ABSTRACT

Title of Dissertation: THE TWINING CHARACTER FORMULA FOR SPLIT GROUPS AND A CELLULAR PAVING FOR QAUSI-SPLIT GROUPS

Jackson Hopper
Doctor of Philosophy, 2024

Dissertation Directed by: Professor Thomas Haines
Department of Mathematics

The dissertation contains two main results. The first is on the twisted Weyl character formula for split groups and the second is a cellular paving result for convolution morphisms in partial affine flag varieties of quasi-split groups.

Let $\widehat{G}$ be a connected reductive group over an algebraically closed field of characteristic 0 with a pinning-preserving outer automorphism σ. Jantzen's twining character formula relates the trace of the action of σ on a highest-weight representation V_μ of $\widehat{G}$ to the character of a corresponding highest-weight representation $(V_\sigma)_\mu$ of a related group $\widehat{G^{\sigma,\circ}}$. This paper extends the methods of Hong's geometric proof for the case $\widehat{G}$ is adjoint, to prove that the formula holds for all split, connected, reductive groups, and examines the role of additional hypotheses. In particular, it is shown that for a disconnected reductive group G, the affine Grassmannian of G is isomorphic to the affine Grassmannian of its neutral component. In the final section, it is explained how these results can be used to draw

conclusions about quasi-split groups over a non-Archimedean local field. This paper thus provides a geometric proof of a generalization of the Jantzen twining character formula, and provides some apparently new results of independent interest along the way.

Now we turn to the context of Chapter 3. Let G be a tamely ramified, quasi-split group over a Laurent series field $K = k((t))$, where k is either finite or algebraically closed. If k is finite of order q and $G_{ad,\tilde{K}}$ contains a factor of type D_4, then we also assume either $3 \mid q$ or $3 \mid q-1$. Given a sequence $(X_{w_i})_i$ of Schubert varieties contained in a fixed partial affine flag variety $\mathscr{F}$ for G, consider the convolution morphism m that maps the twisted product of those Schubert varieties into the partial affine flag variety $\mathscr{F}$. We show that the fibers of m are paved by finite products of affine spaces and punctured affine spaces. This generalizes a result of Haines, which proves a similar result in the case G is split and defined over k. A consequence for structure constants of parahoric Hecke algebras is deduced.

THE TWINING CHARACTER FORMULA FOR SPLIT GROUPS AND A CELLULAR PAVING FOR QUASI-SPLIT GROUPS

by

Jackson Hopper

Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2024

Advisory Committee:
Professor Thomas Haines, Chair/Advisor
Professor Patrick Brosnan
Professor Lawrence Washington
Professor Harry Tamvakis
Professor William Gasarch

Dedication

For Meredith, my inspiration.

Acknowledgments

First and foremost, I am deeply grateful to my advisor Tom Haines for posing the questions that led to this thesis, for his patience in discussing ideas and methods with me, and for his generous feedback on all of my work. I have learned so much from his expertise and his attention to detail, and his support has not only improved this thesis tremendously but has shaped the way I write and think about math.

I am also immensely grateful to Patrick Brosnan, Yihang Zhu, and Niranjan Ramachandran for advice and conversations during my time as a graduate student that have helped with this project. I would also like to thank all of the professors in the mathematics department, especially Harry Tamvakis and Larry Washington for their guidance and Michael Rapoport, Tamás Darvas, and Chris Laskowski for their excellent instruction.

I am profoundly thankful to Paul Pollack, who helped me get started in mathematics. His early mentorship helped me to think of myself as a researcher, and his generous time and support when I was an undergraduate student allowed me to find a way to continue doing math. I would never have made it this far without him.

I would like to extend my gratitude for the financial support I received throughout my program from the University of Maryland. I am especially grateful to Carol Fullerton for her support of my work through the Hauptman Graduate Research Fellowship, as well as to Ralph P. Pass, III for the support I received through the fellowship in his name.

I am grateful to my colleagues at the University of Maryland for their support and friendship, including Shin Song, Haeyun Seo, Arghya Sadhukhan, Chengze Duan, Josue Avila, Geoffrey Sangston, Keith Mills, Yiannis Markakis, Ian Teixeira, Eric Kubischta, Nelson Moll, Haoran Li, Shuo Yan, Yuxiang Ji, Arpith Shanbhag, Prakhar Gupta, Valeria Cherepanova, Avi Schwarzschild, Brooke Herzog, Polo Ji, Jordan Hirsh, Zeynep Kacar, Jermaine McDermott, Ishfaaq Imtiyas, and Sean McLeish. I am especially grateful to Patrick Daniels for his mentorship.

I would also like to thank my colleagues from the University of Cambridge, especially Ross Paterson, Aryan Ghobadi, Emily Roff, Lachlan McPheat, Jef Laga, Morgan Rogers, and Guy Boyde, and from the University of Georgia, Sammy Sbiti and Daniel West, all of whom played an important role in this journey. I am also thankful to my friends in DC, Iowa, and the UK who have been there for me and supported me in so many ways.

Finally, I will be forever indebted my family for all the love and support I have received from them throughout this program and my entire life. Thank you to my parents, Cindy and Doug, who have never doubted me, and to my siblings Rose, Eddie, and Olivia, who are my oldest and closest friends. Thank you to Jamie, Ken, and Marianne, for always looking out for me. Thank you most of all to my partner Meredith, who has been there every step of the way with me, and who has recently agreed to be there with me for many more!

