j}| \leq \sum_{\beta\in\mathcal{I}} b_{\alpha\beta}M_\beta||\nabla\phi||_\infty \delta V(t) \leq \sum_{\beta\in\mathcal{I}} b_{\alpha\beta}M_\beta||\nabla\phi||_\infty\widetilde{\epsilon}. \tag{6.4.7}$$

As is observed in the paper [68], it is crucial to track the dynamics of the eigenvalues of the symmetric part of the velocity gradient matrix ∇u_α. Therefore, we define the following quantities:

$$
\begin{aligned}
S_\alpha =& \frac{\nabla u_\alpha + (\nabla u_\alpha)^T}{2};\\
\nabla u_\alpha =& S_\alpha + \Omega_\alpha, \quad \Omega_\alpha := \begin{bmatrix} 0 & -\frac{1}{2}\omega_\alpha \\ \frac{1}{2}\omega_\alpha & 0 \end{bmatrix}, \quad \omega_\alpha = \partial_1 u_\alpha^2 - \partial_2 u_\alpha^1;\\
\Lambda_\alpha^\ell :=& \lambda_\alpha^\ell(S_\alpha) + \frac{1}{2}\sum_{\beta\in\mathcal{I}} b_{\alpha\beta}\phi * \rho_\beta, \quad \ell = 1, 2.
\end{aligned}
$$

We denote $\lambda_\alpha^\ell(S_\alpha)$ the eigenvalues of the matrix S_α. Recall the vorticity ω_α of the velocity field u_α. From equation (6.4.6), we can derive the time evolution of the matrix S_α:

$$
(S_\alpha)_t + u_\alpha \cdot \nabla S_\alpha + S_\alpha^2 - \frac{\omega_\alpha^2}{4}\mathbf{1}_2 = -\sum_{\beta\in\mathcal{I}} b_{\alpha\beta}\phi * \rho_\beta S_\alpha + R_{\alpha,sym}, \quad R_{\alpha,sym} = \frac{1}{2}(R_\alpha + R_\alpha^T). \tag{6.4.8}
$$

Conjugate both sides of the equation (6.4.8) by the eigenvectors $s_\alpha^\ell, \ell = 1, 2$ of S_α, we obtain the equations governing the eigenvalues of the matrix S_α:

$$
(\partial_t + u_\alpha \cdot \nabla)(\lambda_\alpha^\ell) + (\lambda_\alpha^\ell)^2 = \frac{\omega_\alpha^2}{4} - \sum_{\beta\in\mathcal{I}} b_{\alpha\beta}(\phi * \rho_\beta)\lambda_\alpha^\ell + \langle s_\alpha^\ell, R_{\alpha,sym} s_\alpha^\ell\rangle. \tag{6.4.9}
$$

The crucial observation is that the trace of the R_α has a special structure

$$
\begin{aligned}
TrR_\alpha =& \phi * \nabla \cdot \left(\sum_{\beta\in\mathcal{I}} \rho_\beta u_\beta\right) - u_\alpha \cdot \nabla\left(\sum_{\beta\in\mathcal{I}} \phi * \rho_\beta\right)\\
=& -\left(\sum_{\beta\in\mathcal{I}} \phi * \rho_\beta\right)_t - u_\alpha \cdot \nabla\left(\sum_{\beta\in\mathcal{I}} b_{\alpha\beta}\phi * \rho_\beta\right)\\
=& -(\partial_t + u_\alpha \cdot \nabla)\left(\sum_{\beta\in\mathcal{I}} b_{\alpha\beta}\phi * \rho_\beta\right).
\end{aligned} \tag{6.4.10}
$$

Combining (6.4.9) and (6.4.10) yields the evolution equations for Λ_α^ℓ:

$$
(\partial_t + u_\alpha \cdot \nabla)\Lambda_\alpha^\ell = \frac{\omega_\alpha^2}{4} - (\Lambda_\alpha^\ell)^2 + \frac{(\sum_\beta b_{\alpha\beta}\phi * \rho_\beta)^2}{4} + \langle s_\alpha^\ell, R_{\alpha,sym} s_\alpha^\ell\rangle - \frac{TrR_{\alpha,sym}}{2} \tag{6.4.11}
$$

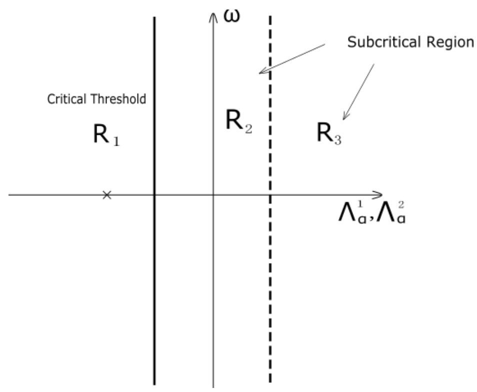

Figure 6.2: Critical Threshold

Note that $\omega_\alpha = -TrJ\nabla u_\alpha$, we derive from (6.4.6) the equation for vorticity

$$(\partial_t + u_\alpha \cdot \nabla)\omega_\alpha = -\Big(\Lambda_\alpha^1 + \Lambda_\alpha^2\Big)\omega_\alpha - TrJR_\alpha. \tag{6.4.12}$$

To study the coupling ODE's (6.4.11) and (6.4.12), we divide the $\mathbb{R}^2$ plane whose horizontal axis denotes the $\Lambda_\alpha^\ell, \ell = 1,2$ variables and whose vertical axis denotes the vorticity ω_α into three parts:

$$\begin{aligned} R_1 :=& \{(x,y)|x \le -\frac{\sum_\beta b_{\alpha\beta}\phi_{min}M_\beta}{3}\}; \quad R_2 := \{(x,y)||x| < \frac{\sum_\beta b_{\alpha\beta}\phi_{min}M_\beta}{3}\}; \\ R_3 :=& \{(x,y)|x \ge \frac{\sum_\beta b_{\alpha\beta}\phi_{min}M_\beta}{3}\}. \end{aligned}$$

Under the assumption of Theorem 15, we have that initially the point $\mathbf{P} := \big((\Lambda_\alpha^1, \omega_\alpha), (\Lambda_\alpha^2, \omega_\alpha)\big)$ is inside the region $(R_2 \cup R_3) \times (R_2 \cup R_3)$. Our goal is to show that the point $\mathbf{P}$ moves to the region $R_3 \times R_3$ and stay bounded for all finite time. We will separate the proof into three cases: case 1, $\mathbf{P} \in R_2 \times R_2$; case 2, $\mathbf{P} \in R_2 \times R_3 \cup R_3 \times R_2$; case 3, $\mathbf{P} \in R_3 \times R_3$.

Case 1: Here we need to show that the point $\mathbf{P}$ moves into region $R_2 \times R_3 \cup R_3 \times R_2$ or $R_3 \times R_3$.

In this case, $|\Lambda_\alpha^\ell| \leq \frac{1}{3}\sum_{\beta\in\mathcal{I}} b_{\alpha\beta}\phi_{min}M_\beta$, we have that

$$(\partial_t + u_\alpha \cdot \nabla)\Lambda_\alpha^\ell \geq \frac{5}{36}\left(\sum_{\beta\in\mathcal{I}} b_{\alpha\beta}\phi_{min}M_\beta\right)^2 - \tilde{\epsilon}. \tag{6.4.13}$$

If we choose the $\tilde{\epsilon}$ to be small enough, we have that

$$\frac{d}{dt}\Lambda_\alpha^\ell \geq \frac{1}{9}\left(\sum_{\beta\in\mathcal{I}} b_{\alpha\beta}\phi_{min}M_\beta\right)^2 > 0. \tag{6.4.14}$$

As a result, we have that

$$\Lambda_\alpha^\ell \geq (\Lambda_\alpha^\ell)_0, \tag{6.4.15}$$

and Λ_α^ℓ reaches R_3 in finite time.

Next we estimate the ω_α. Without loss of generality we assume $\omega_\alpha \geq 0$. Combining (6.4.12) and (6.4.15), we have that

$$(\partial_t + u_\alpha \cdot \nabla)\omega_\alpha \leq -((\Lambda_\alpha^1)_0 + (\Lambda_\alpha^2)_0)\omega_\alpha + \tilde{\epsilon}. \tag{6.4.16}$$

As a result, ω_α is bounded for all finite time. To conclude we have that the point $\mathbf{P}$ cannot blow up to infinity and it must reach $R_2 \times R_3, R_3 \times R_2$ or $R_3 \times R_3$ in finite time.

Case 2: Here we need to show that the point $\mathbf{P}$ goes into region $R_3 \times R_3$. Without loss of generality, we assume that $\Lambda_\alpha^1 \in R_2$ and $\Lambda_\alpha^2 \in R_3$. From the same argument as in the first case, we have that

$$(\partial_t + u_\alpha \cdot \nabla)\Lambda_\alpha^1 \geq \frac{\left(\sum_{\beta\in\mathcal{I}} b_{\alpha\beta}\phi_{min}M_\beta\right)^2}{9}. \tag{6.4.17}$$

Without loss of generality, we assume $\omega_\alpha \geq 0$. Combining (6.4.17) and equation (6.4.12), we have that

$$(\partial_t + u_\alpha \cdot \nabla)\omega_\alpha \leq -(\Lambda_\alpha^1)_0\omega_\alpha + \tilde{\epsilon}. \tag{6.4.18}$$

As a result, ω_α is bounded when $\mathbf{P}$ stays inside $R_2 \times R_3 \cup R_3 \times R_2$. Finally we need to show that Λ_α^2 doesn't blow up to infinity in this time period. Since ω_α is bounded, from the equation (6.4.11), we can see that Λ_α^2 could not goes to infinity. This completes the treatment of case 2.

Case 3: Here we need to show that the point $\mathbf{P}$ stays bounded. From the equation (6.4.12) and the fact that $\Lambda_\alpha^\ell, \ell = 1, 2$ are bounded below, we have that

$$(\partial_t + u_\alpha \cdot \nabla)\omega_\alpha \leq -\frac{2}{3}\sum_{\beta\in\mathcal{I}} b_{\alpha\beta}\phi_{min}M_\beta\omega_\alpha + \tilde{\epsilon}. \tag{6.4.19}$$

Hence ω_α is uniformly bounded in time. Now from the equation (6.4.11), we have that $\Lambda_\alpha^\ell,\ \ \ell = 1, 2$ are uniformly bounded above in time. This completes the proof of the main theorem. $\square$

6.5 Multi-species aggregation equation

In this section, we prove Theorem 16. For the sake of brevity, we omit the proof and refer the interested readers to the paper for further details.

6.6 Second order singular interaction

The equation for the multi-species equation might be the following:

$$(\rho_\alpha)_t + \partial_x(\rho_\alpha u_\alpha) = 0; \tag{6.6.1a}$$

$$(u_\alpha)_t + u_\alpha \partial_x u_\alpha = \sum_\beta b_{\alpha\beta}(\mathcal{L}(\rho_\beta u_\beta) - \mathcal{L}(\rho_\beta)u_\alpha); \tag{6.6.1b}$$

$$\mathcal{L}f = \int_{\mathbb{T}} \phi(|x-y|)(f(y) - f(x))dy. \tag{6.6.1c}$$

This equation has the remarkable quantity $G_\alpha := \partial_x u_\alpha + \sum_{\beta\in\mathcal{I}} b_{\alpha\beta}\mathcal{L}\rho_\beta$.

