THESIS REPORT

Ph.D.

Optimal Decoupling Control

by G. Yudilevitch
Advisor: W.S. Levine

Ph.D. 94-10

ISR
INSTITUTE FOR SYSTEMS RESEARCH

OPTIMAL DECOUPLING CONTROL

by

Gil Yudilevitch

Dissertation submitted to the Faculty of the Graduate School
of The University of Maryland in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
1994

Advisory Committee:

Professor William S. Levine, Chairman/Advisor
Professor André L. Tits
Professor Robert W. Newcomb
Associate Professor Wijesuriya P. Dayawansa
Associate Professor Jewel B. Barlow

Abstract

Title of Dissertation: OPTIMAL DECOUPLING CONTROL

Gil Yudilevitch, Doctor of Philosophy, 1994

Dissertation directed by: Professor William S. Levine

Department of Electrical Engineering

A new approach to optimal decoupling for multi-input multi-output (MIMO), linear, time-invariant control systems, is presented. This general optimal decoupling method is based on the minimization of componentwise performance indices. These indices can be obtained from alternative componentwise mathematical representations of the original system. Furthermore, using these representations, optimal decoupling problems can be written in terms of ordinary optimal control problems. Then one can use classical optimal control techniques such as LQR or $\mathcal{H}_\infty$ to obtain the ideal solutions to the optimal decoupling problems.

However these techniques can not be used to obtain practical closed-loop solutions. In this dissertation, fixed structure control techniques are used to derive the optimal and sub-optimal state feedback control law. For this purpose, a globally convergent numerical algorithm is derived. In addition some other global sub-optimal solutions are studied. This approach is used in the design of an attitude flight control of the UH-60A helicopter in hover.

Dedication

This dissertation is dedicated to the memory of my father-in-law, Zecharia Shalit.

עבודת מחקר זו מוקדשת לזיכרו של חמי, זכריה שליט ז״ל.

/

" כי זרע השלום, הגפן תתן פריה והארץ תתן
את יבולה והשמים יתנו טלם ..."
זכריה ח, י״ב

Acknowledgements

Table of Contents

x

List of Figures

Chapter 1

Introduction

1.1 Strict decoupling

Over the last three decades the problem of achieving decoupling, or noninteraction, in MIMO control systems has been widely studied. In 1963, Morgan [53] was the first to mathematically formulate the decoupling problem using the state space representation. Although his assumptions were very restrictive and although he got only partial solutions, his work was very important because it was the trigger for the on-going theoretical research on this subject. This research has played an important role in the development of MIMO control design techniques.

It is not surprising that Morgan's work, as well as some other previous work on the decoupling problem, was motivated by the needs of flight control designers [57]. The highly coupled nature of aircraft dynamics and the high performance requirements, together with the lack of a good MIMO design procedure made this research essential. Practically, although they only had SISO design techniques, flight control designers used some intuitive methods to reduce the aircraft coupling level. Usually, they interpreted the coupling effects as additional disturbance inputs. Thus, they could consider the MIMO system as a set of decoupled SISO systems with unmeasurable disturbances. In fact, this idea is also used in some of today's practical MIMO control designs, e.g., in helicopter flight control design [63]. Using this approach, the

design usually resulted in high feedback gains. In many practical cases this technique gave satisfactory results. However, in many other cases the remaining closed-loop coupling level was too high. For example, by using this approach in flight control design [63], the inherently high cross-coupling between pitch and roll can be reduced substantially to help the pilot control the aircraft. However, using the same approach for auto-pilot design (no pilot in the closed-loop), the remaining coupling is too high to meet the auto-pilot design requirements. In order to achieve more cross-coupling reduction, cross-feed techniques have often been used, e.g., [8]. These techniques can be used where the system has isolated coupled pairs (e.g., pitch/roll, roll/pitch pair in the flight control system), but they are not suitable for a general coupled system.

Some of these practical techniques have been adopted by control designers in other fields such as: motion control [47], chemical processes [22], etc. However these solutions were not generic and could not guarantee strict decoupling. Moreover in general, they had weak mathematical foundation. The next step was naturally to generalize these solutions using the available SISO design techniques. Therefore, frequency domain approaches were used extensively in the attempt to diagonalize the closed-loop transfer-function matrix (e.g., [18], [10]). In retrospect, these approaches are more appropriate than the state space one, because of the I/O nature of the decoupling problem. Recently, a number of important results were obtained using modern MIMO frequency domain methods (e.g., [13]). However, in the 60's, because of the lack of modern mathematical tools such as coprime factorization, these designs were very complicated. They often ended with high order [33] or noncausal [39] controllers.

Morgan's state space formulation was a good alternative to these methods because it allowed the application of well known state space techniques such as state and output feedback, pole placement, etc. Much research has been done on the strict decoupling problem using the state space representation. Many papers have been published on this subject such as: [14] and [21] on algebraic methods, [77] and [12], on geometrical approaches, etc. This research has been also expanded to nonlinear systems, time-varying systems, digital systems, and some other cases (see [54] for a list of

2

references). Recently the research on decoupling has been returned to the, more natural, I/O (frequency) domain. Research using various techniques such as polynomial coprime factorizations [75], and $\mathcal{RH}_\infty$ coprime factorizations [13] has been published.

1.2 Optimal decoupling

Although, the strict decoupling problem has been successfully solved, the results are not always useful, for the following reasons:

- In most cases strict decoupling is not required. Actually, all practical control designs have to tolerate a certain level of coupling (e.g., [30]).

- For all of the strict decoupling design methods a perfect knowledge of the plant is assumed. In most cases, this information is not available. Using strict decoupling with only approximate models may lead to poor decoupling performance.

- Using strict decoupling, the control effort is not taken into account. That is, strict decoupling may require very large control inputs. Therefore the theoretical strict decoupling solution may not be implementable (e.g., actuator limitation).

- Using strict decoupling, the closed-loop transfer function is diagonalized regardless of the exogenous inputs to the system. For some cases this may be considered as an advantage. However, in several cases we would like to use a priori knowledge about the exogenous input, to improve our design (e.g., the set point of process control system which changes only by steps).

- Generally, for strict decoupling some assumptions on the given plant are needed. In [13] the invertibility of the plant transfer function is assumed for the design of a high order controller. In case of low order control (e.g., constant gain), stronger assumptions may be required (e.g., [21]). These assumptions may not always hold.

Usually these are problems for which optimal control techniques offer solutions. The main difficulty is to define a scalar measure for the coupling level of the system which can be used for optimization. Many measures have been proposed and used in order to reduce the coupling level of the system (e.g., [7,19,38,74]). Some of them have also been used for optimization (e.g., [3,29,78]). Using conventional cost functions such as in [78] leads to weak results. On the other hand using cost functions which contain also the "coupling cost" may lead to numerical solutions which either depend on the exogenous inputs [29], or require high control signals [3].

1.3 Objective and contributions

The main goal of this dissertation is to reduce the optimal decoupling problem to an ordinary optimal control problem which can be solved using standard techniques such as LQR, $\mathcal{H}_\infty$, etc. The contributions of this dissertation can be summarized as follows:

- New linear operators which transform the original system into either "duplicated system" or "simultaneous control" forms, are defined. Using these forms and a quadratic componentwise scalar coupling cost function, the optimal decoupling control (ODC) problem can be reformulated as an optimal control problem.

- Using the above formulation the general ideal optimal decoupling problem is solved. This solution can be used as:

  - a high order optimal control law for stable plants in open and closed loop configurations.

  - a low order LMS sub-optimal closed-loop control law for any plant.

  - the lower bound for the minimum componentwise coupling cost of any other design technique.

- Necessary conditions for optimality for constant state feedback control are derived.

4

- A new numerical algorithm solving the sub-optimal decoupling control (SODC) problem for constant state feedback control with random initial state is derived. This algorithm is globally convergent to a set of stationary points of the componentwise cost function.

- The above solutions are applied to a very challenging decoupling problem: attitude control of the UH-60A helicopter in hover. Using these techniques the roll-to-pitch coupling is reduced with very little change in the other response components.

The contributions of this dissertation to the optimal decoupling theory are shown in the dashed box of Figure 1.1.

1.4 Organization

This dissertation is organized into seven chapters. Chapter 2 contains a comprehensive survey of previous optimal decoupling techniques. In Chapter 3, the optimal decoupling problem is formulated using componentwise performance indices and some special linear operators. Using this formulation the general ideal solution and its applications are given.

In Chapter 4, a constant state feedback control law is assumed. For this case the necessary conditions for optimality are derived using fixed structure control techniques. It is also shown that the optimal solution to this problem depends on the initial state. In Chapter 5, some sub-optimal solutions are presented. The main result is obtained by randomizing the initial state such that the sub-optimal solution no longer depends on the initial state. In order to solve this fixed structure control (or parametric LQ) problem, a numerical algorithm is developed. In general this problem is not convex. Therefore this sub-optimal solution may not be global. Some alternative global sub-optimal solutions are also presented in this chapter.

One application of these results is given in Chapter 6. In this example the
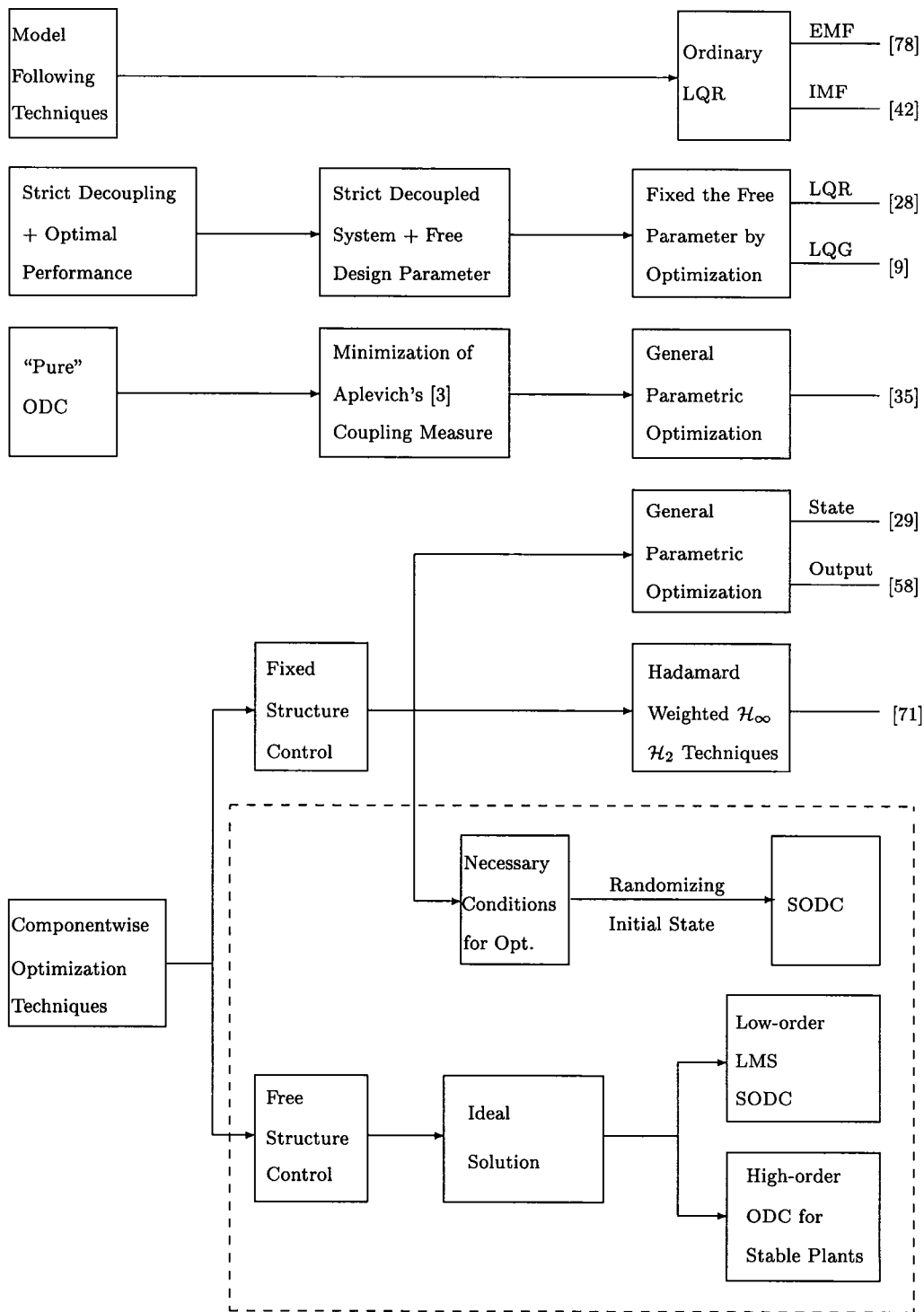
Figure 1.1: The contributions of this dissertation to the optimal decoupling theory (dashed box). Reference numbers indicate other peoples' work.

optimal and sub-optimal decoupling techniques presented in this thesis are applied to the design of an attitude control for the UH-60A helicopter in hover. Finally, in Chapter 7, conclusions are given, and some topics for future research are discussed.

Throughout this thesis, unless otherwise written, the term "system" refers to a linear time-invariant continuous-time causal system. Also when the context makes the meaning clear, we will drop the time domain variable t or the frequency domain variables s or $j\omega$. For example the time signal $x(t)$ is simply written as x, and the transfer function $G(s)$ becomes G.

Chapter 2

Literature survey on optimal decoupling control

2.1 Introduction

Many papers on (strict) decoupling control have been published, and many more on optimal control, but very few deal with optimal decoupling control. The optimal decoupling techniques which have been developed since the late 60's can be divided into the following four groups:

- Optimal model-following.

- Pure optimal decoupling.

- Strict decoupling with optimal performance.

- Componentwise optimization.

In 1968 Yore [78] was the first to use the term "Optimal Decoupling Control" (ODC). His early work is best classified as an optimal model-following technique. It is considered to be the first work on this problem. Model-following is only one example of optimal control techniques in which a very low level of coupling is achieved. In fact in almost any type of high gain MIMO control, including decentralized control, partial

decoupling is achieved. Intuitively this can be explained by breaking the $m \times m$ MIMO system into m independent SISO systems (loops) where the coupling elements are treated, in each loop, as a disturbance input. In this case "partial decoupling" is equivalent to "disturbance rejection" which is a common property of a high gain closed-loop. In modern terms, as we will see later in this survey, these optimal techniques are not considered as ODC. However, because in later works similar ideas have been used, these *optimal model-following* techniques are included in this survey.

In order to solve the optimal decoupling (or minimum coupling) problem one must define a scalar measure of coupling which will then be used as the performance criterion. Such a performance measure was suggested in 1976 by Aplevich [3], and later was used to solve the ODC problem by Hutcheson [35]. This performance measure is completely different from all other performance measures which have been used to solve the ODC problem because it is a rational one and because it has a geometric interpretation. Because this performance measure does not contain any other design requirement, it is refered here as a *pure optimal decoupling* technique.

Using strict decoupling control, there is usually a free parameter which can be used to improve the performance of the decoupled closed-loop system. In some works, e.g., [14], [13], this design degree-of-freedom is used to place the closed-loop poles. In 1972 Hetrakul and Fortmann [28] suggested replacing the pole placement technique, used by Falb and Wolovich [14], to specify the free parameter in their constant gain control, with LQR optimization. Later in 1986 Chen and Kung [9] expanded this idea for LQG systems using $\mathcal{H}_2$ optimization. These works are presented here under the title of *strict decoupling with optimal performance* techniques.

In 1975 Hirzinger [29] suggested solving the ODC problem by assigning different weights to the "coupling" components of the system output, as well as to the "main" output components and the control effort. In his work and later in Sirisena's and Choi's work [58] a dynamic compensator is included to achieve a priori partial (or strict) decoupling. In a later paper by Qi-Jun *et al.* [56] the "pre-decoupling" controller is

10

removed. In a recent work [70] Van-Diggelen and Glover used Hadamard weighting to achieve coupling reduction. In this group of works the output components or the entries of the closed-loop transfer function matrix are weighted separately. Therefore they are called *componentwise optimization* techniques.

2.2 Optimal model-following techniques

The idea of using model-following techniques has been used frequently in the attempts to solve the optimal decoupling problem. Yore [78] slightly modified the standard explicit model-following (EMF) technique in order to achieve better decoupling. However his result is not the solution to the common ODC problem as it has been formulated in later works. Although Yore did not find the complete solution, his work is considered the first serious attempt to formulate and to solve the ODC problem.

Consider a linear time-invariant **square** system with the following minimal state-space realization

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{aligned}
\tag{2.1}
$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{m \times n}$. Let r be an m-vector exogenous piecewise continuous input applied to (2.1). The desired response of (2.1) to r is given as the response y_m of the following controllable **decoupled** state-space realization

$$
\begin{aligned}
\dot{x}_m &= A_m x_m + B_m r \\
y_m &= C_m x_m
\end{aligned}
\tag{2.2}
$$

to r where $A_m \in \mathbb{R}^{n_m \times n_m}$ is asymptotically stable, $B_m \in \mathbb{R}^{n_m \times m}$, $C_m \in \mathbb{R}^{m \times n_m}$.

Define $e = y - y_m$ as shown in Figure 2.1, then the optimization problem is to find a controller which minimizes

$$
J = \int_0^\infty (e^\top Q e + u^\top R u) dt
\tag{2.3}
$$

where Q and R are symmetric positive-definite error and control weighting matrices, respectively. The solution to this problem, for $r = 0$, is obtained by augmenting the

11
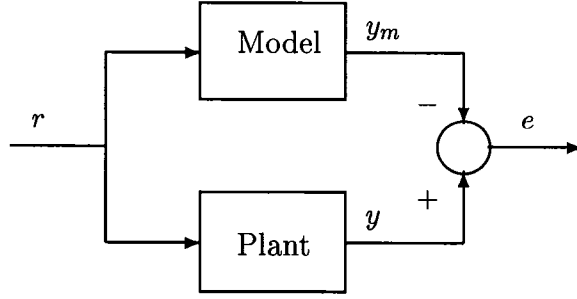
Figure 2.1: The model-following error.

system state with the model state $\bar{x} = \begin{pmatrix} x \\ x_m \end{pmatrix}$. Then the problem is reformulated as a standard LQR problem with

$$J = \int_0^\infty (\bar{x}^\top \overline{Q} \bar{x} + u^\top R u)dt \tag{2.4}$$

where $\overline{Q} = \begin{bmatrix} C^\top Q C & -C^\top Q C_m \\ -C_m^\top Q C & C_m^\top Q C_m \end{bmatrix}$, and

$$\begin{aligned} \dot{\bar{x}} &= \overline{A} \bar{x} + \overline{B} u \\ e &= \overline{C} \bar{x} \end{aligned} \tag{2.5}$$

with $\overline{A} = \begin{bmatrix} A & 0 \\ 0 & A_m \end{bmatrix}$, $\overline{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$ and $\overline{C} = [C \ -C_m]$. The optimal control is then $u = -K\bar{x} = H_m x_m - Hx$ where $K = [H \ -H_m]$ is given by $K = R^{-1}\overline{B}^\top P$ and P is the unique positive-semidefinite solution of the following algebraic Riccati equation (ARE)

$$\overline{A}^\top P + P\overline{A} - P\overline{B}R^{-1}\overline{B}P + \overline{Q} = 0. \tag{2.6}$$

Yore realized that this solution is not good enough for decoupling because it does not take the input r into account. He also realized that Q in (2.3) or $\overline{Q}$ in (2.4) does not contain any explicit "coupling cost".

Remark 2.1 *Indeed Q in (2.3) does not include weights for the coupling components of e. The diagonal entries of Q, q_{ii} are the weights of the quadratic terms e_i^2, and the*

12
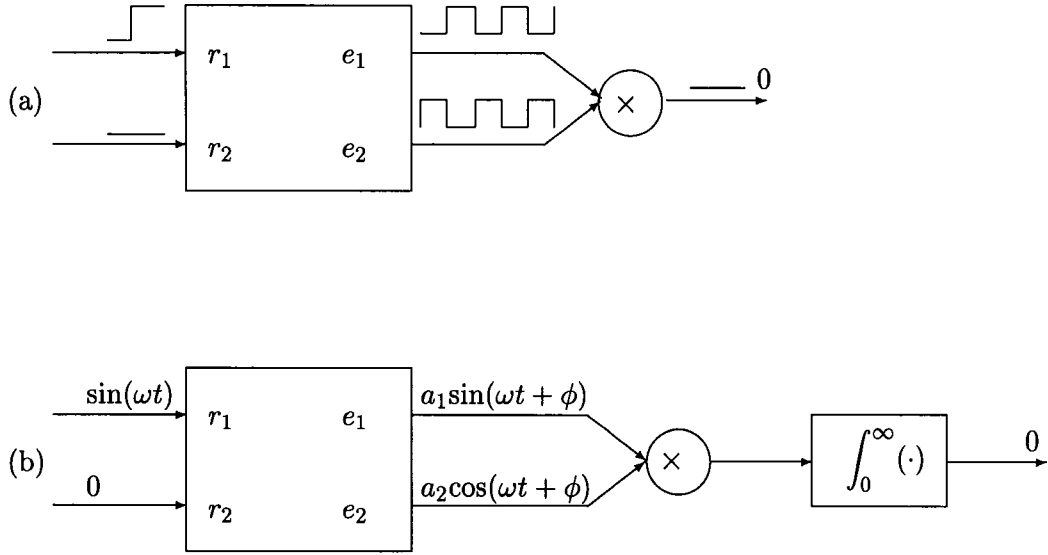
Figure 2.2: Example of a coupled nonlinear system (a) with uncorrelated outputs, and a coupled linear system (b) having outputs with zero correlation measure.

off-diagonal entries q_{ij} are the weights of the correlation terms $e_i e_j$, where neither e_i^2 nor $e_i e_j$ measure coupling. Note that a system output can be coupled but not correlated. For example consider the 2×2 systems of Figure 2.2 where, as a response to $r_1(t) \neq 0$, the errors $e_1(t) \neq 0$, $e_2(t) \neq 0$, but either $e_1(t)e_2(t) = 0 \; \forall \; t \geq 0$, or $\int_0^\infty e_1(t)e_2(t)dt = 0$.

In order to improve decoupling Yore suggested using the input signal in addition to the model state, as shown in Figure 2.3, with the following control law

$$u = G_m r + H_m x_m - H x \tag{2.7}$$

where $K = [H \; -H_m]$ is the standard LQR solution. The additional design parameter G_m is obtained by minimizing $J(K, G_m)$ of (2.3). The particular minimization can be made for a specific input (assuming a known r), or for other requirements (e.g., zero steady-state error). However, he did not find a way to include the explicit "coupling cost" in the design procedure.
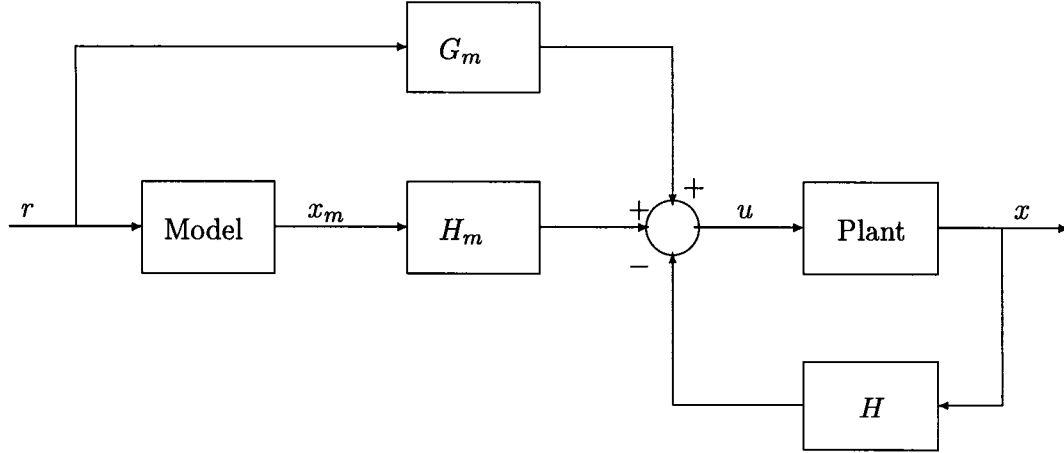
Figure 2.3: ODC using Yore's modified EMF solution.

A similar method was used by Kreindler and Rothschild [42]. They proposed to use the implicit model-following (IMF) approach. In this approach the desired model is used to derive the optimal control but it is not available for the control implementation. This approach allows the use of a constant feedforward gain, as shown in Figure 2.4.

The input reference is known and has the following state-space representation

$$\begin{aligned}\dot{x}_r &= A_r x_r \\ r &= C_r x_r\end{aligned}$$

(2.8)

where $A_r \in \mathbb{R}^{n_r \times n_r}$ is asymptotically stable and $C_r \in \mathbb{R}^{m \times n_r}$. Augmenting the model state (2.2) with the input state (2.8), gives the following autonomous model

$$\begin{aligned}\dot{z} &= \overline{A}_m z \\ y_m &= C_m z\end{aligned}$$

(2.9)

where $z = \begin{pmatrix} x_m \\ x_r \end{pmatrix}$, $\overline{A}_m = \begin{bmatrix} A_m & B_m C_r \\ 0 & A_r \end{bmatrix}$ and $\overline{C}_m = [C_m \ 0]$.

Let $\bar{x} = \begin{pmatrix} x \\ x_r \end{pmatrix}$, then

$$\begin{aligned}\dot{\bar{x}} &= \overline{A}\bar{x} + \overline{B}u \\ y &= \overline{C}\bar{x}\end{aligned}$$

(2.10)

14
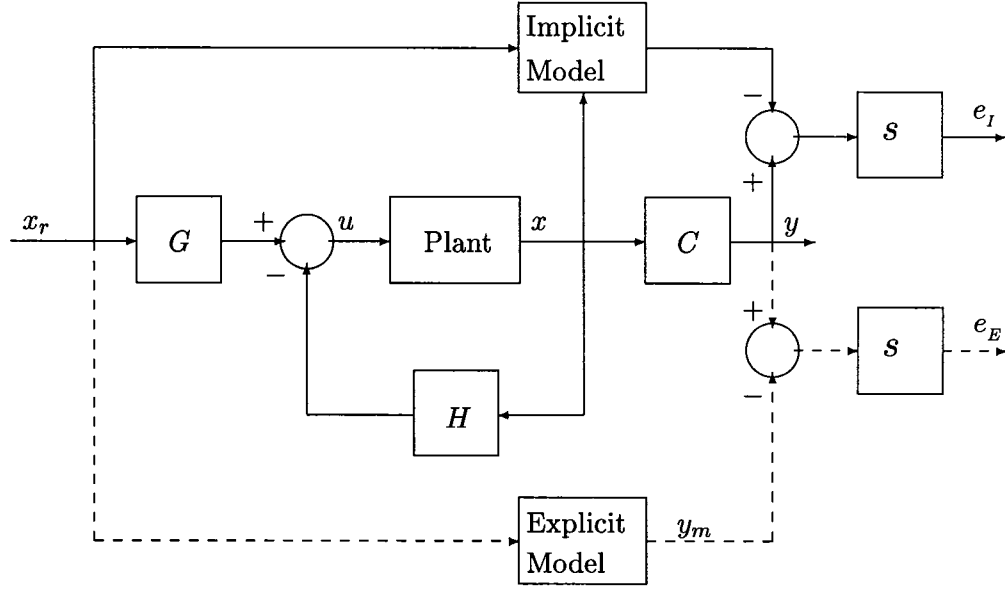
Figure 2.4: Constant gain IMF control. The explicit model is not actually implemented.

with $\overline{A} = \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix}$, $\overline{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$ and $\overline{C} = [C\ 0]$. The IMF error is defined as

$$e = \overline{C}\ \dot{\bar{x}} - \overline{C}_m \overline{A}_m \bar{x}. \tag{2.11}$$

Now e does not depend (explicitly) on x_m, but it does depend on the desired model parameters A_m and C_m. Using the same cost function as before, J in (2.3), gives

$$J = \int_0^\infty (\bar{x}^\top \overline{Q} \bar{x} + 2u^\top S \bar{x} + u^\top R u)dt \tag{2.12}$$

where $\overline{Q} = \begin{bmatrix} \Gamma^\top Q \Gamma & \Gamma^\top Q \Delta \\ \Delta^\top Q \Gamma & \Delta^\top Q \Delta \end{bmatrix}$ and $S = \begin{bmatrix} \Lambda^\top Q \Gamma & \Lambda^\top Q \Delta \end{bmatrix}$, with $\Gamma = CA - C_m A_m$, $\Delta = -C_m B_m C_r$ and $\Lambda = CB$. The optimal control which minimizes this J is

$$u = -K\bar{x} = Gx_r - Hx \tag{2.13}$$

where $K = [H\ -G] = R^{-1}\overline{B}^\top P$ and P is the unique positive-semidefinite solution of the following ARE

$$\widehat{A}^\top P + P\widehat{A} - P\overline{B}R^{-1}\overline{B}P + \widehat{Q} = 0 \tag{2.14}$$

15

where $\widehat{Q} = \overline{Q} - S^\top R^{-1} S$ and $\widehat{A} = \overline{A} - \overline{B}R^{-1}S$.

In both of these approaches, the "decoupling" results from the choice of a decoupled model. To the extent that the controllers achieve good model following the closed-loop systems are decoupled. These approaches have the disadvantages that they are indirect and give no indication as to whether a different model would produce much better decoupling.

2.3 Pure optimal decoupling technique

Matrix and vector coupling measures have been frequently used as design tools in decoupling control design. One example is Bristol's measure, the Relative Gain Array (RGA) [7]

$$\text{RGA} = [\lambda_{ij}] = [t_{ij}\tilde{t}_{ji}] \in \mathbb{R}^{m \times m} \tag{2.15}$$

where t_{ij} and $\tilde{t}_{ij}$ are the $ij^{\underline{th}}$ entries of the system steady-state transfer function gain matrix $T(0)$ and its inverse $T^{-1}(0)$, respectively. This matrix measure can be used as a pairwise design criterion for cross-feed decoupling techniques [4]. However, the RGA and other related nonscalar coupling indices such as the relative dynamic array (RDA) [74], the average relative gain array (ARGA) [19], the direct gain matrix (DGM) [38], etc. (see also [11] and [48]), are not useful for comparison and optimization. In the following we present an ODC design based on a special scalar coupling measure proposed by Aplevich [3]. This measure is derived here only for asymptotically stable systems (a slightly less general case than that developed by Aplevich). We present here the time-domain version, using a state-space representation.

Let $\mathcal{H}^{m \times m}$ be the space of all $m \times m$ matrix valued Lebesgue integrable functions. Let $F(t)$, $T(t) \in \mathcal{H}^{m \times m}$, then the inner product of $F(t)$ and $T(t)$ is defined by

$$\langle F(t), T(t) \rangle = \int_0^\infty \text{tr}\left\{ F^\top(t)T(t) \right\} dt = \int_0^\infty \sum_{i=1}^m \sum_{j=1}^m f_{ij}(t)t_{ij}(t)\, dt \tag{2.16}$$

and the generalized angle between $F(t)$ and $T(t)$ is defined by

$$\cos \theta = \frac{\langle F(t), T(t) \rangle}{\|F(t)\|\, \|T(t)\|} \tag{2.17}$$

16

where $\|F(t)\| = \langle F(t), F(t)\rangle^{1/2}$.

Let $F(t) \in \mathcal{H}^{m \times m}$, let $\mathcal{S} \subset \mathcal{H}^{m \times m}$ be a subspace of $\mathcal{H}^{m \times m}$, and let $F^S(t)$ denote the orthogonal projection of $F(t)$ onto $\mathcal{S}$, then

$$
\begin{aligned}
&F^S(t) \in \mathcal{S} && (a)\\
&F^S(t) \text{ is unique} && (b)\\
&\left\langle F(t) - F^S(t), S(t)\right\rangle = 0, \ \forall S \in \mathcal{S} && (c)\\
&\left\|F(t) - F^S(t)\right\| < \|F(t) - S(t)\| \ \forall S \in \mathcal{S} && (d)
\end{aligned}
\tag{2.18}
$$

i.e., $\left\|F(t) - F^S(t)\right\|$ is the minimum "distance" between $F(t)$ and the subspace $\mathcal{S}$.

Define $\mathcal{D} \subset \mathcal{H}^{m \times m}$ as the diagonal subspace of $\mathcal{H}^{m \times m}$

$$
\mathcal{D} = \{D(t) = [d_{ij}(t)] \in \mathcal{H}^{m \times m} \mid d_{ij}(t) = 0, \ \forall t \geq 0, \ \forall i \neq j, \ i,j = 1,2,\ldots,m\}. \tag{2.19}
$$

Then $F^D(t) = [f_{ij}^D(t)]$ is the orthogonal projection of $F(t)$ onto the diagonal subspace $\mathcal{D}$ with

$$
f_{ij}^D(t) = \begin{cases} 0 & i \neq j \\ f_{ii} & i = j \end{cases} \quad i,j = 1,2,\ldots,m \quad \begin{array}{l} \leftarrow (2.18\text{-}a) \\[6pt] \leftarrow (2.18\text{-}c). \end{array} \tag{2.20}
$$

Let $F(t) = Ce^{At}B \in \mathcal{H}^{m \times m}$ be an impulse response function of the system (2.1), with A asymptotically stable. Then a natural coupling measure for this system is the generalized angle between $F(t)$ and $F^D(t)$. In fact it is better to use, and easier to compute, the following positive measure

$$
\mu(F) = \sin^2\theta = 1 - \cos^2\theta = 1 - \left[\frac{\langle F(t), F^D(t)\rangle}{\|F(t)\|\|F^D(t)\|}\right]^2
$$

$$
= 1 - \frac{\|F^D(t)\|^2}{\|F(t)\|^2} = \frac{\|F(t) - F^D(t)\|^2}{\|F(t)\|^2} = \frac{\sum_{i=1}^m \sum_{j \neq i}^m \|f_{ij}(t)\|^2}{\sum_{i=1}^m \sum_{j=1}^m \|f_{ij}(t)\|^2}
\tag{2.21}
$$

where $\|f_{ij}(t)\|^2 = \int_0^\infty f_{ij}(t)^2 dt = \int_0^\infty (C_i e^{At} B_j)^2 dt$, with $C_i = i^{\underline{th}}$ row of C, and $B_j = j^{\underline{th}}$ column of B.

Since A is asymptotically stable we can write

$$
\|f_{ij}(t)\|^2 = B_j^\top P_i B_j = C_i Q_j C_i^\top \tag{2.22}
$$
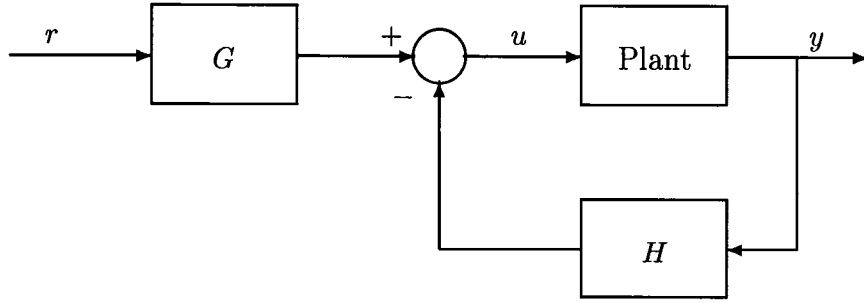
Figure 2.5: Pure ODC using constant output-feedback.

where P_i and Q_j are the unique positive semi-definite solutions to the Lyapunov equations

$$
\begin{aligned}
A^\top P_i &+ P_i A &+ C_i^\top C_i &= 0 \\
AQ_j &+ Q_j A^\top &+ B_j B_j^\top &= 0.
\end{aligned}
\tag{2.23}
$$

Using the definition for the norm of $F(t)$ we obtain

$$
\begin{aligned}
\|F(t)\|^2 &= \sum_{i=1}^{m}\sum_{j=1}^{m}\|f_{ij}(t)\|^2 = \sum_{i=1}^{m}\sum_{j=1}^{m} B_j^\top P_i B_j = \sum_{j=1}^{m} B_j^\top \left(\sum_{i=1}^{m} P_i\right) B_j \\
&= \sum_{j=1}^{m} B_j^\top P B_j = \operatorname{tr}\left(B^\top P B\right)
\end{aligned}
\tag{2.24}
$$

and similarly $\|F(t)\|^2 = \operatorname{tr}\left(CQC^\top\right)$, where $P = \sum_{i=1}^{m} P_i$ and $Q = \sum_{j=1}^{m} Q_j$.

Aplevich also derived a simple expression for $\frac{\partial}{\partial \alpha}\mu(F(\alpha))$ where α is any scalar parameter of the state-space matrices, e.g., let $\{A - BK, B, C\}$ be the closed-loop state-space matrices and let $\alpha = k_{ij}$ be the $ij^{\underline{th}}$ entry of the constant gain controller matrix K.

Three years later, in 1979, Hutcheson [35] used this measure to solve the ODC problem. He suggested using the constant output-feedback control of Figure 2.5

$$
u = Gr - Hy
\tag{2.25}
$$

where

$$
G = -[C(A - BHC)^{-1}B]^{-1}
\tag{2.26}
$$

18

thereby insuring that the closed-loop system has a zero steady-state tracking error $(e = y - r)$.

The closed-loop state-space matrices depend only on H (G is also a function of H). However since now G is a function of the closed-loop matrix $A_H = A - BHC$, we can not use Aplevich's simple expression for $\frac{\partial}{\partial \alpha} \mu(F(\alpha))$.

Remark 2.2 *Apparently we can free G as a design parameter. However since $\mu(F)$ does not contain any requirement on the "main" tracking error (e_i as a response to r_i), the steady-state tracking error may not be zero. Alternatively we can find the ODC assuming $u = -Hy$, then use (2.26) to achieve zero steady-state tracking error. This may lead to a much simpler derivative expression, but it will not give the optimal solution (i.e., it gives a zero steady-state tracking error, but a bigger coupling "angle").*

Skipping mathematical details [35], the derivative expression for the control given by (2.25) and (2.26) is

$$\frac{\partial \mu(F)}{\partial H} = \frac{2\|F^D(t)\|^2}{\|F(t)\|^4} \left[B^\top P(QC^\top - \widetilde{B}) \right] - \frac{2}{\|F(t)\|^2} \left[B^\top \sum_{i=1}^{m} P_i(Q_i C^\top - \widetilde{B}_i e_i^\top) \right] \quad (2.27)$$

where $\widetilde{B} = BG$, and e_i is an m-vector such that $e_i(j) = \{ \begin{smallmatrix} 1 & j=i \\ 0 & j \neq i \end{smallmatrix}$.

The gradient expression (2.27) can be used in a function minimization algorithm, such as the conjugate gradient algorithm, to solve the ODC problem.

Aplevich's measure $\mu(F)$ has nice mathematical, geometrical, and practical properties which make it attractive to be used as a minimization target. However using $\mu(F)$ we do not take into account some important design requirements such as:

(*i*)   Bounds on the control inputs.

(*ii*)   System "main" output performance (see Remark 2.2).

(*iii*)   Assigning different weights to the coupling components.

To meet requirement (*i*), $F(t)$ of (2.21) has to be changed to include the control u in the modified system output. This may require the definition of a modified

performance measure. Alternatively we can use constrained optimization to enforce bounds on u. In order to control the system "main" output (ii), we can try to use constant weighting matrices W_i and W_o to scale $F(t)$ (i.e., $F'(t) = W_oF(t)W_i$). However requirement (iii) which is very important for decoupling design, can not be achieved using $\mu(F)$ since $F(t)$ can not be componentwise scaled [69].

2.4 Strict decoupling with optimal performance

This approach is based on a very common idea used in various modern control techniques. Namely, characterizing a family of controllers, for a given plant, such that for any controller of this family, the closed-loop system has a certain property (e.g., stability). Then, if there is a free parameter in this characterization, it can be used to improve the closed-loop system performance (e.g., pole placement). This idea was used by Hetrakul and Fortmann [28], where the value of the free parameter is derived by optimization.

Consider the system (2.1), and define

$$
O_d = \begin{bmatrix} C_1 A^{d_1} \\ C_2 A^{d_2} \\ \vdots \\ C_m A^{d_m} \end{bmatrix} \tag{2.28}
$$

where C_j is the $j^{\underline{th}}$ row of C, and

$$
d_j = \begin{cases} \min\{i \mid C_j A^i B \neq 0\} & \text{if } C_j A^i B \neq 0 \text{ for some } i \\ n-1 & \text{if } C_j A^i B = 0 \text{ for all } i \end{cases} \quad i = 0, 1, \ldots, n-1. \tag{2.29}
$$

Let

$$
u = Gv - H_0 x \tag{2.30}
$$

where $G = (O_d B)^{-1}$, $H_0 = GO_d A$, and v is a new control input.

Theorem 2.1 *System (2.1) is strictly decoupled using (2.30) if and only if $O_d B$ is nonsingular.*

Proof: [14], [21].

