ABSTRACT

Title of dissertation: MATRIX FIELD THEORY:
APPLICATIONS TO SUPERCONDUCTIVITY

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In this thesis a systematic, functional matrix field theory is developed to describe both clean and disordered s-wave and d-wave superconductors and the quantum phase transitions associated with them. The thesis can be divided into three parts.

The first part includes chapters 1 to 3. In chapter one a general physical introduction is given. In chapters two and three the theory is developed and used to compute the equation of state as well as the number-density susceptibility, spin-density susceptibility, the sound attenuation coefficient, and the electrical conductivity in both clean and disordered s-wave superconductors.

The second part includes chapter four. In this chapter we use the theory to describe the disorder-induced metal - superconductor quantum phase transition. The key physical idea here is that in addition to the superconducting order-parameter fluctuations, there are also additional soft fermionic fluctuations that are important at the transition. We develop a local field theory for the coupled fields describing superconducting and soft fermionic fluctuations. Using simple renormalization group

and scaling ideas, we exactly determine the critical behavior at this quantum phase transition. Our theory justifies previous approaches.

The third part includes chapter five. In this chapter we study the analogous quantum phase transition in disordered d-wave superconductors. This theory should be related to high T_c superconductors. Surprisingly, we show that in both the underdoped and overdoped regions, the coupling of superconducting fluctuations to the soft disordered fermionic fluctuations is much weaker than that in the s-wave case. The net result is that the disordered quantum phase transition in this case is a strong coupling, or described by an infinite disordered fixed point, transition and cannot be described by the perturbative RG description that works so well in the s-wave case. The transition appears to be related to the one that occurs in disordered quantum antiferromagnets.

MATRIX FIELD THEORY: APPLICATIONS TO SUPERCONDUCTIVITY

by

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Chapter 1

Introduction

1.1 Background

For many reasons there is an intense interest in many-electron systems, especially at low or zero temperature. Such systems, with or without quenched disorder, include both high–T_c and conventional superconductors, amorphous alloys, doped semiconductors, itinerant magnetic systems, heavy fermions systems, and quantum Hall systems. The description of the many-electron systems is a difficult problem in modern theoretical physics. The many-particle Schrödinger equation is used to describe the behavior of such systems. In principle, solving the Schrödinger equations leads to the many-body wave functions which contain all the possible information, but this approach is usually unrealistic to implement. Therefore, other techniques are developed to study these problems.

A phenomenological Fermi-liquid theory was introduced by Landau in 1956 [1]. It argues that even in the presence of interactions the low-energy excitations of the many-electron systems can be described in terms of free fermionic quasiparticles with charge e, spin $\frac{1}{2}$ and effective mass m^*. In principle Landau theory has been extended to include the effects of quenched disorder. This theory, though, is difficult to apply to superconducting systems. The use of the microscopic many-body perturbation theory began in the early 1950s [2, 3]. The theory uses ideas from quantum field theory. Feynman diagram techniques are adopted to calculate physical quantities in perturbation theory. Some approximations are always used. The ladder-diagram approximation can be used to describe the system of a dilute Fermi gas with strong short-range repulsive potentials. The random phase approximation has been developed to describe the Fermi liquid phase for real electronic systems, and similar theories have been used to describe magnetic and superconducting phases. The inclusion of disorder, however, has proven to be extremely awkward because a large number of diagrams cancel against each other in perturbation theory. The disorder corrections to Fermi-liquid theory known as "weak-localization effects" [4] were not obtained until the work in the late 1970s and early 1980s [5].

Recently [6, 7], functional Feynman path integral methods have been applied to the many-electron problem, which have certain advantages over the traditional canonical quantization techniques. Coherent state functional integrals provide an economical and physically intuitive formalism which can not only derive the traditional perturbation expansions results but also lead to new insight into nonperturbative problems like quantum phase transitions. One interesting aspect is that it allows for a straightforward application of the renormalization group (RG), implementing an old program of describing the various phases of many-body systems in terms of stable RG fixed points [8]. Corrections to scaling near the disordered Fermi liquid fixed point have been related to the weak localization effects [9].

In this thesis we develop a comprehensive field-theoretical method which can treat both clean and disordered superconducting systems. The theory allows for explicit computations of physical properties such as thermodynamic and transport properties in the superconducting phase, as well as the description of the disorder-induced quantum phase transitions between metal and superconductor phases. Peviously this field theory, or matrix field theory, has been used to describe clean and disordered Fermi liquids, disordered ferromagnetic metals, as well as various other quantum phase transitions [9]. It is the purpose of this thesis to develop the matrix field theory for both clean and disordered s-wave and d-wave spin-singlet superconductors. One can then apply this theory to study transport properties and quantum phase transitions in disordered superconductors.

1.2 Physical picture of disordered electron system

To study transport properties of electrons in solids, the homogeneous electron gas model, or the jellium model, is often introduced [10]. The basic assumption is to treat the atoms or ions of solids as a uniform distributed positive background with a charge density ensuring the overall charge neutrality of the system. The band structure is not important for universal phenomena and is usually neglected. The total energy includes the kinetic energy of electrons, the interactions between electrons including the effective attraction close to the Fermi surface between electrons in case of superconductivity, and the interaction between electrons and impurities.

For simplicity, we only consider uncorrelated non-magnetic random impurities in the system. There are two simple limits of disorder: annealed disorder and quenched disorder [11]. Annealed disorder means the impurities can move randomly from site to site on time-scales short compared with experimental times. Then the partition function will be

$$\begin{aligned} \widetilde{Z}_A &\equiv \mathrm{Tr}_{\{p_i\}} P(\{p_i\}) Z(\{p_i\}) \\ &= \mathrm{Tr}\, P(\{p_i\}) e^{-\beta H(\{p_i\})}, \end{aligned} \tag{1.1}$$

and the free energy is simply given by

$$\widetilde{F}_A = -\frac{1}{\beta} \ln \widetilde{Z}_A. \tag{1.2}$$

$P(\{p_i\})$ is the distribution function which represents the probability of a particular set $\{p_i\}$ of random variables. p_i assumes the value 1 if the ith site is occupied by an impurity ion, or 0 if otherwise. The random variables $\{p_i\}$ are been treated the same as other system variables such as spin. The system is therefore not different from the one without impurities in the view of symmetry. For example, the translational invariance will not be destroyed.

In this thesis we consider quenched disorder, which is more complicated but also more physically interesting. It means that impurities are frozen rigidly over experimental times. The partition function has the form, for a fixed set of $\{p_i\}$,

$$Z(\{p_i\}) \equiv \mathrm{Tr}\, e^{-\beta H(\{p_i\})}, \tag{1.3}$$

and the free energy will be

$$F(\{p_i\}) = -\frac{1}{\beta} \ln Z(\{p_i\}). \tag{1.4}$$

Correlation functions of the systems are then related to the random variables $\{p_i\}$ and will be very difficult to obtain due to the lack of translational invariance in this quenched disorder. However, we imagine that there are m identical systems, except that each system has a different set of $\{p_i\}$ from any other and all of the systems consists of the complete sets of $\{p_i\}$. It is generally believed that the impurity-averaged free energy of one system will give the physical result [12]. That is,

$$\begin{aligned} F(\{p_i\}) \to \widetilde{F}_Q &\equiv \sum_{\{p_i\}} P(\{p_i\}) F(\{p_i\}) \\ &= -\frac{1}{\beta} \sum_{\{p_i\}} P(\{p_i\}) \ln Z(\{p_i\}). \end{aligned} \tag{1.5}$$

We notice that we average over not the partition function but instead the logarithm of the partition function in quenched disorder. Correlation functions are similarly equivalent to their impurity-averaged counterparts. After the impurity averaging the translational invariance is restored and the calculation will be no difference in principle from pure systems. In the present thesis only quenched disorder will be considered.

Now we give the explicit form of the Hamiltonian

$$H = \sum_k T(x_k) + \frac{1}{2}\sum_{k \neq l} v(x_k - x_l) + \sum_k u(x_k) \tag{1.6}$$

where T is the kinetic energy, v is the effective interaction between electrons and u is the impurity-scattering potential from the quenched disorder. The quantity x_k means the coordinates of the kth electron. To compute disorder averages of observables, we imagine a system where the average is over the positions of the N impurity scatters in the system of volume V [13],

$$\{(\ldots)\}_{dis} = \frac{1}{V^N}\int d\mathbf{R_1}\ldots d\mathbf{R_N}\,(\ldots)\,, \tag{1.7}$$

As far as universal properties go, an equivalent problem is one where sites have an impurity with possibility $P(\{p_i\})$. In this case the possibility distribution therefore is

$$P(\{p_i\}) = \prod_i (p\delta_{p_i,1} + (1-p)\delta_{p_i,0}) \tag{1.8}$$

with p the overall impurity concentration. For our case the impurity-scattering potential u is then given by

$$u(\mathbf{x}) = v(\mathbf{x}) - \{v(\mathbf{x})\}_{dis} \tag{1.9a}$$

where

$$v(\mathbf{x}) = \sum_{j=1}^{N} v(\mathbf{x} - \mathbf{R_j}) \tag{1.9b}$$

with, for example,

$$v(\mathbf{x} - \mathbf{R_j}) = \frac{4\pi a}{m}\delta(\mathbf{x} - \mathbf{R_j}). \tag{1.9c}$$

Here a is the s-wave scattering length. We find that

$$\{u(\mathbf{x})\}_{dis} = 0 \tag{1.10a}$$

and

$$\{u(\mathbf{x})u(\mathbf{y})\}_{dis} = \frac{1}{\pi N_F \tau_{\mathrm{e}}}\ \delta(\mathbf{x} - \mathbf{y}) \quad , \tag{1.10b}$$

N_F is the density of states at the Fermi level including both spins, and τ_{e} is the elastic scattering time.

1.3 Some Properties of Superconductors

Kammerlingh Onnes first discovered the phenomenon of superconductivity in 1911. Bardeen, Cooper and Schrieffer proposed the successful microscopic theory (BCS theory) for s-wave superconductors in 1957. The most familiar property of a superconductor is the complete disappearance of the resistance to the flow of electric current. In many respects, the theoretically most important phenomenon is the Meissner effect, or the perfect diamagnetism in superconductors. The normal state of s-wave superconductors is the well-known Fermi liquid state. Other important properties in the superconducting state include the spin susceptibility and the ultrasonic attenuation. As shown in Fig. 1.1, BCS theory predicts that the spin susceptibility of the electron decreases as the temperature is lowered, being zero at $T = 0$. The behavior of the ultrasonic attenuation is similar to the spin susceptibility, i.e., it decreases to zero as the temperature is lowered to zero.

For weak nonmagnetic disorder, the critical temperature T_c is independent of the disorder, and so are all thermodynamic properties of the superconductors. This result is known as Anderson's theorem [14]. For strong non-magnetic disorder, however, numerous experiments [15] have shown that the disorder suppresses superconductivity. At high enough disorder the superconducting critical temperature goes to zero. From Ref. [16], as shown in Fig. 1.2 and Fig. 1.3, the degradation process can be described by

$$T_c = \omega_D \exp\left[-\frac{h}{Z|K_c| - \delta k_c}\right] \quad , \tag{1.11}$$

where K_c is the bare Cooperon (Cooper pair) interaction amplitude, ω_D is the

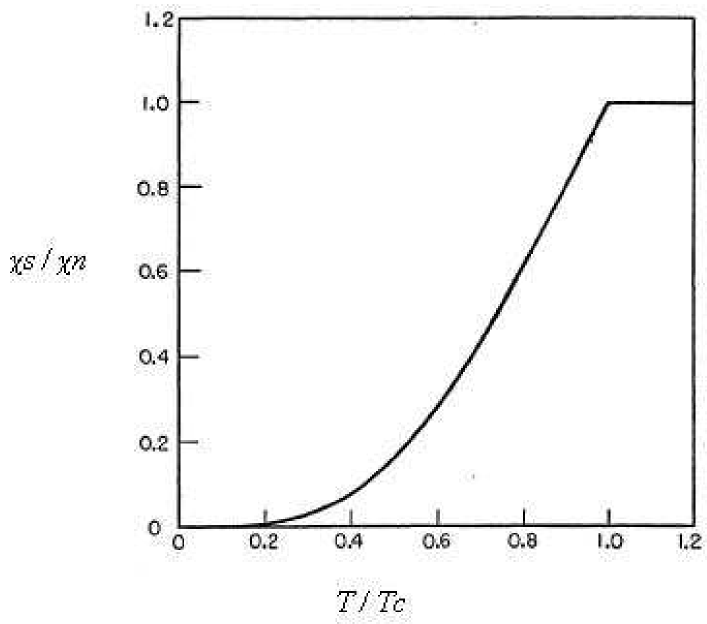

Figure 1.1: Temperature dependence of the paramagnetic spin susceptibility in a BCS superconductor.

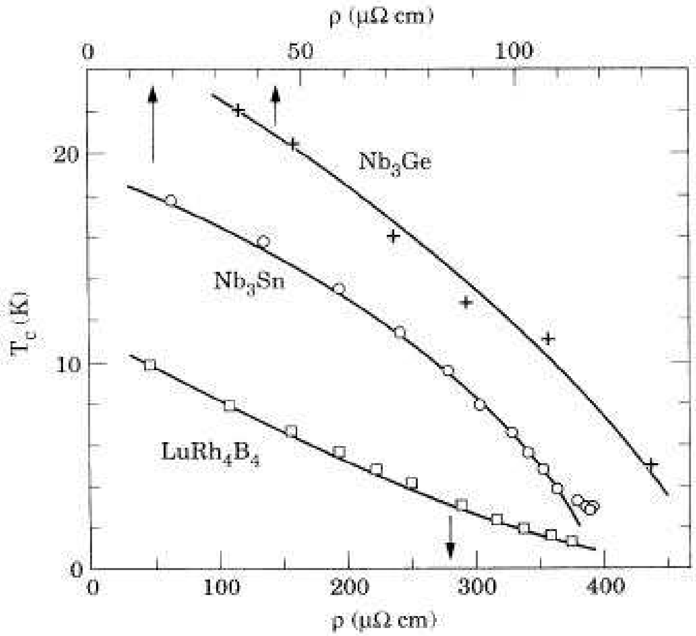

Figure 1.2: Temperature dependence of the nonmagnetic disorder in a BCS superconductor with d=3. The experimental data is from Ref. [17].

upper cutoff frequency on the order of Debye frequency, h and Z are renormalization constants, and δk_c is the (repulsive) Cooperon interaction resulting from the disorder-induced repulsive electron-electron interaction.

Both theory and experiments therefore imply there is a critical disorder where the disorder-induced metal - superconductor phase transition happens at zero temperature. This means we need to treat it as a quantum phase transition. Phase transitions occurring at a nonzero critical temperature can be normally treated

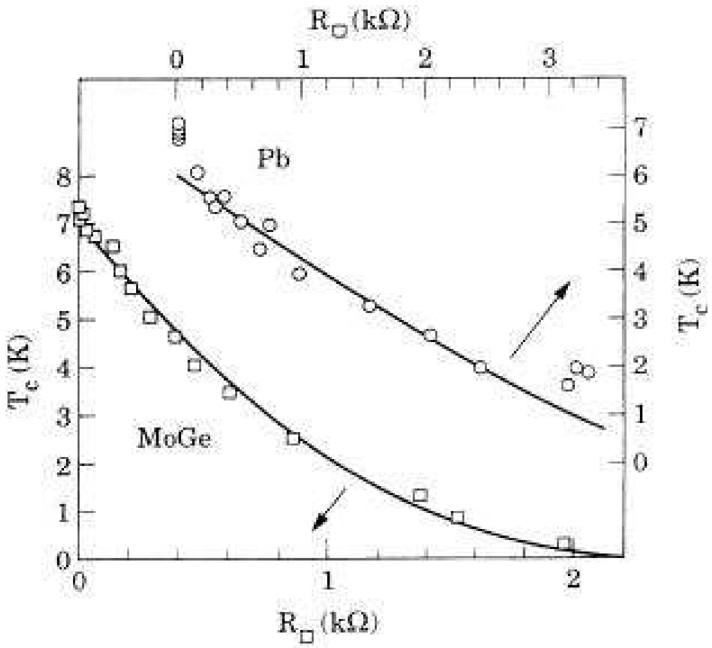

Figure 1.3: Temperature dependence of the nonmagnetic disorder in a BCS superconductor with d=2. The experimental data is from Ref. [18].

within classical statistical mechanics. For a general classical Hamiltonian

$$H(p, q) = H_{\rm kin}(p) + H_{\rm pot}(q) \quad , \tag{1.12a}$$

where p and q are the generalized momenta and positions, and $H_{\rm kin}$ and $H_{\rm pot}$ are the kinetic and potential energy, respectively. The partition function can then be divided into two parts:

$$\begin{aligned} Z &= \int dp\, dq \; e^{-H/k_{\rm B}T} \\ &= \int dp \; e^{-H_{\rm kin}/k_{\rm B}T} \int dq \; e^{-H_{\rm pot}/k_{\rm B}T} \quad , \end{aligned} \tag{1.12b}$$

with one part that depends only on $H_{\rm kin}$ and the other only on $H_{\rm pot}$. As a result, one can study the system's static (thermodynamic) critical behaviors independently from its dynamical ones. In quantum mechanics, however, the partition function has the form of

$$\begin{aligned} Z &= {\rm Tr}\, e^{-\beta(\hat{H}-\mu\hat{N})} \\ &= {\rm Tr}\, e^{-\beta(\hat{H}_{\rm kin}+\hat{H}_{\rm pot}-\mu\hat{N})} \\ &= \int D[\bar{\psi}, \psi] \; e^{S[\bar{\psi},\psi]} \quad , \quad . \end{aligned} \tag{1.13a}$$

where

$$\begin{aligned} S[\bar{\psi}, \psi] &= \int d\mathbf{x} \int_0^{1/k_{\rm B}T} d\tau \; \bar{\psi}(\mathbf{x}, \tau) \left[-\frac{\partial}{\partial \tau} + \mu \right] \psi(\mathbf{x}, \tau) \\ &\quad - \int_0^{1/k_{\rm B}T} d\tau \; H\left(\bar{\psi}(\mathbf{x}, \tau), \psi(\mathbf{x}, \tau) \right) \quad . \end{aligned} \tag{1.13b}$$

with τ the imaginary time. Since the Hamiltonian taken at some imaginary time does not commute with the Hamiltonian taken at another imaginary time, we see that the statics and the dynamics in quantum systems are intrinsically coupled and need to be treated together and simultaneously. We need to describe the disorder-induced metal - superconductor phase transition at $T = 0$ with quantum mechanics since at $T = 0$ the time dimension is infinite in extent and is therefore similar to the spatial dimensions. Physically there is a strong intrinsic coupling between the statics and the dynamics at zero temperature, and this greatly influences the critical behaviors.

Until quite recently the subject of superconductivity was a low-temperature phenomenon. But in 1986 Bednorz and M*ü*ller discovered the high T_c superconducting cuprates [19]. It is now believed that high T_c cuprates are d-wave superconductors [20]. The phase diagram is shown in Fig. 1.4. The carrier doping concentration δ is defined as $\delta = 1 - n$ with n being the carrier number per copper site. The system is the anti-ferromagnetic Mott insulator at half-filling ($\delta = 0$). With a slight amount of hole doping ($\delta > 0$), the AF state is destroyed and an abnormal metallic phase appears. When the amount continues increasing, the superconducting state appears, with the maximum transition temperature $T_c \sim 100K$ in the optimally doped region. Finally the normal metal state will appear in the over-doped region.

In contrast to s-wave superconductors, there are many anomalous behaviors in the underdoped region of cuprates [21]. These deviations from the properties of Fermi liquid theory have been one of the central issues in the understanding of high T_c superconductors. It has been proposed that the abnormal state can be described by some kind of non-Fermi-liquid ground state. But another way is to start from the robust Fermi-liquid state and take into account the additional

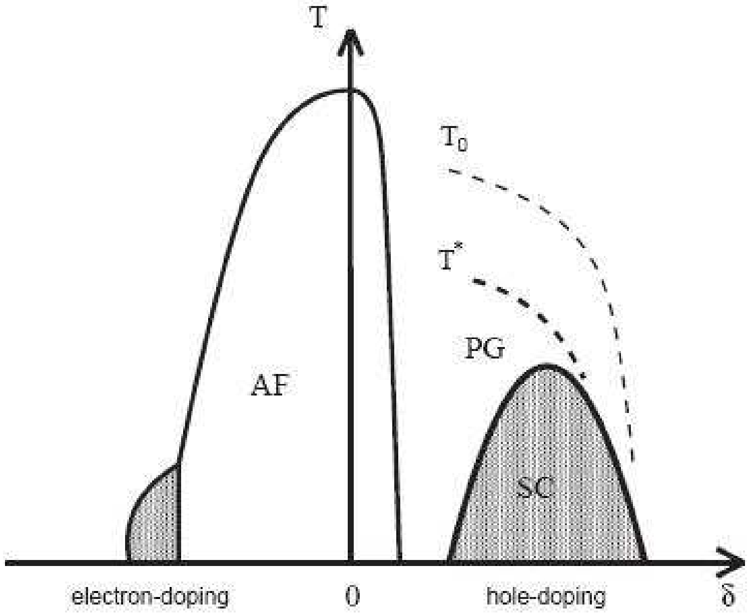

Figure 1.4: The phase diagram of high-T_c superconductors. The horizontal and vertical axes indicate the doping concentration and the temperature, respectively. "AF", "SC", and "PG" denote anti-ferromagnetic, superconducting, and pseudogap state, respectively. The onset curve for the spin fluctuation (T=T_0) and that for the pseudogap formation (T=T^*) show the typical cross-over temperatures [21].

fluctuations that might cause the non-Fermi-liquid behaviors. Fluctuations include the anti-ferromagnetic spin fluctuations and the superconducting fluctuations. The effect of the later ones is believed closely connected to the pseudogap phenomenon. In any case, it is clear that there are very strong antiferromagnetic spin fluctuations and the concept of a pseudogap in the normal state is an important ingredient in the understanding of the underdoped region of the high T_c materials. As in the s-wave case, the disorder-induced metal - superconductor transition at zero temperature is also a quantum phase transition.

1.4 Outline of Thesis

In Chap. 2, the functional matrix field theory is developed for both clean and disordered spin-singlet superconductors. A Feynman path integral formulation of the physical model is given in terms of composite variables that are closely related to the order parameter describing the superconducting phase. The theory is then solved in the saddle point approximation and an explicit equation of state describing the superconducting phase is obtained. Expansion about the saddle point in principle gives the various physical correlation functions.

In Chap. 3, the resulting theory is used to explicitly calculate the physical correlation functions. In particular, the number and spin density susceptibilities, the sound attenuation coefficients, and the electrical conductivity for both clean and disordered s-wave superconductors are obtained. Previously, similar techniques were used to describe spin-triplet, even-parity superconductors [22]. But explicit quantitative expressions for the Gaussian propagators were not obtained. Here we will explicitly determine the correlation functions for the $S = 0$, spin-singlet case. The method can be generalized to evaluate the physical correlation functions in other, more exotic, superconducting states.

In Chap. 4, the field theoretic method is generalized to exactly describe the disorder-induced quantum metal - superconductor phase transitions. A symmetry analysis is performed on the model in order to identify and separate massive modes and massless, or soft ones. By integrating out the massive modes and keep all other soft modes, besides the order parameter fluctuations of the phase transition, one obtains an effective local field action. It can then be analyzed by conventional renormalization methods. We obtain the exact critical behavior at the transition.

In Chap. 5, the local field theory is applied to d-wave superconducting quantum phase transitions. Pseudogap phenomena in the normal metal side are considered. We show that the disorder-induced quantum phase transition from metal to superconductor in this system is a strong coupling (or infinite disorder fixed point) transition and cannot be described with the techniques that exactly solved the transition in the s-wave case.

In Chap. 6, we review the conclusions of the thesis.

In the Appendices, some technical details that are used in the thesis are given.

Chapter 2

Matrix field theory

In the present chapter we will develop a functional matrix field theory for both clean and disordered spin-singlet superconductors. Similar techniques have been applied to describe spin-triplet, even-parity superconductors [22]. But explicit quantitative expressions for the Gaussian propagators had not been obtained. Here we will completely determine the correlation functions for the $S = 0$, spin-singlet case. The method can be generalized to evaluate other physical systems, like spin-triplet superconductors. Our results for the spin-singlet case coincide with earlier ones obtained by conventional methods. Our functional methods, however, have the advantage that they can be easily generalized to describe quantum phase transitions. For example, in later chapters we will use these results to describe a metal - superconductor transition in a dirty metal. The problem will provide new insight and allow us to see if soft modes, other than order parameter fluctuations of the phase transition, will influence the critical behavior. These modes are diffusive in disordered systems or ballistic in clean ones. In a previous treatment the diffusions were integrated out and a non-local field theory for the superconducting order parameter was obtained. This non-local field theory was then used to describe the quantum phase transition. Here we justify that treatment with a local field theory approach.

