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A Game-theoretic Look at the Gaussian Multiaccess Channel

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Abstract

We study the issue of how to fairly allocate communication rate among the users of a Gaussian multiaccess channel. All users are assumed to value rate equally and each is assumed to have no limit on its desired rate. We adopt a cooperative game-theoretic viewpoint, i.e. it is assumed that the users can potentially form coalitions off line to threaten other users with jamming the channel, using this as an argument for deserving a larger share of the rate. To determine the characteristic function of the game, we first determine the capacity region of the Gaussian multiaccess arbitrarily varying channel, with an operational meaning of capacity somewhat modified from the usual one, which is more appropriate to our context and permits time sharing. We then propose a solution concept for the game through a set of natural fairness axioms and prove that there exists a unique fair allocation that satisfies the axioms. Moreover, we demonstrate that the unique allocation is always feasible and lies in the core of the game. It is also shown to possess some intuitively natural qualitative properties as the signal to noise ratio varies.

keywords: Cooperative game theory, Gaussian multiaccess channel, arbitrarily varying channel

1 Introduction

There has been a resurgence of interest in multiuser information theory. Major driving forces behind this are the growth of networking, the increasing penetration of wireless local access (including multiple antenna techniques), and the drive to develop sensor networks using ad hoc wireless networking technology.

Apart from playing an obvious role in meeting the technical challenges needed to develop and deploy such systems, we believe that information theory can also help to provide insight into the important social choice issues that arise from this networking explosion. By this we mean aspects of multiple agent interaction that are normally studied in the economics literature, such as fairness, incentive compatibility, mechanism design, pricing, revenue maximization, budget balance, etc. [3, 6]. When the system is modeled as having a single objective, which may be shared by the agents or imposed from above, some of these issues disappear, and such problems fall within the traditional domain of optimization theory. There is a well established body of such work in the information-theoretic literature. However, much less attention has been paid to the study of situations where the individual agents may have objectives that are in conflict with each other.

In this paper we make a start on the discussion of such issues by considering one of the most basic multiuser information theory models : the Gaussian multiaccess channel. We view the channel from the point of view of the users and ask : what is a fair way for the users to choose an operating point in the capacity region? The answer, of course, depends on what one means by "fair". To this end we adopt the viewpoint of cooperative game theory. First of all, we assume that all users value rate equally and each user has no intrinsic limit on its desired rate. We envision subsets of the users as being able to form contracts off line that enable them to act as coalitions; such a coalition of users can threaten the others with using its power to jam the shared channel and can implicitly demand a larger share of the resource as a payoff to avoid the execution of such a threat. Cooperative game theory addresses this by defining a notion called the "characteristic function" : this associates to each subset of the players (users) a number which is the largest payoff that they are guaranteed even if all the other players form a coalition to work against them. A fair allocation may then be roughly defined as one that is compatible with the characteristic function (precise definitions are given later – what we mean is that the allocation should lie in the core of the game) and satisfies some natural set of fairness axioms. Our goal in this paper is to carry out this cooperative game-theoretic program for the Gaussian multiaccess channel.

There are several aspects that we need to address. First, we need to determine the characteristic function of the game, and for this we need to solve a version of the Gaussian multiaccess arbitrarily varying channel problem. This problem has been known to the community for some time and the major bottleneck to its solution is the ability, in the traditional formulation, of the jammer to use a very skewed power distribution in its jamming sequence, preventing the use of time sharing. In our context it is natural to think of all users as aligned to a common underlying time frame, so we are able to finesse this problem by working with an alternative operational meaning of capacity region (which reduces to the usual definition in the absence of a jammer). This development is carried out in Section 2. Secondly, we need to propose a natural set of fairness axioms for the problem. We do this, with the notion of envy-freeness, in Section 4, after a quick introduction to standard cooperative game-theoretic concepts in Section 3. Finally, we need to demonstrate the existence and uniqueness of a rate allocation that is both compatible with the characteristic function (in the core) and feasible – *i.e.,* it is in the capacity region. We do this in Section 5. We make some concluding remarks in Section 6. Some simple inequalities that we use in the proof are gathered in an Appendix.

2 Capacity region of Gaussian multiaccess arbitrarily varying channel

We first discuss the standard power constrained Gaussian multiaccess channel with an alternative operational interpretation of capacity that yields a capacity region identical to the original one. The reason for doing this will become apparent when we turn to the analysis of the version of the Gaussian multiaccess arbitrarily varying channel that is necessary for us to set up the game-theoretic formulation of the problem we are interested in.

2.1 A mildly different look at the usual Gaussian multiaccess channel

Consider a Gaussian multiple access channel with I users and let $\mathcal{I} = \{1, \ldots, I\}$ denote the set of users. A *coding scheme* at block length n for this channel is a collection of sequences of codes $\{\mathcal{C}_i(k), k \geq 1\}$, one for each user, where $\mathcal{C}_i(k) : M_i(k) \mapsto \mathbf{R}^n$. Here $(M_i(k), k \geq 1)$ is a sequence of positive integers. The coding scheme is assumed to be common knowledge to all users and to the receiver. The coding scheme is said to satisfy power constraints $\underline{\Gamma} = (\Gamma_1, \ldots, \Gamma_I)$ if for each $i \in \mathcal{I}$, each $k \geq 1$, and each $m \in M_i(k)$ we have

$$\sum_{j=1}^{n} x_j^2 \leq n\Gamma_i$$

where $\underline{x} = (x_1, \ldots, x_n) = \mathcal{C}_i(k)(m)$ is the codeword corresponding to message m of user i in block k. We denote a coding scheme at block length n by $\mathcal{C}^{(n)}$.

Let

$$\rho_i = \liminf_{K} \frac{\sum_{k=1}^{K} \log M_i(k)}{nK}$$

where the logarithm is to base 2, and let $\underline{\rho}$ denote $(\rho_1, \ldots, \rho_I)$. We call $\underline{\rho}$ the rate vector of the coding scheme $\mathcal{C}^{(n)}$.

Each user i uses its sequence of codebooks in the coding scheme to encode a sequence of messages block by block, being allowed to send one of $M_i(k)$ messages in block k. Suppose user i sends message $m_i(k)$ in block k. The received vector in block k is

$$\underline{y} = \sum_{i=1}^{I} \underline{x}_i(k) + \underline{z}(k)$$

where $\underline{x}_i(k) = (x_1, \ldots, x_n) = \mathcal{C}_i(k)(m_i(k))$ is the codeword corresponding to message $m_i(k)$ of user i in block k and $\underline{z} = (z_1, \ldots, z_n)$ is a sequence of independent Gaussian random variables of mean zero and variance σ^2. A decoding scheme for the given coding scheme is a sequence $(d(k), k \geq 1)$ where $d(k) : \mathbf{R}^n \mapsto [M_1(k)] \times \ldots \times [M_I(k)] \cup \{*\}$. Here $[M_i(k)]$ denotes the set $\{1, \ldots, M_i(k)\}$. The probability of error of the decoding scheme in block k is defined as

$$e(k) = P(d(k)(\underline{y}) \neq (m_1(k), \ldots, m_I(k))$$

where $\underline{y}$ is the received vector in block k corresponding to the users having transmitted messages $m_i(k)$ in block k and the choices of messages in the message sets of the individual users assumed to be uniformly distributed. The probability of error of the decoding scheme is defined as $e = \sup_{k \geq 1} e(k)$. Finally, for the given coding scheme at block length n, $\mathcal{C}^{(n)}$, let $e(\mathcal{C}^{(n)})$ denote the infimum of the probability of error over all decoding schemes for that coding scheme.

We say coding is achievable at rate vector $\underline{R} = (R_1, \ldots, R_I)$ if there is a sequence of coding schemes $(\mathcal{C}^{(n)}, n \geq 1)$ such that both $e(\mathcal{C}^{(n)}) \to 0$ as $n \to \infty$ and $\liminf_n \underline{\rho}(\mathcal{C}^{(n)}) \geq \underline{R}$. The capacity region of the Gaussian multiaccess channel with I users satisfying power constraints $\underline{\Gamma}$ is defined as the closure of all achievable rate vectors.

Theorem 2.1 *The capacity region of the Gaussian multiaccess channel as defined in the preceding paragraphs is identical to the capacity region of the channel as it is usually defined.*

Proof: The usual definition of coding at block length n differs from our definition precisely in that it presumes a coding scheme where each user uses the same code in each block, *i.e.*, $M_i(k) \equiv M_i$ and $\mathcal{C}_i(k) \equiv \mathcal{C}_i$ for each $k \geq 1$, and also requires that the same decoding rule be used in each block, *i.e.*, $d(k) \equiv d$ for each $k \geq 1$. One then has $e(k) = e$ for each $k \geq 1$ and the rate vector of the code becomes $\underline{\rho} = (\rho_1, \ldots, \rho_I)$ with $\rho_i = \frac{\log M_i}{n}$. Also, in the usual case, one has exactly the same definition for the concept of coding being achievable at rate $\underline{R}$ as we have, only that one has already restricted to coding schemes using the same set of codebooks in each block.

