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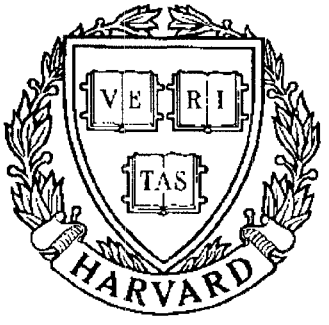

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On Continuity/Discontinuity in Robustness Indicators

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ON CONTINUITY/DISCONTINUITY IN ROBUSTNESS INDICATORS†

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Abstract

Continuity/discontinuity of robustness indicators is reviewed. For the case of real or mixed uncertainty, a regularization of the frequency dependent robustness margin is proposed and its properties are discussed. Implication of this regularization in the case of polynomial families with affine dependency on the uncertainty is pointed out.

1. Background

Roughly speaking, given a nominally stable system and an uncertainty structure, the size of the smallest uncertainty for which the system has a pole at $j\omega$ is given by $1/\mu(\omega)$, where $\mu(\omega)$ is the structured singular value, at $s = j\omega$, of a certain matrix $M(s)$ representing the nominal system. The corresponding "μ-norm", $\sup\{\mu(\omega) : \omega \in [-\infty, \infty]\}$, is thus the inverse of the norm of the smallest destabilizing perturbation in the given structure. In terms of the value set $\mathcal{P}_\delta(\omega)$ of the characteristic polynomial of the system (with uncertain parameters ranging over balls of radius δ), $r(\omega) := 1/\mu(\omega)$ is the smallest $\delta > 0$ such that $\mathcal{P}_\delta(\omega)$ touches the origin. If the coefficients of the characteristic polynomial depend affinely on the uncertainty, $r(\omega)$ and $\mu(\omega)$ have a simple graphical interpretation in relation to $\mathcal{P}_{\delta=1}(\omega)$ ($\mu(\omega)=1/r(\omega)=\ell_1/\ell_2$ on Fig. 1). While in this case $r(\omega)$ and $\mu(\omega)$ can be readily computed, it is often necessary,

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in general, to resort to making use of an upper bound $\hat{\mu}(\omega)$ to $\mu(\omega)$ (e.g., the refined real-mu upper bound obtained in [1]).

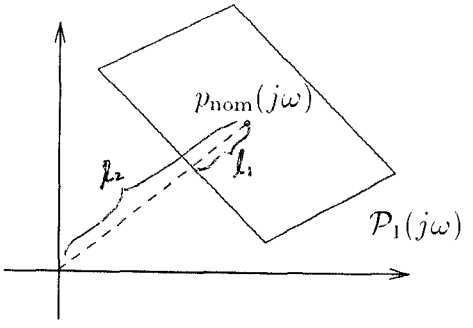

Figure 1.

It is a simply established fact (see, e.g., [2]) that if all uncertain parameters or "blocks" are allowed to take on complex values, then $\mu(\omega)$ is continuous; under mild nondegeneracy assumptions, this implies continuity of the μ-norm, thus of the robustness margin. When real uncertainty is present, it is well known that $\mu(\omega)$ can be discontinuous, and so can the μ norm. Recently, Barmish *et al.* [3] showed that this can occur even in the simple situation of affine dependency of the coefficients of the characteristic polynomial on the uncertainty, with only two uncertain parameters (possible discontinuity of $\mu(\omega)$ as a function of ω in such simple cases had been observed earlier, e.g., [4,5]). Packard [6] and Packard and Pandey [2] then suggested a "regularization" of $\mu(\omega)$ and of the μ-norm, constructed by introducing small complex uncertainties at judiciously selected locations.

In Section 2 below, we briefly review continuity results in the case of complex uncertainty and note that a standard upper bound for this case is continuous. The main contribution is in Section 3, which is devoted to the case where real uncertainty is present (possibly concurrently with complex uncertainty). After reviewing recent results, we propose a regularization for the upper bound $\hat{\mu}(\omega)$ of [1] and discuss its properties. Implications in the context of polynomial families are pointed out.