$\square$

Moreover, there exists a similarity transformation $\hat{x} = Tx$ that puts the closed-loop system into the canonically decoupled (CD) form [21]

$$
\begin{aligned}
\hat{x} &= \widehat{A}\hat{x} + \widehat{B}v \\
y &= \widehat{C}\hat{x}
\end{aligned}
\tag{2.31}
$$

where $\hat{x} = \left(\hat{x}_1^\top \ \hat{x}_2^\top \ \cdots \ \hat{x}_m^\top \ \hat{x}_{m+1}^\top\right)^\top$, $\hat{x}_j \in \mathbb{R}^{n_j}$, $\sum_{j=1}^{m+1} n_j = n$, and $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ are partitioned as follows:

$$
\widehat{A} = T[A - B(O_d B)^{-1} O_d A]T^{-1} =
\begin{bmatrix}
\widehat{A}_1 & 0 & \cdots & 0 & 0 \\
0 & \widehat{A}_2 & & 0 & 0 \\
\vdots & & \ddots & & \vdots \\
0 & 0 & & \widehat{A}_m & 0 \\
\widehat{A}_1^c & \widehat{A}_2^c & \cdots & \widehat{A}_m^c & \widehat{A}_{m+1}^c
\end{bmatrix}
$$

$$
\widehat{B} = TB(O_d B)^{-1} =
\begin{bmatrix}
\widehat{b}_1 & 0 & \cdots & 0 \\
0 & \widehat{b}_2 & & 0 \\
\vdots & & \ddots & \vdots \\
0 & 0 & & \widehat{b}_m \\
\widehat{b}_1^c & \widehat{b}_2^c & \cdots & \widehat{b}_m^c
\end{bmatrix}
\tag{2.32}
$$

and

$$
\widehat{C} = CT^{-1} =
\begin{bmatrix}
\widehat{c}_1 & 0 & \cdots & 0 & 0 \\
0 & \widehat{c}_2 & & 0 & 0 \\
\vdots & & \ddots & & \vdots \\
0 & 0 & \cdots & \widehat{c}_m & 0
\end{bmatrix}.
$$

Thus the subsystems with state vectors $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_m$ are uncoupled from one another and the states $\hat{x}_{m+1}$ are all unobservable.

The class of all decoupling control laws can now be characterized using (2.30)

21
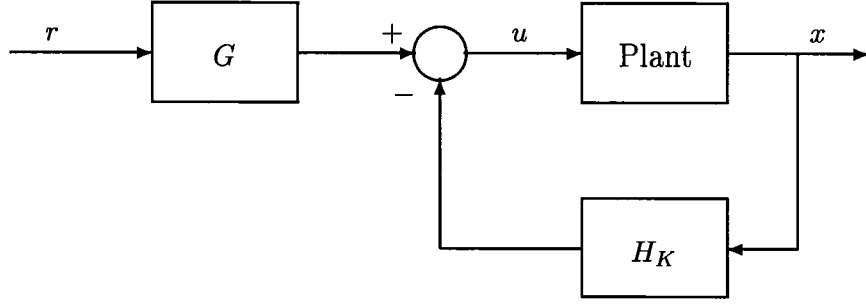
Figure 2.6: Strict decoupling with LQ optimal performance using state-feedback and constant feedforward.

and the new control input v [21]

$$v = r - \widehat{K}\hat{x}. \tag{2.33}$$

Theorem 2.2 *Consider the system (2.1), with $O_d B$ nonsingular, and u as defined in (2.30). Let A, B and C be as in (2.32), and T be such that $\hat{x} = Tx$ satisfies (2.31). Then the closed-loop system is strictly decoupled if and only if*

$$\widehat{K} = \begin{bmatrix} \widehat{k}_1 & 0 & \cdots & 0 & 0 \\ 0 & \widehat{k}_2 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & \widehat{k}_m & 0 \end{bmatrix}. \tag{2.34}$$

Proof: [21].

$\square$

Therefore the control law of Figure 2.6

$$u = Gr - H_K x \tag{2.35}$$

with $H_K = G(O_d A + \widehat{K}T)$, strictly decouples the closed-loop system for any $\widehat{K}$ of the form (2.34). Moreover, suppose $\widehat{A}^c_{m+1}$ is asymptotically stable. If $\widehat{K}$ is chosen such that $\widehat{A}_j - \hat{b}_j \hat{k}_j$ for all $j = 1, 2, \ldots, m$ are asymptotically stable then the closed-loop system is asymptotically stable.

Remark 2.3 *The structure of Figure 2.6 has been widely used in strict-decoupling theory. The choice of state-feedback is clear (guaranteed stability under controllability assumptions). The use of the actual input (instead of its state as is used for optimal tracking) allows the achievement of strict decoupling, [14] and [21].*

Hetrakul and Fortmann tried to find a way to select a feedback control law v that both decouples the closed-loop system and minimizes a quadratic cost functional of the form

$$\widetilde{J} = \int_0^\infty (\hat{x}^\top \widetilde{Q}\hat{x} + v^\top \widetilde{R}v)dt \tag{2.36}$$

where $\widetilde{Q}$ and $\widetilde{R}$ are symmetric, positive-semidefinite and positive-definite matrices, respectively.

In general this problem may have no solution. Suppose that for a particular choice of $\widetilde{Q}$ and $\widetilde{R}$ a strictly decoupling, optimal solution $\widehat{K}$ does exist. Then using the return difference inequality [1], it is easy to show ([28], Theorem 1), that there exists

$$\widehat{J} = \int_0^\infty (\hat{x}^\top \widehat{Q}\hat{x} + v^\top \widehat{R}v)dt \tag{2.37}$$

with $\widehat{R} = \text{diag}(\hat{\rho}_1, \hat{\rho}_2, \ldots, \hat{\rho}_m)$ and $\widehat{Q} = \text{block diag}(\widehat{Q}_1, \widehat{Q}_2, \ldots, \widehat{Q}_m, 0)$, which lead to precisely the same control law $\widehat{K}$. Moreover, using inverse problem results [1], all strictly decoupling optimal solutions $\widehat{K}$ of the form (2.34) may be generated by the proper choice of such diagonal $\widehat{R}$ and block-diagonal $\widehat{Q}$.

In this case we can write

$$\widehat{J} = \sum_{j=1}^{m} \int_0^\infty (\hat{x}_j^\top \widehat{Q}_j \hat{x}_j + \rho_j v_j^2)dt. \tag{2.38}$$

In other words, we can obtain $\widehat{K}$ from the optimal feedback control laws

$$v_j = \hat{k}_j \hat{x}_j, \quad j = 1, 2, \ldots, m \tag{2.39}$$

for the m individual uncoupled subsystems.

This approach is based on the algebraic methods for strict decoupling control. Thus the system has to satisfy some conditions (i.e., $O_d B$ invertible). Moreover,

suppose $O_d B$ is invertible, then the desired cost function is not of the form (2.36) because the original cost

$$J = \int_0^\infty (x^\top Q x + u^\top R u) dt. \tag{2.40}$$

implies that the corresponding cost function given by the CD state $\hat{x}$ is

$$\hat{J} = \int_0^\infty (\hat{x}^\top \hat{Q} \hat{x} + v^\top \hat{R} v - 2\hat{x}^\top \hat{S} v) dt \tag{2.41}$$

where, using (2.30) and the similarity transformation T

$$\hat{Q} = T^{-\top}(Q + H_0^\top R H_0)T^{-1}, \ \hat{R} = G^\top R G \text{ and } \hat{S} = T^{-\top} H_0^\top R G.$$

It is clear that $\hat{J}$ in (2.41) does not necessarily satisfy the conditions given in (2.37).

Moreover even if we start from (2.37) with a proper choice of $\hat{Q}$ and $\hat{R}$, such that $\widehat{K}$ does exist, then the corresponding cost function given by the original state x is

$$J = \int_0^\infty (x^\top Q x + u^\top R u + 2x^\top S u) dt \tag{2.42}$$

where $Q = T^\top \hat{Q} T + H_0^\top G^{-1} \hat{R} G H_0$, $R = G^{-T} \hat{R} G^{-1}$ and $S = H_0^\top G^{-T} \hat{R} G^{-1}$.

Hetrakul and Fortmann did not solve the general design problem, namely how to choose Q and R such that the closed-loop system is both strictly decoupled and optimal. This problem was solved later, in 1986, by Chen and Kung [9]. In fact they solved a slightly more general problem, using the Wiener-Hopf approach (frequency domain technique). They also assumed that in addition to the reference input r the system has two more external inputs, disturbance and noise, which are omitted here for simplicity.

Consider the system

$$y(s) = P(s)u(s) \tag{2.43}$$

where $P(s) = [p_{ij}(s)] \in \mathcal{R}^{m \times m}(s)$ is a real-rational, invertible matrix ($\mathcal{R}^{p \times q}(s)$ is the set of all $p \times q$ real-rational matrices in $s \in \mathbb{C}$).
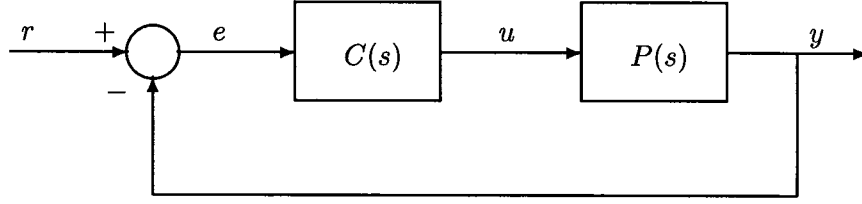
Figure 2.7: Strict decoupling with $\mathcal{H}_2$ optimal performance using one DOF controller.

Remark 2.4 *Theoretically, a proper inverse $P^{-1}(s)$ exists if and only if $P(s)$ has full rank (m) for all but a finite number of $s \in \mathbb{C}$ (including $s = \infty$). Then a strictly proper $P(s)$ leads to an improper inverse. Thus apparently this is a very restrictive assumption and this method can not be used in most cases. However using different strict decoupling parameterization techniques, a "filtered inverse" [79] can be used to create a proper filter that approximates the inverse [13]. This filter, which maintains the strict decoupling property of the closed-loop system, may not maintain the optimality property.*

Define the tracking error as

$$e(s) = r(s) - y(s) \tag{2.44}$$

where $r(s)$ is an external random input with power spectral density $\Sigma_r(s)$. Chen and Kung used the closed-loop structure of Figure 2.7 with a one degree-of-freedom controller

$$u(s) = C(s)e(s) \tag{2.45}$$

where $C(s) \in \mathbb{C}^{m \times m}$ is a real-rational matrix.

Define the sensitivity function of the closed-loop system (2.43)-(2.45) as

$$S(s) = [I + P(s)C(s)]^{-1} \tag{2.46}$$

and the transfer function, or the complementary sensitivity function, of the closed-loop system as

$$T(s) = I - S(s). \tag{2.47}$$

25

Then the closed-loop equations are

$$
\begin{aligned}
e(s) &= S(s)r(s) \\
u(s) &= P^{-1}(s)T(s)r(s) \\
y(s) &= T(s)r(s).
\end{aligned}
\tag{2.48}
$$

Consider the following cost function

$$
J = \frac{1}{2\pi j}E\left\{\int_{-j\infty}^{j\infty}[e^*(s)Q(s)e(s) + u^*(s)R(s)u(s)]ds\right\}
\tag{2.49}
$$

where $x^*(s)$ denotes the complex conjugate transpose of $x(s)$ ($x^*(s) = x(\bar{s})^\top$ where $\bar{s}$ is the complex conjugate of s), and $Q(s)$ and $R(s)$ are Hermitian positive-semidefinite and positive-definite matrices, respectively ($A(j\omega)$ is Hermitian $\Leftrightarrow A^*(j\omega) = A(j\omega)$ for all $\omega \in \mathbb{R}$, $A(j\omega)$ is positive-(semi)definite $\Leftrightarrow x^*A(j\omega)x > (\geq)\ 0$ for all $x \neq 0$ and for all $\omega \in \mathbb{R}$). The cost functional J of (2.49) can be expressed as follows

$$
J = \frac{1}{2\pi j}\text{tr}\left\{\int_{-j\infty}^{j\infty}[Q(s)\Sigma_e(s) + R(s)\Sigma_u(s)]ds\right\}
\tag{2.50}
$$

where $\Sigma_e(s) = S(s)\Sigma_r(s)S^*(s)$ and $\Sigma_u(s) = P(s)^{-1}T(s)\Sigma_r(s)T^*(s)P^{-*}(s)$ (where $P^{-*}(s) = [P^*(s)]^{-1}$).

We consider the problem of designing an optimal controller $C(s)$ by specifying adequate cost weighting matrices $Q(s)$ and $R(s)$ such that the closed-loop transfer matrix is strictly decoupled.

Definition 2.1 *$S(s)$ and $T(s)$ are said to be realizable if the closed-loop system is asymptotically stable for some choice of $C(s)$.*

Lemma 2.1 *Let $P(s) = D_l(s)^{-1}N_l(s) = N_r(s)D_r(s)^{-1}$, where the pairs $N_l(s)$, $D_l(s)$ and $N_r(s)$, $D_r(s)$ are any left and right coprime polynomial factorization of $P(s)$, respectively. Then $S(s)$ and $T(s)$ are realizable if and only if*

$$
S(s) = X(s)D_l(s)
$$

$$
and
\tag{2.51}
$$

$$
T(s) = N_r(s)Y(s)
$$

for some stable rational matrices $X(s)$ and $Y(s)$.

Proof: [9].

□

Suppose $S(s)$ and $T(s)$ are strictly decoupled (diagonal), i.e.,

$$S(s) = \text{diag}(s_1(s), \ s_2(s), \ \ldots, \ s_m(s))$$

and

(2.52)

$$T(s) = \text{diag}(t_1(s), \ t_2(s), \ \ldots, \ t_m(s)).$$

Let $D_l(s)^{-1} = [\hat{d}_{ij}]$ and $N_r(s)^{-1} = [\hat{n}_{ij}]$, then

$$S(s)D_l(s)^{-1} = \begin{bmatrix} s_1(s)\hat{d}_{11} & \cdots & s_1(s)\hat{d}_{1m} \\ \vdots & \ddots & \vdots \\ s_m(s)\hat{d}_{m1} & \cdots & s_m(s)\hat{d}_{mm} \end{bmatrix}$$

and

(2.53)

$$N_r(s)^{-1}T(s) = \begin{bmatrix} t_1(s)\hat{n}_{11} & \cdots & t_m(s)\hat{n}_{1m} \\ \vdots & \ddots & \vdots \\ t_1(s)\hat{n}_{m1} & \cdots & t_m(s)\hat{n}_{mm} \end{bmatrix}.$$

Define

$$\delta_i(s) = \prod_{j=1}^{k_i} (s - \bar{p}_{ij})^{l_{ij}}, \ i = 1, 2, \ldots, m \qquad (2.54)$$

where k_i is the number of distinct RHP poles $\bar{p}_{ij}$ in the $i\underline{th}$ row of $D_l(s)^{-1}$, and l_{ij} is the greatest multiplicity with which each pole $\bar{p}_{ij}$ appears in any element in the $i\underline{th}$ row of $D_l(s)^{-1}$. Similarly, let

$$\eta_j(s) = \prod_{i=1}^{n_j} (s - \bar{z}_{ij})^{m_{ij}}, \ j = 1, 2, \ldots, m \qquad (2.55)$$

where n_j is the number of distinct RHP poles $\bar{z}_{ij}$ in the $j\underline{th}$ column of $N_r(s)^{-1}$, and m_{ij} is the greatest multiplicity with which each pole $\bar{z}_{ij}$ appears in any element in the $j\underline{th}$ column of $N_r(s)^{-1}$.

Definition 2.2 $S(s)$ and $T(s)$ are said to be diagonally realizable if they are realizable and diagonal of the form (2.52).

27

Lemma 2.2 $S(s)$ and $T(s)$ are diagonally realizable if and only if they are diagonal, and for $j = 1, 2, \ldots, m$

1. $s_j(s)$ is analytic in $Re(s) \geq 0$.

2. The numerator of $s_j(s)$ must contain $\delta_j(s)$.

3. The numerator of $t_j(s)$ must contain $\eta_j(s)$.

4. $s_j(s) \neq 0$.

Proof: The proof follows directly from Lemma 2.1 and (2.53).

$\square$

From Lemma 2.2 we conclude that any RHP pole-zero cancellation in $P(s)C(s)$ can be avoided. Lemma 2.2 gives also a full characterization of the class of all strictly decoupling controllers, because if $S(s)$ and $T(s)$ are diagonally realizable then we can obtain $C(s)$, by $C(s) = P(s)^{-1}T(s)S(s)^{-1}$ (see Remark 2.4).

Remark 2.5 From Lemma 2.2, we see that if there exists a polynomial pair $(\delta_j(s), \eta_j(s))$ that is not coprime, then it is impossible to strictly decouple the system. This may be avoided by using $\mathcal{RH}_\infty$ coprime factorization techniques with a two degree-of-freedom control law [13].

Now that we have a full characterization of the strictly decoupling controllers, we can discuss the LQG optimization problem. Let the weighting matrices $Q(s)$ and $R(s)$ of (2.49) be of the following form

$$
\begin{aligned}
Q(s) &= \text{diag}(q_1(s),\ q_2(s),\ \ldots, q_m(s)) \\
R(s) &= R_1^*(s)R_1(s),\ R_1(s) \text{ is invertible}
\end{aligned}
\tag{2.56}
$$

with $R_1^{-1}(s) = \widehat{R}_1(s) = [\hat{r}_{ij}]$, where the $j^{\underline{th}}$ column of $R_1^{-1}(s)$ satisfies the following

$$
\begin{pmatrix}
\hat{r}_{i\,j}(s) \\
\vdots \\
\hat{r}_{j-1\,j}(s) \\
\hat{r}_{j+1\,j}(s) \\
\vdots \\
\hat{r}_{m\,j}(s)
\end{pmatrix}
$$

$$
= -\hat{r}_{jj}(s)
\begin{bmatrix}
p_{1\,1}(s) & \cdots & p_{1\,j-1}(s) & p_{1\,j+1}(s) & \cdots & p_{1\,m}(s) \\
\vdots & & & & & \vdots \\
p_{j-1\,1}(s) & & & & & p_{j-1\,1\,m}(s) \\
p_{j+1\,1}(s) & & & & & p_{j+1\,m}(s) \\
\vdots & & & & & \vdots \\
p_{m\,1}(s) & \cdots & p_{m\,j-1}(s) & p_{m\,j+1}(s) & \cdots & p_{m\,m}(s)
\end{bmatrix}^{-1}
\begin{pmatrix}
p_{i\,j}(s) \\
\vdots \\
p_{j-1\,j}(s) \\
p_{j+1\,j}(s) \\
\vdots \\
p_{m\,j}(s)
\end{pmatrix}
$$

$$
= -\hat{r}_{jj}(s)
\begin{pmatrix}
\tilde{p}_{i\,j}(s) \\
\vdots \\
\tilde{p}_{j-1\,j}(s) \\
\tilde{p}_{j+1\,j}(s) \\
\vdots \\
\tilde{p}_{m\,j}(s)
\end{pmatrix}, \quad i,j = 1,2,\ldots,m, \ i \neq j.
$$

$$(2.57)$$

Lemma 2.3 *If $P(s)$ is invertible, and $R_1(s)$ is defined in (2.57), then*

$$
P(s)R_1(s)^{-1} = \mathrm{diag}(\tilde{p}_1(s), \ \tilde{p}_2(s), \ \ldots, \ \tilde{p}_m(s))
$$

where

$$
\tilde{p}_i(s) = \hat{r}_{ii}(s)\left[p_{ii}(s) - \sum_{j=1}^{m} p_{ij}(s)\tilde{p}_{ji}(s) \right] = \hat{r}_{ii}(s)l_i(s), \ i = 1,2,\ldots,m.
$$

Proof: [9].

$\square$

The only free parameters in $R_1(s)$ are the $\hat{r}_{ii}(s)$ which have to be chosen under the constraint that $\tilde{p}_i(s)$ should contain $\eta_i(s)$ in its numerator and $\delta_i(s)$ in its denominator. From Lemma 2.3, using $R_1(s)$ in (2.57) allows us to diagonalize the plant. Therefore if we choose $C(s)$ to be

$$
C(s) = R_1(s)^{-1}\widetilde{C}(s) \tag{2.58}
$$

then $\widetilde{C}(s)$ can be derived as a decentralized controller. Moreover substituting $Q(s)$ and $R(s)$ of (2.56) into the cost function (2.50), we get

$$
\begin{aligned}
J &= \sum_{i=1}^{m} J_i \\
&= \frac{1}{2\pi j} \sum_{i=1}^{m} \int_{-j\infty}^{j\infty} \sigma_{r_{ii}}(s)[q_i(s)s_i(s)s_i^*(s) + \frac{1}{\tilde{p}_i(s)\tilde{p}_i^*(s)}t_i(s)t_i^*(s)]ds \qquad (2.59) \\
&= \frac{1}{2\pi j} \sum_{i=1}^{m} \int_{-j\infty}^{j\infty} [q_i(s)\Sigma_{\tilde{e}_i\tilde{e}_i}(s) + \Sigma_{\tilde{u}_i\tilde{u}_i}(s)]ds
\end{aligned}
$$

where $\sigma_{r_{ii}}(s)$ are the diagonal entries of $\Sigma_r(s)$. That is, the original cost function J in (2.49) is "decoupled" into m separated cost functions J_i, where each of them depends only on $s_i(s)$ (recall that $t_i(s) = 1 - s_i(s)$).

In the last step we use the Wiener-Hopf technique to specify each $s_i(s)$ independently, as stated in the following theorem

Theorem 2.3 *If $\hat{s}_i(s)$ minimizes the cost function J_i in (2.49), then*

$$
\hat{s}_i(s) = \frac{k_i + \{\frac{\sigma_{r_{ii}}(s)x_i(s)x_i^*(s)}{w_i^*(s)\tilde{p}_i(s)\tilde{p}_i^*(s)}\}_+}{w_i(s)\sigma_{r_{ii}}(s)}
$$

where $x_i(s) = \delta_i(s)\eta_i(s)$, the spectral factorization $w_i(s)$ is given by

$$
w_i(s)w_i^*(s) = x_i(s)x_i^*(s)\left[q_i(s) + \frac{1}{\tilde{p}_i(s)\tilde{p}_i^*(s)}\right]
$$

and $\{A(s)\}_+$ is the part associated with all the finite poles in $Re(s) < 0$ of the partial fraction expansion of the matrix $A(s)$. The remaining design parameter $k_i(s)$ is uniquely determined such that the diagonal realization constraint of Lemma 2.2 is satisfied and such that J_i is finite (i.e., the relative order of the two terms in J_i in (2.59), as $|s| \to \infty$ is at least -2 [80]).

Proof: [9].

$\square$

Theoretically speaking this approach of simultaneous design for strict decoupling and optimal performance is the best we can do. However, strict decoupling

design (when it is applicable) is based on a certain algebraic method in which a perfect knowledge of the system model is assumed. Therefore this design may be too sensitive, in the sense of destroying the strict decoupling, to any plant and controller perturbations. Moreover in this approach the control weight R is completely specified by the need to achieve strict decoupling (first its off diagonal entries are specified in (2.57), then the diagonal entries are chosen to satisfy the diagonal realizability requirement). Therefore we can not really put bounds on the control inputs as may be required. Another disadvantage of the Chen and Kung approach is that the resultant controller may be of high order (see numerical example in [9]).

The above disadvantages in addition to the plant invertibility requirement (see Remark 2.4), reduce the utility of this method. Using a better characterization of the strictly decoupling controllers may lead to better and more practical results.

2.5 Componentwise optimization techniques

The strict decoupling problem has been completely solved using modern coprime factorization techniques [13]. However using this method, the order of the resultant controller is at least the order of the plant. High order controllers are also obtained using classical strict decoupling techniques, such as [14] and [21]. The question, what is the minimal required size of a controller for strict decoupling, has been studied and algebraic criteria have been developed (e.g., the rank criterion [21]). On the other hand, in real life, the controller order can sometimes be constrained. In this case we would like to solve the following problem, given a controller order (or even more generally, a controller structure), find the controller of this order (structure) which minimizes the coupling level of the system.

The first to solve such a problem was Hirzinger [29]. His work is best known for introducing the concept of virtual model-following (VMF). This is an alternative concept to the IMF used by Kreindler and Rothschild [42]. However, although he did not introduce it explicitly, his use of componentwise optimization is a more important
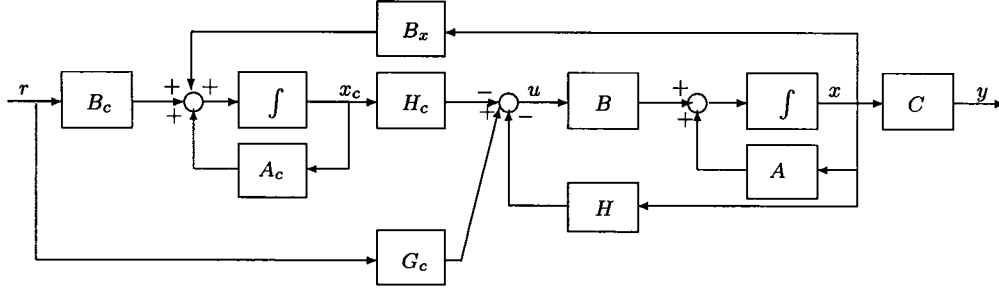
Figure 2.8: ODC using Hirzinger's VMF solution.

contribution to the theory of optimal decoupling.

Consider the system of Figure 2.8, with the following control law

$$u = G_c r - H_c x_c - H x \qquad (2.60)$$

where the plant state space equations are given by (2.1), r is an exogenous input and x_c is the state of the dynamic controller given by

$$\dot{x}_c = A_c x_c + B_c r + B_x x \qquad (2.61)$$

where $A_c \in \mathbb{R}^{n_c \times n_c}$, $B_c \in \mathbb{R}^{n_c \times m}$ and $B_x \in \mathbb{R}^{n_c \times n}$.

Remark 2.6 *Equations (2.60) and (2.61) with $B_x = 0$ define the standard "state feedback + dynamic precompensator" control structure (e.g., model-following [78]). The motivation to use $B_x \neq 0$ is given in Remark 2.8 below.*

The minimal order, strict decoupling problem can be characterized as follows. If the system can be strictly decoupled, then there exists a minimal order n_c^* such that the closed-loop system (2.1), (2.60) and (2.61) has a transfer function of the form

$$T(s) = \text{diag}(t_1(s), \ t_2(s), \ \ldots, \ t_m(s)) = \text{diag}\left(\frac{n_1(s)}{d_1(s)}, \ \frac{n_2(s)}{d_2(s)}, \ \ldots, \ \frac{n_m(s)}{d_m(s)}\right) \qquad (2.62)$$

where $d_j(s)$ and $n_j(s)$ are coprime polynomials of degree δ_j and η_j, respectively, $j = 1, 2, \ldots, m$. The minimal order n_c^* as well as the numbers δ_j and η_j are not known a priori (i.e., they depend on the plant parameters).

32

Using the strict decoupling control structure, the optimal decoupling problem can be defined as follows. Let (2.60), (2.61) be a compensator of a certain order n_c. Let $\tilde{y}(s)$ be the desired decoupled output, i.e., $\tilde{y}(s) = \tilde{T}(s)r(s)$, where $\tilde{T}(s)$ is the desired diagonal transfer function defined similarly to $T(s)$ in (2.62). Define the following cost function

$$J = \sum_{i=1}^{m} J_i \tag{2.63}$$

where each of the individual costs J_i is given by

$$J_i = \int_0^{\infty} \left[(y_i - \tilde{y}_i)^2 q_{ii} + \sum_{j \neq i}^{m} y_j^2 q_{ij} + u^{\top} R_i u \right] dt \tag{2.64}$$

where for all $i, j = 1, 2, \ldots, m$, R_i are positive definite matrices, and $q_{ij} > 0$. Note that i is the input index, i.e., each of these costs arises from a test signal r_i applied at the ith exogenous input when all the other inputs are zero. Choose the n_c-order controller parameters A_c, B_c, G_c, H_c, B_x, and H, such that J in (2.63) is minimal.

Remark 2.7 *As we will see later, it is preferable to use a double subindex to indicate the componentwise outputs y_i (y_i is not the ith output, but it is the output vector arising from r_i). The need of double subindices becomes essential when we consider arbitrary exogenous inputs instead of "test signals".*

Let us represent the ideal transfer functions $\tilde{t}_i(s)$ by state-space models of order ν_i

$$\begin{aligned} \dot{\tilde{x}}_i &= \tilde{A}_i \tilde{x}_i + \tilde{b}_i r_i \\ \tilde{y}_i &= \tilde{c}_i \tilde{x}_i \end{aligned} \tag{2.65}$$

and the ideal transfer function $\tilde{T}(s)$ by the m-augmented model

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}r \\ \tilde{y} &= \tilde{C}\tilde{x} \end{aligned} \tag{2.66}$$

where $\tilde{A} = \text{block diag}(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_m)$, $\tilde{B} = \text{block diag}(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m)$ and $\tilde{C} = \text{block diag}(\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_m)$. The design is done in three steps:

Step 1: Steady-state analysis

To obtain finite J_i for all $i = 1, 2, \ldots, m$, we require

$$y_{s_i} - \tilde{y}_{s_i} = 0 \tag{2.67}$$

where y_{s_i}, and $\tilde{y}_{s_i}$ are the steady-state output, and ideal steady-state output caused by the $i^{\underline{th}}$ exogenous input r_i when all the other inputs are zero, respectively. Using simple steady-state analysis, let $\dot{x}_{s_i} = 0$, then

$$\begin{pmatrix} 0 \\ y_{s_i} \end{pmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{pmatrix} x_{s_i} \\ u_{s_i} \end{pmatrix} = S \begin{pmatrix} x_{s_i} \\ u_{s_i} \end{pmatrix} \tag{2.68}$$

where x_{s_i}, and u_{s_i} are the steady-state, state and control caused by the $i^{\underline{th}}$ input r_i when all the other inputs are zero, respectively. If S is invertible then, from (2.67) and (2.68), $\begin{pmatrix} x_{s_i} \\ u_{s_i} \end{pmatrix} = S^{-1} \begin{pmatrix} 0 \\ \tilde{y}_{s_i} \end{pmatrix}$ (note that nonsingularity of S is a classical requirement for static decoupling [76]).

Using steady-state analysis ($\dot{x}_{s_i} = 0$ and $\dot{x}_{cs_i} = 0$) for (2.60) and (2.61), we can also write

$$\begin{aligned} x_{s_i} &= X_s r_{s_i} & (a) \\ x_{cs_i} &= X_{cs} r_{s_i} & (b) \\ u_{s_i} &= G_c r_{s_i} - H x_{s_i} - H_c x_{c_i} & (c) \\ 0 &= B_c r_{s_i} + B_x x_{s_i} - A_c x_{cs_i} & (d) \end{aligned} \tag{2.69}$$

where X_s and X_{cs} are constant matrices. By assumption r_i is known and so is r_{s_i}. Moreover r_{s_i} has the following form

$$r_{s_i} = \alpha_i e_i \tag{2.70}$$

where $\alpha_i \in \mathbb{R}$ and e_i is an m-vector such that $e_i(j) = \{^{1 \; j=i}_{0 \; j\neq i}$. Substituting (2.70) into (2.69-a) yields

$$X_s = \begin{bmatrix} \dfrac{x_{s1}}{\alpha_1} & \dfrac{x_{s2}}{\alpha_2} & \cdots & \dfrac{x_{sm}}{\alpha_m} \end{bmatrix}. \tag{2.71}$$

Finally using (2.68) and (2.69) we have two new algebraic equations

$$\begin{aligned} BG_c &= BH_c X_{cs} - (A - BH)X_s & (a) \\ B_c &= -A_c X_{cs} - B_x X_s & (b) \end{aligned} \tag{2.72}$$

34

with one additional unknown X_{cs}. This unknown as well as H, H_c, B_x and A_c are the optimization unknowns, where G_c and B_c are obtained from (2.72).

Step 2: Virtual model-following

First we make the following state augmentation. Let $\hat{x} = \begin{pmatrix} x \\ x_c \end{pmatrix}$ and $\hat{u} = -\begin{pmatrix} Hx + H_c x_c \\ B_x x + A_c x_c \end{pmatrix}$, then

$$\dot{\hat{x}} = \widehat{A}\hat{x} + \widehat{B}_1\hat{u} + \widehat{B}_2 r \tag{2.73}$$

where $\widehat{A} = $ block $\text{diag}(A, \ 0)$, $\widehat{B}_1 = $ block $\text{diag}(B, \ I)$ and $\widehat{B}_2 = \begin{bmatrix} BG_c \\ B_c \end{bmatrix}$. The closed-loop system (2.1), (2.60) and (2.61) can now be written as (2.73) with

$$\hat{u} = -\widehat{K}\hat{x} = -\begin{bmatrix} H & H_c \\ B_x & A_c \end{bmatrix}\hat{x}. \tag{2.74}$$

Remark 2.8 *Using the augmented state equation (2.73), the choice of the control structure given by (2.60) and (2.61) becomes clear. Otherwise using ordinary control with $B_x = 0$ implies that $\widehat{K}$ is partly specified. Thus it can not be a free optimization parameter.*

Let $\bar{x} = \begin{pmatrix} \hat{x} \\ \tilde{x} \end{pmatrix}$, then

$$\dot{\bar{x}} = \overline{A}\bar{x} + \overline{B}_1\hat{u} + \overline{B}_2 r \tag{2.75}$$

where $\overline{A} = $ block $\text{diag}(\widehat{A}, \ \tilde{A})$, $\overline{B}_1 = \begin{bmatrix} \widehat{B}_1 \\ 0 \end{bmatrix}$ and $\overline{B}_2 = \begin{bmatrix} \widehat{B}_2 \\ \tilde{B} \end{bmatrix}$. Using the VMF approach only part of the augmented state (the "virtual" state) is available for feedback, $\hat{x} = [I \ 0]\bar{x} = \overline{C}\bar{x}$ then

$$\hat{u} = -\widehat{K}\overline{C}\bar{x}. \tag{2.76}$$

Step 3: Computation of the optimal parameters

Summarizing the results of the previous two steps, we want to minimize the following cost function

$$J = \sum_{i=1}^{m} J_i = \sum_{i=1}^{m} \int_0^\infty (\bar{x}^\top \overline{Q}_i \bar{x} + \hat{u}^\top \widehat{R}_i \hat{u}) dt \tag{2.77}$$

where $\overline{Q}_i$ and $\widehat{R}_i$ are given by the coefficients in (2.64) and the parameters of the state space equation of the augmented state $\bar{x}$

$$
\begin{aligned}
\dot{\bar{x}} &= \overline{A}\bar{x} + \overline{B}_1\hat{u} \\
\hat{u} &= -\widehat{KC}\bar{x}.
\end{aligned}
\tag{2.78}
$$

This problem can be solved numerically to find a local minimum. Hirzinger suggested using the conjugate gradient method, where the gradient of J, with respect to the optimization unknowns, can be easily derived. Because the solution is not global (i.e., it depends on the initial state), we have first to specify the initial augmented state $\bar{x}_{0_i}$, which depends on the test signal r and the desired output $\tilde{y}$, so that $\bar{x}_{0_i} = (x_{0_i}^\top \; x_{c0_i}^\top \; 0^\top \; \cdots \; \tilde{x}_{0_i}^\top \; \cdots \; 0^\top)^\top$.

Remark 2.9 *Hirzinger assumed only two possible input types, step and impulse. However this method can be easily expanded to include any input which is an output of a linear generator of the form*

$$
\begin{aligned}
\dot{x}_r &= A_r x_r \\
r &= C_r x_r
\end{aligned}
\tag{2.79}
$$

where $x_r(0) = x_{r0}$, by augmenting $\bar{x}$ with the input state x_r.

The initial state of the plant x_0 is given, $\tilde{x}_0$ is specified by the desired output, and the initial state of the compensator x_{c0} becomes a part of the optimal solution. Note that for any input, x_{c0} depends on some optimization unknowns (e.g., for step inputs x_{c0} depends on X_{cs}), thus it can not be arbitrary chosen.

Hirzinger [29] presented an alternative solution to the minimal order compensator decoupling problem, which until then had been solved (strictly) using algebraic methods. Instead of using the strict decoupling rank criterion [21], he suggested use of a measure based on the cost function (2.63). Then starting with a low order compensator (say $n_c = 0$), we can iteratively compute the optimal n_c-order compensator. In each iteration this measure is compared to a given criterion, if it is bigger, then the compensator order n_c is increased by 1.

Figure 2.9: ODC using output-feedback and dynamic precompensator.

A slightly different approach was presented two years later by Sirisena and Choi [58]. Actually they solved the following problem. Given a system which can be strictly decoupled by state feedback, find the minimum order compensator such that this system is strictly decoupled by output feedback. Theoretically, this problem is quite different from Hirzinger's, however from the optimal decoupling point of view they are very similar.

Instead of state-feedback, Sirisena and Choi used the output-feedback control of Figure 2.9, where

$$
\begin{aligned}
u &= G_c r - H_c x_c - H y \\
\dot{x}_c &= A_c x_c + B_c r + B_y y.
\end{aligned}
\tag{2.80}
$$

Here G_c is pre-specified as

$$
G_c = (O_d B)^{-1} G_0
\tag{2.81}
$$

where O_d is defined in (2.28), $O_d B$ is invertible and $G_0 = \text{diag}(g_{01}, g_{02}, \ldots, g_{0m})$ is chosen to satisfy the steady-state requirement. They also characterized the strict decoupling output gain matrix H in a similar way to the characterization of the state feedback in (2.35), $H = H_K = (O_d B)^{-1}(P + K)$, where P depends only on the plant and K is the new optimization unknown.

They also used a componentwise cost function to be minimized, of the form

$$
J = \sum_{j=1}^{m} \sum_{\substack{i=1 \\ i \neq j}}^{m} \lim_{t \to \infty} E\{y_{ij}(t)^2\}
\tag{2.82}
$$

37

where $y_{ij}(t)$ is the ith output of the system when the system inputs are given by

$$r_j = \begin{cases} r_i & j = i, \quad E\{r_i(t)r_i(\tau)\} = \delta(t - \tau) \\ 0 & j \neq i. \end{cases} \tag{2.83}$$

Remark 2.10 *Using Parseval's theorem, it is easy to show that the cost function J in (2.82) and (2.83), is exactly the numerator of Aplevich's coupling measure $\mu(F)$ (2.21). Thus this approach can be considered also to be a "pure optimal decoupling" one.*

The remainder of their work is the same as in Hirzinger's [29]. Using Sirisena's and Choi's approach there is a good chance to get very high control amplitudes because of the lack of a control term in the cost function J. Hirzinger simply solved this problem by adding weighted quadratic control terms to the cost function. Moreover his cost function also included the cost of the "main diagonal performance" (i.e., y_{ii}).

In [56], Qi-Jun *et al.* tried to eliminate the need for numerical optimization, and to obtain alternative suboptimal solutions based on standard LQR results. A common way to derive a sub-optimal solution is to make additional assumption(s) such that under these assumption(s) the solution is optimal. Qi-Jun *et al.* [56] made a very strong and wrong assumption (see Remark 2.11 below) which made their solution very simple. However their work is still important, because they used a more general controller structure without any pre-decoupling assumption, which leads to a "standard" componentwise LQR formulation.

The controller, described in their paper, is a digital one, model-following based, and in addition it includes some extra states for integration to improve overall performance [56]. Moreover they used an incremental cost function which leads to a relatively complicated control law. However using the proper augmentation and omitting the disturbance input, (the analog equivalent of) their problem can be formulated as follows. Find the optimal control law of the form

$$u = Gr - Hx. \tag{2.84}$$

38

which minimizes the following componentwise cost function

$$J = \sum_{j=1}^{m} J_j \tag{2.85}$$

where each J_j arises by applying the exogenous input r_j only (all other exogenous inputs are zero) is given by

$$J_j = \int_0^\infty (x^\top Q_j x + u^\top R_j u) dt \tag{2.86}$$

where for all $j = 1, 2, \ldots, m$, $Q_j \geq 0$ and $R_j > 0$ and x is the augmented state.

Qi-Jun *et al.* used a standard LQ tracking result, that the optimal complete state feedback gain is independent of the initial conditions. Under this assumption the overall cost function can be obtained by simply combining all the J_j of (2.86)

$$J = \sum_{j=1}^{m} J_j = \sum_{j=1}^{m} \int_0^\infty (x^\top Q_j x + u^\top R_j u) dt = \int_0^\infty (x^\top Q x + u^\top R u) dt \tag{2.87}$$

where $Q = \sum_{j=1}^{m} Q_j$, and $R = \sum_{j=1}^{m} R_j$.

Remark 2.11 *This standard LQ result is true only for a standard cost function, but not for the cost function of the form (2.85). Because of the summation in (2.85), the optimal solution for this cost function depends on the initial conditions even if we are using complete state feedback. For a detailed discussion of this issue see Section 4.3.3.*

Under the (wrong) assumption that the optimal (feedback) control does not depend on the initial state (exogenous input), the optimal H can be easily derived using a standard ARE solution. The other part of the optimal control law G can be obtained by finding its jth column G_j, where for any $j = 1, 2, \ldots, m$, G_j is the optimal SISO gain which minimizes $J_j(H, G_j)$ for a given input r_j.

In the last and most recent work in this category [70], Van-Diggelen and Glover suggested to use Hadamard (or element-by-element) weighting to achieve robust decoupling.
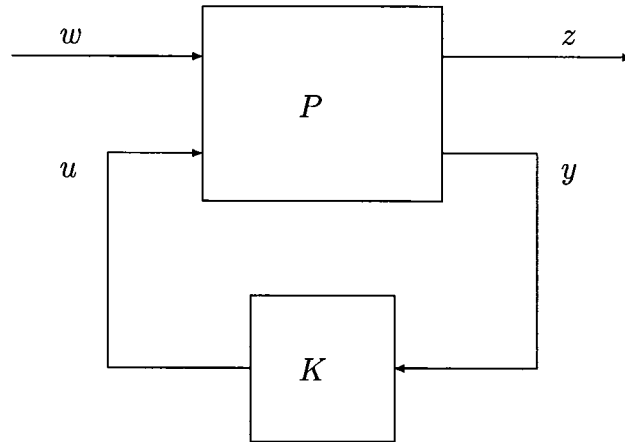
Figure 2.10: Standard block diagram for the Hadamard weighted control problems.

