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Strong D-Stability*

by

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ABSTRACT

A square matrix F is said to be D-stable if the eigenvalues of DF have negative real parts for any diagonal matrix D with positive diagonal elements. The perturbational properties of D-stability and block D-stability, a generalization due to Khalil and Kokotovic, are examined. Notions of 'strong D-stability' and 'strong block D-stability' are introduced, and a general class of strongly block D-stable matrices is identified. Briefly, a matrix is strongly D-stable if it is D-stable and if every sufficiently small perturbation of the matrix is also D-stable. The usefulness of this new concept is illustrated by proving a stability theorem for time-invariant multiparameter singular perturbation problems. The novelty of this theorem is that it applies to two time scale as well as multiple time scale systems, indeed regardless of the relative magnitudes of the singular perturbation parameters. The crucial assumption of the theorem is the strong block D-stability of an associated boundary layer system.

Keywords: D-stability, Perturbation theory, Stability, Singul[illegible]n, Large scale systems, Multiple time scales.

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1. Introduction

The concept of D-stability was studied by Enthoven and Arrow [10], Arrow and McManus [7], Johnson [12] and Khalil and Kokotovic [17]. This concept has proved useful in large-scale systems [26, 22, 18, 24] and multiparameter singular perturbations [1, 5, 4, 17, 19]. Recall [26] that a matrix $F \in R^{m \times m}$ is *D-stable* if the eigenvalues of DF have strictly negative real parts for any diagonal matrix D with strictly positive diagonal elements. That a D-stable matrix is also stable follows easily by considering the case $D = I_m$, the $m \times m$ identity matrix. Stability and D-stability are not equivalent, however.

Given a stable matrix F it is well-known [8, 13] that there is a $\mu > 0$ such that $F+G$ is also stable for any G with $|G| < \mu$. That is, stability is a property robust to small perturbations. Now suppose that F is also a D-stable matrix. Then for any *given* diagonal matrix D with positive diagonal elements, it is clear that both DF and $D(F+G)$ are stable matrices, for all sufficiently small $|G|$. A more interesting question is the following: Does there exist a $\mu > 0$ such that $F+G$ is D-stable whenever $|G| < \mu$? That is, can one be certain that $D(F+G)$ is stable for *any* diagonal matrix D with positive diagonal elements, for all sufficiently small $|G|$? Simple examples show that the answer is no in general. However it is possible to identify classes of D-stable matrices for which small perturbations can give rise only to D-stable matrices. It is useful to introduce the following definition.

Definition 1. The matrix $F \in R^{m \times m}$ is *strongly D-stable* if (i) F is D-stable, and (ii) there is a $\mu > 0$ such that $F+G$ is D-stable for each $G \in R^{m \times m}$ with $|G| < \mu$.

In the next section strong D-stability is shown to be satisfied for a broad class of matrices F. More precisely, this is shown to be the case for the generalized concept of *strong block D-stability,* which is the analogous extension of Khalil and Kokotovic's [17] "block D-stability". In Section 3 these ideas are employed in conjunction with results of [1] to prove a stability theorem for time-invariant multiparameter singular perturbation problems of the form