2.1 Grassmannian field theory

Now we use the functional Feynman path integral formalism to describe a system of interacting, quenched-disordered fermions. The partition function of the system is [23]

$$Z = \mathrm{Tr}\, e^{-\beta(H-\mu N)} = \int D[\bar{\psi},\psi]\; e^{S[\bar{\psi},\psi]} \quad . \tag{2.1}$$

Here the $\bar{\psi}$ and ψ are (anticommuting) Grassmann fields. S is the action which includes three parts:

$$S = S_0 + S_{\mathrm{int}} + S_{\mathrm{dis}} \quad , \tag{2.2a}$$

S_0 describes free electrons with electron mass m and chemical potential μ,

$$S_0 = \int dx \sum_{\sigma} \bar{\psi}_\sigma(x) \left(-\partial_\tau + \frac{\nabla^2}{2m} + \mu\right) \psi_\sigma(x) \quad , \tag{2.2b}$$

where, with a $(d+1)$-vector notation $x = (\mathbf{x},\tau)$ and $\int dx = \int_V d\mathbf{x} \int_0^\beta d\tau$. S_{int} describes a spin-independent two-electron interaction,

$$S_{\mathrm{int}} = -\frac{1}{2}\int dx_1\, dx_2 \sum_{\sigma_1,\sigma_2} v(\mathbf{x}_1-\mathbf{x}_2)\, \bar{\psi}_{\sigma_1}(x_1)\, \bar{\psi}_{\sigma_2}(x_2)\, \psi_{\sigma_2}(x_2)\, \psi_{\sigma_1}(x_1) \quad , \tag{2.2c}$$

and S_{dis} describes the random potential $u(\mathbf{x})$ coupling to the electronic number density,

$$S_{\mathrm{dis}} = -\int dx \sum_{\sigma} u(\mathbf{x})\, \bar{\psi}_\sigma(x)\, \psi_\sigma(x) \quad . \tag{2.2d}$$

For the calculation of physical quantities, we need to average over the disorder distribution. The average is modeled as

$$\{\ldots\}_{\mathrm{dis}} = \int D[u]\; P[u]\; (\ldots) \quad , \tag{2.3a}$$

with the Gaussian distribution $P[u(\mathbf{x})]$

$$P[u(\mathbf{x})] = \frac{\exp\left(-\frac{\pi N_F \tau_e}{2}\int d\mathbf{x}(u(\mathbf{x}))^2\right)}{\int D[u]\exp\left(-\frac{\pi N_F \tau_e}{2}\int d\mathbf{x}(u(\mathbf{x}))^2\right)}. \tag{2.3b}$$

As we discussed in the Sec. 1.3, the quenched-disorder averaging has to be done with the free energy or $\ln Z$. To this end we use the replica trick [11]. By the identity

$$\ln Z = \lim_{N\to 0}\ (Z^N - 1)/N \quad , \tag{2.4}$$

we can average $\ln Z$ in terms of Z^N. With N identical replicas of the system (with N an integer), labeled by the index α, the disorder average of Z^N becomes

$$\begin{aligned}\widetilde{Z} &\equiv \{Z^N\}_{\text{dis}} \\ &= \int D[u]\ P[u] \int \prod_{\alpha=1}^{N} D\left[\bar\psi^\alpha, \psi^\alpha\right] \exp\left[\sum_{\alpha=1}^{N} S^\alpha[\bar\psi^\alpha, \psi^\alpha]\right] \\ &= \int \prod_{\alpha=1}^{N} D\left[\bar\psi^\alpha, \psi^\alpha\right] \exp[\tilde{S}\,] \quad ,\end{aligned} \tag{2.5}$$

where the corresponding action $\tilde{S}$ equals to

$$\tilde{S} = \sum_{\alpha=1}^{N}\ \left(\tilde{S}_0^\alpha + \tilde{S}_{\text{int}}^\alpha + \tilde{S}_{\text{dis}}^\alpha\right) \quad . \tag{2.6}$$

Then we can obtain the disorder-averaged correlation functions as follows,

$$\{<\bar\psi_{\sigma_1}(x_1)\,\psi_{\sigma_2}(x_2)>_Z\}_{\text{dis}} = \lim_{N\to 0} <\bar\psi^{\alpha_1}_{\sigma_1}(x_1)\,\psi^{\alpha_2}_{\sigma_2}(x_2)>_{\widetilde{Z}} \tag{2.7}$$

Here we use a two-point correlation function as an example.

In order to calculate correlation functions, we follow the usual procedure by adding a source to the action,

$$\begin{aligned} S &\to S + \int dx_1 dx_2 \sum_{\sigma_1,\sigma_2} J^{(2)}_{\sigma_1,\sigma_2}(x_1,x_2)\bar\psi_{\sigma_1}(x_1)\,\psi_{\sigma_2}(x_2) \\ S^\alpha &\to S^\alpha + \int dx_1 dx_2 \sum_{\sigma_1,\sigma_2} J^{(2)}_{\sigma_1,\sigma_2}(x_1,x_2)\bar\psi^\alpha_{\sigma_1}(x_1)\,\psi^\alpha_{\sigma_2}(x_2) \quad .\end{aligned} \tag{2.8}$$

We can then differentiate with respect to the source field $J^{(2)}$ and have

$$< \bar{\psi}_{\sigma_1}(x_1)\, \psi_{\sigma_2}(x_2) >_Z = \frac{\delta}{\delta J^{(2)}_{\sigma_1,\sigma_2}(x_1,x_2)} \ln Z|_{J^{(2)}=0} \tag{2.9}$$

and

$$\begin{aligned}
\{< \bar{\psi}_{\sigma_1}(x_1)\, \psi_{\sigma_2}(x_2) >_Z\}_{\text{dis}} &= \frac{\delta}{\delta J^{(2)}_{\sigma_1,\sigma_2}(x_1,x_2)} \left(\int D[u]D[p] \ln Z\right) |_{J^{(2)}=0} \\
&= \frac{\delta}{\delta J^{(2)}_{\sigma_1,\sigma_2}(x_1,x_2)} \left(\int D[u]D[p] \lim_{N\to 0} \frac{Z^N-1}{N}\right) |_{J^{(2)}=0} \\
&= \frac{\delta}{\delta J^{(2)}_{\sigma_1,\sigma_2}(x_1,x_2)} \lim_{N\to 0} \frac{1}{N} \left(\int D[u]D[p] Z^N - 1\right) |_{J^{(2)}=0} \\
&= \frac{\delta}{\delta J^{(2)}_{\sigma_1,\sigma_2}(x_1,x_2)} \lim_{N\to 0} \frac{1}{N} \left(\widetilde{Z} - 1\right) |_{J^{(2)}=0} \\
&= \frac{\delta}{\delta J^{(2)}_{\sigma_1,\sigma_2}(x_1,x_2)} \lim_{N\to 0} \frac{1}{N} \ln \widetilde{Z}|_{J^{(2)}=0} \\
&= \lim_{N\to 0} \frac{1}{N} \frac{1}{N\widetilde{Z}} \int \prod_{\beta=1}^{N} D\left[\bar{\psi}^\beta, \psi^\beta\right] \\
&\qquad \left[\sum_{\alpha=1}^{N} \bar{\psi}^\alpha_{\sigma_1}(x_1) \psi^\alpha_{\sigma_2}(x_2)\right] \exp\left[\widetilde{S}\right] \\
&= \lim_{N\to 0} < \bar{\psi}^{\alpha_1}_{\sigma_1}(x_1)\, \psi^{\alpha_2}_{\sigma_2}(x_2) >_{\widetilde{Z}}
\end{aligned} \tag{2.10}$$

Note that here we make use of

$$\ln \widetilde{Z} \approx \widetilde{Z} - 1 \quad \text{for } N \to 0 \quad , \tag{2.11}$$

which comes from the approximation of $\widetilde{Z} \to 1$ when $N \to 0$.

It is also useful to go to a Fourier representation with wave vectors $\mathbf{k}$ and fermionic Matsubara frequencies $\omega_n = 2\pi T(n + 1/2)$ by the following transformations: [9]

$$\begin{aligned}
\psi_{n\sigma}(\mathbf{x}) &= \sqrt{T} \int_0^\beta d\tau\; e^{i\omega_n \tau}\, \psi_\sigma(x) \quad , \\
\bar{\psi}_{n\sigma}(\mathbf{x}) &= \sqrt{T} \int_0^\beta d\tau\; e^{-i\omega_n \tau}\, \bar{\psi}_\sigma(x) \quad ,
\end{aligned} \tag{2.12a}$$

and

$$\psi_{n\sigma}(\mathbf{k}) = \frac{1}{\sqrt{V}}\int d\mathbf{x}\; e^{-i\mathbf{k}\cdot\mathbf{x}}\,\psi_{n\sigma}(\mathbf{x}) \quad ,$$

$$\bar{\psi}_{n\sigma}(\mathbf{k}) = \frac{1}{\sqrt{V}}\int d\mathbf{x}\; e^{i\mathbf{k}\cdot\mathbf{x}}\,\bar{\psi}_{n\sigma}(\mathbf{x}) \quad . \tag{2.12b}$$

We will have the forms, with a $(d+1)$-vector notation, $k = (\mathbf{k}, \omega_n)$,

$$\tilde{S}_0^\alpha = \sum_{k,\sigma} \bar{\psi}_\sigma^\alpha(k)\,\left[i\omega_n - \mathbf{k}^2/2m + \mu\right]\,\psi_\sigma^\alpha(k) \quad , \tag{2.13a}$$

$$\tilde{S}_{dis}^\alpha = \frac{1}{2\pi N_F \tau_{\mathrm{e}}}\sum_{\beta=1}^{m}\sum_{\{\mathbf{k}_i\}}\sum_{n,m}\sum_{\sigma,\sigma'}\delta_{\mathbf{k}_1+\mathbf{k}_3,\mathbf{k}_2+\mathbf{k}_4} \times\,\bar{\psi}_{n\sigma}^\alpha(\mathbf{k}_1)\,\psi_{n\sigma}^\alpha(\mathbf{k}_2)\,\bar{\psi}_{m\sigma'}^\beta(\mathbf{k}_3)\psi_{m\sigma'}^\beta(\mathbf{k}_4) \quad , \tag{2.13b}$$

and,

$$\tilde{S}_{int}^\alpha = -\frac{T}{2}\sum_{\sigma_1,\sigma_2}\sum_{\{k_i\}}\delta_{k_1+k_2,k_3+k_4}\;v(\mathbf{k}_2-\mathbf{k}_3) \times\bar{\psi}_{\sigma_1}^\alpha(k_1)\,\bar{\psi}_{\sigma_2}^\alpha(k_2)\,\psi_{\sigma_2}^\alpha(k_3)\,\psi_{\sigma_1}^\alpha(k_4) \quad . \tag{2.13c}$$

2.2 Composite variables: Q-matrix

Now we integrate out the Grassmann fields and rewrite the theory in terms of complex-number fields. The resulting model can then be approximately solved by using saddle-point techniques. Physically this step is a mapping from Grassmann variable to physical number density, spin density and Cooperon density variables. In particular, the Cooper density degrees of freedom are directly related to the superconducting order parameters. All of these variables are related to the slow soft modes in the system.

For the reasons that will become clear later, it would be convenient to first define a bispinor [24]

$$\eta_n^\alpha(\mathbf{x}) = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{\psi}_n^\alpha(\mathbf{x}) \\ s_2\, \psi_n^\alpha(\mathbf{x}) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{\psi}_{n\uparrow}^\alpha(\mathbf{x}) \\ \bar{\psi}_{n\downarrow}^\alpha(\mathbf{x}) \\ \psi_{n\downarrow}^\alpha(\mathbf{x}) \\ -\psi_{n\uparrow}^\alpha(\mathbf{x}) \end{pmatrix} \quad , \tag{2.14a}$$

and an adjoint bispinor [25]

$$(\eta^+)_n^\alpha(\mathbf{x}) = ic\eta_n^\alpha(\mathbf{x}) = \frac{i}{\sqrt{2}} \begin{pmatrix} -\psi_{n\uparrow}^\alpha(\mathbf{x}) \\ -\psi_{n\downarrow}^\alpha(\mathbf{x}) \\ \bar{\psi}_{n\downarrow}^\alpha(\mathbf{x}) \\ -\bar{\psi}_{n\uparrow}^\alpha(\mathbf{x}) \end{pmatrix} \quad , \tag{2.14b}$$

with c the charge-conjugation matrix

$$c = \begin{pmatrix} 0 & s_2 \\ s_2 & 0 \end{pmatrix} = i\tau_1 \otimes s_2 \quad . \tag{2.14c}$$

Here we have defined a basis in spin-quaternion space as $\tau_r \otimes s_i$ $(r, i = 0, 1, 2, 3)$, with $\tau_0 = s_0$ the 2×2 identity matrix, and $\tau_j = -s_j = -i\sigma_j$ $(j = 1, 2, 3)$, with σ_j the Pauli matrices.

In terms of the bispinors, the terms on the action $\widetilde{S}$ can be rewritten as follows,

$$\tilde{S}_0 = -i \sum_{\alpha,k} \left(\eta^\alpha(k), [i\omega_n - \mathbf{k}^2/2m + \mu]\, \eta^\alpha(k)\right) \quad , \tag{2.15a}$$

$$\begin{aligned} \tilde{S}_{\text{dis}} &= \frac{-1}{\pi N_F \tau_{\text{e}}} \sum_{\alpha,\beta} \sum_{n,m} \sum_{\mathbf{k},\mathbf{p}} \sum_{\mathbf{q}}{}' \left(\eta_n^\alpha(\mathbf{k}), \eta_n^\alpha(\mathbf{p})\right) \\ &\quad \times \left(\eta_m^\beta(\mathbf{p}+\mathbf{q}), \eta_m^\beta(\mathbf{k}+\mathbf{q})\right) \quad , \end{aligned} \tag{2.15b}$$

and

$$\tilde{S}_{\text{int}} = \tilde{S}_{\text{int}}^{(s)} + \tilde{S}_{\text{int}}^{(t)} + \tilde{S}_{\text{int}}^{(c)} \quad , \tag{2.15c}$$

$$\begin{aligned} \tilde{S}_{\text{int}}^{(s)} &= \frac{T\Gamma^{(s)}}{2} \sum_{\alpha} \sum_{k,p} {\sum_{q}}' \sum_{r=0,3} (-1)^r \\ &\quad \times \big(\eta^\alpha(k), (\tau_r \otimes s_0)\eta^\alpha(k+q)\big) \\ &\quad \times \big(\eta^\alpha(p+q), (\tau_r \otimes s_0)\eta^\alpha(p)\big) \quad , \end{aligned} \tag{2.15d}$$

$$\begin{aligned} \tilde{S}_{\text{int}}^{(t)} &= \frac{T\Gamma^{(t)}}{2} \sum_{\alpha} \sum_{k,p} {\sum_{q}}' \sum_{r=0,3} (-1)^r \sum_{i=1}^{3} \\ &\quad \times \big(\eta^\alpha(k), (\tau_r \otimes s_i)\eta^\alpha(k+q)\big) \\ &\quad \times \big((\eta^\alpha(p+q), (\tau_r \otimes s_i)\eta^\alpha(p)\big) \quad , \end{aligned} \tag{2.15e}$$

$$\begin{aligned} \tilde{S}_{\text{int}}^{(c)} &= \frac{T\Gamma^{(c)}}{2} \sum_{\alpha} \sum_{\mathbf{k},\mathbf{p}} {\sum_{\mathbf{q}}}' \sum_{n_1,n_2,m} \sum_{r=1,2} \\ &\quad \times \big(\eta_{n_1}^\alpha(-\mathbf{k}), (\tau_r \otimes s_0)\, \eta_{-n_1+m}^\alpha(-\mathbf{k}+\mathbf{q})\big) \\ &\quad \times \big(\eta_{-n_2}^\alpha(-\mathbf{p}), (\tau_r \otimes s_0)\, \eta_{n_2+m}^\alpha(-\mathbf{p}-\mathbf{q})\big) \quad . \end{aligned} \tag{2.15f}$$

τ_{e} is the single-particle relaxation time. There is another irrelevant term of the disordered action $\tilde{S}_{\text{dis}}$ which has been neglected for the purpose of the manuscript [26]. The decomposition of the interacting action $\tilde{S}_{\text{int}}$ into three parts comes from the idea that, in long-wavelength low-frequency processes, the possible scattering processes can be divided into three classes: (1) small-angle scattering, (2) large-angle

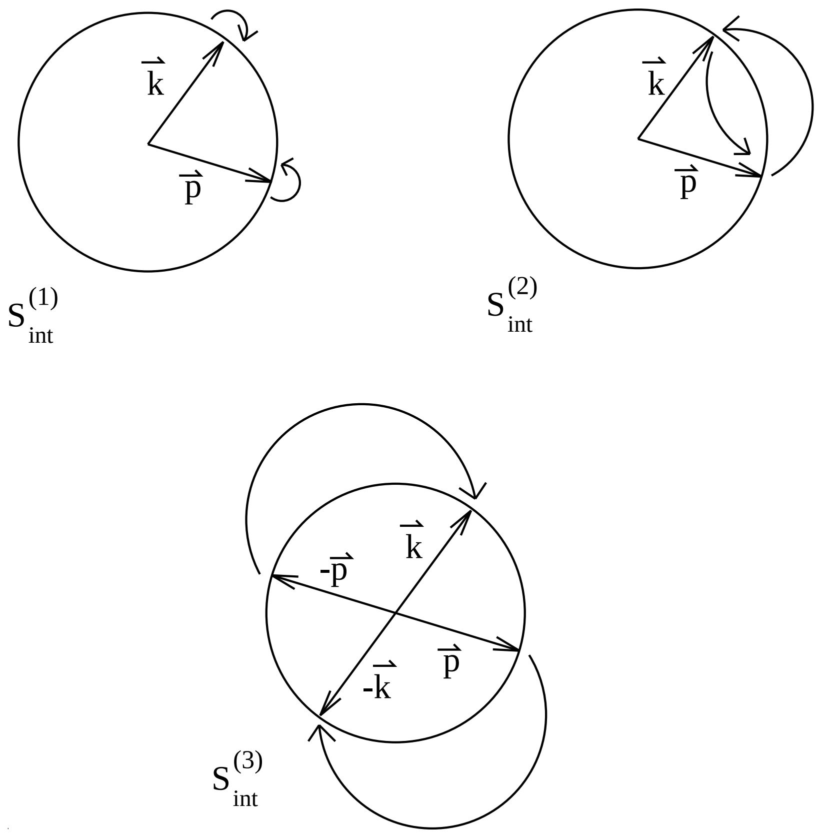

Figure 2.1: Typical small-angle (1), large-angle (2), and $2k_F$-scattering processes (3) near the Fermi surface in $d = 2$.

scattering, and (3) $2k_F$-scattering. These classes are also referred to as the particle-hole channel for classes (1) and (2), and the particle-particle or Cooper channel for class (3), respectively. The corresponding scattering processes are schematically depicted in Fig. 2.1.

In the above equations the prime on the q-summation indicates that only momenta up to some momentum cutoff, λ_c, are integrated over. This restriction is needed to avoid double counting, since each of the three expressions, Eqs. (2.15d) - (2.15f), represent all of $\tilde{S}_{int}^{\alpha}$ if all wave vectors are summed over. λ_c generally has no effect on the long-wavelength low-frequency physics we consider here.

Now we introduce a matrix of bilinear products of the fermion fields,

$$
\begin{aligned}
B_{12}(\mathbf{x}) &= \eta_1^+(\mathbf{x}) \otimes \eta_2(\mathbf{x}) \\
&= \frac{i}{2} \begin{pmatrix} -\psi_{1\uparrow}\bar{\psi}_{2\uparrow} & -\psi_{1\uparrow}\bar{\psi}_{2\downarrow} & -\psi_{1\uparrow}\psi_{2\downarrow} & \psi_{1\uparrow}\psi_{2\uparrow} \\ -\psi_{1\downarrow}\bar{\psi}_{2\uparrow} & -\psi_{1\downarrow}\bar{\psi}_{2\downarrow} & -\psi_{1\downarrow}\psi_{2\downarrow} & \psi_{1\downarrow}\psi_{2\uparrow} \\ \bar{\psi}_{1\downarrow}\bar{\psi}_{2\uparrow} & \bar{\psi}_{1\downarrow}\bar{\psi}_{2\downarrow} & \bar{\psi}_{1\downarrow}\psi_{2\downarrow} & -\bar{\psi}_{1\downarrow}\psi_{2\uparrow} \\ -\bar{\psi}_{1\uparrow}\bar{\psi}_{2\uparrow} & -\bar{\psi}_{1\uparrow}\bar{\psi}_{2\downarrow} & -\bar{\psi}_{1\uparrow}\psi_{2\downarrow} & \bar{\psi}_{1\uparrow}\psi_{2\uparrow} \end{pmatrix} \\
&\cong Q_{12} \quad ,
\end{aligned} \tag{2.16}
$$

where all fields are understood to be taken at position $\mathbf{x}$, and $1 \equiv (n_1, \alpha_1)$ with n_1 denoting a Matsubara frequency and α a replica index, etc. The matrix elements of B commute with one another, and are therefore isomorphic to classical or complex number-valued fields that we denote by Q. We use the notation $a \cong b$ for "a is isomorphic to b". This isomorphism maps the adjoint operation on products of fermionic fields, which is denoted above by an overbar, onto the complex conjugation of the classical fields. We use the isomorphism to constrain B to the classical field Q by means of a functional δ function, and exactly rewrite the partition function [9]

$$
\begin{aligned}
\widetilde{Z} &= \int D[\bar{\psi}, \psi]\, e^{\tilde{S}[\bar{\psi}, \psi]} \int D[Q]\, \delta[Q - B] \\
&= \int D[\bar{\psi}, \psi]\, e^{\tilde{S}[\eta]} \int D[Q]\, D[\widetilde{\Lambda}]\, e^{\mathrm{Tr}\,[\widetilde{\Lambda}(Q-B)]} \\
&\equiv \int D[Q]\, D[\widetilde{\Lambda}]\, e^{\mathcal{A}[Q,\widetilde{\Lambda}]} \quad .
\end{aligned} \tag{2.17}
$$

Here $\widetilde{\Lambda}$ is an auxiliary bosonic matrix field that plays the role of a Lagrange multiplier, and integrates out the fermionic fields.

It is useful to expand the 4×4 matrix in Eq. (4.14) in the spin-quaternion basis,

$$Q_{12}(\mathbf{x}) = \sum_{r,i=0}^{3} (\tau_r \otimes s_i)\, {}_r^i Q_{12}(\mathbf{x}) \tag{2.18}$$

and analogously for $\widetilde{\Lambda}$. In this basis, $i = 0$ and $i = 1,2,3$ describe the spin singlet and the spin triplet, respectively. An explicit calculation reveals that $r = 0,3$ corresponds to the particle-hole channel (i.e., products $\bar{\psi}\psi$), while $r = 1,2$ describes the particle-particle channel (i.e., products $\bar{\psi}\bar{\psi}$ or $\psi\psi$). From the structure of Eq. (4.14) one obtains the following formal symmetry properties of the Q matrices [9],

$$ {}_r^0 Q_{12} = (-)^r\, {}_r^0 Q_{21} \quad , \quad (r = 0,3) \quad , \tag{2.19a}$$

$$ {}_r^i Q_{12} = (-)^{r+1}\, {}_r^i Q_{21} \; , \; (r = 0,3;\; i = 1,2,3) \quad , \tag{2.19b}$$

$$ {}_r^0 Q_{12} = {}_r^0 Q_{21} \quad , \quad (r = 1,2) \quad , \tag{2.19c}$$

$$ {}_r^i Q_{12} = -{}_r^i Q_{21} \quad , \quad (r = 1,2;\; i = 1,2,3) \quad , \tag{2.19d}$$

$$ {}_r^i Q_{12}^* = -{}_r^i Q_{-n_1-1,-n_2-1}^{\alpha_1\alpha_2} \quad . \tag{2.19e}$$

Here the star in Eq. (2.19e) denotes complex conjugation.

Now by using the delta constraint in Eq. (2.17) to rewrite all terms that are quadratic in the fermionic field in terms of Q, we can achieve an integrand that is bilinear in ψ and $\bar{\psi}$. With the help of the following operator identity

$$\int D[\eta] e^{\int dx (\eta(x)|O\eta(x))} = (\det Q)^{1/2} = e^{\int dx \operatorname{tr}(\ln(O(x)))/2} \quad , \tag{2.20}$$

which comes from [23]

$$\int d\psi \exp\left[\sum_{\alpha,\beta} \psi_\alpha m_{\alpha\beta} \psi_\beta\right] = [\det(2m)]^{1/2} \tag{2.21}$$

with m being a complex skew-symmetric matrix, the Grassmannian integral can then be performed exactly, and we obtain for the effective action $\mathcal{A}$

$$\mathcal{A}[Q,\widetilde{\Lambda}] = \mathcal{A}_{\rm int}[Q] + \mathcal{A}_{\rm dis}[Q] + \frac{1}{2}\,{\rm Tr}\,\ln\left(G_0^{-1} - i\widetilde{\Lambda}\right) + \int d{\bf x}\,{\rm tr}\,\left(\widetilde{\Lambda}({\bf x})\,Q({\bf x})\right) \quad . \tag{2.22}$$

Here Tr denotes a trace over all degrees of freedom, including the continuous position variable, while tr is a trace over all those discrete indices that are not explicitly shown. And

$$G_0^{-1} = -\partial_\tau + \partial_{\bf x}^2/2m + \mu \tag{2.23}$$

is the inverse free electron Green operator, with ∂_τ and $\partial_{\bf x}$ derivatives with respect to imaginary time and position, respectively. We can see from the structure of the Tr ln-term in Eq. (4.19) that the physical meaning of the auxiliary field $\widetilde{\Lambda}$ is that of a self-energy. The electron-electron interaction $\mathcal{A}_{\rm int}$ is conveniently decomposed into four pieces that describe the interaction in the particle-hole and particle-particle spin-singlet and spin-triplet channels [9]. For the purposes of the present paper, we need only the particle-particle spin-singlet channel interaction explicitly to describe superconductivity. Similar to the BCS model we ignore the normal Coulomb repulsion in the particle-hole channels, and we also ignore the possibility of triplet superconductivity. Then

$$\begin{aligned} \mathcal{A}_{\rm int}[Q] &= \mathcal{A}_{\rm int}^{(c)} \\ &= \frac{T\Gamma^{(c)}}{2}\int d{\bf x}\sum_{r=1,2}\sum_{n_1,n_2,m}\sum_{\alpha}\left[{\rm tr}\,\left((\tau_r\otimes s_0)\,Q_{n_1,-n_1+m}^{\alpha\alpha}({\bf x})\right)\right] \\ &\quad\times\left[{\rm tr}\,\left((\tau_r\otimes s_0)\,Q_{-n_2,n_2+m}^{\alpha\alpha}({\bf x})\right)\right] \quad , \end{aligned} \tag{2.24}$$

with $\Gamma^{(c)}$ the particle-particle spin-singlet channel interaction amplitude, with $\Gamma^{(c)} <$

0 leading to superconductivity. For the disorder part of the effective action one finds [9]

$$\mathcal{A}_{\text{dis}}[Q] = \frac{1}{\pi N_F \tau_{\text{e}}} \int d\mathbf{x} \ \text{tr} \left(Q(\mathbf{x})\right)^2 \quad , \tag{2.25}$$

with N_{F} the density of states at the Fermi level in saddle-point approximation (see Ref. [9] and Sec. 2.3 below), and τ_{e} the single-particle scattering or relaxation time.