Remark 2.12 *The use of "robust decoupling" in the title of [70] is a little misleading. To be more precise, using only the proposed Hadamard weighted design procedure Van-Diggelen and Glover showed that it is possible to get a trade-off between robust stability and optimal decoupling. In order to achieve robust decoupling (performance) more work is required. In fact they showed ([70], Section 6.4) that the robustness of the coupling response (as well as the main response) is improved substantially by using a μ-synthesis method.*

The design procedure presented in [70] is based on their solution to the, so called, "Hadamard weighted $\mathcal{H}_\infty$-Frobenius" problem presented in [69,70,71,72]. In the following, we will show how they reduced this problem to a standard $\mathcal{H}_\infty$ control problem.

Consider the system described by the block diagram of Figure 2.10 where $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$, $z \in \mathbb{R}^{p_1}$, $y \in \mathbb{R}^{p_2}$, $w \in \mathbb{R}^{m_1}$ and $u \in \mathbb{R}^{m_2}$. Let the transfer function from w to z be given by the following linear fractional transformation (LFT)

$$T_{zw} = \mathcal{F}_l(P, K) = P_{11} + P_{12}K\left(I - P_{22}K\right)^{-1}P_{21}. \tag{2.88}$$

Let $\mathcal{K}$ denote the set of all internally-stabilizing controllers for the system of

40

Figure 2.10. Then the Hadamard weighted $\mathcal{H}_\infty$-Frobenius problem is: find all $K \in \mathcal{K}$ such that

$$\|W \circ \mathcal{F}_l(P, K)\|_{\infty F} < \gamma$$

where W is the (frequency dependend) weighting matrix, $\circ$ denotes the Hadamard product (let $A = [a_{ij}] \in \mathcal{R}^{n \times m}(s)$ and $B = [b_{ij}] \in \mathcal{R}^{n \times m}(s)$, then $A \circ B = [a_{ij} b_{ij}] \in \mathcal{R}^{n \times m}(s)$), and the $\mathcal{H}_\infty$-Frobenius norm is defined as

$$\|A(j\omega)\|_{\infty F} = \sup_\omega \|A(j\omega)\|_F = \sup_\omega \left(\operatorname{tr} \left\{ A^*(j\omega) A(j\omega) \right\} \right)^{1/2}. \tag{2.89}$$

In order to reduce the Hadamard weighted problem to a standard $\mathcal{H}_\infty$ problem, Van-Diggelen and Glover used the vec operator (see definition below). The vec operator is not an isometry on $\mathcal{H}_\infty$ with the $\mathcal{H}_\infty$ norm, but it is an isometry on $\mathcal{H}_\infty$ with the $\mathcal{H}_\infty$-Frobenius norm. Therefore they used the $\mathcal{H}_\infty$-Frobenius norm instead of the standard $\mathcal{H}_\infty$ norm.

Remark 2.13 *Because* $\|\bullet\|_\infty \leq \|\bullet\|_{\infty F}$, *using this technique to achieve robust stability for a given* γ *[70], may lead to a conservative design.*

Remark 2.14 *Considering only the optimal decoupling requirement (for $\mathcal{L}_2$ inputs) and using the formulation given in Chapter 3, the $\mathcal{H}_\infty$-ODC problem can be written as a Hadamard weighted $\mathcal{H}_\infty$-Frobenius problem (see Section 3.6). That is, the $\mathcal{H}_\infty$-Frobenius norm is indeed the right norm to use for $\mathcal{H}_\infty$-ODC problems.*

First we start with the following standard Youla parameterization (2.5) [23]. Let $NM^{-1} = \widetilde{M}^{-1} \widetilde{N}$ be a right (r.c.f) and left coprime factorization (l.c.f) of P_{22}, and let $U_0 V_0^{-1} = \widetilde{V}_0^{-1} \widetilde{U}_0$ be a r.c.f and l.c.f of $K_0 \in \mathcal{K}$. Then all $K \in \mathcal{K}$ are given by

$$K = K_0 + \widetilde{V}_0^{-1} Q \left(I + V_0^{-1} N Q \right)^{-1} V_0^{-1} \tag{2.90}$$

where $Q \in \mathcal{RH}_\infty$, and $\mathcal{F}_l(P, K)$ can be written as

$$F_l(P, K) = R + UQV \tag{2.91}$$

where $R = P_{11} + P_{12}U_0\widetilde{M}P_{21}$, $U = P_{12}M$ and $V = \widetilde{M}P_{21}$.

To proceed we need to define the following operators:

Definition 2.3 *Given a vector $a \in \mathcal{R}^m(s)$ ($\mathcal{R}^m(s)$ is the set of all m real-rational vectors in $s \in \mathbb{C}$) we define*

$$\text{diag}(a) = \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_m \end{bmatrix} \in \mathcal{R}^{m \times m}(s).$$

Definition 2.4 *Given a matrix $A \in \mathcal{R}^{n \times m}(s)$ we define*

$$\text{vec}\{A\} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{pmatrix} \in \mathcal{R}^{nm}(s)$$

where A_j is the $j^{\underline{th}}$ column of A.

Definition 2.5 *The Kronecker product of $A = [a_{ij}] \in \mathcal{R}^{n \times m}(s)$ and $B \in \mathcal{R}^{p \times q}(s)$ is denoted $A \otimes B$ and is given by*

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{bmatrix} \in \mathcal{R}^{np \times mq}(s).$$

The above operators have some useful properties

$$\begin{aligned} \text{vec}\{A \circ B\} &= \text{diag}\left(\text{vec}\{A\}\right)\text{vec}\{B\} \quad &(a) \\ \text{vec}\{AXB\} &= \left(B^\top \otimes A\right)\text{vec}\{X\} \quad &(b) \end{aligned} \tag{2.92}$$

The following equalities show the equivalence of the Hadamard weighted $\mathcal{H}_\infty$-

42

Frobenius problem to the standard $\mathcal{H}_\infty$ control problem.

$$
\begin{aligned}
& \|W \circ \mathcal{F}_l(P, K)\|_{\infty F} && K \in \mathcal{K} \\
= \; & \|W \circ (R + UQV)\|_{\infty F} && Q \in \mathcal{R}\mathcal{H}_\infty \\
= \; & \|\text{vec}\{W \circ (R + UQV)\}\|_\infty && Q \in \mathcal{R}\mathcal{H}_\infty \\
= \; & \|\overline{W} \left(r + \left(V^\top \otimes U \right) q \right)\|_\infty
\end{aligned}
\tag{2.93}
$$

where $\overline{W} = \text{diag}(\text{vec}\{W\})$, $r = \text{vec}\{R\}$ and $q = \text{vec}\{Q\} \in \mathcal{R}\mathcal{H}_\infty$. The first equality follows from the Yola parametrization (2.91). The second holds because vec is an isometry on $\mathcal{H}_\infty$ with the $\mathcal{H}_\infty$-Frobenius norm, and because a vector has only one singular value so its $\mathcal{H}_\infty$ norm and $\mathcal{H}_\infty$-Frobenius norm are equal. The third equality follows from the properties (2.92).

In the sequal, Van-Diggelen and Glover [71] showed that under the standard assumptions on P and some mild assumptions on W there exists a new generalized plant $\tilde{P}$ and a non-singular matrix $\widetilde{W}$ such that

$$
\mathcal{F}_l(\tilde{P}, \widetilde{W}q) = \overline{W} \left(r + \left(V^\top \otimes U \right) q \right).
$$

Now the Hadamard weighted $\mathcal{H}_\infty$-Frobenius problem can be given as the following standard $\mathcal{H}_\infty$ problem: find all $q \in \mathcal{R}\mathcal{H}_\infty$ such that

$$
\|\mathcal{F}_l(\tilde{P}, \widetilde{W}q)\|_\infty < \gamma.
\tag{2.94}
$$

The necessary and sufficient conditions for the existence of a solution to the suboptimal $\mathcal{H}_\infty$-Frobenius problem (2.94) are given in [71] Theorem 4.1, and the conditions for the optimal $\mathcal{H}_\infty$-Frobenius problem (where $<$ in (2.94) is replaced by $\le$) are given in [71] Theorem 5.1. Van-Diggelen and Glover also solved the Hadamard weighted $\mathcal{H}_2$ problem (note that the vec operator is an isometry on $\mathcal{H}_2$ with the $\mathcal{H}_2$ norm).

In their later paper [70], Van-Diggelen and Glover combined the above technique with a loop shaping design method in order to achieve both robust stability and coupling reduction. In fact if we do not care about robustness we can use their technique to obtain the H_∞-ODC (see Section 3.6).

Using the vec operator to reduce the problem to a standard H_∞ problem increases substantially the order of the resultant controller as shown in the following example. Suppose T has the following $n^{\underline{th}}$ order state space realization

$$T = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \stackrel{\text{def}}{=} C(sI - A)^{-1}B + D \in \mathcal{R}^{p \times m}(s).$$

Then the SIMO system vec$\{T\}$ has the following $mn^{\underline{th}}$ order state space realization

$$\text{vec}\{T\} = \left[\begin{array}{ccc|c} A & & & B_1 \\ & \ddots & & \vdots \\ & & A & B_m \\ \hline C & & & D_1 \\ & \ddots & & \vdots \\ & & C & D_m \end{array} \right] = \left[\begin{array}{c|c} I_m \otimes A & \text{vec}\{B\} \\ \hline I_m \otimes C & \text{vec}\{D\} \end{array} \right] \in \mathcal{R}^{mp}(s).$$

In fact Van-Diggelen and Glover used a similarity transformation to obtain an equivalent (but more convenient) state space realization. Given the original problem dimensions $(p_1 + p_2) \times (m_1 + m_2)$, the new generalized plant $\widetilde{P}$ of (2.94) has an order of $(p_2 + m_2)n$. Recall that using standard $\mathcal{H}_\infty$ solutions (without model reduction), the resultant controller $\bar{k}$ (the controller assotiated with $r + \left(V^\top \otimes U \right) q$) has the same order as the generalized plant $\widetilde{P}$. Moreover going back from $\bar{k}$ to K this order is generally not reduced. That is, if we assume an $m \times m$ $n^{\underline{th}}$ order plant the resultant Hadamard weighted $\mathcal{H}_\infty$-Frobenius controller will be of order $2mn$. This is a major disadvantage of this method.

44

Chapter 3

The general optimal decoupling control (ODC) problem

3.1 Introduction

In this chapter we study the general ODC problem. The term "general" is used because we consider the general closed-loop configuration used for decoupling (i.e., tracking system), and we do not make any assumptions about the exogenous reference input. In the next chapters we assume that the exogenous input is deterministic and known, and a state-feedback control law is used. Then the ODC problem will be stated as an LQR type problem. Using the formulation given in this chapter with other assumptions, other ODC problems can also be stated. For example, see the $\mathcal{H}_\infty$ ($\mathcal{H}_2$) ODC problem in Section 3.6.

Trying to solve the ODC problem, we first need a good performance index which can quantitatively measure the coupling level of the system. In this chapter such a measure is defined. Moreover an "imaginary" system, which generates the outputs required to compute this measure, is also defined. This system and its relationships with the original system are the bases for all the new results presented in this thesis.
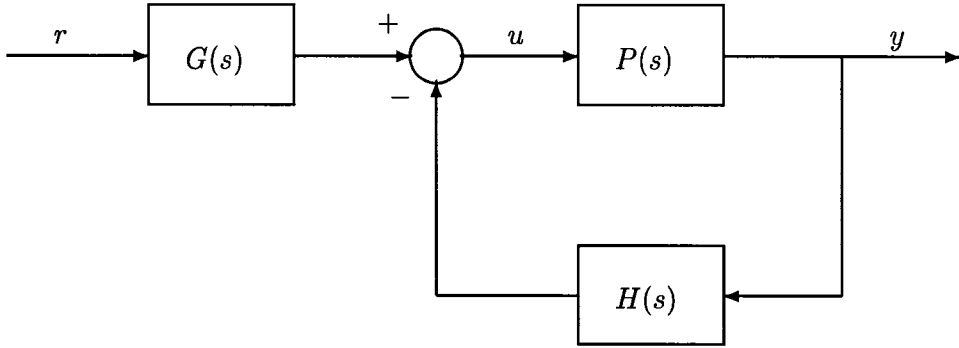
Figure 3.1: Tracking system with two degree of freedom controller.

3.2 Componentwise performance index

Consider the tracking system shown in Figure 3.1. The plant is a linear, time-invariant, finite-dimensional system represented by the real-rational, strictly-proper transfer function matrix $P(s)$. The plant has an m-vector control input u, and an m-vector output y. The control law for the closed-loop system is given by

$$u(s) = G(s)r(s) - H(s)y(s) \tag{3.1}$$

where r is an m-vector exogenous piecewise continuous reference input, and $G(s)$ and $H(s)$ are a two degree of freedom controller. The tracking error is defined by

$$e(s) = y(s) - r(s). \tag{3.2}$$

Let $W_e(s)$ and $W_u(s)$ be the error and control weighting matrices, respectively. Then we define

$$z_e(s) = W_e(s)e(s) \tag{3.3}$$

and

$$z_u(s) = W_u(s)u(s) \tag{3.4}$$

as the weighted error and the weighted control, respectively. The performance index

46

for the optimal tracking problem can be written as

$$J(z_e, z_u) = \left\| \begin{array}{c} z_e \\ z_u \end{array} \right\|_2^2 \tag{3.5}$$

where $\|\cdot\|_2$ denotes the standard $\mathcal{L}_2$ norm. Clearly by minimizing $J(z_e, z_u)$ over $G(s)$ and $H(s)$ we obtain the optimal tracking controller. Finally we can write the following closed-loop relations

$$
\begin{array}{rclcrcl}
y(s) & = & T(s)r(s); & & T(s) & = & [I + P(s)H(s)]^{-1} P(s)G(s) \\
e(s) & = & T_e(s)r(s); & & T_e(s) & = & T(s) - I \\
u(s) & = & T_u(s)r(s); & & T_u(s) & = & G(s) - H(s)T(s).
\end{array} \tag{3.6}
$$

Suppose now that all the inputs in (3.6) except the $j^{\underline{th}}$ one, r_j, are zero. Let $\overline{e}_j$ denote the corresponding error

$$\overline{e}_j = (\overline{e}_{1j}\, \overline{e}_{2j}\, \cdots\, \overline{e}_{mj})^\top. \tag{3.7}$$

This *componentwise error* contains the *main error* $\overline{e}_{jj}$, and the *coupling components* $\overline{e}_{ij}$, $i \neq j$, $i = 1, 2, \ldots, m$. Repeating this test for each input $j = 1, 2, \ldots, m$, we can use all the coupling components to indicate the *coupling level* of the system.

Definition 3.1 *The system (3.6) is said to be coupled if for some $r_j \neq 0$, there exists some $i \neq j$ such that $\overline{e}_{ij} \neq 0$.*

Definition 3.2 *The system (3.6) is said to be decoupled if for all $r_j \neq 0$ and for all $i \neq j$, $\overline{e}_{ij} = 0$.*

Moreover, $\overline{e}_{ij}$ can be used to measure quantitatively the coupling level of a system. Practically, this idea has been widely used to indicate pairwise coupling levels, or to compare between two or more coupling components of the same or different designs (see [7]). However, in order to use this idea for optimization purposes, we have to define a scalar function which measures the overall system coupling level (i.e., takes into account all the coupling components).

Remark 3.1 *The motivation to use "test" inputs is now clear and indeed test inputs (mostly steps) have been used in several works (e.g., [29]). However the theoretical componentwise errors $\bar{e}_j$ are well defined, for any arbitrary input r. Moreover, using superposition, the actual system tracking error can be expressed in terms of these $\bar{e}_j$'s.*

In optimal decoupling design we would like to give higher priority to the coupled part of the tracking error than to the main tracking error. This can be done simply by weighting differently the tracking error components $\bar{e}_{ij}$ (i.e., high weight for $\bar{e}_{ij}$ $i \neq j$ and small weight for e_{ii}). Let $w_{e_{ij}}$ be the componentwise weight of $\bar{e}_{ij}$, then the *weighted componentwise error* $\bar{z}_{e_{ij}}$ is defined as

$$\bar{z}_{e_{ij}}(s) = w_{e_{ij}}(s)\bar{e}_{ij}(s).$$ (3.8)

Let us define the following m^2-vector

$$\bar{e} = (\bar{e}_1^\top \; \bar{e}_2^\top \; \cdots \; \bar{e}_m^\top)^\top = (\bar{e}_{11} \; \bar{e}_{21} \; \cdots \; \bar{e}_{m1} \; \bar{e}_{12} \; \bar{e}_{22} \; \cdots \; \bar{e}_{m2} \; \cdots \; \bar{e}_{1m} \; \bar{e}_{2m} \; \cdots \; \bar{e}_{mm})^\top$$ (3.9)

($\bar{y}$ and $\bar{u}$ are defined similarly) with its diagonal weighting matrix

$$\overline{W}_e = \text{block diag}(\overline{W}_{e_1}, \overline{W}_{e_2}, \ldots, \overline{W}_{e_m})$$ (3.10)

where

$$\overline{W}_{e_j} = \text{diag}(w_{e_{1j}}, w_{e_{2j}}, \ldots, w_{e_{mj}}).$$ (3.11)

Then the m^2-vector weighted componentwise error is defined as

$$\bar{z}_e(s) = \overline{W}_e(s)\bar{e}(s).$$ (3.12)

Replacing z_e in (3.5) by $\bar{z}_e$ of (3.12), we obtain the optimal decoupling performance index

$$J_d(\bar{z}_e, z_u) = \left\| \begin{array}{c} \bar{z}_e \\ z_u \end{array} \right\|_2^2.$$ (3.13)

Similarly to the optimal tracking problem, we can minimize $J_d(\bar{z}_e, z_u)$ over $H(s)$ and $G(s)$ to obtain the optimal decoupling controller. However, using an output feedback

48

control low, the optimal control law depends on $\bar{y}$. Therefore we can not use the original closed-loop form (3.6) to obtain the optimal solution. In order to understand this problem better, we first study the following *duplicated system* representation.

3.3 Duplicated system representation

The duplicated system is an $(m + m^2) \times m$ system which, as a response to r, generates the componentwise system outputs $\bar{y}$, $\bar{u}$ and $\bar{e}$, as well as the original outputs y, u and e. In order to define the duplicated system let us first define the following linear operators:

- The *fragmentation* operator, $\mathbf{F} : \mathcal{R}^m(s) \rightarrow \mathcal{R}^{m^2}(s)$

  Let $r \in \mathcal{R}^m(s)$. Then the elements of $\tilde{r} = \mathbf{F}(r) \in \mathcal{R}^{m^2}(s)$ are defined as

  $$\tilde{r}_i = \begin{cases} r_i & \text{if } i = (m+1)j - m, \ j = 1, 2, \ldots, m \\ 0 & \text{otherwise.} \end{cases} \tag{3.14}$$

  The *fragmentation* operator has the following matrix representation $\tilde{r} = Vr$ where $V \in \mathbb{R}^{m^2 \times m}$ with elements

  $$V_{ij} = \begin{cases} 1 & \text{if } i = (m+1)j - m, \ j = 1, 2, \ldots, m \\ 0 & \text{otherwise} \end{cases} \tag{3.15}$$

  or equivalently

  $$V = [V_1 \ V_2 \ \cdots \ V_m]^\top$$
  $$\text{where } V_j = [v_{j_{kl}}] \in \mathbb{R}^{m \times m}$$
  $$\text{and } v_{j_{kl}} = \begin{cases} 1 & k = l = j \\ 0 & \text{otherwise.} \end{cases} \tag{3.16}$$

  The sub-matrices V_j, $j = 1, 2, \ldots, m$ form an orthogonal basis for the sub-space consisting of all the diagonal matrices in $\mathbb{R}^{m \times m}$, with the following properties

  1. $\sum\limits_{j=1}^{m} V_j = I$

  2. $\forall A \in C^{m \times n} \quad V_j A \ = \ {}^j A \quad \begin{cases} m \times n \text{ zero matrix except its } j^{\underline{th}} \\ \text{row which is the } j^{\underline{th}} \text{ row of } A \end{cases}$

  3. $\forall A \in C^{n \times m} \quad A V_j \ = \ A^j \quad \begin{cases} n \times m \text{ zero matrix except its } j^{\underline{th}} \\ \text{column which is the } j^{\underline{th}} \text{ column of } A \end{cases}$ $\tag{3.17}$

- The *superposition* operator, $\mathbf{S}_n : \mathcal{R}^{mn}(s) \to \mathcal{R}^n(s)$

  Let $\bar{x} = (\bar{x}_1^\top \ \bar{x}_2^\top \ \cdots \ \bar{x}_m^\top)^\top$ where $\bar{x}_i \in \mathcal{R}^n(s) \ \forall \ i = 1, 2, \ldots, m$ and $\bar{x} \in \mathcal{R}^{mn}(s)$ then $x = \mathbf{S}_n(\bar{x}) \in \mathcal{R}^n(s)$ is defined as: $x = \sum\limits_{i=1}^{m} \bar{x}_i$. The *superposition* operator has the following matrix representation: $x = U_n\bar{x}$ where

$$U_n = \begin{bmatrix} \underbrace{I_n \, I_n \, \cdots \, I_n}_{m \ times} \end{bmatrix} \in \mathbb{R}^{n \times mn} \tag{3.18}$$

  where I_n is the n dimensional identity matrix.

  If $m = n$ then, by convention, $\mathbf{S}_n \to \mathbf{S}$, $U_n \to U$ and $I_n \to I$. In this case $\bar{x} \in \mathcal{R}^{m^2}(s)$, $x = U\bar{x} \in \mathcal{R}^n(s)$ with

$$U = \begin{bmatrix} \underbrace{I \, I \, \cdots \, I}_{m \ times} \end{bmatrix} \in \mathbb{R}^{m \times m^2}. \tag{3.19}$$

- The *duplication* operator, $\mathbf{D}_m : \mathcal{R}^{p \times q}(s) \to \mathcal{R}^{mp \times mq}(s)$

  The range space of the linear operator $\mathbf{D}_m(\cdot)$ is denoted by $\widehat{\mathcal{R}}^{p \times q}(s) \subset \mathcal{R}^{mp \times mq}(s)$. Let $M \in \mathcal{R}^{p \times q}(s)$ then $\widehat{M} = \mathbf{D}_m(M) \in \widehat{\mathcal{R}}^{p \times q}(s)$ is defined as

$$\widehat{M} = \text{block diag} \left(\underbrace{M, \, M, \, \cdots, \, M}_{m \ times} \right) = \begin{bmatrix} M & & & \\ & M & & \\ & & \ddots & \\ & & & M \end{bmatrix} \tag{3.20}$$

or

$$\widehat{M}_{kl} = \begin{cases} M_{ij} & \text{if } k = i + rm \text{ and } l = j + rm \\ 0 & \text{otherwise} \end{cases} \quad \begin{aligned} i &= 1, 2, \ldots, p \\ j &= 1, 2, \ldots, q \\ r &= 0, 1, \ldots, m-1 \\ k &= 1, 2, \ldots, mp \\ l &= 1, 2, \ldots, mq. \end{aligned} \tag{3.21}$$

Thus $\widehat{\mathcal{R}}^{p \times q}(s)$ is a subspace contains matrices with special *Repeated Block Diagonal* (RBD) structure. Let $A, B \in \mathcal{R}^{p \times q}(s)$, $C \in \mathcal{R}^{q \times r}(s)$, and let $D \in \mathcal{R}^{n \times n}(s)$ be invertible, then

$$
\begin{aligned}
1. \quad & \widehat{A + B} = \widehat{A} + \widehat{B} \in \widehat{\mathcal{R}}^{p \times q}(s) \subset \mathcal{R}^{mp \times mq}(s) \\
2. \quad & \widehat{AC} = \widehat{A}\widehat{C} \in \widehat{\mathcal{R}}^{q \times r}(s) \subset \mathcal{R}^{mq \times mr}(s) \\
3. \quad & \widehat{A^\top} = \widehat{A}^\top \in \widehat{\mathcal{R}}^{q \times p}(s) \subset \mathcal{R}^{mq \times mp}(s) \\
4. \quad & \widehat{D^{-1}} = \widehat{D}^{-1} \in \widehat{\mathcal{R}}^{n \times n}(s) \subset \mathcal{R}^{mn \times mn}(s).
\end{aligned} \tag{3.22}
$$

Some useful properties of these operators:

- $\mathbf{S}(\cdot)$ is a left inverse of $\mathbf{F}(\cdot)$, i.e., $r = \mathbf{S}(\mathbf{F}(r))$, or

$$UV = I. \tag{3.23}$$

- Let $M \in \mathcal{R}^{p \times q}(s)$ and $\widehat{M} = \mathbf{D}_m(M) \in \widehat{\mathcal{R}}^{p \times q}(s)$ then

$$U_p \widehat{M} = M U_q = \left[\underbrace{M \ M \ \cdots \ M}_{m \ times} \right] \tag{3.24}$$

and if $p = q = m$ then

$$U \widehat{M} = MU \tag{3.25}$$

and

$$U \widehat{M} V = M. \tag{3.26}$$

- Let $\widetilde{M} = \text{block diag}(M_1 \ M_2 \ \cdots \ M_m)$, then

$$U \widetilde{M} V = \sum_{j=1}^{m} M_j V_j = \left[M_1^1 \ M_2^2 \ \cdots \ M_m^m \right] \tag{3.27}$$

where M_j^j is the $j^{\underline{th}}$ column of M_j.

Consider the tracking system shown in Figure 3.2, where $\widehat{P}(s)$, $\widehat{G}(s)$ and $\widehat{H}(s)$ are the m-duplication of $P(s)$, $G(s)$ and $H(s)$, respectively.

The duplicated system closed-loop equations are:

$$
\begin{aligned}
\bar{y}(s) &= \widehat{T}(s)\tilde{r}(s); & \widehat{T}(s) &= \left[\widehat{I} + \widehat{P}(s)\widehat{H}(s) \right]^{-1} \widehat{P}(s)\widehat{G}(s) \\
\bar{e}(s) &= \widehat{T}_e(s)\tilde{r}(s); & \widehat{T}_e(s) &= \widehat{T}(s) - \widehat{I} \\
\bar{u}(s) &= \widehat{T}_u(s)\tilde{r}(s); & \widehat{T}_u(s) &= \widehat{G}(s) - \widehat{H}(s)\widehat{T}(s)
\end{aligned}
\tag{3.28}
$$

where $\tilde{r}(s) = Vr(s)$, $y(s) = U\bar{y}(s)$ and $e(s) = U\bar{e}(s)$. The componentwise control is given by

$$\bar{u}(s) = \widehat{G}(s)\tilde{r}(s) - \widehat{H}(s)\bar{y}(s) = \widehat{G}(s)Vr(s) - \widehat{H}(s)\bar{y}(s) = \overline{G}(s)r(s) - \widehat{H}(s)\bar{y}(s) \tag{3.29}$$
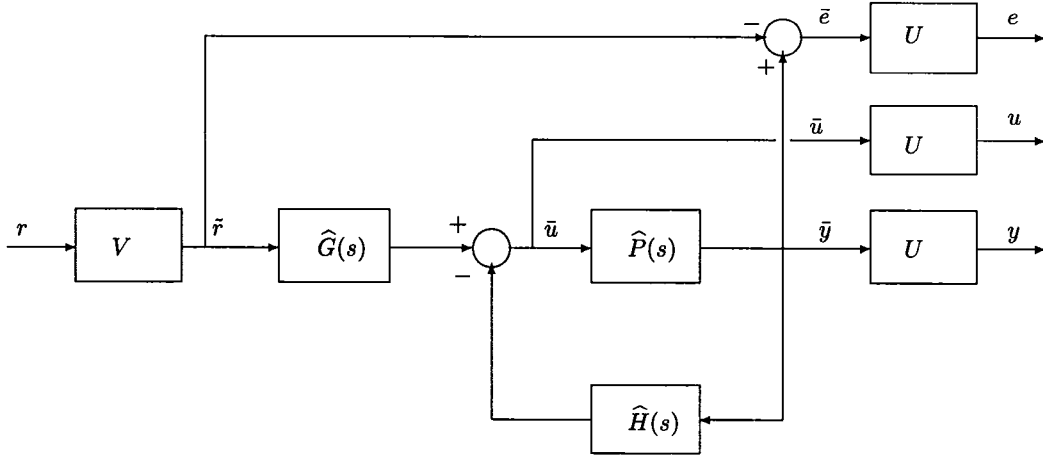
Figure 3.2: The duplicated system representation.

with

$$\overline{G}(s) = \left[\overline{G}_1^\top(s)\ \overline{G}_2^\top(s)\ \cdots\ \overline{G}_m^\top(s) \right]^\top$$

$$= \text{block diag}(G_1(s),\ G_2(s),\ \ldots,\ G_m(s)) \in \mathcal{R}^{m^2 \times m}$$

i.e., $\overline{G}_j(s)$ is a zero matrix except its $j^{\underline{th}}$ column which is the $j^{\underline{th}}$ column of $G(s)$. Using the property (3.25) of U, the actual control is then

$$
\begin{aligned}
u(s) &= U\bar{u}(s) \\
&= U\widehat{G}(s)\tilde{r}(s) - U\widehat{H}(s)\bar{y}(s) \\
&= G(s)U\tilde{r}(s) - H(s)U\bar{y}(s) \\
&= G(s)r(s) - H(s)y(s).
\end{aligned}
\tag{3.30}
$$

This system has exactly the same I/O properties as (3.6), and in addition it generates the desired componentwise outputs. We can now formulate and solve the following ODC problem using the duplicated system set-up and the cost function in (3.13).

$$\min_{\widehat{H}(s),\widehat{G}(s)\in\widehat{\mathcal{S}}^{m\times m}} J_d(\bar{z}_e, z_u) \tag{3.31}$$

where $\widehat{\mathcal{S}}^{m\times m} \subset \widehat{\mathcal{R}}^{m\times m}(s)$ is the set of all RBD controllers which stabilize $\widehat{P}(s)$ (i.e., $T(s)$ of (3.28) is stable). However then, the solution to (3.31) is singular as we next show.

52

To see this, recall from (3.13)

$$J_d(\bar{z}_e, z_u) \;=\; \left\| \begin{array}{c} \bar{z}_e \\[4pt] z_u \end{array} \right\|_2^2 \;=\; \left\| \begin{array}{c} \overline{W}_e(s)\bar{e}(s) \\[4pt] W_u(s)u(s) \end{array} \right\|_2^2$$

$$= \left\| \begin{array}{c} \overline{W}_e(s)\bar{e}(s) \\[4pt] W_u(s)U\bar{u}(s) \end{array} \right\|_2^2 \;=\; \left\| \begin{array}{c} \overline{W}_e(s)\bar{e}(s) \\[4pt] \overline{\overline{W}}_u(s)\bar{u}(s) \end{array} \right\|_2^2$$

(3.32)

where $\overline{\overline{W}}_u(s) = W_u(s)U = W_u(s)\underbrace{\left[I\, I \cdots I\right]}_{m\ times} = \underbrace{\left[W_u(s)\, W_u(s) \cdots W_u(s)\right]}_{m\ times}$. There-
fore $J_d(\bar{z}_e, z_u)$ has quadratic terms of the form $\bar{u}(s)^* R(s)\bar{u}(s)$ where

$$R(s) = \overline{\overline{W}}_u(s)^* \overline{\overline{W}}_u(s) = \begin{bmatrix} W_u(s)^* W_u(s) & \cdots & W_u(s)^* W_u(s) \\ \vdots & \ddots & \vdots \\ W_u(s)^* W_u(s) & \cdots & W_u(s)^* W_u(s) \end{bmatrix}$$

(3.33)

is a singular matrix. This implies that the optimal control u $(u(s) = G(s)r(s) - H(s)y(s))$ may have some unbounded components $\bar{u}_j$.

Therefore we slightly modify (3.13) by replacing $\overline{\overline{W}}_u$ with $\overline{W}_u$ which is defined similarly to $\overline{W}_e$ of (3.10) and (3.11), i.e., $\overline{W}_u$ is a diagonal, nonsingular matrix. Hence z_u in (3.13) is replaced by the weighted componentwise control $\bar{z}_u(s) = \overline{W}_u(s)\bar{u}(s)$. The modified componentwise performance index is now given by

$$J_d(\bar{z}_e, \bar{z}_u) = \left\| \begin{array}{c} \bar{z}_e \\[4pt] \bar{z}_u \end{array} \right\|_2^2 .$$

(3.34)

Remark 3.2 *The components of $\overline{W}_u$ are artificial (generally they have no physical meaning). However, by the right choice we can practically ensure that the control signal u is bounded (recall that because u is a sum of $\bar{u}_j$'s, bounded $\bar{u}$ implies bounded u). Notice that even in ordinary optimal design the choice of the control weighting matrix is not determined uniquely by the physical limitations, and in most cases the designer can use it as a design degree of freedom.*

The general optimal decoupling problem is now given by

$$\min_{\widehat{H}(s),\widehat{G}(s)\in\widehat{S}^{m\times m}} J_d(\overline{z}_e,\overline{z}_u). \tag{3.35}$$

Suppose such $\widehat{H}(s)$ and $\widehat{G}(s)$ exist, then the corresponding $H(s)$ and $G(s)$ are the optimal solution for the original system (3.6).

The optimization problem (3.35) has a structural constraint, namely the solution matrices are restricted to have a special RBD structure. For numerical optimization reasons we prefer to have an unconstrained problem. In fact (3.35) can be reformulated as an unconstrained optimization problem using the *simultaneous control* representation.

3.4   Simultaneous control representation

For the modified problem (3.35) we can alternatively consider the m tracking systems shown in Figure 3.3. Although all the systems have the same input r, each one has a different fragmentation sub-matrix V_j defined in (3.16). Therefore, we have m different SIMO systems, where the $j^{\underline{th}}$ system generates the componentwise output $\bar{y}_j$ and the componentwise error $\bar{e}_j$. The $j^{\underline{th}}$ componentwise control law is now given by

$$\bar{u}_j(s) = G(s)\tilde{r}_j(s) - H(s)\bar{y}_j(s) = G(s)V_jr(s) - H(s)\bar{y}_j(s) = \overline{G}_j(s)r(s) - H(s)\bar{y}_j(s) \tag{3.36}$$

where $H(s)$, $G(s) \in \mathcal{R}^{m\times m}(s)$ have no structural constraints.

Furthermore, using the partitioned error weighting matrix $\overline{W}_{e_j}$ (3.11) and similarly the partitioned control weighting matrix $\overline{W}_{u_j}$, we can define the following weighted componentwise outputs

$$\begin{aligned} \overline{z}_{e_j}(s) &= \overline{W}_{e_j}(s)\bar{e}_j(s) \\ \overline{z}_{u_j}(s) &= \overline{W}_{u_j}(s)\bar{u}_j(s) \end{aligned} \quad j = 1, 2, \ldots, m. \tag{3.37}$$

If we define the $j^{\underline{th}}$ performance index as

$$J_{d_j}(\overline{z}_{e_j}, \overline{z}_{u_j}) = \left\| \begin{array}{c} \overline{z}_{e_j} \\ \overline{z}_{u_j} \end{array} \right\|_2^2 \quad j = 1, 2, \ldots, m. \tag{3.38}$$
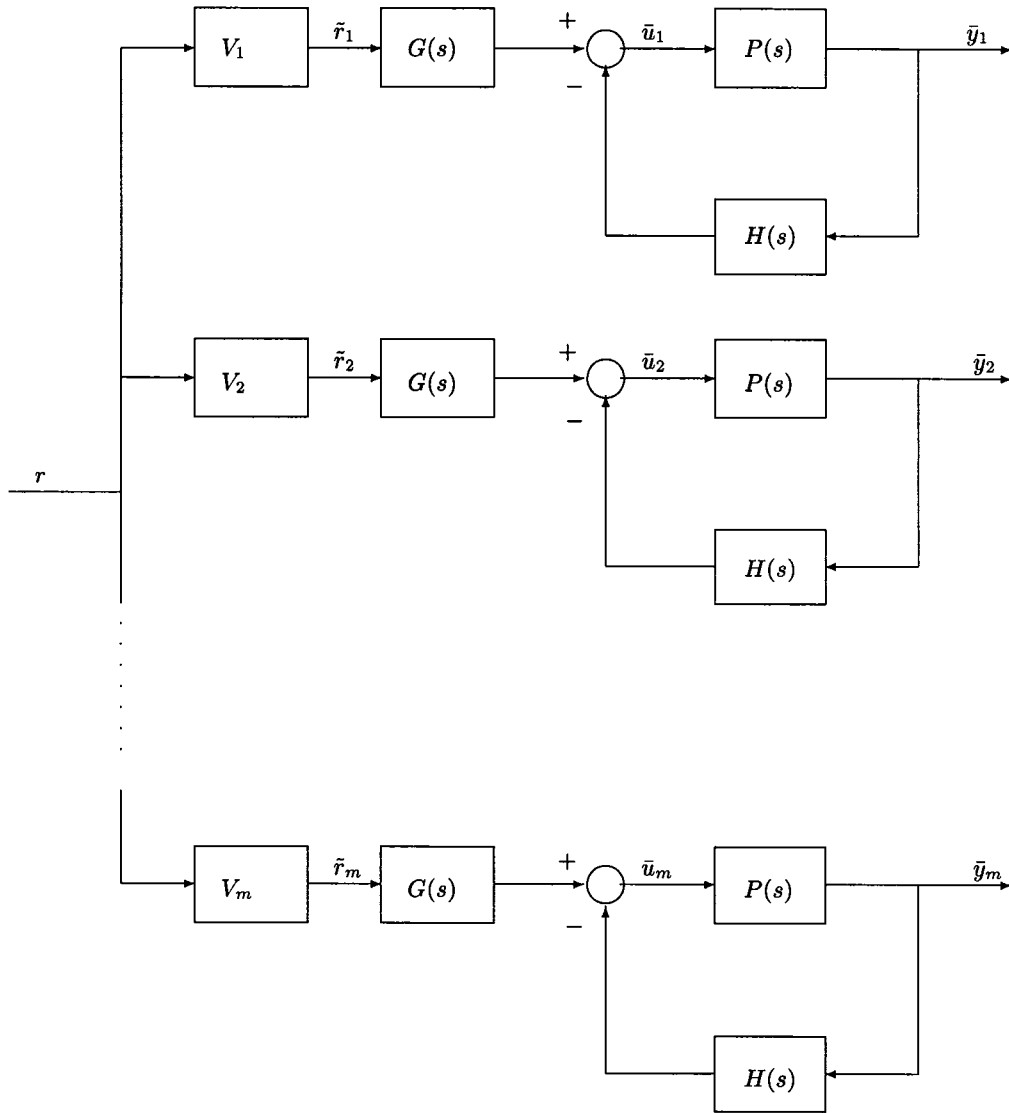
Figure 3.3: The simultaneous control representation.

Then (3.34) can be rewritten as

$$J_d(\overline{z}_e, \overline{z}_u) = \left\| \begin{array}{c} \overline{z}_e \\ \overline{z}_u \end{array} \right\|_2^2 = \sum_{j=1}^{m} J_{d_j}(\overline{z}_{e_j}, \overline{z}_{u_j}) = \sum_{j=1}^{m} \left\| \begin{array}{c} \overline{z}_{e_j} \\ \overline{z}_{u_j} \end{array} \right\|_2^2. \tag{3.39}$$

Remark 3.3 *The original cost function for optimal decoupling (3.13) can not be written, using the simultaneous control representation, as a summation of m independent terms. Then, even for a diagonal control weighting matrix W_u, there should be cross-terms in (3.39). Therefore we consider the simultaneous control representation only for the modified ODC problem (3.35).*

Using the equivalent cost function (3.39) we can replace the structural constraint of (3.35) with the simultaneity requirement, i.e., $H(s)$ and $G(s)$ have to minimize the sum of m different performance indices

$$\min_{H(s), G(s) \in \mathcal{S}^{m \times m}} \sum_{j=1}^{m} J_{d_j}(\overline{z}_{e_j}, \overline{z}_{u_j}) \tag{3.40}$$

where $\mathcal{S}^{m \times m} \subset \mathcal{R}^{m \times m}(s)$ is the set of all the controllers which stabilize $P(s)$ (i.e. $T(s)$ of (3.6) is stable). Now the implementation of the optimal solution is even more direct, namely if there exist $H(s)$ and $G(s)$ which minimize (3.40), they are the optimal solution for the original system.

Remark 3.4 *From (3.36)*

$$\overline{u}_j(s) = G(s)\tilde{r}_j(s) - H(s)\bar{y}_j(s) = \overline{G}_j(s)r(s) - H(s)\bar{y}_j(s).$$

Therefore the minimization is done over the columns of $G(s)$, or over $H(s)$, $\overline{G}_1(s)$, $\overline{G}_2(s)$, ..., $\overline{G}_m(s)$. Apparently $\overline{G}_j(s)$ has a structural constraint, i.e., all its columns except the $j^{\underline{th}}$ one are zeros (3.29). However, using SIMO systems, the optimal $\overline{G}_j(s)$ always has this structure.

Remark 3.5 *Similar formulations, using different approaches for the optimal LQ decoupling problem, were obtained in previous works [29], [58] and [56].*

3.5 The ideal solution and its applications

We have already shown that using componentwise performance indices such as (3.34) the optimal decoupling problem can be formulated as a constrained optimization problem (3.35). Suppose we now remove this structural constraint. Then (3.35) can be written as the following optimization problem

$$\min_{\widetilde{H}(s),\widetilde{G}(s)\in\mathcal{S}^{m^2\times m^2}} J_d(\overline{z}_e,\overline{z}_u) \tag{3.41}$$

where $\mathcal{S}^{m^2\times m^2}\subset\mathcal{R}^{m^2\times m^2}(s)$ is the set of all the controllers which stabilize $\widehat{P}(s)$, i.e., $\widetilde{H}(s)$ and $\widetilde{G}(s)$ have no structural constraint. The equivalent problem using the simultaneous control representation is to find the m optimal solutions for

$$\min_{\overline{H}_j(s),\overline{G}_j(s)\in\mathcal{S}^{m\times m}} J_{d_j}(\overline{z}_{e_j},\overline{z}_{u_j}) \quad j=1,2,\ldots,m \tag{3.42}$$

where $\mathcal{S}^{m\times m}\subset\mathcal{R}^{m\times m}(s)$ is the set of all the controllers which stabilize $P(s)$.