Table of Contents

Chapter 1: Introduction

A fundamental starting point of geometric representation theory is the geometric Satake correspondence. Let G be a split, connected, reductive group over a finite residue field k, and let $F = k(\!(\varpi)\!)$ be a local field with residue field k and ring of integers $\mathscr{O}_F = k[\![\varpi]\!]$. Let $X_*(T)$ be the cocharacter lattice of G, where $T \subset G$ is a maximal torus, and let W be the (finite) Weyl group of G. The Satake isomorphism is an isomorphism of $\mathbb{C}$-algebras between the spherical Hecke algebra $C_c(G(\mathscr{O}_F)\backslash G(F)/G(\mathscr{O}_F), \mathbb{C})$ and the representation ring $\mathbb{C}[X_*(T)]^W$ of the Langlands dual group $\widehat{G}$ over $\mathbb{C}$.

The geometric Satake correspondence geometrizes the Satake isomorphism, and generalizes the setting to an arbitrary residue field k [22], [26]. That is, the geometric Satake correspondence establishes that there is a geometric space—the affine Grassmannian $\mathrm{Gr}_G = LG/L^+G$, which is represented by an infinite-dimensional ind-scheme over k—such that bi-invariant functions on $G(K)$ can be constructed via the sheaf-function correspondence from L^+G-equivariant perverse sheaves on Gr_G. Then the representations of the Langlands dual group $\widehat{G}$ are constructed by taking global cohomology groups of those perverse sheaves, establishing an equivalence of Tannakian categories between representations of $\widehat{G}$ (along with tensor product $\otimes$) and the L^+G-equivariant perverse sheaves, equipped with a convolution operator $\star$.

One reason the geometric Satake correspondence is so useful is that it allows us to deduce theorems of "representation-theoretic" content by investigating the geometry of the affine Grassmannian (and related spaces). In this dissertation two examples of that approach are given. In the first, given in Chapter 2, a formula for the trace character of an operator σ on a representation of a split reductive group $\widehat{G}$ is established by studying the space of σ-invariants for the action of σ on Gr_G, where G is the Langlands dual group of the split group $\widehat{G}$. In the second, given in Chapter 3, the idea of convolution is generalized from the affine Grassmannian to more general partial affine flag varieties of a quasi-split group G. Given a twisted product of a sequence of Schubert varieties in a partial affine flag variety $\mathscr{F}_{\mathbf{f}}$, there is a convolution morphism m that projects that twisted product back into the partial affine flag variety $\mathscr{F}_{\mathbf{f}}$, and in Chapter 3 it is shown that the fibers of m must be paved by a nice class of varieties. This result has consequences for the form of convolutions of certain compactly supported bi-invariant functions on the group G in some cases. We can now discuss the results and methods of Chapter 2 and Chapter 3 in some more detail. Be aware that the contexts, and some of the notations, differ between the two chapters.

Let $\widehat{G}$ be a connected reductive group over an algebraically closed field K of characteristic 0, and let σ be an outer automorphism of $\widehat{G}$ that preserves some Borel pair $\widehat{T} \subset \widehat{B} \subset \widehat{G}$, as well as some set of pinning homomorphisms $\{x_{\alpha^\vee}\}$ for $\widehat{G}$, indexed by simple roots $\alpha^\vee$ for $\widehat{G}$. For a dominant character $\mu \in X^*(\widehat{T})^+$, let V_μ be the irreducible K-representation of $\widehat{G}$ with highest weight μ. If μ is σ-invariant, then there is a canonical action of σ on V_μ fixing the highest weight line pointwise. The primary aim of Chapter 2 is to prove the Jantzen twining character formula holds for the action of σ on σ-invariant irreducible representations, relating the σ-twisted character of V_μ to the character a representation $(V_\sigma)_\mu$ of the related group $\widehat{G^{\sigma,\circ}}$.

Let the character lattice $X^*(\widehat{T})$ be denoted $X^\vee$, let the root system $\Phi(\widehat{G})$ be denoted $\Phi^\vee$, let the cocharacter lattice $X_*(\widehat{T})$ be denoted X, and let the coroot system $\Phi^\vee(\widehat{G})$ be denoted Φ; thus the root datum for $\widehat{G}$ is $(X^\vee, \Phi^\vee, X, \Phi)$. Then we can construct the Chevalley group G over $\mathbb{C}$ using the dual root datum $(X, \Phi, X^\vee, \Phi^\vee)$. Let $T \subset B \subset G$ be the Borel pair corresponding to the same choice of dominant cone in the character lattice, and let $\{x_\alpha\}$ be a pinning determined by the Chevalley basis of G. Since σ acts on Φ preserving the set of simple roots and also acts on $X^\vee$, we can define an action of σ on G as follows: for $\lambda \in X_*(T) = X^\vee$ and $y \in \mathbb{G}_m$, let $\sigma(\lambda(y)) = \sigma(\lambda)(y)$, and for a simple root homomorphism $x_\alpha : \mathbb{G}_a \to G$ and $y \in \mathbb{G}_a$, let $\sigma(x_\alpha(y)) = x_{\sigma(\alpha)}(y)$. Since G is generated by T and U_α for simple roots α, this is sufficient to define an action of σ on G. Let G^σ be the closed σ-invariant subgroup of G, and let $G^{\sigma,\circ}$ be the neutral component. Since σ preserves a pinning, $G^{\sigma,\circ}$ is a split, connected, reductive group over $\mathbb{C}$, and the root datum for $G^{\sigma,\circ}$ can be constructed combinatorially from the action of σ on the root datum of G. Let $\widehat{G^{\sigma,\circ}}$ be the Chevalley group over K with root datum dual to that of $G^{\sigma,\circ}$. In particular, the character lattice of $\widehat{G^{\sigma,\circ}}$ is the set of σ-invariant characters of $\widehat{G}$, $X^*(\widehat{T})^\sigma$. Given a σ-invariant dominant character $\mu \in X^*(\widehat{T})^{+,\sigma}$, let the corresponding irreducible representation of $\widehat{G^{\sigma,\circ}}$ be denoted $(V_\sigma)_\mu$.