It follows then that the capacity region of a Gaussian multiaccess channel, as we have defined it, is no smaller than the capacity region as usually defined. For the converse, suppose we are given a sequence of coding schemes in our sense, $(\mathcal{C}^{(n)}, n \geq 1)$, such that both $e(\mathcal{C}^{(n)}) \to 0$ as $n \to \infty$ and $\liminf_n \underline{\rho}(\mathcal{C}^{(n)}) \geq \underline{R}$. Then, for any $\delta > 0$, for all sufficiently large n we have $\underline{\rho}(\mathcal{C}^{(n)}) \geq \underline{\bar{R}}$ where $\underline{\bar{R}} = (\bar{R}_1, \ldots, \bar{R}_I)$ with $\bar{R}_i = (R_i - \delta)^+$. Given $\epsilon > 0$, however small, we then also have, for all sufficiently large n, that for every block $k \geq 1$ of the coding scheme $\mathcal{C}^{(n)}$ there is a decoding rule for the associated family of codes $\mathcal{C}_i^{(n)}(k), 1 \leq i \leq \mathcal{I}$, resulting in average probability of error at most ϵ. By applying Fano's inequality in the usual way we then conclude that for any subset $\mathcal{J} \subseteq \mathcal{I}$ of users we have

$$\sum_{i \in \mathcal{J}} \frac{1}{n} \log M_i^{(n)}(k) \leq \frac{1}{2} \log(1 + \frac{\sum_{i \in \mathcal{J}} P_i}{\sigma^2}) + D\epsilon$$

where D is a fixed constant that does not depend on n, k, δ or ϵ. From this we conclude that

$$\sum_{i \in \mathcal{J}} \bar{R}_i \leq \frac{1}{2} \log(1 + \frac{\sum_{i \in \mathcal{J}} P_i}{\sigma^2}) + D\epsilon \ .$$

Since this holds for every $\delta > 0$ and every $\epsilon > 0$, we conclude that the capacity region of the channel in our sense is identical to the capacity region of the channel as it is usually defined. □

2.2 A fresh look at the Gaussian multiaccess arbitrarily varying channel

We next turn to study the Gaussian multiaccess channel where a subset of the users form a coalition attempting to limit the ability of the other users to communicate. We call these users the jamming users, while the remaining users are called the communicating users. Assume, for notational convenience, that the jamming users are users $J + 1 \leq i \leq I$, so that the communicating users are users $1 \leq i \leq J$. We assume that communication takes places with all users using codes at block length n to determine their input into the channel. We may then describe the overall communication system via a coding scheme with notation essentially as above, while noting the following crucial difference : since the jamming users are no longer participating in the communication process, we assume that the inputs to the channel of the jamming users are private knowledge to the coalition of jammers. The sequences of codes of the communicating users are assumed to be common knowledge to all users and to the receiver. The communicating users should be able to communicate irrespective of the jamming strategy of the jamming users. With this in mind we may set up the problem as follows :

Since the jamming users do not wish to communicate, we do not ascribe a set of messages to such users, but rather use the notation $\underline{s}_i(k)$ for the input to the channel of the jamming user i, $J+1 \leq i \leq I$, in the block k. Thus a coding scheme $\mathcal{C}^{(n)}$ at block length n for a Gaussian multiaccess channel with I users where users $J+1 \leq i \leq I$ are the jamming users is a collection of sequences of codes $\{\mathcal{C}_i(k), k \geq 1\}$, one for each communicating user, $1 \leq i \leq J$, where $\mathcal{C}_i(k) : M_i(k) \mapsto \mathbf{R}^n$ and a collection of sequences $(\underline{s}_i(k), k \geq 1)$, one for each jamming user i, $J+1 \leq i \leq I$, where $\underline{s}_i(k) \in \mathbf{R}^n$. The coding scheme is said to satisfy power constraints $\underline{\Gamma} = (\Gamma_1, \ldots, \Gamma_I)$ if for each $1 \leq i \leq J$, each $k \geq 1$, and each $m \in M_i(k)$ we have

$$\sum_{j=1}^{n} x_j^2 \leq n\Gamma_i$$

where $\underline{x} = (x_1, \ldots, x_n) = \mathcal{C}_i(k)(m)$ is the codeword corresponding to message m of user i in block k, and further, for each $J+1 \leq i \leq I$ and each $k \geq 1$ we have

$$\sum_{j=1}^{n} s_{ij}(k)^2 \leq n\Gamma_i$$

where $\underline{s}_i(k) = (s_{i1}(k), \ldots, s_{in}(k))$ is the input to the channel of jamming user i. Let

$$\rho_i = \liminf_{K} \frac{\sum_{k=1}^{K} \log M_i(k)}{nK}, \quad 1 \leq i \leq J,$$

where the logarithm is to base 2, and let $\underline{\rho}$ denote $(\rho_1, \ldots, \rho_J)$. We call $\underline{\rho}$ the rate vector of the block coding scheme $\mathcal{C}^{(n)}$.

Suppose communicating user i, $1 \leq i \leq J$, sends message $m_i(k)$ in block k. The received vector in block k is

$$\underline{y} = \sum_{i=1}^{J} \underline{x}_i(k) + \sum_{i=J+1}^{I} \underline{s}_i(k) + \underline{z}(k)$$

where $\underline{x}_i(k) = (x_1, \ldots, x_n) = \mathcal{C}_i(k)(m_i(k))$ is the codeword corresponding to message $m_i(k)$ of communicating user i in block k, and $\underline{z} = (z_1, \ldots, z_n)$ is a sequence of independent Gaussian random variables of mean zero and variance σ^2. A decoding scheme for the given coding scheme is a sequence $(d(k), k \geq 1)$ where $d(k) : \mathbf{R}^n \mapsto [M_1(k)] \times \ldots \times [M_J(k)] \cup \{*\}$. The probability of error of the decoding scheme in block k is defined as

$$e(k) = \sup_{\{\underline{s}_i(k), J+1 \leq i \leq I\}} P(d(k)(\underline{y}) \neq (m_1(k), \ldots, m_I(k)))$$

where $\underline{y}$ is the received vector in block k corresponding to the communicating users having transmitted messages $m_i(k)$ in block k and the choices of messages of the communicating users in the individual message sets assumed to be uniformly distributed. The probability of error of the decoding scheme is defined as $e = \sup_{k \geq 1} e(k)$. Finally, for the given coding scheme at block length n, $\mathcal{C}^{(n)}$, let $e(\mathcal{C}^{(n)})$ denote the infimum of the probability of error over all decoding schemes for that coding scheme.

We say coding is achievable at rate vector $\underline{R} = (R_1, \ldots, R_J)$ if there is a sequence of coding schemes $(\mathcal{C}^{(n)}, n \geq 1)$ such that both $e(\mathcal{C}^{(n)}) \to 0$ as $n \to \infty$ and $\liminf_n \underline{\rho}(\mathcal{C}^{(n)}) \geq \underline{R}$. The capacity region of the Gaussian multiaccess channel with I users where users $J+1 \leq i \leq I$ are jamming users, and satisfying power constraints $\underline{\Gamma}$, is defined as the closure of all achievable rate vectors.

Theorem 2.2 *The capacity region of the Gaussian multiaccess channel with I users where users $J+1 \leq i \leq I$ are jamming users, and satisfying power constraints $\underline{\Gamma}$, is a convex set.*

Proof: Suppose coding is achievable at rate vectors $\underline{R}_0$ and $\underline{R}_1$, where $\underline{R}_0 = (R_1^0, \ldots, R_J^0)$ and $\underline{R}_1 = (R_1^1, \ldots, R_J^1)$, and let $\lambda \in (0, 1)$. We will show that coding is achievable at rate vector $\underline{R}_\lambda$, where $\underline{R}_\lambda = \lambda \underline{R}_1 + (1 - \lambda)\underline{R}_0$. Choose integers $(l_n, n \geq 1)$ such that $\frac{l_n}{n} \to \lambda$ and consider the sequence of coding schemes where the one at block length n is constructed by repeatedly interlacing l_n blocks of the coding scheme at block length n in the sequence that verifies the achievability of $\underline{R}_1$ with $n - l_n$ blocks of the coding scheme at block length n in the sequence that verifies the achievability of $\underline{R}_0$. $\square$

Theorem 2.3 *The capacity region of the Gaussian multiaccess channel with I users where users $J+1 \leq i \leq I$ are jamming users, and satisfying power constraints $\underline{\Gamma}$ is*

$$\begin{aligned} \{(R_1, \ldots, R_J) \ : \ R_i = 0 \ \textit{for all } 1 \leq i \leq J \textit{ such that } \Gamma_i \leq \Lambda \textit{ and} \\ \sum_{i \in A} R_i \leq C(\sum_{i \in \hat{A}} \Gamma_i, \Lambda + \sigma^2) \ \textit{for all } A \subseteq [J] \ \} \end{aligned} \tag{1}$$

where Λ denotes $(\sum_{i=J+1}^{I} \sqrt{\Gamma_i})^2$, $C(\Gamma, \Sigma)$ denotes $\frac{1}{2}\log(1 + \frac{\Gamma}{\Sigma})$, $[J]$ denotes $\{1, \ldots, J\}$, and for $A \subseteq [J]$, $\hat{A}$ denotes $\{i \in A \ : \ \Gamma_i > \Lambda\}$.

Proof: *Proof of the converse* : Since the jamming users are acting as a single coalition we can think of them as a single agent, which we will call the combined jammer. This combined jammer is power limited to Λ, as can be seen from the fact that the jamming users can coherently combine their jamming vectors.

Let $1 \leq i \leq J$ be such that $\Gamma_i \leq \Lambda$. Consider any sequence $(\mathcal{C}^{(n)}, n \geq 1)$ of coding schemes for which $\limsup_n \sup_{k \geq 1} M_i^{(n)}(k) \geq 2$. We claim that $\limsup_n e(\mathcal{C}^{(n)}) \geq \frac{1}{4}$. This proves that no vector of rates $\underline{R} = (R_1, \ldots, R_J)$ with $R_i > 0$ is achievable. Hence the capacity region of the channel is contained in the set of vectors of rates where $R_i = 0$ for all $1 \leq i \leq J$ such that $\Gamma_i \leq \Lambda$.

The proof of this follows Blackwell's argument, which was also used in [1] (see the beginning of the Appendix on page 23 of [1]). Given any $n_0 > 0$ there is $n > n_0$ and at least one block k with $M_i^{(n)}(k) \geq 2$. We now consider the family of jamming strategies for the combined jammer, indexed by $M_i^{(n)}(k)$, with strategy m being to use as jamming sequence (in block k, which is all we are interested in) the codeword $\mathcal{C}_i^{(n)}(k)(m)$ of user i corresponding to his message m. We think of the combined jammer as choosing one of these strategies uniformly at random. Then, for any decoding rule in block k, the error probability in block k for decoding the message of user i, averaged over equiprobable messages and averaged over the strategies of the jammer is at least $\frac{1}{4}$ (in fact very close to $\frac{1}{2}$ if $M_i^{(n)}(k)$ is large) because any decoding rule that is correct when the message is m and the jamming strategy is $\tilde{m}$ ($m \neq \tilde{m}$) is wrong when the message is $\tilde{m}$ and the jamming strategy is m. This means that the jammer has a strategy whose probability of error for decoding the message of user i, averaged over equiprobable messages, is at least $\frac{1}{4}$. This being true in block k, and since the error probability of a coding scheme is given by the supremum of the error probability over its blocks, we have $e(\mathcal{C}^{(n)}) \geq \frac{1}{4}$. Since given arbitrary $n_0 > 0$ there is $n > n_0$ for which this is true, we have $\limsup_n e(\mathcal{C}^{(n)}) \geq \frac{1}{4}$.