Let $\widehat{J}_d$ denote the minimum cost for the optimal solution to (3.35) and let $\widetilde{J}_d$ denote the minimum cost for the optimal solution to (3.41), then

$$\widehat{J}_d\geq\widetilde{J}_d. \tag{3.43}$$

That is, $\widetilde{J}_d$ is the best solution we can obtain. Moreover, because the structural constraint has been removed, the solution to (3.41) is a standard one. Of course this solution is only an *ideal* one because the optimal control law depends on the unavailable output $\overline{y}$

$$\overline{u}(s)=\widetilde{G}(s)\tilde{r}(s)-\widetilde{H}(s)\overline{y}(s). \tag{3.44}$$

The closed-loop equations for the ideal solution are

$$
\begin{aligned}
\overline{y}(s) &= \widetilde{T}(s)\tilde{r}(s); & \widetilde{T}(s) &= \left[\widehat{I}+\widehat{P}(s)\widetilde{H}(s)\right]^{-1}\widehat{P}(s)\widetilde{G}(s) \\
\overline{e}(s) &= \widetilde{T}_e(s)\tilde{r}(s); & \widetilde{T}_e(s) &= \widetilde{T}(s)-\widehat{I} \\
\overline{u}(s) &= \widetilde{T}_u(s)\tilde{r}(s); & \widetilde{T}_u(s) &= \widetilde{G}(s)-\widetilde{H}(s)\widetilde{T}(s).
\end{aligned} \tag{3.45}
$$

From (3.45) we conclude that $\widetilde{H}(s)$ and $\widetilde{G}(s)$ are block diagonal matrices . However they do not satisfy the RBD structural constraint as did $\widehat{H}(s)$ and $\widehat{G}(s)$ in (3.28). In fact the blocks of $\widetilde{H}(s)$ and $\widetilde{G}(s)$ are $\overline{H}_j(s)$ and $\overline{G}_j(s)$ of (3.42), respectively.

The ideal solution $\widetilde{H}(s)$ and $\widetilde{G}(s)$ can be easily found using classical optimal control methods (see Section 4.2 for the state-feedback case). In addition to its limited practical applications presented in Sections 3.5.1 and 3.5.2, the ideal solution can be used to compute $\widetilde{J}_d$. This ideal cost may be used as a design "target" for practical ODC designs (3.43).

Two applications of the ideal solution for stable plants are presented in the following.

3.5.1 Open-loop control implementation

Theoretically the ODC componentwise control law $\bar{u}$ of (3.29) can be generated by an artificial duplicated system. Then from (3.30) we can use the open-loop ODC $u = U\bar{u}$ for the actual system. This idea may be expanded to use the better control law $\bar{u}$ of (3.44), as stated in the following proposition.

Proposition 3.1 *Let $P(s)$ be a transfer function for a linear system with input $u(s)$ and output $y(s)$. Let $\widehat{P}(s)$ be the transfer function for the duplicated system with input $\overline{u}(s)$ and output $\overline{y}(s)$. Suppose that $\overline{u}^\star(s)$ is the optimal control for $\widehat{P}(s)$, i.e., $\overline{u}^\star(s)$ solves (3.41), then $u(s) = U\overline{u}^\star(s)$ is the ODC for $P(s)$.*

Proof: Let $\bar{y}^\star(s)$ denote the optimal output of the duplicated system $\widehat{P}(s)$. Then all we have to show is that using $u(s) = U\overline{u}^\star(s)$, the output of $P(s)$ is $y(s) = U\ \bar{y}^\star(s)$. But this follows directly from (3.25), $P(s)U = U\widehat{P}(s)$ then

$$y(s) = P(s)u(s) = P(s)U\overline{u}^\star(s) = U\widehat{P}(s)\overline{u}^\star(s) = U\bar{y}^\star(s). \qquad (3.46)$$

$\square$

58
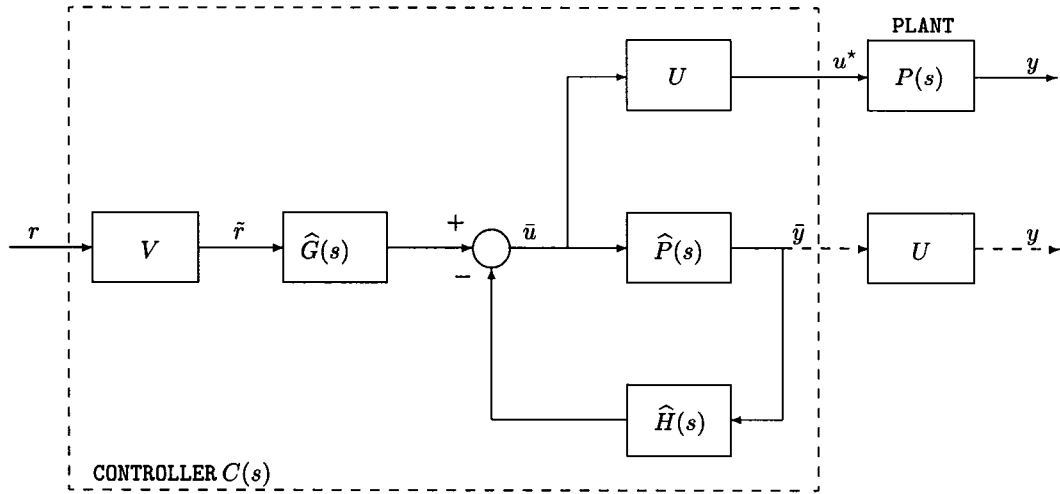
Figure 3.4: Open-loop optimal decoupling.

The implementation of this open-loop ODC is shown in Figure 3.4. The open-loop controller transfer function $C(s)$ can be derived from (3.45)

$$C(s) = U\widetilde{T}_u(s)V = \left[\widetilde{T}_u^1(s) \; \widetilde{T}_u^2(s) \; \cdots \; \widetilde{T}_u^m(s)\right] \qquad (3.47)$$

where $\widetilde{T}_u^j(s)$ is the jth column of the jth block of $\widetilde{T}_u(s)$ in (3.45).

Practically this solution is not very useful. First it is an open-loop control which fails in the presence of disturbances and plant perturbations. Second the resultant controller has a very high order, e.g., if $C(s)$ is obtained from an LQ componentwise state-feedback optimal control law, then it is of order mn. However this simple idea helps to demonstrate the importance of the ideal solution.

3.5.2   Closed-loop control

The ideal solution can be also used in a closed-loop configuration. For this case we introduce two new matrices $\Lambda(s)$ and $\Gamma(s)$ as an additional design degrees of freedom (see Figure 3.5).

59

Figure 3.5: Closed-loop optimal decoupling.

Proposition 3.2 *Consider the closed-loop system shown in Figure 3.1 with stable*

$P(s)$. *If*

$$
\begin{aligned}
\overline{H}(s) &= U \left[\widehat{I} + \widetilde{H}(s)\Gamma(s)\widehat{P}(s) \right]^{-1} \widetilde{H}(s) V \\
\overline{G}(s) &= U \left[\widehat{I} + \widetilde{H}(s)\Gamma(s)\widehat{P}(s) \right]^{-1} \widetilde{G}(s) V
\end{aligned}
\tag{3.48}
$$

where $\widetilde{H}(s)$ and $\widetilde{G}(s)$ are the ideal ODC matrices and $\Lambda(s)U + \Gamma(s) = \widehat{I}$, then $\overline{H}(s)$ and $\overline{G}(s)$ are the ODC matrices for the system in Figure 3.5.

Proof: If $\Lambda(s)U + \Gamma(s) = \widehat{I}$, then the estimated componentwise output of Figure 3.5

$$
\begin{aligned}
\bar{y}_e(s) &= \Lambda(s)y(s) + \Gamma(s)\bar{y}(s) \\
&= \Lambda(s)U\bar{y}(s) + \Gamma(s)\bar{y}(s) \\
&= [\Lambda(s)U + \Gamma(s)]\,\bar{y}(s) \\
&= \bar{y}(s).
\end{aligned}
\tag{3.49}
$$

Because $P(s)$ is stable there are no unstable pole-zero cancellations. Converting the structure of Figure 3.5 to that of Figure 3.1, we obtain (3.48).

□

The closed-loop transfer functions for the nominal case (no system disturbances, measurement noises, or plant perturbations) are exactly the theoretical transfer func-

60

tions (3.45). Moreover, under these assumptions we can choose $\Lambda(s) = 0$, $\Gamma(s) = \widehat{I}$ (i.e., the ideal open-loop solution of (3.47)). In other cases, $\Lambda(s)$ and $\Gamma(s)$ can be chosen to improve the performance of the closed-loop system. For example one natural choice, when no information on system disturbances, measurement noises, or plant perturbations is available, is $\Lambda(s) = V$ and $\Gamma(s) = \widehat{I} - VU$. Practically, for this choice, we assume that the system disturbances, noise and uncertainties, do not affect the output coupling components.

3.6 High order $\mathcal{H}_\infty$ and $\mathcal{H}_2$ ODC

In this section the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ ODC problems are converted to Hadamard weighted problems using the duplicated system representation of Section 3.3. We start with the duplicated tracking system of Figure 3.2. This system can be described by the block diagram of Figure 3.6, with the following generalized plant and controller

$$
\overline{P} = \left[\begin{array}{cc} \left[\begin{array}{c} -\overline{W}_e V \\ 0 \end{array} \right] & \left[\begin{array}{c} \overline{W}_e \widehat{P} \\ \overline{W}_u \end{array} \right] \\ \left[\begin{array}{c} 0 \\ V \end{array} \right] & \left[\begin{array}{c} \widehat{P} \\ 0 \end{array} \right] \end{array} \right] \quad \overline{K} = \left[-\widehat{H} \ \widehat{G} \right].
\tag{3.50}
$$

We assume that the input reference signal is unknown and $\mathcal{L}_2$ bounded (i.e., $\|r\|_2 \leq 1$). We are looking for an internally-stabilizing $\overline{K}$ which minimizes the $\mathcal{L}_2$ norm of the weighted output $\overline{z} = \left(\begin{array}{c} \overline{W}_e \overline{e} \\ \overline{W}_u \overline{u} \end{array} \right)$. It is well known (e.g., [17]) that this problem is equivalent to the following $\mathcal{H}_\infty$ problem:

$$
\inf_{\overline{K} \subset \overline{\mathcal{K}}} \|\mathcal{F}_l(\overline{P}, \overline{K})\|_\infty
\tag{3.51}
$$

where $\mathcal{F}_l(\overline{P}, \overline{K})$ is the LFT given by

$$
\mathcal{F}_l(\overline{P}, \overline{K}) = P_{11} + P_{12} K \left(I - P_{22} K \right)^{-1} P_{21}
\tag{3.52}
$$

and $\overline{\mathcal{K}}$ is the set of all internally-stabilizing $\overline{K}$'s.
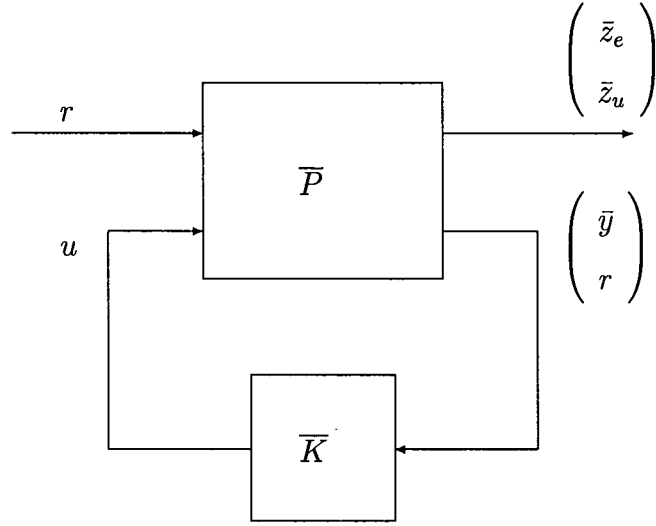
Figure 3.6: Standard block diagram for the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ ODC problems.

In fact the solution to (3.51) is the open-loop $\mathcal{H}_\infty$-ODC. In order to get the closed-loop solution to (3.51) we force $\overline{K}$ to have the RBD structure, i.e.,

$$\overline{K} = \left[- \begin{bmatrix} H & & \\ & \ddots & \\ & & H \end{bmatrix} \begin{bmatrix} G & & \\ & \ddots & \\ & & G \end{bmatrix} \right].$$

Then (3.51) can be written as the following constrained optimization problem

$$\inf_{\widehat{H},\widehat{G}\in\overline{\mathcal{K}}} \|\mathcal{F}_l(\overline{P},\overline{K})\|_\infty. \tag{3.53}$$

We now show that the structurally constrained problem (3.53) can be transformed to an unconstained Hadamard weighted $\mathcal{H}_\infty$-Frobenius problem [71]. Moreover it turns out that the $\mathcal{H}_\infty$-Frobenius norm used by Van-Diggelen and Glover in [71] as an approximation to the standard $\mathcal{H}_\infty$ norm, is indeed the right norm to use in this problem.

First we define the following weighting matrices,

$$W_e = \begin{bmatrix} w_{e11} & \cdots & w_{e1m} \\ \vdots & \ddots & \vdots \\ w_{em1} & \cdots & w_{emm} \end{bmatrix}, \; W_u = \begin{bmatrix} w_{u11} & \cdots & w_{u1m} \\ \vdots & \ddots & \vdots \\ w_{um1} & \cdots & w_{umm} \end{bmatrix} \text{ and } W = \begin{bmatrix} W_e \\ W_u \end{bmatrix}.$$

62

Recall from (3.28) that

$$\mathcal{F}_l(\overline{P}, \overline{K}) = \begin{bmatrix} \overline{W}_e \widehat{T}_e V \\ \overline{W}_u \widehat{T}_u V \end{bmatrix}.$$

(3.54)

Using the definitions for $\overline{W}_e$ and $\overline{W}_u$ (3.10) and (3.11), and for V (3.15) and (3.16)

$$\mathcal{F}_l(\overline{P}, \overline{K}) = \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} \overline{W}_{e_1} & & \\ & \ddots & \\ & & \overline{W}_{e_m} \end{bmatrix} \begin{bmatrix} T_e & & \\ & \ddots & \\ & & T_e \end{bmatrix} \begin{bmatrix} V_1 \\ \vdots \\ V_m \end{bmatrix} \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix} \overline{W}_{u_1} & & \\ & \ddots & \\ & & \overline{W}_{u_m} \end{bmatrix} \begin{bmatrix} T_u & & \\ & \ddots & \\ & & T_u \end{bmatrix} \begin{bmatrix} V_1 \\ \vdots \\ V_m \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} w_{e_{11}} T_{e_{11}} \\ \vdots \\ w_{e_{m1}} T_{e_{m1}} \end{bmatrix} & & \\ & \ddots & \\ & & \begin{bmatrix} w_{e_{1m}} T_{e_{1m}} \\ \vdots \\ w_{e_{mm}} T_{e_{mm}} \end{bmatrix} \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix} w_{u_{11}} T_{u_{11}} \\ \vdots \\ w_{u_{m1}} T_{u_{m1}} \end{bmatrix} & & \\ & \ddots & \\ & & \begin{bmatrix} w_{u_{1m}} T_{u_{1m}} \\ \vdots \\ w_{u_{mm}} T_{u_{mm}} \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

(3.55)

$$= \begin{bmatrix} \text{block diag} \left([W_e \circ T_e]_1, \ldots, [W_e \circ T_e]_m \right) \\ \text{block diag} \left([W_u \circ T_u]_1, \ldots, [W_u \circ T_u]_m \right) \end{bmatrix}$$

where T_e and T_u are defined in (3.6), and $[A \circ B]_j$ is the $j^{\underline{th}}$ column of the Hadamard product matrix $A \circ B$. Finally from the definition of the $\mathcal{H}_\infty$ norm and the $\mathcal{H}_\infty$-

Frobenius norm (2.89)

$$
\begin{aligned}
\|\mathcal{F}_l(\overline{P},\overline{K})\|_\infty &= \sup_\omega \left\{ \sum_{j=1}^m [W_e \circ T_e]_j^* [W_e \circ T_e]_j + [W_u \circ T_u]_j^* [W_u \circ T_u]_j \right\} \\
&= \sup_\omega \left\{ \sum_{j=1}^m [W \circ T]_j^* [W \circ T]_j \right\} \quad\quad (3.56) \\
&= \|W \circ T\|_{\infty F}
\end{aligned}
$$

where $T = \begin{bmatrix} T_e \\ T_u \end{bmatrix}$.

Therefore the $\mathcal{H}_\infty$-ODC problem can be stated as the following Hadamard weighted $\mathcal{H}_\infty$-Frobenius problem

$$
\inf_{K \in \mathcal{K}} \|W \circ T\|_{\infty F} \quad\quad (3.57)
$$

where $K = [-H\ G]$ and $\mathcal{K}$ is the set of all internally-stabilizing controllers for the original tracking system of Figure 3.1. Because now the problem depends only on the original system and not on the duplicated system, the RBD structural constraint is removed. That is, the constrained $\mathcal{H}_\infty$ problem becomes an unconstrained $\mathcal{H}_\infty$-Frobenius problem.

In general there is no explicit solution to (3.57). However, as for the standard $\mathcal{H}_\infty$ problem this problem may be solved by solving iteratively ("γ-iterations") the following sub-optimal $\mathcal{H}_\infty$ problem: find all $K \in \mathcal{K}$ such that

$$
\|W \circ T\|_{\infty F} < \gamma. \quad\quad (3.58)
$$

The sub-optimal problem can be solved using the technique presented in [71] (see also Section 2.5).

Remark 3.6 *Theoretically, the tracking problem of the form of Figure 3.1 is not a standard $\mathcal{H}_\infty$ problem. The D_{21} matrix of the generalized plant state space realization does not have full row rank (i.e., not a "fat" plant). Therefore theoretically the above $\mathcal{H}_\infty$-Frobenius problem can not be solved [71]. In $\mathcal{H}_\infty$ theorey there are some ways to "fix" this problem. First, we can use a minimum phase "squaring-down" compensator*

as proposed by Le and Safonov [45]. Alternatively we can "square-up" the generalized plant by adding some fictitious inputs passing through a gain of vanishingly small size ϵ. The $\mathcal{H}_\infty$ norm of the closed-loop system with the squared-up generalized plant is an upper bound on the $\mathcal{H}_\infty$ norm of the original closed-loop system. Furthermore in [52], Matson, Mita and Anderson showed that this bound is tight as ϵ approaches 0. Same ideas may be used to solve the above $\mathcal{H}_\infty$-ODC problem.

Following the same steps used for the $\mathcal{H}_\infty$-ODC, we can bring the $\mathcal{H}_2$-ODC to the form of a Hadamard weighted $\mathcal{H}_2$ problem. The solution to this problem is also given in [71].

Chapter 4

The state-feedback ODC problem

4.1 Introduction

In this chapter we study the ODC problem under the following assumptions:

(A1) The input reference signal is known and has the following state space realization

$$
\begin{aligned}
\dot{x}_r &= A_r x_r \\
r &= C_r x_r
\end{aligned}
\tag{4.1}
$$

where A_r is asymptotically stable, $x_r \in \mathbb{R}^{n_r}$, $x_r(0) = x_{r_0}$ and $r \in \mathbb{R}^m$, $m \leq n_r$.

(A2) The componentwise cost function is of infinite horizon, with constant state and control weighting matrices.

(A3) The control law is a state-feedback of the form

$$
u = G x_r - H x
\tag{4.2}
$$

where x_r is the state of the input signal, x is the actual or observed system state, and G and H are constant gain matrices of proper dimensions.

Considering only assumptions (A1) and (A2), the ideal ODC is not of the closed-loop form of (A3). First we derive this ideal (free structure) ODC based on standard LQ results. Then the necessary conditions for optimality, for the given closed-loop

structure of (A3), are derived using constant-gain optimal fixed structure control (OFSC) techniques. From these necessary conditions we can obtain the ODC solution. However under assumptions (A1)-(A3), this solution depends on the initial states.

4.2 The ideal (free structure) state-feedback ODC

As we have already seen in Chapter 3 the ideal ODC is actually a classical optimal control of an mn-order tracking system. To derive the explicit state-feedback result we first summarize the well known linear quadratic tracking (LQT) problem.

4.2.1 The optimal linear quadratic tracking (LQT) problem

The standard linear quadratic regulator (LQR) problem

Consider the following minimization problem

$$\min_{u \in \mathcal{U}} \int_0^\infty (x^\top Q x + u^\top R u) dt$$

subject to $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (4.3)

$$\dot{x} = Ax + Bu$$

where $\mathcal{U}$ is the linear space of m-dimensional continuous functions, Q and R are symmetric positive-semidefinite and positive-definite matrices, respectively. $x \in \mathbb{R}^n$, $x(0) = x_0$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^m$ and $m \leq n$. The couple (A, B) is controllable (stabilizable) and the couple $(Q^{1/2}, A)$ is observable (detectable).

The well known global (global in $\mathcal{U}$ and also independent of the initial state) solution to (4.3) is given by

$$u = -R^{-1}B^\top P = -Kx \qquad\qquad\qquad\qquad (4.4)$$

where P is the unique, symmetric, positive-(semi)definite solution to the following algebraic Riccati equation (ARE)

$$PA + A^\top P - PBR^{-1}B^\top P + Q = 0. \qquad\qquad (4.5)$$

The closed-loop matrix $A_c = A - BK$ is asymptotically stable and the minimal cost is given by $x_0^\top P x_0$.

The LQT problem as a standard LQR problem

Consider the tracking problem of Figure 3.1, where $P(s)$ has the following minimal state space realization

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx.
\end{aligned}
\tag{4.6}
$$

Recall that the tracking error is defined as $e = y - r$, where y is given by (4.6), r is given by (4.1) and the LQ optimal tracking problem is

$$
\min_{u \in \mathcal{U}} \int_0^\infty (e^\top Q_e e + u^\top R u) dt
\tag{4.7}
$$

where Q_e and R are symmetric, positive-definite matrices.

The standard solution is given by converting this problem to a standard LQR problem using the following state augmentation $X = \begin{pmatrix} x \\ x_r \end{pmatrix}$. Then

$$
\begin{aligned}
\dot{X} &= \overline{A}X + \overline{B}u \\
e &= \overline{C}X
\end{aligned}
\tag{4.8}
$$

where $\overline{A} = \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix}$, $\overline{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}$, $\overline{C} = [C \ -C_r]$ and $X(0) = X_0 = \begin{pmatrix} x_0 \\ x_{r_0} \end{pmatrix}$. The LQT minimization problem (4.7) can be rewritten as

$$
\min_{u \in \mathcal{U}} \int_0^\infty (X^\top \overline{Q} X + u^\top R u) dt
\tag{4.9}
$$

where

$$
\begin{aligned}
\overline{Q} &= \overline{C}^\top Q_e \overline{C} = \begin{bmatrix} C^\top \\ -C_r^\top \end{bmatrix} Q_e [C \ -C_r] \\
&= \begin{bmatrix} C^\top Q_e C & -C^\top Q_e C_r \\ -C_r^\top Q_e C & C_r^\top Q_e C_r \end{bmatrix} = \begin{bmatrix} \overline{Q}_1 & \overline{Q}_2 \\ \overline{Q}_2^\top & \overline{Q}_3 \end{bmatrix} \geq 0.
\end{aligned}
\tag{4.10}
$$

The augmented realization $(\overline{A}, \overline{B}, \overline{Q}^{1/2})$ is stabilizable and (at least) detectable. Therefore the optimal control is

$$
u = -R^{-1}\overline{B}^\top P = -\overline{K}X
\tag{4.11}
$$

69

where $\overline{P}$ is the unique, symmetric, positive-semidefinite solution of the following ARE

$$\overline{PA} + \overline{A}^{\top}\overline{P} - \overline{PB}R^{-1}\overline{B}^{\top}\overline{P} + \overline{Q} = 0 \qquad (4.12)$$

$\overline{A}_c = A - BK$ is asymptotically stable and the minimal cost is given by $X_0^{\top}\overline{P}X_0$.

In order to apply this solution to the original system, let $K = [H \quad -G]$. The optimal control law is now given by $u = Gx_r - Hx$. Let $\overline{P} = \begin{bmatrix} \overline{P}_1 & \overline{P}_2 \\ \overline{P}_2^{\top} & \overline{P}_3 \end{bmatrix}$, then (4.12) can be replaced by the following matrix equations

$$
\begin{aligned}
\overline{P}_1 A_c + A_c^{\top}\overline{P}_1 + H^{\top}RH + \overline{Q}_1 &= 0 \quad (a) \\
\overline{P}_2 A_r + A_c^{\top}\overline{P}_2 + \overline{Q}_2 &= 0 \quad (b) \\
\overline{P}_3 A_r + A_r^{\top}\overline{P}_3 - G^{\top}RG + \overline{Q}_3 &= 0 \quad (c)
\end{aligned}
\qquad (4.13)
$$

where $\overline{P}_1 > 0$, $\overline{P}_3 \geq 0$, $H = R^{-1}B^{\top}\overline{P}_1$, $G = -R^{-1}B^{\top}\overline{P}_2$, and $A_c = A - BH$ is asymptotically stable. From (4.13) we conclude that H does not depend on the reference input. In fact H is the LQR solution obtained from (4.4) and (4.5) with $Q = \overline{Q}_1$.

4.2.2 The ideal ODC problem as a standard LQT problem

The ideal ODC can be derived by applying the LQT formulation to the duplicated system (Section 3.3) or to the simultaneous control (Section 3.4) state space equations.

First we define the componentwise state as

$$\bar{x} = (\bar{x}_1^{\top} \ \bar{x}_2^{\top} \ \cdots \ \bar{x}_m^{\top})^{\top} \qquad (4.14)$$

with $\bar{x}_j = (\ \bar{x}_{1j} \ \bar{x}_{2j} \ \cdots \ \bar{x}_{nj})^{\top}$, where $\bar{x}_{ij}$ is the ith $(i = 1, 2, \ldots, n)$ element of x, when only the jth $(j = 1, 2, \ldots, m)$ input is active. That is, $x_i = \sum_{j=1}^{m} \bar{x}_{ij}$, and $x = \sum_{j=1}^{m} \bar{x}_j$, or using the superposition operator (3.18), $x = U_n\bar{x}$.

The duplicated system state space equations are

$$
\begin{aligned}
\dot{\bar{x}} &= \widehat{A}\bar{x} + \widehat{B}\bar{u} \\
\bar{y} &= \widehat{C}\bar{x}
\end{aligned}
\qquad (4.15)
$$

where $\bar{x} \in \mathbb{R}^{mn}$, $\bar{u} \in \mathbb{R}^{m^2}$ and $\bar{y} \in \mathbb{R}^{m^2}$, $m \leq n$.

70

Remark 4.1 *There is some freedom in the choice of the initial componentwise state*
$\bar{x}(0)$. *Theoretically any $\bar{x}(0)$ which satisfies $x(0) = U_n \bar{x}(0)$ is acceptable. However, the*
"best" (in the LMS sense) one can do is to assume $\bar{x}_j(0) = \frac{x_0}{m}$, $\forall\, j = 1, 2, \ldots, m$ (see
Section 5.3.2 for more details). Note that in decoupling theory we usually consider only
the zero-state (I/O) response of the system (i.e., $x(0) = 0$). Then naturally, $\bar{x}(0) = 0$.

Some useful relations between the original system state space realization prop-
erties (4.6), and those of the duplicated system (4.15), are stated in the following
lemma.

Lemma 4.1 *Consider the two state-space realizations (4.6) and (4.15), then*

(i)   *The triple $(\widehat{A}, \widehat{B}, \widehat{C})$ is minimal if and only if (A, B, C) is a minimal triple.*

(ii)   *The couple (A, BU) is controllable if and only if (A, B) is a controllable pair.*

(iii)   *The couple $(U_n, \widehat{A})$ is not completely observable.*

Proof: Follows from the definition of the duplication operator (3.20) and the matrices
U and U_n. (*i*), Using the Cayley-Hamilton theorem, the rank of the controllability and
observability matrices of the duplicated system (4.15) is reduced to m times the rank
of those matrices of the original system (4.6). (*ii*), From the definition of U (3.19),
the rank of the controllability matrix of the couple (A, BU) is the same as that of the
couple (A, B). (*iii*), From the definition of U_n (3.18), the rank of the observability
matrix of the couple $(U_n, \widehat{A})$ is n. Thus a realization with this couple has $(m - 1)n$
unobservable states.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Let us define the componentwise tracking error as

$$\bar{e} = \bar{y} - \tilde{r} = \widehat{C}\bar{x} - VC_r x_r$$

Then the ODC problem can be stated as

$$\min_{\bar{u} \in \mathcal{U}} \int_0^\infty (\bar{e}^\top \widetilde{Q}_e \bar{e} + \bar{u}^\top \widetilde{R}\bar{u}) dt, \qquad\qquad (4.16)$$

where

$$\widetilde{Q}_e = \text{block diag}(\widetilde{Q}_{e_1}, \ \widetilde{Q}_{e_2}, \ \dots \ , \widetilde{Q}_{e_m}) \ \text{ and } \ \widetilde{R} = \text{block diag}(\widetilde{R}_1, \ \widetilde{R}_2, \ \dots \ , \widetilde{R}_m)$$

are the (diagonal) positive-definite error and control weighting matrices, respectively. This minimization problem can also be written as a standard LQR problem using state augmentation of the componentwise state $\bar{x}$ and the input reference x_r.

Let $\widetilde{X} = \begin{pmatrix} \bar{x} \\ x_r \end{pmatrix}$, then

$$\begin{aligned}
\dot{\widetilde{X}} &= \widetilde{A}\widetilde{X} + \widetilde{B}\bar{u} \\
\bar{e} &= \widetilde{C}\widetilde{X}
\end{aligned} \tag{4.17}$$

where $\widetilde{A} = \begin{bmatrix} \widehat{A} & 0 \\ 0 & A_r \end{bmatrix}$, $\widetilde{B} = \begin{bmatrix} \widehat{B} \\ 0 \end{bmatrix}$, $\widetilde{C} = \begin{bmatrix} \widehat{C} & -VC_r \end{bmatrix}$ and $\widetilde{X}(0) = \widetilde{X}_0 = \begin{pmatrix} \bar{x}_0 \\ x_{r_0} \end{pmatrix}$. The minimization problem (4.16) becomes

$$\min_{\bar{u}\in\overline{\mathcal{U}}} \int_0^\infty (\widetilde{X}^\top \widetilde{Q}\widetilde{X} + \bar{u}^\top \widetilde{R}\bar{u})dt \tag{4.18}$$

where $\overline{\mathcal{U}}$ is the m^2-dimensional space of continuous functions and

$$\begin{aligned}
\widetilde{Q} &= \widetilde{C}^\top \widetilde{Q}_e \widetilde{C} = \begin{bmatrix} \widehat{C}^\top \\ -C_r^\top V^\top \end{bmatrix} \widetilde{Q}_e[\widehat{C} \ -VC_r] \\
&= \begin{bmatrix} \widehat{C}^\top \widetilde{Q}_e \widehat{C} & -\widehat{C}^\top \widetilde{Q}_e VC_r \\ -C_r^\top V^\top \widetilde{Q}_e \widehat{C} & C_r^\top V^\top \widetilde{Q}_e VC_r \end{bmatrix} = \begin{bmatrix} \widetilde{Q}_1 & \widetilde{Q}_2 \\ \widetilde{Q}_2^\top & \widetilde{Q}_3 \end{bmatrix} \geq 0.
\end{aligned} \tag{4.19}$$

From Lemma 4.1, the triple $(\widehat{A}, \widehat{B}, \widetilde{Q}^{1/2})$ is stabilizable and detectable. Therefore the optimal control is

$$\bar{u} = -\widetilde{R}^{-1}\widetilde{B}^\top \widetilde{P} = -\widetilde{K}\widetilde{X} \tag{4.20}$$

where $\widetilde{P}$ is the unique, symmetric, positive-semidefinite solution of the following ARE

$$\widetilde{P}\widetilde{A} + \widetilde{A}^\top \widetilde{P} - \widetilde{P}\widetilde{B}\widetilde{R}^{-1}\widetilde{B}^\top \widetilde{P} + \widetilde{Q} = 0 \tag{4.21}$$

$\widetilde{A}_c = \widetilde{A} - \widetilde{B}\widetilde{K}$ is asymptotically stable and the minimal cost is given by $\widetilde{X}_0\widetilde{P}\widetilde{X}_0$.

Recall from Lemma 4.1 that the componentwise state $\bar{x}$ can not be observed from the actual state x ($x = U_n\bar{x}$, but $(U_n, \widehat{A})$ is not observable). Therefore this

72

solution can only be applied to stable plants in high-order open-loop or closed-loop configuration. See Section 4.2.3 below for these implementations. For this case, let $\widetilde{K} = [\widetilde{H} \; -\widetilde{G}]$. The optimal free structure control law is now given by

$$\bar{u} = \widetilde{G}x_r - \widetilde{H}\bar{x} \tag{4.22}$$

where $\widetilde{P} = \begin{bmatrix} \widetilde{P}_1 & \widetilde{P}_2 \\ \widetilde{P}_2^\top & \widetilde{P}_3 \end{bmatrix}$, and (4.21) is replaced by the following matrix equations

$$
\begin{aligned}
\widetilde{P}_1\widetilde{A}_c + \widetilde{A}_c^\top \widetilde{P}_1 + \widetilde{H}^\top \widetilde{R}\widetilde{H} + \widetilde{Q}_1 &= 0 \quad (a) \\
\widetilde{P}_2 A_r + \widetilde{A}_c^\top \widetilde{P}_2 + \widetilde{Q}_2 &= 0 \quad (b) \\
\widetilde{P}_3 A_r + A_r^\top \widetilde{P}_3 - \widetilde{G}^\top \widetilde{R}\widetilde{G} + \widetilde{Q}_3 &= 0 \quad (c)
\end{aligned}
\tag{4.23}
$$

where $\widetilde{P}_1 > 0$, $\widetilde{P}_3 \geq 0$, $\widetilde{H} = \widetilde{R}^{-1}\widehat{B}^\top \widetilde{P}_1 \in \mathbb{R}^{m(m \times n)}$, $\widetilde{G} = -\widetilde{R}^{-1}\widehat{B}^\top \widetilde{P}_2 \in \mathbb{R}^{m^2 \times n_r}$, and $\widetilde{A}_c = \widehat{A} - \widehat{B}\widetilde{H}$ is asymptotically stable.

Alternatively, we can use the simultaneous control representation. The $j^{\underline{th}}$ simultaneous control state-space equations are

$$
\begin{aligned}
\dot{\bar{x}}_j &= A\bar{x}_j + B\bar{u}_j \\
\bar{y}_j &= C\bar{x}_j
\end{aligned}
\quad j = 1, 2, \ldots, m
\tag{4.24}
$$

where the $j^{\underline{th}}$ componentwise tracking error is

$$\bar{e}_j = \bar{y}_j - \tilde{r}_j = C\bar{x}_j - V_j C_r x_r. \tag{4.25}$$

The ODC problem is given by

$$\min_{u_1 \cdots u_m \in \mathcal{U}} \sum_{j=1}^{m} \int_0^\infty (\bar{e}_j^\top \widetilde{Q}_j \bar{e}_j + \bar{u}_j^\top \widetilde{R}_j \bar{u}_j)dt \tag{4.26}$$

or, because the m SIMO systems are independent, for all $j = 1, 2, \ldots, m$

$$\min_{u_j \in \mathcal{U}} \int_0^\infty (\bar{e}_j^\top \widetilde{Q}_j \bar{e}_j + \bar{u}_j^\top \widetilde{R}_j \bar{u}_j)dt. \tag{4.27}$$

Let $\overline{X}_j = \begin{pmatrix} \bar{x}_j \\ x_r \end{pmatrix}$, then

$$
\begin{aligned}
\dot{\overline{X}}_j &= \overline{A}\,\overline{X}_j + \overline{B}\bar{u} \\
\bar{e}_j &= \overline{C}_j \overline{X}_j
\end{aligned}
\tag{4.28}
$$

73

where $\overline{A} = \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix}$, $\overline{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$, $\overline{C}_j = [C \quad -V_j C_r]$ and $\overline{X}_j(0) = \overline{X}_{0_j} = \begin{pmatrix} \tilde{x}_{0_j} \\ x_{r_0} \end{pmatrix}$. The minimization problem (4.27) becomes

$$\min_{\bar{u}_j \in \mathcal{U}} \int_0^\infty (\overline{X}_j^\top \overline{Q}_j \overline{X}_j + \bar{u}_j^\top \tilde{R}_j \bar{u}_j) dt \tag{4.29}$$

where

$$\begin{aligned} \overline{Q}_j &= \overline{C}_j \tilde{Q}_{e_j} \overline{C}_j = \begin{bmatrix} C^\top \\ -C_r^\top V_j \end{bmatrix} \tilde{Q}_{e_j} [C \quad -V_j C_r] \\ &= \begin{bmatrix} C^\top \tilde{Q}_{e_j} C & -C^\top \tilde{Q}_{e_j} V_j C_r \\ -C_r^\top V_j \tilde{Q}_{e_j} C & C_r^\top V_j \tilde{Q}_{e_j} V_j C_r \end{bmatrix} = \begin{bmatrix} \overline{Q}_{1_j} & \overline{Q}_{2_j} \\ \overline{Q}_{2_j}^\top & \overline{Q}_{3_j} \end{bmatrix} \geq 0. \end{aligned} \tag{4.30}$$

For all $j = 1, 2, \ldots, m$ the triples $(\overline{A}, \overline{B}, \overline{Q}_j^{1/2})$ are stabilizable and detectable, and the optimal componentwise control is

$$\bar{u}_j = -\tilde{R}_j^{-1} \overline{B}^\top \overline{P}_j = -\overline{K}_j \overline{X}_j \tag{4.31}$$

where $\overline{P}_j$ are the unique, symmetric, positive-semidefinite solutions of the following $j^{\underline{th}}$ ARE

$$\overline{P}_j \overline{A} + \overline{A}_j^\top \overline{P} - \overline{P}_j \overline{B} \tilde{R}_j^{-1} \overline{B}^\top \overline{P}_j + \overline{Q}_j = 0 \tag{4.32}$$

$\overline{A}_{c_j} = \overline{A} - \overline{B} \overline{K}_j$ is asymptotically stable, the minimal componentwise cost is given by $\overline{X}_{0_j}^\top \overline{P}_j \overline{X}_{0_j}$ and the minimal cost is $\sum_{j=1}^m \overline{X}_{0_j}^\top \overline{P}_j \overline{X}_{0_j}$.

Again we can partition the solution, let $\overline{K}_j = [\overline{H}_j \quad -\overline{G}_j]$. The optimal componentwise control law is now given by $\bar{u}_j = \overline{G}_j x_r - \overline{H}_j \tilde{x}_j$. Let $\overline{P}_j = \begin{bmatrix} \overline{P}_{1_j} & \overline{P}_{2_j} \\ \overline{P}_{2_j}^\top & \overline{P}_{3_j} \end{bmatrix}$, then (4.32) can be replaced by the following m sets of matrix equations

$$\begin{aligned} \overline{P}_{1_j} A_{c_j} + A_{c_j}^\top \overline{P}_{1_j} + \overline{H}_j^\top \tilde{R}_j \overline{H}_j + \overline{Q}_{1_j} &= 0 \quad (a) \\ \overline{P}_{2_j} A_r + A_{c_j}^\top \overline{P}_{2_j} + \overline{Q}_{2_j} &= 0 \quad (b) \\ \overline{P}_{3_j} A_r + A_r^\top \overline{P}_{3_j} - \overline{G}_j^\top \tilde{R}_j \overline{G}_j + \overline{Q}_{3_j} &= 0 \quad (c) \end{aligned} \tag{4.33}$$

where $\overline{P}_{1_j} > 0$, $\overline{P}_{3_j} \geq 0$, $\overline{H}_j = \tilde{R}_j^{-1} B^\top \overline{P}_{1_j}$, $\overline{G}_j = -\tilde{R}_j^{-1} B^\top \overline{P}_{2_j}$, and $A_{c_j} = A - B\overline{H}_j$ is asymptotically stable.
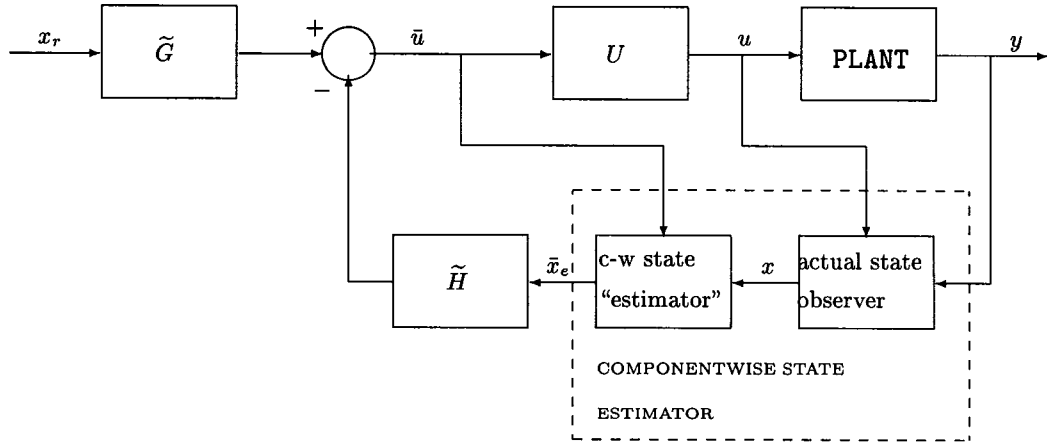
Figure 4.1: Closed-loop ODC using componentwise state feedback estimator.