Given a σ-invariant dominant character $\mu \in X^*(\widehat{T})^{+,\sigma}$, the action of σ on V_μ permutes the weight spaces $V_\mu(\lambda) \subset V_\mu$ for $\lambda \in X^*(\widehat{T})$. In particular, if λ is a σ-invariant character of $\widehat{G}$ and a nonzero weight of V_μ, then the action of σ on $V_\mu(\lambda)$ has a possibly-nonzero trace,

$\operatorname{tr}(\sigma \mid V_\mu(\lambda))$. These traces are used to define the σ-twisted character

$$\operatorname{ch}_\sigma(V_\mu) := \sum_{\substack{\lambda \in Wt(\mu) \\ \sigma(\lambda)\lambda}} \operatorname{tr}(\sigma \mid V_\mu(\lambda)) e^\lambda,$$

where e^λ are basis elements of the character algebra $\mathbb{C}[X^*(\widehat{T})]$.

The main theorem of the second chapter relates $\operatorname{ch}_\sigma(V_\mu)$ to the ordinary character $\operatorname{ch}((V_\sigma)_\mu)$, implying there is a Weyl-type formula for the action of σ on V_μ.

Theorem 1.0.1 (Jantzen twining character formula)**.** *Let $\widehat{G}$, G, and σ be as above. Let μ be a σ-invariant dominant character in $X^*(\widehat{T})$ and let λ be a nonzero, σ-invariant weight of V_μ. Then we have the following equality for the trace of σ:*

$$\operatorname{tr}(\sigma \mid V_\mu(\lambda)) = \dim((V_\sigma)_\mu(\lambda)). \tag{1.1}$$

The Weyl character formula for $\widehat{G^{\sigma,\circ}}$ thus implies a twining formula for the σ-twisted character of V_μ:

$$\operatorname{ch}_\sigma(V_\mu) = \operatorname{ch}((V_\sigma)_\mu) = \sum_{w \in W^\sigma} w \left(\prod_{\alpha^\vee \in (\Phi_\sigma^\vee)^+} \frac{1}{1 - e^{-\alpha^\vee}} \right) e^{w(\mu)},$$

where W is the Weyl group of $\widehat{G}$, $W^\sigma \subset W$ is the σ-invariant subgroup as well as the Weyl group of $\widehat{G^{\sigma,\circ}}$, and $\Phi_\sigma^\vee$ is the root system of $\widehat{G^{\sigma,\circ}}$. Note that $\Phi_\sigma^\vee$ can be calculated combinatorially from the action of σ on $\Phi^\vee$ [11, Theorem 6.8].

The proof is geometric in nature and follows closely after the proof in the case $\widehat{G}$ is semisimple, simply connected, and adjoint (or equivalently that G is simply connected) found in [15]. Although the result for general split groups can be derived, as in [6], from cohomology calculations in [19], most proofs assume at least that G is semisimple and simply connected [16], [21].

There are two primary reasons for assuming G is simply connected when taking a geometric approach. One is that the connected components of the affine Grassmannian Gr_G are indexed by the fundamental group of G, and so Gr_G is connected if and only if G is simply connected. Due to the reductions to "stable AMV cycles" in the proof of [15], this consideration is not a significant obstacle. The second reason is that the classification of split reductive groups only applies to connected reductive groups. Known examples make it clear that the correct group to appear in the statement of Theorem 1.0.1 is $\widehat{G^{\sigma,\circ}}$. However, it is not *a priori* obvious how the affine Grassmannian of Gr_{G^σ} relates to $\mathrm{Gr}_{G^{\sigma,\circ}}$. We show that in fact the two functors are isormorphic over any field of characteristic 0. To prove this result we use as a lemma that if G is an étale group scheme over a field of characteristic 0, then the three functors G, LG, and L^+G are all equal. These results, though not strictly necessary, seem to be new and interesting in their own right.

The overall strategy is to construct the affine Grassmannian Gr_G for the group G and understand the natural action of σ on Mirković–Vilonen (MV) cycles contained in the affine Grassmannian Gr_G. In particular, there is a natural closed immersion of functors over $\mathbb{C}$ given by $\mathrm{Gr}_{G^\sigma} \to \mathrm{Gr}_G$, and the reduced structure of the image of Gr_{G^σ} in Gr_G is exactly the σ-fixed-point subfunctor $(\mathrm{Gr}_G)^\sigma \subset \mathrm{Gr}_G$.

The geometric Satake correspondence of Mirković–Vilonen [22] establishes that, for a dominant cocharacter μ and a nonzero weight λ of the representation V_μ, the vector space spanned by MV cycles of coweight (λ, μ) is exactly the weight space $V_\mu(\lambda) \subset V_\mu$. Thus, the key step in proving the equality (1.1) is to establish that, if μ and λ are σ-invariant, then the σ-fixed point subvarieties of the σ-invariant MV cycles of coweight (λ, μ) are exactly the MV cycles of coweight (λ, μ) contained in Gr_{G^σ}; and that if μ or λ is not σ-invariant, then the σ-fixed point subvarieties are all empty.

Given a dominant cocharacter $\mu \in X_*(T)^+ = X^*(\widehat{T})^+$ and a weight cocharacter $\lambda \in X_*(T)$, an MV cycle of coweight (λ, μ) is an irreducible component of the intersection of the Schubert variety $\overline{\mathrm{Gr}_G^\mu}$ with the semi-infinite cell $S_{w_0}^\lambda := LU^- \cdot \varpi^\lambda e_0$. In order to introduce an action of $X_*(T)$, the class of MV cycles is enlarged to the class of Anderson–Mirković–Vilonen (AMV) cycles. Rather than an irreducible component of $S_{w_0}^\lambda \cap \overline{\mathrm{Gr}_G^\mu}$, an AMV cycle is an irreducible component of $S_{w_0}^\lambda \cap S_e^\mu$. Then by normality of U^- in B^- and normality of U in B, cocharacters act by translation on these intersections of semi-infinite cells. For example, given a triple of cocharacters $\lambda, \mu, \nu \in X_*(T)$ (μ not be need be dominant for the coweight of an AMV cycle), we have $\varpi^\nu \cdot (S_{w_0}^\lambda \cap S_e^\mu) = S_{w_0}^{\lambda+\nu} \cap S_e^{\mu+\nu}$. This can be used to translate the problem of studying arbitrary MV cycles to studying "stable" AMV cycles, or those of coweight $(\lambda - \mu, 0)$. Because stable AMV cycles are contained in the neutral component of Gr_G, the proof method is not obstructed by the possibility that Gr_G may not be connected. Using a criterion for determining which AMV cycles have dense intersection with MV cycles due to [1], we can study the action of σ on AMV cycles, and deduce the relevant consequences for MV cycles.