We will focus on the matrix elements ${}^0_0Q$ and ${}^0_1Q$ in disordered superconductivity states. From Eqs. (4.14) and (2.18) we find

$${}^0_0Q_{12}(\mathbf{x}) \cong \frac{i}{8} \left[-\psi_{1\uparrow}(\mathbf{x})\bar{\psi}_{2\uparrow}(\mathbf{x}) - \psi_{1\downarrow}(\mathbf{x})\bar{\psi}_{2\downarrow}(\mathbf{x}) + \bar{\psi}_{1\downarrow}(\mathbf{x})\psi_{2\downarrow}(\mathbf{x}) + \bar{\psi}_{1\uparrow}(\mathbf{x})\psi_{2\uparrow}(\mathbf{x})\right] \quad , \tag{2.26a}$$

$${}^0_1Q_{12}(\mathbf{x}) \cong \frac{-1}{8} \left[-\psi_{1\uparrow}(\mathbf{x})\psi_{2\downarrow}(\mathbf{x}) + \psi_{1\downarrow}(\mathbf{x})\psi_{2\uparrow}(\mathbf{x}) + \bar{\psi}_{1\downarrow}(\mathbf{x})\bar{\psi}_{2\uparrow}(\mathbf{x}) - \bar{\psi}_{1\uparrow}(\mathbf{x})\bar{\psi}_{2\downarrow}(\mathbf{x})\right] \quad . \tag{2.26b}$$

Note that ${}^0_2Q_{12}$ has a similar structure with ${}^0_1Q_{12}$. So it is also correct if we use ${}^0_2Q_{12}$ instead of ${}^0_1Q_{12}$. Physically, ${}^0_0Q_{12}$ is related to the single particle density of states, while ${}^0_1Q_{12}$ is basically the superconducting order parameter.

2.3 The saddle-point method

We now look for a saddle-point solution of the field theory derived in the previous section. The saddle-point condition is [9, 27]

$$\left.\frac{\delta \mathcal{A}}{\delta Q}\right|_{Q_{\text{sp}},\widetilde{\Lambda}_{\text{sp}}} = \left.\frac{\delta \mathcal{A}}{\delta \widetilde{\Lambda}}\right|_{Q_{\text{sp}},\widetilde{\Lambda}_{\text{sp}}} = 0 \quad . \tag{2.27}$$

According to Eqs. (2.26), the saddle point values of both Q and $\widetilde{\Lambda}$ in singlet superconductivity-like phases have the structures

$$
\begin{aligned}
{}_r^i Q_{12}(\mathbf{x})\Big|_{\rm sp} &= \delta_{\alpha_1\alpha_2}\,\delta_{i0}\,[\delta_{n_1,-n_2}\,\delta_{r1}\,Q_{n_1} + \delta_{n_1,n_2}\,\delta_{r0}\,\Lambda_{n_1}] \quad , \qquad (2.28a)\\
{}_r^i \widetilde{\Lambda}_{12}(\mathbf{x})\Big|_{\rm sp} &= \delta_{\alpha_1\alpha_2}\,\delta_{i0}\,[\delta_{n_1,-n_2}\,\delta_{r1}\,(iq_{n_1}) + \delta_{n_1,n_2}\,\delta_{r0}\,(-i\lambda_{n_1})] \quad , \qquad (2.28b)
\end{aligned}
$$

where we assume $\Lambda_n = -\Lambda_{-n}$, $\lambda_n = -\lambda_{-n}$ which is equivalent to a redefinition of the chemical potential [27], and set $Q_n = Q_{-n}$, $q_n = q_{-n}$ which follows from Eqs. (4.28) and (2.19c). Substituting this into Eqs. (4.19) - (2.25), and using the saddle-point condition Eq. (2.27), we obtain the saddle-point equations

$$
\begin{aligned}
\Lambda_n &= \frac{i}{2V}\sum_{\mathbf{k}} \mathcal{G}_n(\mathbf{k}) \quad , \qquad (2.29a)\\
Q_n &= \frac{-i}{2V}\sum_{\mathbf{k}} \mathcal{F}_n(\mathbf{k}) \quad , \qquad (2.29b)\\
\lambda_n &= \frac{-2i}{\pi N_{\rm F}\tau_{\rm e}}\,\Lambda_n \quad , \qquad (2.29c)\\
q_n &= \frac{2i}{\pi N_{\rm F}\tau_{\rm e}}\,Q_n - 4i\,\Gamma^{(c)}\,T\sum_m Q_m \quad . \qquad (2.29d)
\end{aligned}
$$

Here

$$
\begin{aligned}
\mathcal{G}_n(\mathbf{k}) &= \frac{-(i\omega_n - \lambda_n) - \xi_{\mathbf{k}}}{-(i\omega_n - \lambda_n)^2 + \xi_{\mathbf{k}}^2 + q_n^2} \quad , \qquad (2.30a)\\
\mathcal{F}_n(\mathbf{k}) &= \frac{q_n}{-(i\omega_n - \lambda_n)^2 + \xi_{\mathbf{k}}^2 + q_n^2} \quad , \qquad (2.30b)
\end{aligned}
$$

are Green functions with $\xi_{\mathbf{k}} = \mathbf{k}^2/2m - \mu$.

From Eqs. (4.29), it is easy to find

$$
\begin{aligned}
\lambda_n &= \frac{1}{\pi N_{\rm F}\tau_{\rm e}}\,\frac{1}{V}\sum_{\mathbf{k}} \mathcal{G}_n(\mathbf{k}) \quad , \qquad (2.31a)\\
q_n &= \frac{1}{\pi N_{\rm F}\tau_{\rm e}}\,\frac{1}{V}\sum_{\mathbf{k}} \mathcal{F}_n(\mathbf{k}) - 2\,\Gamma^{(c)}\,T\,\frac{1}{V}\sum_{\mathbf{k}}\sum_m \mathcal{F}_m(\mathbf{k}) \quad . \qquad (2.31b)
\end{aligned}
$$

We now define a gap function Δ by [28]

$$q_n = \bar{q}_n + \Delta \equiv \eta_n \Delta \quad , \tag{2.32a}$$

with

$$\bar{q}_n = \frac{1}{\pi N_{\rm F} \tau_{\rm e}} \frac{1}{V} \sum_{\mathbf{k}} \mathcal{F}_n(\mathbf{k}) \quad , \tag{2.32b}$$

and it can be shown that

$$\eta_n \omega_n = i\lambda_n + \omega_n \quad . \tag{2.32c}$$

We then obtain the gap equation,

$$\begin{aligned} \Delta &= -2\,\Gamma^{(c)}\,T\,\frac{1}{V}\sum_{\mathbf{k}}\sum_{n}\frac{\eta_n\Delta}{(\eta_n\omega_n)^2+\xi_{\mathbf{k}}^2+(\eta_n\Delta)^2} \\ &= -2\,\Gamma^{(c)}\,T\sum_{n}N(0)\int d\xi_{\mathbf{k}}\,\frac{\Delta}{\omega_n^2+\xi_{\mathbf{k}}^2+\Delta^2} \end{aligned} \tag{2.33}$$

with $N(0) = \frac{N_F}{2}$ the density of states per spin at the Fermi surface. A remarkable aspect of this gap equation is that in this approximation the gap Δ and the critical temperature T_c are independent of the (nonmagnetic) disorder, and so are all thermodynamic properties in superconductivity. This result is known as Anderson's theorem [14].

We next obtain the density of states. From Eq. (4.25a) it follows,

$$N(\epsilon_F + \omega) = \frac{4}{\pi}\,\mathrm{Re}\left\langle {}^0_0Q_{nn}(\mathbf{x})\right\rangle\Big|_{i\omega_n\to\omega+i0} \quad . \tag{2.34}$$

In the saddle point approximation, we have for the density of states

$$\begin{aligned} N(\epsilon_F+\omega) &= \frac{-2}{\pi}\frac{1}{V}\sum_{\mathbf{k}}\mathrm{Im}\,\mathcal{G}_n(\mathbf{k}, i\omega_n\to\omega+i0) \\ &= N_F\,\frac{\omega}{\sqrt{\omega^2-\Delta^2}} \quad \text{for } \omega>\Delta \\ &= 0 \quad \text{for } \omega<\Delta \quad . \end{aligned} \tag{2.35}$$

For later reference we also define a matrix saddle-point Green function

$$G_{\rm sp} = \left(G_0^{-1} - i\widetilde{\Lambda}\right)^{-1}\Big|_{\rm sp} \quad , \tag{2.36a}$$

whose matrix elements are given by

$$(G_{\rm sp})_{nm}(\mathbf{k}) = \delta_{nm}\,\mathcal{G}_n(\mathbf{k})\,(\tau_0 \otimes s_0) - \,\delta_{n,-m}\,\mathcal{F}_n(\mathbf{k})\,(\tau_1 \otimes s_0) \quad . \tag{2.36b}$$

Note that the above results are the standard ones.

2.4 Gaussian approximation

We next set up the calculation of the Gaussian fluctuations about the saddle point discussed above. In the following section these results will be used to compute the physical correlation functions in the disordered superconducting phase. To obtain this, we write Q and $\widetilde{\Lambda}$ in Eqs. (4.19) - (2.25) as,

$$Q = Q_{\rm sp} + \delta Q \quad , \tag{2.37a}$$

$$\widetilde{\Lambda} = \widetilde{\Lambda}_{\rm sp} + \delta\widetilde{\Lambda} \quad , \tag{2.37b}$$

and then expand to second or Gaussian order in the fluctuations δQ and $\delta\widetilde{\Lambda}$. Denoting the constant saddle point contribution to the effective action by $\mathcal{A}_{\rm sp}$, and the Gaussian action by $\mathcal{A}_G$, we have, to the Gaussian order, that

$$\mathcal{A}[Q,\widetilde{\Lambda}] = \mathcal{A}_{\rm sp} + \mathcal{A}_G[Q,\widetilde{\Lambda}] \quad , \tag{2.38}$$

with

$$\mathcal{A}_G[Q,\widetilde{\Lambda}] = \mathcal{A}_{\rm int}[\delta Q] + \mathcal{A}_{\rm dis}[\delta Q] + \frac{1}{4}\,{\rm Tr}\,\left(G_{\rm sp}\delta\widetilde{\Lambda}G_{\rm sp}\,\delta\widetilde{\Lambda}\right) + \int d\mathbf{x}\;{\rm tr}\,\left(\delta\widetilde{\Lambda}(\mathbf{x})\,\delta Q(\mathbf{x})\right) \quad , \tag{2.39}$$

For the quadratic part we find

$$\frac{1}{4}\mathrm{Tr}\,\left(G_{\rm sp}\,\delta\widetilde{\Lambda}\,G_{\rm sp}\,\delta\widetilde{\Lambda}\right) = \frac{1}{V}\sum_{\mathbf{k}}\sum_{1,2,3,4}\sum_{r,s}\sum_{i,j}{}^{i}_{r}(\delta\widetilde{\Lambda})_{12}(\mathbf{k})\;{}^{ij}_{rs}A_{12,34}(\mathbf{k})\,{}^{j}_{s}(\delta\widetilde{\Lambda})_{34}(-\mathbf{k}) \quad . \tag{2.40a}$$

Here

$$\begin{aligned}
{}^{ij}_{rs}A_{12,34}(\mathbf{k}) &= \delta_{13}\,\delta_{24}\,\varphi^{00}_{12}(\mathbf{k})\,N^{00}_{rs}\,\delta_{ij}\,{}^{i}_{r}I_{12} \\
&\quad +\delta_{13}\,\delta_{2,-4}\,\varphi^{01}_{12}(\mathbf{k})\,N^{01}_{rs}\,\delta_{ij}\,{}^{i}_{r}I_{12} \\
&\quad +\delta_{1,-3}\,\delta_{24}\,\varphi^{10}_{12}(\mathbf{k})\,N^{10}_{rs}\,\delta_{ij}\,{}^{i}_{r}I_{12} \\
&\quad +\delta_{1,-3}\,\delta_{2,-4}\,\varphi^{11}_{12}(\mathbf{k})\,N^{11}_{rs}\,\delta_{ij}\,{}^{i}_{r}I_{12}, \\
&\equiv {}^{ij}_{rs}A^{(0)}_{12,34}(\mathbf{k})\,{}^{i}_{r}I_{12} \quad ,
\end{aligned} \tag{2.40b}$$

with 4×4 matrices

$$\begin{aligned}
N^{00} &= \begin{pmatrix} i\tau_3 & 0 \\ 0 & -i\tau_3 \end{pmatrix} \quad , \quad N^{01} = \begin{pmatrix} -i\tau_1 & 0 \\ 0 & -i\tau_1 \end{pmatrix} \quad , \\
N^{10} &= \begin{pmatrix} -i\tau_1 & 0 \\ 0 & i\tau_1 \end{pmatrix} \quad , \quad N^{11} = \begin{pmatrix} -i\tau_3 & 0 \\ 0 & -i\tau_3 \end{pmatrix} \quad ,
\end{aligned} \tag{2.40c}$$

and

$${}^{i}_{r}I_{12} = 1 + \delta_{12}\left[-1 + \begin{pmatrix} + \\ + \\ + \\ - \end{pmatrix}_r \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_i\right] \quad , \tag{2.40d}$$

where $\begin{pmatrix} + \\ + \\ + \\ - \end{pmatrix}_r = \delta_{r0} + \delta_{r1} + \delta_{r2} - \delta_{r3}$, etc. and

$$\varphi^{00}_{nm}(\mathbf{k}) = \frac{1}{V}\sum_{\mathbf{p}}\mathcal{G}_n(\mathbf{p})\,\mathcal{G}_m(\mathbf{p}+\mathbf{k}) \quad , \tag{2.40e}$$

and φ^{01}, φ^{10}, and φ^{11} defined similarly with $\mathcal{G}\mathcal{G}$ in Eq. (2.40e) replaced by $(-1)\mathcal{G}\mathcal{F}$, $(-1)\mathcal{F}\mathcal{G}$, and $\mathcal{F}\mathcal{F}$, respectively.

In a similar way, the term that couples $\delta\widetilde{\Lambda}$ and δQ can be written

$$\text{Tr}\left(\delta\widetilde{\Lambda}\,\delta Q\right) = 4\sum_{1,2,3,4}\frac{1}{V}\sum_{\mathbf{k}}\sum_{r,i}{}^{i}_{r}(\delta\widetilde{\Lambda})_{12}(\mathbf{k})\,{}^{i}_{r}B_{12}(\mathbf{k})\,{}^{i}_{r}(\delta Q)_{12}(-\mathbf{k})\ , \tag{2.41a}$$

where

$${}^{i}_{r}B_{12}(\mathbf{k}) = {}^{i}_{r}I_{12}\begin{pmatrix} + \\ - \\ - \\ + \end{pmatrix}_r \quad . \tag{2.41b}$$

Q and $\widetilde{\Lambda}$ can now be decoupled by shifting and scaling the $\widetilde{\Lambda}$ field. If we define a new field $\bar{\Lambda}$ by

$${}^{i}_{r}(\delta\widetilde{\Lambda})_{12}(\mathbf{k}) = 2\,{}^{ij}_{rs}(A^{-1})_{12,34}(\mathbf{k})\,\left({}^{j}_{s}(\delta\bar{\Lambda})_{34}(\mathbf{k}) - {}^{j}_{s}(\delta Q)_{34}(\mathbf{k})\right){}^{j}_{s}B_{34}\ , \tag{2.42}$$

with A^{-1} being the inverse of the matrix A defined in Eq. (2.40b), then $\bar{\Lambda}$ and Q decouple. Integrating out $\delta\bar{\Lambda}$ and the Gaussian action is remained completely in terms of Q,

$$\begin{aligned}\mathcal{A}_{\text{G}}[Q] \;\; &= \;\; -\frac{4}{V}\sum_{\mathbf{k}}\sum_{1234}\sum_{rs}\sum_{ij}{}^{i}_{r}(\delta Q)_{12}(\mathbf{k})\,{}^{ij}_{rs}(A^{-1})_{12,34}(\mathbf{k})\,{}^{i}_{r}B_{12}\,{}^{j}_{s}B_{34}\,{}^{j}_{s}(\delta Q)_{34}(-\mathbf{k}) \\ &\quad + \mathcal{A}_{\text{int}}[\delta Q] + \mathcal{A}_{\text{dis}}[\delta Q] \quad ,\end{aligned} \tag{2.43}$$

It is convenient to rewrite this result as

$$\mathcal{A}_{\text{G}}[Q] = \frac{-4}{V}\sum_{\mathbf{k}}\sum_{1234}\sum_{rs}\sum_{ij}{}^{i}_{r}(\delta Q)_{12}(\mathbf{k})\,{}^{ij}_{rs}M_{12,34}(\mathbf{k})\,{}^{j}_{s}(\delta Q)_{34}(-\mathbf{k}) \quad , \tag{2.44a}$$

where

$$\begin{aligned}{}^{ij}_{rs}M_{12,34}(\mathbf{k}) \;\; &= \;\; {}^{ij}_{rs}(A^{-1})_{12,34}(\mathbf{k})\,{}^{i}_{r}B_{12}\,{}^{j}_{s}B_{34} \\ &\quad -2T\Gamma^{(c)}\,\delta_{ij}\delta_{rs}\delta_{1+2,3+4}\begin{pmatrix} 0 \\ + \\ + \\ 0 \end{pmatrix}_r \begin{pmatrix} + \\ 0 \\ 0 \\ 0 \end{pmatrix}_i \\ &\quad -\frac{1}{\pi N_F \tau_{\text{e}}}\,{}^{i}_{r}B_{12}\,\delta_{ij}\delta_{rs}\delta_{13}\delta_{24}\ .\end{aligned} \tag{2.44b}$$

Chapter 3

Physical correlation functions

3.1 Ultrasonic attenuation by saddle-point approximation

We now use the results of the preceding sections to calculate transverse ultrasonic attenuation in both clean and disordered superconductors. The ultrasonic attenuation is defined by the power dissipated per unit energy flux, as

$$\alpha = \frac{P}{\frac{1}{2}\rho_{\mathrm{ion}}\upsilon^2 c_{\mathrm{s}}} \tag{3.1}$$

where P is the power dissipated by the sound wave per unit volume and time, ρ_{ion} the mass density of the material, υ the phonon velocity amplitude, and c_{s} the velocity of sound. The dissipated power comes from the electron-phonon interaction:

$$P = -\frac{1}{2V}\frac{\delta\langle H_{\mathrm{ep}}\rangle}{\delta t} \tag{3.2}$$

where the electron-phonon interaction has the form of

$$H_{\mathrm{ep}} = \sum_{i,j}\int d\mathrm{r}\tau_{ij}(\mathrm{r})\nabla_{\mathrm{i}}\mathrm{u}_{\mathrm{j}}(\mathrm{r}) \tag{3.3}$$

with the stress tensor of the electronic system τ_{ij}

$$\tau_{ij}(\mathrm{r}) = \frac{1}{4\mathrm{m_e}}\sum_{\sigma}(\nabla-\nabla')_{\mathrm{i}}(\nabla-\nabla')_{\mathrm{j}}\bar{\psi}_{\sigma}(\mathrm{r})\psi_{\sigma}(\mathrm{r}')|_{\mathrm{r}=\mathrm{r}'} \quad . \tag{3.4}$$

Here $u_j(\mathrm{r})$ is the phonon displacement field and $\upsilon = \frac{\partial u_j}{\partial t}$.

As shown in Ref. [29], the sound attenuation coefficient has the expression, with the help of linear-response theory, that

$$\alpha(\omega) = \lim_{k\to 0} \frac{\omega}{\rho_{\rm ion} c_{\rm s}^3} \,{\rm Im}\chi(\mathbf{k}, i\omega_n \to \omega + i0) \quad , \tag{3.5}$$

where the stress-stress spectral function, with $D_x \equiv \partial_{x_1}\partial_{x_2}$,

$$\begin{aligned}\chi(\mathbf{k}, i\omega_n) &= \frac{1}{m_e^2}\frac{1}{V}\int d\mathbf{x}d\mathbf{x}'d\mathbf{y}d\mathbf{y}' \exp\left(-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})\right)\sum_{\sigma_1,\sigma_2}\delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{y}-\mathbf{y}')D_x D_y \\ &\times \frac{1}{\beta}\sum_{\omega_1,\omega_2}\langle \bar{\psi}^\alpha_{\omega_1,\sigma_1}(\mathbf{x})\psi^\alpha_{\omega_1-\omega_n,\sigma_1}(\mathbf{x}')\bar{\psi}^\alpha_{\omega_2,\sigma_2}(\mathbf{y})\psi^\alpha_{\omega_2+\omega_n,\sigma_2}(\mathbf{y}')\rangle \quad .\end{aligned} \tag{3.6}$$

By introducing a source term of the form

$$\delta\tilde{S}^\alpha = \int d\mathbf{x}\sum_{\omega_n} h(\omega_n,\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\sum_{\omega,\sigma} \bar{\psi}^\alpha_{\omega,\sigma}(\mathbf{x})\, D_x\, \psi^\alpha_{\omega+\omega_n,\sigma}(\mathbf{x}), \tag{3.7}$$

we can obtain

$$\chi(\mathbf{k}=0, i\omega_n) = \frac{1}{m_e^2\beta V}\frac{\partial^2\widetilde{Z}}{\partial h(\omega_n,\mathbf{k})\partial h(\omega_{-n},-\mathbf{k})}\bigg|_{h=0} \tag{3.8}$$

with the third term of the right side of the Eq. (4.19) becoming

$$\begin{aligned}\mathcal{A}_3 &= \frac{1}{2}\,{\rm Tr}\,\ln\left(G_0^{-1} - i\widetilde{\Lambda} + D\right) \\ &= \frac{1}{2}\,{\rm Tr}\,\ln\left(G_0^{-1} - i(\widetilde{\Lambda}_{sp} + \delta\widetilde{\Lambda}) + D\right) \\ &= \frac{1}{2}\,{\rm Tr}\,\ln\left(1 + DG_{sp} - i\delta\widetilde{\Lambda}G_{sp}\right) + \frac{1}{2}\,{\rm Tr}\,\ln\left(G_{sp}^{-1}\right)\end{aligned} \tag{3.9}$$

and $D \equiv \sum_{\omega_n}\delta(\omega_1 - \omega_2 + \omega_n)h\exp\left(-i\mathbf{k}\cdot\mathbf{x}\right)D_x$.

In the saddle-point approximation, we neglect the $\delta\widetilde{\Lambda}$ item and have

$$\mathcal{A}_3 = \frac{-1}{4}\,{\rm Tr}\,\left(DG_{sp}DG_{sp}\right) + const. \tag{3.10}$$

We then obtain the ultrasonic attenuation coefficient, for small frequency,

$$\alpha_s(\omega) = \alpha_n\,\frac{2}{1+\exp\left(\beta\Delta\right)} \tag{3.11}$$

for both clean and disordered superconductors. Here α_n is the attenuation coefficient of the normal metal [30]. In the clean metal it has

$$\alpha_{n,clean} = \frac{k_f^4 \omega^2}{30\pi q \rho_{ion} c_s^3} \tag{3.12a}$$

with the usual conditions $\omega < qv_f < \Delta$ satisfied. In the disordered case

$$\alpha_{n,disordered} = \frac{2N(0)k_f^4 \omega^2 \tau}{15m^2 \rho_{ion} c_s^3}, \tag{3.12b}$$

where the approximation of $\tau_e \Delta \ll 1$ is assumed, which is called the dirty limit [31]. The above result confirms Levy's prediction which was obtained by Boltzmann's transport equation [32]. It should be noted that no Greens function method has been used to obtain this result before.

The above method can be used to obtain other physical properties, like longitudinal electrical conductivity. In that case, higher-order corrections must be included due to the gauge invariance problem [33]. Below we show how to correctly obtain the conductivity by using the Gaussian fluctuations about the saddle point [34].

3.2 Gaussian propagators

We now use the results of the preceding sections to calculate some correlation functions of physical interest. We find in Appendix A that the number density susceptibility, χ_n, and the spin density susceptibility, χ_s, can be expressed in terms of the Q-correlation functions,

$$\chi^{(i)}(\mathbf{k}, \omega_n) = \frac{16T}{V} \sum_{1,2} \sum_{r=0,3} \left\langle {}^i_r(\delta Q)_{1+n,1}(\mathbf{k}) \, {}^i_r(\delta Q)_{2+n,2}(-\mathbf{k}) \right\rangle \quad , \tag{3.13}$$

with $\chi^{(0)} = \chi_n$ and $\chi^{(1,2,3)} = \chi_s$. Here the Gaussian propagators in the Eq. (3.13) are given in terms of the inverse of the matrix M defined in Eq. (2.44b) by

$$\left\langle {}^i_r(\delta Q)_{12}(\mathbf{k}_1)\, {}^j_s(\delta Q)_{34}(\mathbf{k}_2) \right\rangle_G = \frac{V}{8}\, \delta_{\mathbf{k}_1,-\mathbf{k}_2}\, {}^{ij}_{rs}M^{-1}_{12,34}(\mathbf{k}_1)\ , \tag{3.14}$$

where $\langle \ldots \rangle_G$ denotes an average with the Gaussian action $\mathcal{A}_G$. We find from Eqs. (3.13) and (3.14) that M^{-1} determines the correlation functions within Gaussian approximation.