6.7 Conclusions

In this section, we introduced the multi-species concept into the hydrodynamic flocking models. We introduced the concept of essentially negative matrices and give explicit criterion to guarantee global well-posedness of the multi-species hydrodynamic flocking systems (6.1.2) and multi-species aggregation systems (6.1.16).

Chapter 7: Multi-species Patlak-Keller-Segel system

7.1 Overview

In this paper, we consider the multi-species parabolic-elliptic Patlak-Keller-Segel systems subject to cross chemical interraction which model social phenomena involving multiple bacteria species

$$(n_\alpha)_t+\nabla\cdot(\nabla c_\alpha n_\alpha)=\Delta n_\alpha,\quad \alpha\in\mathcal{I} \tag{7.1.1a}$$

$$-\Delta c_\alpha=\sum_{\beta\in\mathcal{I}} b_{\alpha\beta}n_\beta, \tag{7.1.1b}$$

$$n_\alpha(t=0)=(n_\alpha)_0,\quad x\in\mathbb{R}^2. \tag{7.1.1c}$$

Here n_α, c_α denote the bacteria and the chemical densities respectively. The parameters $\alpha,\beta\in\mathcal{I}$ indicate the bacteria and chemicals' species. The total number of species, which is denoted $|\mathcal{I}|$ throughout the paper, is assumed to be finite. The first equation in the system (7.1.1) describes the time evolution of the bacteria density n_α subject to chemical density distribution c_α and diffusion. The second equation governs the evolution of the chemical density c_α, which is determined by the collective effect of different species of bacteria n_β. The *chemical generation coefficients* $b_{\alpha\beta}$ represent the relative impact of the bacteria density n_β on the chemical distribution (c_α).

In the last few years, social interaction within biofilms - a special form of bacteria colonies - intrigued increasing interest among the biology and biophysics community, [45]. In a biofilm, billions of bacteria of different species live together and create hard-to-remove infections. Different cells in the biofilm specialize in various tasks, acquiring food, defending the colony and reserving genetic information included. Chemical signals and ion signals are generated to commute information within these bacteria colonies. The multi-species PKS model (7.1.1) serves as an attempt to understand the biofilm. Moreover, in the Chemotaxis experiment, the bacteria involved have large genetic variation. For example, E.coli only share 30% of their genes. The equation (7.1.1) also serves as a more accurate model taking into account the possible genetic variation appeared in the experiments.

We recall the large literature on the single species Patlak-Keller-Segel model (7.1.1) ($|\mathcal{I}| = 1$), referring the interested reader to the review [73] and the following works [22], [23], [75], [72], [101], [100], [43], [42], [72], [114], [31], [82], [44], [28], [27], [25], [30], [15], [17], [14], [7]. We summarize the essential results here. The equation (7.1.1) with one species is L^1 critical in dimension two. The maximum principle and divergence structure of the equation yield that the solutions n preserve positivity and L^1 norm $M := ||n(t)||_{L^1} = ||n_0||_{L^1}$. If the intitial data n_0 has subcritical mass $M < 8\pi$ and finite second moment, the unique global smooth solutions exist for all time, [28], [34], [52]. If $M > 8\pi$ and the second moment is finite, solutions blow up in finite time, [75], [99], [28]. If $M = 8\pi$, solution aggregates to a dirac mass as time tends to infinity, [27].

The multi-species PKS equation (7.1.1) has attracted increasing interest in the last decade. Its study originates in Wolansky's work [119]. Since then, a lot of research was carried out in the specific case of two interacting species, [40], [3], [83], [2], [57], [55]. Even in the two species case, the behavior of the multi-species PKS system differs from the single species one. Consider a two-species model with chemical generation coefficients $b_{11} = b_{22} = 0$ and $b_{12} = b_{21} = 1$. If one species has mass strictly less than 4π, the mass of the other species can be arbitrarily large without yielding finite time blow-up. In this paper, we rigorously quantify the subcritical mass condition for the multi-species PKS model.

Before stating the main theorems, we list the basic assumptions and terminologies. In the most part of this paper, we assume the chemical generation coefficient matrix $b_{\alpha\beta}$ to be symmetric

$$b_{\alpha\beta} = b_{\beta\alpha}. \tag{7.1.2}$$

Moreover, the following initial conditions are always satisfied

$$\sum_{\alpha\in\mathcal{I}}(n_\alpha)_0(1+|x|^2) \in L^1(\mathbb{R}^2); \quad (n_\alpha)_0 \log(n_\alpha)_0 \in L^1(\mathbb{R}^2), \quad \forall \alpha \in \mathcal{I}. \tag{7.1.3}$$

We introduce the function Q acting on subsets $\mathcal{J}$ of the index set $\mathcal{I}$, i.e., $\mathcal{J} \subset \mathcal{I}$,

$$Q_{\mathbf{B},\mathbf{M}}[\mathcal{J}] = \frac{\sum_{\alpha,\beta\in\mathcal{J}} b_{\alpha\beta} M_\alpha M_\beta}{\sum_{\alpha\in\mathcal{J}} M_\alpha}, \quad \mathbf{B} := (b_{\alpha\beta})_{\alpha,\beta\in\mathcal{I}}, \quad \mathbf{M} := (M_\alpha := ||n_\alpha||_1)_{\alpha\in\mathcal{I}}. \tag{7.1.4}$$

If $\mathcal{J} = \mathcal{I}$, $Q_{\mathbf{B},\mathbf{M}}[\mathcal{J}]$ has a simple matrix representation:

$$Q_{\mathbf{B},\mathbf{M}}[\mathcal{I}] = \frac{\langle \mathbf{B}\mathbf{M}, \mathbf{M}\rangle}{|\mathbf{M}|}. \tag{7.1.5}$$

where $\langle \cdot, \cdot \rangle$, $|\cdot|$ denote the vector inner product and the vector L^1 norm, respectively.

Same as in the single species case, there exists natural dissipated free energy for the system (7.1.1)

$$E[\mathbf{n}] = \sum_{\alpha\in\mathcal{I}} \int n_\alpha \log n_\alpha dx + \sum_{\alpha,\beta\in\mathcal{I}} \frac{b_{\alpha\beta}}{4\pi} \int n_\alpha(x) \log|x-y| n_\beta(y) dxdy, \quad \mathbf{n} := (n_\alpha)_{\alpha\in\mathcal{I}}. \tag{7.1.6}$$

The proof of the dissipation of (7.1.6) is postponed to the next section. We solve the equation (7.1.1) in the distribution sense with free energy dissipation constraint.

Definition 6 (Free energy solutions). *For any distributional solutions* $\mathbf{n}$ *to the equation* (7.1.1) *subject to initial data* $\mathbf{n}_0$*, they are the free energy solutions to* (7.1.1) *if the following free energy dissipation inequality holds on some maximal time interval* $[0, T_\star)$

$$E[\mathbf{n}(t)] + \sum_{\alpha\in\mathcal{I}} \int_0^t \int_{\mathbb{R}^2} n_\alpha |\nabla \log n_\alpha - \nabla c_\alpha|^2 dxds \le E[\mathbf{n}_0], \quad \forall t \in [0, T_\star). \tag{7.1.7}$$

The existence and blow-up theorems of (7.1.1) are stated as follows.

Theorem 17 (Global existence: subcritical mass). *Consider the equation* (7.1.1) *subject to initial conditions* (7.1.3). *If the chemical generation matrix* $\mathbf{B}$ *and the mass vector* $\mathbf{M}$ *satisfy that*

$$Q_{\mathbf{B},\mathbf{M}}[\mathcal{J}] < Q_{\mathbf{B},\mathbf{M}}[\mathcal{I}] < 8\pi \text{ for all } \emptyset \neq \mathcal{J} \subsetneqq \mathcal{I}, \tag{7.1.8}$$

then the free energy solutions to (7.1.1) *exist for all finite time.*

Theorem 18 (Blow-up: supercritical mass). *Consider the equation* (7.1.1) *subject to smooth initial data* $n_\alpha \in H^s, s \geq 2$ *with finite second moment. If* $Q_{\mathbf{B},\mathbf{M}}[\mathcal{I}] > 8\pi$*, the solution blows up in finite time.*

Remark 26. *It is not clear whether the subcritical mass conditions provided in Theorem 17 is sharp in general. However, if $|\mathcal{I}| = 1, 2$, we can show that the condition is sharp.*

Remark 27. *It can be shown that for the two species PKS with chemical generation coefficients $b_{11} = b_{22} = 0$, $b_{12} = b_{21} = 1$, the condition in Theorem 17 is equivalent to $({M_1}^{-1} + {M_2}^{-1})^{-1} < 4\pi$. If either M_1 or M_2 is strictly less than 4π, the inequality is always true. As a result, we prove the claim in the introduction.*

To formulate the smoothness and uniqueness theorems, we need further physical restriction on the free energy solutions. First, the physical solutions to equation (7.1.1) should satisfy the conservation of mass:

$$||n_\alpha(t)||_1 \equiv ||n_\alpha(0)||_1 = M_\alpha, \quad \forall \alpha \in \mathcal{I}. \tag{7.1.9a}$$

Moreover, by formal computation, which is postponed to the next section, we have that the total second moment grows linearly

$$\sum_{\alpha\in\mathcal{I}} V_\alpha(t) := \sum_{\alpha\in\mathcal{I}} n_\alpha(t)|x|^2 dx = \left(\sum_{\alpha} 4M_\alpha\right)\left(1 - \frac{Q_{\mathbf{B},\mathbf{M}}[\mathcal{I}]}{8\pi}\right)t + \sum_{\alpha\in\mathcal{I}} V_\alpha(0). \tag{7.1.9b}$$

Finally, since it is well-known that the boundedness of the entropy $S[n_\alpha] := \int n_\alpha \log n_\alpha$ is closely related to existence of smooth solutions, we consider free energy solutions subject to bounded entropy and free energy dissipation,

$$\begin{aligned}\mathcal{A}_t[\mathbf{n}] := \sup_{s\in[0,t]} &\left\{\sum_{\alpha\in\mathcal{I}} \int n_\alpha(x,s)\log^+ n_\alpha(x,s)dx\right\} \\ &+ \sum_{\alpha\in\mathcal{I}} \int_0^t \int n_\alpha(x,s)|\nabla \log n_\alpha(x,s) - \nabla c_\alpha(x,s)|^2 dxds < \infty, \quad \forall t < T_\star,\end{aligned} \tag{7.1.9c}$$

where $T_\star$ denotes the maximal existing time and $\log^+$ denotes the positive part of the function log. A similar quantity is defined in the paper [52]. We say that a free energy solution is *physically relevant* if it satisfies physical constraints (7.1.9a),(7.1.9b) and (7.1.9c).