In order to establish the bijection between σ-invariant AMV cycles of Gr_G and all AMV cycles of Gr_{G^σ}, as well as to establish the isomorphisms of the latter with the σ-fixed point subvarieties of the former, we make use of an indexing of AMV cycles due to [17] by "$\mathbf{i}$-Lusztig data". These data are useful because they reflect in a transparent way the action of σ on the AMV cycles they index, allowing us to prove the following dichotomy of taking σ-fixed points: if the **i**-Lusztig datum of an AMV cycle is σ-invariant in a suitable sense, then its σ-fixed point subvariety is a corresponding AMV cycle for G^σ; and if it is not σ-invariant, then its σ-fixed point subvariety is empty.

Finally, Theorem 1.0.1 is used to deduce a result of [9] in the more general context of split, connected, reductive groups, removing a hypothesis of semisimplicity.

Now let us motivate and describe the contents of Chapter 3. Note that the notation will differ for this section.

The geometric Satake correspondence is just one example of the usefulness of geometrizing convolution of functions in partial affine flag varieties. For a reductive group G defined over a Laurent series field $K = k(\!(t)\!)$, every parahoric group scheme is associated to a partial affine flag variety. If G is a constant group, isomorphic to a base change of a group defined over k, as in Chapter 2 (note the change in notation; the residue field k for G in Chapter 2 was the complex numbers $\mathbb{C}$ and the uniformizer for the Laurent series field was denoted ϖ rather than t), then the base change $G_{k[\![t]\!]}$ is one such parahoric group scheme and the affine Grassmannian Gr_G is the corresponding partial affine flag variety. In that case, given a sequence of dominant cocharacters $\mu_\bullet = (\mu_1, \ldots, \mu_p)$, the convolution morphism of the corresponding Schubert varieties

$$m_{\mu_\bullet} : \overline{\mathrm{Gr}_G^{\mu_1}} \,\tilde{\times}\, \cdots \,\tilde{\times}\, \overline{\mathrm{Gr}_G^{\mu_p}} \to \overline{\mathrm{Gr}_G^{|\mu_\bullet|}}$$

is used to define the convolution product of L^+G-equivariant perverse sheaves on Gr_G. From this construction we can derive the geometric Satake equivalence, which is an essential tool for studying representations of reductive groups, as is seen in Chapter 2.

Another example of the use of a convolution morphism is the Demazure resolution of singularities of the full affine flag variety. If we again let G be a reductive group defined over K but instead take the parahoric group scheme to be an Iwahori $\mathscr{I}$, then the corresponding partial affine flag variety is the full affine flag variety $\mathscr{F}$. Schubert varieties of $\mathscr{F}$ are indexed by elements of the Iwahori–Weyl group W for G. Then, given a sequence of simple reflections $s_\bullet = (s_1, \ldots, s_p)$ forming a reduced word for the product $w = s_1 \cdots s_p$, the corresponding twisted product of Schubert varieties is a smooth k-variety and the convolution morhpism

$$m_{s_\bullet} : X_{s_1} \tilde{\times} \cdots \tilde{\times} X_{s_p} \to X_w$$

is a resolution of singularities of the Schubert variety X_w.

Lastly, in the case that $k = \mathbb{F}_q$ is a finite field, then for every parahoric subgroup $\mathscr{P} \subset G(K)$, convolution morphisms help us to understand the convolution product in the parahoric Hecke algebra $\mathscr{H}(G//\mathscr{P}) = C_c(\mathscr{P}\backslash G(K)/\mathscr{P}, \mathbb{C})$. Let $\mathscr{F}_\mathscr{P}$ be the partial affine flag variety corresponding to $\mathscr{P}$; Schubert varieties of $\mathscr{F}_\mathscr{P}$ are indexed by double cosets $w \in W_\mathscr{P}\backslash W/W_\mathscr{P}$, where W is the Iwahori–Weyl group of G and $W_\mathscr{P}$ is the parabolic subgroup corresponding to $\mathscr{P}$. If we normalize a Haar measure on G such that $\mathscr{P}$ has volume 1, then for a pair of basis characteristic functions $(1_{\mathscr{P}w_1\mathscr{P}}, 1_{\mathscr{P}w_2\mathscr{P}})$ let the $\mathscr{P}$-Demazure product ${}^\mathscr{P}w_1^\mathscr{P} * {}^\mathscr{P}w_2^\mathscr{P}$ be the Demazure product of the maximal-length representatives of w_1 and w_2 in W. Then we can use the fibers of the convolution morphism

$$m_{(w_1,w_2)} : X_{w_1} \tilde{\times} X_{w_2} \to X_{{}^\mathscr{P}w_1^\mathscr{P} * {}^\mathscr{P}w_2^\mathscr{P}}$$

to determine the structure constants $c^v_{w_1,w_2}(q)$ in the formula

$$1_{\mathscr{P}w_1\mathscr{P}} * 1_{\mathscr{P}w_2\mathscr{P}} = \sum_{v^{\mathscr{P}} \leq^{\mathscr{P}} w_1^{\mathscr{P}} *^{\mathscr{P}} w_2^{\mathscr{P}}} c^v_{w_1,w_2}(q) 1_{\mathscr{P}v\mathscr{P}},$$