In the following section we will be interested in the number density susceptibility χ_n. Other correlation functions can be obtained similarly by applying the technique introduced below. From the expression of Q in terms of the fermionic fields, Eq. (4.14), it is easy to see that the contributions to Eq. (3.13) from $r = 0$ and $r = 3$ are identical for $\omega_n \neq 0$. We can therefore write

$$\chi_n(\mathbf{k}, \omega_n) = 4T \sum_{1,2} {}^{00}_{33}M^{-1}_{1+n,1;2+n,2}(\mathbf{k}) \quad , \tag{3.15}$$

To find $\sum_{1,2} {}^{00}_{33}M^{-1}_{1+n,1;2+n,2}$, we rewrite M as

$$\begin{aligned} {}^{ij}_{rs}M_{12,34}(\mathbf{k}) &\equiv {}^{ij}_{rs}(A^{-1})_{12,34}(\mathbf{k})\, {}^i_r B_{12}\, {}^j_s B_{34} - {}^{ij}_{rs}D_{12,34} \\ &\equiv {}^{ij}_{rs}(C^{-1})_{12,34}(\mathbf{k}) - {}^{ij}_{rs}D_{12,34} \quad . \end{aligned} \tag{3.16}$$

Then we find

$$M^{-1} = \left(C^{-1} - D \right)^{-1} \quad . \tag{3.17}$$

It is convenient to write the inverse of the matrix M as an integral equation,

$$M^{-1} = C + C\, D\, M^{-1} \quad , \tag{3.18}$$

with

$$ {}^{ij}_{rs}C_{12,34} = {}^{ij}_{rs}A^{(0)}_{12,34} \begin{pmatrix} + \\ - \\ - \\ + \end{pmatrix}_r {}^{j}_{s}B_{34} \quad . \tag{3.19} $$

For further simplicity, we set $\Gamma = 2T\Gamma^{(c)}$, $\tau^0 = \pi N_F \tau_{\rm e}$ and ${}^{i}_{r}I_{12} = 1$ for $\omega_n \neq 0$. Expanding Eq. (3.18) we have

$$ \begin{aligned} {}^{00}_{33}M^{-1}_{12,34} &= {}^{00}_{33}C_{12,34} - \Gamma \sum_{56,78} {}^{00}_{32}A^{(0)}_{12,56}\, \delta_{5+6,7+8}\, {}^{00}_{23}M^{-1}_{78,34} + \frac{1}{\tau^0} \sum_{js} \sum_{56} {}^{0j}_{3s}A^{(0)}_{12,56}\, {}^{j0}_{s3}M^{-1}_{56,34} \\ &= {}^{00}_{33}A^{(0)}_{12,34} - \Gamma \begin{pmatrix} -\varphi^{01}_{12} \sum_{78} \delta_{1-2,7+8}\, {}^{00}_{23}M^{-1}_{78,34} \\ +\varphi^{10}_{12} \sum_{78} \delta_{-1+2,7+8}\, {}^{00}_{23}M^{-1}_{78,34} \end{pmatrix} \\ &\quad + \frac{1}{\tau^0} \begin{pmatrix} +\varphi^{00}_{12}\, {}^{00}_{33}M^{-1}_{1,2;3,4} \\ -\varphi^{01}_{12}\, {}^{00}_{23}M^{-1}_{1,-2;3,4} \\ +\varphi^{10}_{12}\, {}^{00}_{23}M^{-1}_{-1,2;3,4} \\ +\varphi^{11}_{12}\, {}^{00}_{33}M^{-1}_{-1,-2;3,4} \end{pmatrix} \quad , \end{aligned} \tag{3.20} $$

where we have used the structures of B, Eq. (2.41b) and $A^{(0)}$, Eq. (2.40b). ${}^{10}_{23}M^{-1}$ in turn obeys the integral equation

$$ \begin{aligned} {}^{00}_{23}M^{-1}_{12,34} &= -{}^{00}_{23}A^{(0)}_{12,34} + \Gamma \begin{pmatrix} -\varphi^{00}_{12} \sum_{78} \delta_{1+2,7+8}\, {}^{00}_{23}M^{-1}_{78,34} \\ -\varphi^{11}_{12} \sum_{78} \delta_{-1-2,7+8}\, {}^{00}_{23}M^{-1}_{78,34} \end{pmatrix} \\ &\quad - \frac{1}{\tau^0} \begin{pmatrix} -\varphi^{00}_{12}\, {}^{00}_{23}M^{-1}_{1,2;3,4} \\ -\varphi^{01}_{12}\, {}^{00}_{33}M^{-1}_{1,-2;3,4} \\ +\varphi^{10}_{12}\, {}^{00}_{33}M^{-1}_{-1,2;3,4} \\ -\varphi^{11}_{12}\, {}^{00}_{23}M^{-1}_{-1,-2;3,4} \end{pmatrix} \quad . \end{aligned} \tag{3.21} $$

Similar results can be obtained for ${}^{00}_{33}M^{-1}_{-1,-2;3,4}$ and ${}^{00}_{23}M^{-1}_{-1,-2;3,4}$. By using these four matrix elements of M^{-1}, one finds

$$ {}^{00}_{23}M^{-1}_{12,34} = X^{(2)}_{12,34} + Y^{(2)}_{12} \sum_{78} \delta_{1+2,7+8}\, {}^{00}_{23}M^{-1}_{78,34} + Z^{(2)}_{12} \sum_{78} \delta_{-1-2,7+8}\, {}^{00}_{23}M^{-1}_{78,34}, \tag{3.22a} $$

$$ \begin{aligned} {}^{00}_{23}M^{-1}_{-1,-2;3,4} &= X^{(2)}_{-1,-2;3,4} + Y^{(2)}_{-1,-2} \sum_{78} \delta_{-1-2,7+8}\, {}^{00}_{23}M^{-1}_{78,34} \\ &\quad + Z^{(2)}_{-1,-2} \sum_{78} \delta_{1+2,7+8}\, {}^{00}_{23}M^{-1}_{78,34} \end{aligned} \tag{3.22b} $$

and

$$ {}^{00}_{33}M^{-1}_{12,34} = X^{(3)}_{12,34} + Y^{(3)}_{12} \sum_{78} \delta_{1-2,7+8}\, {}^{00}_{23}M^{-1}_{78,34} + Z^{(3)}_{12} \sum_{78} \delta_{-1+2,7+8}\, {}^{00}_{23}M^{-1}_{78,34} \tag{3.22c} $$

The parameters $X^{(2)}_{12,34}$, $Y^{(2)}_{12}$, $Z^{(2)}_{12}$, $X^{(3)}_{12,34}$, $Y^{(3)}_{12}$ and $Z^{(3)}_{12}$ will be specified in Appendix B.

We are now ready to calculate the $\sum_{1,2} {}^{00}_{33}M^{-1}_{1+n,1;2+n,2}$. We first need to obtain $\sum_{1,2} {}^{00}_{23}M^{-1}_{1+n,-1;2+n,2}$ and $\sum_{1,2} {}^{00}_{23}M^{-1}_{-1-n,1;2+n,2}$ from Eq. (3.22c). Summing both sides of Eqs. (3.22a) and (3.22b) with $\sum_{1,2}$, we can get

$$\sum_{1,2} {}^{00}_{23}M^{-1}_{1+n,-1;2+n,2} = \frac{(1-\sum_1 Y^{(2)}_{-1-n,1})\sum_{12} X^{(2)}_{1+n,-1;2+n,2} + \sum_1 Z^{(2)}_{1+n,-1}\sum_{12} X^{(2)}_{-1-n,1;2+n,2}}{(1-\sum_1 Y^{(2)}_{1+n,-1})(1-\sum_1 Y^{(2)}_{-1-n,1}) - \sum_1 Z^{(2)}_{1+n,-1}\sum_1 Z^{(2)}_{-1-n,1}}, \tag{3.23a}$$

$$\sum_{1,2} {}^{00}_{23}M^{-1}_{-1-n,1;2+n,2} = \frac{(1-\sum_1 Y^{(2)}_{1+n,-1})\sum_{12} X^{(2)}_{-1-n,1;2+n,2} + \sum_1 Z^{(2)}_{-1-n,1}\sum_{12} X^{(2)}_{1+n,-1;2+n,2}}{(1-\sum_1 Y^{(2)}_{1+n,-1})(1-\sum_1 Y^{(2)}_{-1-n,1}) - \sum_1 Z^{(2)}_{1+n,-1}\sum_1 Z^{(2)}_{-1-n,1}}. \tag{3.23b}$$

Then it is easily shown that

$$\begin{aligned}\sum_{1,2} {}^{00}_{33}M^{-1}_{1+n,1;2+n,2} &= \sum_{12} X^{(3)}_{1+n,1;2+n,2} + \sum_1 Y^{(3)}_{1+n,1}\sum_{12} {}^{00}_{23}M^{-1}_{1+n,-1;2+n,2} \\ &\quad + \sum_1 Z^{(3)}_{1+n,1}\sum_{12} {}^{00}_{23}M^{-1}_{-1-n,1;2+n,2}\end{aligned} \tag{3.24}$$

We can now obtain the number density susceptibility χ_n. Note that by substituting Eqs. (3.23) into Eqs. (3.22) we can also obtain the explicit forms of ${}^{00}_{23}M^{-1}_{12,34}$ and ${}^{00}_{33}M^{-1}_{12,34}$. This technique can then be generalized to obtain all elements of M^{-1}, which in turn gives the Gaussian propagators completely.

3.3 The clean limit

In this section we discuss the clean limit, or the non-impurity electron gas. Let us perform the clean limit, $\tau_e \to \infty$. $\mathcal{A}_{\rm dis}$ then vanishes. That also means $\lambda_n \to 0$,

$\Lambda_n \to 0$ and $q_n = \Delta$. It is easy to show that

$$\sum_{1,2} {}^{00}_{23}M^{-1}_{1+n,-1;2+n,2} =$$
$$\frac{(1+\Gamma\sum_1 \varphi^{00}_{-1-n,1})(\sum_1 \varphi^{01}_{1+n,-1}) + \Gamma\sum_1 \varphi^{11}_{1+n,-1}(\sum_1 \varphi^{10}_{-1-n,1})}{(1+\Gamma\sum_1 \varphi^{00}_{1+n,-1})(1+\Gamma\sum_1 \varphi^{00}_{-1-n,1}) - \Gamma\sum_1 \varphi^{11}_{1+n,-1}\,\Gamma\sum_1 \varphi^{11}_{-1-n,1}}, \qquad (3.25a)$$

$$\sum_{1,2} {}^{00}_{23}M^{-1}_{-1-n,1;2+n,2} =$$
$$\frac{(1+\Gamma\sum_1 \varphi^{00}_{1+n,-1})(-\sum_1 \varphi^{10}_{-1-n,1}) - \Gamma\sum_1 \varphi^{11}_{-1-n,1}(\sum_1 \varphi^{01}_{1+n,-1})}{(1+\Gamma\sum_1 \varphi^{00}_{1+n,-1})(1+\Gamma\sum_1 \varphi^{00}_{-1-n,1}) - \Gamma\sum_1 \varphi^{11}_{1+n,-1}\,\Gamma\sum_1 \varphi^{11}_{-1-n,1}} \qquad (3.25b)$$

and

$$\begin{aligned}\sum_{1,2} {}^{00}_{33}M^{-1}_{1+n,1;2+n,2} &= \sum_1 \varphi^{00}_{1+n,1} + \Gamma\sum_1 \varphi^{01}_{1+n,1}\sum_{12} {}^{00}_{23}M^{-1}_{1+n,-1;2+n,2} \\ &\quad -\Gamma\sum_1 \varphi^{10}_{1+n,1}\sum_{12} {}^{00}_{23}M^{-1}_{-1-n,1;2+n,2}. \qquad (3.25c)\end{aligned}$$

Using the results in Appendix C, we obtain the number density susceptibility of clean superconductor, for small $|\mathbf{k}|$ and ω_n,

$$\chi_n(\mathbf{k},\omega_n) = -N_F\,\frac{\frac{v_f^2}{3}\mathbf{k}^2}{\omega_n^2 + \frac{v_f^2}{3}\mathbf{k}^2} \quad . \qquad (3.26)$$

The electrical conductivity σ is determined by χ_n via [35]

$$\sigma(\mathbf{k},\omega) = ie^2\frac{\omega}{\mathbf{k}^2}\,\chi_n(\mathbf{k}, i\omega_n \to \omega + i0) \quad . \qquad (3.27)$$

In particular, the real part of the conductivity as a function of real frequencies has a delta-function contribution, for small ω,

$$\begin{aligned}\text{Re}\ \sigma(\omega) &= -\lim_{k\to 0} e^2\frac{\omega}{\mathbf{k}^2}\,\text{Im}\chi_n(\mathbf{k}, i\omega_n \to \omega + i0) \\ &= \frac{e^2\,N_F\,\pi\,v_f^2}{3}\,\delta(\omega) \\ &= \frac{n\,\pi\,e^2}{m}\,\delta(\omega) \quad , \qquad (3.28)\end{aligned}$$

with $n = \frac{k_f^3}{3\pi^2}$ the particle number density. This coincides with the known result [36, 37, 38].

Similar procedure can be applied to obtain the spin density susceptibility, by noting that

$$\begin{aligned}\chi_s(\mathbf{k}, \omega_n = 0) &= \frac{16T}{V} \sum_{1,2} \left\langle \tfrac{1}{3}(\delta Q)_{1,1}(\mathbf{k}) \, \tfrac{1}{3}(\delta Q)_{2,2}(-\mathbf{k}) \right\rangle \\ &= 2T \sum_{1,2} {}^{11}_{33}M^{-1}_{1,1;2,2} \end{aligned} \tag{3.29a}$$

and

$$\begin{aligned}\sum_{1,2} {}^{11}_{33}M^{-1}_{1+n,1;2+n,2} &= \sum_{1,2} {}^{11}_{33}A^{(0)}_{1+n,1;2+n,2} \\ &= \sum_{1} \varphi^{00}_{1+n,1} + \delta_{n,0} \sum_{1} \varphi^{11}_{1+n,1} \\ &= \frac{-N_F}{2T} \frac{n_n}{n} \\ &\quad \text{for } \omega_n = 0, \ |\mathbf{k}| \to 0, \end{aligned} \tag{3.29b}$$

where $n = n_s + n_n$, with n_s the density of superconducting electrons, n_n the density of normal electrons [39],

$$n_n = n \int_{-\infty}^{\infty} d\xi_{\mathbf{p}} \frac{\exp\left(\frac{\sqrt{\xi_{\mathbf{p}}{}^2 + \Delta^2}}{T}\right)}{T\left(1 + \exp\left(\frac{\sqrt{\xi_{\mathbf{p}}{}^2 + \Delta^2}}{T}\right)\right)^2}. \tag{3.29c}$$

The result $\chi_s(k \to 0, \omega_n = 0) = -N_F \frac{n_n}{n}$ is consistent with Yosida's [40].

The above result means $\chi_s = 0$ at zero temperature. This is because a BCS superconductor is a perfect diamagnet at $T = 0$ [40]. The non-zero part comes from the contribution of normal electrons at finite temperature [41], since some Cooper pairs are broken into normal electrons at $T \neq 0$.

3.4 The disordered case

Now we turn to the disordered case. The approximation of $\tau_e\Delta \ll 1$ is assumed. This is called the dirty limit [31]. Calculations in Appendix D show that in the limit of long wavelength and low frequency,

$$\chi_n(\mathbf{k},\omega_n) = -N_F \, \frac{\frac{\pi\Delta\tau_e v_f^2}{3}\mathbf{k}^2}{\omega_n^2 + \frac{\pi\Delta\tau_e v_f^2}{3}\mathbf{k}^2} \quad , \tag{3.30}$$

and the real part of the conductivity as a function of real frequencies has also a delta-function contribution

$$\begin{aligned} \text{Re}\,\sigma(\omega) &= -\lim_{k\to 0} e^2 \frac{\omega}{\mathbf{k}^2}\,\text{Im}\chi_n(\mathbf{k}, i\omega_n \to \omega + i0) \\ &= \frac{e^2\, N_F\, \Delta\, \tau_e\, \pi^2\, v_f^2}{3}\,\delta(\omega) \quad . \end{aligned} \tag{3.31}$$

This coincides with the result already known, too [38].

Again, a similar procedure can be applied to obtain the spin density susceptibility. We find in Appendix D that, at $T = 0$

$$\chi_s = 0 \quad \text{for } \omega_n = 0,\ |\mathbf{k}| \to 0 \quad , \tag{3.32}$$

That means the spin response in the nonmagnetic disordered case is the same as that in the clean limit. This is consistent with Devereaux and Belitz's argument [42], which has shown that the nonmagnetic disorder has no effect on the spin-flip pair breaking rate.

Chapter 4

Quantum Metal - Superconductor Transition: A Local Field Theory Approach

As we showed in Chapter 2, small amounts of nonmagnetic disorder has a vanishing effect on the superconducting transition temperature, T_c. Historically this is known as Anderson's theorem. At large amounts of disorder, however, the superconducting critical temperature vanishes, $T_c \to 0$. At $T_c = 0$, therefore, there is a disorder-induced metal - superconductor transition. Physically, the long range behavior of the non-order-parameter fluctuations leads to an effective long range interaction between the order parameter fluctuations. The net result is that the long range interactions suppress fluctuation effect and makes the theory exactly soluble.

In this chapter this quantum phase transition is studied. An effective local field theory is developed that keeps all soft modes or fluctuations explicitly. Renormalization group analysis on the resulting local field theory is then used to exactly determine the quantum critical behavior at this transition.

4.1 Introduction

Recently there has been much interest in quantum phase transitions. Occurring at $T = 0$, these transitions provide new insight into the possible physical phases of systems at low temperature [43]. The first quantum phase transition studied in detail was the ferromagnetic transition in an itinerant electron system at zero temperature. Hertz argued in 1976 that the transition was mean-field-like in the physically interesting dimension $d = 3$ [44]. This simple mean-field description was later shown to be incorrect [45, 46, 47]. The reason for this breakdown is the existence of soft or massless modes other than the order parameter fluctuations. These modes were neglected in Hertz's theory. In both clean and disordered systems these modes are coupled to the order parameter fluctuations and modify the critical behavior [46]. Technically, if these additional soft modes are integrated out, they lead to a long-ranged interaction and a nonlocal field theory. For the disordered case it was argued that once this effect is taken into account, all other fluctuation effects are suppressed by the long-range nature of the interactions and that the critical behavior is governed by a fixed point that is Gaussian, but does not yield mean-field exponents [47].

A similar argument was used to describe the normal metal to superconductor quantum phase transition at $T = 0$ [48]. In this case the usual finite temperature superconducting phase transition is driven to zero temperature by nonmagnetic disorder [49], where the additional soft modes come from particle-hole excitations. Again, it was argued that the critical behavior found at this quantum phase transition [48] could be exactly determined using the same technique as in Refs. [46, 47].

The theory developed in Refs. [46, 47, 48], however, suffered from one major drawback: Since the additional soft modes were integrated out in order to obtain a description entirely in terms of the order parameter fluctuations, the effective field theory that was derived was nonlocal [50] and not suitable for conventional perturbative renormalization group treatment. The analysis in Refs. [46, 47, 48] was therefore restricted to power counting arguments at tree level to show that all non-Gaussian terms are irrelevant in a RG sense. However, relying entirely on tree-level power counting can be dangerous. Later on, logarithmic corrections were found in the description of the disordered quantum ferromagnetic transition [51].

It is the purpose of this chapter to keep all the relevant soft modes and to construct an effective local field theory for the metal - superconductor transition so that the exact behavior at this quantum phase transition can be determined using conventional renormalization group methods. Unlike the quantum ferromagnetic transition discussed above, we will show that the previous results for the metal - superconductor transition, though from a nonlocal field theory, are still valid. The reason for this is explained in detail.

This chapter is organized as follows. In Sec. 4.2 we use methods developed in Refs. [9, 34] to derive an effective local theory for disordered electron systems that explicitly separates massive modes from soft ones, and keeps all of the latter. In Sec. 4.3 we give a renormalization group analysis of this model. In Sec. 4.4 we discuss our results.

4.2 Effective Local Field Theory

A local field theory will be developed in this section to describe the normal metal to superconductor quantum phase transition at $T = 0$. All relevant soft modes will be contained in this field theory. We start from a general model of interacting electrons with quenched disorder and an attractive Cooperon interaction amplitude. We then introduce the superconducting order parameter and separate massive and soft modes. After integrating out the massive modes, we obtain a effective local field theory that describe the coupling between the superconducting fluctuations and the soft or massless diffusive modes.

4.2.1 Composite field theory

The general partition function of the interacting, disordered electrons has been given in the form of Grassmann fields $\bar{\psi}$ and ψ chapter 2:

$$Z = \int D[\bar{\psi}, \psi]\ e^{S[\bar{\psi},\psi]} \quad . \tag{4.1a}$$

with the action S being

$$\begin{aligned} S \quad &= \quad - \int_0^\beta d\tau \int d\mathbf{x} \sum_\sigma\ \bar{\psi}_\sigma(\mathbf{x},\tau)\, \frac{\partial}{\partial \tau}\, \psi_\sigma(\mathbf{x},\tau) \\ &\quad - \int_0^\beta d\tau\ H(\tau)\ . \end{aligned} \tag{4.1b}$$

We denote the spatial position by $\mathbf{x}$, and the imaginary time by τ. $H(\tau)$ is the Hamiltonian in imaginary time representation, $\beta = 1/T$ is the inverse temperature, $\sigma = \uparrow, \downarrow$ denotes spin labels, and units such that $\hbar = k_B = 1$ are assumed as before.

The Hamiltonian H includes three parts:

$$H = H_0 + H_{\text{int}} + H_{\text{dis}} \quad . \tag{4.2}$$

Here H_0 describes a free electron fluid with chemical potential μ. H_{int} describes a spin-independent two-electron interaction with potential $v(\mathbf{x}_1 - \mathbf{x}_2)$, which as in Refs. [9] can be conveniently divided into three parts: particle- hole spin-singlet channel with interaction amplitude $\Gamma^{(s)}$, particle- hole spin-triplet channel with $\Gamma^{(t)}$, and particle-particle spin- singlet channel (or the Cooper channel) with $\Gamma^{(c)}$. $\Gamma^{(c)} < 0$ leads to superconductivity. The particle-particle spin-triplet channel (or the triplet superconductivity channel) is neglected here [52]. H_{dis} describes a static random potential $u(\mathbf{x})$ coupling to the electronic number density, with the assumption of this potential $u(\mathbf{x})$ being delta-correlated and Gaussian distributed. Since the system contains quenched disorder, the replica trick [11] is then introduced to perform the disorder average.

As what we have done in previous chapters and papers [9, 34], we next integrate out the Grassmann fields and rewrite the theory in terms of complex-number fields Q and $\widetilde{\Lambda}$. With the help of the following isomorphism,

$$\begin{aligned} B_{12} &= \frac{i}{2} \begin{pmatrix} -\psi_{1\uparrow}\bar{\psi}_{2\uparrow} & -\psi_{1\uparrow}\bar{\psi}_{2\downarrow} & -\psi_{1\uparrow}\psi_{2\downarrow} & \psi_{1\uparrow}\psi_{2\uparrow} \\ -\psi_{1\downarrow}\bar{\psi}_{2\uparrow} & -\psi_{1\downarrow}\bar{\psi}_{2\downarrow} & -\psi_{1\downarrow}\psi_{2\downarrow} & \psi_{1\downarrow}\psi_{2\uparrow} \\ \bar{\psi}_{1\downarrow}\bar{\psi}_{2\uparrow} & \bar{\psi}_{1\downarrow}\bar{\psi}_{2\downarrow} & \bar{\psi}_{1\downarrow}\psi_{2\downarrow} & -\bar{\psi}_{1\downarrow}\psi_{2\uparrow} \\ -\bar{\psi}_{1\uparrow}\bar{\psi}_{2\uparrow} & -\bar{\psi}_{1\uparrow}\bar{\psi}_{2\downarrow} & -\bar{\psi}_{1\uparrow}\psi_{2\downarrow} & \bar{\psi}_{1\uparrow}\psi_{2\uparrow} \end{pmatrix} \\ &\cong Q_{12} \quad , \end{aligned} \tag{4.3}$$

where all fields are understood to be taken at position $\mathbf{x}$, and $1 \equiv (n_1, \alpha_1)$ with n_1

denoting a Matsubara frequency and α a replica index, etc, we exactly rewrite the partition function as

$$\begin{aligned} Z &= \int D[\bar{\psi},\psi]\; e^{S[\bar{\psi},\psi]} \int D[Q]\, \delta[Q-B] \\ &= \int D[\bar{\psi},\psi]\; e^{S[\bar{\psi},\psi]} \int D[Q]\, D[\widetilde{\Lambda}]\; e^{\mathrm{Tr}\,[\widetilde{\Lambda}(Q-B)]} \\ &\equiv \int D[Q]\, D[\widetilde{\Lambda}]\; e^{\widetilde{\mathcal{A}}[Q,\widetilde{\Lambda}]} \quad . \end{aligned} \tag{4.4}$$

Here $\widetilde{\Lambda}$ is an auxiliary bosonic matrix field that plays the role of a Lagrange multiplier. The reason to do so is that the rewritten action is particularly suited for the separation of massive and soft modes. It is helpful to expand the 4×4 matrix in Eq. (4.3) in a spin-quaternion basis, as in Chap. 2,

$$Q_{12}(\mathbf{x}) = \sum_{r,i=0}^{3} (\tau_r \otimes s_i)\, {}_r^i Q_{12}(\mathbf{x}) \tag{4.5}$$

and analogously for $\widetilde{\Lambda}$. Here $\tau_0 = s_0$ is the 2×2 unit matrix, and $\tau_j = -s_j = -i\sigma_j$, $(j = 1,2,3)$, with $\sigma_{1,2,3}$ the Pauli matrices. In this basis, $i = 0$ and $i = 1,2,3$ describe the spin singlet and the spin triplet, respectively. An explicit calculation reveals that $r = 0,3$ corresponds to the particle-hole channel (i.e., products $\bar{\psi}\psi$), while $r = 1,2$ describes the particle-particle channel (i.e., products $\bar{\psi}\bar{\psi}$ or $\psi\psi$).