Theorem 19 (Smoothnness of the free energy solutions)**.** *Consider the equation* (7.1.1) *subject to initial condition* (7.1.3)*. The physically relevant free energy solutions* $(n_\alpha)_{\alpha\in\mathcal{I}}$ *are smooth, i.e.,* $n_\alpha \in C^\infty((0,T_\star)\times\mathbb{R}^2)$, $\forall\alpha\in\mathcal{I}$*, where* $T_\star$ *is the maximal existence time. If the subcritical mass condition* (7.1.8) *is satisfied,* $T_\star=\infty$*. Furthermore, the equality holds in* (7.1.7)*.*

Theorem 20 (Uniqueness of the free energy solutions)**.** *Consider the equation* (7.1.1) *subject to initial condition* (7.1.3)*. There exists at most one physically relevant free energy solution.*

If the chemical generation matrix $\mathbf{B}$ is non-symmetric, the analysis applied to prove the theorems above faces significant difficulties. However, we can prove the global existence and uniqueness result for a special class of multi-species PKS systems which are called *essentially dissipative*. The definition is as follows:

Definition 7. *Define the sequences of subsets* $\mathcal{I}^{(0)} \subset \mathcal{I}^{(1)} \subset ... \subset \mathcal{I}^{(|\mathcal{I}|)}$ *of* $\mathcal{I}$ *as follows:*

$$\mathcal{I}^{(0)} := \{\alpha \in \mathcal{I} | b_{\alpha\beta} \leq 0, \quad \forall \beta \in \mathcal{I}\};$$

$$...;$$

$$\mathcal{I}^{(k)} := \{\alpha \in \mathcal{I} | b_{\alpha\beta} \leq 0, \quad \forall \beta \in \mathcal{I} \backslash \mathcal{I}^{(k-1)}\};$$

$$...;$$

$$\mathcal{I}^{(|\mathcal{I}|)} := \{\alpha \in \mathcal{I} | b_{\alpha\beta} \leq 0, \quad \forall \beta \in \mathcal{I} \backslash \mathcal{I}^{(|\mathcal{I}|-1)}\}.$$

If $\mathcal{I}^{(|\mathcal{I}|)} = \mathcal{I}$, *we called the matrix* $\mathbf{B}$ *essentially dissipative.*

The theorem corresponding to the multi-species PKS model (7.1.1) subject to essentially dissipative $\mathbf{B}$ is as follows.

Theorem 21. *Consider the multi-species PKS system* (7.1.1) *subject to initial condition* $(n_\alpha)_0 \in H^s$, $\forall \alpha \in \mathcal{I}, s \geq 2$. *Assume that the chemical generation matrix* $\mathbf{B}$ *is essentially dissipative. Then there exists a global solution to the equation* (7.1.1).

The paper is organized as follows: in section 2, we give preliminaries and the proof of Theorem 18; in section 3, we prove the existence of global free energy solutions with subcritical mass; in section 4, we prove the smoothness of the free energy solutions; in section 5, we prove the uniqueness of the free energy solutions; in the last section, we discuss the non-symmetric case.

7.2 Preliminary

Two quantities are crucial in the analysis of the long time behavior of the multi-species PKS dynamics (7.1.1), i.e., the free energy $E[\mathbf{n}]$ (7.1.6) and the second moment $\sum_\alpha V_\alpha$ (7.1.9b). In this section, we calculate the time evolution of these two quantities formally and give the proof of Theorem 18.

Same as in the single species case, the free energy $E[\mathbf{n}]$ (7.1.6) is formally dissipated under the equation (7.1.1).

Lemma 7.2.1. *Consider smooth solutions $\mathbf{n}$ to the equation (7.1.1) subject to initial data $\mathbf{n}_0$, the free energy $E[\mathbf{n}]$ (7.1.6) is deceasing and it satisfies the following free energy dissipation relation*

$$E[\mathbf{n}(t)] = E[\mathbf{n}_0] - \sum_{\alpha\in\mathcal{I}} \int_0^t \int n_\alpha |\nabla \log n_\alpha - \nabla c_\alpha|^2 dx ds =: E[\mathbf{n}_0] - \int_0^t \mathcal{D}[\mathbf{n}(s)] ds. \tag{7.2.1}$$

Proof. We apply the equation (7.1.1) and the symmetric condition (7.1.2) to calculate the time evolution of the free energy $E[\mathbf{n}]$

$$\begin{aligned}\frac{d}{dt}E[\mathbf{n}] =& \sum_\alpha \int (n_\alpha)_t \log n_\alpha dx - \sum_\alpha \int \frac{c_\alpha (n_\alpha)_t}{2} dx \\ &+ \sum_{\alpha,\beta} \frac{b_{\alpha\beta}}{4\pi} \iint (n_\beta)_t(x) \log|x-y| n_\alpha(y) dxdy \\ =& \sum_\alpha \int (n_\alpha)_t \log n_\alpha dx - \sum_\alpha \int \frac{c_\alpha (n_\alpha)_t}{2} dx \\ &+ \sum_{\alpha,\beta} \frac{b_{\alpha\beta}}{4\pi} \iint (n_\alpha)_t(x) \log|x-y| n_\beta(y) dxdy \\ =& \sum_\alpha \int (n_\alpha)_t (\log n_\alpha - c_\alpha) dx. \end{aligned} \tag{7.2.2}$$

Since the equation (7.1.1) can be rewritten as

$$(n_\alpha)_t = \nabla \cdot (n_\alpha (\nabla \log n_\alpha - \nabla c_\alpha)),$$

applying integration by parts on the time evolution of $E[\mathbf{n}]$ (7.2.1) yields

$$\frac{d}{dt}E[\mathbf{n}] = -\sum_\alpha \int n_\alpha |\nabla \log n_\alpha - \nabla c_\alpha|^2 dx \leq 0.$$

Now by integration in time, we obtain (7.2.1). $\square$

Next we give the time evolution of the second moment.

Lemma 7.2.2. *Consider smooth solutions* $\mathbf{n}$ *to the equation* (7.1.1) *subject to smooth initial data* $\mathbf{n}_0$*. The time evolution of the total second moment* $\sum_{\alpha\in\mathcal{I}} V_\alpha$ *satisfies the following equality*

$$\frac{d}{dt}\sum_\alpha V_\alpha = \left(\sum_\alpha 4M_\alpha\right)\left(1 - \frac{Q_{\mathbf{B},\mathbf{M}}[\mathcal{I}]}{8\pi}\right). \tag{7.2.3}$$

Proof. Applying the equation (7.1.1) and the symmetry condition (7.1.2), we calculate the time evolution of the total second moment as follows

$$\begin{aligned}
\frac{d}{dt}\sum_{\alpha\in\mathcal{I}} V_\alpha =& \sum_\alpha 4M_\alpha - \sum_{\alpha,\beta} b_{\alpha\beta}\frac{1}{2\pi}\iint \frac{2x\cdot(x-y)}{|x-y|^2} n_\beta(y)n_\alpha(x)dxdy \\
=& \sum_\alpha 4M_\alpha - \sum_{\alpha,\beta} b_{\alpha\beta}\frac{1}{4\pi}\iint \frac{2(x-y)\cdot(x-y)}{|x-y|^2} n_\beta(y)n_\alpha(x)dxdy \\
=& \left(\sum_\alpha 4M_\alpha\right)\left(1 - \frac{Q_{\mathbf{B},\mathbf{M}}[\mathcal{I}]}{8\pi}\right).
\end{aligned}$$

This completes the proof of the lemma. $\square$

Proof of Theorem 18. Suppose that the solution $\mathbf{n}$ is smooth for all time. By the assumption $Q_{\mathbf{B},\mathbf{M}}[\mathcal{I}] > 8\pi$, we have that the time evolution (7.2.3) is a strictly negative constant. As a result, the total second moment will decrease to zero at a finite time $T_\star$ while the L^1 norm of the solution $\sum_{\alpha\in\mathcal{I}} ||n_\alpha||_1$ is preserved. At time $T_\star$, the smoothness assumption of the solution will be contradicted. Hence the solution must lose H^s regularity before $T_\star$. $\square$

7.3 Global existence for subcritical data

7.3.1 A priori estimate on entropy

Same as in the classical Patlak-Keller-Segel equation analysis, the a priori estimate of the free energy (7.1.7) is combined with the logarithmic Hardy-Littlewood-Sobolev inequality to recover a uniform in time a priori bound on the entropy, which in turn yields existence of free energy solution for all time. In stead of the classical logarithmic Hardy-Littlewood-Sobolev inequality, we use the log-Hardy-Littlewood-Sobolev inequality for systems proven in the paper [107]. We recall some definition from the paper [107]. Let $\mathcal{I} := \{1, ..., N\}$ be the index set and $\mathcal{J}$ be a subset of $\mathcal{I}$, and define the following quantity:

$$\Lambda_{\mathcal{J}}(\mathbf{M}) := 8\pi \sum_{\alpha\in\mathcal{J}} M_\alpha - \sum_{\alpha,\beta\in\mathcal{J}} a_{\alpha\beta} M_\alpha M_\beta, \quad \mathbf{M} := (M_\alpha)_{\alpha\in\mathcal{I}}. \tag{7.3.1}$$

the function space

$$\begin{aligned}\Gamma_{\mathbf{M}}(\mathbb{R}^2) = \{(n_\alpha)_{\alpha\in\mathcal{I}} | n_\alpha \geq 0, \int_{\mathbb{R}^2} n_\alpha |\log n_\alpha| dx < \infty,\\ \int n_\alpha dx = M_\alpha, \int n_\alpha \log(1+|x|^2) dx < \infty, \forall \alpha \in \mathcal{I}\},\end{aligned} \tag{7.3.2}$$

and the functional

$$\Psi((n_\alpha)_{\alpha\in\mathcal{I}}) = \sum_{\alpha\in\mathcal{I}} \int n_\alpha \log n_\alpha dx + \frac{1}{4\pi} \sum_{\alpha,\beta\in\mathcal{I}} a_{\alpha\beta} \iint n_\alpha(x) \log|x-y| n_\beta(y) dx dy. \tag{7.3.3}$$

We summarize the inequality from the paper [107] in the following theorem (Theorem 4 in [107]):

Theorem 22. *Let $A = (a_{\alpha\beta})_{\alpha,\beta\in\mathcal{I}}$ be symmetric matrix with entry $a_{\alpha\beta} \geq 0$ and $\mathbf{M} = (M_1, M_2, ...M_N) \in \mathbb{R}^N_+$. Then $\Lambda_{\mathcal{I}}(\mathbf{M}) = 0$ and*

$$\begin{cases} \Lambda_{\mathcal{J}}(\mathbf{M}) \geq 0, \quad \forall \emptyset \neq \mathcal{J} \subset \mathcal{I}, \\ \text{if } \Lambda_{\mathcal{J}}(M) = 0 \text{ for some } \mathcal{J}, \text{ then } a_{\alpha\alpha} + \Lambda_{\mathcal{J}\setminus\{\alpha\}}(\mathbf{M}) > 0, \quad \forall \alpha \in \mathcal{J}, \end{cases} \tag{7.3.4}$$

are the necessary and sufficient conditions for the boundedness from below of the functional $\Psi((n_\alpha)_{\alpha=1}^N)$ on $\Gamma_{\mathbf{M}}(\mathbb{R}^2)$.