Next consider a sequence of coding schemes where for all sufficiently large n we have $M_i^{(n)}(k) = 1$ for all $k \geq 1$ and for all $1 \leq i \leq J$ such that $\Gamma_i \leq \Lambda$. Given $\tilde{\Lambda} < \Lambda$ consider the strategy of the combined jammer for the coding scheme at block length n which is to pick a jamming sequence whose components are independent Gaussian random variables of mean zero and variance $\tilde{\Lambda}$ and to use the same jamming sequence in each block. The probability that the jamming sequence does not meet the power constraint goes to zero as $n \to \infty$. In any block of the coding scheme at block length n the average probability of error, averaged over equiprobable messages and the randomly chosen jamming sequence, is bounded above by the sum of the probability that the randomly chosen jamming sequence does not meet the power constraint and the maximum over jamming sequences that do meet the power constraint of the average probability of error, averaged over equiprobable messages. Since this goes to zero as $n \to \infty$ it must be the case that in each block the rate vectors asymptotically satisfy the capacity constraints for a Gaussian multiaccess channel with noise power $\tilde{\Lambda} + \sigma^2$. We may now let $\tilde{\Lambda} \to \Lambda$ to complete the proof of the converse.

Proof of achievability :

We first need to verify that everything in the paper of Csiszar and Narayan goes through with the following noise model : instead of the noise vector $\underline{V} = (V_1, \ldots, V_n)$ in a block having i.i.d Gaussian coordinates, it is modeled as

$$\underline{V} = \underline{W} + \underline{U}_1 + \ldots + \underline{U}_L$$

where $\underline{W} = (W_1, \ldots, W_n)$ has i.i.d Gaussian coordinates, $\underline{W}, \underline{U}_1, \ldots, \underline{U}_L$ are independent, and each $\underline{U}_l$ is chosen uniformly at random from a sphere in $\mathbf{R}^n$ of fixed radius.

Let $\underline{R} = (R_1, \ldots, R_J)$ be a corner point of the rate region given in the theorem. To clarify what this means let us assume, for notational convenience, that $\Gamma_i > \Lambda$ for $1 \leq i \leq K$ and $\Gamma_i \leq \Lambda$ for $K + 1 \leq i \leq J$ and let us consider the corner point corresponding to the decoding order : decode user 1, followed by user 2, and so on ending with user K. By this we mean the rate vector $\underline{R} = (R_1, \ldots, R_J)$ with

$$\begin{aligned} R_1 &= C(\Gamma_1, \sum_{i=2}^{K} \Gamma_i + \Lambda + \sigma^2) \\ R_2 &= C(\Gamma_2, \sum_{i=3}^{K} \Gamma_i + \Lambda + \sigma^2) \end{aligned}$$

$$
\begin{aligned}
&\vdots \\
R_K &= C(\Gamma_K, \Lambda + \sigma^2) \\
R_i &= 0 \ \text{ for } K+1 \le i \le J \ .
\end{aligned}
$$

Fix $\delta > 0$ with $\delta < \min_{1 \le i \le K} R_i$. We will demonstrate the achievability of the rate vector $\underline{\bar{R}}$ where $\underline{\bar{R}} = (\bar{R}_1, \dots, \bar{R}_J)$ with $\bar{R}_i = (R_i - \delta)$ for $1 \le i \le K$ and $\bar{R}_i = 0$ for $K+1 \le i \le J$. From the result on the convexity of the capacity region that we have proved earlier, on letting $\delta \to 0$, this will complete the proof.

We will show the existence of a sequence of coding schemes that verify the achievability of $\underline{\bar{R}}$ by a random coding argument. The coding schemes that we will show the existence of will be coding schemes in the usual sense, *i.e.*, they will be constant in each block. Thus we focus on any one block of the coding scheme to be constructed at block length n and drop the notation for the specific block under consideration.

Following Csiszar and Narayan, we change the normalization of power so that the additive Gaussian noise in the channel has mean zero and variance $\frac{\sigma^2}{n}$ in each coordinate and so that each codeword of each communicating user i and the jamming sequence of each jamming user i have norm at most $\sqrt{\Gamma_i}$ respectively.

Suppose user 1 chooses a random codebook $\mathcal{C}_1$ as follows : he first picks $M_1 = 2^{n\bar{R}_1}$ independent vectors in $\mathbf{R}^n$, each uniformly distributed on the sphere of radius 1, call them $\underline{z}_{11}, \dots, \underline{z}_{1M_1}$. These are scaled by $\sqrt{\Gamma_1}$ to give the codewords $\underline{x}_{1i_1} = \sqrt{\Gamma_1}\underline{z}_{1i_1}, 1 \le i_i \le M_1$. Users 2 through K choose their random codebooks by an analogous procedure.

The average probability of error of maximum likelihood decoding for user K, assuming that users 1 through $K-1$ have already been correctly decoded, when the jamming users use jamming sequences $\underline{s}_i$, $J+1 \le i \le I$ and users $K+1 \le i \le J$ send the zero vector, is given by

$$
\begin{aligned}
\bar{e}_K(\underline{s}) = \frac{1}{M_K} \sum_{i_K=1}^{M_K} P(\|\underline{x}_{Ki_K} + \sum_{i=J+1}^{I} \underline{s}_i + \underline{W} - \underline{x}_{Kj_K}\|^2 \le \| \sum_{i=J+1}^{I} \underline{s}_i + \underline{W}\|^2 \\
\text{for some } j_K \ne i_K \ ) \ .
\end{aligned}
$$

This should be thought of as a random variable, which is a function of the realization of the random codebook $\mathcal{C}_K$ as well as $\underline{s} = (\underline{s}_i, J+1 \le i \le I)$. Here $\underline{W} = (W_1, \dots, W_n)$ with coordinates that are i.i.d. Gaussian with mean zero and variance $\frac{\sigma^2}{n}$.

Fix $1 \le k < K$. The average probability of error of maximum likelihood decoding for user k, assuming that users 1 through $k-1$ (this set is empty if $k=1$) have already been correctly decoded, when the jamming users use jamming sequences $\underline{s}_i$, $J+1 \le i \le I$ and users $K+1 \le i \le J$ send the zero vector is given by

$$
\begin{aligned}
\bar{e}_k(\underline{s}) = \frac{1}{M_k} \sum_{i_k=1}^{M_k} \quad & P(\ \|\underline{x}_{ki_k} + \sum_{l=k+1}^{K} \underline{x}_{li_l} + \sum_{i=J+1}^{I} \underline{s}_i + \underline{W} - \underline{x}_{kj_k}\|^2 \\
& \le \| \sum_{l=k+1}^{K} \underline{x}_{li_l} + \sum_{i=J+1}^{I} \underline{s}_i + \underline{W}\|^2 \ \text{ for some } j_k \ne i_k) \ .
\end{aligned}
$$

Here i_l for $k+1 \le l \le K$ are thought of as random, uniformly distributed on $[M_l]$ respectively. Also, $\bar{e}_k(\underline{s})$ is thought of as a random variable which is a function of the realizations of the random codebooks $\mathcal{C}_k$ through to $\mathcal{C}_K$ as well as of $\underline{s} = (\underline{s}_i, J+1 \le i \le I)$.

For each $1 \leq k \leq K$, let $\bar{e}_k$ denote $\sup_{\underline{s}} \bar{e}_k(\underline{s})$. Again these are thought of as random variables dependent on the realization of the random codebooks.

We now consider the expectations of each of these quantities, taken over the distribution of the random codebooks. Consider first the expectation of $\bar{e}_K$. On the event that the random codebook $\mathcal{C}_K$ is derived from scaling a set of unit norm vectors that satisfy the conditions of Lemma 1 of Csiszar and Narayan, $\bar{e}_K$ is pointwise dominated by γ_n for some $\gamma_n \to 0$ as $n \to \infty$. Further, the complement of this event has probability bounded above by τ_n for some $\tau_n \to 0$ as $n \to \infty$. Hence the expectation of $\bar{e}_K$ is bounded above by some α_n where $\alpha_n \to 0$ as $n \to \infty$.

Next consider any $1 \leq k < K$. On the event that the codebook $\mathcal{C}_k$ is derived from scaling a set of unit norm vectors that satisfy the conditions of Lemma 1 of Csiszar and Narayan, $\bar{e}_K$ is pointwise dominated by γ_n for some $\gamma_n \to 0$ as $n \to \infty$, where in making this claim we appeal to the extension of the results in Csiszar and Narayan to noise models which are of the extended type discussed above, the salient point being that we now treat the entire vector $\sum_{l=k+1}^{K} \underline{x}_{l i_l} + \underline{W}$ as a noise vector (where we recall that for each $k+1 \leq l \leq K$ we think of i_l as a random variable drawn uniformly at random from $[M_l]$). As before the complement of this event has probability bounded above by τ_n for some $\tau_n \to 0$ as $n \to \infty$. Hence the expectation of $\bar{e}_k$ is bounded above by some α_n where $\alpha_n \to 0$ as $n \to \infty$.

We may therefore find realizations of the codebooks $\mathcal{C}_k, 1 \leq k \leq K$ such that for these realizations each of the quantities $\bar{e}_k, 1 \leq k \leq K$ is bounded above by α_n, where $\alpha_n \to 0$ as $n \to \infty$. This completes the proof. $\square$

3 Cooperative games

In Section 2 we have characterized the capacity region of a Gaussian multiaccess channel when a subset of users form a coalition of jamming users. The formation of such coalitions will depend on the offered rates to each individual user by proposed coalitions. In other words, a user will join a coalition of communicating users only if it feels that it is getting a fair share of the available rate based on its perception. A natural question that arises in such a scenario is how these users should allocate the available rate among themselves, given the possibility of forming different coalitions among themselves. Providing a plausible answer to this question for the Gaussian multiaccess channel is the main aim of this paper. We approach this question via cooperative game theory. In this section we therefore first briefly review the language of cooperative game theory [4].

Let $\mathcal{I} = \{1, 2, \cdots, I\}$ be the set of players. The problem is to decide how to allocate some fungible quantity between the users (in our case this quantity is communication rate), which we will call value. A key assumption is that all the users have a shared notion of value. A non-empty subset of $\mathcal{I}$ is called a coalition. The basic mathematical object that describes the game between the players is its *characteristic function*. This is a function v defined on the subsets of $\mathcal{I}$, which associates to each $S \subset \mathcal{I}$ the maximum value that the players in S can acquire from the game regardless of what the players in $\mathcal{I} \setminus S$ do. Namely, the players in S, acting as a coalition, can guarantee themselves as a group the value $v(S)$ even if the players in $\mathcal{I} \setminus S$, acting as a coalition, work to minimize the total value achieved by the players in S.