We then decouple the particle-particle spin-singlet interaction by means of a standard Hubbard-Stratonovich transformation. Denoting the Hubbard-Stratonovich field by Ψ, the partition function becomes

$$Z = \int D[Q,\widetilde{\Lambda},\Psi]\; e^{\widetilde{\mathcal{A}}[Q,\widetilde{\Lambda},\Psi]} \quad , \tag{4.6a}$$

where the action

$$\begin{aligned}
\widetilde{\mathcal{A}}[Q,\widetilde{\Lambda},\Psi] &= \mathcal{A}_{\mathrm{dis}}[Q] + \mathcal{A}_{\mathrm{int}}^{(s)}[Q] + \mathcal{A}_{\mathrm{int}}^{(t)}[Q] \\
&\quad + \frac{1}{2}\,\mathrm{Tr}\,\ln\left(G_0^{-1} - i\widetilde{\Lambda}\right) + \mathrm{Tr}\,\left(\widetilde{\Lambda} Q\right) \\
&\quad - \int d\mathbf{x} \sum_{\alpha}\sum_{n}\sum_{r=1,2} {}^{0}_{r}\Psi^{\alpha}_{n}(\mathbf{x})\, {}^{0}_{r}\Psi^{\alpha}_{n}(\mathbf{x}) \\
&\quad + i\sqrt{2T|\Gamma^{(c)}|}\int d\mathbf{x} \sum_{\alpha}\sum_{n}\sum_{r=1,2} {}^{0}_{r}\Psi^{\alpha}_{n}(\mathbf{x}) \\
&\quad \times \sum_{m} \mathrm{tr}\,\left[(\tau_r \otimes s_0)\; Q^{\alpha\alpha}_{m,-m+n}(\mathbf{x})\right] \quad .
\end{aligned} \tag{4.6b}$$

with Tr denoting a trace over all degrees of freedom, including the continuous real space position, and tr a trace over all discrete degrees of freedom that are not summed over explicitly. $\Gamma^{(c)}$ is the attractive Cooperon interaction amplitude. The first three terms in Eq. (4.6b) have the following forms,

$$\mathcal{A}_{\mathrm{dis}}[Q] = \frac{1}{\pi N_{\mathrm{F}}\tau_{\mathrm{e}}}\int d\mathbf{x}\;\mathrm{tr}\;(Q(\mathbf{x}))^2 \quad , \tag{4.6c}$$

$$\begin{aligned}
\mathcal{A}_{\mathrm{int}}^{(s)} &= \frac{T\Gamma^{(s)}}{2}\int d\mathbf{x}\sum_{r=0,3}(-1)^r \sum_{n_1,n_2,m}\sum_{\alpha} \\
&\quad \times \left[\mathrm{tr}\;\left((\tau_r\otimes s_0)\,Q^{\alpha\alpha}_{n_1,n_1+m}(\mathbf{x})\right)\right] \\
&\quad \times \left[\mathrm{tr}\;\left((\tau_r\otimes s_0)\,Q^{\alpha\alpha}_{n_2+m,n_2}(\mathbf{x})\right)\right] \quad ,
\end{aligned} \tag{4.6d}$$

$$\begin{aligned}
\mathcal{A}_{\mathrm{int}}^{(t)} &= \frac{T\Gamma^{(t)}}{2}\int d\mathbf{x}\sum_{r=0,3}(-1)^r \sum_{n_1,n_2,m}\sum_{\alpha}\sum_{i=1}^{3} \\
&\quad \times \left[\mathrm{tr}\;\left((\tau_r\otimes s_i)\,Q^{\alpha\alpha}_{n_1,n_1+m}(\mathbf{x})\right)\right] \\
&\quad \times \left[\mathrm{tr}\;\left((\tau_r\otimes s_i)\,Q^{\alpha\alpha}_{n_2+m,n_2}(\mathbf{x})\right)\right] \quad ,
\end{aligned} \tag{4.6e}$$

Finally,

$$G_0^{-1} = -\partial_\tau + \nabla^2/2m + \mu \quad , \tag{4.6f}$$

is the inverse bare Green operator.

Physically, the Hubbard-Stratonovich field Ψ can be related to the superconducting, or Cooper pair, order parameter, $\Psi \sim \psi\psi$.

4.2.2 Soft modes

Now we are ready to separate the massive and soft modes. The separation follows the similar procedure in previous papers [9]. As in Eq. (4.3) the matrix elements of Q are bilinear in the fermionic fields, so Q-Q correlation functions describe two-fermion excitations. Symmetry analysis shows that the Q_{nm}-fluctuations in a Fermi liquid are massive or soft, respectively, depending on whether $nm > 0$ or $nm < 0$. Group theory is then used to argue that Q can be generally written as

$$Q = \mathcal{S}\, P\, \mathcal{S}^{-1} \quad , \tag{4.7}$$

where P is block-diagonal in Matsubara frequency space,

$$P = \begin{pmatrix} P^{>} & 0 \\ 0 & P^{<} \end{pmatrix} \quad , \tag{4.8}$$

with $P^{>}$ and $P^{<}$ matrices with elements P_{nm} where $n, m > 0$ and $n, m < 0$, respectively. The matrices $\mathcal{S}$ are elements of $\mathrm{USp}(8Nn, \mathcal{C})/\mathrm{USp}(4Nn, \mathcal{C}) \times \mathrm{USp}(4Nn, \mathcal{C})$, a homogeneous space with N replicas and n Matsubara frequencies in a system. This achieves the desired separation of soft and massive modes. The massive modes are represented by the matrix P, while the soft ones are represented by the transformations $\mathcal{S}$. The same procedure can be applied to $\widetilde{\Lambda}$ as follows

$$\widetilde{\Lambda}(\mathbf{x}) = \mathcal{S}(\mathbf{x})\, \Lambda(\mathbf{x})\, \mathcal{S}^{-1}(\mathbf{x}) \quad . \tag{4.9}$$

Λ can also be shown to be massive.

The next step is to integrate out the massive modes. We expand the massive modes about their saddle-point values,

$$P = \langle P \rangle + \Delta P \quad , \quad \Lambda = \langle \Lambda \rangle + \Delta \Lambda \quad . \tag{4.10}$$

A new matrix field is then introduced as

$$\hat{Q}(\mathbf{x}) = \frac{4}{\pi N_F} \mathcal{S}(\mathbf{x}) \langle P \rangle \mathcal{S}^{-1}(\mathbf{x}) \quad . \tag{4.11a}$$

Since $\hat{Q}$ is subject to the following constraints

$$\hat{Q}^2(\mathbf{x}) \equiv \tau_0 \otimes \tau_0 \quad , \quad \hat{Q}^\dagger = \hat{Q} \quad , \quad \mathrm{tr}\, \hat{Q}(\mathbf{x}) = 0 \quad , \tag{4.11b}$$

it can be written in a block matrix form analogous to that used in Eq. (4.8) as

$$\hat{Q} = \begin{pmatrix} \sqrt{1 - qq^\dagger} & q \\ q^\dagger & -\sqrt{1 - q^\dagger q} \end{pmatrix} \tag{4.11c}$$

with the matrix q having elements q_{nm} whose frequency labels are restricted to $n \geq 0$, $m < 0$. Symmetry analysis with Ward identities ensures that the matrix q are massless or soft, which are diffusive in disordered systems.

Following the procedure used to derive the generalized nonlinear sigma model [53] and also considering the leading corrections to the model, we obtain the effective action

$$
\begin{aligned}
\widetilde{\mathcal{A}}[q,\Psi,\Delta P,\Delta\Lambda] \;&=\; \mathcal{A}_{\mathrm{NL}\sigma\mathrm{M}}[q] + \delta\mathcal{A}[\Delta P,\Delta\Lambda,q] \\
&\quad - \int d\mathbf{x} \sum_{\alpha}\sum_{n}\sum_{r=1,2} {}^{0}_{r}\Psi^{\alpha}_{n}(\mathbf{x})\, {}^{0}_{r}\Psi^{\alpha}_{n}(\mathbf{x}) \\
&\quad + i\sqrt{\pi T|K^{(c)}|} \int d\mathbf{x} \sum_{\alpha}\sum_{n}\sum_{r=1,2} {}^{0}_{r}\Psi^{\alpha}_{n}(\mathbf{x}) \\
&\quad \times \sum_{m} \mathrm{tr}\,(\tau_r \otimes s_0)\, \Big[\hat{Q}^{\alpha\alpha}_{m,-m+n}(\mathbf{x}) \\
&\quad + \frac{4}{\pi N_{\mathrm{F}}}\, \big(\mathcal{S}\Delta P\mathcal{S}^{-1}\big)^{\alpha\alpha}_{m,-m+n}(\mathbf{x})\Big] \quad .
\end{aligned}
\tag{4.12}
$$

Here $K^{(c)} = \pi N_{\mathrm{F}}^{2}\Gamma^{(c)}/8$. $\mathcal{A}_{\mathrm{NL}\sigma\mathrm{M}}$ is the known action of the nonlinear sigma model,

$$
\begin{aligned}
\mathcal{A}_{\mathrm{NL}\sigma\mathrm{M}} \;&=\; \mathcal{A}^{(s)}_{\mathrm{int}}[\pi\, N_{\mathrm{F}}\hat{Q}/4] + \mathcal{A}^{(t)}_{\mathrm{int}}[\pi\, N_{\mathrm{F}}\hat{Q}/4] \\
&\quad + \frac{-1}{2G}\int d\mathbf{x}\; \mathrm{tr}\, \Big(\nabla\hat{Q}(\mathbf{x})\Big)^{2} \\
&\quad + 2H\int d\mathbf{x}\; \mathrm{tr}\, \Big(\Omega\,\hat{Q}(\mathbf{x})\Big) \quad ,
\end{aligned}
\tag{4.13a}
$$

with $\mathcal{A}^{(s)}_{\mathrm{int}}$ from Eq. (4.6d), $\mathcal{A}^{(t)}_{\mathrm{int}}$ from Eq. (4.6e), and Ω a frequency matrix with elements

$$
\Omega_{12} = (\tau_0 \otimes s_0)\; \delta_{12}\, \omega_{n_1} \quad . \tag{4.13b}
$$

The coupling constants G and H are proportional to the inverse conductivity, $G \propto 1/\sigma$, and the specific heat coefficient, $H \propto \gamma \equiv \lim_{T\to 0} C_V/T$, respectively [27, 54].

$\delta\mathcal{A}$ contains the corrections to the nonlinear sigma model that were given in Ref. [9]. We list explicitly the terms that are bilinear in the massive fluctuations ΔP and $\Delta\Lambda$, but do not contain couplings between the massive modes and q which are irrelevant from the view of the renormalization group analysis. The terms of higher order in ΔP are also neglected, which would produce terms of higher order in Ψ. For our purposes it will suffice to keep the terms of order Ψ^2.

$$\begin{aligned} \delta\mathcal{A}^{(2)} &= \mathcal{A}_{\rm dis}[\Delta P] + \int d\mathbf{x}\,{\rm tr}\,\left(\Delta\Lambda(\mathbf{x})\Delta P(\mathbf{x})\right) \\ &+\frac{1}{4}\int d\mathbf{x}d\mathbf{y}\,{\rm tr}\,\left(G_{\rm sp}(\mathbf{x}-\mathbf{y})\,\Delta\Lambda(\mathbf{y})\,G_{\rm sp}(\mathbf{y}-\mathbf{x})\,\Delta\Lambda(\mathbf{x})\right) \end{aligned} \tag{4.14}$$

with $\mathcal{A}_{\rm dis}^{(s)}$ from Eq. (4.6c) and the saddle-point Green function $G_{\rm sp}$

$$\begin{aligned} G_{\rm sp}(\mathbf{k},\omega_n) &= \left[G_0^{-1} - i\langle\Lambda\rangle\right]^{-1} \\ &\approx \left[i\omega_n - \frac{\mathbf{k}^2}{2m} + \mu + \frac{i}{2\tau_{\rm el}}\,{\rm sgn}\,\omega_n\right]^{-1} . \end{aligned} \tag{4.15}$$

Note that for our purposes it suffices to keep only the disorder contribution to the self energy in self-consistent Born approximation, and neglect the Hartree-Fock interaction contribution.

The remaining task is to integrate out ΔP and $\Delta\Lambda$. The matrix $\mathcal{S}$ can be treated as $\mathcal{S} = 1$ when we neglect the coupling between massless modes q and those massive fluctuations ΔP and $\Delta\Lambda$, which would produce another term of $O(\Psi^4)$ irrelevant for the purpose of the current paper [51]. An additional quadratic contribution in terms of the order parameter field Ψ will obtained from Eq. (4.14) and the last term in Eq. (4.12). Combining it with the Ψ^2 term in Eq. (4.12) yields a term

$$\begin{aligned} \mathcal{A}_{\rm G}[\Psi] &= -\sum_{\mathbf{k}}\sum_{\alpha}\sum_{n}\sum_{r=1,2} {}^0_r\Psi^\alpha_n(\mathbf{k}) \\ &\quad\times\left[1 + 2\Gamma_{(c)}\tilde{\chi}(\mathbf{k},\Omega_n)\right]\,{}^0_r\Psi^\alpha_n(-\mathbf{k}) \end{aligned} \tag{4.16a}$$

where

$$\tilde{\chi}(\mathbf{k},\Omega_n) = T\sum_{n_1,n_2}\,\Theta(n_1n_2)\,\delta_{n_1+n_2,n}\,\mathcal{D}_{n_1n_2}(\mathbf{k}) \quad , \tag{4.16b}$$

is given in terms of

$$\mathcal{D}_{nm}(\mathbf{k}) = \varphi_{nm}(\mathbf{k}) \left[1 - \frac{1}{2\pi N_{\mathrm{F}}\tau_{\mathrm{el}}}\,\varphi_{nm}(\mathbf{k})\right]^{-1} \tag{4.16c}$$

with

$$\varphi_{nm}(\mathbf{k}) = \frac{1}{V}\sum_{\mathbf{p}} G_{\mathrm{sp}}(\mathbf{p},\omega_n)\, G_{\mathrm{sp}}(\mathbf{p}+\mathbf{k},\omega_m) \quad . \tag{4.16d}$$

The Theta-function in Eq. (4.16b), which restricts the frequency sum to frequencies that both have the same sign. For small frequencies and wavenumbers, the calculation shows that

$$\mathcal{A}_{\mathrm{G}}[\Psi] = -\sum_{\mathbf{k}}\sum_{\alpha}\sum_{n}\sum_{r=1,2} {}^{0}_{r}\Psi^{\alpha}_{n}(\mathbf{k})\, u_2\, {}^{0}_{r}\Psi^{\alpha}_{n}(-\mathbf{k}) \quad , \tag{4.17a}$$

with

$$u_2 = 1 + O(\mathbf{k}^2, \Omega_n) \quad . \tag{4.17b}$$

Below we will see the wavenumber and frequency corrections indicated in Eq. (4.17b) are irrelevant for the critical behavior. Note that the standard BCS or Cooper logarithmic item actually arises from the coupling term, $\mathcal{A}_{\mathrm{c}}$, given below. The vertex in Eq. (4.16a) is simply a number in the long wavelength, low frequency limit, as is indicated by Eq. (4.17b).

Now we can write an effective local action including only soft modes and the superconducting order-parameter fluctuations. The action has the form of

$$\widetilde{\mathcal{A}}_{\mathrm{eff}}[\Psi, q] = \mathcal{A}_{\mathrm{G}}[\Psi] + \mathcal{A}_{\mathrm{NL}\sigma\mathrm{M}}[q] + \mathcal{A}_{\mathrm{c}}[\Psi, q] \quad . \tag{4.18a}$$

Here the nonlinear sigma model part of the action, $\mathcal{A}_{\mathrm{NL}\sigma\mathrm{M}}$, has been given in Eqs.

(4.13), and $\mathcal{A}_{\mathrm{c}}$ represents the coupling between Ψ and q,

$$\begin{aligned}\mathcal{A}_{\mathrm{c}}[\Psi, q] &= i\sqrt{\pi T|K^{(c)}|}\int d\mathbf{x}\sum_{\alpha}\sum_{n}\sum_{r=1,2}{}^{0}_{r}\Psi^{\alpha}_{n}(\mathbf{x}) \\ &\quad\times\sum_{m}\mathrm{tr}\left[(\tau_r\otimes s_0)\;\hat{Q}^{\alpha\alpha}_{m,-m+n}(\mathbf{x})\right].\end{aligned} \tag{4.18b}$$

For the simplicity, we rewrite the coupling action as

$$\mathcal{A}_{\mathrm{c}}[\Psi, q] = i\sqrt{\pi T|K^{(c)}|}\int d\mathbf{x}\;\mathrm{tr}\;\left(b(\mathbf{x})\,\hat{Q}(\mathbf{x})\right) \quad . \tag{4.19}$$

Here we define a field

$$b_{12}(\mathbf{x}) = \sum_{r=1,2}(\tau_r\otimes s_0)\,{}^{0}_{r}b_{12}(\mathbf{x}) \quad , \tag{4.20a}$$

with components

$${}^{0}_{r}b_{12}(\mathbf{k}) \;=\; \delta_{\alpha_1\alpha_2}\sum_{n}\delta_{n,n_1+n_2}{}^{0}_{r}\Psi^{\alpha_1}_{n}(\mathbf{k}) \quad . \tag{4.20b}$$

Using Eq. (4.11b) in Eq. (4.19), it leads to a series of terms coupling Ψ and q, Ψ and q^2, etc. We thus obtain $\mathcal{A}_{\mathrm{c}}[\Psi, q]$ in form of a series

$$\mathcal{A}_{\mathrm{c}}[\Psi, q] = \mathcal{A}_{\Psi-q} + \mathcal{A}_{\Psi-q^2} + \ldots \tag{4.21a}$$

The first term in this series is obtained by just replacing Q by q in Eq. (4.19),

$$\mathcal{A}_{\Psi-q} = ic_1 T^{1/2}\int d\mathbf{x}\;\mathrm{tr}\;(b(\mathbf{x})\,q(\mathbf{x})) \tag{4.21b}$$

with $c_1 = \sqrt{\pi|K^{(c)}|}$. The next term in this expansion yields

$$\mathcal{A}_{\Psi-q^2} = ic_2\sqrt{T}\int d\mathbf{x}\;\mathrm{tr}\;\left(b(\mathbf{x})\,q(\mathbf{x})\,q^{\dagger}(\mathbf{x})\right) \tag{4.21c}$$

with $c_2 = c_1/16$. Higher order terms in q in this expansion will turn out to be irrelevant for determining the critical behavior at the quantum phase transition.

4.3 Renormalization group analysis

In this section, we explore the effective local action obtained in the above section and examine the critical behavior by renormalization group analysis. We first determine the Gaussian, or second order, action. The moment-shell technique is then employed to find possible corrections to the previous treatment for the quantum metal - superconductor transition.

4.3.1 Gaussian Action

For the purpose of the following renormalization group analysis, we first need to determine the Gaussian or second-order action. It can be obtained from the effective local action $\widetilde{\mathcal{A}}_{\text{eff}}[\Psi, q]$ as follows,

$$\begin{aligned}\mathcal{A}^{(2)}[\Psi, q] &= -\sum_{\mathbf{k}} \sum_{n} \sum_{\alpha} \sum_{r=1,2} {}^{0}_{r}\Psi^{\alpha}_{n}(\mathbf{k})\, u_2(\mathbf{k})\, {}^{0}_{r}\Psi^{\alpha}_{n}(-\mathbf{k}) \\ &\quad -\frac{4}{G} \sum_{\mathbf{k}} \sum_{1,2,3,4} \sum_{i,r} {}^{i}_{r}q_{12}(\mathbf{k})\, {}^{i}_{r}\Gamma^{(2)}_{12,34}(\mathbf{k})\, {}^{i}_{r}q_{34}(-\mathbf{k}) \\ &\quad -8i\sqrt{\pi T |K^{(c)}|} \sum_{\mathbf{k}} \sum_{12} \sum_{r=1,2} {}^{0}_{r}q_{12}(\mathbf{k})\, {}^{0}_{r}b_{12}(-\mathbf{k}),\end{aligned} \tag{4.22a}$$

where the bare two-point q vertexes come from the nonlinear sigma model $\mathcal{A}_{\mathrm{NL}\sigma\mathrm{M}}$ and have the forms of

$$
\begin{aligned}
{}^{0}_{1,2}\Gamma^{(2)}_{12,34}(\mathbf{k}) &= -\delta_{13}\delta_{24}\left(\mathbf{k}^2 + GH\Omega_{n_1-n_2}\right) + \delta_{1+2,3+4} \\
&\quad \times \delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_3}\, 4\pi TG\delta k_c, \qquad (4.22\mathrm{b}) \\
{}^{0}_{0,3}\Gamma^{(2)}_{12,34}(\mathbf{k}) &= \delta_{13}\delta_{24}\left(\mathbf{k}^2 + GH\Omega_{n_1-n_2}\right) + \delta_{1-2,3-4} \\
&\quad \times \delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_3}\, 4\pi TGK_s, \qquad (4.22\mathrm{c}) \\
{}^{1,2,3}_{0,3}\Gamma^{(2)}_{12,34}(\mathbf{k}) &= \delta_{13}\delta_{24}\left(\mathbf{k}^2 + GH\Omega_{n_1-n_2}\right) + \delta_{1-2,3-4} \\
&\quad \times \delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_3}\, 4\pi TGK_t, \qquad (4.22\mathrm{d})
\end{aligned}
$$

with $K_s = -\pi\, N_{\mathrm{F}}^2\, \Gamma^{(s)}/8$ and $K_t = -\pi\, N_{\mathrm{F}}^2\, \Gamma^{(t)}/8$. Note that there is an additional repulsive interaction, δk_c, in Eq. (5.22b), which comes from the one-loop disorder renormalization of the action [49]. We choose to take this effect into account at Gaussian order. Alternately, it would arise as a higher-order disorder effect. For a complete discussion of this term we refer elsewhere [27]. Here we note that it is this term that drives the superconducting transition temperature to zero, and leads to a quantum metal - superconductor phase transition.

As an aside, we note that if the fermionic q fields are integrated out, an effective action containing only the superconducting order parameter can be derived. In the long wavelength and low frequency limit that action is,

$$\begin{aligned}\mathcal{A}^{(2)}[\Psi] &= -\sum_{\mathbf{k}}\sum_{n}\sum_{\alpha}\sum_{r=1,2} {}_r^0\Psi_n^\alpha(\mathbf{k}) \\ &\quad \left(u_2(\mathbf{k}) + \frac{\frac{-|K^{(c)}|}{H}\ln\frac{\Omega_0}{|\Omega_n|+\mathbf{k}^2/GH}}{1+\frac{\delta k_c}{H}\ln\frac{\Omega_0}{|\Omega_n|+\mathbf{k}^2/GH}}\right) {}_r^0\Psi_n^\alpha(-\mathbf{k}) \\ &\simeq -\sum_{\mathbf{k}}\sum_{n}\sum_{\alpha}\sum_{r=1,2} {}_r^0\Psi_n^\alpha(\mathbf{k}) \\ &\quad \left(t + \frac{|K^{(c)}|}{\delta k_c{}^2}\frac{1}{\ln\frac{\Omega_0}{|\Omega_n|+\mathbf{k}^2/GH}}\right) {}_r^0\Psi_n^\alpha(-\mathbf{k}) \quad .\end{aligned} \tag{4.23}$$

Here Ω_0 is a frequency cutoff on the order of the Debye frequency, and $t = u_2 - \frac{|K^{(c)}|}{\delta k_c}$ denotes the distance from the mean field or Gaussian critical point. Note the crucial point, it is the disorder (δk_c) that allows t to change signs and therefore leads to a metal - superconductor quantum phase transition. $\mathcal{A}^{(2)}[\Psi]$ is the Gaussian order parameter field theory that was considered in Ref. [48].

It must be stressed, however, that the whole point at our procedure is to not integrate out the fermionic degrees of freedom. Only then, will the starting action be local in space and time and be amendable to the standard renormalization group treatment.

For the coupled field theory it is straightforward to calculate the two-point correlation functions. For the superconducting order parameter correlations we obtain

$$\langle {}_r^0\Psi_n^\alpha(\mathbf{k})\, {}_s^0\Psi_m^\beta(\mathbf{p})\rangle = \delta_{\mathbf{k},-\mathbf{p}}\,\delta_{n,m}\,\delta_{rs}\,\delta_{\alpha\beta}\,\frac{1}{2}\,\mathcal{M}_n(\mathbf{k}) \quad , \tag{4.24a}$$

with

$$\mathcal{M}_n(\mathbf{k}) = \frac{1}{t + \frac{|K^{(c)}|}{\delta {k_c}^2}\frac{1}{\ln\frac{\Omega_0}{|\Omega_n| + \mathbf{k}^2/GH}}} \quad . \tag{4.24b}$$

Similarly, we find the fermionic propagators

$$\langle {}^i_r q_{12}(\mathbf{k})\, {}^j_s q_{34}(\mathbf{p})\rangle = \delta_{\mathbf{k},-\mathbf{p}}\, \delta_{ij}\, \frac{G}{8}\, {}^i_r\Gamma^{(2)\,-1}_{12,34}(\mathbf{k}) \quad , \tag{4.25a}$$

where

$$\begin{aligned} {}^{i}_{0,3}\Gamma^{(2)\,-1}_{12,34}(\mathbf{k}) &= \delta_{13}\delta_{24}\mathcal{D}_{n_1-n_2}(\mathbf{k}) - \delta_{1-2,3-4}\delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_3} \\ &\quad \times 2\pi TGK^{(i)}\, \mathcal{D}_{n_1-n_2}(\mathbf{k})\, \mathcal{D}^{(\mathrm{i})}_{n_1-n_2}(\mathbf{k}) \ , \end{aligned} \tag{4.25b}$$

and

$$\begin{aligned} {}^{0}_{1,2}\Gamma^{(2)\,-1}_{12,34}(\mathbf{k}) &= -\delta_{13}\delta_{24}\mathcal{D}_{n_1-n_2}(\mathbf{k}) + \delta_{1+2,3+4}\delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_3} \\ &\quad \times \frac{4\pi TGK^{(c)}\, \mathcal{D}_{n_1-n_2}(\mathbf{k})\, \mathcal{D}_{n_3-n_4}(\mathbf{k})}{1 + 4\pi TGK^{(c)} \ln\frac{\Omega_0}{|\Omega_n| + \mathbf{k}^2/GH}} \ . \end{aligned} \tag{4.25c}$$

Here $\mathcal{D}^{(\mathrm{i})}$ is the spin-singlet propagator, which in the limit of long wavelengths and small frequencies reads [27]

$$\mathcal{D}^{(\mathrm{i})}_n(\mathbf{k}) = \frac{1}{\mathbf{k}^2 + G(H + K^{(i)})\Omega_n} \quad . \tag{4.25d}$$

In writing Eq. (4.25c), we have for simplicity put the additional repulsive interaction, δk_c to zero.