The following two Lemmas are used in the paper.

Lemma 1. (see, e.g., [7, Theorem 1.5.2]) Let $f(x, y)$ be continuous on the direct product of

compact sets $Z = X \times Y$. Then $\max_{x \in X} f(x,y)$ is continuous. □

Lemma 2. Let X be any set and, for any $x \in X$, let $f(x,y)$ be continuous in y. Then $\inf_{x \in X} f(x,y)$ is upper semicontinuous and $\sup_{x \in X} f(x,y)$ is lower semicontinuous.

Proof: Let $y_k \to \hat{y}$. For any $\tilde{x} \in X$,

$$f(\tilde{x}, y_k) \geq \inf_{x \in X} f(x, y_k) \qquad \forall\, k.$$

Since $f(\tilde{x}, \cdot)$ is continuous,

$$f(\tilde{x}, \hat{y}) = \lim_{k \to \infty} f(\tilde{x}, y_k) \geq \limsup_{k \to \infty} \inf_{x \in X} f(x, y_k).$$

Since this holds for any $\tilde{x} \in X$,

$$\inf_{x \in X} f(x, \hat{y}) \geq \limsup_{k \to \infty} \inf_{x \in X} f(x, y_k).$$

Thus the first claim holds. The second claim follows from the fact that

$$\sup_{x \in X} f(x,y) = - \inf_{x \in X} [-f(x,y)].$$

□

2. Complex uncertainty

The structured singular value of a complex square matrix M with respect to a given complex uncertainty structure [8] can be defined, e.g., as

$$\mu(M) = \max\{\rho(M\Delta) : \Delta \in \mathcal{X}, \bar{\sigma}(\Delta) \leq 1\} \tag{1}$$

where ρ denotes the spectral radius, $\bar{\sigma}$ the spectral norm, and $\mathcal{X}$ the set of all block-diagonal matrices with fixed block sizes corresponding to the uncertainty structure. Continuity of $\mu(M)$ in the entries of M is a direct consequence of Lemma 1. Robust stability of a nominally stable system characterized by a matrix transfer function $M(s)$, for all uncertainty of size

(H_∞ norm) at most 1 and of structure $\mathcal{X}$, is then equivalent to satisfaction of the inequality $\|M\|_\mu < 1$, where $\|M\|_\mu$ is the "μ-norm" of $M(s)$ (not a norm), given by

$$\|M\|_\mu = \sup_{\omega \in \mathbf{R}} \mu(M(j\omega))$$

(Small μ Theorem [9]; see also [10]). Suppose that $M(s)$ depends on some parameter d ("problem data"); denote this by $M(d,s)$. Suppose that, for any ω, $M(\cdot, j\omega)$ is continuous in a neighborhood of some $\hat{d}$. Then, by Lemma 2, $\|M(d)\|_\mu$ is lower semicontinuous around $\hat{d}$. If $M(\cdot, s)$ is also rational and strictly proper in a neighborhood of $\hat{d}$, then it is a consequence of Lemma 1 that $\|M(d)\|_\mu$ is continuous around $\hat{d}$. If $M(d,s)$ is merely proper though, this result is not true in general (take $M(d,s) = ds/(d^2 s + 1)$, with $\hat{d} = 0$) but still holds under a certain no-degree-dropping condition (see [11] for an analysis in the context of polynomial families).

We conclude this section by noting that, when uncertainty is nonrepeated, the well known complex upper bound to $\mu(M)$ can be shown to be continuous in the entries of M. Here, $\mathcal{D}$ is the set of positive definite diagonal Hermitian matrices, denoted as $\mathrm{diag}(d^1, \cdots, d^n)$, that commute with $\mathcal{X}$, and for any matrix A, A^{ij} denotes its (i,j)th element.