In the Gaussian multiaccess channel viewed as a game, $v(S)$ for a subset of users $S \subseteq \mathcal{I}$ is seen, from Theorem 2.3, to be

$$v(S) = C(\Gamma_{\hat{S}}, \Lambda_{\mathcal{I}\setminus S} + \sigma^2) \tag{2}$$

where $\Lambda_{\mathcal{I}\setminus S} = (\sum_{i\in\mathcal{I}\setminus S}\sqrt{\Gamma_i})^2$, $\hat{S} = \{i \in S \mid \Gamma_i > \Lambda_{\mathcal{I}\setminus S}\}$, and $\Gamma_{\hat{S}} = \sum_{i\in\hat{S}}\Gamma_i$.

It is generally assumed that the characteristic function should satisfy the following two properties :

(i) The value of characteristic function for an empty set is zero, *i.e.*, $v(\emptyset) = 0$.

(ii) Suppose that S and T are two disjoint coalitions, *i.e.*, $S \cap T = \emptyset$. Then,

$$v(S) + v(T) \le v(S \cup T) \qquad \text{(superadditivity)} \tag{3}$$

In our case, these properties are apparent. The truth of the second property is most easily seen from the obvious fact that the sum of the sum rates achievable by two disjoint coalitions of users each acting separately when its complementary set of users acts to jam it is no bigger than the sum rate that can be achieved when both these sets of users group together to form a single coalition whose complement is then acting to jam it.

An imputation for an I-person game v is a vector $x = (x_1, \cdots, x_I)$ that satisfies (i) $\sum_{i\in\mathcal{I}} x_i = v(\mathcal{I})$, and (ii) $x_i \ge v(\{i\})$ for all $i \in \mathcal{I}$. One can argue that any proposed allocation of value among the users that is not an imputation is unreasonable. This is because, first of all there is no point in leaving some value unallocated so the first condition had better be satisfied, and second, if some user is given an allocation that is less than what he/she could get by acting unilaterally even when everybody else forms a coalition to work against him/her, this user will not be satisfied with the proposed allocation.

We say that an imputation x dominates another imputation y through a coalition S if (i) $x_i > y_i$ for all $i \in S$, and (ii) $\sum_{i\in S} x_i \le v(S)$. Also, we say that x dominates y if there exists some coalition S such that x dominates y through S. One can argue that any proposed allocation of value among the users that is dominated by another allocation is unreasonable. This is because if the proposed allocation (necessarily an imputation), say y, is dominated by some other imputation, say x, through a subset of users S, then the users in S will find it to their advantage to disregard the proposed allocation, since they can act as a coalition to guarantee each of them a strictly better allocation even when the users in $\mathcal{I}\setminus S$ work as a coalition against them.

The set of all undominated imputations is called the *core* of the game, denoted by $C(v)$. With the axioms we have assumed on the characteristic function, one can show that $C(v)$ is given by the set of imputations that satisfy $\sum_{i\in S} x_i \ge v(S)$ for all $S \subset \mathcal{I}$. Also, what we have argued so far suggests that any proposed allocation of value among the users must lie in the core of the game if it is to be considered reasonable.

Here it is necessary, unfortunately, to point out a key problem with cooperative game theory : there are examples of games where the core can be empty. Such games would then seem to have no reasonable solution. On the other hand, in many examples the core is nonempty and for such games (of which, as we will prove, the Gaussian multiaccess game is one) the core provides a natural starting point in the search for reasonable solutions. However, for most games, including the one we are interested in, the core has more than one element. Thus one needs some additional natural set of axioms to further restrict the set of reasonable solutions. We will carry out this program for our game in the next two sections. To end this section, we mention another very popular axiomatic approach to the search for a notion of solution for a cooperative game, due to Shapley. Our purpose in describing the Shapley approach is only for completeness : axiom (a3) in the axiomatic formulation below is not natural for the class of games we consider, since there is no natural way in which the sums of the characteristic functions of two of our games is physically relevant. Indeed, the Shapley value also suffers from another problem, in that it can result in an allocation that is not in the core, as we will later show it does in our game. Nevertheless, being such a popular concept, it merits some discussion.

Let π be a permutation of the set of players $\mathcal{I}$. Then, we denote by πv the new game u such that, for all $S = \{i_1, \cdots, i_s\} \subseteq \mathcal{I}$, $u(\{\pi(i_1), \pi(i_2), \cdots, \pi(i_s)\}) = v(S)$. Clearly, this new game is the game v with the roles of the players interchanged by the permutation π.

Shapley proposed to study an allocation of value $\phi[v]$ in game v that satisfies the following set of axioms:

a1. Additive partition of the value of the game: $\sum_{i \in N} \phi_i[v] = v(\mathcal{I})$.

a2. Invariance under permutation: for any permutation π and $i \in \mathcal{I}$, $\phi_{\pi(i)}[\pi v] = \phi_i[v]$.

a3. If u and v are any two games, $\phi_i[u + v] = \phi_i[u] + \phi_i[v]$.

He showed that there is a unique function ϕ defined on all games that satisfies the axioms above. It is given by

$$\phi_i[v] = \sum_{T \subset \mathcal{N}, i \in T} \frac{(t-1)!(n-t)!}{n!} \left(v(T) - v(T - \{i\})\right) , \tag{4}$$

where $t = |T|$. This allocation is called the *Shapley value* of the game.

4 An envy-free allocation of rate

In the preceding section we have already implicitly introduced our notion of the Gaussian multiaccess game through definition of its characteristic function, given in equation (2). Communication rate plays the role of value in our game, and the maximum value that a coalition of users can get even when the other users work as a coalition against them is the maximum sum capacity of the corresponding Gaussian multiaccess arbitrarily varying channel, whose capacity region we characterized in Theorem 2.3.

In this section we first discuss the core of our game. The core can be immediately written down as the set of allocations satisfying a set of linear inequalities determined from the characteristic function, as was mentioned in Section 3 – all this uses is that the characteristic function of our game is superadditive. The most important point for us is that the core contains all feasible imputations, as demonstrated in subsection 4.1. We then propose a natural set of fairness axioms which we will work with in Section 5 to propose an allocation in the game that is uniquely defined as the one satisfying our axioms and will turn out to be both feasible and in the core. This is done in subsection 4.2. Finally in subsection 4.3 we briefly return to the Shapley value and give a simple example of a Gaussian multiaccess game where the Shapley value is neither feasible nor in the core.

4.1 The core of a Gaussian multiaccess game

Recall that the capacity region for a Gaussian multiaccess channel, which can be thought of as the capacity region corresponding to the game when all the users form a single communicating coalition, is given by the set of rate vectors satisfying

$$\sum_{i \in A} R_i \leq \quad C(\Gamma_A, \sigma^2) \qquad \text{for all } A \subseteq \mathcal{I} \ . \tag{5}$$

We denote the set of rate vectors $\underline{R}$ that satisfy (5) by $\mathcal{C}$. The main result about the core of our game that interests us is the following :

Proposition 1 *The core of the game $C(v)$ contains $\{\underline{R} \in \mathcal{C} \mid \sum_{i \in \mathcal{I}} R_i = C(\Gamma_{\mathcal{I}}, \sigma^2)\}$.*

Proof: We show that if $\underline{R}$ belongs to the capacity region and $\sum_{i \in \mathcal{I}} R_i = C(\Gamma_{\mathcal{I}}, \sigma^2)$, then for all $S \subseteq \mathcal{I}$,

$$\sum_{i \in S} R_i \geq C(\Gamma_S, \lambda_{\mathcal{I} \setminus S} + \sigma^2) \geq v(S) \ .$$

This can be easily shown from the following:

$$\begin{aligned} \sum_{i \in S} R_i &= C(\Gamma_{\mathcal{I}}, \sigma^2) - \sum_{j \in \mathcal{I} \setminus S} R_j \\ &\geq C(\Gamma_{\mathcal{I}}, \sigma^2) - C(\Gamma_{\mathcal{I} \setminus S}, \sigma^2) \\ &= C(\Gamma_S, \lambda_{\mathcal{I} \setminus S} + \sigma^2) \ , \end{aligned}$$

where the inequality follows from the capacity constraint in (5) with $A = \mathcal{I} \setminus S$. $\square$

One sees from this result that the core of our game is not empty. In fact, *all* imputations in $\mathcal{C}$ belong to the core, and so it is also the case that the core is not unique. Therefore, to arrive at a reasonable allocation that serves as a solution for the game, we need to propose a natural system of axioms that one can argue the allocation should satisfy. This allocation should, of course, also be feasible for it to make sense. We now turn to this issue.

4.2 Fair allocation

We now propose an axiomatic system to define a fair allocation in our game. The first two axioms of Shapley (axioms a1 and a2) which require efficiency and invariance under permutation respectively, are both natural requirements : there is no point in leaving some rate unallocated, and any solution concept that yields different values for the same player if they are labelled differently is hardly natural. Hence, we impose the following two axioms:

(f1) Efficiency or Pareto optimality: $\sum_{i \in \mathcal{I}} \psi_i(v) = C(\Gamma_{\mathcal{I}}, \sigma^2)$.

(f2) Invariance under permutation: For any permutation π and $i \in \mathcal{I}$, $\psi_i(v) = \psi_{\pi(i)}(\pi v)$, where πv is the permuted game.

To these, we add a third axiom based on the notion of envy [2].[1] The second axiom of Shapley states that two users with the same power constraint should receive the same rate. Now suppose that we take two users i and j such that $\Gamma_i > \Gamma_j$. Then, clearly user i can act as if its power constraint were also Γ_j instead of Γ_i by not consuming all of its power constraint. Let this new game with user i's power constraint reduced to Γ_j be denoted by $v^{i,j}$. We say that user i envies user j if $\psi_j(v) \neq \psi_i(v^{i,j})$, and we measure it by the difference $\psi_j(v) - \psi_i(v^{i,j})$. Based on this definition of envy we impose the following fairness constraint:

(f3) Fairness (or envy-free allocation)

$$\sum_{\{j \neq i | \Gamma_j \leq \Gamma_i\}} \psi_j(v) - \psi_i(v^{i,j}) = \sum_{\{j \neq i | \Gamma_j < \Gamma_i\}} \psi_j(v) - \psi_i(v^{i,j}) = 0 \text{ for all } i \in \mathcal{I} ,$$

where the first equality is a consequence of axiom (f2).