4.2.3 Implementation of the state-feedback ideal solution

As for the general case of Section 3.5, we can apply the ideal solution only to stable systems using high order (mn-order) controllers, in either open-loop (Figure 3.4) or closed-loop configuration. For example, if the actual system state is available for feedback, we can build an mn-order "componentwise state estimator" and use the following control law

$$u = U\bar{u} = U\left[\widetilde{G}x_r - \widetilde{H}(\Lambda x + \Gamma\bar{x}_e)\right] \tag{4.34}$$

where $\bar{x}_e$ is the computed componentwise state, $\widetilde{G}$ and $\widetilde{H}$ are the ideal solution matrices of (4.13), and $\Lambda U + \Gamma = I_{mn}$. Note that $\bar{x}_e$ is computed in "open-loop", i.e., practically $x_e = U_n\bar{x}_e$ may not converge to the actual state x. Because theoretically $x = \sum_{j=1}^{m}\bar{x}_j$, we can use a reduced order $(m-1)n$ "estimator". For example, obtaining the $m\underline{th}$ computed componentwise state $\bar{x}_{e_m}$ from the other $m-1$ componentwise states and the actual system state ($\bar{x}_{e_m} = x - \sum_{j=1}^{m-1}\bar{x}_{e_j}$). In other cases where the system state is not available for feedback, an additional standard n-order observer may be used. The closed-loop system for this case is presented in Figure 4.1.

75

4.3   The optimal decoupling fixed structure control problem

The ideal ODC solutions can not be applied to general (unstable) plants. The next natural step is to search for closed-loop, low-order solutions. We consider the following two equivalent state-feedback control laws. The duplicated system control law

$$\bar{u} = \widetilde{G}x_r - \widehat{H}\bar{x} \tag{4.35}$$

where $\widehat{H} \in \widehat{\mathbb{R}}^{m \times n}$ has a RBD structural constraint, or

$$\bar{u}_j = \overline{G}_j x_r - H\bar{x}_j \tag{4.36}$$

where H is identical for all $j = 1, 2, \ldots, m$.

These ODC laws can be implemented as

$$
\begin{aligned}
u = U\bar{u} &= U\widetilde{G}x_r - U\widehat{H}\bar{x} \\
&= Gx_r - HU_n\bar{x} \quad \text{from (3.25)} \\
&= Gx_r - Hx \quad \text{from (4.14)}
\end{aligned} \tag{4.37}
$$

where $G = U\widetilde{G}$, or equivalently

$$
\begin{aligned}
u = \sum_{j=1}^{m} \bar{u}_j &= \sum_{j=1}^{m} \overline{G}_j x_r - \sum_{j=1}^{m} H\bar{x}_j \\
&= Gx_r - H\sum_{j=1}^{m} \bar{x}_j \\
&= Gx_r - Hx \quad \text{from (4.14)}
\end{aligned} \tag{4.38}
$$

where $G = \sum_{j=1}^{m} \overline{G}_j$.

Remark 4.2 *Contrary to the general case (3.36), using "state-feedback" for the input signal, the general simultaneous control formulation is now made of m identical MIMO systems with common H and different $\overline{G}_j$'s. Therefore $\overline{G}_j$ has an $m \times n_r$ unconstrained structure. However in many practical cases where the state of the input is not coupled (i.e., $A_r = $ block diag $(A_{r_1}, A_{r_2}, \ldots A_{r_m})$, $A_{r_j} \in \mathbb{R}^{n_{r_j} \times n_{r_j}}$), $\overline{G}_j$ has $n_r - n_{r_j}$ zero columns and G has a columnwise structure, $G = [G_1 \ G_2 \ \cdots G_m]$ where $G_j \in \mathbb{R}^{m \times n_{r_j}}$.*

76

Note that because the control structure is fixed, the ODC can only be derived from one of the following parametric optimization problems. The constrained duplicated system problem

$$\min_{\widehat{H} \in \widehat{\mathcal{S}}, \widetilde{G}} \int_0^\infty (\bar{e}^\top \widetilde{Q}_e \bar{e} + \bar{u}^\top \widetilde{R} \bar{u}) dt \tag{4.39}$$

$\widehat{\mathcal{S}} = \left\{ \widehat{H} = \text{block diag}(H, H, \ldots, H) \in \mathbb{R}^{m(m \times n)} \mid H \in \mathcal{S} \right\}$, or the simultaneous control problem

$$\min_{H \in \mathcal{S}, \overline{G}_1, \ldots, \overline{G}_m} \sum_{j=1}^m \int_0^\infty (\bar{e}_j^\top \widetilde{Q}_{e_j} \bar{e}_j + \bar{u}_j^\top \widetilde{R}_j \bar{u}_j) dt \tag{4.40}$$

where $\mathcal{S} = \{ H \in \mathbb{R}^{m \times n} \mid A - BH \text{ is asymptotically stable} \}$.

Because of the structural constraint on the state feedback matrices, in (4.39), or equivalently the summation in (4.40), these LQ problems can not be solved using classical LQR solutions such as (4.20)-(4.23) or (4.31)-(4.33). In the sequel we derive the necessary conditions for optimality required to solve this problem using the optimal fixed structure control (OFSC) formulation.

4.3.1 Deriving the necessary conditions for optimality for the standard LQR problem using the OFSC formulation

Levine and Athans considered and solved an OFSC problem in the late 60's [46]. In fact they considered the optimal constant output feedback control problem, with $u = -Ky$ where the optimal K may be obtained from the necessary conditions for optimality.

In the following we present the necessary conditions for optimality for the standard LQR problem (4.3). These conditions are usually not used to solve (4.3) (see Remark 4.3 below). We will use this result to derive the necessary conditions for optimality for the ODC problem.

We start with the following definitions:

Definition 4.1 *Let $M \in \mathbb{R}^{n \times n}$. Then the trace of M is given by*

$$\operatorname{tr}\{M\} = \sum_{i=1}^{n} M_{ii}. \tag{4.41}$$

Let $M, N \in \mathbb{R}^{n \times n}$, $P \in \mathbb{R}^{r \times n}$, $Q \in \mathbb{R}^{n \times r}$, and $m, n \in \mathbb{R}$. Then the trace operator has the following useful properties

$$\begin{aligned}
\operatorname{tr}\left\{M^{\top}\right\} &= \operatorname{tr}\{M\} \\
\operatorname{tr}\{mM + nN\} &= m \operatorname{tr}\{M\} + n \operatorname{tr}\{N\} \\
\operatorname{tr}\{PQ\} &= \operatorname{tr}\{QP\}.
\end{aligned} \tag{4.42}$$

Definition 4.2 *Let $Y, Z \in \mathbb{R}^{r \times n}$. Then the inner product $\langle Y, Z \rangle : \mathbb{R}^{r \times n} \longmapsto \mathbb{R}$, is given by*

$$\langle Y, Z \rangle = \operatorname{tr}\left\{Y^{\top} Z\right\}. \tag{4.43}$$

Definition 4.3 *Let $F(Z)$ be a continuously differentiable mapping of Z into $\mathbb{R}^{n \times n}$. Then $f(Z) = \operatorname{tr}\{F(Z)\}$ is a trace function of Z.*

The computation of $\dfrac{\partial f(Z)}{\partial Z}$, the gradient matrix of $f(Z)$, is given by the following lemma

Lemma 4.2 *Let $f(Z)$ be a trace function of Z. Then*

$$f(Z + \alpha \Delta Z) - f(Z) = \alpha \operatorname{tr}\left\{\left[\frac{\partial f(Z)}{\partial Z}\right]^{\top} \Delta Z\right\} + o(\alpha \Delta Z) \tag{4.44}$$

where $o(\alpha \Delta Z)$ is a function satisfying $\dfrac{o(\alpha \Delta Z)}{\alpha \|\Delta Z\|} \to 0$ as $\alpha \to 0$.

Proof: From Definitions 4.3 and 4.1

$$\begin{aligned}
f(Z + \alpha \Delta Z) - f(Z) &= \operatorname{tr}\{F(Z + \alpha \Delta Z)\} - \operatorname{tr}\{F(Z)\} \\
&= \operatorname{tr}\{F(Z + \alpha \Delta Z) - F(Z)\} \\
&= \sum_{i=1}^{n} \{F_{ii}(Z + \alpha \Delta Z) - F_{ii}(Z)\} \\
&= \sum_{i=1}^{n} \left[\left\langle \frac{\partial F_{ii}(Z)}{\partial Z}, \alpha \Delta Z \right\rangle + o_i(\alpha \Delta Z)\right]
\end{aligned}$$

78

where $o_i(\alpha \Delta Z)$ is a function satisfying $\frac{o_i(\alpha \Delta Z)}{\alpha \|\Delta Z\|} \to 0$ as $\alpha \to 0$.

Using Definition 4.2 and the trace properties (4.42)

$$
\begin{aligned}
f(Z + \alpha \Delta Z) - f(Z) &= \alpha \sum_{i=1}^{n} \left[\mathrm{tr} \left\{ \left[\frac{\partial F_{ii}(Z)}{\partial Z} \right]^{\mathsf{T}} \Delta Z \right\} + o_i(\alpha \Delta Z) \right] \\
&= \alpha \, \mathrm{tr} \left\{ \left[\frac{\partial \sum_{i=1}^{n} F_{ii}(Z)}{\partial Z} \right]^{\mathsf{T}} \Delta Z \right\} + \sum_{i=1}^{n} o_i(\alpha \Delta Z) \\
&= \alpha \, \mathrm{tr} \left\{ \left[\frac{\partial \, \mathrm{tr}\{F(Z)\}}{\partial Z} \right]^{\mathsf{T}} \Delta Z \right\} + o(\alpha \Delta Z) \\
&= \alpha \, \mathrm{tr} \left\{ \frac{\partial f(Z)}{\partial Z}^{\mathsf{T}} \Delta Z \right\} + o(\alpha \Delta Z)
\end{aligned}
$$

where $o(\alpha \Delta Z) = \sum_{i=1}^{n} o_i(\alpha \Delta Z)$ is a function satisfying $\frac{o(\alpha \Delta Z)}{\alpha \|\Delta Z\|} \to 0$ as $\alpha \to 0$.

$\square$

Consider the following minimization problem

$$
\min_{K \in \mathcal{S}} = \int_0^{\infty} \left(x^{\mathsf{T}} Q x + u^{\mathsf{T}} R u \right) dt
$$

subject to

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
u &= -Kx
\end{aligned}
$$

(4.45)

where $\mathcal{S} = \{K \mid A - BK \text{ is asymptotically stable}\}$, $x(0) = x_0$, and (A, B) is controllable.

Theorem 4.1 *The gradient matrix of the cost function*

$$
J = \int_0^{\infty} \left(x^{\mathsf{T}} Q x + u^{\mathsf{T}} R u \right) dt
$$

of the minimization problem (4.45), over $\mathcal{S}$ is given by

$$
\frac{\partial J(K, x_0)}{\partial K} = 2 \left[RK - B^{\mathsf{T}} P \right] L
$$

(4.46)

where P satisfies

$$
PA + A^{\mathsf{T}} P + Q + K^{\mathsf{T}} RK = 0
$$

(4.47)

and L satisfies

$$
LA^{\mathsf{T}} + AL + x_0 x_0^{\mathsf{T}} = 0.
$$

(4.48)

Proof: Let $K_0 \in \mathcal{S}$ (recall that from the controllability of the pair (A, B), $\mathcal{S}$ is nonempty), then

$$
\begin{aligned}
J_0(K_0) &= \int_0^\infty \left(x^\top Q x + u^\top R u \right) dt \\
&= \int_0^\infty x^\top \left(Q + K_0^\top R K_0 \right) x \, dt \\
&= x_0^\top \left[\int_0^\infty e^{A_0^\top t} \left(Q + K_0^\top R K_0 \right) e^{A_0 t} dt \right] x_0 \\
&= x_0^\top P_0 x_0 \\
&= \operatorname{tr} \left\{ x_0^\top P_0 x_0 \right\} \\
&= \operatorname{tr} \left\{ P_0 \left[x_0 x_0^\top \right] \right\} \\
&= \operatorname{tr} \left\{ P_0 \Pi \right\}
\end{aligned}
\tag{4.49}
$$

where $\Pi = x_0 x_0^\top$ and $P_0^\top = P_0 = \int_0^\infty e^{A_0^\top t} \left(Q + K_0^\top R K_0 \right) e^{A_0 t} dt \geq 0$ satisfies the Lyapunov equation

$$
A_0^\top P_0 + P_0 A_0 + Q + K_0^\top R K_0 = 0.
\tag{4.50}
$$

Let $K = K_0 + \alpha \Delta K \in \mathcal{S}$ and the associated $P = P_0 + \alpha \Delta P \geq 0$, then

$$
J(K) = \int_0^\infty x^\top \left(Q + K^\top R K \right) x \, dt = \operatorname{tr} \left\{ P \Pi \right\}
\tag{4.51}
$$

where $P^\top = P$ satisfies the Lyapunov equation

$$
\begin{aligned}
(A_0 - \alpha B \Delta K)^\top (P_0 + \alpha \Delta P) + (P_0 + \alpha \Delta P)(A_0 - \alpha B \Delta K) & \\
+ Q + (K_0 + \alpha \Delta K)^\top R (K_0 + \alpha \Delta K) &= 0.
\end{aligned}
\tag{4.52}
$$

Using (4.50) yields

$$
A_0^\top \Delta P + \Delta P A_0 - \Delta K^\top B^\top P_0 - P_0 B \Delta K + K_0^\top R \Delta K + \Delta K^\top R K_0 + o_1(\alpha \Delta K) = 0 \tag{4.53}
$$

where $o_1(\alpha \Delta K) = \alpha \left(\Delta K^\top R \Delta K - \Delta K^\top B^\top \Delta P - \Delta P B \Delta K \right)$. From (4.49) and (4.51) we also obtain

$$
\Delta J = J(K) - J_0(K_0) = \alpha \operatorname{tr} \left\{ \Delta P \Pi \right\}.
\tag{4.54}
$$

We are looking for an expression of the form $\Delta J = f(\Delta K)$. Therefore we use

80

the alternative form to (4.49)

$$
\begin{aligned}
J_0(K_0) &= \int_0^\infty \left(x^\top Q x + u^\top R u \right) dt \\
&= \int_0^\infty x^\top \left(Q + K_0^\top R K_0 \right) x dt \\
&= \mathrm{tr} \left\{ \int_0^\infty x^\top \left(Q + K_0^\top R K_0 \right) x dt \right\} \\
&= \mathrm{tr} \left\{ \left(Q + K_0^\top R K_0 \right) \int_0^\infty x x^\top dt \right\} \\
&= \mathrm{tr} \left\{ \left(Q + K_0^\top R K_0 \right) \int_0^\infty e^{A_0 t} \left[x_0 x_0^\top \right] e^{A_0^\top t} dt \right\} \\
&= \mathrm{tr} \left\{ \left(Q + K_0^\top R K_0 \right) L_0 \right\}
\end{aligned}
\tag{4.55}
$$

where $L_0^\top = L_0 = \int_0^\infty e^{A_0 t} x_0 x_0^\top e^{A_0^\top t} dt \geq 0$ satisfies the Lyapunov equation

$$
A_0 L_0 + L_0 A_0^\top + \Pi = 0.
\tag{4.56}
$$

Again, let $K = K_0 + \alpha \Delta K \in S$ and the associated $L = L_0 + \alpha \Delta L \geq 0$, then

$$
(A_0 - \alpha B \Delta K)(L_0 + \alpha \Delta L) + (L_0 + \alpha \Delta L)(A_0 - \alpha B \Delta K)^\top + \Pi = 0
\tag{4.57}
$$

and using (4.56)

$$
A_0 \Delta L + \Delta L A_0^\top - B \Delta K L_0 - L_0 \Delta K^\top B^\top + o_2(\alpha \Delta K) = 0
\tag{4.58}
$$

where $o_2(\alpha \Delta K) = -\alpha (B \Delta K \Delta L + \Delta L \Delta K^\top B^\top)$.

The associated cost increment is given by

$$
\begin{aligned}
\Delta J &= J(K) - J_0(K_0) \\
&= \mathrm{tr} \left\{ \left[Q + (K_0 + \alpha \Delta K)^\top R (K_0 + \alpha \Delta K) \right] (L_0 + \alpha \Delta L) \right\} \\
&\quad - \mathrm{tr} \left\{ \left(Q + K_0^\top R K_0 \right) L_0 \right\} \\
&= 2\alpha \, \mathrm{tr} \left\{ L_0 K_0^\top R \Delta K \right\} + \alpha \, \mathrm{tr} \left\{ \left(Q + K_0^\top R K_0 \right) \Delta L \right\} + o_3(\alpha \Delta K)
\end{aligned}
\tag{4.59}
$$

where $o_3(\alpha \Delta K) = \alpha^2 \, \mathrm{tr} \left\{ \Delta K^\top R \Delta K \left(L_0 + \alpha \Delta L \right) + 2 \Delta L K_0^\top R \Delta K \right\}$.

In order to eliminate ΔP and ΔL from the expressions for ΔJ in (4.54) and (4.59), we first postmultiply (4.50) by $\alpha \Delta L$ and take the trace to obtain

$$
2\alpha \, \mathrm{tr} \left\{ P_0 A_0 \Delta L \right\} + \alpha \, \mathrm{tr} \left\{ \left(Q + K_0^\top R K_0 \right) \Delta L \right\} = 0
\tag{4.60}
$$

81

where we used properties of the trace (4.42). Then we premultiply (4.58) by αP_0 and take the trace to obtain

$$2\alpha \operatorname{tr}\{P_0 A_0 \Delta L\} - 2\alpha \operatorname{tr}\{L_0 P_0 B \Delta K\} + o_4(\alpha \Delta K) = 0 \tag{4.61}$$

where $o_4(\alpha \Delta K) = -2\alpha^2 \operatorname{tr}\{\Delta L P_0 B \Delta K\}$. From (4.60) and (4.61) we obtain

$$\alpha \operatorname{tr}\left\{\left(Q + K_0^\top R K_0\right) \Delta L\right\} = -2\alpha \operatorname{tr}\{L_0 P_0 B \Delta K\} + o_4(\alpha \Delta K). \tag{4.62}$$

Substituting (4.62) into (4.59) yields

$$\Delta J(\alpha \Delta K) = \alpha \operatorname{tr}\left\{2L_0 \left[K_0^\top R - P_0 B\right] \Delta K\right\} + o(\alpha \Delta K) \tag{4.63}$$

where $o(\alpha \Delta K) = o_3(\alpha \Delta K) + o_4(\alpha \Delta K) = \alpha^2 \operatorname{tr}\left\{\Delta K^\top R \Delta K (L_0 + \alpha \Delta L) + 2\Delta L \left[K_0^\top R\right.\right.$ $\left.\left.- P_0 B\right] \Delta K\right\}$ is a trace function satisfying $\frac{o(\alpha \Delta K)}{\alpha \|\Delta K\|} \to 0$ as $\alpha \to 0$.

From Lemma 4.2 the expression for the gradient of $J(K)$ at K_0 is given by

$$\frac{\partial J(K, x_0)}{\partial K}\Big|_{K=K_0} = 2\left[RK_0 - B^\top P_0(K_0)\right] L_0(K_0) \tag{4.64}$$

where $P_0(K_0)$ and $L_0(K_0)$ satisfy (4.50) and (4.56), respectively.

□

Corollary 4.1 *The necessary condition for optimality for the minimization problem (4.45) is*

$$\left[RK - B^\top P\right] L = 0 \tag{4.65}$$

where L and P satisfy (4.56) and (4.50), respectively.

Proof: Follows directly from Theorem 4.1. K which satisfies (4.65) is a stationary point of the cost function $J(K)$ to be minimized.

□

Remark 4.3 *It is clear that the necessary conditions of (4.65) do not simply lead to the LQR solution (4.4). In order to obtain (4.4), in addition to the standard conditions $Q \geq 0$ and $R > 0$, L should be positive definite. Recall from the Lyapunov theorem that L of (4.56) is positive definite if and only if, for all $x_0 \in \mathbb{R}^n$, either $x_0 x_0^\top > 0$ or $(A - BK, x_0)$ is controllable. Of course none of these assumption holds.*

4.3.2 The OFSC-LQT as a standard OFSC-LQR problem

In order to convert the LQT problem to an LQR problem, we again use the augmented state $X = \begin{pmatrix} x \\ x_r \end{pmatrix}$ and the control law $u = Gx_r - Hx$.

Define

$$\overline{K} = [H \quad -G] \tag{4.66}$$

and let $\overline{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \overline{A}_c = \overline{A} - \overline{BK}, \overline{Q} = \begin{bmatrix} \overline{Q}_1 & \overline{Q}_2 \\ \overline{Q}_2^\top & \overline{Q}_3 \end{bmatrix}, \overline{P} = \begin{bmatrix} \overline{P}_1 & \overline{P}_2 \\ \overline{P}_2^\top & \overline{P}_3 \end{bmatrix}, \overline{L} = \begin{bmatrix} \overline{L}_1 & \overline{L}_2 \\ \overline{L}_2^\top & \overline{L}_3 \end{bmatrix},$

and $\overline{\Pi} = \begin{bmatrix} \overline{\Pi}_1 & \overline{\Pi}_2 \\ \overline{\Pi}_2^\top & \overline{\Pi}_3 \end{bmatrix}$. Then the necessary conditions for optimality are

$$\frac{\partial J(\overline{K}, x_0)}{\partial \overline{K}} \;=\; 2\left[R\overline{K} - \overline{B}^\top \overline{P} \right] \overline{L} \;=\; 0 \tag{4.67}$$

where $\overline{P}$ and $\overline{L}$ satisfy

$$\begin{aligned}
\overline{A}_c^\top \overline{P} + \overline{P}\overline{A}_c + \overline{K}^\top R\overline{K} + \overline{Q} &= 0 \quad (a) \\
\overline{A}_c \overline{L} + \overline{L}\overline{A}_c^\top + \overline{\Pi} &= 0 \quad (b)
\end{aligned} \tag{4.68}$$

We can also use the partitioned form

$$\begin{aligned}
\frac{\partial J(H, G, x_0)}{\partial H} &= 2\left\{ \left[RH - B^\top \overline{P}_1 \right] \overline{L}_1 - \left[RG + B^\top \overline{P}_2 \right] \overline{L}_2^\top \right\} = 0 \quad (a) \\
\frac{\partial J(H, G, x_0)}{\partial G} &= 2\left\{ \left[RH - B^\top \overline{P}_1 \right] \overline{L}_2 - \left[RG + B^\top \overline{P}_2 \right] \overline{L}_3 \right\} = 0 \quad (b)
\end{aligned} \tag{4.69}$$

where

$$\begin{aligned}
\overline{P}_1(A - BH) + (A - BH)^\top \overline{P}_1 + H^\top RH + \overline{Q}_1 &= 0 \quad (a) \\
\overline{P}_2 A_r + (A - BH)^\top \overline{P}_2 + \overline{P}_1 BG - H^\top RG + \overline{Q}_2 &= 0 \quad (b) \\
\overline{P}_3 A_r + A_r^\top \overline{P}_3 + \overline{P}_2^\top BG + G^\top B^\top \overline{P}_2 + G^\top RG + \overline{Q}_3 &= 0 \quad (c) \\
\overline{L}_1(A - BH)^\top + (A - BH)\overline{L}_1 + \overline{L}_2 G^\top B^\top + BG\overline{L}_2^\top + \overline{\Pi}_1 &= 0 \quad (d) \\
\overline{L}_2 A_r^\top + (A - BH)\overline{L}_2 + BG\overline{L}_3 + \overline{\Pi}_2 &= 0 \quad (e) \\
\overline{L}_3 A_r^\top + A_r \overline{L}_3 + \overline{\Pi}_3 &= 0 \quad (f)
\end{aligned} \tag{4.70}$$

4.3.3 The state-feedback ODC problem as an OFSC-LQT problem

We can use both the duplicated system and the simultaneous control representations to derive the necessary conditions for optimality for the ODC problem. However using

the duplicated system representation, the stability of the duplicated closed-loop system is not always guaranteed. Recall that the ODC is given by

$$
\begin{aligned}
u &= U\bar{u} \\
&= U\left[\tilde{G}x_r - \hat{H}\tilde{x}\right] \\
&= Gx_r - U\hat{H}\tilde{x} \\
&= Gx_r - HU_n\bar{x}.
\end{aligned}
$$

This is an "output-feedback" control law. Thus a stable solution does not always exist. On the other hand, the controllability of (A, B) implies that a stable ODC does always exists. This "contradiction" is caused because, using the OFSC formulation, the RBD structural constraint is changed such that the componentwise states $\bar{x}_j$ are no longer isolated (i.e., they can affect each other). The simple way to avoid this problem is to use the alternative simultaneous control representation. Then the componentwise control laws are indeed isolated state-feedback control laws.

Theorem 4.2 *The gradient matrices of the cost function for the ODC problem (4.40) are*

$$
\frac{\partial J_d}{\partial H} = 2\sum_{j=1}^{m}\left\{\left[\tilde{R}_j H - B^\top \overline{P}_{1_j}\right]\overline{L}_{1_j} - \left[\tilde{R}_j \overline{G}_j + B^\top \overline{P}_{2_j}\right]\overline{L}_{2_j}^\top\right\} \quad (a)
$$

and for all $j = 1, 2, \ldots, m$

$$
\frac{\partial J_d}{\partial \overline{G}_j} = -2\left\{\left[\tilde{R}_j H - B^\top \overline{P}_{1_j}\right]\overline{L}_{2_j} - \left[\tilde{R}_j \overline{G}_j + B^\top \overline{P}_{2_j}\right]\overline{L}_3\right\} \quad (b)
$$

$$(4.71)$$

where

$$
\begin{aligned}
\overline{P}_{1_j}(A - BH) + (A - BH)^\top \overline{P}_{1_j} + H^\top \tilde{R}_j H + \overline{Q}_{1_j} &= 0 \quad (a) \\
\overline{P}_{2_j} A_r + (A - BH)^\top \overline{P}_{2_j} + \overline{P}_{1_j} B\overline{G}_j - H^\top \tilde{R}_j \overline{G}_j + \overline{Q}_{2_j} &= 0 \quad (b) \\
\overline{P}_{3_j} A_r + A_r^\top \overline{P}_{3_j} + \overline{P}_{2_j}^\top B\overline{G}_j + \overline{G}_j^\top B^\top \overline{P}_{2_j} + \overline{G}_j^\top \tilde{R}_j \overline{G}_j + \overline{Q}_{3_j} &= 0 \quad (c) \\
\overline{L}_{1_j}(A - BH)^\top + (A - BH)\overline{L}_{1_j} + \overline{L}_{2_j}\overline{G}_j^\top B^\top + B\overline{G}_j\overline{L}_{2_j}^\top + \overline{\Pi}_{1_j} &= 0 \quad (d) \\
\overline{L}_{2_j} A_r^\top + (A - BH)\overline{L}_{2_j} + B\overline{G}_j\overline{L}_{3_j} + \overline{\Pi}_{2_j} &= 0 \quad (e) \\
\overline{L}_3 A_r^\top + A_r \overline{L}_3 + \overline{\Pi}_3 &= 0 \quad (f)
\end{aligned}
$$

$$(4.72)$$

or, alternatively, in a compact form

$$
\begin{aligned}
\overline{P}_j \overline{A}_{cj} + \overline{A}_{cj}^\top \overline{P}_j + \overline{K}_j^\top \tilde{R}_j \overline{K}_j + \overline{Q}_j &= 0 \quad (a) \\
\overline{L}_j \overline{A}_{cj}^\top + \overline{A}_{cj}\overline{L}_j + \overline{\Pi}_j &= 0 \quad (b).
\end{aligned}
$$

$$(4.73)$$

where $\overline{Q}_{1_j}$, $\overline{Q}_{2_j}$, *and* $\overline{Q}_{3_j}$ *are given by (4.30),* $\overline{\Pi}_j = \begin{bmatrix} \overline{\Pi}_{1_j} & \overline{\Pi}_{2_j} \\ \overline{\Pi}_{2_j}^\top & \overline{\Pi}_3 \end{bmatrix} = \overline{X}_{0_j} \overline{X}_{0_j}^\top$ *and* $A - BH$ *is asymptotically stable.*

Proof: Using the simultaneous control representation, of (4.40) the the cost function is given by

$$J_d = \sum_{j=1}^m J_{d_j} = \sum_{j=1}^m \int_0^\infty (\bar{e}_j^\top \tilde{Q}_{e_j} \bar{e}_j + \bar{u}_j^\top \tilde{R}_j \bar{u}_j) dt. \tag{4.74}$$

Similarly to (4.66), define $\overline{K}_i = [H \ -\overline{G}_i]$. Then from Theorem 4.1 the $i^{\underline{th}}$ gradient of J_d is given by

$$\frac{\partial J_d}{\partial \overline{K}_i} = \sum_{j=1}^m \frac{\partial J_{d_j}(\overline{K}_j, \overline{X}_{0_j})}{\partial \overline{K}_i} = \sum_{j=1}^m \left[\frac{\partial J_{d_j}(H, \overline{G}_j, \overline{X}_{0_j})}{\partial H} \ -\frac{\partial J_{d_j}(H, \overline{G}_j, \overline{X}_{0_j})}{\partial \overline{G}_i} \right]. \tag{4.75}$$

Let $\overline{P}_j = \begin{bmatrix} \overline{P}_{1_j} & \overline{P}_{2_j} \\ \overline{P}_{2_j}^\top & \overline{P}_{3_j} \end{bmatrix}$, $\overline{L}_j = \begin{bmatrix} \overline{L}_{1_j} & \overline{L}_{2_j} \\ \overline{L}_{2_j}^\top & \overline{L}_3 \end{bmatrix}$, and $\overline{\Pi}_j = \begin{bmatrix} \overline{\Pi}_{1_j} & \overline{\Pi}_{2_j} \\ \overline{\Pi}_{2_j}^\top & \overline{\Pi}_3 \end{bmatrix}$, then from (4.69)

$$\frac{\partial J_{d_j}(H, \overline{G}_j, \overline{X}_{0_j})}{\partial H} = 2\left\{ \left[\tilde{R}_j H - B^\top \overline{P}_{1_j}\right] \overline{L}_{1_j} - \left[\tilde{R}_j \overline{G}_j + B^\top \overline{P}_{2_j}\right] \overline{L}_{2_j}^\top \right\} \tag{4.76}$$

and

$$\frac{\partial J_{d_j}(H, \overline{G}_j, \overline{X}_{0_j})}{\partial \overline{G}_i} = \begin{cases} -2\left\{ \left[\tilde{R}_j H - B^\top \overline{P}_{1_j}\right] \overline{L}_{2_j} - \left[\tilde{R}_j \overline{G}_j + B^\top \overline{P}_{2_j}\right] \overline{L}_3 \right\} & i = j \\ 0 & i \neq j. \end{cases} \tag{4.77}$$

Therefore for all $j = 1, 2, \ldots, m$, the gradient of J_d is given by

$$\frac{\partial J_d}{\partial \overline{K}_j} = \left[\frac{\partial J_d}{\partial H} \ -\frac{\partial J_d}{\partial \overline{G}_j} \right] = \left[\sum_{j=1}^m \frac{\partial J_{d_j}(H, \overline{G}_j, \overline{X}_{0_j})}{\partial H} \ -\frac{\partial J_{d_j}(H, \overline{G}_j, \overline{X}_{0_j})}{\partial \overline{G}_j} \right] = 0 \tag{4.78}$$

where

$$\begin{aligned} \frac{\partial J_d}{\partial H} &= 2\sum_{j=1}^m \left\{ \left[\tilde{R}_j H - B^\top \overline{P}_{1_j}\right] \overline{L}_{1_j} - \left[\tilde{R}_j \overline{G}_j + B^\top \overline{P}_{2_j}\right] \overline{L}_{2_j}^\top \right\} \quad (a) \\ \frac{\partial J_d}{\partial \overline{G}_j} &= -2\left\{ \left[\tilde{R}_j H - B^\top \overline{P}_{1_j}\right] \overline{L}_{2_j} - \left[\tilde{R}_j \overline{G}_j + B^\top \overline{P}_{2_j}\right] \overline{L}_3 \right\} \quad (b) \end{aligned} \tag{4.79}$$

and

$$\overline{P}_{1_j}(A - BH) + (A - BH)^\top \overline{P}_{1_j} + H^\top \tilde{R}_j H + \overline{Q}_{1_j} = 0 \quad (a)$$

$$\overline{P}_{2_j} A_r + (A - BH)^\top \overline{P}_{2_j} + \overline{P}_{1_j} B \overline{G}_j - H^\top \tilde{R}_j \overline{G}_j + \overline{Q}_{2_j} = 0 \quad (b)$$

$$\overline{P}_{3_j} A_r + A_r^\top \overline{P}_{3_j} + \overline{P}_{2_j}^\top B \overline{G}_j + \overline{G}_j^\top B^\top \overline{P}_{2_j} + \overline{G}_j^\top \tilde{R}_j \overline{G}_j + \overline{Q}_{3_j} = 0 \quad (c)$$

$$\overline{L}_{1_j}(A - BH)^\top + (A - BH)\overline{L}_{1_j} + \overline{L}_{2_j} \overline{G}_j^\top B^\top + B \overline{G}_j \overline{L}_{2_j}^\top + \overline{\Pi}_{1_j} = 0 \quad (d)$$

$$\overline{L}_{2_j} A_r^\top + (A - BH)\overline{L}_{2_j} + B \overline{G}_j \overline{L}_{3_j} + \overline{\Pi}_{2_j} = 0 \quad (e)$$

$$\overline{L}_3 A_r^\top + A_r \overline{L}_3 + \overline{\Pi}_3 = 0 \quad (f)$$

or,

$$\overline{P}_j \overline{A}_{cj} + \overline{A}_{cj}^\top \overline{P}_j + \overline{K}_j^\top \tilde{R}_j \overline{K}_j + \overline{Q}_j = 0 \quad (a)$$

$$\overline{L}_j \overline{A}_{cj}^\top + \overline{A}_{cj} \overline{L}_j + \overline{\Pi}_j = 0 \quad (b).$$

$\square$

Corollary 4.2 *The necessary conditions for optimality for the ODC problem (4.40) are*

$$\sum_{j=1}^m \left\{ \left[\tilde{R}_j H - B^\top \overline{P}_{1_j} \right] \overline{L}_{1_j} - \left[\tilde{R}_j \overline{G}_j + B^\top \overline{P}_{2_j} \right] \overline{L}_{2_j}^\top \right\} = 0 \quad (a)$$

and for all $j = 1, 2, \ldots, m$

$$\left[\tilde{R}_j H - B^\top \overline{P}_{1_j} \right] \overline{L}_{2_j} - \left[\tilde{R}_j \overline{G}_j + B^\top \overline{P}_{2_j} \right] \overline{L}_3 = 0 \quad (b)$$

(4.80)

where $\overline{P}_{1_j}, \overline{P}_{2_j}, \overline{L}_{1_j}, \overline{L}_{2_j}$ and $\overline{L}_3$ satisfy (4.72).

Proof: Follows directly from Theorem 4.2 by letting all the gradient matrices be zero.

$\square$

As for the OFSC-LQR necessary conditions for optimality of Corollary 4.1, the ODC conditions for optimality given by Corollary 4.2 are neither actually implementable nor are they truly a solution to the real problem. First, the solution which is obtained from (4.79) and (4.72) depends on the initial state. Moreover, for a given initial state, there may exist several local minima for the state-feedback ODC problem (4.39) or (4.40).

Actually, such a local minimum may be found, using a standard optimization algorithm, without using explicitly the conditions of Corollary 4.2. Then, the gradient

information provided by (4.79) and (4.72) may be used in the optimization process. For example, recall Hirzinger's suggestion to solve a similar problem (for step and impulse inputs) using the conjugate gradient method [29].

Although the results obtained in this chapter are not generally implementable, they are used to derive the sub-optimal (but more "global") solutions presented in the next chapter.

Chapter 5

Sub-optimal solutions to the state-feedback ODC problem

5.1 Introduction

The ideal global solutions presented in Section 3.5 and Section 4.2 are not totally general because they can be applied only to stable plants. Furthermore they require high-order controllers. On the other hand the state-feedback fixed structure ODC of Section 4.3.3 depends on the system initial state and, thus, it is not totally general either. In this chapter we present some sub-optimal state-feedback solutions to the ODC problem.

In order to obtain the sub-ODC (SODC) solutions of this chapter we add some assumptions and/or use approximations. Therefore these solutions are no longer "optimal". Intuitively speaking, all the different SODC solutions of this chapter have some "average" properties.

The first SODC, average initial state, solution is obtained by randomizing the initial augmented state. This solution is more "global" than the one which may be obtained from the necessary conditions of Corollary 4.2, because it no longer depends explicitly on the system initial state. However it is still "local" because it is obtained from (other) necessary conditions for optimality. The second sub-optimal solution,

based on average weights, is not really an ODC solution because it is derived as an ordinary LQT solution for the average W_{e_j}'s and W_{u_j}'s. The third sub-optimal solution, based on average gain, is a LMS solution, based on the ideal state-feedback (free structure) ODC solutions of Section 4.2. The last two SODC are completely global solutions (satisfy the global minimum of a sub-optimal cost function and do not depend explicitly on the initial states) and are much easier to compute.

5.2 SODC for randomized initial state

In order to remove the explicit dependence of the optimal solution on the initial states, we assume that the initial (system and input) states are random and normally distributed. This idea was proposed by Levine and Athans [46] and has been widely used to solve OFSC problems (for example see [25], [5], [50]). Using the same approach, under some additional mild assumptions, the necessary conditions for optimality of Corollary 4.1 transform to a set of equations from which the average initial state SODC can be obtained.

5.2.1 The state-feedback SODC problem for random initial state

The state-feedback SODC problem can now be written as the following optimization problem

$$\min_{\overline{K}_1 \cdots \overline{K}_m \in \mathcal{K}} J_d(\overline{K}_1 \cdots \overline{K}_m) \tag{5.1}$$

where

$$J_d(\overline{K}_1 \cdots \overline{K}_m) = \sum_{j=1}^{m} J_{d_j}(\overline{K}_j) = \sum_{j=1}^{m} E\left\{ \int_0^\infty (\overline{X}_j^\top \overline{Q}_j \overline{X}_j + \bar{u}_j^\top \widetilde{R}_j \bar{u}_j) dt \right\} \tag{5.2}$$

subject to the minimal state space realizations

$$\begin{aligned} \dot{\bar{x}}_j &= A\bar{x}_j + B\bar{u}_j \\ \bar{y}_j &= C\bar{x}_j \end{aligned} \quad j = 1, 2, \ldots, m \tag{5.3}$$

and to the input reference signal dynamics

$$\begin{aligned} \dot{x}_r &= A_r x_r \\ r &= C_r x_r \end{aligned}$$ (5.4)

where A_r is asymptotically stable. The initial augmented states $\overline{X}_{0_j} = \begin{pmatrix} \bar{x}_j(0) \\ x_r(0) \end{pmatrix}$ are assumed to be normally distributed with zero mean and covariance $\overline{\Pi}_j = E\left\{\overline{X}_{0j}\overline{X}_{0j}^\top\right\} > 0$.

We also add the following assumptions.

(A1) $\tilde{R}_1 = \tilde{R}_2 = \cdots = \tilde{R}_m = R > 0$.

(A2) $\overline{Q}_j > 0$, for all $j = 1, 2, \ldots, m$.

Remark 5.1 *(A1) is a fair assumption (see also Remark 3.2). However assumption (A2) seems to be a strong one. Although it is always made for the regulator problems (e.g., the constant-gain, output-feedback, OFSC problem [46], [66], [25], etc.), it does not hold for general tracking problems (then it is true if and only if $m = n = n_r$ and C and C_r are invertible). Therefore theoretically it does not hold for the general ODC problem. However practically $\overline{Q}_j$ may be replaced with a positive definite approximation, say $\overline{Q}_j + \epsilon_j I$ where ϵ_j is positive and arbitrary small. Then instead of minimizing the original cost function J_d we minimize its upper bound $J_\epsilon = J_d + \sum_{j=1}^m \epsilon_j \int_0^\infty \overline{X}_j^\top \overline{X}_j dt > J_d$. Theoretically J_ϵ may not be arbitrary tight, i.e., $\epsilon_j \int_0^\infty \overline{X}_j^\top \overline{X}_j dt$ may not go to zero as $\epsilon_j \to 0$. Note that $\overline{Q}_j > 0$ is used to prove convergence of a numerical algorithm, which for many cases converges with $\overline{Q}_j \geq 0$. Therefore it is not a necessary condition. Moreover for its uses in the sequel, (A2) can be replaced by a weaker assumption, "$\overline{Q}_j > 0$, for **some** $j = 1, 2, \ldots, m$", which makes J_ϵ a tighter bound.*

Theorem 5.1 *Consider the m simultaneous control tracking systems given by (5.3) and (5.4) with random initial state. For any stabilizing H and $\overline{G}_1 \cdots \overline{G}_m$ let the average*

componentwise cost J_d be given by (5.2) with assumption (A1). Then

$$
\begin{aligned}
H &= R^{-1}B^\top P_1^{\#} + \tfrac{1}{2}R^{-1}\Xi\Upsilon^{-1} \\
\overline{G}_j &= R^{-1}B^\top P_{2_j}^{\#} + \tfrac{1}{2}R^{-1}\frac{\partial J_d}{\partial \overline{G}_j}\overline{L}_3^{-1} \quad j=1,2,\ldots,m
\end{aligned}
$$

with $\Upsilon = \sum_{j=1}^m L_j^{\#} > 0$, $P_1^{\#} = \sum_{j=1}^m \overline{P}_{1_j}L_j^{\#}\Upsilon^{-1}$, $\Xi = \frac{\partial J_d}{\partial H} + \sum_{j=1}^m \frac{\partial J_d}{\partial \overline{G}_j}\overline{L}_3^{-1}\overline{L}_{2_j}^\top$ and for all $j = 1,2,\ldots,m$, $L_j^{\#} = \overline{L}_{1_j} - \overline{L}_{2_j}\overline{L}_3^{-1}\overline{L}_{2_j}^\top > 0$, $P_{2_j}^{\#} = \left[P_1^{\#} - \overline{P}_{1_j}\right]\overline{L}_{2_j}\overline{L}_3^{-1} - \overline{P}_{2_j}\cdot$, where $\overline{P}_{1_j} > 0$ and $\overline{P}_{2_j}$ satisfy (4.72-a,b) and $\overline{L}_j > 0$ satisfies (4.72-b) with $\overline{\Pi}_j = E\left\{\overline{X}_{0j}\overline{X}_{0j}^\top\right\} > 0$.