Recall set function Q (7.1.4). The above theorem yields the following proposition:

Proposition 13. *Consider the equation (7.1.1) subject to smooth initial data. If for all nonempty set $\mathcal{J} \subsetneqq \mathcal{I}$, we have that $Q_{\mathbf{B},\mathbf{M}}[\mathcal{J}] < Q_{\mathbf{B},\mathbf{M}}[\mathcal{I}] < 8\pi$, then the entropy $\sum_\alpha \int n_\alpha \log n_\alpha dx$ is bounded for all time.*

Proof. First we rewrite the free energy dissipation relation (7.2.1) as follows

$$\begin{aligned} E[\mathbf{n}] \geq E[\mathbf{n}_0] =& (1-\theta)\sum_{\alpha\in\mathcal{I}} \int n_\alpha \log n_\alpha + \theta\Big(\sum_{\alpha\in\mathcal{I}} \int n_\alpha \log n_\alpha dx \\ & + \frac{1}{4\pi} \sum_{\alpha,\beta\in\mathcal{I}} \frac{(b_{\alpha\beta})_+}{\theta} \iint n_\alpha(x) \log|x-y| n_\beta(y) dxdy\Big) \\ & - \sum_{\alpha,\beta} \frac{(b_{\alpha\beta})_-}{4\pi}(M_\alpha V_\beta + M_\beta V_\alpha). \end{aligned} \tag{7.3.5}$$

Define $a_{\alpha\beta} := b_{\alpha\beta}/\theta$, $0 < \theta < 1$.

In order to apply Theorem 22, we need to check two conditions, i.e., $\Lambda_I(\mathbf{M}) = 0$ and (7.3.4). By choosing θ properly, we make $\Lambda_I(\mathbf{M}) = 0$. Calculation yields

$$\Lambda_I(\mathbf{M}) = 0 \Leftrightarrow \theta = \frac{\sum_{\alpha,\beta\in\mathcal{I}} b_{\alpha\beta} M_\alpha M_\beta}{8\pi \sum_{\beta\in\mathcal{I}} M_\beta} = \frac{Q_{\mathbf{B},\mathbf{M}}[\mathcal{I}]}{8\pi}.$$

Note that the assumption $Q(I) < 8\pi$ guarantees that $\theta < 1$.

Next we check the second condition (7.3.4). Recalling the definition of θ and $Q_{\mathbf{B},\mathbf{M}}[\mathcal{J}]$, the following equivalent stronger conditions guarantees the existence of a lower bound of Ψ

$$\begin{aligned}
&Q_{\mathbf{B},\mathbf{M}}[\mathcal{I}] > Q_{\mathbf{B},\mathbf{M}}[\mathcal{J}], \quad \forall \emptyset \neq \mathcal{J} \subset \mathcal{I},\\
&\quad \Leftrightarrow \Lambda_{\mathcal{J}}(\mathbf{M}) = 8\pi \sum_{\beta\in\mathcal{J}} M_\beta - \frac{8\pi \sum_{\beta\in\mathcal{I}} M_\beta}{\sum_{\alpha,\beta\in\mathcal{I}} b_{\alpha\beta} M_\alpha M_\beta} \sum_{\alpha,\beta\in\mathcal{J}} b_{\alpha\beta} M_\alpha M_\beta > 0, \quad \forall \emptyset \neq \mathcal{J} \subsetneqq \mathcal{I},\\
&\quad \Leftrightarrow \Lambda_{\mathcal{J}}(\mathbf{M}) > 0, \quad \forall \emptyset \neq \mathcal{J} \subsetneqq \mathcal{I},
\end{aligned}$$

which yields the condition (7.3.4). Once the two conditions (7.3.4) are checked, combining Theorem 22 and the fact that $0 < \theta < 1$ yields that

$$\begin{aligned}
E[\mathbf{n}] \geq & E[\mathbf{n}_0] \geq (1-\theta) \sum_{\mathcal{I}} \int n_\alpha \log n_\alpha - \theta C,\\
\Rightarrow & \sum_{\alpha\in\mathcal{I}} \int n_\alpha \log n_\alpha dx \leq \frac{E[\mathbf{n}_0] + \theta C}{1-\theta} < \infty.
\end{aligned}$$

This completes the proof. □

7.3.2 Local existence and extension theorems

The following two propositions are the main local existence theorems.

Proposition 14. *(Criterion for Local Existence) Let $(n_\alpha^\epsilon)_{\alpha\in\mathcal{I}}$ be the solutions to the regularized multi-species PKS system (7.3.6) on $[0,T]$ subject to initial condition (7.1.3). If $\sum_\alpha S[n_\alpha^\epsilon(t)]$ is bounded from above uniformly in ϵ for $t \in [0,T]$, then the cluster points of $\{(n_\alpha^\epsilon)_{\alpha\in\mathcal{I}}\}_{\epsilon>0}$, in the $L_t^2 L_x^2$ strong topology, are non-negative free-energy solutions of the multi-species Patlak-Keller-Segel system (7.1.1) with initial data $(n_\alpha)_0$ on $[0,T]$.*

Proposition 15. *(Maximal Free-energy Solutions) Consider the multi-species PKS system (7.1.1) subject to initial condition (7.1.3). There exists a maximal existence time $T^* > 0$ of a free-energy solution to the system (7.1.1). Moreover, if $T^* < \infty$ then $\exists \alpha \in \mathcal{I}$, such that*

$$\lim_{t \to T^*} \int_{\mathbb{R}^2} n_\alpha \log n_\alpha dx = \infty.$$

Proof of proposition 14. The proof is divided into several steps.

First, we derive the existence of the approximate solution, which is obtained by appropriately truncate the singularity in the convolution kernel $K = (-\Delta)^{-1}$:

$$\begin{aligned} K^\epsilon(z) :=& K^1\left(\frac{|z|}{\epsilon}\right) - \frac{1}{2\pi}\epsilon; \\ K^1(|z|) :=& -\frac{1}{2\pi}\log|z|, \quad |z| \geq 4, \\ K^1(|z|) :=& 0, \quad |z| \leq 1. \end{aligned}$$

Since $||\nabla K^\epsilon||_\infty$ is bounded for any fixed positive ϵ, under appropriate conditions, we have the global solution in $L^2((0,T], H^1) \cap C((0,T], L^2)$ for the regularized version of the PKS system with cross chemical attraction

$$(n_\alpha^\epsilon)_t + \nabla \cdot (\nabla c_\alpha^\epsilon n_\alpha^\epsilon) = \Delta n_\alpha^\epsilon, \quad \alpha \in \mathcal{I} \tag{7.3.6a}$$

$$c_\alpha^\epsilon = K^\epsilon * \left(\sum_{\beta \in \mathcal{I}} b_{\alpha\beta} n_\beta\right), \tag{7.3.6b}$$

$$n_\alpha^\epsilon(t=0) = (n_\alpha)_0, \quad x \in \mathbb{R}^2. \tag{7.3.6c}$$

The following Gagliardo-Nirenberg-Sobolev inequality is useful in the sequel:

$$||u||_{L^p}^2 \leq C_{GNS}(p)||\nabla u||_{L^2}^{2-4/p}||u||_{L^2}^{4/p}, \forall u \in H^1, \forall p \in [2, \infty).$$

Step 1- A priori estimates on n^ϵ and c^ϵ. First define the second moment V_α by:

$$V_\alpha := \int_{\mathbb{R}^2} |x|^2 n_\alpha dx, i = 1, 2.$$

Similar to the calculation in [27], we have the following:

$$\frac{d}{dt}\big(\sum_\alpha V_\alpha\big) \le 4 \sum_\alpha M_\alpha, \tag{7.3.7}$$

from which $(1 + |x|^2)n_\alpha^\epsilon \in L^\infty((0, T), L^1)$ uniformly in ϵ.

Moreover, the following lemma can be proved without change:

Lemma 7.3.1. *For any g such that $(1 + |x|^2)g \in L^1_+(\mathbb{R}^2)$, we have $g \log^- g \in L^1(\mathbb{R}^2)$ and*

$$\int_{\mathbb{R}^2} g \log^- g dx \le \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 g(x)dx + \log(2\pi) \int_{\mathbb{R}^2} g(x)dx + \frac{1}{e}. \tag{7.3.8}$$

Proof. The proof of the lemma can be found in the paper [28] and [27]. We refer the interested readers to these papers for further details. □

The lemma yields that

$$\int |n_\alpha^\epsilon \log n_\alpha^\epsilon| dx \le \int n_\alpha^\epsilon(\log n_\alpha^\epsilon + |x|^2)dx + 2\log(2\pi)M_\alpha + \frac{2}{e}.$$

The entropy term verifies $n_\alpha^\epsilon \log n_\alpha^\epsilon \in L^\infty((0, T), L^1)$ uniformly in ϵ whenever $(1 + |x|^2)n_\alpha^\epsilon \in L^\infty((0, T), L^1)$ uniformly in ϵ and $S[\mathbf{n}^\epsilon(t)]$ is bounded from above uniformly in ϵ and $t \in (0, T)$.

Using that $n_\alpha^\epsilon \log n_\alpha^\epsilon \in L^\infty((0, T), L^1)$, $c_\alpha^\epsilon = K^\epsilon * n_\alpha^\epsilon$ and applying the Young's inequality, we deduce that

$$\begin{aligned}|c_\alpha(x)| \leq& \frac{1}{2\pi}\sum_{\beta\in\mathcal{I}}|b_{\alpha\beta}|\int_{|x-y|\leq 1}|K^\epsilon(|x-y|)n_\beta(y)|dy + \sum_{\beta\in\mathcal{I}}|b_{\alpha\beta}|\int_{|x-y|\geq 1}K^\epsilon(|x-y|)n_\beta(y)dy\\ \leq&\sum_{\beta\in\mathcal{I}}|b_{\alpha\beta}|M_\beta||K^\epsilon(\cdot)||_{L^1(B_1)} + \sum_{\beta\in\mathcal{I}}|b_{\alpha\beta}|\int(\log(1+|x|)+\log(1+|y|))n_\beta(y)dy\\ \lesssim&\sum_{\beta\in\mathcal{I}}|b_{\alpha\beta}|(M_\beta+V_\beta+M_\beta\log(1+|x|)).\end{aligned}$$

Combining it with the second moment $\sum_\alpha V_\alpha$ control (7.3.7), we have that $\int n_\alpha(t)c_\alpha(t)dx$ is bounded independent of ϵ on time interval $[0,T]$:

$$\int n_\alpha(t)c_\alpha(t)dx \lesssim \sum_{\beta\in\mathcal{I}}|b_{\alpha\beta}|(M_\beta+V_\beta)M_\alpha + \sum_{\beta\in\mathcal{I}}|b_{\alpha\beta}|M_\beta V_\alpha < \infty, \quad t\in[\delta,T].$$