We will show in the following section that axiom (f3) is equivalent to the stronger requirement that for all i and j such that $\Gamma_i > \Gamma_j$

$$\psi_j(v) - \psi_i(v^{i,j}) = 0 . \tag{6}$$

Also in that section we will show that there is a unique allocation satisfying our axioms and this allocation is feasible and in the core.

4.3 About the Shapley value

To end this section we briefly discuss the popular concept of Shapley value, even though we have argued that the axiom system underlying this concept is not appropriate for our game. We give an example of a Gaussian multiaccess game where the Shapley value is not feasible and does not lie in the core.

Consider the following example: there are three users with $\Gamma_1 = 10.1, \Gamma_2 = 10$, and $\Gamma_3 = 1$. The noise power is given by $\sigma^2 = 10$. In this example, from (4), the Shapley value of user 3 is

$$\begin{aligned} \phi_3[v] &= \frac{1}{3}\left(v(\{3\}) - v(\emptyset)\right) + \frac{1}{6}\left(\left(v(\{1,3\}) - v(\{1\})\right) + \left(v(\{2,3\}) - v(\{2\})\right)\right) \\ &\quad + \frac{1}{3}\left(v(\{1,2,3\}) - v(\{1,2\})\right) \\ &= \frac{1}{3}(0-0) + \frac{1}{6}((C(10.1,20) - 0) + (0-0)) + \frac{1}{3}(C(21.1,10) - C(20.1,11)) \\ &= 0.07206 . \end{aligned}$$

However, the rate of user 3 cannot be larger than $C(\Gamma_3, \sigma^2) = C(1, 10) = 0.06875$ from the capacity constraint in (5). Therefore, the Shapley value of this game does not belong to $\mathcal{C}$.

In addition, one can see that the Shapley value of the game does not belong to the core of the game by realizing that $\phi_1[v] + \phi_2[v] = C(21.1, 10) - \phi_1[v] < C(20.1, 11) = v(\{1, 2\})$.

[1]The definition of envy is slightly modified in our context from the reference.

5 Existence, uniqueness and properties of a fair allocation

In this section we first show that there exists a unique function ψ, defined on all games, which satisfies axioms (f1) - (f3) in Section 4.2, and then prove that the unique allocation always lies in the capacity region, *i.e.,* it is feasible. Since the allocation is efficient it then follows from Proposition 1 that it is also in the core. Without loss of generality we assume that users are ordered by decreasing power constraint Γ_i throughout the rest of this paper.

5.1 Existence of a fair allocation

Theorem 5.1 *There exists a unique rate allocation that satisfies axioms (f1) through (f3).*

Proof: Consider the following rate allocation: for all $i \in \mathcal{I}$

$$\varphi_i(v) = \frac{C(i\Gamma_i + \sum_{j=i+1}^{I} \Gamma_j, \sigma^2) - \sum_{j=i+1}^{I} \varphi_j(v)}{i} \; . \tag{7}$$

One can see that such an allocation can be easily computed recursively, starting with user I. Axiom (f1) is trivially satisfied by construction. Suppose that there are two users i and $i+1$ such that $\Gamma_i = \Gamma_{i+1}$. We show that $\varphi_i(v) = \varphi_{i+1}(v)$. In order to show this we prove that for all $k = 1, \cdots, I-1$,

$$\varphi_k(v) - \varphi_{k+1}(v) = \frac{C(k\Gamma_k + \sum_{j=k+1}^{I} \Gamma_j, \sigma^2) - C(k\Gamma_{k+1} + \sum_{j=k+1}^{I} \Gamma_j, \sigma^2)}{k} \; . \tag{8}$$

This can be proved by showing that the sum of the right-hand side of (8) and $\varphi_{k+1}(v)$ from (7) equals $\varphi_k(v)$ as follows.

$$\begin{aligned}
&\varphi_{k+1}(v) + \frac{C(k\Gamma_k + \sum_{j=k+1}^{I} \Gamma_j, \sigma^2) - C(k\Gamma_{k+1} + \sum_{j=k+1}^{I} \Gamma_j, \sigma^2)}{k} \\
&= \frac{C(k\Gamma_{k+1} + \sum_{j=k+1}^{I} \Gamma_j, \sigma^2) - \sum_{j=k+2}^{I} \varphi_j(v)}{k+1} \\
&\quad + \frac{C(k\Gamma_k + \sum_{j=k+1}^{I} \Gamma_j, \sigma^2) - C(k\Gamma_{k+1} + \sum_{j=k+1}^{I} \Gamma_j, \sigma^2)}{k} \\
&= \frac{1}{k}\left(C(k\Gamma_k + \sum_{j=k+1}^{I} \Gamma_j, \sigma^2) - \frac{C((k+1)\Gamma_{k+1} + \sum_{j=k+2}^{I} \Gamma_j, \sigma^2) + k\sum_{j=k+2}^{I} \varphi_j(v)}{k+1} \right) \\
&= \frac{C(k\Gamma_k + \sum_{j=k+1}^{I} \Gamma_j, \sigma^2)}{k} \\
&\quad - \frac{1}{k}\left(\frac{C((k+1)\Gamma_{k+1} + \sum_{j=k+2}^{I} \Gamma_j, \sigma^2) - \sum_{j=k+2}^{I} \varphi_j(v) + (k+1)\sum_{j=k+2}^{I} \varphi_j(v)}{k+1} \right) \\
&= \frac{C(k\Gamma_k + \sum_{j=k+1}^{I} \Gamma_j, \sigma^2)}{k} \\
&\quad - \frac{1}{k}\left(\frac{C((k+1)\Gamma_{k+1} + \sum_{j=k+2}^{I} \Gamma_j, \sigma^2) - \sum_{j=k+2}^{I} \varphi_j(v)}{k+1} + \sum_{j=k+2}^{I} \varphi_j(v) \right) \\
&= \frac{C(k\Gamma_k + \sum_{j=k+1}^{I} \Gamma_j, \sigma^2) - \left(\varphi_{k+1}(v) + \sum_{j=k+2}^{I} \varphi_j(v)\right)}{k} \\
&= \varphi_k(v) \; .
\end{aligned}$$

From (8) one can see that if $\Gamma_i = \Gamma_{i+1}$, then $\varphi_i(v) - \varphi_{i+1}(v) = 0$. Hence, this proves that the allocation given by (7) satisfies axiom (f2).

In order to show that axiom (f3) is satisfied, we first prove the following Lemma.

Lemma 1 *For all i and j such that $\Gamma_i > \Gamma_j$, $\varphi_i(v^{i,j}) = \varphi_j(v)$.*

Proof: This can be easily shown from

$$\begin{aligned}\varphi_i(v^{i,j}) &= \frac{C(j\Gamma_j + \sum_{k=j+1}^{I}\Gamma_k, \sigma^2) - \sum_{k=j+1}^{I}\varphi_k(v^{i,j})}{j} \\ &= \frac{C(j\Gamma_j + \sum_{k=j+1}^{I}\Gamma_k, \sigma^2) - \sum_{k=j+1}^{I}\varphi_k(v)}{j} \\ &= \varphi_j(v)\ ,\end{aligned}$$

where the first equality is a consequence of axiom (f2), and the second equality follows because for all $k > j$, $\varphi_k(v) = \varphi_k(v^{i,j})$, *i.e.*, the rates of users $k > j$ are not affected by the replacement of power constraint of user i by that of user j as can be seen from (7). $\square$

Now axiom (f3) holds trivially from

$$\sum_{\{j>i|\Gamma_j<\Gamma_i\}} (\varphi_j(v) - \varphi_i(v^{i,j})) = \sum_{\{j>i|\Gamma_j<\Gamma_i\}} (0) = 0\ .$$

Note that, combined with Lemma 1, axiom (f3) implies the stronger condition given in (6) under $\varphi(\cdot)$.

5.2 Uniqueness of the fair allocation

We now prove the uniqueness of rate allocation that satisfies axioms (f1) - (f3). For $i = 2, \cdots, I$, let $u^{i,j}$, $j < i$, be the game where the power constraints of the users are given by $\tilde{\Gamma}_k$

$$\tilde{\Gamma}_k = \begin{cases} \Gamma_i & j \le k \le i \\ \Gamma_k & \text{otherwise} \end{cases}\ .$$

Let $i = I$. From the fairness constraint in axiom (f3), we have $\psi_{I-1}(u^{I,I-1}) = \psi_I(v) = \psi_I(u^{I,I-1})$. Starting with $j = I - 1$ repeatedly applying the fairness requirement to user $j - 1$ in game $u^{I,j}$ with decreasing j, $j = 2, \cdots, I - 1$, yields

$$\psi_{j-1}(u^{I,j-1}) = \psi_j(u^{I,j-1}) = \cdots = \psi_{I-1}(u^{I,I-1}) = \psi_I(v)\ ,$$

which, for $j = 2$, gives us $\psi_I(v) = \frac{C(I\cdot\Gamma_I,\sigma^2)}{I}$ from axiom (f2) since all users have the same power constraint Γ_I in game $u^{I,1}$. Following similar steps with $v^{j,I}$, $j = 1, \cdots, I-1$, also leads to $\psi_j(v^{j,I}) = \psi_I(v) = \frac{C(I\cdot\Gamma_I,\sigma^2)}{I}$.