4.3.2 Momentum-shell renormalization group analysis

From the discussion above, the following effective local action with all the relevant soft modes can be used to exactly determine the behavior of quantum metal - superconductor transition,

$$\begin{aligned}\widetilde{\mathcal{A}}_{\text{eff}} &= \mathcal{A}^{(2)}[\Psi] + \mathcal{A}_{\Psi-q} + \mathcal{A}_{\Psi-q^2} \\ &- \frac{4}{G}\sum_{\mathbf{k}}\sum_{1,2,3,4}\sum_{i,r} {}^i_r q_{12}(\mathbf{k})\, {}^i_r\Gamma^{(2)}_{12,34}(\mathbf{k})\, {}^i_r q_{34}(-\mathbf{k}). \end{aligned} \tag{4.26}$$

The standard momentum-shell renormalization group (RG) technique [8, 55, 56] on this local field theory is employed here. The parameters t, G, H, c_1, c_2 as well as the fields Ψ and q in the theory defined above will be renormalized. We use b as the RG length rescaling factor, and we rescale the wavenumber and two fields straightforwardly via

$$\mathbf{k} \rightarrow \mathbf{k}'/b \quad , \tag{4.27a}$$

$$\mathbf{\Psi}_n(\mathbf{k}) \rightarrow b^{(2-\eta_\Psi)/2}\mathbf{\Psi}'_n(\mathbf{k}') \quad , \tag{4.27b}$$

$$q_{nm}(\mathbf{k}) \rightarrow b^{(2-\eta_q)/2}q'_{nm}(\mathbf{k}') \quad . \tag{4.27c}$$

The rescaling of imaginary time, frequency, or temperature is less straightforward. In general, there are two different time scales in the problem, namely, one that is associated with the critical order-parameter fluctuations, and one that is associated with the soft fermionic fluctuations. Therefore, we allow for two different dynamical exponents, z_Ψ and z_q. The temperature may then get rescaled via

$$T \rightarrow b^{-z_\Psi}T' \quad , \tag{4.27d}$$

or via

$$T \to b^{-z_q} T' \quad , \tag{4.27e}$$

How these various exponents should be chosen is discussed below.

In the tree, or zero-loop, approximation the RG equations for the parameters in our field theory are determined as

$$\begin{aligned}
t' &= b^{2-\eta_\Psi} t \quad , && (4.28a)\\
\frac{1}{G' H' T'_\Psi} &= \frac{b^{-2}}{G H T_\Psi} \quad , && (4.28b)\\
\frac{1}{G'} &= \frac{b^{-\eta_q}}{G} \quad , && (4.28c)\\
H' T'_q &= b^{2-\eta_q} H T_q \quad , && (4.28d)\\
c'_1 T'^{1/2} &= c_1 T^{1/2} b^{\frac{4-\eta_\Psi-\eta_q}{2}} \quad , && (4.28e)\\
c'_2 T'^{1/2} &= c_2 T^{1/2} b^{\frac{-d+6-\eta_\Psi-2\eta_q}{2}} \quad , && (4.28f)
\end{aligned}$$

Note that in giving Eqs. (4.28e) and (4.28f), the particular choice of T was not yet specified because it is not obvious if a z_q or a z_Ψ should be used for these terms that describe a coupling between q and Ψ fields.

If we assume the Fermi-liquid degrees of freedom to be at a stable Fermi-liquid fixed point, we must choose G and H to be marginal, which implies

$$\begin{aligned}
\eta_q &= 0 \quad , && (4.29a)\\
z_\Psi &= 2 \quad , && (4.29b)\\
z_q &= 2 \quad . && (4.29c)
\end{aligned}$$

Here we find that two dynamical exponents, z_Ψ and z_q, have the same value, which is different from the ferromagnetic systems [51]. We further choose

$$\eta_{\Psi} = 2 \quad , \tag{4.29d}$$

which is implied by the logarithmic structure of Eq. (4.24b). With these choices, we find that

$$c_2' = b^{\frac{-d+2}{2}} c_2 \quad , \tag{4.29e}$$

As in the ferromagnetic case, there is a critical fixed point where c_1 is marginal, and the fermions are diffusive, with exponents given by Eqs. (4.29). However, in contrast to the magnetic case, the coupling constant c_2 of the term $\mathcal{A}_{\Psi-q^2}$ is RG irrelevant for all $d > 2$, and so are all higher order terms in the expansion in powers of q. We therefore conclude that the Gaussian critical behavior is exact [48]. No additional logarithmic corrections exist here. The most important technical difference that leads to the irrelevance of c_2 for this quantum phase transition, while for the quantum ferromagnetic transition it was marginal, is that the time scales for the order-parameter fluctuations and the fermions, respectively, are the same [57]. This renders inoperative the mechanism that led the possibility of c_2 being marginal as in the ferromagnetic case. Physically, the very long range interaction between the order-parameter fluctuations stabilizes the Gaussian critical behavior. This is in agreement with the fact that long-ranged order parameter correlations in classical systems stabilize mean-field critical behavior [58].

As noted above, Eq. (4.29e) implies that the Gaussian theory gave the exact critical behavior. For completeness, the critical exponents, including logarithmic terms, are

$$\eta_\Psi = 2 - \frac{\ln\ln b^2}{\ln b} \quad , \tag{4.30a}$$

$$\nu = \frac{\ln b}{\ln\ln b^2} \quad , \tag{4.30b}$$

$$\gamma = 1 \quad . \tag{4.30c}$$

Formally, when $b \to \infty$ we have $\eta_\Psi = 2$ and $\nu = \infty$. Physically, for example, Eq. (4.30b) implies the wavelength length behaves as

$$\xi \simeq \xi_0 e^{1/2t} \tag{4.31}$$

with ξ_0 the microscopic coherence length.

4.4 Conclusion

We have investigated the quantum metal - superconductor phase transition in the present paper on the basis of an effective local field theory [59]. With a simple renormalization group analysis, we have determined the critical behavior at the quantum metal - superconductor phase transition. In contrast to the disordered ferromagnetic case studied earlier, we showed that the previous results obtained with a nonlocal field theory are correct. The reason is that the two dynamical exponents, z_Ψ and z_q, are exactly the same for the disordered metal - superconductor quantum phase transition. This point is further discussed in Refs. [60].

Chapter 5

Pseudogap Effect on d-wave Superconducting Quantum Phase Transition

In this chapter we study the disorder-induced quantum phase transition from metal to d-wave superconductor phase transition. An effective local field theory is developed that keeps all soft modes or fluctuations explicitly. Renormalization group analysis is then used to study the quantum critical behavior at this transition. We reach the surprising conclusion that the quantum phase transition is a strong coupling (or infinite disorder fixed point) transition independent of the hole doping, that is, pseudogap effects in the normal state do not seem to have any effect.

5.1 Introduction

In the last chapter an effective local field theory for the metal to s-wave superconductor transition was developed by keeping all the relevant soft modes. The exact behavior at this quantum phase transition was determined. We found that the coupling to non-order-parameter soft fluctuations was so strong that once these fluctuations were taken into account all others could be (exactly) ignored. The net result was that a Gaussian field theory exactly described the quantum critical behavior.

The situation in d-wave superconductors, however, may be different from that of conventional superconductors. Indeed, we show that the d-wave symmetry of the superconducting state makes the coupling between the order parameter fluctuations and additional soft modes weaker than in the s-wave case. Because of this, the extra soft modes have a much weaker effect on the metal - superconductor transition. The net result of this weaker coupling is that higher order fluctuation effects are not suppressed. We conclude that for the d-wave case, the metal - superconductor transition is likely described by an infinite disorder fixed point, similar to the case of disordered quantum antiferromagnets.

In this chapter we study the quantum d-wave superconducting phase transition with an effective local field theory. We divide our study into two parts: the case of overdoped region where the normal state can be treated as a normal Fermi liquid and the case of underdoped region where the normal state is believed as a pseudogap state.

5.2 Effective Local Field Theory

A local field theory will be developed in this section to describe the metal to d-wave superconductor quantum phase transition at $T = 0$. All relevant soft modes will be contained in this field theory. We start from a general model of interacting electrons with quenched disorder and attractive d-wave symmetry Cooperon interaction amplitude. We then introduce the d-wave superconducting order parameter and separate massive and soft modes. After integrating out the massive modes, we obtain an effective local field theory that describes the coupling between the superconducting fluctuations and the soft or massless diffusive modes.

5.2.1 Composite field theory

Similar to the last chapter, the general partition function of the interacting, disordered electrons can be given in the form of Grassmann fields $\bar{\psi}$ and ψ [23]

$$\widetilde{Z} = \int D[\bar{\psi}, \psi] \; e^{\tilde{S}[\bar{\psi},\psi]} \quad . \tag{5.1a}$$

with the action $\tilde{S}$ being

$$\begin{aligned} \tilde{S} &= -\int_0^\beta d\tau \int d\mathbf{x} \sum_\sigma \; \bar{\psi}_\sigma^\alpha(\mathbf{x},\tau) \frac{\partial}{\partial\tau} \psi_\sigma^\alpha(\mathbf{x},\tau) \\ &\quad -\int_0^\beta d\tau \; H(\tau) \; . \end{aligned} \tag{5.1b}$$

The Hamiltonian H includes three parts:

$$H = H_0 + H_{\text{dis}} + H_{\text{int}} \quad . \tag{5.2}$$

H_{int} includes the d-wave particle-particle spin-singlet channel (or the Cooper channel). The part of d-wave Cooper channel can be transformed into the action $\tilde{S}$ and has the form of

$$\begin{aligned} \tilde{S}_{\text{int}}^{(d)}[\bar{\psi},\psi] &= -\int_0^\beta d\tau \; H_{\text{int}}^{(d)}(\tau) \\ &= T \sum_{\omega,\nu,\nu'} \sum_{\vec{k},\vec{k}',\vec{p},\alpha,\sigma} \frac{V_{\vec{k},\vec{k}'}}{2} \bar{\psi}_\sigma^\alpha(\vec{k}+\vec{p},\nu+\omega) \\ &\quad \times \bar{\psi}_{-\sigma}^\alpha(-\vec{k},-\nu)\psi_{-\sigma}^\alpha(-\vec{k}',-\nu')\psi_\sigma^\alpha(\vec{k}'+\vec{p},\nu'+\omega), \end{aligned} \tag{5.3}$$

with index α denoting replicas and σ spin. Because of the d-wave symmetry we assume $V_{\vec{k},\vec{k}'} = V_0 \cos 2\theta_{\vec{k}} \cos 2\theta_{\vec{k}'}$, with $V_0 > 0$ (attraction), and $\theta_{\vec{k}}$, $\theta_{\vec{k}'}$ the angle of the two momenta.

We can exactly rewrite the partition as function as

$$
\begin{aligned}
\widetilde{Z} &= \int D[\bar{\psi},\psi]\; e^{\tilde{S}[\bar{\psi},\psi]} \int D[Q]\,\delta[Q-B] \\
&= \int D[\bar{\psi},\psi]\; e^{\tilde{S}[\bar{\psi},\psi]} \int D[Q]\,D[\widetilde{\Lambda}]\; e^{\mathrm{Tr}\,[\widetilde{\Lambda}(Q-B)]} \\
&\equiv \int D[\bar{\psi},\psi]\,D[Q]\,D[\widetilde{\Lambda}]\; e^{\widetilde{\mathcal{A}}[Q,\widetilde{\Lambda},\bar{\psi},\psi]} \quad . \qquad (5.4a)
\end{aligned}
$$

Here

$$
\begin{aligned}
\widetilde{\mathcal{A}}[Q,\widetilde{\Lambda},\bar{\psi},\psi] &= \mathcal{A}_{\mathrm{dis}}[Q] + \mathcal{A}_{\mathrm{int}}^{(s)}[Q] + \mathcal{A}_{\mathrm{int}}^{(t)}[Q] \\
&\quad + \tilde{S}_0[\bar{\psi},\psi] + \tilde{S}_{\mathrm{int}}^{(d)}[\bar{\psi},\psi] + \mathrm{Tr}\,[\widetilde{\Lambda}(Q-B)] \qquad (5.4b)
\end{aligned}
$$

with

$$
\tilde{S}_0 = \int_0^\beta d\tau \int d\mathbf{x} \sum_{\alpha\sigma} \bar{\psi}_\sigma^\alpha(\mathbf{x},\tau)\,(-\frac{\partial}{\partial\tau} + \frac{\nabla^2}{2m} + \mu)\,\psi_\sigma^\alpha(\mathbf{x},\tau), \qquad (5.4c)
$$

and

$$
\mathcal{A}_{\mathrm{dis}}[Q] = \frac{1}{\pi N_{\mathrm{F}}\tau_{\mathrm{e}}} \int d\mathbf{x}\ \mathrm{tr}\ (Q(\mathbf{x}))^2 \quad , \qquad (5.4d)
$$

$$
\begin{aligned}
\mathcal{A}_{\mathrm{int}}^{(s)} &= \frac{T\Gamma^{(s)}}{2} \int d\mathbf{x} \sum_{r=0,3} (-1)^r \sum_{n_1,n_2,m} \sum_{\alpha} \\
&\quad \times \left[\mathrm{tr}\ \left((\tau_r \otimes s_0)\, Q_{n_1,n_1+m}^{\alpha\alpha}(\mathbf{x})\right)\right] \\
&\quad \times \left[\mathrm{tr}\ \left((\tau_r \otimes s_0)\, Q_{n_2+m,n_2}^{\alpha\alpha}(\mathbf{x})\right)\right] \quad , \qquad (5.4e)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_{\mathrm{int}}^{(t)} &= \frac{T\Gamma^{(t)}}{2} \int d\mathbf{x} \sum_{r=0,3} (-1)^r \sum_{n_1,n_2,m} \sum_{\alpha} \sum_{i=1}^{3} \\
&\quad \times \left[\mathrm{tr}\ \left((\tau_r \otimes s_i)\, Q_{n_1,n_1+m}^{\alpha\alpha}(\mathbf{x})\right)\right] \\
&\quad \times \left[\mathrm{tr}\ \left((\tau_r \otimes s_i)\, Q_{n_2+m,n_2}^{\alpha\alpha}(\mathbf{x})\right)\right] \quad , \qquad (5.4f)
\end{aligned}
$$

We then decouple the d-wave particle-particle spin-singlet interaction by means of a standard Hubbard-Stratonovich transformation. Denoting the Hubbard-Stratonovich field by $\Delta^{\alpha}_{n\sigma}(\vec{p})$, we obtain

$$\begin{aligned} e^{\tilde{S}^{(d)}_{\rm int}} &= \int D[\bar{\Delta},\Delta]\exp\Bigg(-\sum_{n,\vec{p},\sigma,\alpha}\bar{\Delta}^{\alpha}_{n\sigma}(\vec{p})\Delta^{\alpha}_{n\sigma}(-\vec{p}) \\ &\quad -\sqrt{\frac{TV_0}{2}}\sum_{n,\vec{p},\sigma,\alpha}\bar{\Delta}^{\alpha}_{n\sigma}(\vec{p}) \\ &\qquad \times\sum_{m,\vec{k}}\psi^{\alpha}_{-\sigma}(-\vec{k},-m)\psi^{\alpha}_{\sigma}(\vec{k}+\vec{p},m+n)\cos 2\theta_{\vec{k}} \\ &\quad -\sqrt{\frac{TV_0}{2}}\sum_{n,\vec{p},\sigma,\alpha}\Delta^{\alpha}_{n\sigma}(\vec{p}) \\ &\qquad \times\sum_{m,\vec{k}}\bar{\psi}^{\alpha}_{\sigma}(\vec{k}+\vec{p},m+n)\bar{\psi}^{\alpha}_{-\sigma}(-\vec{k},-m)\cos 2\theta_{\vec{k}}\Bigg) \end{aligned} \tag{5.5}$$

The partition function becomes

$$\widetilde{Z} = \int D[Q,\widetilde{\Lambda},\Psi]\; e^{\widetilde{\mathcal{A}}[Q,\widetilde{\Lambda},\Psi]} \quad , \tag{5.6a}$$

where the action

$$\begin{aligned} \widetilde{\mathcal{A}}[Q,\widetilde{\Lambda},\Psi] &= \mathcal{A}_{\rm dis}[Q]+\mathcal{A}^{(s)}_{\rm int}[Q]+\mathcal{A}^{(t)}_{\rm int}[Q] \\ &\quad +{\rm Tr}\left(\widetilde{\Lambda}Q\right)-\sum_{n,\vec{p},\sigma,\alpha}\bar{\Delta}^{\alpha}_{n\sigma}(\vec{p})\Delta^{\alpha}_{n\sigma}(-\vec{p}) \\ &\quad +\frac{1}{2}\,{\rm Tr}\,\ln\left(G_0^{-1}-i\widetilde{\Lambda}-iM\right). \end{aligned} \tag{5.6b}$$

with

$$G_0^{-1} = -\partial_\tau + \nabla^2/2m + \mu \quad , \tag{5.6c}$$

being the inverse bare Green operator, and

$$M = \frac{\sqrt{2TV_0}}{k_F^2}\delta_{n1+n2,n}\Delta^{\alpha}_{n}(\partial_x^2-\partial_y^2)(\tau_2\otimes s_0) \tag{5.6d}$$

Here Tr denotes a trace over all degrees of freedom, including the continuous real space position, and tr a trace over all discrete degrees of freedom that are not summed over explicitly.

Physically, the Hubbard-Stratonovich field Δ can be related to the superconducting, or Cooperon, order parameter.

5.2.2 Soft modes

Now we are ready to separate the massive and soft modes. The separation will need to take advantage of two different procedures in previous papers [9, 27]. First, it was argued that Q can be generally written as

$$Q = \mathcal{S}\, P\, \mathcal{S}^{-1} \quad , \tag{5.7}$$

and $\widetilde{\Lambda}$ as

$$\widetilde{\Lambda}(\mathbf{x}) = \mathcal{S}(\mathbf{x})\, \Lambda(\mathbf{x})\, \mathcal{S}^{-1}(\mathbf{x}) \quad . \tag{5.8}$$

Λ can also be shown to be massive.

The next step is to integrate out the massive modes. We expand the massive modes about their saddle-point values,

$$P = \langle P \rangle + \Delta P \quad , \quad \Lambda = \langle \Lambda \rangle + \Delta\Lambda \quad . \tag{5.9a}$$

with

$$\langle \Lambda \rangle = \frac{-2}{\pi N_F \tau_{\mathrm{e}}} \langle P \rangle \tag{5.9b}$$

A matrix field is introduced as

$$\begin{aligned}\hat{Q}(\mathbf{x}) &= \frac{4}{\pi N_F}\mathcal{S}(\mathbf{x})\langle P\rangle\mathcal{S}^{-1}(\mathbf{x}) \\ &= \begin{pmatrix} \sqrt{1-qq^\dagger} & q \\ q^\dagger & -\sqrt{1-q^\dagger q}\end{pmatrix} \quad .\end{aligned} \tag{5.10}$$

with the matrix q having elements q_{nm} whose frequency labels are restricted to $n \geq 0$, $m < 0$. Symmetry analysis with Ward identities ensures that the matrices q are massless or soft, which are diffusive in disordered systems.

Note that the corrections from ΔP and $\Delta\Lambda$ are irrelevant in superconducting case, we first have

$$\widetilde{\Lambda}(\mathbf{x}) = \mathcal{S}(\mathbf{x})\langle\Lambda\rangle(\mathbf{x})\mathcal{S}^{-1}(\mathbf{x}) = \frac{-1}{2\tau_{\rm e}}\hat{Q} \quad . \tag{5.11}$$

By defining

$$\hat{Q} = \hat{Q}_{\rm sp} + \tilde{Q} + 2\tau_{\rm e}\Omega \tag{5.12}$$

with

$$\Omega_{12} = (\tau_0 \otimes s_0)\delta_{12}\omega, \tag{5.13}$$

we then obtain

$$\begin{aligned}\frac{1}{2}\text{Tr}\ln\left(G_0^{-1} - i\widetilde{\Lambda} - iM\right) &= \frac{1}{2}\text{Tr}\ln(G_{\rm sp}^{-1}) \\ &-\frac{1}{4\tau_{\rm e}}\text{Tr}\,[M(G_{\rm sp}^{-1})\tilde{Q}(G_{\rm sp}^{-1})] \\ &+\frac{1}{4\tau_{\rm e}}\text{Tr}\,(\Omega(G_{\rm sp}^{-1})\tilde{Q}(G_{\rm sp}^{-1})) \\ &+\frac{1}{16\tau_{\rm e}^2}\text{Tr}\,(\tilde{Q}(G_{\rm sp}^{-1})\tilde{Q}(G_{\rm sp}^{-1})) \\ &+\frac{1}{4}\text{Tr}\,(M(G_{\rm sp}^{-1})M(G_{\rm sp}^{-1})) \\ &+\frac{i}{8\tau_{\rm e}^2}\text{Tr}\,[M(G_{\rm sp}^{-1})\tilde{Q}(G_{\rm sp}^{-1})\tilde{Q}(G_{\rm sp}^{-1})] + ...\end{aligned} \tag{5.14a}$$

and

$$\begin{aligned}\mathcal{A}_{\rm dis}[Q] + {\rm Tr}\left(\widetilde{\Lambda}Q\right) &= \frac{\pi N_F}{16\tau_{\rm e}}{\rm Tr}\,(\tilde{Q}\tilde{Q}) + \frac{-\pi N_F}{8\tau_{\rm e}}{\rm Tr}\,(\tilde{Q}\tilde{Q}) + ... \\ &= \frac{-\pi N_F}{16\tau_{\rm e}}{\rm Tr}\,(\tilde{Q}\tilde{Q}) + ... \end{aligned} \tag{5.14b}$$

Combining the fourth item of Eq. (5.14a) and Eq. (5.14b) we now have

$$\begin{aligned}\frac{1}{16\tau_{\rm e}^2}{\rm Tr}\,(\tilde{Q}(G_{\rm sp}^{-1})\tilde{Q}(G_{\rm sp}^{-1})) + \frac{-\pi N_F}{16\tau_{\rm e}}{\rm Tr}\,(\tilde{Q}\tilde{Q}) &= \\ & \frac{-\pi N_F}{16\tau_{\rm e}}\sum_{\vec{p}} {\rm tr}\,(\tilde{Q}(\vec{p})\tilde{Q})(-\vec{p}) \\ & \times(1 - \frac{1}{\pi N_F\tau_{\rm e}}\sum_{\vec{k}} G_{\rm sp}(\vec{k})G_{\rm sp})(\vec{k}+\vec{p}) \\ &= \frac{-\pi N_F}{16\tau_{\rm e}}\sum_{\vec{p}} {\rm tr}\,(\tilde{Q}(\vec{p})\tilde{Q})(-\vec{p})\tau_{\rm e}Dk^2 \\ &= \frac{-\pi N_F D}{16}{\rm Tr}\,(\vec{\nabla}\tilde{Q})^2 \end{aligned} \tag{5.14c}$$

Finally, with the help of Eqs. (2.5) and Eqs. (5.14), we obtain the following effective local action

$$\widetilde{\mathcal{A}}[q,\Psi,\Delta P,\Delta\Lambda] = \mathcal{A}_{\rm NL\sigma M}[\tilde{Q}] + \mathcal{A}_{\rm G}[\Delta] + \mathcal{A}_{\rm c}[\Delta,\tilde{Q}] \tag{5.15}$$

Here $\mathcal{A}_{\rm NL\sigma M}$ is the known action of the nonlinear sigma model

$$\begin{aligned}\mathcal{A}_{\rm NL\sigma M}[\tilde{Q}] &= \mathcal{A}^{(s)}_{\rm int}[\frac{\pi N_F}{4}\tilde{Q}] + \mathcal{A}^{(t)}_{\rm int}[\frac{\pi N_F}{4}\tilde{Q}] \\ &+ \frac{-1}{2G}\int d\mathbf{x}\ {\rm tr}\left(\nabla\tilde{Q}(\mathbf{x})\right)^2 + 2H\int d\mathbf{x}\ {\rm tr}\left(\Omega\,\tilde{Q}(\mathbf{x})\right)\end{aligned} \tag{5.16}$$

with $G = \frac{8}{\pi N_F D}$ and $H = \frac{\pi N_F}{8}$,

$$\mathcal{A}_{\rm G}[\Delta] = -\sum_{n,\vec{p},\sigma,\alpha}\bar{\Delta}^{\alpha}_{n\sigma}(\vec{p})\Delta^{\alpha}_{n\sigma}(-\vec{p}) + \frac{1}{4}{\rm Tr}\,(M(G_{\rm sp}^{-1})M(G_{\rm sp}^{-1})) + \mathcal{A}^{(4)}_{\rm G}[\Delta] \tag{5.17}$$

with $\mathcal{A}_{\mathrm{G}}^{(4)}$ being the normal term of order Δ^4 including the off-diagonal element, and

$$\begin{aligned}\mathcal{A}_{\mathrm{c}}[\Delta, \tilde{Q}] &= -\frac{1}{4\tau_{\mathrm{e}}}\mathrm{Tr}\,(M(G_{\mathrm{sp}}^{-1})\tilde{Q}(G_{\mathrm{sp}}^{-1})) \\ &\quad + \frac{i}{8\tau_{\mathrm{e}}^2}\mathrm{Tr}\,(M(G_{\mathrm{sp}}^{-1})\tilde{Q}(G_{\mathrm{sp}}^{-1})\tilde{Q}(G_{\mathrm{sp}}^{-1}))\end{aligned} \tag{5.18}$$

The saddle-point Green function in the overdoped region is given by G_{sp}

$$\begin{aligned}G_{\mathrm{sp}}(\mathbf{k}, \omega_n) &= \left[G_0^{-1} - i\langle\Lambda\rangle\right]^{-1} \\ &\approx \left[i\omega_n - \frac{\mathbf{k}^2}{2m} + \mu + \frac{i}{2\tau_{\mathrm{el}}}\,\mathrm{sgn}\,\omega_n\right]^{-1}.\end{aligned} \tag{5.19}$$

Note that for our purposes it suffices to keep only the disorder contribution to the self energy in self-consistent Born approximation, and neglect the Hartree-Fock interaction contribution.