where the sum runs over $v \in W_{\mathscr{P}}\backslash W/W_{\mathscr{P}}$. Specifically, we have the formula

$$c^v_{w_1,w_2}(q) = \# m^{-1}_{(w_1,w_2)}(v\mathscr{P})(\mathbb{F}_q).$$

A standard setting for studying the geometry of partial affine flag varieties is the one in which we have a split, reductive, constant group G over a Laurent series field, or at least that the residue field k is algebraically closed. However, for many applications—including the geometric Langlands program, the study of Shimura varieties and local models, and the calculation of structure constants for parahoric Hecke algebras—a more general context is necessary. In Chapter 3 it is assumed that G is a connected, reductive group over a Laurent series field $K = k((t))$ where the residue field k is either algebraically closed or finite, and suppose G is quasi-split over K and splits over a tamely ramified extension $\tilde{K}/K$. We also assume that if k is finite of order q and the split adjoint form $G_{ad,\tilde{K}}$ of G contains a factor of type D_4, then either $3 \mid q$ or $3 \mid q-1$.

The main goal of Chapter 3 is to prove a suitable cellular paving result holds for all fibers of convolution morphisms, and for all partial affine flag varieties, when G is as above. This generalizes a result of [10], which proves a similar paving result in the case G is a split, constant group and k is algebraically closed.

Fix a maximal K-split torus $S \subset G$ and let T be the centralizer of S in G. Since G

is quasi-split, T is a maximal torus. Fix also a Borel subgroup $B \subset G$ defined over K and containing T. Let $\mathscr{O} := k[\![t]\!]$ be the ring of integers for K, and let $\tilde{K} = \tilde{k}(\!(u)\!)$ be a minimal Galois field extension over which G splits.

Let $\mathscr{A}$ be the apartment for G with origin $\mathbf{0} \in \mathscr{A}$ and let Iwahori–Weyl group of G over K be denoted W. Note that $W = \Omega \ltimes W_{\mathrm{af}}$ is a quasi-Coxeter group, with subgroup W_{af} generated by simple reflections S_{af} and subgroup Ω preserving the fundamental alcove $\mathbf{a} \subset \mathscr{A}$.

For every facet $\mathbf{f}$ in the fundamental apartment there is a corresponding parahoric group scheme $\mathscr{P}_{\mathbf{f}}$ over $\mathscr{O}$ and parahoric subgroup $\mathscr{P}_{\mathbf{f}}(\mathscr{O}) \subset G(K)$. Furthermore, since parahoric group schemes are schemes over $\mathscr{O}$, for each $\mathbf{f}$ we have the parahoric positive loop group functor $L^+\mathscr{P}_{\mathbf{f}}$ on Aff_k that maps a k-algebra $R \mapsto \mathscr{P}_{\mathbf{f}}(R[\![t]\!])$. Then for each parahoric group scheme the functor $L^+\mathscr{P}_{\mathbf{f}}$ is a smooth, connected affine group scheme over k, though it is not of finite type. Similarly, let LG be the functor on Aff_k defined by $R \mapsto G(R(\!(t)\!))$; then LG is represented by an ind-affine ind-scheme.

Let $\mathscr{B}(G, K)$ be the Bruhat–Tits building of G. Then the apartment $\mathscr{A}$ is the fundamental apartment of the building, and $G(K)$ acts on $\mathscr{B}(G, K)$. Each parahoric subgroup $\mathscr{P}_{\mathbf{f}}(\mathscr{O})$ fixes the corresponding facet $\mathbf{f}$ pointwise. If $\mathbf{f}$ is in the closure of $\mathbf{a}$, then let $W_{\mathbf{f}} \subset W_{\mathrm{af}}$ be the subgroup of W fixing $\mathbf{f}$ pointwise. Then $W_{\mathbf{f}}$ is a parabolic subgroup of W_{af}, and the Bruhat order $\leq$ is well-defined on the coset space $W/W_{\mathbf{f}}$ as well as the double coset space $W_{\mathbf{f}}\backslash W/W_{\mathbf{f}}$.

Given a facet $\mathbf{f}$ in the closure of $\mathbf{a}$ the functor $\mathscr{F}_{\mathbf{f}}$ on Aff_k is defined by taking the étale sheafification of the presheaf functor $R \mapsto LG(R)/L^+\mathscr{P}_{\mathbf{f}}(R)$. Then $\mathscr{F}_{\mathbf{f}}$ is called a partial affine flag variety corresponding to $\mathbf{f}$, and is represented by an ind-projective ind-scheme. A fundamental result about a partial affine flag variety $\mathscr{F}_{\mathbf{f}}$ is that it is stratified by $L^+\mathscr{P}_{\mathbf{f}}$-orbits indexed by the double coset space $W_{\mathbf{f}}\backslash W/W_{\mathbf{f}}$ [24]. If $\dot{w} \in G(K)$ is a lift of a double coset w, then the $L^+\mathscr{P}_{\mathbf{f}}$-orbit of the point $\dot{w}e_{\mathbf{f}}$ in $\mathscr{F}_{\mathbf{f}}$ is called the Schubert cell corresponding to w, and denoted $Y_{w,\mathbf{f}}$. The reduced closure $X_{w,\mathbf{f}} := \overline{Y_{w,\mathbf{f}}}$ is called the Schubert variety corresponding to w. For a pair of double cosets v, w, we have $Y_{v,\mathbf{f}} \subset X_{w,\mathbf{f}}$ if and only if $v \leq w$.