Proposition 1. Suppose $\mathcal{D}$ consists of diagonal matrices (i.e., uncertainty is nonrepeated) then $\bar{\mu}(M) := \inf_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1})$ is continuous in the entries of M.

Proof: Upper semicontinuity follows from Lemma 2. To prove lower semicontinuity, let $M_k \to \hat{M}$ and, given $\epsilon > 0$, let $D_k := \mathrm{diag}(d_k^1, \cdots, d_k^n)$ be such that $\bar{\sigma}(D_k M_k D_k^{-1}) \leq \inf_{D \in \mathcal{D}} \bar{\sigma}(D M_k D^{-1}) + \epsilon$. Thus $\bar{\sigma}(D_k M_k D_k^{-1})$ is bounded. Since $(D_k M_k D_k^{-1})^{ij} = \frac{d_k^i}{d_k^j} M_k^{ij}$, $\frac{d_k^i}{d_k^j}$ must be bounded for all i, j such that $\hat{M}^{ij} \neq 0$. Thus $\frac{d_k^i}{d_k^j}(M_k^{ij} - \hat{M}^{ij}) \to 0$, i.e., $(D_k(M_k - \hat{M})D_k^{-1})^{ij} \to 0 \quad \forall\ i,\ j$ such that $\hat{M}^{ij} \neq 0$. Thus there exist sequences $\{S_k\}$ and $\{T_k\}$ such that

$$D_k(M_k - \hat{M})D_k^{-1} = S_k + T_k$$

with $T_k \to 0$ as $k \to \infty$, and $S_k^{ij} = 0$ if $\hat{M}^{ij} \neq 0$. Then, for all k,

$$\bar{\sigma}(D_k \hat{M} D_k^{-1}) \leq \bar{\sigma}(D_k \hat{M} D_k^{-1} + S_k)$$

and thus

$$\begin{aligned}\bar{\mu}(\hat{M}) &= \inf_{D\in\mathcal{D}} \bar{\sigma}(D\hat{M}D^{-1}) \leq \liminf_{k\to\infty} \bar{\sigma}(D_k\hat{M}D_k^{-1}) \\ &\leq \liminf_{k\to\infty} \bar{\sigma}(D_k\hat{M}D_k^{-1} + S_k) \\ &= \liminf_{k\to\infty} \bar{\sigma}(D_kM_kD_k^{-1} - T_k).\end{aligned}$$

Since all components of $D_kM_kD_k^{-1}$ are bounded and $T_k \to 0$,

$$\begin{aligned}\liminf_{k\to\infty} \bar{\sigma}(D_kM_kD_k^{-1} - T_k) &= \liminf_{k\to\infty} \bar{\sigma}(D_kM_kD_k^{-1}) \\ &\leq \liminf_{k\to\infty} \bar{\mu}(M_k) + \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this completes the proof. □

This implies that when $M(d,s)$ is rational and strictly proper in a neighborhood of $\hat{d}$, $\sup_{\omega\in\mathbf{R}} \inf_{D\in\mathcal{D}} \bar{\sigma}(DM(d,j\omega)D^{-1})$ is continuous in d around $\hat{d}$. It is conjectured that Propostion 1 holds without the restriction on $\mathcal{D}$.

3. Real or mixed uncertainty

When some of the blocks in $\mathcal{X}$ are constrained to be real (this is the case when parametric uncertainty is present) expression (1) is no longer valid (see, e.g., [1]). The following upper bound to $\mu(M)$ is in most cases lower than the "complex" upper bound of Proposition 1 (see [1,12]):

$$\mu(M) \leq \hat{\mu}(M) := \sqrt{\max\left\{0, \inf_{D\in\mathcal{D},G\in\mathcal{G}} \bar{\lambda}(M^HDM + j(GM - M^HG), D)\right\}} \qquad (2)$$