Next we estimate $||\sqrt{n_\alpha^\epsilon}\nabla c_\alpha^\epsilon||_{L^2((\delta,T)\times\mathbb{R}^2)}$. For the sake of simplicity, we do the formal estimation here. Further details can be found in [28]. Before going into detailed calculations, we note that the total mass in the superlevel set can be controlled in terms of the bound on the entropy

$$\sum_{\alpha\in\mathcal{I}}\int_{n_\alpha\geq K}n_\alpha dx \leq \frac{1}{\log(K)}\sum_\alpha\int|n_\alpha\log n_\alpha|dx \leq \frac{C_{L\log L}}{\log(K)} =: \eta(K), \tag{7.3.9}$$

where $C_{L\log L}$ is the bound of the total entropy $\sum_\alpha n_\alpha\log n_\alpha \in L^\infty((\delta,T),L^1)$. The time evolution of the $S[\mathbf{n}]$ can be calculated using the equation (7.1.1) as follows

$$\frac{d}{dt}\sum_{\alpha\in\mathcal{I}}S[n_\alpha](t) = -4\sum_{\alpha\in\mathcal{I}}\int|\nabla\sqrt{n_\alpha}|^2(t)dx + \sum_{\alpha,\beta\in\mathcal{I}}b_{\alpha\beta}\int n_\alpha(t)n_\beta(t)dx. \tag{7.3.10}$$

The second term on the right hand side of (7.3.10) can be estimated as follows:

$$
\begin{aligned}
&\sum_{\alpha,\beta\in\mathcal{I}} b_{\alpha\beta}\int n_\alpha(t)n_\beta(t)dx \\
&\quad = \sup_{\alpha,\beta} b_{\alpha\beta}\sum_\alpha ||n_\alpha||_2 \sum_\beta ||n_\beta||_2 \\
&\quad \leq \sup_{\alpha,\beta} b_{\alpha\beta}\left(\sum_\alpha ||n_\alpha \mathbf{1}_{n_\alpha\geq K}||_2 + \sum_\alpha M_\alpha^{1/2}K^{1/2}\right)^2 \\
&\quad \lesssim \sup_{\alpha,\beta} b_{\alpha\beta}\left(\sum_\alpha ||n_\alpha \mathbf{1}_{n_\alpha\geq K}||_1^{1/4}||n_\alpha||_3^{3/4}\right)^2 + \sup_{\alpha,\beta} b_{\alpha\beta}|\mathcal{I}|K\sum_\alpha M_\alpha \\
&\quad \lesssim \eta(K)^{1/2}\sup_{\alpha,\beta}|b_{\alpha\beta}|\left(\sum_\alpha M_\alpha\right)^{1/2}\left(\sum_\alpha ||\nabla\sqrt{n_\alpha}||_2^2\right) + \sup_{\alpha,\beta} b_{\alpha\beta}|\mathcal{I}|K\sum_\alpha M_\alpha.
\end{aligned}
\tag{7.3.11}
$$

Combining (7.3.10) and (7.3.11), we have the following estimate on the time evolution of $\sum_\alpha S[n_\alpha]$:

$$
\begin{aligned}
&\frac{d}{dt}\sum_{\alpha\in\mathcal{I}} S[n_\alpha](t) \\
&\quad \leq -\left(4-\eta(K)^{1/2}\sup_{\alpha,\beta} b_{\alpha\beta}\left(\sum_\alpha M_\alpha\right)^{1/2} C_{L\log L}\right)||\nabla\sqrt{n_\alpha}||_2^2 + \sup_{\alpha,\beta} b_{\alpha\beta}|\mathcal{I}|K\sum_\alpha M_\alpha.
\end{aligned}
\tag{7.3.12}
$$

The factor $(4-\eta(K)^{1/2}\sup_{\alpha,\beta} b_{\alpha\beta}(\sum_\alpha M_\alpha)^{1/2}C_{L\log L})$ is negative for K large enough. Therefore, for large enough fixed K, we have the following estimate on $||\nabla\sqrt{n_\alpha}||_{L_t^2L_x^2}$:

$$
\sum_\alpha\int_\delta^T\int|\nabla\sqrt{n_\alpha(t)}|^2dxdt \leq \frac{S[\mathbf{n}(\delta)]-S[\mathbf{n}(T)]+C\sup_{\alpha,\beta} b_{\alpha\beta}m(I)K\sum_\alpha M_\alpha T}{(4-\eta(K)^{1/2}\sup_{\alpha,\beta} b_{\alpha\beta}(\sum_\alpha M_\alpha)^{1/2}C_{L\log L})}.
\tag{7.3.13}
$$

It follows that $\nabla\sqrt{n_\alpha}$ is bounded in $L^2([\delta,T]\times\mathbb{R}^2)$.

As a consequence of the $L^2([\delta,T]\times\mathbb{R}^2)$ estimate on $\nabla\sqrt{n_\alpha^\epsilon}$ and of the computation

$$\frac{d}{dt}S[\mathbf{n}^\epsilon(t)] = -4\sum_\alpha \int |\nabla\sqrt{n_\alpha^\epsilon}|^2 dx + 2\sum_{\alpha\in\mathcal{I}} \int n_\alpha^\epsilon(-\Delta c_\alpha^\epsilon)dx$$

together with the fact that $-\Delta c_\alpha^\epsilon \geq 0$, we have that the function $b_{\alpha\beta} n_\alpha^\epsilon \Delta c_\beta^\epsilon \in L^1([\delta, T]\times\mathbb{R}^2)$. Now we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\sum_\alpha \int n_\alpha^\epsilon c_\alpha^\epsilon dx =& \frac{1}{2}\sum_\alpha \int (n_\alpha^\epsilon)_t c_\alpha^\epsilon dx + \frac{1}{2}\sum_\alpha \int n_\alpha^\epsilon (c_\alpha^\epsilon)_t dx \\ =& \sum_\alpha \int n_\alpha^\epsilon \Delta c_\alpha^\epsilon + \sum_\alpha \int |\nabla c_\alpha^\epsilon|^2 n_\alpha^\epsilon dx,\end{aligned}$$

implies that

$$\sum_\alpha \int_\delta^T \int n_\alpha^\epsilon |\nabla c_\alpha^\epsilon|^2 dx dt \leq C < \infty.$$

We have that $\sqrt{n_\alpha^\epsilon}|\nabla c_\alpha^\epsilon| \in L^2([\delta, T]\times\mathbb{R}^2)$ where the bounded-ness of (**??**) was used. In this way, we obtained estimates on the two terms appearing in the dissipation of the free energy.

Step 2- Passing to the limit in $L_t^2([\delta, T]; L^2)$. Thanks to the equation (7.1.1), we obtain that the $\partial_t n_\alpha^\epsilon$ is in $L_t^2([\delta, T]; H_x^{-1})$ (Check!). Here we would like to use the Aubin-Lions compactness lemma. Therefore, we need to show the bounds on the sequence

$$||n_\alpha^\epsilon||_{L_t^2([\delta,T],L_x^2)} \leq C < \infty,$$

$$||n_\alpha^\epsilon||_{L_t^2([\delta,T],H^1)} \leq C < \infty, \quad \forall \alpha \in \mathcal{I},$$

where the constant C's are independent of ϵ.

$||n_\alpha^\epsilon||_{L_t^2([\delta,T];L_x^2)}$ **estimate:** Here we prove the following lemma

Lemma 7.3.2. *Consider the system (7.3.6) subject to initial condition* $(n_\alpha)_0 \in L^p$, *for all* $\alpha \in \mathcal{I}$ *and* $p \in (1, \infty)$. *The solution to (7.3.6) is bounded in* $L^p, p \in (1, \infty)$.

Proof. We do the L^p energy estimate formally and refer the interested readers to the paper [28] for detailed justifications. During the calculation, we will use the following natural implication of the GNS inequality

$$\begin{aligned}
&\int (f-K)_+^{p+1} dx \\
&\leq C_{GNS} \int (f-K)_+ dx \int |\nabla (f-K)_+^{p/2}|^2 dx \leq C_{GNS} \frac{||f \log f||_1}{\log K} \int |\nabla (f-K)_+^{p/2}|^2 dx \\
&=: C_{GNS} \eta(K) \int |\nabla (f-K)_+^{p/2}|^2 dx. \qquad (7.3.14)
\end{aligned}$$

Here note that if $||f \log f||_1$ is bounded,

$$\eta_\alpha(K) = \frac{||f \log f||_1}{\log K}$$

is small if one choose K large. Now we estimate the time evolution of $\sum_\alpha ||(n_\alpha - K)_+||_p^p$

with (7.3.14) as follows

$$
\begin{aligned}
&\frac{1}{p}\sum_{\alpha}\frac{d}{dt}\int (n_\alpha - K)_+^p dx \\
=& -4\frac{p-1}{p^2}\sum_{\alpha}\int |\nabla (n_\alpha - K)_+^{p/2}|^2 dx + \sum_{\alpha}\frac{p-1}{p}\int \nabla c_\alpha \cdot \nabla (n_\alpha - K)_+^p dx \\
& - \sum_{\alpha}\int \Delta c_\alpha n_\alpha (n_\alpha - K)_+^{p-1} dx \\
\leq& -4\frac{p-1}{p^2}\sum_{\alpha}\int |\nabla (n_\alpha - K)_+^{p/2}|^2 dx \\
& + \sup_{\alpha,\beta} b_{\alpha\beta}\left(\sum_{\alpha} ||(n_\alpha - K)_+||_1^p\right)^{1/(p+1)}\left(\sum_{\alpha} ||(n_\alpha - K)_+||_1^{1/p}\right)^{p/(p+1)} \times \\
& \times \left(\sum_{\alpha} ||\nabla (n_\alpha - K)_+^{p/2}||_2^2\right) \\
& + C(K, (b_{\alpha\beta})_{\alpha,\beta\in\mathcal{I}}, \mathbf{M})||(n_\alpha - K)_+||_p^p + C(K, (b_{\alpha\beta})_{\alpha,\beta\in\mathcal{I}}, \mathbf{M}) \\
\leq& \left(-\frac{4(p-1)}{p^2} + \eta(K)^{1/(p+1)}\sup b_{\alpha\beta}\left(\sum_{\alpha} M_\alpha^p\right)^{1/(p+1)}\right)\sum_{\alpha}\int |\nabla (n_\alpha - K)_+^{p/2}|^2 dx \\
& + C(K, (b_{\alpha\beta})_{\alpha,\beta\in\mathcal{I}}, \mathbf{M})||(n_\alpha - K)_+||_p^p + C(K, (b_{\alpha\beta})_{\alpha,\beta\in\mathcal{I}}, \mathbf{M}).
\end{aligned}
$$

If $\eta(K)$ is small enough, the leading order term is negative, and the estimate can be further simplified as follows:

$$
\frac{d}{dt}\sum_{\alpha} ||(n_\alpha - K)_+||_p^p \leq C_p(K, (b_{\alpha\beta})_{\alpha,\beta\in\mathcal{I}}, \mathbf{M})\sum_{\alpha} ||(n_\alpha - K)_+||_p^p + C_p(K, (b_{\alpha\beta})_{\alpha,\beta\in\mathcal{I}}, \mathbf{M}).
\tag{7.3.15}
$$

Now we see that for any finite time interval $[0, T]$, the L^p norm is bounded uniformly independent of ϵ. $\square$