Given a sequence $w_\bullet = (w_1, \dots, w_p) \in (W_{\mathbf{f}}\backslash W/W_{\mathbf{f}})^p$, let the $\mathbf{f}$-Demazure product ${}^{\mathbf{f}}w_*^{\mathbf{f}}$ be the Demazure product of the maximal representatives ${}^{\mathbf{f}}w_i^{\mathbf{f}} \in W$ of the respective double cosets w_i [7]. Then we have a twisted products of Schubert cells and Schubert varieties formed as the reduced closed subschemes of $X_{{}^{\mathbf{f}}w_*^{\mathbf{f}}}$ whose $\bar{k}$-points are given respectively by

$$Y_{w_\bullet}(\bar{k}) := Y_{w_1} \tilde{\times} \cdots \tilde{\times} Y_{w_p}(\bar{k}) = \{(g_1e_{\mathbf{f}}, \dots, g_pe_{\mathbf{f}}) \in \mathscr{F}_{\mathbf{f}}^p \mid g_{i-1}^{-1}g_ie_{\mathbf{f}} \in Y_{w_i}(\bar{k}) \text{ for all } i\}$$

$$X_{w_\bullet}(\bar{k}) := X_{w_1} \tilde{\times} \cdots \tilde{\times} X_{w_p}(\bar{k}) = \{(g_1e_{\mathbf{f}}, \dots, g_pe_{\mathbf{f}}) \in \mathscr{F}_{\mathbf{f}}^p \mid g_{i-1}^{-1}g_ie_{\mathbf{f}} \in X_{w_i}(\bar{k}) \text{ for all } i\}.$$

We also have the convolution morphisms defined on $\bar{k}$-points by

$$\begin{aligned} m^\circ_{w_\bullet,\mathbf{f}} : Y_{w_\bullet} &\to X_{{}^{\mathbf{f}}w_*^{\mathbf{f}}} & \qquad m_{w_\bullet,\mathbf{f}} : X_{w_\bullet} &\to X_{{}^{\mathbf{f}}w_*^{\mathbf{f}}} \\ (g_1e_{\mathbf{f}}, \dots, g_pe_{\mathbf{f}}) &\mapsto g_pe_{\mathbf{f}}, & (g_1e_{\mathbf{f}}, \dots, g_pe_{\mathbf{f}}) &\mapsto g_pe_{\mathbf{f}}. \end{aligned}$$

The main results of Chapter 3 are pavings for the fibers of $m_{w_\bullet,\mathbf{f}}$. Given a class of k-varieties $\mathscr{C}$, we say a k-scheme X is paved by varieties in $\mathscr{C}$, or simply paved by $\mathscr{C}$, if there is a finite exhaustion of closed subvarieties $\varnothing = X_0 \subset X_1 \subset \cdots \subset X_n = X$ such that each locally closed difference $X_i - X_{i-1}$ is isomorphic to a member of $\mathscr{C}$.

Theorem 1.0.2. *Let G be a quasi-split group as above over a field K, and let $s_\bullet = (s_1, \ldots, s_p)$ be a sequence of simple reflections in S_{af}. Then the fibers of the convolution morphism*

$$m_{s_\bullet,\mathbf{a}} : X_{s_\bullet,\mathbf{a}} \to X_{s_*,\mathbf{a}}$$

are paved by finite affine spaces.

Theorem 1.0.3. *Let G be a quasi-split group as above over a field K, let $\mathbf{f}$ be a facet in the closure of $\mathbf{a}$, and let $w_\bullet = (w_1, \ldots, w_p)$ be a sequence of elements of $W_{\mathbf{f}} \backslash W / W_{\mathbf{f}}$. Let $\mathscr{C}$ be the class of finite products of affine spaces and punctured affine spaces. Then the fibers of the convolution morphism*

$$m_{w_\bullet,\mathbf{f}} : X_{w_\bullet,\mathbf{f}} \to X_{\mathbf{f}w^{\mathbf{f}}_*}$$

are paved by $\mathscr{C}$.

The key idea of the proofs of both theorems is to work inductively. Specifically, for a given codimension 1 facet $\mathbf{f}$ in the closure of an alcove $\mathbf{a}'$ in the building $\mathscr{B}(G, K)$, the set of alcoves other than $\mathbf{a}'$ containing $\mathbf{f}$ in their closure is indexed by the k-points of a unipotent group scheme $\mathsf{U}_{\alpha+r}$. Moreover, the corresponding Schubert cell $wY_{s_{\alpha+r},\mathbf{a}}$ is isomorphic to $\mathsf{U}_{\alpha+r}$ as a variety over k, which is isomorphic to an affine space $\mathbb{A}^d_k$ for some $d \leq 3$. Furthermore, the punctured Schubert cell $wY_{s_{\alpha+r},\mathbf{a}} - \{wse_{\mathbf{a}}\}$ is isomorphic to $\mathbf{U}_{\alpha+r} - \{1\}$, which is isomorphic to $\mathbb{A}^d_k - \{0\}$.

We show that a fiber of $m_{w_\bullet}$ has a locally closed decomposition where each locally closed subscheme is isomorphic to a product of two spaces known to have a paving of the desired form. The first is the fiber of a "smaller" convolution morphism, which is assumed by induction to have the desired paving, and the second is shown to be isomorphic to $\mathsf{U}_{\alpha+r}$ (or $\mathsf{U}_{\alpha+r} - \{1\}$ in the proof of Theorem 1.0.3).

An essential lemma for this induction is called stratified triviality. This result states that, for any fixed choice of facet $\mathbf{f}$ in the closure of $\mathbf{a}$, the morphism $m_{w_\bullet,\mathbf{f}}$ is trivial over the intersection of its image with an $L^+\mathscr{I}$-orbit. This allows us to deduce the product decomposition of locally closed subschemes of the fiber as mentioned above.

Stratified triviality is proved as a consequence of an Iwahori-type decomposition. This Iwahori-type decomposition is itself a consequence of an analogous decomposition for split groups proved in [7], along with a comparison of affine root subgroups of G to those of the split group $G_{\tilde{K}}$.