Proof: Using the gradient expressions of Theorem 4.2 with $\overline{\Pi}_j = E\left\{\overline{X}_{0j}\overline{X}_{0j}^\top\right\} > 0$, we first consider equation (4.72-f), where A_r is asymptotically stable and $\overline{\Pi}_3 = E\left\{x_{r_0}x_{r_0}^\top\right\} > 0$. Thus L_3 is the unique positive definite solution to (4.72-f). Then from (4.71-b) we can write,

$$
\widetilde{R}_j\overline{G}_j + B^\top\overline{P}_{2_j} = \left[\widetilde{R}_jH - B^\top\overline{P}_{1_j}\right]\overline{L}_2\,\overline{L}_3^{-1} + \frac{1}{2}\frac{\partial J_d}{\partial \overline{G}_j}\overline{L}_3^{-1}. \tag{5.5}
$$

Substituting (5.5) into (4.71-a) yields

$$
\begin{aligned}
\sum_{j=1}^m\Big\{ \left[\widetilde{R}_jH - B^\top\overline{P}_{1_j}\right]\overline{L}_{1_j} - & \\
\left[\widetilde{R}_jH - B^\top\overline{P}_{1_j}\right]\overline{L}_{2_j}\overline{L}_3^{-1}\overline{L}_{2_j}^\top - \tfrac{1}{2}\tfrac{\partial J_d}{\partial\overline{G}_j}\overline{L}_3^{-1}\overline{L}_{2_j}^\top\Big\} &= \tfrac{1}{2}\tfrac{\partial J_d}{\partial H} &\Rightarrow \\
\sum_{j=1}^m\Big\{\left[\widetilde{R}_jH - B^\top\overline{P}_{1_j}\right]\left[\overline{L}_{1_j} - \overline{L}_{2_j}\overline{L}_3^{-1}\overline{L}_{2_j}^\top\right]\Big\} &= \tfrac{1}{2}\left[\tfrac{\partial J_d}{\partial H} + \sum_{j=1}^m\tfrac{\partial J_d}{\partial\overline{G}_j}\overline{L}_3^{-1}\overline{L}_{2_j}^\top\right] &\Rightarrow \\
\sum_{j=1}^m\Big\{\left[\widetilde{R}_jH - B^\top\overline{P}_{1_j}\right]L_j^{\#}\Big\} &= \tfrac{1}{2}\Xi &\Rightarrow \\
\sum_{j=1}^m\widetilde{R}_jHL_j^{\#} &= B^\top\sum_{j=1}^m\overline{P}_{1_j}L_j^{\#} + \tfrac{1}{2}\Xi
\end{aligned}
$$

$$\tag{5.6}$$

where $L_j^{\#}$ is the Schur complement of $\overline{L}_j$ (i.e., $L_j^{\#} = \overline{L}_{1_j} - \overline{L}_{2_j}\overline{L}_3^{-1}\overline{L}_{2_j}^\top$), and $\Xi = \frac{\partial J_d}{\partial H} + \sum_{j=1}^m \frac{\partial J_d}{\partial\overline{G}_j}\overline{L}_3^{-1}\overline{L}_{2_j}^\top$. Using assumption (A1) yields

$$
RH\sum_{j=1}^m L_j^{\#} = B^\top\sum_{j=1}^m\overline{P}_{1_j}L_j^{\#} + \frac{1}{2}\Xi. \tag{5.7}
$$

By assumption, for all $j = 1, 2, \ldots, m$, A_{c_j} is asymptotically stable and $\overline{\Pi}_j > 0$. Therefore $\overline{L}_j$ is the unique positive definite solution to (4.73-b), which implies that $L_j^{\#} > 0$ (the Schur complement is a symmetric positive definite matrix [32]). Hence equation (5.7) can be rewritten as

$$
\begin{aligned}
H &= R^{-1}B^{\mathsf{T}} \sum_{j=1}^{m} \overline{P}_{1_j} L_j^{\#} \left[\sum_{j=1}^{m} L_j^{\#} \right]^{-1} + \tfrac{1}{2} R^{-1} \Xi \left[\sum_{j=1}^{m} L_j^{\#} \right]^{-1} \\
&= R^{-1}B^{\mathsf{T}} \sum_{j=1}^{m} \overline{P}_{1_j} L_j^{\#} \Upsilon^{-1} + \tfrac{1}{2} R^{-1} \Xi \Upsilon^{-1} \\
&= R^{-1}B^{\mathsf{T}} P_1^{\#} + \tfrac{1}{2} R^{-1} \Xi \Upsilon^{-1}
\end{aligned}
\tag{5.8}
$$

where $\Upsilon = \left[\sum_{j=1}^{m} L_j^{\#} \right]$ and $P_1^{\#} = \sum_{j=1}^{m} \overline{P}_{1_j} L_j^{\#} \Upsilon^{-1}$ (i.e., $P_1^{\#}$ is a "weighted average" of P_{1_j}'s).

Finally, the expression for G_j is derived from (4.71-b)

$$
\begin{aligned}
\overline{G}_j &= R^{-1} \left[RH\overline{L}_{2_j} - B^{\mathsf{T}} \overline{P}_{2_j} \overline{L}_3 - B^{\mathsf{T}} \overline{P}_{1_j} \overline{L}_{2_j} + \frac{\partial J_d}{\partial \overline{G}_j} \right] \overline{L}_3^{-1} \\
&= \left[H - R^{-1}B^{\mathsf{T}} \overline{P}_{1_j} \right] \overline{L}_{2_j} \overline{L}_3^{-1} - R^{-1}B^{\mathsf{T}} \overline{P}_{2_j} + R^{-1} \frac{\partial J_d}{\partial \overline{G}_j} \overline{L}_3^{-1} \\
&= R^{-1}B^{\mathsf{T}} \left\{ \left[P_1^{\#} - \overline{P}_{1_j} \right] \overline{L}_{2_j} \overline{L}_3^{-1} - \overline{P}_{2_j} \right\} + R^{-1} \frac{\partial J_d}{\partial \overline{G}_j} \overline{L}_3^{-1} \\
&= R^{-1}B^{\mathsf{T}} P_{2_j}^{\#} + R^{-1} \frac{\partial J_d}{\partial \overline{G}_j} \overline{L}_3^{-1}
\end{aligned}
\tag{5.9}
$$

where $P_{2_j}^{\#} = \left[P_1^{\#} - \overline{P}_{1_j} \right] \overline{L}_{2_j} \overline{L}_3^{-1} - \overline{P}_{2_j}$.

$\square$

Corollary 5.1 *Consider the SODC problem given by (5.1)-(5.4) with random initial state and with assumption (A1). Then any stabilizing H and $\overline{G}_1 \cdots \overline{G}_m$ which satisfy the necessary conditions for optimality for this problem are given by*

$$
\begin{aligned}
H &= R^{-1}B^{\mathsf{T}} P_1^{\#} \\
\overline{G}_j &= R^{-1}B^{\mathsf{T}} P_{2_j}^{\#} \quad j = 1, 2, \ldots, m.
\end{aligned}
$$

Proof: Follows directly from Theorem 5.1 by letting all the gradient matrices be zero ($\Rightarrow$ also $\Xi = 0$).

$\square$

Substituting H and $\overline{G}_1 \cdots \overline{G}_m$ of Corollary 5.1 into (4.73), we get a set of $2m$ coupled, $(n+nr)$-order, nonlinear matrix equations with $2m$ unknowns $\overline{P}_1, \overline{P}_2, \ldots, \overline{P}_m$ and $\overline{L}_1, \overline{L}_2, \ldots, \overline{L}_m$. Recall that $\overline{L}_3$ depends only on the input reference signal (it is obtained a-priori by solving (4.72-f) once), and that the $\overline{P}_{3_j}$'s are not required for the solution. Therefore this set can be replaced by the set of partitioned, nonlinear, coupled, matrix equations (4.72-a, b, d, e) with the $4m$ unknowns, $\overline{P}_{1_1}, \overline{P}_{1_2}, \ldots, \overline{P}_{1_m}$, $\overline{P}_{2_1}, \overline{P}_{2_2}, \ldots, \overline{P}_{2_m}, \overline{L}_{1_1}, \overline{L}_{1_2}, \ldots, \overline{L}_{1_m}$ and $\overline{L}_{2_1}, \overline{L}_{2_2}, \ldots, \overline{L}_{2_m}$.

Remark 5.2 *Replacing H by arbitrary H_j, (4.72) becomes the standard OFSC coupled Lyapunov/Riccati type set of equations. In fact these equations, for arbitrary H_j, resemble the more complicated, decentralized OFSC case (e.g., [20] and [67]). Of course, the H_j's are not arbitrary ($H_j = H$, $\forall\, j = 1, 2, , m$) in the ODC problem, therefore H and $\overline{G}_j$ of Corollary 5.1 cause an additional "coupling" between the different j's. This coupling makes the solution to the ODC problem more complicated.*

5.2.2   Solution using a descent Anderson and Moore type algorithm

Standard OFSC problems can be solved using several different types of numerical algorithms, such as: the Levine-Athans algorithm [25,46,50,66], the Anderson-Moore algorithm [1,25,49,50], Newton's method [5,6,50] and others (for a detailed survey see [5] and [50]).

In the following we introduce an algorithm based on the descent Anderson and Moore (DAM) algorithm for solving the above sets of equations [5]. The original algorithm proposed by Anderson and Moore [1] has no convergence properties because it is based on a successive approximation approach (i.e., fixed step size) [25]. Later Mäkilä [49] showed that the successor used by Anderson and Moore is in fact a direction of descent for the minimized quadratic cost function. Therefore with a proper step size (obtained by a line search), the algorithm converges. A similar idea is used in the following algorithm to solve the set of nonlinear coupled matrix equations (4.72) with H and $\overline{G}_j$ of Corollary 5.1.

Remark 5.3 *Other algorithms with probably better convergence rates may be obtained by modifying other OFSC algorithms [5], to solve the ODC problem. However the main goal of this dissertation is to show that the ODC problem can be practically solved using the new formulations presented in the previous chapters. Therefore this relatively simple algorithm, which at each iteration, requires only solution of a set of linear matrix equations, was chosen.*

The DAM type algorithm

- **Step 0:**

    - Set $k = 0$.

    - Solve the Lyapunov equation (4.72-f) to obtain $\overline{L}_3$.

    - Choose $H^0 \in \mathcal{S}, \overline{G}_1^0, \overline{G}_2^0, \ldots, \overline{G}_m^0$.

- **Step 1:**

    - Solve equations (4.72-a, b, d, e) for $H^k, \overline{G}_1^k, \overline{G}_2^k, \ldots, \overline{G}_m^k$ to obtain $\overline{P}_{1_1}^k, \overline{P}_{1_2}^k,$
    $\ldots, \overline{P}_{1_m}^k, \overline{P}_{2_1}^k, \overline{P}_{2_2}^k, \ldots, \overline{P}_{2_m}^k, \overline{L}_{1_1}^k, \overline{L}_{1_2}^k, \ldots, \overline{L}_{1_m}^k$ and $\overline{L}_{2_1}^k, \overline{L}_{2_2}^k, \ldots, \overline{L}_{2_m}^k$.

    - Compute directions of descent (see proof of Theorem 5.2 below)

$$\Delta H^k = R^{-1} B^\top P_1^{\#\,k} - H^k \tag{5.10}$$

and

$$\Delta \overline{G}_j^k = R^{-1} B^\top P_{2_j}^{\#\,k} - \overline{G}_j^k. \tag{5.11}$$

    - Check the termination condition. If $\left\| \left[\Delta H^k \ \Delta \overline{G}_1^k \ \cdots \ \Delta \overline{G}_m^k \right] \right\|_F \leq \delta$, stop ($\delta$ is a small positive tolerance parameter). Else, go to Step 2.

- **Step 2:**

    - Perform a line search to find the "best" step size α^k, i.e.,

$$\alpha^k = \arg \min_{\alpha^k > 0} J(H^k + \alpha^k \Delta H^k, \overline{G}_1^k + \alpha^k \Delta \overline{G}_1^k, \ldots, \overline{G}_m^k + \alpha^k \Delta \overline{G}_m^k). \tag{5.12}$$

95

- **Step 3:**

  - Set $H^{k+1} = H^k + \alpha^k \Delta H^k$, $\overline{G}_1^{k+1} = \overline{G}_1^k + \alpha^k \Delta \overline{G}_1^k$, $\ldots, \overline{G}_m^{k+1} = \overline{G}_m^k + \alpha^k \Delta \overline{G}_m^k$.

  - Set $k = k + 1$.

  - Go to Step 1.

5.2.3 Global convergence of the numerical algorithm to a stationary point of the cost function

The main difficulty in analyzing the convergence properties of the above algorithm is that the stabilizing set over which the minimization is performed is, in general, unbounded and not closed. Therefore we start this discussion, by introducing the following level set

$$\widetilde{\mathcal{K}}_{\overline{K}_1^0 \cdots \overline{K}_m^0} = \left\{ \begin{array}{c} \overline{K}_j \in \mathcal{K} \\ \forall j = 1,2,\ldots,m \end{array} \mid J_d(\overline{K}_1 \cdots \overline{K}_m) \leq J_d(\overline{K}_1^0 \cdots \overline{K}_m^0) \right\} \qquad (5.13)$$

where $\overline{K}_j = [H \ -\overline{G}_j]$ and $\mathcal{K} = \left\{ \overline{K}_j \in \mathbb{R}^{m \times (n+n_r)} \mid A - BH \text{ is asymptotically stable} \right\}$. (Note that if $\widetilde{\mathcal{K}}_{\overline{K}_j}^j = \left\{ \overline{K}_j \in \mathcal{K} \mid J_{d_j}(\overline{K}_j) \leq J_{d_j}(\overline{K}_j^0) \right\}$, then the cartesian product $\widetilde{\mathcal{K}}_{\overline{K}_1^0}^1 \times \widetilde{\mathcal{K}}_{\overline{K}_2^0}^2 \times \cdots \times \widetilde{\mathcal{K}}_{\overline{K}_m^0}^m \subseteq \widetilde{\mathcal{K}}_{\overline{K}_1^0 \cdots \overline{K}_m^0}$).

Lemma 5.1 *Let $U \in \mathbb{R}^{p \times p}$ and $V \in \mathbb{R}^{q \times q}$ be symmetric positive definite matrices. Then for all $X \in \mathbb{R}^{p \times q}$ there exist $a, b > 0$ such that*

$$a \|X\| \leq \left[\mathrm{tr} \left\{ X^\top U X V \right\} \right]^{1/2} \leq b \|X\|.$$

Proof: Let $Y, Z \in \mathbb{R}^{p \times q}$ and define $\langle X, Y \rangle = \text{tr} \left\{ X^\top U Y V \right\}$, then using the properties of the trace operator (4.42)

$$
\begin{aligned}
(a) \qquad \langle X, Y \rangle &= \text{tr} \left\{ X^\top U Y V \right\} = \text{tr} \left\{ V^{\top/2} X^\top U^{1/2} U^{\top/2} Y V^{1/2} \right\} \\
&= \text{tr} \left\{ V^{\top/2} Y^\top U^{1/2} U^{\top/2} X V^{1/2} \right\} = \text{tr} \left\{ Y^\top U X V \right\} \\
&= \langle Y, X \rangle .
\end{aligned}
$$

$$
\begin{aligned}
(b) \quad \langle \alpha X + \beta Z, Y \rangle &= \text{tr} \left\{ (\alpha X + \beta Z)^\top U Y V \right\} = \alpha \, \text{tr} \left\{ X^\top U Y V \right\} \\
&\quad + \beta \, \text{tr} \left\{ Z^\top U Y V \right\} \\
&= \alpha \langle X, Y \rangle + \beta \langle Z, Y \rangle \text{ for scalars } \alpha \text{ and } \beta.
\end{aligned}
$$

$$
\begin{aligned}
(c) \qquad \langle X, X \rangle &= \text{tr} \left\{ X^\top U X V \right\} = \text{tr} \left\{ V^{\top/2} X^\top U^{1/2} U^{\top/2} X V^{1/2} \right\} \\
&= \text{tr} \left\{ A^\top A \right\} \geq 0, \text{ where } A = U^{\top/2} X V^{1/2}.
\end{aligned}
$$

$$
\begin{aligned}
(d) \qquad \langle X, X \rangle &= 0 \text{ if and only if } X = 0. \\
X = 0 \quad &\rightarrow \quad \langle X, X \rangle = 0. \\
\langle X, X \rangle = 0 \quad &\rightarrow \quad 0 = \text{tr} \left\{ A^\top A \right\} \geq \left\| A^\top A \right\| = \| A \|^2 \rightarrow \| A \| = 0 \rightarrow \\
&\qquad A = 0 \rightarrow X = U^{-\top/2} A V^{-1/2} = 0.
\end{aligned}
$$

By (a)-(d), $\langle X, Y \rangle$ is an inner product for $\mathbb{R}^{p \times q}$ and $[\langle X, X \rangle]^{1/2} = \left[\text{tr} \left\{ X^\top U X V \right\} \right]^{1/2}$ is the norm induced by it. The assertion of the lemma is a well-known equivalence property of norms on finite dimensional real spaces.

$\square$

Lemma 5.2 *Consider the ODC problem given by (5.1)-(5.4) with assumption (A2). Then there exists an $\alpha < 0$ such that $\left\{ \overline{K}_1, \overline{K}_2, \ldots, \overline{K}_m \right\} \in \tilde{\mathcal{K}}_{\overline{K}_1^0 \cdots \overline{K}_m^0}$ implies $\max_i \Re \left\{ \lambda_i (\overline{A} - \overline{B} \overline{K}_j) \right\} \leq \alpha < 0$ for all $j = 1, 2, \ldots, m$.*

Proof: From (4.49) $J_{d_j}(\overline{K}_j) = \text{tr} \left\{ \overline{P}_j(\overline{K}_j) \overline{\Pi}_j \right\} = \text{tr} \left\{ \overline{P}_j^{\top/2}(\overline{K}_j) \overline{\Pi}_j \overline{P}_j^{1/2}(\overline{K}_j) \right\}$. For all $j = 1, 2, \ldots, m$, $\overline{\Pi}_j > 0$. Then from Lemma 5.1 there exists $a_j > 0$ such that $J_{d_j}(\overline{K}_j) \geq a_j \left\| \overline{P}_j^{1/2}(\overline{K}_j) \right\|$.

Then $\left\{ \overline{K}_1, \overline{K}_2, \ldots, \overline{K}_m \right\} \in \tilde{\mathcal{K}}_{\overline{K}_1^0 \cdots \overline{K}_m^0}$ implies

$$
\begin{aligned}
J_d(\overline{K}_1^0 \cdots \overline{K}_m^0) &\geq J_d(\overline{K}_1 \cdots \overline{K}_m) = \sum_{j=1}^m J_{d_j}(\overline{K}_j) \\
&\geq J_{d_j}(\overline{K}_j) = \text{tr} \left\{ \overline{P}_j(\overline{K}_j) \overline{\Pi}_j \right\} \geq a_j \left\| \overline{P}_j^{1/2}(\overline{K}_j) \right\|
\end{aligned}
$$

or for all $j = 1, 2, \ldots, m$

$$\rho_j \geq \left\| \overline{P}_j(\overline{K}_j) \right\| = \left\| \overline{P}_j \right\|$$

where ρ_j is a finite constant.

Let $\overline{A}_{c_j} = \overline{A} - \overline{B}\overline{K}_j = \begin{bmatrix} A - BH & B\overline{G}_j \\ 0 & A_r \end{bmatrix}$. Then $\overline{A}_{c_j}$ has the same eigenvalues λ_i $i = 1, 2, \ldots, n$, for all $j = 1, 2, \ldots, m$. Let λ_m be the eigenvalue of $\overline{A}_{c_j}$ with the largest real part (i.e., $\Re\{\lambda_m\} = \max_i \Re\left\{\lambda_i(\overline{A} - \overline{B}\overline{K}_j)\right\}$). Let u be the normalized ($\|u\|^2 = 1$) eigenvector corresponding to λ_m, and let $M_j = \overline{K}_j^\top R \overline{K}_j + \overline{Q}_j$. Then from (4.73) for all $j = 1, 2, \ldots, m$

$$\overline{P}_j \overline{A}_{c_j} + \overline{A}_{c_j}^\top \overline{P}_j + M_j = 0$$

and

$$
\begin{aligned}
\left\langle u, \overline{P}_j \overline{A}_{c_j} u \right\rangle + \left\langle u, \overline{A}_{c_j}^\top \overline{P}_j u \right\rangle + \langle u, M_j u \rangle &= \\
\left\langle \overline{P}_j u, \overline{A}_{c_j} u \right\rangle + \left\langle \overline{A}_{c_j} u, \overline{P}_j u \right\rangle + \langle u, M_j u \rangle &= \\
2\Re\{\lambda_m\} \left\langle u, \overline{P}_j u \right\rangle + \langle u, M_j u \rangle &= 0
\end{aligned}
$$

where by assumption (A2) $M_j \geq \overline{Q}_j > 0$ and $\langle u, M_j u \rangle \geq \lambda_{\min}(M_j) > 0$. Therefore

$$\rho_j \geq \left\| \overline{P}_j \right\| = \left\| u^\top \right\| \left\| \overline{P}_j \right\| \|u\| \geq \left\langle u, \overline{P}_j u \right\rangle = -\frac{\langle u, M_j u \rangle}{2\Re\{\lambda_m\}} > 0$$

or

$$\Re\{\lambda_m\} \leq -\frac{\langle u, M_j u \rangle}{2\rho_j} = \alpha_j < 0$$

for all $j = 1, 2, \ldots, m$, or

$$\Re\{\lambda_m\} \leq \alpha$$

where $\alpha = \max_j \alpha_j < 0$.

$\square$

Lemma 5.3 *Consider the ODC problem given by (5.1)-(5.4) with assumption (A2). Then $\widetilde{\mathcal{K}}_{\overline{K}_1^0 \cdots \overline{K}_m^0}$ is a compact set.*

Proof: The compactness property is established by showing that $\widetilde{\mathcal{K}}_{\overline{K}_1^0 \cdots \overline{K}_m^0}$ is bounded and closed. From (4.73), by the positive definiteness of $\overline{L}_j$, and by Lemma 5.1, for all

$j = 1, 2, \ldots, m$ there exists $a_j > 0$ such that $\overline{K}_j \in \tilde{\mathcal{K}}_{\overline{K}_1^0 \cdots \overline{K}_m^0}$ implies

$$
\begin{aligned}
J_d(\overline{K}_1^0 \cdots \overline{K}_m^0) \;\geq\; J_d(\overline{K}_1 \cdots \overline{K}_m) \quad &=\; \textstyle\sum_{j=1}^m J_{d_j}(\overline{K}_j) \quad \geq\; J_{d_j}(\overline{K}_j) \\
=\; \mathrm{tr}\left\{ \left(\overline{K}_j^\top R\overline{K}_j + \overline{Q}_j\right)\overline{L}_j\right\} \;&>\; \mathrm{tr}\left\{\overline{K}_j^\top R\overline{K}_j \overline{L}_j\right\} \;\geq\; a_j \left\|\overline{K}_j\right\|.
\end{aligned}
$$

Therefore, for all $j = 1, 2, \ldots, m$

$$
\left\|\overline{K}_j\right\| < \frac{J_d(\overline{K}_1^0 \cdots \overline{K}_m^0)}{a_j} = \kappa_j \leq \kappa
$$

where $\kappa = \max\limits_j \kappa_j$ is a finite constant, i.e., $\tilde{\mathcal{K}}_{\overline{K}_1^0 \cdots \overline{K}_m^0}$ is bounded.

Define the set

$$
\tilde{\mathcal{K}}(\alpha) = \left\{ K \in \mathcal{K} \;\middle|\; \Re\left\{\lambda_i(\overline{A} - \overline{B}K)\right\}_{\forall\, i=1,2,\ldots,n} \leq \alpha < 0\right\}.
$$

Since the eigenvalues $\lambda_i(\overline{A} - \overline{B}\overline{K}_j)$ are continuous functions of $\overline{K}_j$, the set $\tilde{\mathcal{K}}(\alpha)$ is closed. From Lemma 5.2

$$
\tilde{\mathcal{K}}_{\overline{K}_1^0 \cdots \overline{K}_m^0} = \left\{ \overline{K}_j \in \tilde{\mathcal{K}}(\alpha)_{\forall\, j=1,2,\ldots,m} \;\middle|\; \sum_{j=1}^m J_{d_j}(\overline{K}_j) \leq \sum_{j=1}^m J_{d_j}(\overline{K}_j^0)\right\}
$$

$\overline{P}_j(\overline{K}_j)$, $\overline{L}_j(\overline{K}_j)$ and $J_{d_j}(\overline{K}_j)$ are all continuously differentiable on $\tilde{\mathcal{K}}(\alpha)$, thus $\tilde{\mathcal{K}}_{\overline{K}_1^0 \cdots \overline{K}_m^0}$ is closed.

$\square$

Theorem 5.2 *Consider the ODC problem given by (5.1)-(5.4) with assumptions (A1) and (A2). Then for any $\left\{\overline{K}_1^0,\ \overline{K}_2^0,\ \ldots,\ \overline{K}_m^0\right\} \in \mathcal{K}$ the above DAM type algorithm, with the termination test excluded, generates a sequence $\left\{\overline{K}_1^k,\ \overline{K}_2^k,\ \ldots,\ \overline{K}_m^k\right\} \in \tilde{\mathcal{K}}_{\overline{K}_1^0 \cdots \overline{K}_m^0}$ such that $\left\{\overline{K}_1^k,\ \overline{K}_2^k,\ \ldots,\ \overline{K}_m^k\right\} \to \widehat{\mathcal{K}}$ where $\widehat{\mathcal{K}}$ is a set of stationary points of the cost function (5.2) (i.e., all the points in $\widehat{\mathcal{K}}$ satisfy the necessary conditions for optimality of Corollary 5.1.*

Proof: From Theorem 5.1 any $\overline{K}_j \in \mathcal{K}$ can be written as $\overline{K}_j = [H\ -\overline{G}_j]$, with

$$
H = R^{-1}B^\top P_1^\# + \frac{1}{2}R^{-1}\Xi\Upsilon^{-1} \tag{5.14}
$$

and

$$\overline{G}_j = R^{-1}B^\top P_2^\# + \frac{1}{2}R^{-1}\frac{\partial J_{d_j}}{\partial \overline{G}_j}\overline{L}_3^{-1} \tag{5.15}$$

where $\Xi = \frac{\partial J_d}{\partial H} + \sum_{j=1}^m \frac{\partial J_d}{\partial \overline{G}_j}\overline{L}_3^{-1}\overline{L}_{2_j}^\top$ and $\Upsilon = \sum_{j=1}^m L_j^\#$.

Omitting the iteration index k, and substituting (5.14) into (5.10) and (5.15) into (5.11) yields

$$\Delta H = R^{-1}B^\top P_1^\# - H = -\frac{1}{2}R^{-1}\Xi\Upsilon^{-1} \tag{5.16}$$

and

$$
\begin{aligned}
\Delta \overline{G}_j = R^{-1}B^\top P_2^\# - \overline{G}_j &= -\tfrac{1}{2}R^{-1}\left(\Xi\Upsilon^{-1}\overline{L}_{2_j} + \frac{\partial J_{d_j}}{\partial \overline{G}_j}\right)\overline{L}_3^{-1} \\
&= \Delta H \overline{L}_{2_j}\overline{L}_3^{-1} - \tfrac{1}{2}R^{-1}\frac{\partial J_{d_j}}{\partial \overline{G}_j}\overline{L}_3^{-1}
\end{aligned}
\tag{5.17}
$$

Recall from (4.63) that

$$
\begin{aligned}
\Delta J_d &= \sum_{j=1}^m \Delta J_{d_j} = \sum_{j=1}^m \Delta J_{d_j}(\alpha\Delta\overline{K}_j) \\
&= \alpha\sum_{j=1}^m \operatorname{tr}\left\{2\overline{L}_j\left[\overline{K}_j^\top R - \overline{P}_j\overline{B}\right]\Delta\overline{K}_j\right\} + \sum_{j=1}^m o_j\left(\alpha\Delta\overline{K}_j\right)
\end{aligned}
\tag{5.18}
$$

and from Theorem 4.1

$$
\begin{aligned}
\Delta J_d &= \alpha\sum_{j=1}^m \operatorname{tr}\left\{\frac{\partial J_{d_j}}{\partial \overline{K}_j}^\top \Delta\overline{K}_j\right\} + \sum_{j=1}^m o_j\left(\alpha\Delta\overline{K}_j\right) \\
&= \alpha\sum_{j=1}^m \operatorname{tr}\left\{\left[\frac{\partial J_{d_j}}{\partial H} - \frac{\partial J_{d_j}}{\partial \overline{G}_j}\right]^\top\left[\Delta H - \Delta\overline{G}_j\right]\right\} \\
&\quad + o\left(\alpha\Delta H, \alpha\Delta\overline{G}_1, \dots, \alpha\Delta\overline{G}_m\right) \\
&= \alpha\sum_{j=1}^m \operatorname{tr}\left\{\begin{bmatrix} \frac{\partial J_{d_j}}{\partial H}^\top \Delta H & -\frac{\partial J_{d_j}}{\partial H}^\top \Delta\overline{G}_j \\ -\frac{\partial J_{d_j}}{\partial \overline{G}_j}^\top \Delta H & \frac{\partial J_{d_j}}{\partial \overline{G}_j}^\top \Delta\overline{G}_j \end{bmatrix}\right\} \\
&\quad + o\left(\alpha\Delta H, \alpha\Delta\overline{G}_1, \dots, \alpha\Delta\overline{G}_m\right) \\
&= \alpha\operatorname{tr}\left\{\frac{\partial \sum_{j=1}^m J_{d_j}}{\partial H}^\top \Delta H\right\} + \alpha\sum_{j=1}^m \operatorname{tr}\left\{\frac{\partial J_{d_j}}{\partial \overline{G}_j}^\top \Delta\overline{G}_j\right\} \\
&\quad + o\left(\alpha\Delta H, \alpha\Delta\overline{G}_1, \dots, \alpha\Delta\overline{G}_m\right) \\
&= \alpha\operatorname{tr}\left\{\frac{\partial J_d}{\partial H}^\top \Delta H\right\} + \alpha\sum_{j=1}^m \operatorname{tr}\left\{\frac{\partial J_{d_j}}{\partial \overline{G}_j}^\top \Delta\overline{G}_j\right\} \\
&\quad + o\left(\alpha\Delta H, \alpha\Delta\overline{G}_1, \dots, \alpha\Delta\overline{G}_m\right)
\end{aligned}
\tag{5.19}
$$

100

Substituting (5.17) into (5.19) yields

$$
\begin{aligned}
\Delta J_d \;=&\; \alpha \operatorname{tr}\left\{ \tfrac{\partial J_d}{\partial H}^{\top} \Delta H \right\} + \alpha \sum_{j=1}^{m} \operatorname{tr}\left\{ \tfrac{\partial J_{d_j}}{\partial \overline{G}_j}^{\top} \left(\Delta H \overline{L}_{2_j} \overline{L}_3^{-1} - \tfrac{1}{2} R^{-1} \tfrac{\partial J_{d_j}}{\partial \overline{G}_j} \overline{L}_3^{-1} \right) \right\} \\
&\; + o\left(\alpha \Delta H, \alpha \Delta \overline{G}_1, \dots, \alpha \Delta \overline{G}_m \right) \\
=&\; \alpha \operatorname{tr}\left\{ \tfrac{\partial J_d}{\partial H}^{\top} \Delta H \right\} + \alpha \sum_{j=1}^{m} \operatorname{tr}\left\{ \left(\tfrac{\partial J_{d_j}}{\partial \overline{G}_j} \overline{L}_3^{-1} \overline{L}_{2_j}^{\top} \right)^{\top} \Delta H \right\} - \\
&\; \tfrac{\alpha}{2} \sum_{j=1}^{m} \operatorname{tr}\left\{ \tfrac{\partial J_{d_j}}{\partial \overline{G}_j}^{\top} R^{-1} \tfrac{\partial J_{d_j}}{\partial \overline{G}_j} \overline{L}_3^{-1} \right\} + o\left(\alpha \Delta H, \alpha \Delta \overline{G}_1, \dots, \alpha \Delta \overline{G}_m \right) \\
=&\; \alpha \operatorname{tr}\left\{ \tfrac{\partial J_d}{\partial H}^{\top} \Delta H \right\} + \alpha \operatorname{tr}\left\{ \sum_{j=1}^{m} \left(\tfrac{\partial J_{d_j}}{\partial \overline{G}_j} \overline{L}_3^{-1} \overline{L}_{2_j}^{\top} \right)^{\top} \Delta H \right\} - \\
&\; \tfrac{\alpha}{2} \sum_{j=1}^{m} \operatorname{tr}\left\{ \tfrac{\partial J_{d_j}}{\partial \overline{G}_j}^{\top} R^{-1} \tfrac{\partial J_{d_j}}{\partial \overline{G}_j} \overline{L}_3^{-1} \right\} + o\left(\alpha \Delta H, \alpha \Delta \overline{G}_1, \dots, \alpha \Delta \overline{G}_m \right) \\
=&\; \alpha \operatorname{tr}\left\{ \left[\tfrac{\partial J_d}{\partial H} + \sum_{j=1}^{m} \tfrac{\partial J_{d_j}}{\partial \overline{G}_j} \overline{L}_3^{-1} \overline{L}_{2_j}^{\top} \right]^{\top} \Delta H \right\} \\
&\; - \tfrac{\alpha}{2} \sum_{j=1}^{m} \operatorname{tr}\left\{ \tfrac{\partial J_{d_j}}{\partial \overline{G}_j}^{\top} R^{-1} \tfrac{\partial J_{d_j}}{\partial \overline{G}_j} \overline{L}_3^{-1} \right\} + o\left(\alpha \Delta H, \alpha \Delta \overline{G}_1, \dots, \alpha \Delta \overline{G}_m \right) .
\end{aligned}
$$

$$(5.20)$$

and using (5.16) yields

$$
\begin{aligned}
\Delta J_d \;=&\; -\tfrac{\alpha}{2} \operatorname{tr}\left\{ \Xi^{\top} R^{-1} \Xi \Upsilon^{-1} \right\} - \tfrac{\alpha}{2} \sum_{j=1}^{m} \operatorname{tr}\left\{ \tfrac{\partial J_{d_j}}{\partial \overline{G}_j}^{\top} R^{-1} \tfrac{\partial J_{d_j}}{\partial \overline{G}_j} L_3^{-1} \right\} \\
&\; + o\left(\alpha \Delta H, \alpha \Delta \overline{G}_1, \dots, \alpha \Delta \overline{G}_m \right) \\
=&\; -\tfrac{\alpha}{2} \operatorname{tr}\left\{ \left(\Xi \Upsilon^{-\frac{1}{2}} \right)^{\top} R^{-1} \left(\Xi \Upsilon^{-\frac{1}{2}} \right) \right\} \\
&\; - \tfrac{\alpha}{2} \sum_{j=1}^{m} \operatorname{tr}\left\{ \left(\tfrac{\partial J_{d_j}}{\partial \overline{G}_j} L_3^{-\frac{1}{2}} \right)^{\top} R^{-1} \left(\tfrac{\partial J_{d_j}}{\partial \overline{G}_j} L_3^{-\frac{1}{2}} \right) \right\} \\
&\; + o\left(\alpha \Delta H, \alpha \Delta \overline{G}_1, \dots, \alpha \Delta \overline{G}_m \right) .
\end{aligned}
$$

$$(5.21)$$

The first two terms in (5.21) are negative for $\frac{\partial J}{\partial H} \neq 0$, $\frac{\partial J_j}{\partial \overline{G}_j} \neq 0$ and $\frac{o(\alpha \Delta K)}{\alpha \|\Delta K\|} \to 0$ as $\alpha \to 0$. Therefore for any iteration k there always exists $\bar{\alpha}$ such that for any $\alpha \in (0, \bar{\alpha}]$ (obtained by the line search of Step 2), $J_d^{k+1} < J_d^{k}$.

Hence $\left\{ J_d^{k} \right\} = \left\{ J_d\left(\overline{K}_1^{k}, \overline{K}_2^{k}, \dots, \overline{K}_m^{k} \right) \right\}$ is a monotone decreasing sequence, bounded from below ($J_d \geq 0$). Thus

$$
\left\{ J_d^{k+m} - J_d^{k} \right\} \to 0 \text{ for any } m \geq 0.
$$

101

In particular $\left\{\Delta J_d^k\right\} = \left\{J_d^{k+1} - J_d^k\right\} \to 0$.

Recall from (4.63) that $o = o\left(\alpha\Delta H, \alpha\Delta\overline{G}_1, \ldots, \alpha\Delta\overline{G}_m\right)$ can be written as

$$o = \alpha^{k\,2} \sum_{j=1}^m \operatorname{tr}\left\{\Delta\overline{K}_j^k{}^\top R\Delta\overline{K}_j^k\overline{L}_j^k + 2\Delta L_j^k\left[\overline{K}_j^k{}^\top R - \overline{P}_j^k\overline{B}\,\right]\Delta\overline{K}_j^k\right\}$$

$$= \alpha^{k\,2}\beta_k$$

where $\beta_k = \sum_{j=1}^m \operatorname{tr}\left\{\Delta\overline{K}_j^k{}^\top R\Delta\overline{K}_j^k\overline{L}_j^k + 2\Delta L_j^k\left[\overline{K}_j^k{}^\top R - \overline{P}_j^k\overline{B}\,\right]\Delta\overline{K}_j^k\right\}$. By the compactness of $\widetilde{\mathcal{K}}_{K_1^0\cdots K_m^0}$ and by the continuity of the trace operator and the solution to the Lyapunov equation, for all $k \geq 0$, β_k is bounded from above, i.e., there exists $\beta > 0$ such that

$$\beta_k \leq \beta, \ \forall k \geq 0.$$

Suppose $\frac{\partial J}{\partial H}$ and $\frac{\partial J_j}{\partial\overline{G}_j}$ are not zero. Let

$$M_k = -\tfrac{1}{2}\operatorname{tr}\left\{\left(\Xi\Upsilon^{-\frac{1}{2}}\right)^\top R^{-1}\left(\Xi\Upsilon^{-\frac{1}{2}}\right)\right\}$$

$$-\tfrac{1}{2}\sum_{j=1}^m \operatorname{tr}\left\{\left(\frac{\partial J_{d_j}}{\partial\overline{G}_j}L_3^{-\frac{1}{2}}\right)^\top R^{-1}\left(\frac{\partial J_{d_j}}{\partial\overline{G}_j}L_3^{-\frac{1}{2}}\right)\right\} < 0.$$

Now, the expression for the cost increment can be written as

$$\Delta J_d^k = \alpha^k M_k + \alpha^{k\,2}\beta_k \leq \alpha^k M_k + \alpha^{k\,2}\beta.$$

We continue by contradiction. Suppose that for all $k \geq 0$, M_k is bounded away from zero, i.e., there exists $M < 0$ such that

$$M_k \leq M < 0, \ \forall k \geq 0.$$

Then using the descent property (5.21), for all $k \geq 0$

$$\Delta J_d \leq \alpha^k M + \alpha^{k\,2}\beta < 0. \tag{5.22}$$

Recall that for any $k \geq 0$, α^k is a positive scalar. Then from (5.22) we conclude that any $\alpha^k \in \left(0, \frac{-M}{\beta}\right)$ is acceptable. For all $k \geq 0$, let us choose $\alpha^k = \alpha = \frac{-M}{2\beta}$, then

$$\Delta J_d^k = \Delta J_d \leq -\frac{M^2}{4\beta} < 0, \ \forall k \geq 0$$

102

which implies that $\left\{ J_d^k \right\} \to -\infty$. This contradicts the fact that $J_d \geq 0$. Therefore $M_k \to 0$ regardless of α^k. Hence,

$$\begin{aligned}
\left\{ \left\| \frac{\partial J^k}{\partial H^k} \right\| \right\} &\to 0 \\
\left\{ \left\| \frac{\partial J_j^k}{\partial \overline{G}_j^k} \right\| \right\} &\to 0 \text{ for all } j = 1, 2, \ldots, m
\end{aligned}$$

and

$$\begin{aligned}
\left\{ \frac{\partial J^k}{\partial H^k} \right\} &\to 0 \\
\left\{ \frac{\partial J_j^k}{\partial \overline{G}_j^k} \right\} &\to 0 \text{ for all } j = 1, 2, \ldots, m.
\end{aligned} \tag{5.23}$$

That is, any accumulation point of the infinite sequence $\left\{ \overline{K}_1^k \cdot \overline{K}_m^k \right\}$ generated by the above algorithm is stationary.