where $\mathcal{G}$ is the set of Hermitian matrices with zero entries at positions corresponding to uncertainty allowed to take on complex values, and where, given Hermitian matrices A and B with $B > 0$, $\bar{\lambda}(A,B) = \max\{\alpha : \det(A - \alpha B) = 0\}$. This upper bound to $\mu(M)$ is computationally attractive in view of the fact that $\bar{\lambda}(\cdot,\cdot)$ is quasi-convex whenever its second argument is positive definite [12] and that powerful algorithms are available for minimization of such functions [12–14]. In general, by Lemma 2, $\hat{\mu}(M)$ is upper semicontinuous, but

because $\mathcal{D}$ and $\mathcal{G}$ are not compact, it is not always continuous in the entries of M (e.g., take the case of a complex scalar M). Note however that, for the case when M is real and the real uncertainty is nonrepeated, it has been shown that an optimal choice for G in (2) is $G = 0_n$, the zero matrix, so that $\hat{\mu}(M) = \inf_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1})$ [15]; and, in view of Proposition 1 above, under the additional assumption that the complex uncertainty is also nonrepeated, it is continuous. To circumvent the discontinuity problem in the general case, consider, for $\epsilon \in (0, 1]$, the quantity

$$\hat{\mu}(M, \epsilon) := \sqrt{\max\left\{0, \min_{D \in \mathcal{D}_\epsilon, G \in \mathcal{G}_\epsilon} \bar{\lambda}(M^H D M + j(GM - M^H G), D)\right\}}$$

where $\mathcal{D}_\epsilon = \{D \in \mathcal{D} : ||D|| \leq \frac{1}{\epsilon},\ D \geq \epsilon I\}$ and $\mathcal{G}_\epsilon = \{G \in \mathcal{G} : ||G|| \leq \frac{1}{\epsilon}\}$, where $||\cdot||$ is any given norm. Also let $\hat{\mu}(M, 0) = \hat{\mu}(M)$. Since both $\mathcal{D}_\epsilon$ and $\mathcal{G}_\epsilon$ are convex, $\hat{\mu}(M, \epsilon)$ is still efficiently computable. Moreover, the following holds.

Proposition 2. For any M, $\hat{\mu}(M, \cdot)$ is continuous over $[0, 1]$ and monotonically decreasing as $\epsilon \searrow 0$. For any $\epsilon \in (0, 1]$, $\hat{\mu}(\cdot, \epsilon)$ is continuous.

Proof: Continuity of $\hat{\mu}(M, \cdot)$ at any $\epsilon \in (0, 1]$ follows from continuity of $\bar{\lambda}(\cdot, \cdot)$ and from compactness and continuity in ϵ of $\mathcal{D}_\epsilon$ and $\mathcal{G}_\epsilon$ in the Hausdorff metric. Monotonicity in ϵ and continuity of $\hat{\mu}(M, \cdot)$ at $\epsilon = 0$ directly follow from the facts that $\mathcal{D} = \bigcup_{\epsilon > 0} \mathcal{D}_\epsilon$, $\mathcal{G} = \bigcup_{\epsilon > 0} \mathcal{G}_\epsilon$ and, given $\epsilon_1 > \epsilon_2 > 0$, $\mathcal{D}_{\epsilon_1} \subset \mathcal{D}_{\epsilon_2}$ and $\mathcal{G}_{\epsilon_1} \subset \mathcal{G}_{\epsilon_2}$. Finally, continuity of $\hat{\mu}(\cdot, \epsilon)$ directly follows from Lemma 1. □