Lemma 7.3.3. *Consider (7.1.1) subject to initial data $\mathbf{n}_0$ satisfying (7.1.3) and $Q_{\mathbf{B},\mathbf{M}}[\mathcal{J}] < Q_{\mathbf{B},\mathbf{M}}[\mathcal{I}] < 8\pi, \emptyset \neq \mathcal{J} \subsetneqq \mathcal{I}$. Then there exists $h_p(t) \in C(0, \infty)$ such that for almost any $t > 0$, $||n(\cdot, t)||_p \leq h_p(t)$.*

Proof. The proof is similar to the corresponding proof in [28] with some modifications. For the sake of completeness, we sketch the proof. First, we fix $t > 0$ and $1 < p < \infty$, and define

$$q(s) := 1 + (p-1)\frac{s}{t}. \tag{7.3.16}$$

Next, we define the following quantities:

$$\mathbb{F}_\alpha = \left(\int_{\mathbb{R}^2} (n_\alpha - K)_+^{q(s)} dx \right)^{1/q(s)}, \tag{7.3.17}$$

$$\mathbb{F} = \left(\sum_\alpha \mathbb{F}_\alpha^{q(s)} \right)^{1/q(s)}. \tag{7.3.18}$$

By taking the time derivative of $\mathbb{F}$, combining the log-Sobolev inequality we have the following relation

$$\begin{aligned}\mathbb{F}^{q-1}\frac{d}{dt}\mathbb{F} =& \frac{q'}{q^2}\sum_\alpha \int (n_\alpha - K)_+^{q(s)} \log\frac{(n_\alpha - K)_+^q}{\mathbb{F}^q} dx + \sum_\alpha \int (n_\alpha - K)_+^{q-1}\partial_t n_\alpha dx \\ \leq& \sum_\alpha \left(\frac{2\sigma q'}{q^2} - 4\frac{q-1}{q^2} + C((b_{\alpha\beta})_{\alpha,\beta\in\mathcal{I}})\frac{q-1}{q}K(q)\eta \right) ||\nabla(n_\alpha - K)_+^{q/2}||_2^2 \\ &- \sum_\alpha (2 + \log(2\pi\sigma))\frac{q'}{q^2}\int (n_\alpha - K)_+^q dx. \end{aligned} \tag{7.3.19}$$

Now by taking σ small, we have that

$$\frac{d}{dt}\mathbb{F} \leq -\frac{q'}{q^2}(2 + \log(2\pi\sigma))\mathbb{F}.$$

By integrating in time, noting that $\mathbb{F}(0)$ is finite, we have that $\mathbb{F} \leq h_p(t)$. This finishes the proof of the lemma. ∎

$||\nabla n^\epsilon||_{L_t^2([\delta,T];L_x^2)}$ **estimate:** In order to get the $L^2([\delta, T] \times \mathbb{R}^2)$ control of the ∇n^ϵ, we first calculate the time evolution of the L^2 norm of n_α^ϵ $\sum ||n_\alpha^\epsilon||_2^2$:

$$\begin{aligned}\frac{d}{dt}\sum_\alpha \int |n_\alpha^\epsilon|^2 dx =& 2\sum_\alpha \int n_\alpha^\epsilon(\Delta n_\alpha^\epsilon - \nabla\cdot(\nabla c_\alpha^\epsilon n_\alpha^\epsilon)dx \\ =& -\sum_\alpha \int |\nabla n_\alpha^\epsilon|^2 dx + 2\sum_\alpha \int \nabla n_\alpha^\epsilon \cdot \nabla c_\alpha^\epsilon n_\alpha^\epsilon dx.\end{aligned}$$

Integration in time yields that

$$\sum_\alpha ||n_\alpha^\epsilon(T)||_2^2 - \sum_\alpha ||n_\alpha^\epsilon(\delta)||_2^2 + 2\sum_\alpha ||\nabla n_\alpha^\epsilon||^2_{L_t^2([\delta,T];L_x^2)} \leq \sum_\alpha ||n_\alpha^\epsilon \nabla c_\alpha^\epsilon||^2_{L_t^2([\delta,T];L_x^2)}. \tag{7.3.20}$$

From (7.3.20) we see that since n_α^ϵ is bounded independent of ϵ in $L_t^\infty([\delta,T];L_x^2)$, if the quantity $||n_\alpha^\epsilon \nabla c_\alpha^\epsilon||_{L_t^2([0,\delta,T];L_x^2)}$ is bounded, ∇n_α^ϵ is bounded in $L_t^2([\delta,T];L^2)$ independent of ϵ. By the HLS inequality, we have that

$$||\nabla c_\alpha^\epsilon||_4 \leq C_{HLS} \sum_{\beta\in\mathcal{I}} |b_{\alpha\beta}|||n_\beta^\epsilon||_{4/3}.$$

As a result, we have that

$$||n_\alpha^\epsilon \nabla c_\alpha^\epsilon||_2 \leq ||n_\alpha^\epsilon||_4 ||\nabla c_\alpha^\epsilon||_4 \leq \sum_\beta C_{HLS}|b_{\alpha\beta}|||n_\alpha^\epsilon||_4||n_\beta^\epsilon||_{4/3}.$$

Since n_α^ϵ is bounded independent of ϵ in the space $L_t^\infty([\delta,T];L_x^p)$, $p \in (1,\infty)$, we have that $n^\epsilon \nabla c^\epsilon$ is bounded on $L_t^\infty([\delta,T];L_x^2)$. Combining this fact and the estimate (7.3.20), we have the bound on ∇n_α^ϵ.

Define the space V as $H^1 \cap \{f| \int f|x|^2dx < \infty\}$. A bounded set in the space V is precompact in L^2. Combining the second moment bound (7.3.7) and the H^1 bound of $(n_\alpha^\epsilon)_{\alpha\in\mathcal{I}}$, we have that the set $(n_\alpha^\epsilon)_{\epsilon\geq 0}, \quad \forall \alpha \in \mathcal{I}$ lies in a compact subspace of L^2 for almost every $t \in [\delta,T]$.

Step 3- Proof of the free energy estimates (7.1.7). Since the solution to the regularized multi-species PKS system has a decreasing free energy $E[\mathbf{n}^\epsilon]$, we have that

$$E[\mathbf{n}_0] \geq E[\mathbf{n}^\epsilon(\delta)] \geq E[\mathbf{n}^\epsilon(t)] + \sum_\alpha \int_\delta^t \int n_\alpha^\epsilon |\nabla \log n_\alpha^\epsilon - \sum_\beta b_{\alpha\beta}\nabla c_\beta^\epsilon|^2 dxdt, \quad \forall t\in[\delta,T]. \tag{7.3.21}$$

In order to show (7.1.7), we need to show proper convergence for each single term in (7.3.21). We first decompose the free energy dissipation term as follows:

$$\int_\delta^T \int n_\alpha^\epsilon |\nabla \log n_\alpha^\epsilon - \sum_\beta b_{\alpha\beta} \nabla c_\beta^\epsilon|^2 dxdt$$
$$= 4 \iint_{(\delta,T)\times\mathbb{R}^2} |\nabla \sqrt{n_\alpha^\epsilon}|^2 dxdt + \iint_{[\delta,T]\times\mathbb{R}^2} n_\alpha^\epsilon |\nabla c_\alpha^\epsilon|^2 dxdt - 2\sum_{\alpha,\beta} \iint_{(\delta,T)\times\mathbb{R}^2} b_{\alpha\beta} n_\alpha^\epsilon n_\beta^\epsilon dxdt. \tag{7.3.22}$$

By the convexity of $f \to \int_{\mathbb{R}^2} |\nabla\sqrt{f}|^2 dx$, weak semi-continuity and the strong convergence of n_α^ϵ in $L_t^2([\delta, T]; L_x^2)$, we have that the first two terms in (7.3.22) satisfies the following inequalities

$$\iint_{(\delta,T)\times\mathbb{R}^2} |\nabla\sqrt{n_\alpha}|^2 dxdt \le \liminf_{\epsilon\to 0_+} \iint_{(\delta,T)\times\mathbb{R}^2} |\nabla \sqrt{n_\alpha^\epsilon}|^2 dxdt \tag{7.3.23}$$

$$\iint_{(\delta,T)\times\mathbb{R}^2} n_\alpha |\nabla c_\alpha|^2 dxdt = \lim_{\epsilon\to 0_+} \iint_{(\delta,T)\times\mathbb{R}^2} n_\alpha^\epsilon |\nabla c_\alpha^\epsilon|^2 dxdt. \tag{7.3.24}$$

Since the $(n_\alpha^\epsilon)_{\epsilon>0}$ converges strongly in the $L^2([\delta, T]\times\mathbb{R}^2)$ space. The last term on the right hand side of (7.3.22) converges. Moreover, it can be checked that $S[n_\alpha^\epsilon(t)] \to S[n_\alpha(t)]$ for almost every $t \in [\delta, T]$. The argument is similar to the one used in [28] Lemma 4.6. As a result, combining these facts and (7.3.21), (7.3.22), (7.3.23) and (7.3.24) yields that

$$E[\mathbf{n}_0] \ge E[\mathbf{n}(\delta)] \ge E[\mathbf{n}(t)] + \sum_\alpha \int_\delta^t \int n_\alpha |\nabla \log n_\alpha - \nabla c_\alpha|^2 dxds.$$

Now by the monotone convergence theorem, we have proven (7.1.7).

□

Proof of proposition 15. We prove by contradiction. Assume that at time $T_\star < \infty$, the entropy $\sum_\alpha S[n_\alpha^\epsilon(T_\star)]$ is uniformly bounded with respect to ϵ.

First, from the equation (7.3.6), we directly calculate the time evolution of the entropy:

$$\begin{aligned}&\frac{d}{dt}\sum_\alpha\int n_\alpha^\epsilon\log n_\alpha^\epsilon\\&=-\sum_\alpha 4\int\nabla\sqrt{n^\epsilon}-\sum_{\alpha,\beta}b_{\alpha\beta}\int_{n_\alpha^\epsilon\leq K}n_\alpha^\epsilon\Delta(K^\epsilon*n_\beta^\epsilon)-\sum_{\alpha,\beta}b_{\alpha\beta}\int_{n_\alpha^\epsilon>K}n_\alpha^\epsilon\Delta(K^\epsilon*n_\beta^\epsilon)\\&=:-\sum_\alpha 4\int\nabla\sqrt{n^\epsilon}+I+II.\end{aligned}\tag{7.3.25}$$

The term I in (7.3.25) can be estimated as follows:

$$I\leq\sum_{\alpha,\beta}K|b_{\alpha\beta}|||\Delta K^\epsilon||_1M_\beta.\tag{7.3.26}$$

Recall that $||\Delta K^\epsilon||_1$ is bounded independent of ϵ, so term I is bounded independent of ϵ. For the second term II in (7.3.25), we estimated it using the Hölder's inequality, Gagliardo-Nirenberg-Sobolev inequality and Young's inequality as follows:

$$\begin{aligned}II&\leq\sum_{\alpha,\beta}|b_{\alpha\beta}|\left(\left(\int_{n_\alpha^\epsilon\geq K}n_\alpha^\epsilon dx\right)^{1/2}||n_\alpha^\epsilon||_3^{3/2}+||\Delta K^\epsilon||_1^2\left(M_\beta K+\int_{n_\beta^\epsilon\geq K}(n_\beta^\epsilon)^2\right)\right)\\&\leq\sum_{\alpha,\beta}|b_{\alpha\beta}|C_{GNS}(1+||\Delta K^\epsilon||_1^2)\frac{S^+(n_\alpha)}{\log K}M_\alpha^{1/2}||\nabla\sqrt{n_\alpha^\epsilon}||_2^2+\sum_{\alpha,\beta}|b_{\alpha\beta}|||\Delta K^\epsilon||_1^2M_\alpha K.\end{aligned}\tag{7.3.27}$$