Further, from (5.16) and (5.17) it follows that also

$$\left\{ \Delta \overline{K}_j^k \right\} = \left\{ \left[\Delta H^k - \Delta \overline{G}_j^k \right] \right\} \to 0 \text{ for all } j = 1, 2, \ldots, m.$$

Let $\widehat{\mathcal{K}} = \left\{ \overline{K}_1 \cdots \overline{K}_m \in \widetilde{\mathcal{K}}_{\overline{K}_1^0 \cdots \overline{K}_m^0} \ \middle| \ \frac{\partial J_d}{\partial \overline{K}_j} = 0 \ \forall j = 1, 2, \ldots, m \right\}$ be a set of stationary points of the cost function $J_d \left(\overline{K}_1 \cdots \overline{K}_m \right)$. By the continuity of $\frac{\partial J_d \left(\overline{K}_1 \cdots \overline{K}_m \right)}{\partial \overline{K}_j}$ in $\overline{K}_1 \cdots \overline{K}_m$, $\widehat{\mathcal{K}}$ is closed. Moreover because $\widehat{\mathcal{K}} \subset \widetilde{\mathcal{K}}_{\overline{K}_1^0 \cdots \overline{K}_m^0}$, $\widehat{\mathcal{K}}$ is compact.

We show by contradiction that $\left\{ \overline{K}_1^k \cdots \overline{K}_m^k \right\} \to \widehat{\mathcal{K}}$, or that the distance between $\left\{ \overline{K}_1^k \cdots \overline{K}_m^k \right\}$ and $\widehat{\mathcal{K}}$ $\mathrm{d} \left(\left\{ \overline{K}_1^k \cdots \overline{K}_m^k \right\}, \widehat{\mathcal{K}} \right) \to 0$. Let

$$\mathrm{d} \left(\left\{ \overline{K}_1^k \cdots \overline{K}_m^k \right\}, \widehat{\mathcal{K}} \right) = \min_{K_1 \cdots K_m \in \widehat{\mathcal{K}}} \sum_{j=1}^m \left\| \overline{K}_j^k - K_j \right\|.$$

Then from the compactness of $\widehat{\mathcal{K}}$, $\mathrm{d} \left(\left\{ \overline{K}_1^k \cdots \overline{K}_m^k \right\}, \widehat{\mathcal{K}} \right)$ is a continuous function in $\overline{K}_1 \cdots \overline{K}_m$.

Suppose $\mathrm{d} \left(\left\{ \overline{K}_1^k \cdots \overline{K}_m^k \right\}, \widehat{\mathcal{K}} \right) \not\to 0$. Then there exists $\delta > 0$ and an infinite set of nonnegative indices $\mathcal{I}$, such that $\mathrm{d} \left(\left\{ \overline{K}_1^k \cdots \overline{K}_m^k \right\}, \widehat{\mathcal{K}} \right) \geq \delta$ for all $k \in \mathcal{I}$. On the other hand recall that for all $k \geq 0$ $\left\{ \overline{K}_1^k \cdots \overline{K}_m^k \right\}$ is bounded (in a compact set). Thus there exists an infinite set of nonnegative indices $\widehat{\mathcal{I}} \subset \mathcal{I}$ and an accumulation point $\overline{K}_1^* \cdots \overline{K}_m^*$ such that the sub-sequence $\left\{ \overline{K}_1^k \cdots \overline{K}_m^k \right\} \to \overline{K}_1^* \cdots \overline{K}_m^*$ for all $k \in \widehat{\mathcal{I}}$.

Using the continuity of $\mathrm{d}\left(\left\{\overline{K}_1^k \cdots \overline{K}_m^k\right\}, \widehat{\mathcal{K}}\right)$ leads to the following contradiction

$$\mathrm{d}\left(\left\{\overline{K}_1^* \cdots \overline{K}_m^*\right\}, \widehat{\mathcal{K}}\right) \geq \delta > 0 \Rightarrow \overline{K}_1^* \cdots \overline{K}_m^* \notin \widehat{\mathcal{K}}.$$

Hence, $\left\{\overline{K}_1^k \cdots \overline{K}_m^k\right\} \to \widehat{\mathcal{K}}.$

$\square$

5.3 Global sub-optimal solutions

The solution to the state-feedback ODC problem presented in Section 5.2 does not depend explicitly on the initial state. However it depends on its covariance matrix. Furthermore this solution does not guarantee convergence to a global minimum of the cost function (generally $J_d(K)$ is not convex in K on $\mathcal{K}$). On the other hand solving numerically the above DAM type algorithm may require many iterations (much CPU time). In this section two alternative SODC solutions are presented. It turns out that they are both related to the following average approximation

$$\bar{x}_1 \simeq \bar{x}_2 \simeq \cdots \simeq \bar{x}_m \simeq \frac{x}{m} \tag{5.24}$$

for all $t \geq 0$.

5.3.1 The average weight solution

Intuitively we can get the average weight solution for the ODC problem, by replacing all the weighting matrices with their averages, e.g., $W_{e_j}' = \frac{1}{m} \sum_{j=1}^m W_{e_j}$. In fact this solution was mistakenly obtained as the optimal solution to the ODC problem in [56] using a wrong assumption (see Remark 2.11). Indeed their assumption is true only if we require the $\overline{G}_j$ to be also identical for all $j = 1, 2, \ldots, m$ (i.e., the same structural constraint as for H). Then, assumptions (A1) and (A2) of Section 5.2, can be removed.

Proposition 5.1 *Let* $\overline{Q}_j \geq 0$, $\widetilde{R}_j > 0$, *and* $\bar{x}_{0_1} = \bar{x}_{0_2} = \cdots = \bar{x}_{0_m} = \frac{x_0}{m}$. *Suppose that all the* $\overline{G}_j$*'s in (5.1) are forced to be equal. Then the ODC is given by*

$$K = [H \quad -G] = R^{-1}\overline{B}^\top P \tag{5.25}$$

104

where P is the unique positive-semidefinite solution of

$$P\overline{A} + \overline{A}P - P\overline{B}R^{-1}\overline{B}P + Q = 0 \qquad (5.26)$$

where $Q = \frac{1}{m}\sum_{j=1}^{m}\overline{Q}_j$ and $R = \frac{1}{m}\sum_{j=1}^{m}\widetilde{R}_j$.

Proof: Let $\overline{G}_1 = \overline{G}_2 = \cdots = \overline{G}_m = \frac{1}{m}G'$, then for all $t \geq 0$, and for all $j = 1, 2, \ldots, m$, $\bar{u}_j = \frac{1}{m}u = \frac{1}{m}G'x_r - H\frac{x}{m} = G\frac{x_r}{m} - H\frac{x}{m}$. Define $\overline{X}_j = \begin{pmatrix} \frac{x}{m} \\ \frac{x_r}{m} \end{pmatrix}$, then the ODC componentwise cost function is given by

$$
\begin{aligned}
J_d &= \sum_{j=1}^{m} \int_0^\infty \left(\overline{X}_j^\top \overline{Q}_j \overline{X}_j + \bar{u}_j^\top \widetilde{R}_j \bar{u}_j \right) dt \\
&= \int_0^\infty \left\{ \frac{X}{m}^\top \left[\sum_{j=1}^m \overline{Q}_j \right] \frac{X}{m} + \frac{u}{m}^\top \left[\sum_{j=1}^m \widetilde{R}_j \right] \frac{u}{m} \right\} dt \qquad (5.27) \\
&= \frac{1}{m} \int_0^\infty \left(X^\top Q X + u^\top R u \right) dt
\end{aligned}
$$

where $X = \begin{pmatrix} x \\ x_r \end{pmatrix}$, $Q = \frac{1}{m}\sum_{j=1}^{m}\overline{Q}_j$ and $R = \frac{1}{m}\sum_{j=1}^{m}\widetilde{R}_j$. Minimizing J_d (or mJ_d) leads to the ordinary LQT solution (5.25), (5.26).

$\square$

Using an average Q the advantage of componentwise costs vanishes. Thus this solution is not really an ODC. However the simple average solution of Proposition 5.1 has several advantages. First it is very easy to compute. Moreover it does not depend on the system or input initial states or their statistics. The optimal cost of this solution can be used as the "ordinary LQR" comparative measure to any other ODC cost function. Recall that the lower bound to any ODC solution is the ideal solution obtained in Section 4.2. The average weights solution is generally not an upper bound (e.g., using the average initial state ODC with extreme initial states may lead to higher cost). It may be used as an initial guess for the DAM type algorithm of Section 5.2.
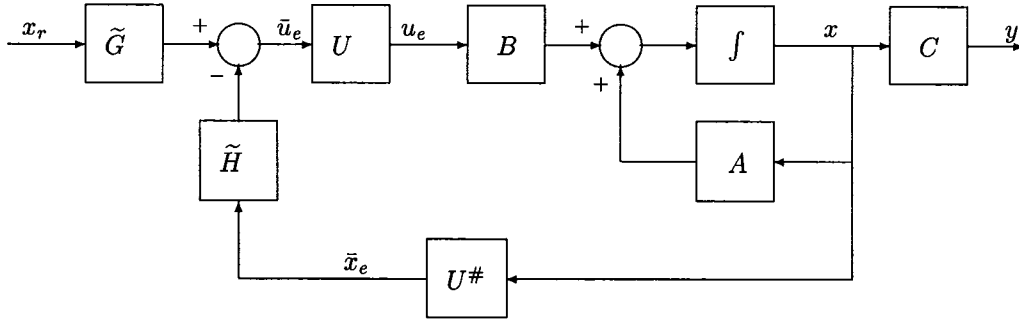
Figure 5.1: The global LMS sub-optimal solution.

5.3.2 The LMS componentwise state-feedback (average gains) solution

This solution is based on the ideal (free structure) results of Section 4.2. Recall from Lemma 4.1 that $(\widehat{A}, U_n)$ is not completely observable, thus the componentwise state $\bar{x}$ is not measurable and the ideal solution can not be implemented. However using the following simple idea, the ideal solution can be applied to the least mean square (LMS) estimation of $\bar{x}$.

Suppose that there exists a constant matrix $U^{\#} \in \mathbb{R}^{mn \times n}$ such that $\bar{x}_e = U^{\#}x$ is the (static) LMS estimation of the unmeasurable componentwise state $\bar{x}$ (i.e., $U^{\#} = \arg \min_{U' \in \mathbb{R}^{mn \times n}} (\bar{x} - U'x)^{\top}(\bar{x} - U'x))$. If the closed-loop system shown in Figure 5.1 is stable and the couple $(\widetilde{H}, \widetilde{G})$ is the ideal ODC, then the couple $H = U\widetilde{H}U^{\#}$ and $G = U\widetilde{G}$ is the LMS-SODC.

Conjecture 5.1 *Consider the set of unique, symmetric, positive definite solutions* $\{P_1, P_2, \ldots, P_m\}$ *to the set of AREs*

$$P_j A + A^{\top} P_j + P_j B R^{-1} B^{\top} P_j + Q_j = 0$$

where $R > 0$, (A, B) *is controllable and for all* $j = 1, 2, \ldots, m$, $Q_j \geq 0$ *and* $(A, Q_j^{1/2})$ *is detectable. Let* $H_j = R^{-1}B^{\top}P_j$ *and* $H = \frac{1}{m}\sum_{j=1}^{m} H_j$, *then* $A - BH$ *is asymptotically stable.*

106

Remark 5.4 *The assertion of Conjecture 5.1 is based the following observations. First, it generally holds for the scalar case. Second, no counter example was found for the many cases which have been checked throughout this research. In order to prove this conjecture (and make it a lemma), one may try using the "return difference" inequality (e.g., [2]) to derive the LQR "gain margin" condition for this case. Other possible proof approaches using: Lyapunov theorem, Hamiltonian matrix, ARE operator, etc., may also be used.*

Proposition 5.2 *The LMS-SODC solution is given by $H = U\widetilde{H}U_n^\dagger = \frac{1}{m}\sum_{j=1}^m \overline{H}_j$ and $G = U\widetilde{G} = \sum_{j=1}^m \overline{G}_j$, where $(\widetilde{H}, \widetilde{G})$ is the ideal (free structure) solution (4.22).*

Proof: Recall that by definition $x = U_n\bar{x}$. Then $\bar{x}_e$, the LMS static estimator for $\bar{x}$ given x, is given by [32]

$$\bar{x}_e = U_n^\dagger x \tag{5.28}$$

where $U_n^\dagger$ is the Moore-Penrose pseudo-inverse of U_n, given by

$$U_n^\dagger = \frac{1}{m}U_n^\top. \tag{5.29}$$

Substituting (5.28) and (5.29) into the ideal ODC (4.22), and using $u_e = U\bar{u}_e$ we finally obtain

$$
\begin{aligned}
u_e &= U\bar{u}_e \\
&= U(\widetilde{G}x_r - \widetilde{H}\bar{x}_e) \\
&= U\widetilde{G}x_r - U\widetilde{H}U_n^\dagger x \\
&= Gx_r - Hx
\end{aligned}
\tag{5.30}
$$

where $H = U\widetilde{H}U_n^\dagger = \frac{1}{m}U\widetilde{H}U_n^\top = \frac{1}{m}\sum_{j=1}^m \overline{H}_j$ and $G = U\widetilde{G} = \sum_{j=1}^m \overline{G}_j$. From Conjecture 5.1, $A - BH$ is asymptotically stable, which completes the proof.

$\square$

Chapter 6

Numerical example

6.1 Introduction

In this chapter we apply the ODC techniques presented in this thesis to the design of an attitude control system for a helicopter in hover. This particular example was chosen in order to demonstrate an application of the ODC design.

Decoupled attitude control in hover is required for almost all helicopters. Military and civilian helicopters have very complicated hovering missions. For example, military army helicopter missions requiring precise nap-of-the-earth (NOE) tasks such as landing airborne troops, or mask/unmask tasks such as target tracking, etc. [31, 36]. Civilian helicopters also have hovering tasks such as police, rescue, industrial, TV reports, etc. For these tasks the helicopter and its pilot are required to perform "precision hover" in very dangerous situations such as, very low flight levels, close to steep mountain sides or canyon walls, or complex environments (cities, army bases, ships, hospitals, industrial facilities, etc.). In these situations good decoupled pilot/helicopter operation is required (e.g., [30]).

From its dynamics and aerodynamics it can be seen that a helicopter with its flight control system (FCS) off is a highly coupled MIMO system, even in hover conditions [37]. Moreover, the coupling levels of most helicopters with their FCS on are still too high for the above hovering tasks. Then the pilot has to correct the

Figure 6.1: The NASA Ames UH-60A "Black Hawk" helicopter.

coupling effects by entering proper commands to the coupled channels. This requires from the pilot to give his full attention to flight operations. As a result either the pilot/helicopter performance level is reduced, or additional crew members are required in order to complete the helicopter tasks.

In the following example we show that, using ODC, the control system designer has a simple and direct design method which allows him to reduce one or more coupling components (in this case roll-to-pitch attitude coupling), almost without changing the other error components.

6.2 Attitude control of the UH-60A "Black Hawk" helicopter in hover

The UH-60A helicopter of Figure 6.1 is used by NASA and the US army as a demonstrator for advanced flight control laws [43]. It was chosen as the example for this thesis mainly because it is one of the relatively highly coupled helicopters [8]. As can be seen from Figure 6.1 it has a relatively long tail and a tilted tail rotor (to balance its tail weight) which increase the coupling of its dynamics.

110

6.2.1 Linearized model for the UH-60A in hover

The helicopter dynamics and aerodynamics are represented by a set of nonlinear coupled ordinary differential equations (ODE). The model used in this chapter is based on the full helicopter model suggested by Kim [40]. This mathematical model includes a total of 39 states including 9 states for the 6 degree-of-freedom fuselage motion, 8 for the rotor blades' flap motion, 8 for lag, and 2 for torsion motion, 2 states for the rotor rotational speed and 3 for its inflow dynamics, 1 state for the tail rotor inflow dynamics, 2 states for the delay effect of the fuselage wake on the empennage, 1 state for the gas generator speed, and 3 states for the main engine thermodynamic effects. This set of equations can be solved using the software simulation package UMGenHel [41]. Using the finite difference method (about the heave trim solution), this package can generate a linearized model about a given flight condition. In addition, using model reduction techniques, it is possible to eliminate states from the full order linearized model.

For the purpose of this dissertation we consider such a linearized (about hover conditions), reduced order (9 states) model. Using only the 9 fuselage states, the coupling effects (contained also in the other 30 states), are reduced. However from the use of similar linearized and reduced models, [63], [8] and [81], it turns out that this 9 state model represents most of the coupling effects.

The state space equations for the attitude control of the helicopter in hover are

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{aligned}
\tag{6.1}
$$

where $x = (u\ v\ w\ p\ q\ r\ \phi\ \theta\ \psi)^{\top}$. Using the standard rotorcraft notation, u, v, w - stand for the fuselage longitudinal, lateral and vertical linear velocity, respectively, p, q, r - stand for the fuselage angular rates about its body axes x_b, y_b, z_b and ϕ, θ, ψ - stand for angular rotation about these axes. The attitude of the helicopter is usually given by three Euler angles where, in hover, we assume them to be the fuselage body angles. Therefore $y = (\theta,\ \phi, \psi)^{\top}$ are the helicopter pitch, roll and yaw attitude angles.

The pilot controls these angles by using the following corresponding commands $u = (\delta_\theta, \ \delta_\phi, \ \delta_\psi)^\top$. See Appendix A for the UMGenHel A and B matrices.

6.2.2 Design methods for rotorcraft flight control systems (FCS) and their decoupling properties.

Three main design methods have been used for FCS:

- **High gain decentralized control** - This classical design method has been used for many rotorcraft FCS (e.g., [44,63,65]). Using high gain decentralized feedback control, the off-axis response is treated as an unmeasurable disturbance input. Mathematical models of helicopter dynamics give good prediction of the helicopter on-axis response, but usually poor prediction of its off-axis response [40], [26]. Therefore this method has been mainly used in cases where other information (e.g., system identification models [64]) are not available. The main advantages of this method are: simple design procedure (SISO techniques) and simple implementation. The main disadvantages are: poor robustness (low stability margins) and high control signals (saturated actuators).

- **Cross-feed techniques** - These design techniques are used to reduce the I/O coupling "directly". That is, the coupling effect is reduced by feeding "opposite" control signals to the coupled channels. These methods are very useful for systems with coupled and/or cross-coupled I/O pairs. Helicopters have two main coupled pairs, the cross-coupled pitch and roll and the yaw-to-heave coupled pair. Therefore these methods are used for rotorcraft FCS design [36], [8]. The main advantages of these methods are: directly affects the desired coupling components, relatively simple design procedure (e.g., using an approximate scalar inverse of the coupling component as the cross-feed transfer function). The main disadvantages are: requires fair knowledge of the helicopter off-axes response, becomes very complicated (even impossible) for more general (not pairwise) coupled systems.

112

- **MIMO design techniques** - In modern rotorcraft FCS design classical and modern MIMO control design methods have recently been used. These methods including optimal techniques such as LQR [24], $\mathcal{H}_2/\mathcal{H}_\infty$ [59,61], parametric optimization [15,68,81], etc., and other MIMO design techniques such as eigenstructure assignment [34], QFT [27], Nyquist array [73], etc. (see [51] for partial review). The main disadvantage of the MIMO techniques is that using precisely the theoretical design procedure it is either impossible to meet all the design requirements or many design iterations are required. Therefore many ad hoc design methods, based on theoretical MIMO techniques, are also used. Using those techniques (e.g., [26]) the design process is usually made in two phases. In some of these two-phase methods (e.g., [34] and [73]), in order to make the design process simpler, decoupling techniques are used in the preliminary design phase. In some other works (e.g., [15,81]) the decoupling requirement is taken into account. However most of those and the other MIMO design methods do not give a special priority to the decoupling requirement.

6.2.3 A preliminary LQR design example

In order to make a fair comparison between ODC techniques and a standard optimal control technique, we need a proper preliminary LQR design. Recall from Section 5.3.1 that the "average weights" solution is not an ODC. However because its cost function is the average of the m componentwise cost functions, it can be used as a fair comparison to other ODCs. In order to emphasis the importance of the decoupling requirement and the optimal decoupling design, the preliminary LQR design was made such that one of the coupling components (roll-to-pitch) is very high. This highly coupled design is set to be the "starting point" for all the different design techniques compared in Section 6.3.6.

In order to make the roll-to-pitch coupling component higher, the LQR preliminary design parameters were chosen as follows. First, in order to make this example
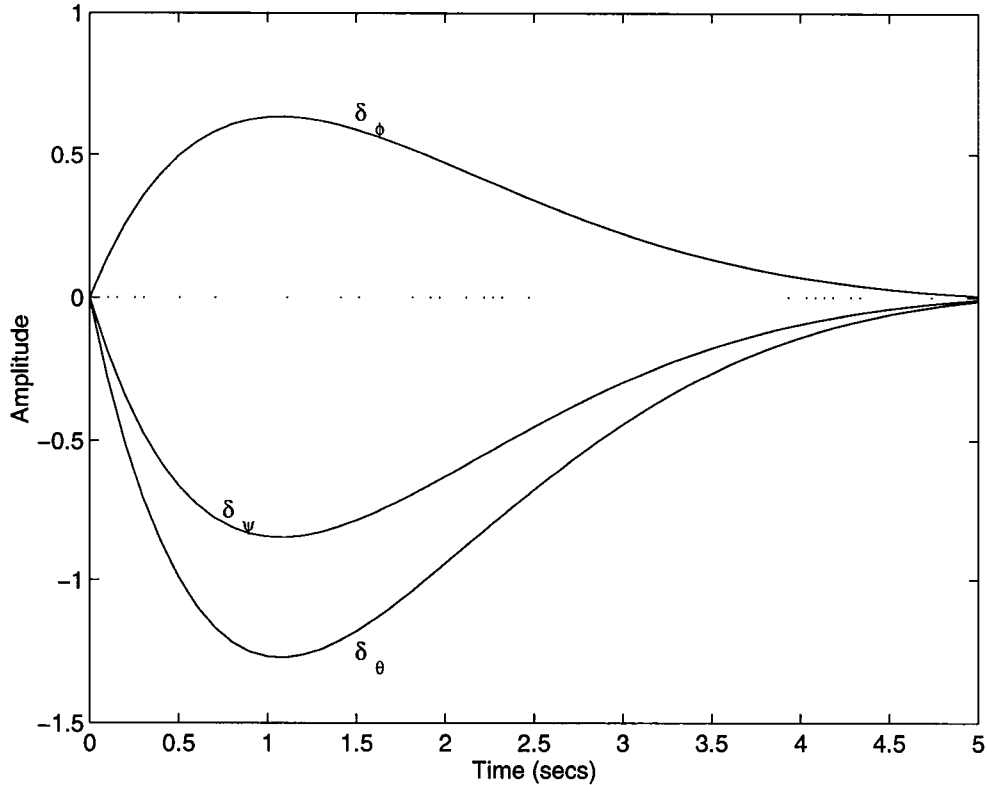
Figure 6.2: Typical set of pilot commands.

as real as possible, the error and control weighting matrices (W_e and W_u) were tuned such that their final gains (those which are implemented) have about the same values as those in [63]. After several design iterations we obtained, $W_e = \text{diag}(1, 1.2, 1)$ and $W_u = \text{diag}(0.1, 0.5, 0.1)$. Then, the weigh of the roll error was reduced to 0.3. Later at the end of the design process, this weight will be increased back to its "real" value.

We assume that at $t = 0$ the helicopter is in trim conditions (hover), thus $x_0 = 0$. The input commands used for this example are an approximation of a pilot's slow attitude change commands in hover . The input for each channel is given by a state response of a second order system with $1\frac{rad}{sec}$ natural frequency and 0.8 damping ratio. Note that the test inputs defined in [30] and [60] can not be used for this example because they can not be obtained as the state response of a stable autonomous
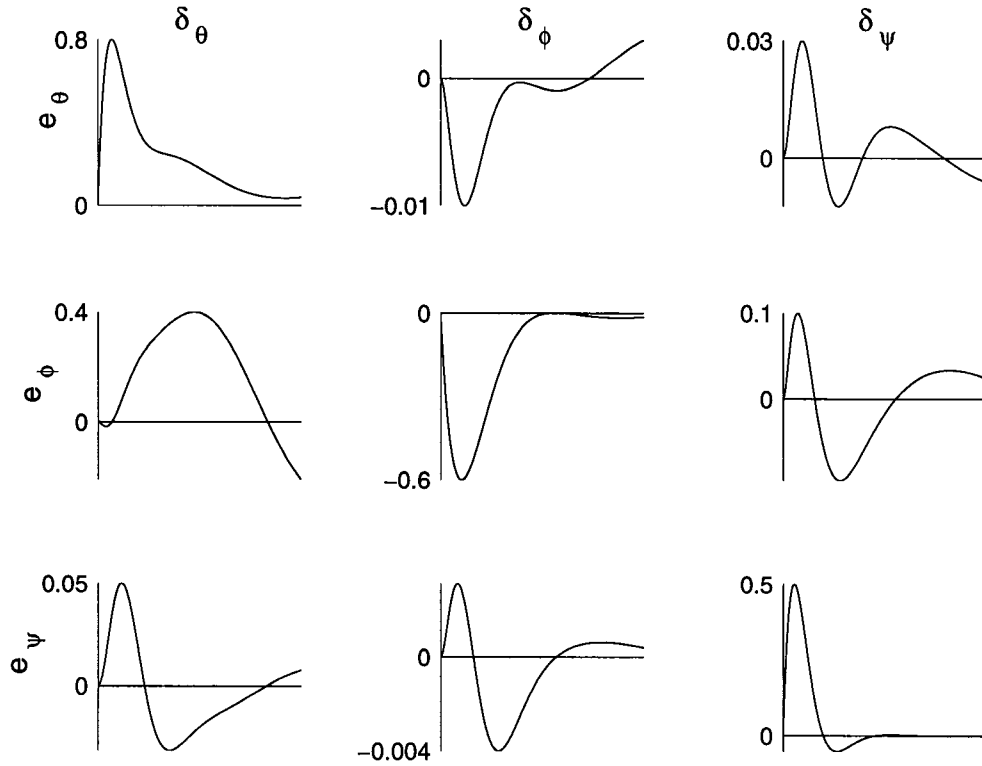
Figure 6.3: Attitude error for the UH-60A in hover with preliminary LQR control law. Horizontal axes - time [0 to 10 sec], vertical axes - attitude error.

system (i.e., in this case A_r is not asymptotically stable). For each channel input model, the initial "position" state (x_{r_1}) is set to be zero, and the "rate" state (x_{r_2}) is randomly chosen. This choice of the initial system and input states can be considered as a reasonable random choice of initial augmented state normally distributed with zero mean and covariance I. A typical pilot command set for the choice $x_{r_2}(0) = (3, -1.5, 2)$, is shown in Figure 6.2.

The attitude change of the UH-60A helicopter (from 0 to 10 seconds) with such a closed-loop FCS in response to the above typical set of inputs is shown in Figure 6.3. Obviously the above choice of W_e and W_u lead to high roll-to-pitch coupling response (see $e_\phi/\delta_\theta = e_{21}$ in Figure 6.3). However usually (e.g., [65]) the roll-to-pitch coupling response is relatively high (higher than the pitch-to-roll response), due to the

115

relatively low fuselage moment of inertia about its roll (x_b) axis.

6.3 Reducing roll-to-pitch coupling using ODC techniques

Our goal is to reduce the roll-to-pitch component (e_ϕ/δ_θ of Figure 6.3) and keep the main and other coupling error components (almost) unchanged. For this purpose we check the following four different design techniques presented in this thesis:

- The theoretical solution of Section 4.2.

- The average gain solution of Section 5.3.2.

- The average weights solution of Section 5.3.1.

- The average initial state solution of Section 5.2.

For all these techniques, the componentwise error and control weighting matrices are

$$
\begin{aligned}
\overline{W}_e &= \text{diag}\,(1, \overline{w}_{e_{21}}, 1,\ 1, 0.3, 1,\ 1, 0.3, 1) \\
\overline{W}_u &= \text{diag}\,(0.1, 0.5, 0.1,\ 0.1, 0.5, 0.1,\ 0.1, 0.5, 0.1)
\end{aligned}
\tag{6.2}
$$

where $\overline{w}_{e_{21}}$ is the only decoupling design degree-of-freedom. Using each of the above ODC techniques we obtain three different solutions for the following values of $\overline{w}_{e_{21}}$, 0.3, 1 and 3. Note that the first value (0.3) leads to equal $\overline{W}_{e_j}$'s (i.e., $\overline{W}_{e1} = \overline{W}_{e2} = \overline{W}_{e3} = \text{diag}\,(1, 0.3, 1)$). Therefore we expect the corresponding solution to be equal for all the above design methods (the same "starting point").

6.3.1 The theoretical (free structure) solution

Recall that the theoretical solution may be applied only to stable plants in practice. The open-loop helicopter (without the FCS) is unstable. Therefore this free structure solution is only a theoretical one. However it still can be used for comparison because its minimum cost function is the lower bound for all the cost functions of the other design methods. The theoretical solution is computed using standard LQR routines
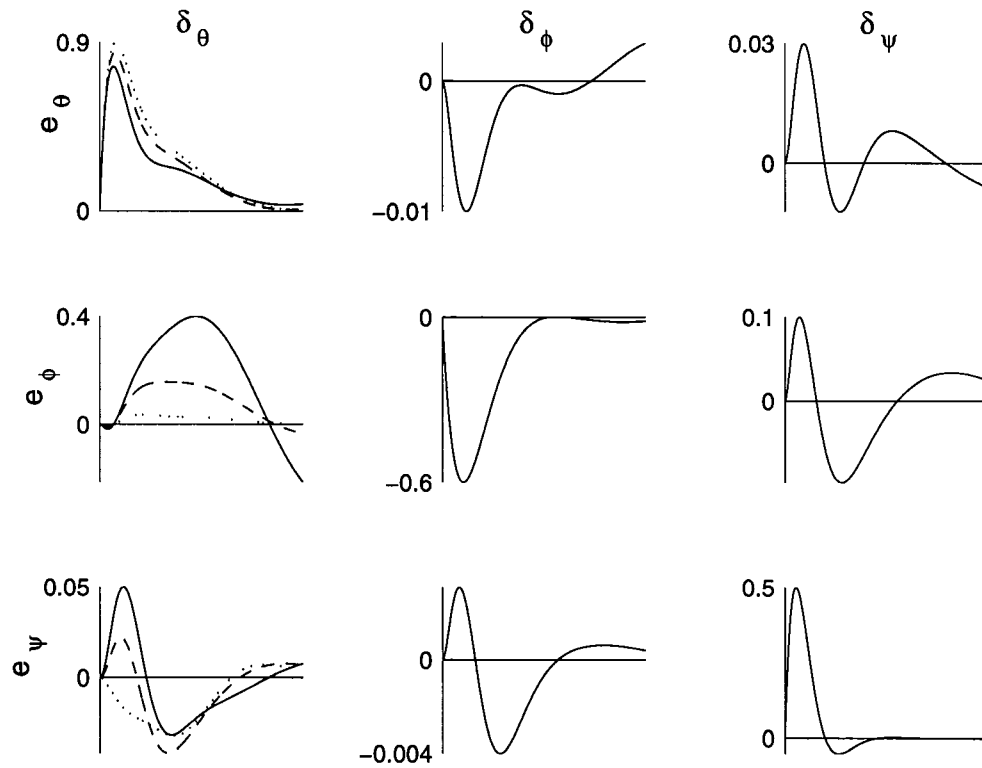
116

Figure 6.4: Attitude error for the UH-60A in hover with theoretical ODC designed for $\overline{w}_{e_{21}} = 0.3$ (solid line), $\overline{w}_{e_{21}} = 1$ (dashed line) and $\overline{w}_{e_{21}} = 3$ (dotted line). Horizontal axes - time [0 to 10 sec], vertical axes - attitude error.

117

(e.g., MATLAB's "lqr" function [62]). The attitude change of the UH-60A helicopter with the theoretical solution is shown in Figure 6.4.

The main observation is that the theoretical solution gives almost the perfect desired response. That is, increasing $\overline{w}_{e_{21}}$ reduces the $\bar{e}_{21}$ (roll-to-pitch) coupling component. Moreover it hardly affects the other $\bar{e}_{ij}$ error components. In fact the $\bar{e}_2$ and $\bar{e}_3$ error components are not affected at all and $\bar{e}_{11}$ and $\bar{e}_{31}$ are affected very little. Recall that using the simultaneous control representation, changing only one weighting component $\overline{w}_{e_{ij}}$ implies that only the $j\underline{th}$ sub-system (the first one in this example) is affected. The theoretical cost is computed by

$$J_{d_{\min}} = \bar{x}_{0_1}^\top \overline{P}_1 \bar{x}_{0_1} + \bar{x}_{0_2}^\top \overline{P}_2 \bar{x}_{0_2} + \bar{x}_{0_3}^\top \overline{P}_3 \bar{x}_{0_3}. \tag{6.3}$$

The computed values for $\overline{w}_{e_{21}} = 0.3, 1$, and 3, are 1.934, 2.243 and 2.442, respectively.

6.3.2 The average gain (LMS) solution

This practical SODC solution is obtained directly from the above theoretical solution, where $H = \frac{1}{3}\left[\overline{H}_1 + \overline{H}_2 + \overline{H}_3\right]$ and $G = \overline{G}_1 + \overline{G}_2 + \overline{G}_3$. From Figure 6.5 we conclude that the results are not as good as for the theoretical solution. Moreover, contrary to the theoretical solution, the other error components are slightly affected. The average gain costs are computed by integrating

$$J_{d_{\min}} = \int_0^{T_f} \left(\bar{e}_1^\top \overline{Q}_1 \bar{e}_1 + \bar{u}_1^\top R\bar{u}_1 + \bar{e}_2^\top \overline{Q}_2 \bar{e}_2 + \bar{u}_2^\top R\bar{u}_2 + \bar{e}_3^\top \overline{Q}_3 \bar{e}_3 + \bar{u}_3^\top R\bar{u}_3 \right) dt. \tag{6.4}$$

with $T_f \gg \tau_{\max}$ where $\tau_{\max}$ is the largest time constant of the closed loop system (i.e., practically $T_f = \infty$). The minimum costs are 1.934, 2.366 and 3.027. Note that the minimum cost for $\overline{w}_{e_{21}} = 0.3$ is exactly the same as for the theoretical solution and it is higher for $\overline{w}_{e_{21}} = 1$ and for the final design value 3.

6.3.3 The average weight solution

As we have already pointed out, this solution is not an ODC. It is used for comparison and as the initial guess of the next two ODC solutions. The solution is simply a LQR
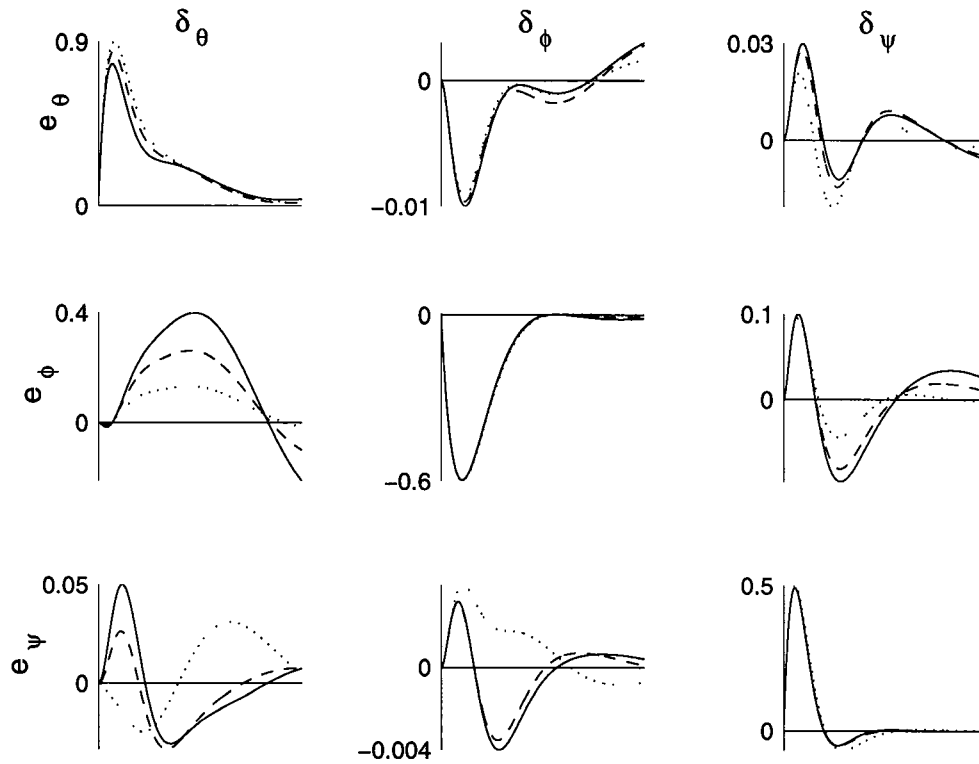
Figure 6.5: Attitude error for the UH-60A in hover with average gains ODC designed for $\overline{w}_{e_{21}} = 0.3$ (solid line), $\overline{w}_{e_{21}} = 1$ (dashed line) and $\overline{w}_{e_{21}} = 3$ (dotted line). Horizontal axes - time [0 to 10 sec], vertical axes - attitude error.

119

Figure 6.6: Attitude error for the UH-60A in hover with average weights ODC designed for $\overline{w}_{e_{21}} = 0.3$ (solid line), $\overline{w}_{e_{21}} = 1$ (dashed line) and $\overline{w}_{e_{21}} = 3$ (dotted line). Horizontal axes - time [0 to 10 sec], vertical axes - attitude error.
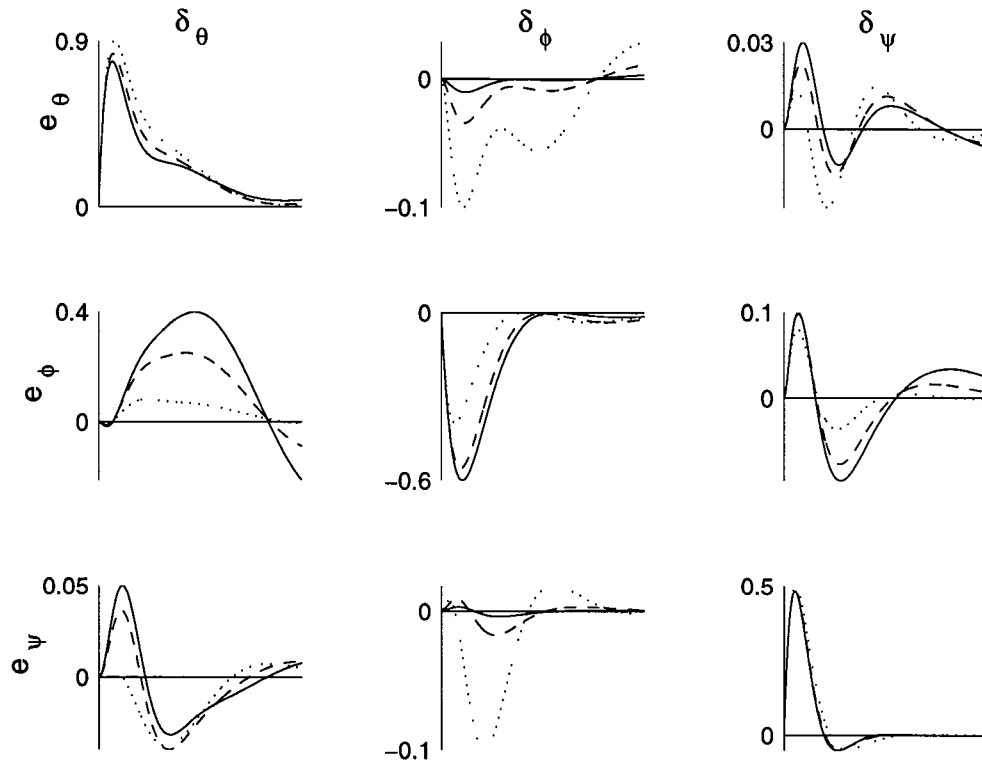
solution with the average error weighting matrix $W_e = \frac{1}{3}\left[\overline{W}_{e_1} + \overline{W}_{e_2} + \overline{W}_{e_3}\right]$. Note that for $\overline{w}_{e_{21}} = 0.3$ it is also the preliminary LQR solution. The attitude change for this solution is shown in Figure 6.6. Although it gives fair reduction of the roll-to-pitch coupling, this solution affects the other error components. In fact it increases substantially two other coupling components $\bar{e}_{12}$ and $\bar{e}_{32}$. This demonstrate the claim that it is not really an ODC. The average weights cost is computed by integration (6.4). The minimum costs are 1.934, 2.348 and 2.973. Note that although this is not an ODC, its cost for $\overline{w}_{e_{21}} = 1$ is less (better) than this of the average gain SODC solution.

6.3.4   The average initial state solution

This solution is obtained by using the DAM type algorithm of Section 5.2.2. See Appendix A for MATLAB code solving this algorithm with Armijo line search., normalized covariance matrices ($\Pi_1 = \Pi_2 = \Pi_3 = I$) and stopping criterion $\delta = 10^{-5}$. The above average weight solution was taken as the initial guess for the algorithm. The attitude change for this solution is shown in Figure 6.7. The minimum average costs 8.4280, 10.5186, and 18.9254 are obtained from the numerical algorithm after 5, 16, and 130 iterations, respectively. The minimum actual costs 1.9345, 2.2791, and 2.4957 are computed by numerical integration (6.4).