To prove Theorem 1.0.2, induction is carried out according to the length of the sequence of simple reflections. Fix a sequence $s_\bullet = (s_1, \ldots, s_p)$ and a point $we_{\mathbf{a}}$ in the image of $m_{s_\bullet}$. Then let $s'_\bullet = (s_1, \ldots, s_{p-1})$ and let $m_{s'_\bullet,\mathbf{a}}$ be the corresponding convolution morphism. We use the branching in the building to show that, for a fixed choice of x in the image of $m_{s'_\bullet,\mathbf{a}}$ and the fiber lying over $we_{\mathbf{a}}$, the space of points in the intersection of the image of $m_{s_\bullet,\mathbf{a}}$ with x as the $(p-1)$th coordinate is isomorphic to one of the following three spaces: either $\mathsf{U}_{\alpha+r}(k)$ where s_p is a simple reflection of type $\alpha + r$, a single point, or an empty set. Then by stratified triviality, the fiber over $we_{\mathbf{a}}$ is a product of the smaller fiber and one of the three spaces listed, which is sufficient to prove the fiber is paved by affine spaces.

Next Theorem 1.0.3 is proved in three steps, which are fundamentally similar to the proof of Theorem 1.0.2. However, in order to achieve full generality, it is necessary to use the stratification of the twisted Schubert variety $X_{w_\bullet}$ by twisted Schubert cells $Y_{v_\bullet} \subset X_{w_\bullet}$.

The restriction of the convolution morphism $m_{w_\bullet}$ is called $m^\circ_{w_\bullet}$. In the first step we assume, as in the proof of Theorem 1.0.2, that $w_\bullet = s_\bullet$ is a sequence of simple reflections and that $\mathbf{f} = \mathbf{a}$. Then we proceed as before, fixing a point $we_{\mathbf{a}}$ in the image of $m^\circ_{s_\bullet}$ and a point x in the image of $m^\circ_{s'_\bullet}$ lying over $we_{\mathbf{a}}$, and describing the space of points in the intersection of the $L^+\mathscr{I}$-orbit of $we_{\mathbf{a}}$ with x as the $(p-1)$th component of the fiber. However, for $m^\circ_{s_\bullet}$ there are now four possibilities for what the isomorphism class of this space may be: it may be $\mathsf{U}_{\alpha+r}$, a single point, empty, or $\mathsf{U}_{\alpha+r} - \{1\}$. In all cases we deduce the desired paving result using stratified triviality.

In the second step, we assume that $w_\bullet$ is an arbitrary sequence of elements of W but still work with the fixed facet $\mathbf{f} = \mathbf{a}$, and show that this convolution morphism is isomorphic the convolution corresponding to a longer sequence of simple reflections. Finally, we carefully choose representatives of double cosets $w_i \in W_{\mathbf{f}} \backslash W / W_{\mathbf{f}}$ and show the desired paving result for a twisted product of Schubert varieties in $\mathscr{F}_{\mathbf{f}}$ can be deduced as a special case.

After the proof of the main theorems, we deduce a consequence for structure constants of parahoric Hecke algebras.

Chapter 2: The Twining Character Formula for Reductive Groups

Jantzen's twining character formula is a twisted version of the Weyl character formula. Given a pinning-preserving outer automorphism σ on a connected, reductive group scheme $\widehat{G}$ over an algebraically closed field, the formula describes the twisted character of σ on a highest-weight representation of $\widehat{G}$ in purely combinatorial terms, and can be calculated using the σ-action on the root datum of $\widehat{G}$. It was first proved by Jantzen, [16], with alternative proofs provided by [21], [15], and [6].

Most of the above proofs share an assumption that the group $\widehat{G}$ is connected, semisimple, and adjoint, and some impose additional hypotheses. However, the proof of [6] holds for connected, reductive groups, following some cohomology calculations in [19]. In this paper I will make the same assumptions: that $\widehat{G}$ is a connected, and reductive group.

I will follow quite closely the geometric proof of Hong [15]. I will outline the structure, then reproduce the proof in a fairly self-contained way to make clear where the stronger hypotheses might be convenient—and why they are unnecessary. For one thing, Hong's proof applies equally well as written whether $\widehat{G}$ is assumed to be adjoint or simply connected. For the more general case, Proposition 2.3.1 will be useful. Although Proposition 2.3.1 is not necessary to prove Theorem 2.0.1, as shown in [6], I believe it is interesting in its own right and have not seen it elsewhere in the literature.

Let $\widehat{G}$ be a connected, reductive group over an algebraically closed field K of characteristic 0, and fix a root datum of $\widehat{G}$. In particular, fix a maximal torus and Borel $\widehat{T} \subset \widehat{B} \subset \widehat{G}$. Let G be the complex group with dual root datum, and with a corresponding choice of maximal torus and Borel $T \subset B \subset G$. Let σ be an automorphism of $\widehat{G}$ preserving the root datum and a pinning, and consider its induced action on G, which preserves the dual root datum and a pinning. Let G^{σ} be the fixed-point subgroup of G, and let $G^{\sigma,\circ}$ be the neutral component of G^{σ}. Then $G^{\sigma,\circ}$ is a connected, reductive group [32] (see also [12]), and a closed subgroup of G, with maximal torus and Borel $T^{\sigma,\circ} \subset B^{\sigma,\circ} \subset G^{\sigma,\circ}$. The cocharacter lattice $X_*(T^{\sigma,\circ})$ is a subgroup of $X_*(T)$, and other components of the root datum of $G^{\sigma,\circ}$ can also be determined combinatorially. Let $\widehat{G^{\sigma,\circ}}$ be the dual of $G^{\sigma,\circ}$ over K.