$$\dot{x} = Ax + By \tag{1a}$$

$$\epsilon_i \dot{y}_i = C_i x + D_i y, \quad i = 1,\ldots,M. \tag{1b}$$

The theorem asserts that the null solution of (1) is asymptotically stable for all sufficiently small $|\epsilon|$, $\epsilon_i > 0$, $i = 1,\ldots,M$ if (i) the reduced system obtained from (1) by formally setting $\epsilon_i = 0$, $i = 1,\ldots,M$ is asymptotically stable, and (ii) an associated boundary layer system matrix is strongly block D-stable. The novelty of this stability theorem is that it does not require any restriction on the relative magnitudes of the ratios ϵ_i/ϵ_j. Therefore the two time scale case and the multiple time scale case are treated as special cases in this framework. Recall [1, 5, 4, 17, 21] that Eq. (1) possesses two time scales if the ratios ϵ_i/ϵ_j are bounded, and it possesses multiple time scales otherwise. The multiple time scales set-up is typified by the assumption [27, 28, 11, 21, 23] $\epsilon_{i+1}/\epsilon_i \to 0$ as $\epsilon_i \to 0$, $i = 1,\ldots,M-1$. Study of singularly perturbed systems containing several small parameters under the bounded mutual ratios assumption was initiated by Khalil and Kokotovic in [15] and [17]. They refer to this set-up as the "multiparameter problem". In the present paper and the related works [4, 1, 5], however, the term *multiparameter singular perturbation problem* is intended to apply in the broader sense that no *a priori* restriction is imposed on the relative magnitudes of the small parameters ϵ_i.

Section 4 contains a further generalization of D-stability, namely D-hyperbolicity. A comparison of the results on multiparameter singular perturbations obtained here and similar results [14, 1] is also given in Section 4.

In the sequel $\sigma(F)$ for a square matrix F denotes the spectrum or set of eigenvalues of F, and R_+^M denotes the positive orthant of R^M, i.e. the set $\{ \epsilon \in R^M : \epsilon_i > 0,\ i = 1,...,M \}$.

2. Perturbation analysis

That D-stability need not be a robust property is best illustrated by an example. The matrix

$$A_0 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

is D-stable, while for arbitrarily small $\mu > 0$ the perturbed matrix

$$A_\mu = \begin{pmatrix} \mu & 1 \\ -1 & -1 \end{pmatrix}$$

is not D-stable. To see this, let $D = \text{diag}\,(d_1, d_2)$ and note that the characteristic polynomial of DA_μ is then

$$\lambda^2 + (d_2 - d_1\mu)\lambda + d_1 d_2.$$

When μ=0 the coefficients in the characteristic polynomial are positive for any positive d_1, d_2. Hence A_0 is D-stable. However for any μ>0, no matter how small, there exists a pair of positive numbers d_1, d_2 such that the coefficient of λ is nonpositive, implying that A_μ is not D-stable for any μ>0.

Even though D-stability is not a robust property in general, it is important to identify classes of D-stable matrices for which D-stability is a robust property. The value of such an identification will be clear from the results of the next section, where general stability results for multiparameter singular perturbation problems are derived based on the perturbational aspects of D-stability.

The following generalization of D-stability has been proposed in [17] (cf. also Khalil [16] and Khalil and Kokotovic [14]).

Definition 2. The matrix $F \in R^{m \times m}$ is *block D-stable* (relative to the multi-index $(m_1,...,m_M)$) if for all $d_i > 0$, $i = 1,...,M$,

$$\text{Re}\ \sigma(D(d)F) < 0 \tag{2}$$

where

$$D(d) := \text{block diag}\,(d_1 I_{m_1}, \ldots, d_M I_{m_M}). \tag{3}$$

The class of block D-stable matrices of interest in this paper is specified in the next definition.

Definition 3. The matrix $F \in R^{m \times m}$ is *strongly block D-stable* (relative to the multi-index $\overline{m} := (m_1,...,m_M)$) if ($i$) F is block D-stable (relative to $\overline{m}$), and (ii) there is a $\mu > 0$ such that $F+G$ is block D-stable (relative to $\overline{m}$) for each $G \in R^{m \times m}$ with $|G| < \mu$.

Note that strong D-stability is merely strong block D-stability relative to the multi-index $(m_1=1,...,m_M=1)$. It therefore suffices to find general classes of strongly block D-stable matrices.

Matrices F satisfying the following hypothesis are known to be block D-stable. For proofs and more details, see [12, 17].

(P) There exists a block diagonal positive definite matrix P, i.e.

$$P = \text{block diag } [P_1, \ . \ . \ . \ , P_M] \tag{4}$$

with dim $P_i = m_i$, such that Q given by

$$PF + F' P = -Q \tag{5}$$

is positive definite.

The following result asserts that matrices F satisfying this hypothesis are strongly block D-stable.