Decrease i by one, *i.e.*, let $i = I - 1$. Following the above argument starting with $j = i$ and successively applying the fairness requirement to $u^{i,j-1}$ with decreasing j yields

$$\psi_i(v) = \frac{1}{i}\left(C(i\Gamma_i + \sum_{k=i+1}^{I}\Gamma_k, \sigma^2) - \sum_{k=i+1}^{I}\psi_k(v)\right) = \varphi_i(v)\ . \tag{9}$$

Repeating this procedure after decrementing i by one until $i = 1$ leads to (9) for all $i = 1, \cdots, I$. This completes the proof. $\square$

5.3 On the structure of the fair allocation

We now provide an intuitive way of understanding the structure of the fair allocation whose existence and uniqueness was demonstrated in the last two subsections. Consider the following scenario. Initially all users start with the same power constraint Γ_I, where each user receives a rate of $\frac{C(I\cdot\Gamma_I,\sigma^2)}{I}$. Now suppose that user $I-1$ decides to unilaterally increase its power constraint to Γ_{I-1}. Then, since user $I-1$ was willing to pay the price of increasing its power constraint (assuming that it finds this decision beneficial), the increase in the total rate, namely $C(\Gamma_{I-1}+(I-1)\cdot\Gamma_I,\sigma^2)-C(I\Gamma_I,\sigma^2)$, should be assigned to user $I-1$, while the rates of the other users remain the same. Suppose now user $I-2$ pays the price to increase its power constraint to Γ_{I-1}. Then, the gain enjoyed by user $I-1$ before should be shared with user $I-2$ as well. More specifically, the increase in the total rate $C(2\Gamma_{I-1}+(I-2)\cdot\Gamma_I,\sigma^2)-C(I\Gamma_I,\sigma^2)$, should be equally shared by users $I-2$ and $I-1$. Similar arguments can be made when another user unilaterally changes its power constraint. The rate allocation in (7) the allocation scheme resulting from this thought process.

5.4 Feasibility and reasonableness of the fair allocation

We now demonstrate that the rate allocation $\varphi(v)$ lies in the capacity region, i.e. it feasible, and then show that it is in the core of the game.

Theorem 5.2 *The rate allocation given by (7) lies in the capacity region, i.e., for all $S\subseteq\mathcal{I}$, $\sum_{i\in S}\varphi_i(v)\le C\left(\sum_{i\in S}\Gamma_i,\sigma^2\right)$.*

Proof: Before we prove the theorem we note that several simple inequalities will be used in the proof, all of which can be obtained from the strict concavity of log function. These are gathered in the Appendix.

We prove the theorem by induction. If there is only one user, the theorem is true. Now suppose that the theorem holds for any set of k users, $k=1,\cdots,K$. We now show that the theorem is true for any set of $K+1$ users.

Let $\mathcal{I}^{K+1}=\{1,\cdots,K+1\}$ be the set of users and denote the game by v. We let $S^j=\mathcal{I}^{K+1}\setminus\{j\}$, where $j\in\mathcal{I}^{K+1}$, and denote the game only with the users in S^j by v^j. In order to prove that theorem holds with $\mathcal{I}^{K+1}$, it suffices to show that, for all $j\in\mathcal{I}^{K+1}$

$$\varphi_i(v^j)>\varphi_i(v),\ i\neq j\ . \tag{10}$$

Eq. (10) tells us that when a new user enters the game, the rate of the existing users strictly decreases. Therefore, from the induction hypothesis, for all strict subsets S of $\mathcal{I}^{K+1}$ we will have $\sum_{i\in S}\varphi_i(v)<C\left(\sum_{i\in S}\Gamma_i,\sigma^2\right)=\sum_{i\in S}\varphi_i(v^S)$, where v^S is the game only with users in S. Thus, the theorem will follow.

We consider the following three cases: (1) $\Gamma_j=\Gamma_{K+1}$, (2) $\Gamma_j=\Gamma_1$, and (3) $\Gamma_1<\Gamma_j<\Gamma_{K+1}$. case (1). $\Gamma_j=\Gamma_{K+1}$: In this case, from axiom (f2), without loss of generality we assume that $j=K+1$. First, from (7) and (8)

$$\begin{aligned}\varphi_K(v)&=\varphi_{K+1}(v)+(\varphi_K(v)-\varphi_{K+1}(v))\\&=\frac{C((K+1)\Gamma_{K+1},\sigma^2)}{K+1}+\frac{C(KP_K+\Gamma_{K+1},\sigma^2)-C((K+1)\Gamma_{K+1},\sigma^2)}{K}\end{aligned}$$

$$= \frac{C(KP_K + \Gamma_{K+1}, \sigma^2)}{K} - \frac{C((K+1)\Gamma_{K+1}, \sigma^2)}{K(K+1)}$$

$$\begin{aligned}
&\varphi_K(v^{K+1}) - \varphi_K(v) \\
&= \frac{1}{K}\left(C(K \cdot \Gamma_K, \sigma^2) - C(K \cdot \Gamma_K + \Gamma_{K+1}, \sigma^2) + \frac{1}{K+1}C((K+1)\Gamma_{K+1}, \sigma^2)\right) \\
&\geq \frac{1}{K}\left(\frac{1}{K+1}C((K+1)\Gamma_{K+1}, \sigma^2) - \Big(C((K+1)\Gamma_{K+1}, \sigma^2) - C(K\Gamma_{K+1}, \sigma^2)\Big)\right) \\
&> 0 \ ,
\end{aligned}$$

where the first inequality follows from (23) (with $x = K \cdot \Gamma_{k+1}, y = K \cdot \Gamma_k$, and $z = \Gamma_{k+1}$), and the second inequality from (24) (with $x = \Gamma_{k+1}$, and $z = 0$). From (8) and (26), one can show that for all $i = 1, \cdots, K-1$,

$$\begin{aligned}
\varphi_i(v) - \varphi_{i+1}(v) &= \frac{1}{i}\left(C(i\Gamma_i + \sum_{j=i+1}^{K+1}\Gamma_j, \sigma^2) - C(i\Gamma_{i+1} + \sum_{j=i+1}^{K+1}\Gamma_j, \sigma^2)\right) \\
&\leq \frac{1}{i}\left(C(i\Gamma_i + \sum_{j=i+1}^{K}\Gamma_j, \sigma^2) - C(i\Gamma_{i+1} + \sum_{j=i+1}^{K}\Gamma_j, \sigma^2)\right) \\
&= \varphi_i(v^{K+1}) - \varphi_{i+1}(v^{K+1})
\end{aligned}$$

where the equality holds only if $\Gamma_i = \Gamma_{i+1}$. Eq. (10) now follows from the above two inequalities.

case (2). $\Gamma_j = \Gamma_1$: Again, from axiom (f2), without loss of generality we assume that $j = 1$. This case follows directly from (8) and (25) because, for all $i = 2, \cdots, K$,

$$\begin{aligned}
\varphi_i(v) - \varphi_{i+1}(v) &= \frac{1}{i}\left(C(i\Gamma_i + \sum_{k=i+1}^{K+1}\Gamma_k, \sigma^2) - C(i\Gamma_{i+1} + \sum_{k=i+1}^{K+1}\Gamma_k, \sigma^2)\right) \\
&\leq \frac{1}{i-1}\left(C((i-1)\Gamma_i + \sum_{k=i+1}^{K+1}\Gamma_k, \sigma^2) - C((i-1)\Gamma_{i+1} + \sum_{k=i+1}^{K+1}\Gamma_k, \sigma^2)\right) \\
&= \varphi_i(v^1) - \varphi_{i+1}(v^1) \ .
\end{aligned} \tag{11}$$

and $\varphi_{K+1}(v) = \frac{1}{K+1}C((K+1)\Gamma_{K+1}, \sigma^2) < \frac{1}{K}C(K\Gamma_{K+1}, \sigma^2) = \varphi_{K+1}(v^1)$.

case (3). $\Gamma_1 < \Gamma_j < \Gamma_{K+1}$: Similarly as in (11), we have for all $i = j+1, \cdots, K$,

$$\varphi_i(v) - \varphi_{i+1}(v) \leq \varphi_i(v^j) - \varphi_{i+1}(v^j) \ \text{ and } \ \varphi_{K+1}(v) < \varphi_{K+1}(v^j) \ . \tag{12}$$

We first show that, for all $\Gamma_j \in [\Gamma_{j+1}, \Gamma_{j-1}]$

$$\varphi_{j-1}(v) - \varphi_{j+1}(v) \leq \varphi_{j-1}(v^j) - \varphi_{j+1}(v^j) \ . \tag{13}$$

Suppose that $\Gamma_{j-1} = \Gamma_{j+1}$. Then, (13) is trivially true from axiom (f2). Hence, we assume that $\Gamma_{j-1} > \Gamma_{j+1}$. One can show from (7) that $\varphi_{j-1}(v) - \varphi_{j+1}(v)$ is strictly decreasing in Γ_j over the range $[\Gamma_{j+1},\ \Gamma_{j-1}]$. This can be shown as follows. Suppose that $\Gamma_j < \Gamma_{j-1}$ and Δ is some positive constant such that $0 < \Delta \leq \Gamma_{j-1} - \Gamma_j$. Then, one can show that

$$\begin{aligned}
&\varphi_{j-1}(v) - \varphi_{j+1}(v) \\
&= \frac{1}{j-1}\left(C((j-1)\Gamma_{j-1} + \sum_{i=j}^{I}\Gamma_i, \sigma^2) - C((j-1)\Gamma_j + \sum_{i=j}^{I}\Gamma_i, \sigma^2)\right) \\
&\quad + \frac{1}{j}\left(C(j \cdot \Gamma_j + \sum_{i=j+1}^{I}\Gamma_i, \sigma^2) - C(j \cdot \Gamma_{j+1} + \sum_{i=j+1}^{I}\Gamma_i, \sigma^2)\right) \\
&= \frac{C((j-1)\Gamma_{j-1} + \sum_{i=j}^{I}\Gamma_i, \sigma^2)}{j-1} - \frac{C((j-1)\Gamma_j + \sum_{i=j}^{I}\Gamma_i, \sigma^2)}{j(j-1)} \\
&\quad - \frac{C(j \cdot \Gamma_{j+1} + \sum_{i=j+1}^{I}\Gamma_i, \sigma^2)}{j} \\
&> \frac{C((j-1)\Gamma_{j-1} + \Delta + \sum_{i=j}^{I}\Gamma_i, \sigma^2)}{j-1} - \frac{C((j-1)\Gamma_j + j \cdot \Delta + \sum_{i=j}^{I}\Gamma_i, \sigma^2)}{j(j-1)} \\
&\quad - \frac{C(j \cdot \Gamma_{j+1} + \sum_{i=j+1}^{I}\Gamma_i, \sigma^2)}{j} \\
&= \frac{1}{j-1}\left(C((j-1)\Gamma_{j-1} + \Delta + \sum_{i=j}^{I}\Gamma_i, \sigma^2) - C((j-1)(\Gamma_j + \Delta) + \Delta + \sum_{i=j}^{I}\Gamma_i, \sigma^2)\right) \\
&\quad + \frac{1}{j}\left(C(j \cdot (\Gamma_j + \Delta) + \sum_{i=j+1}^{I}\Gamma_i, \sigma^2) - C(j \cdot \Gamma_{j+1} + \sum_{i=j+1}^{I}\Gamma_i, \sigma^2)\right)
\end{aligned}$$

where the inequality follows from the strict concavity of log function. Hence, it suffices to show that (13) is true with $\Gamma_j = \Gamma_{j+1}$ as follows:

$$\begin{aligned}
&\varphi_{j-1}(v) - \varphi_{j+1}(v) = \varphi_{j-1}(v) - \varphi_j(v) \\
&\quad = \frac{1}{j-1}\left(C((j-1)\Gamma_{j-1} + \sum_{k=j}^{K+1}\Gamma_k, \sigma^2) - C((j-1)\Gamma_j + \sum_{k=j}^{K+1}\Gamma_k, \sigma^2)\right) \\
&\quad < \frac{1}{j-1}\left(C((j-1)\Gamma_{j-1} + \sum_{k=j+1}^{K+1}\Gamma_k, \sigma^2) - C((j-1)\Gamma_{j+1} + \sum_{k=j+1}^{K+1}\Gamma_k, \sigma^2)\right) \\
&\quad = \varphi_{j-1}(v^j) - \varphi_{j+1}(v^j)\ ,
\end{aligned}$$

where the inequality follows from (26).