6.3.5   The direct minimization solution

A direct minimization has been used for comparison. The constrained optimization problem was solved using the CONSOL-OPTCAD (C-O) optimization package [16] with MATLAB interface (to evaluate the componentwise cost function). See Appendix A for C-O and MATLAB codes. Recall that this solution depends on the initial system and input states. The direct minimization has been performed only for $\bar{w}_{e_{21}} = 3$.
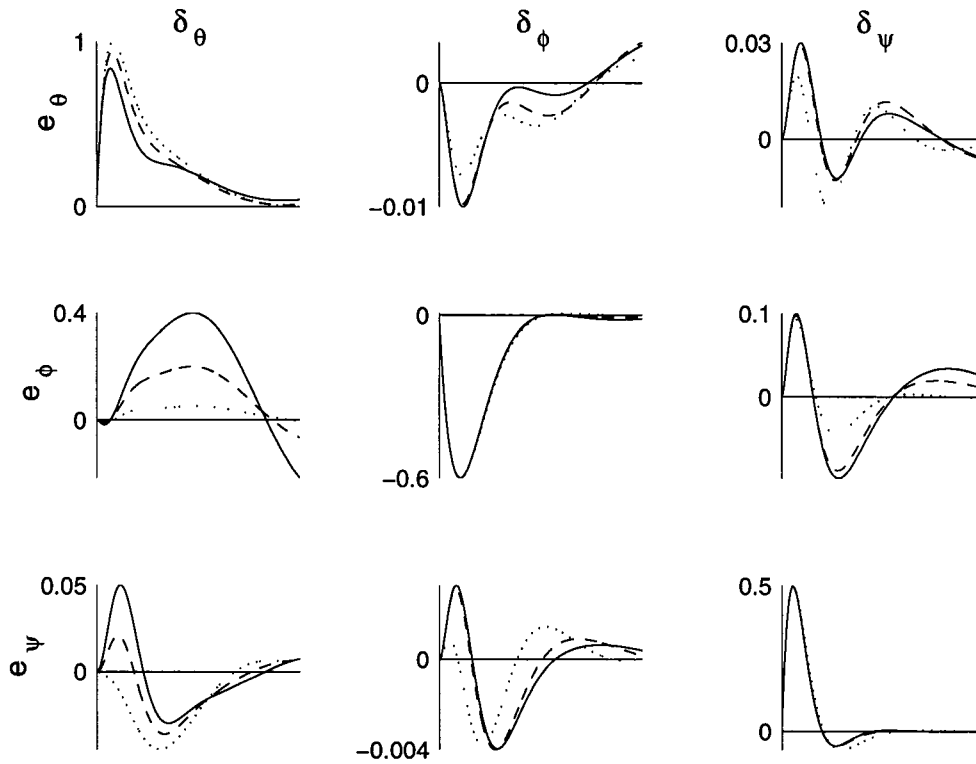
Figure 6.7: Attitude error for the UH-60A in hover with average initial state ODC designed for $\overline{w}_{e_{21}} = 0.3$ (solid line), $\overline{w}_{e_{21}} = 1$ (dashed line) and $\overline{w}_{e_{21}} = 3$ (dotted line). Horizontal axes - time [0 to 10 sec], vertical axes - attitude error.
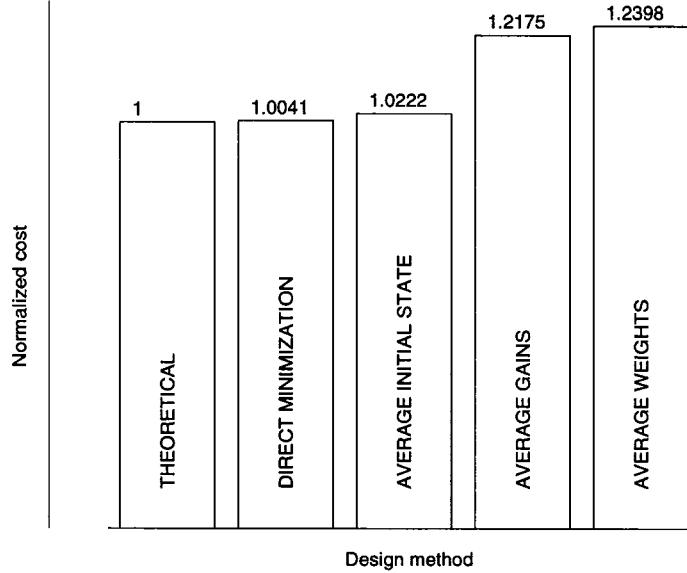
Figure 6.8: Comparison between the minimum costs of the different ODC design methods.

6.3.6 Comparison between the different ODC design methods

Using each of the four different ODC methods, it is possible to reduce the roll-to-pitch (e_{21}) error coupling component. The reduction level is proportional to the corresponding weight $(\bar{w}_{e_{21}})$. For all the ODC methods, except the average weight, the other error coupling components are almost not affected at all.

Because the target is to minimize the componentwise cost function, the best mathematical comparison measure is the minimum cost achieved by using each one of the above methods. The minimum normalized costs (divided by the theoretical minimum cost) for $\bar{w}_{e_{21}} = 3$ are shown in Figure 6.8.

Using direct minimization for a specific input (initial states) it is possible to achieve almost (+0.4%) the theoretical (lowest) cost. However, this solution may not be good for any possible input (initial states). The best SODC cost (+2.2%) is obtained by using the average-initial-state solution. This controller does not depend on the specific initial state of the input reference signal. The minimum costs obtained
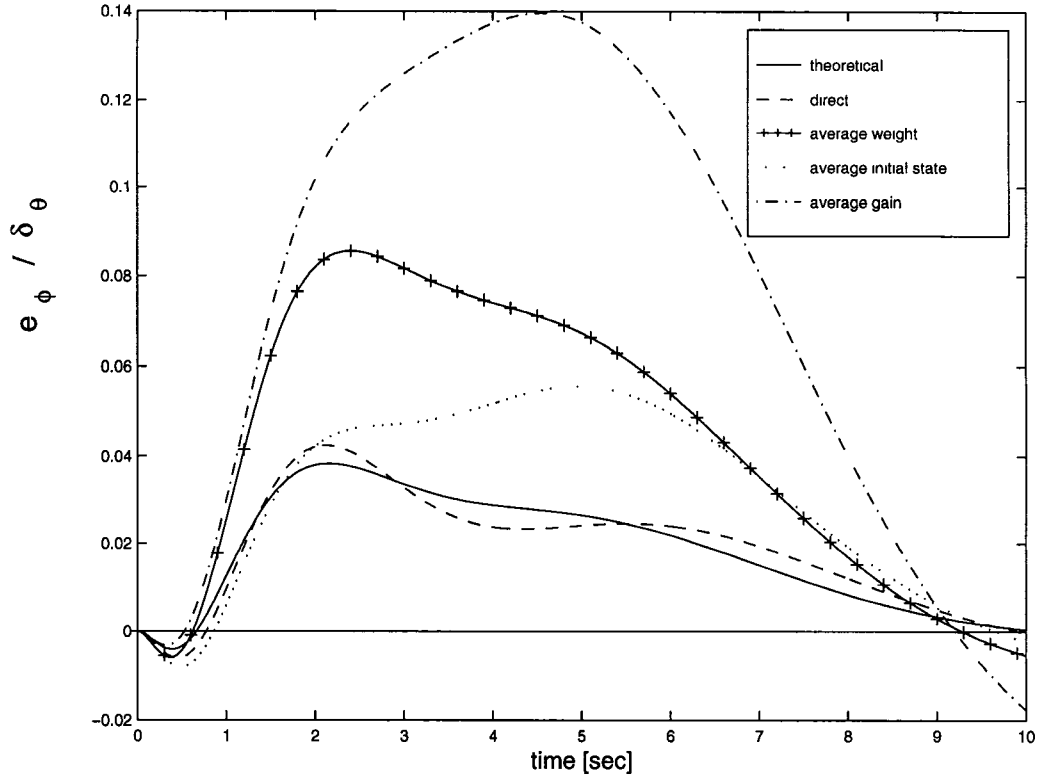
Figure 6.9: Comparison between the e_ϕ/δ_θ error component of the different ODC design methods.

by using the other SODC solutions are about 20% higher than the theoretical one.

Another way to compare between the different methods is by looking at the time response of the e_{21} (e_ϕ/δ_θ) error coupling component (Figure 6.9). The time responses for direct minimization and average initial state are close to the theoretical time response. The other two SODC solutions having much higher error coupling component. The average weight error coupling component is better than that of the average gain. However recall that the e_{12} and e_{32} error coupling components of the average weight solution are much higher.

Chapter 7

Conclusions and future research

7.1 Conclusions

In this dissertation a new approach to optimal decoupling control (ODC) is developed. This research has been motivated by the need to have alternative design methods to the strict decoupling design techniques. ODC techniques are not only theoretically important but also practically implementable.

All the various solutions to the ODC problem derived in this dissertation are based on componentwise representations of the system and its control laws. These representations allow us to perform a more detailed analysis of the coupling of the system, than can be done using standard mathematical representations. Furthermore, using these representations, we can define a scalar quadratic coupling measure for minimization. The particular minimization problems depend on the specific design requirements.

Using this idea, the theoretical ODC can be simply obtained from standard optimal control solutions. This theoretical ODC has two major disadvantages. First, it may be applied only to stable plants. Moreover the theoretical ODC has high order. It is also shown that high order stabilizing ODC controllers can be obtained by using this formulation and the Hadamard weighted design techniques [71].

In order to find low order solutions to the ODC problem, a constant state

feedback control law is considered. It is shown that the optimal solution in this case depends on the initial state. This dependence can be removed by randomizing the initial state. A globally convergent algorithm for the new sub-optimal problem is derived. In addition some other global sub-optimal solutions are presented.

Finally it is shown that, at least in some cases, using these componentwise control design techniques, one (or more) component of the system coupling error can be reduced almost without changing the other coupling components. This property of the ODC design technique is demonstrated by using this approach in the design of the attitude control of the UH-60A helicopter in hover.

7.2 Future research

The solutions to the ODC, presented in this thesis, are either (very) high order control, or (very) low order control. High order controllers are usually not practical (i.e., too expensive, too "big", etc.). On the other hand, for most practical cases, constant gain ODC's are not satisfactory. For these cases the "best" solution is probably a "low" order dynamic controller. The first question to be asked is: how low should be the order of the ODC ? Note that similar questions were studied by Hirzinger [29] and by Sirisena and Choi [58].

The answer to this question may be found by going in the following two opposite directions:

- "Going up" from the constant gain solutions of Chapter 5 by adding dynamics. This can be done using similar ideas to those which are used in [55] for the design of optimal fixed structure control (OFSC) dynamic output feedback.

- "Going down" from the high order ODC of Section 3.6 For this purpose, model reduction techniques may be studied.

It is also very useful to study the performance of two controllers, with the same given order, obtained by the above two different methods.

126

One interesting open question is the conjecture used in Section 5.3.2 (see Remark 5.4). The proof of this claim may be also useful for other optimal control applications. Moreover, if it is proved it may be extended to the case of dynamic controllers.

Another possible future research topic is to improve the numerical algorithm for the constant state feedback SODC problem of Section 5.2.2 (make it converge faster, include additional dynamics, etc.). Furthermore, under some additional assumptions, this algorithm can be extended to find the SODC for constant output feedback control. See [5] for improved OFSC algorithms for the optimal constant output feedback problem.

One interesting question for future research would be to determine the class of systems for which the coupling error can be greatly reduced without affecting other aspects of performance. Topics for further research can also be suggested by expanding the ideas of this dissertation to the case of digital systems, time-varying systems, nonlinear systems, etc.

Appendix A

Some codes used for the numerical example

A.1 Linearized A and B matrices for the UH-60A in hover

```
A = [0.0162   0.0013   0.0116  -0.7495   1.0102  -0.3027         0  -32.0681  0
    -0.0068  -0.0458  -0.0008   0.3325  -0.6139  -1.1648   32.0231    0.1381  0
    -0.0456  -0.0052  -0.1692  -0.1387   2.3348   2.0203    1.6973   -2.6048  0
     0.0362  -0.0404   0.0021  -5.4470  -0.9290  -0.1352         0         0  0
     0.0014   0.0031   0.0031   0.0346  -0.7940  -0.0721         0         0  0
     0.0060   0.0021  -0.0012  -0.3009  -0.1079  -0.2535         0         0  0
          0        0        0   1.0000  -0.0043   0.0812         0         0  0
          0        0        0        0   0.9986   0.0529         0         0  0
          0        0        0        0  -0.0531   1.0019         0         0  0];

B = [-1.5621   -0.0666    1.0216
     -0.0966    0.4041   -0.7989
     -0.0826    0.0001    0.4177
      0.1493    1.3605   -0.5341
      0.3711    0.0113   -0.1096
      0.0103    0.0876    0.3247
           0         0         0
           0         0         0
           0         0         0];
```

A.2 The DAM type algorithm with Armijo line search

```
function [H,G,f0] = dam(A,B,C,Ar,Cr,Web,Wu,Pi,H0,G0)

% DAM    Solves the LQ-SODC problem using DAM (Descent Anderson and Moore)
%        type algorithm. H and G are the constant system and input
%        state feedback gains. f0 is the final cost.
%
%        - (A,B,C) are the m x m strictly-proper system state-space matrices.
%        - (Ar,Cr) are the m input state-space matrices.
%        - Web is an m^2 x m^2 nonsingular diagonal componentwise error
%          weigthing matrix, and Wu is an m x m control weighting matrix.
%        - Pi is the initial augmented state covariance matrix.
%        - H0,G0 is the initial guess, if they are not specified then the
%          "average weight" solution is taken.
%
%        Gil Yudilevitch
%        April-24-94

% Initialization
```

```
% --------------

i = 0;
[n,m]=size(B); nr=length(Ar);
M =1:m^2; M2=1:m+1:m2;
for i=1:m,          U(M,M+m*(i-1)) = eye(m); end; % computing U
 V=zeros(m^2,m ); V(M2,M )        = eye(m);        % computing V
 Ab=[A,zeros(n,nr);zeros(nr,n),Ar];               % augmented sys.
 Bb=[B;zeros(nr,m)];
Qb=Web'*Web; R=Wu'*Wu;

% --- if no H0,G0 are specified, the initial
%     guess is the average weight solution

if nargin==9,
[H,G,P] = lqt(A,B,C,Ar,Cr,U*Qb*U'/m,R);
else, H=H0; G=G0; end;

%                                    _
% --- computing Q

for j=1:m,
  js=num2str(j);
  Vj=V((j-1)*m+1:j*m,:);
  eval(['G_',js,' = G/m;']);
  Qbj = Qb((j-1)*m+1:j*m,(j-1)*m+1:j*m);
  Q1 =     C'*Qbj*C;
  Q2 =    -C'*Qbj*Vj*Cr;
  Q3 = Cr'*Vj*Qbj*Vj*Cr;
  eval(['Q_',js,' = [Q1,Q2;Q2'',Q3];']);
end;

% Main loop (START)
% ----------------

while 1 == 1,

 % Finding directions of descent
 %
 % ----------------------------
 %

  f0=0; LM=zeros(n,n); LL=zeros(n,n);
  PL=zeros(n,n); dJdHH=zeros(m,n+nr);
  for j=1:m,
    js=num2str(j);
    eval(['K_',js,' = [ H ,-G_',js,'];']);
    eval(['Ac_',js,' = Ab-Bb*K_',js,';']);
    eval(['P    = lyap(Ac_',js,''',K_',js,'''*R*K_',js,'+Q_',js,');']);
    f0=f0+trace(P*Pi);
    eval(['P1_',js,' = P(1:n      , 1:n   );']); %      | P1 | P2 |
%
    eval(['P2_',js,' = P(1:n      ,n+1:n+nr);']); % P = | ---|--- |
%
            P3       = P(n+1:n+nr,n+1:n+nr);      %      | P2'| P3 |
%
    eval(['L = lyap(Ac_',js,',Pi);']);
               L1 = L(1:n    , 1:n      );        %      | L1 | L2 |
%
    eval(['L2_',js,' = L(1:n    ,n+1:n+nr);']);   % L = | ---|--- |
%
    if j==1,     L3  = L(n+1:n+nr,n+1:n+nr); end;% | L2'| L3 |
%
    eval(['dJdK=2*(R*K_',js,'-Bb''*P)*L;']);
```

130

```
      eval(['dJdG_',js,' = -dJdK(:,n+1:n+nr);']);
      dJdHH=dJdHH+dJdK;
      eval(['LS = L1-L2_',js,'*inv(L3)*L2_',js,''';']);
      LL = LL+LS;
      eval(['PL = PL+P1_',js,'*LS;']);
   end;
   dJdH  = dJdHH(:,1:n);

   Hn = inv(R)*B'*PL*inv(LL); DH = Hn-H; DK = DH;
   for j = 1:m,
      js=num2str(j);
      eval(['Gn  = (Hn-inv(R)*B''*P1_',js,')*L2_',js,...
                    '*inv(L3)-inv(R)*B''*P2_',js,';']);
      eval(['DG_',js,' = Gn-G_',js,';']);
      eval(['DK = [DK,DG_',js,'];']);
      eval(['DK_',js,' = [DH,-DG_',js,'];']);
   end;

% --- checking termination condition
%

   DKn = norm(DK,'fro')/m^2;
   if DKn < 1e-5,
      disp(['Local min. after ',num2str(i),' iterations']);
      break;
   end;

% Armijo line search algorithm
%
% --------------------------
%

% --- initialization of line search
%

   K_G=zeros(nr,nr);
   for j = 1:m,
      js=num2str(j);
      eval(['K_G = K_G+dJdG_',js,'''*DG_',js,';']);
   end;
   KKn=trace(dJdH'*DH) + trace(K_G);
   delF=1; alflag = 0; KKc=0;
   Alfa=0.01; Beta=0.9; k=-2;% line search parameters (can be changed)
%

   while delF >= KKc, % Armijo criterion
%
      k=k+1;
      alfa  = Beta^k;  % updating step size
%

   % --- cheking step size and stability the algorithm stop
%
   %      if the step size is too small or the closed-loop is unstable
%

      if alfa < 1e-10,
         alflag = 1;
         disp('The step size is less than 1e-10 !');
         break;
      end;
      for j=1:m,
         js=num2str(j);
         eval(['k_',js,' = K_',js,'+alfa*DK_',js,';']);
```

```
        eval(['Ac_',js,' = Ab-Bb*k_',js,';']);
        eval(['Eig=max(real(eig(Ac_',js,')));']);
        while Eig >= 0,
          k=k+1;
          alfa  = Beta^k;
          if alfa < 1e-10,
            alflag = 1;
            disp('Can not find a stable solution !');
            break;
          end;
        end;
      if alflag==1, break; end;
      end;

    % --- evaluating the cost function
  %

      f1 = 0;
      for j=1:m,
        js=num2str(j);
        eval(['P = lyap(Ac_',js,''',k_',js,'''*R*k_',js,'+Q_',js,');']);
        f1 = f1+trace(P*Pi);
      end;
      delF = f1-f0;
      KKc=Alfa*Beta^k*KKn;

  end;% of Armijo line search
%


% Update H and G's
%
% ----------------
%

  H = H+alfa*DH;  % computing H
%
  for j=1:m,      % computing G's
%
    js=num2str(j);
    eval(['G_',js,' = G_',js,'+alfa*DG_',js,';']);
  end;

  % --- printing algorithm information
  %

  if i > 0,
    fprintf('i =%g,  |dK| = %7.4g, dJ = %7.4g, alfa = %7.4g\n',...
  %
            i,      DKn,          delF,       alfa); end;

  i = i+1;

end; % Main loop
 %

% --- computing G

G = zeros(m,nr); for j=1:m, eval(['G = G + G_',num2str(j),';']); end;
```

A.3 C-O/MATLAB files for direct minimization

A.3.1 The C-O Problem Description File (PDF)

```
/*======================================================================*/
/* C-O/MATLAB - OPTIMAL DECOUPLING - PDF                                */
/* Direct minimization of the componentwise quadratic cost function */
/* To be run with SIMU.M and INIT.M                                     */
/* Gil Yudilevitch                                                      */
/* May-13-94                                                            */
/*======================================================================*/

/* --- design parameters and their initial values (average weight)
       C-O does not work with matrices  */

design_parameter h11 init =  -0.1234
design_parameter h12 init =   0.0995
design_parameter h13 init =   0.0017
design_parameter h14 init =  -0.5631
design_parameter h15 init =   6.1277
design_parameter h16 init =   0.7893
design_parameter h17 init =  -2.3985
design_parameter h18 init =  12.1081
design_parameter h19 init =   1.1815
design_parameter h21 init =   0.0130
design_parameter h22 init =  -0.0093
design_parameter h23 init =  -0.0005
design_parameter h24 init =   0.4941
design_parameter h25 init =  -0.2279
design_parameter h26 init =   0.6848
design_parameter h27 init =   3.1004
design_parameter h28 init =   0.0395
design_parameter h29 init =   0.3535
design_parameter h31 init =   0.0407
design_parameter h32 init =  -0.0131
design_parameter h33 init =  -0.0037
design_parameter h34 init =  -0.3867
design_parameter h35 init =  -0.7414
design_parameter h36 init =   7.3236
design_parameter h37 init =  -1.8386
design_parameter h38 init =  -0.8363
design_parameter h39 init =   9.7714
design_parameter g11 init =  -1.5140
design_parameter g12 init =   2.7649
design_parameter g13 init =   2.0019
design_parameter g14 init =   0.7473
design_parameter g15 init =  -0.4458
design_parameter g16 init =  -0.1185
design_parameter g21 init =   0.0501
design_parameter g22 init =   0.0686
design_parameter g23 init =  -0.9551
design_parameter g24 init =   0.5976
design_parameter g25 init =  -0.0944
design_parameter g26 init =  -0.0167
design_parameter g31 init =  -0.0656
design_parameter g32 init =  -0.4583
design_parameter g33 init =  -0.5195
design_parameter g34 init =  -3.5588
design_parameter g35 init =  -3.4731
design_parameter g36 init =   1.9150

global double getout();

/* --- the objective (JD) and the constraint (MR) are computed
```

```
        by simulation (see SIMU.M) */
objective "coupling D"
        minimize {
        return getout("JD", 1);
        }
   good_value = 0.000
   bad_value  = 0.002

constraint "stability" hard
        {
        return getout("MR", 1);
        }
   <=
   good_value =  0.00000
   bad_value  =  0.00001
```

A.3.2   MATLAB initialization file INIT.M

```
% C-O/MATLAB initialization m-file
% for OPTDEC and SIMU.M

% Gil Yudilevitch
% May-16-94

% --- loading system matrices

load A -ascii; load B -ascii;
[n,m]=size(B); M =1:m^2; M2=1:m+1:m2;
V=zeros(m^2,m ); V(M2,M ) = eye(m);

% --- pilot's typical input
%     Wn and zeta are human factors

Wn=1; zeta=0.8; nr=6; a=Wn^2; b=2*zeta*Wn;
Ari=[0 1;-a -b];
Ar=[Ari,zeros(2,4);zeros(2,2),Ari,zeros(2,2);zeros(2,4),Ari];
Cr=[0 a 0 0 0 0;0 0 0 a 0 0;0 0 0 0 0 a];
x0=[3,-1.5,2];          % initial stse for the pilot's specific input
 x01=[zeros(1,n),x0(1),0,0,0,0,0];
x02=[zeros(1,n),0,0,x0(2),0,0,0];
x03=[zeros(1,n),0,0,0,0,x0(3),0];

% --- componentwise error and control weighting matrices

Web=diag([1 3 1  1 0.3 1  1 0.3 1]);
Wu=0.1*diag([1,5,1]);% Wub = block diag(Wu,Wu,Wu)
 %
```

A.3.3   MATLAB simulation file SIMU.M

```
% C-O/MATLAB initialization m-file
% for OPTDEC and SIMU.M

% Gil Yudilevitch
% May-16-94

% --- packing d.p.'s into matrices
```

134

```
H = [h11 h12 h13 h14 h15 h16 h17 h18 h19
     h21 h22 h23 h24 h25 h26 h27 h28 h29
     h31 h32 h33 h34 h35 h36 h37 h38 h39];

G = [g11 g12 g13 g14 g15 g16
     g21 g22 g23 g24 g25 g26
     g31 g32 g33 g34 g35 g36];

Ac  = [A-B*H,B*G;zeros(nr,n),Ar];% closed-loop matrix
 MR  = max(real(eig(Ac)));        % stability criterion (C-0 constraint)
 %

% --- computing the cost function JD (C-0 objective)

JD = 0;
for j=1:3,
  js = num2str(j);
  Cc = [C,-V((j-1)*m+1:j*m,:)*Cr];
  We = Web((j-1)*m+1:j*m,(j-1)*m+1:j*m);
  P  = lyap(Ac',[H,-G]'*Wu'*Wu*[H,-G]+Cc'*We'*We*Cc);
  eval(['JD = JD+x0',js,'*P*x0',js,''';']);
end;

save simu JD MR;
```

References

1. ANDERSON, B. D. O. AND MOORE, J. B. *Linear Optimal Control*. Prentice-Hall, 1971.

2. _____. *Optimal Control: Linear Quadratic Methods*. Prentice-Hall, 1990.

3. APLEVICH, J. D. A Geometrically-Motivated Scalar Measure of System Interaction. Presented at *The IFAC Conf. on Large Scale Systems Theory and Appl.*, Udine Italy (June, 1976).

4. ARGOUN, M. B. AND VEGTE, J. V. DE. Output Feedback Decoupling in the Frequency Domain. *Int. J. Control Vol. 31* (1980), 665–675.

5. BESELER, J. W. An Implementation of Newton's Method for The Optimal Output and Decentralized Feedback Problems. Rensselaer Polytechnic Institute, Master's Thesis, July, 1990.

6. BINGULAC, S. P., CUK, N. M. AND CALOVIC, M. S. Calculation of Optimum Feedback Gains for Output-Constrained Regulators. *IEEE Trans. Aut. Control Vol. 20* (1975), 164–166.

7. BRISTOL, E. G. On a New Measure of Interaction for Multivariable Process Control. *IEEE Trans. Aut. Control Vol. 11, No. 1* (1966), 133–134.

8. CATAPANG, D. R., TISCHLER, M. B. AND BIEZAD, D. J. Robust Crossfeed Design for Hovering Rotorcraft. Presented at *Symposium and tutorial on Quantitative Feedback Theory*, Dayton, Ohio (Aug., 1992).

9. CHEN, B-S. AND KUNG, C-C. On the Solution of the Decoupling Problem in LQG Optimal Systems. *Opt. Control Appl. and Methods Vol. 7* (1986), 135–152.

10. CHEN, K., MATHIAS, R. A. AND SAUTER, D. M. Design of Noninteracting Control Systems using Bode Diagrams. *Trans. AIEE Vol. 80 (Appl. and Ind.)* (Jan., 1962), 336–346.

11. DAVISON, E. J. AND MAN, F. T. Interaction Index for Multivariable Control Systems. *Proc. IEE Vol. 117, No. 2* (Feb., 1970), 459–462.

12. DESCUSE, J., LAFAY, J. F. AND KUCERA, V. Decoupling by Restricted Static State Feedback: The General Case. *IEEE Trans. Aut. Control Vol. 24* (1984), 79–81.

13. DESOER, C. A. AND GUNDES, A. N. Decoupling Linear Multiinput Multioutput Plants by Dynamic Feedback: An Algebraic Theory. *IEEE Trans. Aut. Control Vol. 31, No. 8* (Aug., 1986), 744–750.

14. FALB, P. L. AND WOLOVICH, W. A. Decoupling in the Design and Synthesis of Multivariable Control Systems. *IEEE Trans. Aut. Control Vol. 12, No. 6* (Dec., 1967), 651–659.

15. FAN, M. K. H., TITS, A. L., BARLOW, J., TSING, N-K., TISCHLER, M. AND TAKAHASHI, M. On the Design of Decoupling Controllers for Advanced Rotorcraft in the Hover Case. Presented at *AIAA 29th Aerospace Sciences Meeting* , Reno, Nevada (Jan., 1991).

16. FAN, M. K. H., TITS, A. L., ZHOU, J. L., WANG, L. AND KONINCKX, J. CONSOL - OPTCAD, User's Manual. SRC, Univ. of Maryland at College Park , Tech. Report TR 87-212r2a, Aug., 1991.

17. FRANCIS, B. A. *A Course in $\mathcal{H}_\infty$ Control Theory. Lecture Notes in Control and Information Sciences.* vol. Vol. 88, Springer-Verlag, Berlin, 1987.

18. FREEMAN, H. A Synthesis Method for Multipole Control Systems. *Trans. AIEE Vol. 76 (Appl. and Ind.)* (March, 1957), 28–31.

19. GAGNEPAIN, J-P. AND SEBORG, D. E. Analysis of Process Interactions with Application to Multiloop Control System Design. *Ind. Eng. Chem. Process Des. Dev. Vol. 21* (1982), 5–11.

20. GEROMEL, J. C. AND BERNUSSOU, J. An algorithm for Optimal Decentralized Control of Dynamic Systems. *Automatica* Vol. 15 (1979), 489–491.

21. GILBERT, E. G. The Decoupling of Linear Systems by State Feedback. *SIAM J. Control* Vol. 7, No. 1 (1969), 50–63.

22. GILLES, E. D., RETZBACH, B. AND SILBERBERGER, F. Modeling, Simulation and Control of an Extractive Distillation Column. *Computer Applications to Chem. Eng. (ACS Symp. Ser No. 124)* (1980), 481–492.

23. GLOVER, K. AND MCFARLANE, D. C. *Robust controller design using normalized coprime factor plant design.* Springer-Verlag, Berlin; New York, 1989.

24. GRIBBLE, J. J. AND MURRAY-SMITH, D. J. Command Following Control Law Design by Linear Quadratic Optimisation. Presented at *The 16-th European Rotorcraft Forum, Paper III.5.3*, Glasgow (Sept., 1990).

25. GRIMBLE, M. J. AND JOHNSON, M. A. *Optimal Control and Stochastic Estimation - Theory and Applications.* vol. 1, John Wiley and Sons.

26. GRÜNHAGEN, W. V., BOUWER, G., PAUSDER, H-J., HENSCHER, F. AND KALETKA, J. A High Bandwidth Control System for a Helicopter In-Flight Simulator. *Int. J. Control* Vol. 59, No. 1 (1994), 239–261.

27. HESS, R. A. AND GORDER, P. J. Quntitative Feedback Theory Applied to the Design of a Rotorcraft Flight Control System. *J. Guidance, Control and Dynamics* Vol. 16, No. 4 (July, 1993), 748–753.

28. HETRAKUL, P. AND FORTMANN, T. E. Optimal Decoupling of LInear Systems. *IEEE Trans. Aut. Control* Vol. 17 (1972), 375–377.

29. HIRZINGER, G. Decoupling Multivariable Systems by Optimal Control Techniques. *Int. J. Control* Vol. 22, No. 2 (1975), 157–168.

30. HOH, R. H., MITCHELL, D. G. AND APONSO, B. L. Handling Qualities Requirements for Military Rotorcraft. US Army ASC, Aeronautical Design Standard, ADS-33C, Aug., 1989.

31. HOH, R. H., MITCHELL, D. G., APONSO, B. L., KEY, D. L. AND BLANKEN, C. L. Background Information and User's Guide for Handling Qualities Requirements for Military Rotorcraft. US Army ASC, USAAVSCOM TR 89-A-8, Dec., 1989.

32. HORN, R. A. AND JOHNSON, C. R. *Marix Analysis*. Cambridge, 1985.

33. HOROWITZ, I. M. Synthesis of Linear Multivariable Feedback Control Systems. *IRE Trans. Aut. Control Vol. AC-5* (June, 1960), 94–105.

34. HUGHES, G., MANNESS, M. A. AND MURRAY-SMITH, D. J. Eigenstructure Assignment for Handling Qualities in Helicopter Flight Control Law Design. Presented at *The 16-th European Rotorcraft Forum, Paper III.10.2*, Glasgow (Sept., 1990).

35. HUTCHESON, W. J. Decoupling of Multivariable Systems by Function Optimization. *Int. J. Control Vol. 30, No. 6* (1979), 1001–1012.

36. JEWELL, W. F. AND CLEMENT, W. F. Crossfeed Compensation Techniques for Decoupling Rotorcraft Response to Control Inputs. Systems Technology Inc., TR-1229-1, Sept., 1985.

37. JOHNSON, W. *Helicopter Theory*. Princeton University Press, Princeton, New Jersey, 1980.

38. JOHNSTON, R. D. AND BARTON, G. W. Structural Interaction Analysis. *Int. J. Control Vol. 41* (1985), 1005–1013.

39. KAVANAGH, R. J. Multivariable Control System Synthesis. *Trans. AIEE Vol. 77 (Appl. and Ind.)* (Nov., 1958), 425–429.

40. KIM, F. D. Formulation and Validation of High-Order Mathematical Models of Helicopter Flight Dynamics. University of Maryland, Ph.D. Thesis, Dept. of Aerospace Engineering, College Park, MD, 1991.

41. KIM, F. D., TURNOUR, S. R. AND CELI, R. *UMGenHel - Version 1.2.* Center for Rotorcraft Education and Research, Department of Aerospace Engineering, University of Maryland, College Park, MD , Sept., 1991.

42. KREINDLER, E. AND ROTHSCHILD, D. Design of Optimal Decoupling Controllers: Application to VTOL. *Int. J. Systems Sc. Vol. 10, No. 8* (1979), 873–882.

43. LANDIS, K. H., DAVIS, J. M., DABUNDO, C. AND KELLER, J. F. Advanced Flight Control Technology Achievements at Boeing Helicopters. *Int. J. Control Vol. 59, No. 1* (1994), 263–290.

44. LANDIS, K. H. AND GLUSMAN, S. I. Development of ADOCS Controllers and Control Laws Volume 2 - Literature Review and Preliminary Analysis. NASA Ames Research Center, US Army ASC, NASA Contractor Report 177339, USAAVSCOM TR 84-A-7, 1984.

45. LE, V. X. AND SAFONOV, M. G. Rational Matrix GCD's and the Design of Squaring-Down Compensators - A State Space Theory. *IEEE Trans. Aut. Control Vol. 37 No. 3* (March, 1992), 384–392.

46. LEVINE, W. S. AND ATHANS, M. On The Determination of The Optimal Constant Output Feedback Gains for Linear Multivariable System. *IEEE Trans. Aut. Control Vol. 15, No. 1* (Feb., 1970), 44–48.

47. LOH, N. K., CHEOK, K. C. AND BECK, R. R. Modern Control Design for Gun-Turret Control System. Southcon/85 Convention Record, Atlanta GA, 1983.

48. MACFARLANE, A. G. J. AND BELLERTRUTTI, J. J. The Characteristic Locus Design Method . *Automatica Vol. 9* (1973), 575–588.

49. MÄKILÄ, P. M. On The Anderson-Moore Method for Solving The Optimal Output Feedback Problem. *IEEE Trans. Aut. Control Vol. 29* (Aug., 1984), 834–836.

50. MÄKILÄ, P. M. AND TOIVONEN, H. T. Computational Methods for Parametric LQ Problems - A Survey. *IEEE Trans. Aut. Control* *Vol. 32, No. 8* (Aug., 1987), 658-671.

51. MANNESS, M. A., GRIBBLE, J. J. AND MURRAY-SMITH, D. J. Multivariable Methods for Helicopter Flight Control Law Design : A Review. Presented at *The 16-th European Rotorcraft Forum, Paper III.5.2*, Glasgow (Sept., 1990).

52. MATSON, J. B., MITA, T. AND ANDERSON, B. D. O. Existence Conditions for a Nonstandard $\mathcal{H}_\infty$ Problem. Presented at *The American Control Conf.*, San-Francisco (June, 1993).

53. MORGAN, B. S. The Synthesis of Linear Multivariable System by State-Variable Feedback. *IEEE Trans. Aut. Control Vol. 9* (1964), 405-411.

54. PARASKEVOPOULOS, P. N. AND KOUMBOULIS, F. N. A New Approach to the Decoupling Problem of Linear Time-Invariant Systems. *J. Franklin Inst. Vol. 31, No. 8* (1992), 347-369.

55. POTVIN, A. F. A Unified Solution to Constrained Configuration Control Law Design. Massachuestts Institute of Technology, Master's Thesis, Oct., 1991.

56. QI-JUN, X., YOU-XIAN, S., CHUN-HUI, Z. AND YI-QUN, Y. An Optimal Decoupling Controller with Application to the Control of Paper Machine Headboxes. *The IFAC CAD in Control Sys. Symp. Vol. 24* (1988), 307-312.

57. SECKEL, E. *Stability and Control of Airplanes and Helicopters*. Academic Press, New York, 1964.

58. SIRISENA, H. R. AND CHOI, S. S. Minimal Order Compensators for Decoupling and Arbitrary Pole Placement in Linear Multivariable Systems. *Int. J. Control Vol. 25, No. 5* (1977), 755-767.

59. TAKAHASHI, M. D. Design of Flight-Control Laws for a UH-60 Helicopter in Hover Using an $\mathcal{H}_2$ Loop-Shaping Synthesis Method. NASA, US Army ASC, NASA Tech. Memo. 103834, USAAVSCOM TR 91-A-006, Dec., 1991.

60. ———. Flight-Control Design Using Rotor-State Feedback for an Articulated Rotor Helicopter in Hover. NASA, US Army ASC, NASA Tech. Memo. 103967, US-AAVSCOM TR 92-A-012, Jan., 1993.

61. ———. Design and Comparison of Pitch-Roll $\mathcal{H}_\infty$ Control Laws With and Without Rotor-State Feedback for a Hovering Helicopter. NASA, US Army ASC, NASA Tech. Memo. 108793, USAAVSCOM TR 93-A-013, Jan., 1994.

62. THE MATH WORK INC. *Control Toolbox for Use with MATLAB - User's Guide.*

63. TISCHLER, M. B. Digital Control of Highly Augmented Combat Rotorcraft. NASA, US Army ASC, NASA Tech. Memo. 88346, USAAVSCOM TR 87-A-5, May, 1987.

64. TISCHLER, M. B. AND CAUFFMAN, M. G. Frequency-Response Method for Rotorcraft System Identification: Flight Applications to BO 105 Coupled Rotor/Fuselage Dynamics. *J. American Helicopter Society Vol. 37, No. 3* (July, 1992), 3–17.

65. TISCHLER, M. B., FLETCHER, J. W., PATRICK, P. M., MORRIS, M. AND TUCKER, G. E. Flying Analysis and Flight Evaluation of a Highly Augmented Combat Rotorcraft. *J. of Guidance, Control, and Dynamics Vol. 14, No. 5* (Oct., 1991), 954–963.

66. TOIVONEN, H. T. A Globally Convergent Algorithm for The Optimal Constant Output Feedback Problem. *Int. J. Control Vol. 41* (1985), 1589–1599.

67. TOIVONEN, H. T. AND MÄKILÄ, P. M. A Convergent Anderson-Moore Algorithm for Optimal Decentralized Control. *Automatica Vol. 21* (1985), 743–744.

68. TSING, N-K. Computer-Based Techniques for Control System Design, with applications to Rotorcraft Control. University of Maryland , Ph.D. Thesis, Dept. of Electrical Engineering, College Park, MD, 1992.

69. VAN-DIGGELEN, F. Hadamard Weighting in Robust Control. University of Cambridge, Ph.D. Thesis, Engineering Dept. , Cambridge, UK, 1992.

70. VAN-DIGGELEN, F. AND GLOVER, K. A Hadamard Weighted Loop Shaping Design Procedure for Robust Decoupling. *Automatica Vol. 30, No. 5* (1994), 831–845.

71. ———. State-space Solutions to Hadamard Weighted $\mathcal{H}_\infty$ and $\mathcal{H}_\in$ Control Problems. *Int. J. Control Vol. 59, No. 2* (1994), 357–394.

72. ———. Element-by-element Weighted $\mathcal{H}_\infty$-Frobenius and $\mathcal{H}_\in$ Norm Problems. Presented at *The 30th Conf. on Decision and Control*, Brighton, England (Dec., 1991).

73. WILLIAMS, S. J. The Application of Frequency Response Multivariable Design Techniques to VSTOL Aircraft. Cambridge Control Ltd., Report No. 5, May, 1988.

74. WITCHER, M. F. AND McAVOY, T. J. Interaction Control Systems: Steady State and Dynamic Measurement of Interaction. *ISA Trans. Vol. 16, No. 3* (1977), 35–41.

75. WOLOVICH, W. A. Multipurpose Controllers for Multivariable Systems. *IEEE Trans. Aut. Control Vol. 26, No. 1* (Feb., 1981), 162–170.

76. ———. Static Decoupling. *IEEE Trans. Aut. Control Vol. 18, No. 5* (Oct., 1973), 536–537.

77. WONHAM, W. M. AND MORSE, A. S. Decoupling and Pole Assignment in Linear Multivariable Systems: A Geometric Approach. *SIAM J. Control Vol. 8* (1970), 1–18.

78. YORE, E. Optimal Decoupling Control. Presented at *The JACC* (1968).

79. YOSHIKAWA, T. AND SUGIE, T. Filtered Inverse Systems. *Int. J. Control Vol. 43, No. 6* (1986), 1661–1671.

80. YOULA, D. C., JABR, H. A. AND JR., J. J. B. Modern Wiener-Hopf Design of Optimal Controllers, Part II: The Multivariable Case. *IEEE Trans. Aut. Control Vol. 21* (1976), 319–338.

81. YUDILEVITCH, G. AND LEVINE, W. S. Rotorcraft Flight Control System Design Using CONSOL-OPTCAD - Final Report. Institute for Systems Research, University of Maryland at College Park , ISR TR 94-54, 1994.