Proposition 1. *Let $F \in R^{m \times m}$ satisfy hypothesis (P). Then F is strongly block D-stable.*

Proof: The proof is standard. Letting G denote a small perturbation of F, one shows that (P) applies also to $F+G$, with the same block diagonal positive definite matrix P. The new Q matrix will be

$$Q_G := Q + PG + G' P.$$

To see that Q_G is positive definite for all sufficiently small $|G|$, recall that positive definiteness of Q is equivalent to the existence of a $c > 0$ such that $x' Qx \geq c |x|^2$ for all $x \in R^m$. Therefore

$$x' Q_G x \geq c|x|^2 + x' (PG + G' P)x$$

$$\geq (c - 2|P||G|) |x|^2$$

and the positive definiteness of Q_G for sufficiently small $|G|$ clearly follows from the last inequality. Note that the proof assumes that the matrix norm is symmetric (such as the spectral or Frobenius norms), but this in no way affects the conclusion.

Matrices satisfying hypothesis (P) have been useful in the stability analysis of multiparameter singular perturbation problems; see for instance [4, 19]. These matrices have been discussed here only to illustrate one important subclass of the class of strongly block D-stable matrices. Other interesting classes of such matrices can be constructed using results from large scale system theory [22, 26], though this is not pursued here.

3. Stability of multiparameter singular perturbation problems

The importance of the foregoing definitions and results will now be illustrated by proving a general stability theorem for time-invariant multiparameter singular perturbation problems. (For comparisons with related theorems previously announced in [1] and [14] see Section 4.) The novelty of the theorem is that it presupposes no constraint on the relative magnitudes of the singular perturbation parameters ϵ_i of Eq. (1), which is repeated here for convenience:

$$\dot{x} = Ax + By \tag{6a}$$

$$\epsilon_i \dot{y}_i = C_i x + D_i y, \quad i=1,\ldots,M. \tag{6b}$$

Here $x \in R^n$, $y = (y_1, \ldots, y_M) \in R^m$, $y_i \in R^{m_i}$, $\epsilon = (\epsilon_1, \ldots, \epsilon_M)$ with each $\epsilon_i > 0$ a small parameter, A, B, C_i, D_i are real matrices of appropriate dimension, and the dot denotes differentiation with respect to time t.

As in [1], it is useful to define matrices C and D by

$$C = \text{block diag}\,(C_1,\ldots,C_M)$$

$$D = \text{block col}\,(D_1,\ldots,D_M)$$

and rewrite (6) as

$$\dot{x} = Ax + By \tag{7a}$$

$$E(\epsilon)\dot{y} = Cx + Dy \tag{7b}$$

where $E(\epsilon) := \text{block diag}\,(\epsilon_1 I_{m_1},\ldots,\epsilon_M I_{m_M})$.

The eigenvalues of (7) (or (6)) are of course the eigenvalues of the Jacobian matrix

$$J(\epsilon) = \begin{bmatrix} A & B \\ E^{-1}(\epsilon)C & E^{-1}(\epsilon)D \end{bmatrix} \tag{8}$$

of (7).

The next lemma was introduced in [1]. It gives an algebraic matrix Riccati equation whose solution is useful in exhibiting a transformation which decouples the fast and slow modes of (7).

Lemma 1. *Suppose* det $D \neq 0$ *and denote* $E = E(\epsilon)$, $A_0 := A - BD^{-1}C$. *Then the Riccati equation*

$$D\Gamma + E(D^{-1}C - \Gamma)A_0 - E\Gamma B\Gamma + ED^{-1}CB\Gamma = 0 \tag{9}$$

for the $m \times n$ *matrix* Γ *has a locally unique solution* $\Gamma(\epsilon)$ *near* $0 \in R^{m \times n}$ *for* $|\epsilon|$ *sufficiently small.*

The next theorem gives an exact expression for the eigenvalues of $J(\epsilon)$ in terms of the eigenvalues of matrices associated with appropriate fast and slow subsystems of (6). This theorem was derived in [1] by using Lemma 1 to exhibit a similarity transformation rendering $J(\epsilon)$ in block upper triangular form.