5.2.3 d-wave symmetry

The d-wave symmetry automatically affects the form of the local field action. For the action, $\mathcal{A}_{\mathrm{G}}[\Delta]$, we have

$$\begin{aligned}\mathcal{A}_{\mathrm{G}}[\Delta] &= -\sum_{n,\vec{p},\sigma,\alpha} \bar{\Delta}_{n\sigma}^{\alpha}(\vec{p}) \Big(1 - \frac{V_0 N_F}{2}\ln(2\epsilon_F \tau_{\mathrm{e}}) \\ &\quad + a_1 p^2 + a_2\omega + O(\omega^2, q^4)\Big)\, \Delta_{n\sigma}^{\alpha}(-\vec{p}) \quad .\end{aligned} \tag{5.20}$$

The standard ultraviolet divergence in the second item of Eq. (5.17) is regularized here by employing a high-energy cutoff $\Omega \approx \epsilon_F \gg 1/\tau_{\mathrm{e}}$.

The main effect from d-wave symmetry appears in the part of the action,

$\mathcal{A}_{\rm c}[\Delta, \tilde{Q}]$. We have

$$\begin{aligned}\mathcal{A}_{\rm c}[\Delta, \tilde{Q}] &= c1\sqrt{T} \sum_{n1,n2,\vec{p},\alpha} {}^0_2q^{\alpha}_{n1n2}(\vec{p})\, p^2 \sum_{n} \delta_{n1+n2,n}\Delta^{\alpha}_{n}(\vec{p}) \\ &+c2\sqrt{T} \sum_{n1,n2,m,\vec{p},\vec{k},\alpha} {}^0_2q^{\alpha}_{n1m}(-\vec{p}){}^0_0q^{\alpha}_{n2m}(-\vec{p}-\vec{k}) \\ &\times p^2 \sum_{n} \delta_{n1+n2,n}\Delta^{\alpha}_{n}(\vec{p}) \end{aligned} \tag{5.21}$$

with $c1$, $c2$ two nonzero constants. Here Eq. (4.11c) has been used. The critical point is the extra (compared to the s-wave case) factor of p^2 on the right side of Eq. (5.21), which is due to integrations over the angles of the momenta. It greatly weakens the effects of the extra soft modes on the quantum phase transition.

To make our point clearer, we need the q form of the the nonlinear sigma model $\mathcal{A}_{\rm NL\sigma M}$

$$\mathcal{A}_{\rm NL\sigma M}[q] = -\frac{4}{G}\sum_{\mathbf{k}}\sum_{1,2,3,4}\sum_{i,r} {}^i_rq_{12}(\mathbf{k})\,{}^i_r\Gamma^{(2)}_{12,34}(\mathbf{k})\,{}^i_rq_{34}(-\mathbf{k}) \tag{5.22a}$$

with

$$\begin{aligned} {}^0_{1,2}\Gamma^{(2)}_{12,34}(\mathbf{k}) &= -\delta_{13}\delta_{24}\left(\mathbf{k}^2 + GH\Omega_{n_1-n_2}\right) + \delta_{1+2,3+4} \\ &\quad\times\delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_3}\, 4\pi TG\delta k_c, \qquad (5.22\text{b}) \\ {}^0_{0,3}\Gamma^{(2)}_{12,34}(\mathbf{k}) &= \delta_{13}\delta_{24}\left(\mathbf{k}^2 + GH\Omega_{n_1-n_2}\right) + \delta_{1-2,3-4} \\ &\quad\times\delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_3}\, 4\pi TGK_s, \qquad (5.22\text{c}) \\ {}^{1,2,3}_{0,3}\Gamma^{(2)}_{12,34}(\mathbf{k}) &= \delta_{13}\delta_{24}\left(\mathbf{k}^2 + GH\Omega_{n_1-n_2}\right) + \delta_{1-2,3-4} \\ &\quad\times\delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_3}\, 4\pi TGK_t, \qquad (5.22\text{d}) \end{aligned}$$

with $K_s = -\pi\, N_{\rm F}^2\,\Gamma^{(s)}/8$ and $K_t = -\pi\, N_{\rm F}^2\,\Gamma^{(t)}/8$. Note that there is an additional repulsive interaction, $\delta k_c = \delta k_c^0 + \delta k_c^1\mathbf{k}^{d-2}$ with d the dimension, in Eq. (5.22b), which comes from the one-loop renormalization of the action [49]. We choose to take this effect into account at Gaussian order. Alternately, it would arise as a higher order disorder effect. For a complete discussion of this term we refer elsewhere [27]. Here we note that it is this term that drives the superconducting transition temperature to zero, and leads to a quantum metal - superconductor phase transition.

Now if the fermionic q fields are integrated out, an effective action containing only the superconducting order parameter can be derived. In the long wavelength and low frequency limit,

$$\begin{aligned}\mathcal{A}^{(2)}[\Delta] &= -\sum_{n,\vec{p},\sigma,\alpha} \bar{\Delta}^{\alpha}_{n\sigma}(\vec{p}) \\ &\qquad \left(t + a_1 p^2 + a_2\omega + O(\omega^2, p^4 \ln(p))\right) \Delta^{\alpha}_{n\sigma}(-\vec{p}) \quad , \end{aligned} \tag{5.23}$$

with $t = 1 - \frac{V_0 N_F}{2} \ln(2\epsilon_F \tau_e)$.

The procedure in this chapter is similar to that in the s-wave case. The effective local action we obtained under the d-wave symmetry, given by Eq. (5.15), has the same structure as in s-wave case which was given by Eq. (4.18a). The difference is that the Cooperon potential is now in the d-wave symmetry, given by Eq. (5.3). It will greatly change the effective action, as shown in Eq. (5.21). Therefore it will greatly affect the behavior of the system. Our above result shows that the s-wave logarithmic singularity at Eq. (4.23) has been demoted to irrelevant term $p^4 \ln(p)$ due to the d-wave symmetry.

5.3 Renormalization group analysis in the overdoped region

Since there are no relevant logarithmic Cooper channel singularities as in s-wave superconductors or q^{2-d} term as in ferromagnets here, the critical behavior will be similar to that in the disordered iterant anti-ferromagnets. In particular, perturbation around the Gaussian fixed points shows that there are RG relevant nonlinear terms. If one examines these nonlinearities with an ϵ-expansion then one finds that no perturbative fixed points exist. That means there is no critical fixed point in the perturbative renormalization analysis. The result is consistent with the previous work [61].

5.4 Properties in the underdoped region

The difference in the underdoped region is the anomalous normal state, in which pseudogap phenomena have prevented a simple Fermi-liquid description. We here adopted the idea of pre-formed Cooper pairs. Then the normal state can be described by a Fermi liquid with strong d-wave superconducting fluctuations. From ref. [21], we assume the self-energy

$$\Sigma^R(\mathbf{k},\omega) = \frac{\Delta^2\phi_{\mathbf{k}}^2}{\omega+\epsilon(\mathbf{k})+i\delta} \tag{5.24}$$

with

$$\phi_{\mathbf{k}} = \cos 2\theta_{\mathbf{k}}. \tag{5.25}$$

Then the Greens function has the form,

$$G^R(\mathbf{k},\omega) = \frac{\omega+\epsilon(\mathbf{k})}{(\omega+\epsilon(\mathbf{k}))(\omega-\epsilon(\mathbf{k}))-\Delta^2\phi_{\mathbf{k}}^2} \tag{5.26}$$

Following the same procedure as in the overdoped region by using the above expressions, our calculation shows the same result as in the overdoped region. That means that the quantum critical behavior is independent on the pseudogap phenomena.

5.5 Conclusion

We have investigated the quantum d-wave superconducting phase transition on the basis of an effective local field theory in this chapter. With a simple renormalization group analysis, we have determined that the critical behavior at the quantum d-wave metal - superconductor phase transition is similar to the case in the antiferromagnets and appears to be the same in both the underdoped and overdoped regions. In both cases the quantum critical points are related to an infinite disorder fixed point. Further investigation is still needed.

Chapter 6

Conclusion

In this thesis we have given a systematic, functional field theory approach to describe both clean and disordered s-wave and d-wave superconductors and the quantum phase transitions from metal to superconducting states. In chapters two and three the theory was developed and used to compute the equation of state as well as the number density susceptibility, spin density susceptibility, the sound attenuation coefficient, and the electrical conductivity in both clean and disordered s-wave superconductors. In the appropriate limits, we recover all of the known previous results, but now within a single formalism.

In chapter four we considered the disorder-induced metal - superconductor quantum phase transition in s-wave superconductors. The key physical idea here is that in addition to the superconducting order parameter fluctuations, there are also soft fermionic fluctuations that are important at this transition. In a previous theory for this quantum phase transition these additional soft modes were integrated out so that the resulting order parameter field theory was nonlocal. We instead demanded a local field theory that involved a coupled field theory describing both superconducting and soft fermionic fluctuations. Using simple renormalization group and scaling ideas, we exactly determined the critical behavior at this quantum phase transition. Our theory justifies the previous approach.

In chapter five we studied the analogous quantum phase transition in disordered d-wave superconductors. This work should be relevant for high T_c materials. Surprisingly, we showed that in both the underdoped and overdoped regions, the coupling of superconducting fluctuations to the soft disordered fermionic fluctuations is much weaker than that in the s-wave case. The net result is that the disordered quantum phase transition in this case is a strong coupling, or described by an infinite disordered fixed point, transition and cannot be described by the perturbative RG description that works so well in the s-wave case. In fact, this quantum phase transition appears to be related to the one that occurs in $O(2)$ disordered quantum antiferromagnets [61].

Appendix A

Correlation functions in terms of Q matrices

The real number density susceptibility has the following form[2]

$$X^R(\mathbf{x}_1 t_1, \mathbf{x}_2 t_2) = -i\theta(t_1 - t_2)\langle[\tilde{n}(\mathbf{x}_1 t_1), \tilde{n}(\mathbf{x}_2 t_2)]\rangle \tag{A.1}$$

where

$$\tilde{n} = n - \langle n \rangle \tag{A.2}$$

with n the number density operator. It is inconvenient to calculate it directly. Instead, we introduce a corresponding temperature function that depends on the imaginary–time variables

$$\chi_n(\mathbf{x}_1 \tau_1, \mathbf{x}_2 \tau_2) = -\langle T_\tau[\tilde{n}(\mathbf{x}_1 \tau_1)\tilde{n}(\mathbf{x}_2 \tau_2)]\rangle \tag{A.3}$$

where we have the following relation between Eqs. (A.1) and (A.3) with the Lehmann representation

$$X^R(\mathbf{k}, \omega) = \chi_n(\mathbf{k}, i\omega_n \to \omega + i0). \tag{A.4}$$

The time–order indication T_τ of Eq. (A.3) will disappear in the functional integral form,[23] which is the case in the present paper.

Next we notice that

$$ {}^0_0Q_{n_1 n_2} \cong \frac{i}{8}\sum_\sigma \left(\bar{\psi}_{n_1,\sigma}\psi_{n_2,\sigma} + \bar{\psi}_{n_2,\sigma}\psi_{n_1,\sigma}\right), \tag{A.5a}$$

$$ {}^0_3Q_{n_1 n_2} \cong \frac{1}{8}\sum_\sigma \left(\bar{\psi}_{n_1,\sigma}\psi_{n_2,\sigma} - \bar{\psi}_{n_2,\sigma}\psi_{n_1,\sigma}\right). \tag{A.5b}$$

By using Eqs. (A.3) and (A.5) we can then obtain

$$\chi_n(\mathbf{k},\omega_n) = 16T \sum_{1,2} \sum_{r=0,3} \left\langle {}^{0}_{r}(\delta Q)_{1+n,1}(\mathbf{k})\, {}^{0}_{r}(\delta Q)_{2+n,2}(-\mathbf{k}) \right\rangle . \tag{A.6}$$

Similar analysis can be applied to find the spin density susceptibility. With the spin density

$$\mathbf{n}_s(\mathbf{k},\omega_n) = \sqrt{\frac{T}{V}} \sum_{\mathbf{p},\omega} \big(\psi(\mathbf{p},\omega), \sigma\, \psi(\mathbf{p}+\mathbf{k},\omega+\omega_n)\big) \tag{A.7}$$

we can obtain Eq. (3.13).

Appendix B

Useful coefficients for the matrix M

In Sec. 3.2, we have introduced some parameters, including $X^{(2)}_{12,34}$, $Y^{(2)}_{12}$, $Z^{(2)}_{12}$, $X^{(3)}_{12,34}$, $Y^{(3)}_{12}$ and $Z^{(3)}_{12}$. Here we give their definitions respectively as follows,

$$X^{(2)}_{12,34} = \frac{(1 - J^{(1)}_{-1,-2})\, X^{(1)}_{12,34} + K^{(1)}_{12}\, X^{(1)}_{-1,-2;3,4}}{(1 - J^{(1)}_{12})(1 - J^{(1)}_{-1,-2}) - K^{(1)}_{12} K^{(1)}_{-1,-2}} \quad , \tag{B.1a}$$

$$Y^{(2)}_{12} = \frac{(1 - J^{(1)}_{-1,-2})\, Y^{(1)}_{12} + K^{(1)}_{12}\, Z^{(1)}_{-1,-2}}{(1 - J^{(1)}_{12})(1 - J^{(1)}_{-1,-2}) - K^{(1)}_{12} K^{(1)}_{-1,-2}} \quad , \tag{B.1b}$$

$$Z^{(2)}_{12} = \frac{(1 - J^{(1)}_{-1,-2})\, Z^{(1)}_{12} + K^{(1)}_{12}\, Y^{(1)}_{-1,-2}}{(1 - J^{(1)}_{12})(1 - J^{(1)}_{-1,-2}) - K^{(1)}_{12} K^{(1)}_{-1,-2}} \quad , \tag{B.1c}$$

where

$$X^{(1)}_{12,34} = -{}^{00}_{23}A^{(0)}_{12,34} + \frac{1}{\tau^0}\varphi^{01}_{12}\, X_{1,-2;3,4} - \frac{1}{\tau^0}\varphi^{10}_{12}\, X_{-1,2;3,4} \quad , \tag{B.2a}$$

$$J^{(1)}_{12} = \frac{1}{\tau^0}\varphi^{00}_{12} + \frac{1}{\tau^0}\varphi^{01}_{12}\, J_{1,-2} - \frac{1}{\tau^0}\varphi^{10}_{12}\, K_{-1,2} \quad , \tag{B.2b}$$

$$K^{(1)}_{12} = \frac{1}{\tau^0}\varphi^{11}_{12} + \frac{1}{\tau^0}\varphi^{01}_{12}\, K_{1,-2} - \frac{1}{\tau^0}\varphi^{10}_{12}\, J_{-1,2} \quad , \tag{B.2c}$$

$$Y^{(1)}_{12} = -\Gamma\varphi^{00}_{12} + \frac{1}{\tau^0}\varphi^{01}_{12}\, Y_{1,-2} - \frac{1}{\tau^0}\varphi^{10}_{12}\, Z_{-1,2} \quad , \tag{B.2d}$$

$$Z^{(1)}_{12} = -\Gamma\varphi^{11}_{12} + \frac{1}{\tau^0}\varphi^{01}_{12}\, Z_{1,-2} - \frac{1}{\tau^0}\varphi^{10}_{12}\, Y_{-1,2} \quad , \tag{B.2e}$$

with

$$X_{12,34} = \frac{(1 - \frac{1}{\tau^0}\varphi^{00}_{-1,-2})\,{}^{00}_{33}A^{(0)}_{12,34} + \frac{1}{\tau^0}\varphi^{11}_{12}\,{}^{00}_{33}A^{(0)}_{-1,-2;3,4}}{(1 - \frac{1}{\tau^0}\varphi^{00}_{12})(1 - \frac{1}{\tau^0}\varphi^{00}_{-1,-2}) - \frac{1}{\tau^0}\varphi^{11}_{12}\,\frac{1}{\tau^0}\varphi^{11}_{-1,-2}} \quad , \tag{B.3a}$$

$$J_{12} = \frac{(1 - \frac{1}{\tau^0}\varphi^{00}_{-1,-2})(-\frac{1}{\tau^0}\varphi^{01}_{12}) + \frac{1}{\tau^0}\varphi^{11}_{12}(\frac{1}{\tau^0}\varphi^{10}_{-1,-2})}{(1 - \frac{1}{\tau^0}\varphi^{00}_{12})(1 - \frac{1}{\tau^0}\varphi^{00}_{-1,-2}) - \frac{1}{\tau^0}\varphi^{11}_{12}\,\frac{1}{\tau^0}\varphi^{11}_{-1,-2}} \quad , \tag{B.3b}$$

$$K_{12} = \frac{(1-\frac{1}{\tau^0}\varphi^{00}_{-1,-2})(\frac{1}{\tau^0}\varphi^{10}_{12}) + \frac{1}{\tau^0}\varphi^{11}_{12}(-\frac{1}{\tau^0}\varphi^{01}_{-1,-2})}{(1-\frac{1}{\tau^0}\varphi^{00}_{12})(1-\frac{1}{\tau^0}\varphi^{00}_{-1,-2}) - \frac{1}{\tau^0}\varphi^{11}_{12}\,\frac{1}{\tau^0}\varphi^{11}_{-1,-2}} \quad , \tag{B.3c}$$

$$Y_{12} = \frac{(1-\frac{1}{\tau^0}\varphi^{00}_{-1,-2})(\Gamma\varphi^{01}_{12}) + \frac{1}{\tau^0}\varphi^{11}_{12}(-\Gamma\varphi^{10}_{-1,-2})}{(1-\frac{1}{\tau^0}\varphi^{00}_{12})(1-\frac{1}{\tau^0}\varphi^{00}_{-1,-2}) - \frac{1}{\tau^0}\varphi^{11}_{12}\,\frac{1}{\tau^0}\varphi^{11}_{-1,-2}} \quad , \tag{B.3d}$$

$$Z_{12} = \frac{(1-\frac{1}{\tau^0}\varphi^{00}_{-1,-2})(-\Gamma\varphi^{10}_{12}) + \frac{1}{\tau^0}\varphi^{11}_{12}(\Gamma\varphi^{01}_{-1,-2})}{(1-\frac{1}{\tau^0}\varphi^{00}_{12})(1-\frac{1}{\tau^0}\varphi^{00}_{-1,-2}) - \frac{1}{\tau^0}\varphi^{11}_{12}\,\frac{1}{\tau^0}\varphi^{11}_{-1,-2}} \quad . \tag{B.3e}$$

We also have

$$X^{(3)}_{12,34} = X_{12,34} + J_{12}\,X^{(2)}_{1,-2;3,4} + K_{12}\,X^{(2)}_{-1,2;3,4} \quad , \tag{B.4a}$$

$$Y^{(3)}_{12} = Y_{12} + J_{12}\,Y^{(2)}_{1,-2} + K_{12}\,Z^{(2)}_{-1,2} \quad , \tag{B.4b}$$

$$Z^{(3)}_{12} = Z_{12} + J_{12}\,Z^{(2)}_{1,-2} + K_{12}\,Y^{(2)}_{-1,2} \quad . \tag{B.4c}$$

For $\omega_n \neq 0$ the following equations are useful:

$$\sum_2 {}^{00}_{23}A^{(0)}_{1+n,-1;2+n,2} = -\varphi^{01}_{1+n,-1} \quad , \tag{B.5a}$$

$$\sum_2 {}^{00}_{23}A^{(0)}_{-1-n,1;2+n,2} = \varphi^{10}_{-1-n,1} \quad , \tag{B.5b}$$

$$\sum_2 {}^{00}_{33}A^{(0)}_{1+n,1;2+n,2} = \varphi^{00}_{1+n,1} \quad , \tag{B.5c}$$

$$\sum_2 {}^{00}_{33}A^{(0)}_{-1-n,-1;2+n,2} = \varphi^{11}_{-1-n,-1} \quad . \tag{B.5d}$$

Appendix C

Detailed calculations in the clean limit

We will show in this appendix that, in the limit of long wavelength and low frequency,

$$\sum_1 \varphi^{00}_{1+n,1} = m_0 + m_1\,\mathbf{k}^2 + m_2\,\omega_n^2, \tag{C.1a}$$

$$\sum_1 \varphi^{01}_{1+n,\pm1} = im_3\,\omega_n, \tag{C.1b}$$

$$\sum_1 \varphi^{10}_{1+n,1} = \sum_1 \varphi^{01}_{1-n,1}, \tag{C.1c}$$

$$\sum_1 \varphi^{10}_{1+n,-1} = \sum_1 \varphi^{01}_{1+n,-1}, \tag{C.1d}$$

$$\Gamma\sum_1 \varphi^{00}_{1+n,-1} = -(1 + a + 2b\,\mathbf{k}^2 + 2c\,\omega_n^2) \tag{C.1e}$$

and

$$\Gamma\sum_1 \varphi^{11}_{1+n,-1} = a - b\,\mathbf{k}^2 - c\,\omega_n^2 \tag{C.1f}$$

where $m_0 = -\frac{N(0)}{2T}$, $m_1 = -\frac{N(0)v_f^2}{36T\Delta^2}$, $m_2 = \frac{N(0)}{12T\Delta^2}$, $m_3 = \frac{N(0)}{4T\Delta}$, $a = \Gamma^{(c)}N(0)$, $b = \frac{\Gamma^{(c)}N(0)v_f^2}{18\Delta^2}$ and $c = \frac{\Gamma^{(c)}N(0)}{6\Delta^2}$ with $N(0) = \frac{N_F}{2}$ the density of states per spin at the Fermi surface and $v_f = \frac{k_f}{m}$ the Fermi velocity.

C.1 Method I

Now we demonstrate how to obtain the results of Eqs. (C.1). First we show how to get Eq. (C.1a). Similar to the calculations in the section 52 of Ref. [2], we find, with Eq. (2.40e), that

$$\sum_1 \varphi^{00}_{1+n,1}(\mathbf{k}) = \frac{1}{2T}\int \frac{d\mathbf{p}}{(2\pi)^3}$$
$$((\tanh\frac{E_+}{2T} + \tanh\frac{E_-}{2T})$$
$$\times(\frac{-1}{2}(1-\frac{\xi_+\xi_-}{E_+E_-})\frac{E_+ + E_-}{\omega_n^2+(E_+ + E_-)^2} + \frac{-1}{2}(\frac{\xi_+}{E_+} - \frac{\xi_-}{E_-})\frac{i\omega_n}{\omega_n^2+(E_+ + E_-)^2})$$
$$-(\tanh\frac{E_+}{2T} - \tanh\frac{E_-}{2T})$$
$$\times(\frac{1}{2}(1+\frac{\xi_+\xi_-}{E_+E_-})\frac{E_+ - E_-}{\omega_n^2+(E_+ - E_-)^2} + \frac{1}{2}(\frac{\xi_+}{E_+} + \frac{\xi_-}{E_-})\frac{i\omega_n}{\omega_n^2+(E_+ - E_-)^2})). \quad \text{(C.2)}$$

Here $E_\pm = \sqrt{\xi_\pm^2 + \Delta^2}$ with $\xi_\pm = \xi_\mathbf{p} \pm a/2$ and $a = |\mathbf{k}|v_f z$. For simplicity, we set $\omega = 0$ first. Then

$$\sum_1 \varphi^{00}_{1,1}(\mathbf{k}) = \frac{-N(0)}{4T}\int_{-1}^{1} dz \int_{-\omega_\mathrm{D}}^{\omega_\mathrm{D}} d\xi_\mathbf{p}\, ((\frac{\tanh\frac{E_+}{2T}}{E_+} - \frac{\tanh\frac{E_-}{2T}}{E_-})\frac{\Delta^2}{E_+^2 - E_-^2}$$
$$+(\frac{\xi_+\tanh\frac{E_+}{2T}}{E_+} - \frac{\xi_-\tanh\frac{E_-}{2T}}{E_-})\frac{1}{\xi_+ - \xi_-}) \quad \text{(C.3)}$$

with the Debye frequency $\omega_\mathrm{D} \gg \Delta$. An examination of the case in the normal state, which means $\Delta = 0$, shows that

$$\begin{aligned}\sum_1 \varphi^{00}_{1,1}(\mathbf{k}, \Delta = 0) &= \frac{-N(0)}{4T}\int_{-1}^{1} dz \int_{-\infty}^{\infty} d\xi_\mathbf{p}\, \frac{\tanh\frac{\xi_+}{2T} - \tanh\frac{\xi_-}{2T}}{\xi_+ - \xi_-} \\ &= \frac{-N(0)}{T},\end{aligned} \quad \text{(C.4)}$$

where we set ω_D equal to ∞, since the integral converges. The evaluation of the superconducting case may be simplified by considering the difference between Eqs.

(C.3) and (C.4)

$$\begin{aligned}\sum_1 \varphi^{00}_{1,1} &= \frac{-N(0)}{T} + (\sum_1 \varphi^{00}_{1,1} - \sum_1 \varphi^{00}_{1,1}(\Delta = 0)) \\ &= \frac{-N(0)}{2T} - \frac{N(0)}{4T}\int_{-1}^{1} dz \int_{-\infty}^{\infty} d\xi_{\mathbf{p}}\, (\frac{\tanh\frac{E_+}{2T}}{E_+} - \frac{\tanh\frac{E_-}{2T}}{E_-})\frac{\Delta^2}{E_+^2 - E_-^2} \\ &= \frac{-N(0)}{2T} - \frac{N(0)v_f^2}{36T\Delta^2}\mathbf{k}^2 \qquad \text{(C.5)}\end{aligned}$$

Here $|\mathbf{k}|v_f \ll \pi\Delta$ and $\tanh\frac{\Delta}{2T} = 1$ have been assumed. The latter assumption means $T \to 0$. And the difference between Eqs. (C.2) and (C.3) can be obtained by setting $|\mathbf{k}| = 0$

$$\begin{aligned}\sum_1 \varphi^{00}_{1+n,1} - \sum_1 \varphi^{00}_{1,1} &= \frac{N(0)\omega_n^2}{4T}\int_{-\infty}^{\infty} d\xi_{\mathbf{p}}\, (\frac{1-\xi_{\mathbf{p}}^2/E^2}{4E^3}\tanh\frac{E}{2T}) \\ &= \frac{N(0)}{12T\Delta^2}\omega_n^2. \qquad \text{(C.6)}\end{aligned}$$

Finally, by Eqs. (C.5) and (C.6), we get the result of Eq. (C.1a).