The Small μ Theorem still holds in the real/mixed uncertainty case. However, since $\mu(M(s))$ may now be discontinuous at $s = \infty$, the "μ-norm" must be defined as [2]

$$\|M\|_\mu = \max\{\sup_{\omega \in \mathbf{R}} \mu(M(j\omega)), \mu(M(\infty))\} = \sup_{\omega \in [-\infty, \infty]} \mu(M(j\omega))$$

where $M(\infty) = \lim_{s \to \infty} M(s)$. Thus a *sufficient* condition for robust stability of the corresponding system under structured uncertainty of size no more than 1 is that, for some $\epsilon \in (0, 1]$,

$$\sup_{\omega \in [-\infty, \infty]} \hat{\mu}(M(d, j\omega), \epsilon) < 1 \tag{3}$$

where d is again some "problem data". If $M(d,s)$ is continuous in s (e.g., rational), continuity of $\hat{\mu}(\cdot,\epsilon)$ implies that the "sup" can be taken over $(-\infty,\infty)$. If $M(d,s)$ is continuous in (d,s) and is rational and strictly proper, the left hand side in (3) is a continuous function of d. The next proposition shows that not much is lost by taking $\epsilon \neq 0$. Again, the strict properness condition can be relaxed.

Proposition 3. Let $M(s)$ be rational and strictly proper. Then

$$\sup_{\omega} \hat{\mu}(M(j\omega),\epsilon) \to \sup_{\omega} \hat{\mu}(M(j\omega)) \quad \text{as} \quad \epsilon \searrow 0.$$

Proof. That the lim inf of the left hand side, as $\epsilon \searrow 0$, is no smaller than the right hand side follows from monotonicity in ϵ (Proposition 2). We prove by contradiction that the lim sup of the left hand side is no larger than the right hand side. Thus, suppose that there exist $\delta > 0$, $\epsilon_i \searrow 0$ such that, for all i,

$$\sup_{\omega} \hat{\mu}(M(j\omega),\epsilon_i) > \sup_{\omega} \hat{\mu}(M(j\omega)) + \delta.$$

By upper semicontinuity and strict properness, both suprema are achieved, say at ω_i and ω^* respectively, with $\{\omega_i\}$ bounded (since, for any $\epsilon \in [0,1]$, $\hat{\mu}(M(j\omega),\epsilon) \leq \bar{\sigma}(M(j\omega))$ for all ω). Without loss of generality, assume that, for some $\hat{\omega}$, $\omega_i \to \hat{\omega}$ as i goes to ∞. We can write

$$\hat{\mu}(M(j\omega_i),\epsilon_i) > \hat{\mu}(M(j\omega^*)) + \delta \geq \hat{\mu}(M(j\hat{\omega})) + \delta.$$

The value $\hat{\mu}(M(j\omega_i),\epsilon_i)$ is increased when we replace ϵ_i by ϵ_k with $k < i$, thus,

$$\hat{\mu}(M(j\omega_i),\epsilon_k) > \hat{\mu}(M(j\hat{\omega})) + \delta \qquad \forall\ i,\ k \quad \text{with} \quad k \leq i.$$

Taking the limit as $i \to \infty$, we obtain, in view of Proposition 2,

$$\hat{\mu}(M(j\hat{\omega}),\epsilon_k) > \hat{\mu}(M(j\hat{\omega})) + \delta \qquad \forall\ k.$$

This contradicts continuity of $\hat{\mu}(M(j\hat{\omega}),\cdot)$ over $[0,1]$ (Proposition 2). □

Compared to the regularization proposed by Packard and Pandey [2], $\hat{\mu}(M,\epsilon)$ has the advantage that its computation does not involve any augmented uncertainty structure (note

that, in general, the regularization of [2] is not efficiently computable, and an upper bound to it–such as $\hat{\mu}$–must be used).

Finally, it is known that robustness problem for polynomial families with affine dependency on the real uncertain parameters can be formulated as rank 1 μ problem. In such cases $\mu(M(j\omega))$ can be computed exactly [17] (but may be discontinuous). Young *et al.* [18] recently showed that if $\text{rank}(M) = 1$, then $\hat{\mu}(M) = \mu(M)$. This implies that, in the rational strictly proper rank 1 case,

$$\sup_{\omega} \hat{\mu}(M(j\omega), \epsilon) \to ||M||_{\mu} \quad \text{as} \quad \epsilon \to 0.$$

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