Theorem 1. *Let* $|\epsilon|$ *be sufficiently small so that Lemma 1 applies. Then*

$$\sigma(J(\epsilon)) = \sigma(A - BD^{-1}C + B\Gamma(\epsilon)) \cup \sigma(E^{-1}(\epsilon)D + D^{-1}CB - \Gamma(\epsilon)B) \quad (10)$$

if $\epsilon_i \neq 0$, $i = 1,\dots,M$.

Theorem 1 motivates the introduction of a boundary layer system, usually taken as

$$E(\epsilon)\dot{z} = Dz.$$

The difficulties which characterize multiparameter singular perturbations can be traced to the fact that, in any time scale, the boundary layer system depends on the vector ϵ of singular perturbation parameters.

As was noted in [1], the exact representation (10) for the eigenvalues of (7) allows an immediate proof of the following theorem, a time-varying generalization of which was proved in [17]. The theorem states that under the assumption that the mutual ratios ϵ_i/ϵ_j are bounded, D-stability of the matrix D (or even a weaker condition) suffices to guarantee stability for arbitrarily small $|\epsilon|$, $\epsilon_i > 0$.

Theorem 2. *Let* H *denote the set*

$$H = \{ \epsilon \in R_+^M : c_{ij} \leq \frac{\epsilon_i}{\epsilon_j} \leq C_{ij}\ , i,j=1,\dots,M \}$$

where the c_{ij} *and* C_{ij} *are fixed positive numbers. Then the null solution of (7) is asymptotically stable for all* $\epsilon \in H$ *with* $|\epsilon|$ *sufficiently small if: (i) the reduced system obtained by formally setting* $\epsilon = 0$ *is asymptotically stable, i.e.*

$$\operatorname{Re} \sigma (A - BD^{-1}C) < 0,$$

and (ii)

$$\operatorname{Re} \sigma (E^{-1}(\epsilon)D) < 0$$

for all $\epsilon \in H$.

Proof: Condition (i) shows that the first set of eigenvalues in the right side of Eq. (10) have negative real parts for all sufficiently small $|\epsilon|$. Applying (ii) along each ray connecting the origin in R^M to the points of H on a sphere in R^M of sufficiently small radius, one sees that stability is ensured for sufficiently small $|\epsilon|$ along these rays. (One is dealing with a single parameter perturbation problem on each such ray.) Compactness now implies that an upper bound on $|\epsilon|$ exists uniformly for the whole set H.

Theorem 1 will now be used along with the concept of strong block D-stability to prove the following general stability result for the multiparameter singular perturbation problem (6) (or (7)). Note that Theorem 3 does not restrict the allowable perturbations as did Theorem 2.

Theorem 3. *Suppose that all eigenvalues of* $A_0 = A - BD^{-1}C$ *have strictly negative real parts, and let* D *be strongly block* D*-stable, relative to the multi-index* $(m_1,\dots,m_M)$. *Then there is a* $\mu > 0$ *such that the null solution of system (6) (or, equivalently, of (7)) is asymptotically stable for all* $\epsilon = (\epsilon_1, \dots, \epsilon_M)$ *with* $|\epsilon| < \mu$ *and* $\epsilon_i > 0$, $i = 1,\dots,M$.

Proof: By Theorem 1, the eigenvalues of $J(\epsilon)$ separate into two distinct groups for $|\epsilon|$ sufficiently small. The first group consists of the 'slow' eigenvalues, i.e. those of $A_0+B\Gamma(\epsilon)$. By hypothesis A_0 is a stable matrix, Lemma 1 implies $\Gamma(\epsilon) = O(|\epsilon|)$. The continuity of the eigenvalues of of a matrix with respect to parameters now implies that the slow eigenvalues will certainly have negative real parts for sufficiently small $|\epsilon|$. Now consider the 'fast' eigenvalues, i.e. those of