Now we show that for all $i = 1, \cdots, j-2$,

$$\varphi_i(v) - \varphi_{i+1}(v) \leq \varphi_i(v^j) - \varphi_{i+1}(v^j)\ . \tag{14}$$

This follows trivially because

$$\varphi_i(v) - \varphi_{i+1}(v) = \frac{1}{i}\left(C(i \cdot \Gamma_i + \sum_{k=i+1}^{I}\Gamma_k, \sigma^2) - C(i \cdot \Gamma_{i+1} + \sum_{k=i+1}^{I}\Gamma_k, \sigma^2)\right)$$

$$\leq \frac{1}{i}\left(C(i\cdot\Gamma_i+\sum_{k=i+1,k\neq j}^{I}\Gamma_k,\sigma^2)-C(i\cdot\Gamma_{i+1}+\sum_{k=i+1,k\neq j}^{I}\Gamma_k,\sigma^2)\right)$$
$$=\varphi_i(v^j)-\varphi_{i+1}(v^j)\ ,$$

where the inequality is true from (26). Eq. (10) now follows from (12) - (14). □

Theorem 5.3 *The rate allocation $\varphi(v)$ is in the core of the game.*

Proof: This is a simple consequence of the fact that the core contains all imputations that are in $\mathcal{C}$ and $\varphi(v)$ is an imputation (from axiom (f1)) and lies in $\mathcal{C}$ from Theorem 5.2. □

5.5 Qualitative properties of the fair allocation

Finally, we investigate how the allocation $\varphi(v)$ changes as one varies the noise power σ^2. In order to make the dependence on σ^2 explicit we denote by $\varphi^\sigma(v)$ the rate allocation given the noise power σ^2.

Theorem 5.4 *The rate allocation $\varphi^\sigma(v)$ satisfies the following: for all $i, j \in \mathcal{I}$ such that $\Gamma_j > \Gamma_i$*

(1) $\frac{\varphi_j^\sigma(v)}{\varphi_i^\sigma(v)}$ *is strictly increasing in $\sigma, \sigma > 0$.*

(2) $\lim_{\sigma^2\downarrow 0}\frac{\varphi_j^\sigma(v)}{\varphi_i^\sigma(v)}=1$.

(3) $\lim_{\sigma^2\uparrow\infty}\frac{\varphi_j^\sigma(v)}{\varphi_i^\sigma(v)}=\frac{\Gamma_j}{\Gamma_i}$.

From (1) - (3) for any i and j such that $\Gamma_j \geq \Gamma_i$, we have

$$1\leq\frac{\varphi_j(v)}{\varphi_i(v)}\leq\frac{\Gamma_j}{\Gamma_i}\ .$$

Proof: The first property can be proved by showing that for all $k = 1, \cdots, I-1$, and $\epsilon > 0$,

$$\frac{\varphi_k^{\sigma+\epsilon}(v)-\varphi_{k+1}^{\sigma+\epsilon}(v)}{\varphi_k^{\sigma}(v)-\varphi_{k+1}^{\sigma}(v)}\geq\frac{\varphi_{k+1}^{\sigma+\epsilon}(v)-\varphi_{k+2}^{\sigma+\epsilon}(v)}{\varphi_{k+1}^{\sigma}(v)-\varphi_{k+2}^{\sigma}(v)} \tag{15}$$

where Γ_{I+1} and $\varphi_{I+1}^\sigma(v)$ are defined to be zero. Here we assume that $\Gamma_k > \Gamma_{k+1} > \Gamma_{k+2}$ and show that a strict inequality holds. The other cases can be handled similarly. After some algebra we get

$$\frac{\varphi_k^{\sigma+\epsilon}(v)-\varphi_{k+1}^{\sigma+\epsilon}(v)}{\varphi_k^{\sigma}(v)-\varphi_{k+1}^{\sigma}(v)}=\frac{\left[C\left(k\Gamma_k+\sum_{j=k+1}^{I}\Gamma_j,(\sigma+\epsilon)^2\right)-C\left(k\Gamma_{k+1}+\sum_{j=k+1}^{I}\Gamma_j,(\sigma+\epsilon)^2\right)\right]}{\left[C\left(k\Gamma_k+\sum_{j=k+1}^{I}\Gamma_j,\sigma^2\right)-C\left(k\Gamma_{k+1}+\sum_{j=k+1}^{I}\Gamma_j,\sigma^2\right)\right]}$$
$$=\frac{\log\left(\frac{k(\Gamma_k-\Gamma_{k+1})+(\sigma+\epsilon)^2+k\Gamma_{k+1}+\sum_{j=k+1}^{I}\Gamma_j}{(\sigma+\epsilon)^2+k\Gamma_{k+1}+\sum_{j=k+1}^{I}\Gamma_j}\right)}{\log\left(\frac{k(\Gamma_k-\Gamma_{k+1})+\sigma^2+k\Gamma_{k+1}+\sum_{j=k+1}^{I}\Gamma_j}{\sigma^2+k\Gamma_{k+1}+\sum_{j=k+1}^{I}\Gamma_j}\right)} \tag{16}$$

and

$$\begin{aligned}
&\frac{\varphi_{k+1}^{\sigma+\epsilon}(v)-\varphi_{k+2}^{\sigma+\epsilon}(v)}{\varphi_{k+1}^{\sigma}(v)-\varphi_{k+2}^{\sigma}(v)} \\
&=\frac{\left[C\left((k+1)\Gamma_{k+1}+\sum_{j=k+2}^{I}\Gamma_j,(\sigma+\epsilon)^2\right)-C\left((k+1)\Gamma_{k+2}+\sum_{j=k+2}^{I}\Gamma_j,(\sigma+\epsilon)^2\right)\right]}{\left[C\left((k+1)\Gamma_{k+1}+\sum_{j=k+2}^{I}\Gamma_j,\sigma^2\right)-C\left((k+1)\Gamma_{k+2}+\sum_{j=k+2}^{I}\Gamma_j,\sigma^2\right)\right]} \\
&=\frac{\log\left(\frac{(k+1)(\Gamma_{k+1}-\Gamma_{k+2})+(\sigma+\epsilon)^2+(k+1)\Gamma_{k+2}+\sum_{j=k+2}^{I}\Gamma_j}{(\sigma+\epsilon)^2+(k+1)\Gamma_{k+2}+\sum_{j=k+2}^{I}\Gamma_j}\right)}{\log\left(\frac{(k+1)(\Gamma_{k+1}-\Gamma_{k+2})+\sigma^2+(k+1)\Gamma_{k+2}+\sum_{j=k+2}^{I}\Gamma_j}{\sigma^2+(k+1)\Gamma_{k+2}+\sum_{j=k+2}^{I}\Gamma_j}\right)} \\
&=\frac{\log\left(\frac{(k+1)\Gamma_{k+1}+(\sigma+\epsilon)^2+\sum_{j=k+2}^{I}\Gamma_j}{(\sigma+\epsilon)^2+(k+1)\Gamma_{k+2}+\sum_{j=k+2}^{I}\Gamma_j}\right)}{\log\left(\frac{(k+1)\Gamma_{k+1}+\sigma^2+\sum_{j=k+2}^{I}\Gamma_j}{\sigma^2+(k+1)\Gamma_{k+2}+\sum_{j=k+2}^{I}\Gamma_j}\right)}\,. \qquad (17)
\end{aligned}$$

Let $\gamma = k(\Gamma_k - \Gamma_{k+1})$, $\alpha = (k+1)(\Gamma_{k+1} - \Gamma_{k+2})$, and $\beta = (k+1)\Gamma_{k+2} + \sum_{j=k+2}^{I}\Gamma_j$. Then,

$$\frac{\varphi_{k}^{\sigma+\epsilon}(v)-\varphi_{k+1}^{\sigma+\epsilon}(v)}{\varphi_{k}^{\sigma}(v)-\varphi_{k+1}^{\sigma}(v)}=\frac{\log\left(\frac{\gamma+(\sigma+\epsilon)^2+\alpha+\beta}{(\sigma+\epsilon)^2+\alpha+\beta}\right)}{\log\left(\frac{\gamma+\sigma^2+\alpha+\beta}{\sigma^2+\alpha+\beta}\right)} \qquad (18)$$

$$\frac{\varphi_{k+1}^{\sigma+\epsilon}(v)-\varphi_{k+2}^{\sigma+\epsilon}(v)}{\varphi_{k+1}^{\sigma}(v)-\varphi_{k+2}^{\sigma}(v)}=\frac{\log\left(\frac{(\sigma+\epsilon)^2+\alpha+\beta}{(\sigma+\epsilon)^2+\beta}\right)}{\log\left(\frac{\sigma^2+\alpha+\beta}{\sigma^2+\beta}\right)} \qquad (19)$$