Here S^+ denote the positive part of the entropy, i.e., $S^+[f]=\int f\log^+fdx$. Combining the estimates (7.3.25), (7.3.26) with (7.3.27), we end up with

$$\begin{aligned}\frac{d}{dt}\sum_\alpha S[n_\alpha^\epsilon]\leq&\underbrace{\sum_\alpha\left(-4+\sum_\beta|b_{\alpha\beta}|C_{GNS}(1+||\Delta K^\epsilon||_1^2)\frac{S^+(n_\alpha^\epsilon)}{\log K}M_\alpha^{1/2}\right)}_{=:A(t)}||\nabla\sqrt{n_\alpha^\epsilon}||_2^2\\&+\sum_{\alpha,\beta}(1+|b_{\alpha\beta}|)(1+||\Delta K^\epsilon||_1^2)M_\alpha K.\end{aligned}\tag{7.3.28}$$

Since the negative part of the entropy is bounded (7.3.8) and the second moment control (7.3.7), we have that $A(t)$ can be estimated as follows:

$$\begin{aligned} A(t) \leq & -4|\mathcal{I}| + \frac{1}{\log K}\sum_{\beta}|b_{\alpha\beta}|(1+||\Delta K^{\epsilon}||_1^2)\sum_{\alpha}\Big(S[n_\alpha^\epsilon(t)] + V[n_\alpha^\epsilon(T_\star)] \\ & + 4M_\alpha(t-T_\star) + 2\log(2\pi)M_\alpha + 2e^{-1}\Big) \end{aligned} \tag{7.3.29}$$

Since the entropy $\sum_\alpha S[n_\alpha^\epsilon]$ is uniformly bounded independent of ϵ at time $T_\star$. We could take the K large such that $A(t) \leq -2$ at time $T_\star$. By continuity, there is a small time τ_ϵ such that for $\forall t \in [T_\star, T_\star + \tau_\epsilon)$,

$$\begin{aligned} &\sum_\alpha S[n_\alpha^\epsilon(t)] \\ &\quad \leq \sum_\alpha S[n_\alpha^\epsilon(T_\star)] + (t-T_\star)\sum_{\alpha,\beta}(1+|b_{\alpha\beta}|)(1+||\Delta K^\epsilon||_1^2)M_\alpha K, \ \ \forall t \in [T_\star, T_\star+\tau_\epsilon]. \end{aligned} \tag{7.3.30}$$

But then we can pick τ independent of ϵ such that

$$A(t) \lesssim -4|\mathcal{I}| + \frac{1}{\log K}\Big(\sum_\alpha S[n_\alpha^\epsilon](T_\star) + K\tau + 1\Big) \leq 0.$$

The solution τ to the above inequality is independent of the choice of ϵ, and $[T_\star, T_\star + \tau) \subset [T_\star, T_\star + \tau_\epsilon)$ for any ϵ. Therefore, by Proposition 14, we can extend the free energy solution pass the $T_\star$, contradicting the maximality of $T_\star$. As a result, we have completed the proof of the proposition. □

7.4 Smoothness of the free energy solutions

For the sake of brevity, we skip the proof of Theorem 19 and refer the interested readers to the paper [66].

7.5 Uniqueness of the free energy solutions

After proving the smoothness theorem for the system (7.1.1), we are ready to prove the uniqueness of the free energy solution $(n_\alpha)_{\alpha\in\mathcal{I}}$. We will organize the proof into several lemmas.

Lemma 7.5.1. *Consider the free energy solution* $\mathbf{n}$ *to the system* (7.1.1)*. The following holds*

$$\lim_{t\to 0^+} t^{1/4}||n_\alpha(t)||_{4/3} = 0. \tag{7.5.1}$$

Proof. The proof is similar to the one carried out in the paper [52]. For the sake of completeness, we give the proof below. Standard L^2 energy estimate yields

$$\frac{d}{dt}\sum_\alpha ||n_\alpha||_2^2 + 2\sum_\alpha ||\nabla n_\alpha||_2^2 = \sum_{\alpha,\beta\in\mathcal{I}} b_{\alpha\beta}\int n_\alpha^2 n_\beta dx. \tag{7.5.2}$$

Applying the Nash inequality, we estimate the right hand side as follows

$$\begin{aligned}
&\frac{d}{dt}\sum_\alpha ||n_\alpha||_2^2\\
&\leq -\sum_\alpha ||\nabla n_\alpha||_2^2 + \sum_{\alpha,\beta}|b_{\alpha\beta}|||n_\beta||_3^3\\
&\lesssim -\sum_\alpha ||\nabla n_\alpha||_2^2 + \sum_{\alpha,\beta}|b_{\alpha\beta}|\left(K^2M_\beta + \frac{\sup_{t\in[0,T]} S[\mathbf{n}(t)]^{1/3}}{\log^{1/3}K}||n_\beta||_1^{2/3}||\nabla n_\beta||_2^2\right)\\
&\lesssim -\frac{\sum_\alpha ||n_\alpha||_2^4}{C_{GNS}\sup_\alpha M_\alpha^2} + \sum_{\alpha,\beta}|b_{\alpha\beta}|K^2M_\beta,
\end{aligned} \tag{7.5.3}$$

where K is a big number. Now by solving a super equation

$$\frac{d}{dt}f = -\frac{f^2}{C_{GNS}\sup_\alpha M_\alpha^2} + C'K^2\sum_{\alpha,\beta}|b_{\alpha\beta}|M_\beta, \quad f(0) = \infty,$$

where C' is chosen large compared to the implicit constants appeared in (7.5.3). We have that

$$\sum_{\alpha} ||n_\alpha(t)||_2^2 t \leq C < \infty, \tag{7.5.4}$$

for $\forall t \in [0,T]$.

By the Hölder's inequality and the boundedness of the entropy, we have that

$$\begin{aligned}\int n_\alpha^{4/3} dx \leq & \left(\int n_\alpha(\log^+ n_\alpha + 2)dx\right)^{2/3} \left(\int n_\alpha^2 (2+\log^+ n_\alpha)^{-2} dx\right)^{1/3} \\ \leq & C\left(\int n_\alpha^2(2+\log^+ n_\alpha)^{-2}dx\right)^{1/3}.\end{aligned} \tag{7.5.5}$$

We can separate the domain into $n_\alpha \leq R$ and $n_\alpha \geq R$ case and use the increasing property of the function $s/(2+\log^+ s)^2$, the conservation of mass and (7.5.4) to get

$$\begin{aligned} t\int n_\alpha^2(2+\log^+ n_\alpha)^{-2} \leq & t\int_{n_\alpha \leq R} n_\alpha^2(2+\log^+ n_\alpha)^{-2} + \int t \int_{n_\alpha < R} n_\alpha^2 (2+\log^+ n_\alpha)^{-2} \\ \leq & t\frac{MR}{(2+\log^+ R)^2} + \frac{K}{(2+\log^+ R)^2}. \end{aligned}$$

Now set $R := 1/t$, we have

$$t\int n_\alpha^2(2+\log^+ n_\alpha)^{-2} \leq \frac{M+K}{(2+\log^+ 1/t)^2} \to 0, t \to 0_+. \tag{7.5.6}$$

Combining this with (7.5.5) yields the result. □

Now consider the equation in the mild form. Since we have smoothness of the free energy solution, we have that the two formulation are equivalent. Suppose that $(n_{\alpha,1})_{\alpha\in\mathcal{I}}, (n_{\alpha,2})_{\alpha\in\mathcal{I}}$ are two solutions subject to the same initial data $(n_\alpha)_0$, $\alpha \in \mathcal{I}$, their difference satisfies:

$$\begin{aligned} n_{\alpha,2} - n_{\alpha,1} = & \int_0^t e^{(t-s)\Delta}\nabla((\nabla c_{\alpha,2}(s) - \nabla c_{\alpha,1}(s))n_{\alpha,2}(s))ds \\ & + \int_0^t e^{(t-s)\Delta}\nabla(\nabla c_{\alpha,1}(s)(n_{\alpha,2}(s) - n_{\alpha,1}(s)))ds. \end{aligned}$$

Define the following quantities:

$$Z_{\alpha,\ell}(t) := \sup_{0<s\leq t} s^{1/4}||n_{\alpha,\ell}(s)||_{4/3}, \quad \ell = \{1,2\}; \tag{7.5.7}$$

$$\Delta_\alpha(t) := \sup_{0<s\leq t} s^{1/4}||n_{\alpha,2}(s) - n_{\alpha,1}(s)||_{4/3}. \tag{7.5.8}$$

From the estimate (7.5.1), we have that $\lim_{t\to 0_+} Z_{\alpha,\ell}(t) = 0$. The $\Delta_\alpha(t)$ can be further decomposed as follows:

$$\begin{aligned}\Delta_\alpha(T) \leq & \sup_{0\leq t\leq T} t^{1/4}\left|\left|\int_0^t e^{(t-s)\Delta}\nabla((\nabla c_{\alpha,2}(s) - \nabla c_{\alpha,1}(s))n_{\alpha,2}(s))ds\right|\right|_{4/3} \\ & + \sup_{0\leq t\leq T} t^{1/4}\left|\left|\int_0^t e^{(t-s)\Delta}\nabla(\nabla c_{\alpha,1}(s)(n_{\alpha,2}(s) - n_{\alpha,1}(s)))ds\right|\right|_{4/3} \\ =: & \sup_{0\leq t\leq T} J_{\alpha,1}(t) + \sup_{0\leq t\leq T} J_{\alpha,2}(t). \end{aligned} \tag{7.5.9}$$

Now we can estimate the $J_{2,\alpha}$ term in (7.5.9) using the Hölder inequality, Minkowski integral inequality and heat semigroup estimate

$$\begin{aligned} J_{2,\alpha} \leq & t^{1/4}\int_0^t \frac{C}{(t-s)^{3/4}}||\nabla c_{\alpha,1}||_{4/3}||n_{\alpha,2} - n_{\alpha,1}||_{4/3}ds \\ \leq & \int_0^t C\frac{t^{1/4}}{s^{1/2}(t-s)^{3/4}}ds\sum_{\beta\in\mathcal{I}}|b_{\alpha\beta}|Z_{\beta,1}\Delta_\alpha(t). \end{aligned} \tag{7.5.10}$$

Similarly, we can estimate the $J_{\alpha,1}$ term as follows:

$$J_{1,\alpha} \leq C\sum_\beta |b_{\alpha\beta}|\Delta_\beta(t)Z_{2,\alpha}(t). \tag{7.5.11}$$

Combining (7.5.9),(7.5.11),(7.5.10) and (7.1.2), we have that

$$\begin{aligned}\sum_\alpha \Delta_\alpha(T) = & \sum_\alpha \sup_{0\leq t\leq T} t^{1/4}||n_{\alpha,2}(t) - n_{\alpha,1}(t)||_{4/3} \\ \lesssim & \sum_{\alpha,\beta}|b_{\alpha\beta}|\sup_{0\leq t\leq T}\Delta_\alpha(t)(Z_{\beta,1}(t) + Z_{\beta,2}(t)) \lesssim \sum_\alpha \Delta_\alpha(T)\sum_\beta\sum_{\ell=1}^2 |b_{\alpha\beta}|Z_{\beta,\ell}(T).\end{aligned}$$

Now thanks to the fact that (7.5.1), we have that

$$\sum_{\alpha} \Delta_\alpha(T) \leq \frac{1}{2} \sum_{\alpha} \Delta_\alpha(T), \quad T \in [0, T_\star], \tag{7.5.12}$$

for some small $T_\star > 0$. So we have $\Delta_\alpha \equiv 0, \forall \alpha \in \mathcal{I}, \forall t \in [0, T_\star]$. Now the uniqueness follows if we iterate this argument.