$$E^{-1}(\epsilon)D + D^{-1}CB - \Gamma(\epsilon)B. \tag{11}$$

Note that, since D is by assumption a strongly block D-stable matrix, any sufficiently small perturbation of D will also be block D-stable. In particular, the matrix

$$D + E(\epsilon)D^{-1}CB - E(\epsilon)\Gamma(\epsilon)B \tag{12}$$

is block D-stable (relative to the multi-index $(m_1,...,m_M)$) for all sufficiently small $|\epsilon|$. Therefore, $E^{-1}(\epsilon)$ being a candidate diagonal matrix with positive diagonal elements, multiplication of the matrix of Eq. (12) from the left by $E^{-1}(\epsilon)$ yields a matrix whose eigenvalues have strictly negative real parts. The resulting matrix is of course given by Eq. (11). Recalling that the eigenvalues of this matrix are precisely the 'fast' system eigenvalues completes the proof.

4. Concluding remarks

An immediate extension of the results of Section 2 can be made. Consider the following definitions [5].

Definition 4. A matrix $F \in R^{n \times n}$ is *D-hyperbolic* if, for any real $n \times n$ diagonal matrix D with nonzero diagonal elements, the eigenvalues of DF have nonzero real parts.

Definition 5. A matrix $F \in R^{n \times n}$ is *strongly D-hyperbolic* if it is D-hyperbolic and if every sufficiently small perturbation of F is also D-hyperbolic.

The techniques of Section 3 can be used with only minor changes to prove the following theorem.

Theorem 4. *Consider a multiparameter singular perturbation problem (7) with D a strongly block D-hyperbolic matrix. Then none of the fast eigenvalues of $J(\epsilon)$ lie on the imaginary axis, for all sufficiently small $|\epsilon|$, $\epsilon_i > 0$, $i = 1,...,M$.*

A further extension to a concept of 'strong block D-hyperbolicity' is clearly straightforward. Theorem 4 is useful in the analysis of some local nonlinear multiparameter singular perturbation problems [5].

Some remarks are in order regarding the results of Section 3. First, it should be noted that an important implication of Theorem 3 is that, under the strong block D-stability condition, it may be possible to find explicit upper bounds on the small parameters ϵ_i such that stability is ensured if these bounds are respected, regardless of the relative magnitudes of the ϵ_i. Such a uniformly valid upper bound is obtained in [4, 3] using ideas presented in [6], and upon invoking hypothesis (P). It does not appear likely that a generally applicable upper bound can be obtained without invoking some such additional assumption leading to the strong block D-stability of the boundary layer system. Further work is needed on this problem. Second, a theorem similar to Theorem 3 was announced by the author as Theorem 4 of [1]. In that theorem it was assumed that ϵ approaches 0 in some subset H of R_+^M. In the light of the results obtained in the present paper, it is now clear that Theorem 4 of [1] is correct for *generic* systems (7) satisfying the stated hypotheses, but *not* for all such systems. The corrected version follows.

Theorem 5. *Suppose* A_0 *is a stable matrix, and let the set* $H \subset R_+^M$ *be such that*

$$\mathrm{Re}\,\sigma(E^{-1}(\epsilon)D) < 0 \tag{13}$$

for all $\epsilon \in H$. *Moreover, assume that (13) also holds if* D *is replaced by any sufficiently small perturbation of* D, *for all* $\epsilon \in H$. *Then the null solution of (7) is asymptotically stable for all* $\epsilon \in H$ *with* $|\epsilon|$ *sufficiently small.*

Theorem 3 results from Theorem 5 upon setting $H = R_+^M$. Theorem 5 follows by the same arguments used to prove Theorem 3.

The relationship between stability of two time scale and multiple time scale singular perturbation problems has also been studied in [14]. Conditions were derived which, when satisfied along with the block D-stability hypothesis, imply satisfaction of hypotheses which by the results of Hoppensteadt [11] guarantee stability even under the multiple time scales set-up. An interesting question which deserves further investigation is to study the conditions obtained there (cf. Theorem 2 of [14]) in relation to strong block D-stability.

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