One can show that (18) is a strictly increasing function of γ and

$$\lim_{\gamma\downarrow 0}\frac{\log\left(\frac{\gamma+(\sigma+\epsilon)^2+\alpha+\beta}{(\sigma+\epsilon)^2+\alpha+\beta}\right)}{\log\left(\frac{\gamma+\sigma^2+\alpha+\beta}{\sigma^2+\alpha+\beta}\right)}=\frac{\sigma^2+\alpha+\beta}{(\sigma+\epsilon)^2+\alpha+\beta}$$

by L'Hôpital's rule. In order to prove (15) we now show that

$$\frac{\sigma^2+\alpha+\beta}{(\sigma+\epsilon)^2+\alpha+\beta}\geq\frac{\varphi_{k+1}^{\sigma+\epsilon}(v)-\varphi_{k+2}^{\sigma+\epsilon}(v)}{\varphi_{k+1}^{\sigma}(v)-\varphi_{k+2}^{\sigma}(v)} \qquad (20)$$

From (19)

$$\begin{aligned}
\frac{\varphi_{k+1}^{\sigma+\epsilon}(v)-\varphi_{k+2}^{\sigma+\epsilon}(v)}{\varphi_{k+1}^{\sigma}(v)-\varphi_{k+2}^{\sigma}(v)}&=\frac{\log\left(\frac{(\sigma+\epsilon)^2+\alpha+\beta}{(\sigma+\epsilon)^2+\beta}\right)}{\log\left(\frac{\sigma^2+\alpha+\beta}{\sigma^2+\beta}\right)} \\
&=\frac{\log((\sigma+\epsilon)^2+\alpha+\beta)-\log((\sigma+\epsilon)^2+\beta)}{\log(\sigma^2+\alpha+\beta)-\log(\sigma^2+\beta)} \qquad (21)
\end{aligned}$$

From (20) and (21) it suffices to show that for all $\alpha', 0 \leq \alpha' \leq \alpha$ we have

$$\frac{\frac{\partial}{\partial\alpha'}\log((\sigma+\epsilon)^2+\beta+\alpha')}{\frac{\partial}{\partial\alpha'}\log(\sigma^2+\beta+\alpha')}=\frac{\sigma^2+\beta+\alpha'}{(\sigma+\epsilon)^2+\beta+\alpha'}\leq\frac{\sigma^2+\beta+\alpha}{(\sigma+\epsilon)^2+\beta+\alpha}$$

because

$$(21) \leq \sup_{0\leq\alpha'\leq\alpha} \frac{\sigma^2+\beta+\alpha'}{(\sigma+\epsilon)^2+\beta+\alpha'} \leq \frac{\sigma^2+\beta+\alpha}{(\sigma+\epsilon)^2+\beta+\alpha} \,. \tag{22}$$

Since the first inequality in (22) is strict for any positive α and (18) is strictly increasing in γ, under the assumption that $\Gamma_k > \Gamma_{k+1} > \Gamma_{k+2}$, the inequality in (15) is strict. This completes the proof of the first property.

We now prove that $\lim_{\sigma^2\downarrow 0} \frac{\varphi_j^\sigma(v)}{\varphi_i^\sigma(v)} = 1$. In order to show this, it suffices to show that, for all $k = 1, \cdots, I-1$, $\frac{\varphi_k^\sigma(v)-\varphi_{k+1}^\sigma(v)}{\varphi_I^\sigma(v)} \to 0$ as $\sigma \downarrow 0$. This can be easily shown as follows. From (7) $\varphi_I^\sigma(v) = \frac{C(I\cdot\Gamma_I,\sigma^2)}{I} \to \infty$ as $\sigma\downarrow 0$. On the other hand, from (8)

$$\begin{aligned}
\varphi_k^\sigma(v) - \varphi_{k+1}^\sigma(v) &= \frac{C(k\Gamma_k + \sum_{j=k+1}^I \Gamma_j, \sigma^2) - C(k\Gamma_{k+1} + \sum_{j=k+1}^I \Gamma_j, \sigma^2)}{k} \\
&= \frac{C(k(\Gamma_k - \Gamma_{k+1}), \sigma^2 + k\Gamma_{k+1} + \sum_{j=k+1}^I \Gamma_j)}{k} \\
&\to \quad \frac{C(k(\Gamma_k - \Gamma_{k+1}), k\Gamma_{k+1} \ + \sum_{j=k+1}^I \Gamma_j)}{k} \qquad \text{as } \sigma\downarrow 0 \,.
\end{aligned}$$

Thus, the difference $\varphi_k^\sigma(v) - \varphi_{k+1}^\sigma(v)$ is decreasing in σ and $\lim_{\sigma\downarrow 0}\left(\varphi_k^\sigma(v) - \varphi_{k+1}^\sigma(v)\right)$ is finite for all $k = 1, \cdots, I-1$, whereas $\lim_{\sigma\downarrow 0}\varphi_I^\sigma(v) = \infty$. Therefore, $\frac{\varphi_k^\sigma(v)}{\varphi_I^\sigma(v)} \downarrow 1$ as $\sigma\downarrow 0$ for all $k = 1, \cdots, I$.

To prove the last property we show that for all $k = 1, \cdots, I-1$,[2]

$$\frac{\varphi_k^\sigma(v) - \varphi_{k+1}^\sigma(v)}{\varphi_{k+1}^\sigma(v) - \varphi_{k+2}^\sigma(v)} \to \frac{\Gamma_k - \Gamma_{k+1}}{\Gamma_{k+1} - \Gamma_{k+2}} \qquad \text{as } \sigma\uparrow\infty \,.$$

The rest can be easily proved from that $\frac{d}{dx}\log(1+x) = \frac{1}{(1+x)\ln 2}$.[3] From (8)

$$\begin{aligned}
\frac{\varphi_k^\sigma(v) - \varphi_{k+1}^\sigma(v)}{\varphi_{k+1}^\sigma(v) - \varphi_{k+2}^\sigma(v)} &= \frac{(k+1)\cdot C\left(k(\Gamma_k - \Gamma_{k+1}), \sigma^2 + k\cdot\Gamma_{k+1} + \sum_{j=k+1}^I \Gamma_j\right)}{k\cdot C\left((k+1)(\Gamma_{k+1} - \Gamma_{k+2}), \sigma^2 + (k+1)\Gamma_{k+2} + \sum_{j=k+2}^I \Gamma_j\right)} \\
&\to \frac{\Gamma_k - \Gamma_{k+1}}{\Gamma_{k+1} - \Gamma_{k+2}} \qquad \text{as } \sigma\uparrow\infty \ \text{ by L'Hôpital's rule}
\end{aligned}$$

This completes the proof of the theorem. □

Theorem 5.4 tells us that when the signal-to-noise-ratio (SNR) is high for all users, their rates are roughly the same, while when the SNR is low, their rates are approximately proportional to their power constraints. In the two users case when the noise power is very small, *i.e.,* high SNR, the constraint that $R_1 + R_2 \leq C\left(\Gamma_1 + \Gamma_2, \sigma^2\right)$ becomes the dominant constraint. Hence, the allocation is mostly governed by this constraint and both users receive roughly the same rate. On the other hand, if the power constraints of the two users are quite different and the noise power is high,

[2] Here we assume that $\Gamma_{k+1} > \Gamma_{k+2}$.

[3] Recall that the logarithm is to base 2 in this paper.

i.e., low SNR, then the constraint on the rate of the user with smaller power constraint becomes more active. Therefore, the rate allocation is close to the intersection of these two constraints. Similar intuition carries over to multiple user cases. Thus, in conclusion, we have also demonstrated some pleasing and natural properties of the fair rate allocation as the noise power varies.

6 Concluding remarks

We have studied the issue of how to fairly allocate communication rate among the users of a Gaussian multiaccess channel. We adopted a cooperative game-theoretic viewpoint which presumed the ability to make contracts offline based on threat strategies and the potential formation of coalitions. Contributions of the paper included (i) determining the capacity region of the Gaussian multiaccess arbitrarily varying channel (albeit with an alternative operational definition of capacity which avoids the main technical difficulty of the usual approach); (ii) determining the characteristic function of the Gaussian multiaccess game; (iii) proposing a natural set of axioms that a fair allocation should satisfy; (iv) proving the existence and uniqueness of a fair allocation that satisfies these axioms; (v) demonstrating that this fair allocation is feasible and reasonable, in the sense that it lies in the core of the game; and finally (vi) demonstrating some nice qualitative properties of the fair allocation as the noise power varies. We also briefly touched upon the Shapley value as a solution concept, even though we argued that the axioms underlying this concept are not natural for our game.

Our main motivation for undertaking this study was to explore the intersection of the two rich fields of cooperative game theory and information theory. We believe that investigations in this intersection can have a valuable role to play in bringing information-theoretic ideas to bear on the social choice issues that arise in multiuser communication systems where the agents might have diverse objectives.

7 Appendix

Here we gather some simple inequalities which the reader will have no trouble verifying in a few minutes.

- If $y \geq x \geq 0$ and $z > 0$, then

$$C(x+z,\sigma^2) - C(x,\sigma^2) \geq C(y+z,\sigma^2) - C(y,\sigma^2)\ , \tag{23}$$

  and the equality holds only if $y = x$.

- If $x > 0$ and $z \geq 0$, then

$$C((k+1)x+z,\sigma^2) > (k+1)\left(C((k+1)x+z,\sigma^2) - C(kx+z,\sigma^2)\right) \tag{24}$$

  where $k \in \mathcal{N} = \{1, 2, \cdots\}$.

- If $y \geq x \geq 0$ and $z \geq 0$, then for $k = 2, 3, \cdots$,

$$\begin{aligned}&\frac{1}{k}\Big(C(ky+z,\sigma^2) - C(kx+z,\sigma^2)\Big)\\&\qquad \leq \frac{1}{k-1}\Big(C((k-1)y+z,\sigma^2) - C((k-1)x+z,\sigma^2)\Big)\end{aligned} \tag{25}$$

  and the equality holds only if $x = y$.

- If $z_1 > z_2 \geq 0$ and $y \geq x \geq 0$, then for all $k \in \mathcal{N}$

$$C(ky+z_1,\sigma^2) - C(kx+z_1,\sigma^2) \leq C(ky+z_2,\sigma^2) - C(kx+z_2,\sigma^2)\,, \tag{26}$$

  and the equality holds only if $y = x$.

- If $y > x \geq 0$ and $z \geq 0$, then

$$\begin{aligned}&C(ky+z,\sigma^2) - C(kx+z,\sigma^2)\\&> k\Big(C((k-1)y+y+z,\sigma^2) - C((k-1)y+x+z,\sigma^2)\Big)\ .\end{aligned} \tag{27}$$

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