To get Eq. (C.1e), we use the same procedure as above, with somewhat different techniques.

$$\begin{aligned}&\sum_1 \varphi^{00}_{1+n,-1}(\mathbf{k}) = \frac{1}{2T}\int \frac{d\mathbf{p}}{(2\pi)^3} \\ &((\tanh\frac{E_+}{2T} + \tanh\frac{E_-}{2T}) \\ &\times (\frac{1}{2}(1 + \frac{\xi_+\xi_-}{E_+E_-})\frac{E_+ + E_-}{\omega_n^2 + (E_+ + E_-)^2} + \frac{1}{2}(\frac{\xi_+}{E_+} + \frac{\xi_-}{E_-})\frac{i\omega_n}{\omega_n^2 + (E_+ + E_-)^2}) \\ &+ (\tanh\frac{E_+}{2T} - \tanh\frac{E_-}{2T}) \\ &\times (\frac{1}{2}(1 - \frac{\xi_+\xi_-}{E_+E_-})\frac{E_+ - E_-}{\omega_n^2 + (E_+ - E_-)^2} + \frac{1}{2}(\frac{\xi_+}{E_+} - \frac{\xi_-}{E_-})\frac{i\omega_n}{\omega_n^2 + (E_+ - E_-)^2})). \qquad \text{(C.7)}\end{aligned}$$

Again, we set $\omega = 0$ first. We find

$$\begin{aligned}
\sum_1 \varphi^{00}_{1,-1}(\mathbf{k}) &= \frac{N(0)}{4T}\int_{-1}^{1} dz \int_{-\omega_D}^{\omega_D} d\xi_{\mathbf{p}}\,((\frac{\tanh\frac{E_+}{2T}}{E_+} - \frac{\tanh\frac{E_-}{2T}}{E_-})\frac{\Delta^2}{E_+^2 - E_-^2} \\
&\quad + (\frac{\xi_+\tanh\frac{E_+}{2T}}{E_+} + \frac{\xi_-\tanh\frac{E_-}{2T}}{E_-})\frac{1}{\xi_+ + \xi_-}) \\
&= \frac{N(0)}{4T}\int_{-1}^{1} dz \int_{-\infty}^{\infty} d\xi_{\mathbf{p}}\,((\frac{\tanh\frac{E_+}{2T}}{E_+} - \frac{\tanh\frac{E_-}{2T}}{E_-})\frac{\Delta^2}{E_+^2 - E_-^2}) \\
&\quad + \frac{N(0)}{4T}\int_{-1}^{1} dz \int_{-\infty}^{\infty} d\xi_{\mathbf{p}}\,(\frac{a}{4\xi_{\mathbf{p}}}(\frac{\tanh\frac{E_+}{2T}}{E_+} - \frac{\tanh\frac{E_-}{2T}}{E_-})) \\
&\quad + \frac{N(0)}{4T}\int_{-1}^{1} dz \int_{-\omega_D}^{\omega_D} d\xi_{\mathbf{p}}\,(\frac{1}{2}(\frac{\tanh\frac{E_+}{2T}}{E_+} + \frac{\tanh\frac{E_-}{2T}}{E_-})) \\
&= (\frac{-N(0)}{2T} + \frac{N(0)v_f^2}{36T\Delta^2}\mathbf{k}^2) + (\frac{-N(0)v_f^2}{12T\Delta^2}\mathbf{k}^2) + (\frac{-1}{2T\Gamma^{(c)}}) \\
&= \frac{-1}{2T\Gamma^{(c)}} + \frac{-N(0)}{2T} + \frac{-N(0)v_f^2}{18T\Delta^2}\mathbf{k}^2, \qquad \text{(C.8)}
\end{aligned}$$

where the gap equation Eq. (2.33) has been used to obtain the first item of the last equation.[2] Similarly, the difference between Eqs. (C.7) and (C.8) can be obtained by setting $|\mathbf{k}| = 0$

$$\begin{aligned}
\sum_1 \varphi^{00}_{1+n,-1} - \sum_1 \varphi^{00}_{1,-1} &= \frac{-N(0)\omega_n^2}{4T}\int_{-\infty}^{\infty} d\xi_{\mathbf{p}}\,(\frac{1+\xi_{\mathbf{p}}^2/E^2}{4E^3}\tanh\frac{E}{2T}) \\
&= \frac{-N(0)}{6T\Delta^2}\omega_n^2. \qquad \text{(C.9)}
\end{aligned}$$

Combining Eqs. (C.8) and (C.9), we get the result of Eq. (C.1e). Other results of Eqs. (C.1) can be analogously obtained by the methods used here for Eqs. (C.1a) and (C.1e).

C.2 Method II

The same results for Eqs. (C.1) can also be obtained by first calculating the integration over ξ and then summing over the frequency. Here we only show how to calculate the zero frequency and zero momentum parts of $\sum_m \varphi^{00}_{m+n,m}(\mathbf{k})$ and $\sum_m \varphi^{00}_{m+n,-m}(\mathbf{k})$. The additional parts for small $|\mathbf{k}|$ and ω_n, like $\sum_m \varphi^{00}_{m+n,-m}(\mathbf{k}) - \sum_m \varphi^{00}_{m,-m}(\mathbf{k}=0)$, are relatively easier to evaluate. In Appendix D we will show the complete calculations in the disordered case. With $T \to 0$,

$$\begin{aligned}
\sum_m \varphi^{00}_{m,m}(\mathbf{k}=0) &= N(0)\sum_m \int_{-\omega_\mathrm{D}}^{\omega_\mathrm{D}} d\xi_\mathbf{p}\,(\frac{-i\omega_m-\xi_\mathbf{p}}{\omega_m^2+\xi_\mathbf{p}^2+\Delta^2}\,\frac{-i\omega_m-\xi_\mathbf{p}}{\omega_m^2+\xi_\mathbf{p}^2+\Delta^2}) \\
&= N(0)\int_{-\infty}^{\infty}\frac{d\omega_m}{2\pi T}(\frac{-\omega_\mathrm{D}(2\omega_m^2+\Delta^2)}{(\omega_m^2+\Delta^2)(\omega_\mathrm{D}^2+\omega_m^2+\Delta^2)} \\
&\quad +\frac{\Delta^2}{(\omega_m^2+\Delta^2)^{3/2}}\arctan\frac{\omega_\mathrm{D}}{\sqrt{\omega_m^2+\Delta^2}}) \\
&= \frac{N(0)}{2\pi T}((-2\pi)+(\pi)) \\
&= \frac{-N(0)}{2T}.
\end{aligned} \tag{C.10}$$

Note that we cannot set ω_D equal to ∞ before the summation over the frequency, otherwise a wrong result of $\frac{N(0)}{2T}$ will be obtained. And

$$\begin{aligned}
\sum_m \varphi^{00}_{m,-m}(\mathbf{k}=0) &= N(0)\sum_m \int_{-\omega_\mathrm{D}}^{\omega_\mathrm{D}} d\xi_\mathbf{p}\,(\frac{-i\omega_m-\xi_\mathbf{p}}{\omega_m^2+\xi_\mathbf{p}^2+\Delta^2}\,\frac{i\omega_m-\xi_\mathbf{p}}{\omega_m^2+\xi_\mathbf{p}^2+\Delta^2}) \\
&= N(0)\int_{-\infty}^{\infty}\frac{d\omega_m}{2\pi T}(\frac{-\omega_\mathrm{D}\Delta^2}{(\omega_m^2+\Delta^2)(\omega_\mathrm{D}^2+\omega_m^2+\Delta^2)} \\
&\quad -\frac{\Delta^2}{(\omega_m^2+\Delta^2)^{3/2}}\arctan\frac{\omega_\mathrm{D}}{\sqrt{\omega_m^2+\Delta^2}} \\
&\quad +\frac{2}{\sqrt{\omega_m^2+\Delta^2}}\arctan\frac{\omega_\mathrm{D}}{\sqrt{\omega_m^2+\Delta^2}}) \\
&= \frac{N(0)}{2\pi T}((0)+(-\pi)+(\int_{-\frac{\omega_\mathrm{D}}{\Delta}}^{\frac{\omega_\mathrm{D}}{\Delta}} d\omega_m\,\frac{\pi}{\sqrt{\omega_m^2+1}})) \\
&= \frac{-N(0)}{2T}+\frac{N(0)}{T}\ln\frac{2\omega_\mathrm{D}}{\Delta} \\
&= \frac{-1}{2T\Gamma^{(c)}}-\frac{N(0)}{2T},
\end{aligned} \tag{C.11}$$

where the equality

$$\int_{-\infty}^{\infty} d\omega\,(\frac{1}{\sqrt{\omega^2+1}}\arctan\frac{D}{\sqrt{\omega^2+1}}) \equiv \int_{-D}^{D} d\omega\,(\frac{\pi}{2}\,\frac{1}{\sqrt{\omega^2+1}}), \tag{C.12a}$$

has been adopted, which can be proven by using the general expansion[62]

$$\arctan x = \frac{x}{1+x^2}(1+\frac{2}{3}\frac{x^2}{1+x^2}+\frac{2\times 4}{3\times 5}(\frac{x^2}{1+x^2})^2+\frac{2\times 4\times 6}{3\times 5\times 7}(\frac{x^2}{1+x^2})^3+\ldots). \tag{C.12b}$$

The last equation of Eq. (C.11) is obtained by using some results from the section 51 of Ref. [2].

We also find in the calculation of χ_n that only the zero frequency and zero momentum part of Eq. (C.1a) will contribute to it, which means we just need to obtain m_0. This gives us another way to find Eq. (C.1a), or m_0. By using the compressibility sum rule of free electrons (i.e. $\Gamma^{(c)} = 0$), we have[27, 63]

$$\lim_{\mathbf{k}\to 0}\lim_{\omega_n\to 0}\chi_n = -N_F, \tag{C.13}$$

which in turn gives $m_0 = -\frac{N(0)}{2T}$.

C.3 Exact expressions at finite temperature

To get Eq. (3.29b), we need to obtain the exact solution at $T \neq 0$. Setting $\omega_n = 0$ first and then $k \to 0$, we find that, by using Eq. (C.2),

$$\begin{aligned}\sum_1 \varphi^{00}_{1,1}(\mathbf{k}\to 0) &= \frac{N(0)}{2T}\int d\xi_{\mathbf{p}}(\frac{-\Delta^2}{2E^3}\tanh\frac{E}{2T}-\frac{1}{2}(1+\frac{\xi^2_{\mathbf{p}}}{E^2})\frac{\partial(\tanh\frac{E}{2T})}{\partial E}) \\ &= \frac{-N(0)}{2T}\int d\xi_{\mathbf{p}}(\frac{-\Delta^2}{2E}\frac{\partial\frac{\tanh\frac{E}{2T}}{E}}{\partial E}+\frac{\partial(\tanh\frac{E}{2T})}{\partial E}).\end{aligned} \tag{C.14a}$$

Similar calculations can be used to obtain the following expressions:

$$\sum_1 \varphi^{11}_{1,1}(\mathbf{k}\to 0) = \frac{N(0)}{2T}\int d\xi_{\mathbf{p}}(\frac{-\Delta^2}{2E}\frac{\partial\frac{\tanh\frac{E}{2T}}{E}}{\partial E}), \tag{C.14b}$$

$$\sum_{1} \varphi^{00}_{1,-1}(\mathbf{k} \to 0) = \frac{-1}{2T\Gamma^{(c)}} + \frac{-N(0)}{2T} \int d\xi_{\mathbf{p}} (\frac{-\Delta^2}{2E} \frac{\partial \frac{\tanh \frac{E}{2T}}{E}}{\partial E}), \tag{C.14c}$$

$$\sum_{1} \varphi^{01}_{1,-1}(\mathbf{k} \to 0) = 0, \tag{C.14d}$$

Appendix D

Detailed calculations in the disordered case

In this appendix we demonstrate how to obtain the result of Eq. (3.30). First we deal with $\varphi^{00}_{m+n,m}(\mathbf{k})$. For $\omega_n = 0$, we find

$$
\begin{aligned}
\varphi^{00}_{m,m}(\mathbf{k}) = {} & \frac{N(0)}{2}\int_{-1}^{1} dz \int_{-\omega_{\mathrm{D}}}^{\omega_{\mathrm{D}}} d\xi_{\mathbf{p}} \Big(\frac{-i\eta_m\omega_m - \xi_+}{(\eta_m\omega_m)^2 + \xi_+^2 + (\eta_m\Delta)^2}\frac{-i\eta_m\omega_m - \xi_-}{(\eta_m\omega_m)^2 + \xi_-^2 + (\eta_m\Delta)^2}\Big) \\
= {} & N(0)\int_{-1}^{1} dz \Big(\frac{(\eta_m\Delta)^2}{\sqrt{(\eta_m\omega_m)^2 + (\eta_m\Delta)^2}}\frac{\arctan\frac{\omega_{\mathrm{D}}-\frac{a}{2}}{\sqrt{(\eta_m\omega_m)^2+(\eta_m\Delta)^2}} + \arctan\frac{\omega_{\mathrm{D}}+\frac{a}{2}}{\sqrt{(\eta_m\omega_m)^2+(\eta_m\Delta)^2}}}{a^2 + 4((\eta_m\omega_m)^2 + (\eta_m\Delta)^2)} \\
& + \frac{\ln\left(4((\eta_m\omega_m)^2 + (\eta_m\Delta)^2) + (2\omega_{\mathrm{D}} - a)^2\right) - \ln\left(4((\eta_m\omega_m)^2 + (\eta_m\Delta)^2) + (2\omega_{\mathrm{D}} + a)^2\right)}{2a} \\
& \times \frac{a^2 + 4(\eta_m\omega_m)^2 + 2(\eta_m\Delta)^2}{a^2 + 4((\eta_m\omega_m)^2 + (\eta_m\Delta)^2)}\Big) \\
= {} & N(0)\int_{-1}^{1} dz \Big(\Big(\frac{(\eta_m\Delta)^2\arctan\frac{\omega_{\mathrm{D}}}{\sqrt{(\eta_m\omega_m)^2+(\eta_m\Delta)^2}}}{2((\eta_m\omega_m)^2 + (\eta_m\Delta)^2)^{3/2}} \\
& - \frac{\omega_{\mathrm{D}}}{(\eta_m\omega_m)^2 + (\eta_m\Delta)^2 + \omega_{\mathrm{D}}^2}\frac{2(\eta_m\omega_m)^2 + (\eta_m\Delta)^2}{2((\eta_m\omega_m)^2 + (\eta_m\Delta)^2)}\Big) \\
& + \frac{a^2}{2}\Big(\frac{-\pi(\eta_m\Delta)^2}{8((\eta_m\omega_m)^2 + (\eta_m\Delta)^2)^{5/2}}\Big)\Big) \\
= {} & N(0)\Big(\frac{(\eta_m\Delta)^2\arctan\frac{\omega_{\mathrm{D}}}{\sqrt{(\eta_m\omega_m)^2+(\eta_m\Delta)^2}}}{((\eta_m\omega_m)^2 + (\eta_m\Delta)^2)^{3/2}} \\
& - \frac{\omega_{\mathrm{D}}}{(\eta_m\omega_m)^2 + (\eta_m\Delta)^2 + \omega_{\mathrm{D}}^2}\frac{2(\eta_m\omega_m)^2 + (\eta_m\Delta)^2}{(\eta_m\omega_m)^2 + (\eta_m\Delta)^2}\Big) \\
& + \frac{N(0)v^2\mathbf{k}^2}{3}\Big(\frac{-\pi(\eta_m\Delta)^2}{8((\eta_m\omega_m)^2 + (\eta_m\Delta)^2)^{5/2}}\Big).
\end{aligned}
\tag{D.1}
$$

Here

$$
\eta_m = 1 + \frac{1}{2\tau_{\mathrm{e}}\sqrt{\omega_m^2 + \Delta^2}},
\tag{D.2}
$$

which is given by Eqs. (2.32). The last item of the result in Eq. (D.1), i.e., for the expansion on small $|\mathbf{k}|$, has been obtained by using the fact that $\omega_{\rm D} \gg \Delta$. Now if we set $|\mathbf{k}| = 0$ instead, it yields analogously

$$
\begin{aligned}
\varphi^{00}_{m+n,m} = & \frac{2N(0)}{((\eta_{m+n}\omega_{m+n})^2 + (\eta_{m+n}\Delta)^2) - ((\eta_m\omega_m)^2 + (\eta_m\Delta)^2)} (\\
& (\sqrt{(\eta_{m+n}\omega_{m+n})^2 + (\eta_{m+n}\Delta)^2} \arctan \frac{\omega_{\rm D}}{\sqrt{(\eta_{m+n}\omega_{m+n})^2 + (\eta_{m+n}\Delta)^2}} \\
& - \sqrt{(\eta_m\omega_m)^2 + (\eta_m\Delta)^2} \arctan \frac{\omega_{\rm D}}{\sqrt{(\eta_m\omega_m)^2 + (\eta_m\Delta)^2}}) \\
& + \eta_{m+n}\omega_{m+n}\eta_m\omega_m(\\
& \frac{1}{\sqrt{(\eta_{m+n}\omega_{m+n})^2 + (\eta_{m+n}\Delta)^2}} \arctan \frac{\omega_{\rm D}}{\sqrt{(\eta_{m+n}\omega_{m+n})^2 + (\eta_{m+n}\Delta)^2}} \\
& - \frac{1}{\sqrt{(\eta_m\omega_m)^2 + (\eta_m\Delta)^2}} \arctan \frac{\omega_{\rm D}}{\sqrt{(\eta_m\omega_m)^2 + (\eta_m\Delta)^2}})) \\
= & \varphi^{00}_{m,m} + \omega_n \frac{-\pi\tau_{\rm e}\Delta^2\omega_m(1 + 3\tau_{\rm e}\sqrt{\omega_m^2 + \Delta^2})}{(\omega_m^2 + \Delta^2)^2\,(1 + 2\tau_{\rm e}\sqrt{\omega_m^2 + \Delta^2})^2} \\
& + \frac{\omega_n^2}{2} \frac{\pi\tau_{\rm e}\Delta^2\,(3\omega_m^2 + \tau_{\rm e}\sqrt{\omega_m^2 + \Delta^2}(-\Delta^2 + 14\omega_m^2) + \tau_{\rm e}^2(-2\Delta^4 + 16\Delta^2\omega_m^2 + 18\omega_m^4))}{(\omega_m^2 + \Delta^2)^3\,(1 + 2\tau_{\rm e}\sqrt{\omega_m^2 + \Delta^2})^3}.
\end{aligned}
\tag{D.3}
$$

Therefore a combination of Eqs. (D.1) and (D.3) gives $\varphi^{00}_{m+n,m}(\mathbf{k})$ up to the second-order expansions on small $|\mathbf{k}|$ and ω_n. Other items, including $\varphi^{00}_{m+n,-m}(\mathbf{k})$, $\varphi^{01}_{m+n,m}(\mathbf{k})$, $\varphi^{10}_{m+n,m}(\mathbf{k})$ and $\varphi^{11}_{m+n,m}(\mathbf{k})$, can be evaluated by the same techniques. Through lengthy but not difficult calculations, we obtain

$$\sum_1 Y^{(2)}_{1+n,-1} = \sum_1 Y^{(2)}_{-1-n,1} = 1 + N(0)\Gamma^{(c)} + \frac{\Gamma^{(c)}\tau^0 v_f^2}{16\Delta}\mathbf{k}^2 + \frac{\Gamma^{(c)}N_F}{6\Delta^2}\omega_n^2, \tag{D.4a}$$

$$\sum_1 Z^{(2)}_{1+n,-1} = \sum_1 Z^{(2)}_{-1-n,1} = -N(0)\Gamma^{(c)} + \frac{\Gamma^{(c)}\tau^0 v_f^2}{48\Delta}\mathbf{k}^2 + \frac{\Gamma^{(c)}N_F}{12\Delta^2}\omega_n^2, \tag{D.4b}$$

$$\sum_{12} X^{(2)}_{1+n,-1;2+n,2} = \sum_{12} X^{(2)}_{-1-n,1;2+n,2} = \frac{iN_F}{8T\Delta}\omega_n, \tag{D.4c}$$

$$\sum_{1} Y^{(3)}_{1+n,1} = \sum_{1} Z^{(3)}_{1+n,1} = \frac{i\Gamma N_F}{8T\Delta}\omega_n \tag{D.4d}$$

and

$$\begin{aligned}
\sum_{12} X^{(3)}_{1,1;2,2} &= \frac{\tau^0}{2\pi T}\int_{-\infty}^{\infty} d\omega \\
&\quad \Big(\frac{-2\tau_e\omega_D(2\omega^2+\Delta^2)}{(\omega^2+\Delta^2)(\pi(1+2\tau_e\sqrt{\omega^2+\Delta^2})^2+4\tau_e\omega_D(1+\pi\tau_e\omega_D))} \\
&\quad +\frac{\Delta^2\arctan\frac{2\tau_e\omega_D}{1+2\tau_e\sqrt{\omega^2+\Delta^2}}}{(\omega^2+\Delta^2)(\pi(1+2\tau_e\sqrt{\omega^2+\Delta^2})-2\arctan\frac{2\tau_e\omega_D}{1+2\tau_e\sqrt{\omega^2+\Delta^2}})}\Big) \\
&= \frac{\tau^0}{2\pi T}\int_{-\infty}^{\infty} d\omega\,\Big(\frac{-2\tau_e\omega_D(2\omega^2+\Delta^2)}{(\omega^2+\Delta^2)(\pi(2\tau_e\sqrt{\omega^2+\Delta^2})^2+4\tau_e\omega_D(\pi\tau_e\omega_D))} \\
&\quad +\frac{\pi}{2}\frac{\Delta^2}{(\omega^2+\Delta^2)(\pi(2\tau_e\sqrt{\omega^2+\Delta^2}))}\Big) \\
&= \frac{-N_F}{4T}.
\end{aligned} \tag{D.4e}$$

Now by using Eqs. (3.15) and (3.24) we obtain the number density susceptibility.

For the spin density susceptibility, we find

$$ {}^{11}_{33}M^{-1}_{12,34} = \frac{L^{(1)}_{12,34} + E^{(1)}_{12}L^{(1)}_{-1,-2;3,4}}{1-E^{(1)}_{12}E^{(1)}_{-1,-2}}, \tag{D.5a}$$

where

$$L^{(1)}_{12,34} = \frac{{}^{11}_{33}A^{(0)}_{12,34} - \frac{1}{\tau^0}\varphi^{01}_{12}L_{1,-2;3,4} + \frac{1}{\tau^0}\varphi^{10}_{12}L_{-1,2;3,4}}{1-\frac{1}{\tau^0}\varphi^{00}_{12}+\frac{1}{\tau^0}\varphi^{01}_{12}E_{1,-2}-\frac{1}{\tau^0}\varphi^{10}_{12}F_{-1,2}}, \tag{D.5b}$$

$$E^{(1)}_{12} = \frac{\frac{1}{\tau^0}\varphi^{11}_{12} - \frac{1}{\tau^0}\varphi^{01}_{12}F_{1,-2} + \frac{1}{\tau^0}\varphi^{10}_{12}E_{-1,2}}{1-\frac{1}{\tau^0}\varphi^{00}_{12}+\frac{1}{\tau^0}\varphi^{01}_{12}E_{1,-2}-\frac{1}{\tau^0}\varphi^{10}_{12}F_{-1,2}}, \tag{D.5c}$$

with

$$L_{1,-2;3,4} = \frac{-(1-\frac{1}{\tau^0}\varphi^{00}_{-1,2}){}^{11}_{23}A^{(0)}_{1,-2;3,4} - \frac{1}{\tau^0}\varphi^{11}_{1,-2}\,{}^{11}_{23}A^{(0)}_{-1,2;3,4}}{(1-\frac{1}{\tau^0}\varphi^{00}_{1,-2})(1-\frac{1}{\tau^0}\varphi^{00}_{-1,2})-\frac{1}{\tau^0}\varphi^{11}_{1,-2}\,\frac{1}{\tau^0}\varphi^{11}_{-1,2}}, \tag{D.5d}$$

$$E_{1,-2} = \frac{(1-\frac{1}{\tau^0}\varphi^{00}_{-1,2})\frac{1}{\tau^0}\varphi^{01}_{1,-2} - \frac{1}{\tau^0}\varphi^{11}_{1,-2}\,\frac{1}{\tau^0}\varphi^{10}_{-1,2}}{(1-\frac{1}{\tau^0}\varphi^{00}_{1,-2})(1-\frac{1}{\tau^0}\varphi^{00}_{-1,2})-\frac{1}{\tau^0}\varphi^{11}_{1,-2}\,\frac{1}{\tau^0}\varphi^{11}_{-1,2}}, \tag{D.5e}$$

and

$$F_{1,-2} = \frac{-(1-\frac{1}{\tau^0}\varphi^{00}_{-1,2})\frac{1}{\tau^0}\varphi^{10}_{1,-2} + \frac{1}{\tau^0}\varphi^{11}_{1,-2}\,\frac{1}{\tau^0}\varphi^{01}_{-1,2}}{(1-\frac{1}{\tau^0}\varphi^{00}_{1,-2})(1-\frac{1}{\tau^0}\varphi^{00}_{-1,2})-\frac{1}{\tau^0}\varphi^{11}_{1,-2}\,\frac{1}{\tau^0}\varphi^{11}_{-1,2}}. \tag{D.5f}$$

We find in the Eqs. (D.5) that there is no diffusive structure in the form of ${}^{11}_{33}M^{-1}_{12,34}$. That means the spin density susceptibility in the limit of $\omega_n = 0$ and $|\mathbf{k}| = 0$ will be the focus of attention. The expansions on small $|\mathbf{k}|$ and ω_n are not so important. Similar to the calculations in the Eqs. (D.4), we obtain

$$\sum_{1,2} {}^{11}_{33}M^{-1}_{1,1;2,2} = 0. \tag{D.6}$$

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