7.6 Multi-species PKS subject to non-symmetric $\mathbf{B}$ matrix

7.6.1 symmetrizable case

In general, the chemical generation coefficient matrix $(b_{\alpha\beta})_{\alpha,\beta\in\mathcal{I}}$ is nonsymmetric. This introduces new challenges in the analysis. We will not cover the general situation in this paper. However, in certain cases, one can symmetrize the system. First recall the *sign* function:

$$sign(f) = f/|f|, \quad f \neq 0; \quad sign(f) = 0, \quad f = 0. \tag{7.6.1}$$

If $sign(b_{\alpha\beta}) = sign(b_{\beta\alpha})$ and the matrix $\mathbf{B}$ is three diagonal, i.e., $b_{\alpha\beta} \neq 0$ only if $|\alpha - \beta| \leq 1$, the system can always be symmetrized.To show the method, we consider system (7.1.1) subject to general 3-by-3 matrix

$$\partial_t n_\alpha + \sum_{\beta\in\{1,2,3\}} \nabla \cdot (b_{\alpha\beta}(-\nabla\Delta^{-1})n_\beta n_\alpha) = \Delta n_\alpha, \quad sign(b_{\alpha\beta}) = sign(b_{\beta\alpha}), \quad b_{13} = b_{31} = 0.$$

First we can multiply the equation corresponds to n_2 by b_{12}/b_{21} and redefine $\tilde{n}_2 := \frac{b_{12}}{b_{21}} n_2$ and obtain

$$\partial_t n_1 + \nabla \cdot (b_{11}(-\nabla\Delta^{-1})n_1 n_1 + b_{21}(-\nabla\Delta^{-1})\tilde{n}_2 n_1) = \Delta n_1;$$
$$\partial_t \tilde{n}_2 + \nabla \cdot \left(b_{21}(-\nabla\Delta^{-1})n_1\tilde{n}_2 + \frac{b_{21}b_{22}}{b_{12}}(-\nabla\Delta^{-1})\tilde{n}_2\tilde{n}_2 + b_{23}(-\nabla\Delta^{-1})n_3\tilde{n}_2\right) = \Delta\tilde{n}_2.$$

Now we can do the same trick on the third equation, multiplying the it by $\frac{b_{12}b_{23}}{b_{32}b_{21}}$ and redefine $\tilde{n}_3 := \frac{b_{12}b_{23}n_3}{b_{32}b_{21}}$, we obtain that

$$\partial_t\tilde{n}_2+\nabla\cdot\left(b_{21}(-\nabla\Delta^{-1})n_1\tilde{n}_2+\frac{b_{21}b_{22}}{b_{12}}(-\nabla\Delta^{-1})\tilde{n}_2\tilde{n}_2+\frac{b_{32}b_{21}}{b_{12}}(-\nabla\Delta^{-1})\tilde{n}_3\tilde{n}_2\right)=\Delta\tilde{n}_2,$$
$$\partial_t\tilde{n}_3+\nabla\cdot\left(\frac{b_{32}b_{21}}{b_{12}}(-\nabla\Delta^{-1})\tilde{n}_2\tilde{n}_3+\frac{b_{32}b_{21}b_{33}}{b_{12}b_{23}}(-\nabla\Delta^{-1})\tilde{n}_3\tilde{n}_3\right)=\Delta\tilde{n}_3.$$

Now we see that the new coefficient matrix is symmetric. For general tridiagonal matrix with $sign(b_{\alpha\beta})=sign(b_{\beta\alpha})$, the symmetrization is similar.

Furthermore, if we have the relation $b_{\alpha\beta}=-b_{\beta\alpha}$ and $b_{\alpha\beta}\geq 0,\quad \alpha>\beta$, the solution is global in time without restriction on the mass M_α.

7.6.2 Essentially dissipative case

In this section, we prove Theorem 21.

Proof of Theorem 21. If $\mathcal{I}^{|\mathcal{I}|}=\mathcal{I}$, $\mathcal{I}^{(0)}\neq\emptyset$. Otherwise $\mathcal{I}^{(|\mathcal{I}|)}$ is an empty set, which is a contradiction. For any finite time interval $[0,T]$, we prove that $||n_\alpha||_{H^s}\leq C_{H^s}<\infty$ for all $t\in[0,T]$.

First we use the maximum principle of parabolic equation to prove the L^∞ bound of the n_α's. First we pick all the $\alpha^0\in\mathcal{I}^{(0)}$, and calculate the time evolution of the maximum of n_{α^0}

$$\frac{\partial}{\partial_t}n_{\alpha^0}(x_\star,t)=\Delta n_{\alpha^0}(x_\star,t)-\nabla c_{\alpha^0}(x_\star,t)\cdot\nabla n_{\alpha^0}(x_\star,t)+\sum_\beta b_{\alpha\beta}n_\beta(x_\star,t)n_{\alpha^0}(x_\star,t)\leq 0. \tag{7.6.2}$$

As a result, we obtain that the n_{α^0} is bounded from above. Since $n_{\alpha^0}\geq 0$, we have that $||n_{\alpha^0}||_\infty\leq C_{\mathcal{I}^{(0)}}<\infty$. Next we look at all the α^1's in the set $\mathcal{I}^{(1)}$. Apply the maximum principle, we have that

$$\frac{\partial}{\partial_t} n_{\alpha^1}(x_\star, t) = \Delta n_{\alpha^1}(x_\star, t) - \nabla c_{\alpha^1}(x_\star, t) \cdot \nabla n_{\alpha^1}(x_\star, t) + \sum_\beta b_{\alpha\beta} n_\beta(x_\star, t) n_{\alpha^0}(x_\star, t)$$
$$\leq \sum_{\beta \in \mathcal{I}^{(0)}} b_{\alpha\beta} ||n_\beta||_\infty n_{\alpha^1}.$$

Since $||n_\beta||_\infty < C_{\mathcal{I}^{(0)}} < \infty$, we have that $||n_{\alpha^1}(t)||_\infty \leq C_{\mathcal{I}^{(1)}} < \infty$, $\forall t \in [0, T]$. By the same arguement, we have that

$$||n_\alpha(t)||_\infty \leq C_\infty < \infty, \quad \forall \alpha \in \mathcal{I}^{(|\mathcal{I}|)} \tag{7.6.3}$$

Since $\mathbf{B}$ is essentially dissipative, $\mathcal{I}^{(|\mathcal{I}|)} = \mathcal{I}$, we have that $||n_\alpha||_\infty \leq C_\infty$ for all $\alpha \in \mathcal{I}$. This completes the first part of the proof.

Next we estimate the H^s, $2 \leq s \in \mathbb{N}$ norm of the solution. Assume that we have already obtained the H^{s-1} estimate, i.e.,

$$||n_\alpha(t)||_{H^{s-1}} \leq C_{H^{s-1}} < \infty, \quad t \in [0, T]. \tag{7.6.4}$$

We estimate the time evolution of $\sum_\alpha ||\nabla^s n_\alpha||_2^2$ as follows:

$$\frac{d}{dt} \sum_\alpha ||\nabla^s n_\alpha||_2^2$$
$$\leq -\sum_\alpha ||\nabla^{s+1} n_\alpha||_2^2 + \sum_\alpha ||\nabla c_\alpha||_\infty^2 ||\nabla^s n_\alpha||_2^2 + \sum_\alpha \sum_{\ell=2}^{s+1} ||\nabla^\ell c_\alpha||_4^2 ||\nabla^{s+1-\ell} n_\alpha||_4^2$$
$$\lesssim \sum_\alpha ||\nabla^s n_\alpha||_2^2.$$

As a result, we have that $\sum_\alpha ||n_\alpha(t)||_{H^s} \leq C_{H^s} < \infty$ for all $t \in [0, T]$. This completes the proof of the theorem. $\square$

7.7 Conclusion

In this section, we introduced the multi-species concept into the PKS equation and continued a project initiated by Wolansky. We used the log-HLS inequality associated to systems to give explicit criterion to guarantee global well-posedness of the multi-species PKS system (7.1.1) subject to symmetric chemical generation matrix **B**. We also analysed some specific cases where the chemical generation matrix is non-symmetric.

Chapter 8: Conclusion

8.1 Chemotaxis in moving fluid

I focus on the parabolic-elliptic Patlak-Keller-Segel equation with additional advection modelling Chemotaxis in a moving fluid:

$$\begin{cases} \partial_t n = \Delta n - \nabla \cdot (\nabla c n) - A u \cdot \nabla n, \\ \qquad - \Delta c = n, \\ \nabla \cdot u = 0; \;\; n(\cdot, 0) = n_0. \end{cases} \tag{8.1.1}$$

I address the problem that whether there exists a simple vector field u to suppress the blow-up of the system subject to supercritical mass. In Chapter 2 and 3, I exploit the shear flow enhanced dissipation effect and hyperbolic flow fast splitting effect to suppress the blow-up of the system. In Chapter 4, I further extend some of these results to the parabolic-parabolic PKS system.

8.2 Flocking

The hydrodynamic flocking model describes the evolution of the population density ρ and the agent velocity u subject to alignment forces determined by the influence function ϕ:

$$\begin{cases} u_t + u \cdot \nabla u = \{\phi * (\rho u) - \phi * \rho u\}, \\ \rho_t + \nabla \cdot (\rho u) = 0, \quad (\rho, u)(0) = (\rho_0, u_0). \end{cases} \tag{8.2.1}$$

In Chapter 5, I give explicit global well-posedness criteria in dimension two. The result clarifies the roles played by stretching and vorticity in the dynamics. It is worth noting that these two effects are not present in the one dimensional analysis.

8.3 Multi-species PKS systems and multi-species hydrodynamic flocking models

In Chapter 6 and 7, I introduce multi-species concepts into the PKS models and Hydrodynamic flocking models. I gave explicit global well-postedness criteria. These criteria have strong relations to the interaction matrix $\mathbf{B} = (b_{\alpha\beta})_{\alpha,\beta\in\mathcal{I}}$ involved in the system, which cannot be observed in the single-species models.

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