Note that σ acts on the lattice of cocharacters of T (i.e. characters of $\widehat{T}$). If μ is any σ-invariant dominant cocharacter of G, then we are interested in the action of σ on the irreducible highest-weight representation V_μ of $\widehat{G}$, as well as the action of σ on weight spaces $V_\mu(\lambda)$, where λ is a nonzero weight of V_μ. The set of such weights is denoted $Wt(\mu)$. Up to a scalar, there is a unique vector space automorphism $\sigma : V_\mu \to V_\mu$ commuting with the action of $\widehat{G}$ on V_μ. We can normalize this automorphism by assuming σ not only stabilizes the highest-weight line $V_\mu(\mu)$, but fixes it pointwise, uniquely determining an action of σ. The irreducible representation of $\widehat{G^{\sigma,\circ}}$ of highest weight μ is denoted $(V_\sigma)_\mu$. Then we have the following theorem relating V_μ and $(V_\sigma)_\mu$.

Theorem 2.0.1 (Jantzen's twining character formula)**.** *Let $\widehat{G}$, G, and σ be as above. Let μ be a σ-invariant dominant character in $X^*(\widehat{T})$ and let $\lambda \in Wt(\mu)$ be a σ-invariant weight* *of V_μ. Then σ preserves $V_\mu(\lambda)$, and we have the following equality:*

$$\operatorname{tr}(\sigma \mid V_\mu(\lambda)) = \dim((V_\sigma)_\mu(\lambda)). \tag{2.1}$$

The Weyl character formula for $\widehat{G^{\sigma,\circ}}$ thus implies a twining formula for the twisted character of σ:

$$\sum_{\substack{\lambda \in Wt(\mu) \\ \sigma(\lambda)=\lambda}} \operatorname{tr}(\sigma \mid V_\mu(\lambda)) e^\lambda = \operatorname{ch}((V_\sigma)_\mu) = \sum_{w \in W^\sigma} w \left(\prod_{\alpha \in N'_\sigma(\Phi)^+} \frac{1}{1-e^{-\alpha}} \right) e^{w(\mu)}.$$

Here $N'_\sigma(\Phi)$ is a root system explicitly determined by the σ-action on Φ and is the root system of the group $\widehat{G^{\sigma,\circ}}$.

See Section 2.7 for details on how to determine $N'_\sigma(\Phi)$ using the root datum of $\widehat{G}$. Notation and proofs there are drawn from [11].

There are two apparent justifications for the stronger hypotheses taken in previous proofs of Theorem 2.0.1. First, unless $\widehat{G}$ is semisimple and either simply connected or adjoint, G^σ may not be connected. This turns out to be immaterial, as the affine Grassmannian of G^σ (and the category of sheaves on it) "forgets" any disconnectedness of G^σ, cf. Proposition 2.3.1.

Second, the root lattice of $\widehat{G}$ is a strict sublattice of $X^*(\widehat{T})$ in the case $\widehat{G}$ is not semisimple and adjoint. In this case the affine Grassmannian of G is disconnected, and the dimensions of Schubert varieties vary in differing components. However, it turns out that, due to the use of Anderson's polytope calculus and normalization to stable AMV cycles, disconnectedness of the affine Grassmannian is also immaterial.

In Section 2.1, I establish conventions used throughout the paper. In Section 2.2, I outline the proof of Theorem 2.0.1, omitting some details, to see how the stronger hypotheses are used in the literature. Sections 2.3 through 2.6 comprise a complete proof of Theorem 2.0.1: Section 2.3 describes the σ-action on subvarieties of the affine Grassmannian of G, relating them to corresponding varieties in the affine Grassmannian of Gr_{G^σ}; Section 2.4 considers the action of σ on $\mathbf{i}$-Lusztig strata, establishing a condition for invariance; Section 2.5 establishes a coweight-preserving bijection between the σ-invariant MV cycles of G and all MV cycles of $G^{\sigma,\circ}$; and Section 2.6 completes the proof of Theorem 2.0.1 by showing that σ fixes basis vectors corresponding to σ-invariant MV cycles, implying that the trace of σ is exactly the number of preserved MV cycles. Section 2.7 deals explicitly with root data and uses Theorem 2.0.1 to prove a Theorem 7.7, which is stated in [11] without reference to a fully general proof. Finally, Appendix A is a complete list of results from [22] used in this paper.

2.1 Notation

Here I will establish some notation and conventions. Throughout, $\widehat{G}$ is a connected, reductive group over an algebraically closed field K of characteristic 0. I primarily work with its complex dual group G. If I write "character," "coweight," or other similar term, without specifying which group I am referring to, I intend to refer to G.

Fix a maximal torus and Borel $T \subset B \subset G$. We also have the corresponding maximal unipotent subgroup $U \subset B$. Let the set of simple roots be denoted Π and the simple coroots denoted $\Pi^\vee$. Let the set of roots be denoted Φ and the set of coroots denoted $\Phi^\vee$. Let the character lattice be denoted $X^*(T)$ and the cocharacter lattice denoted $X_*(T)$.

Fix also a maximal torus and Borel $\widehat{T} \subset \widehat{B} \subset \widehat{G}$, with corresponding maximal unipotent $\widehat{U}$. Then the character lattice for $\widehat{G}$ is exactly $X^*(\widehat{T}) = X_*(T)$, and the cocharacter lattice is $X_*(\widehat{T}) = X^*(T)$; the set of roots of $\widehat{G}$ is $\Phi^\vee$, and the set of coroots is Φ; the set of simple roots of $\widehat{G}$ is $\Pi^\vee$, and the set of simple coroots is Π.

Let G^{der} be the derived subgroup of G, and G^{sc} the universal cover of G^{der}. Then G^{sc} and G^{der} have the same set of coroots and simple coroots as G, and there is a natural embedding $X_*(T^{sc}) \hookrightarrow X_*(T)$, with $X_*(T^{sc})$ generated by $\Pi^\vee$. Given any two cocharacters $\mu, \lambda \in X_*(T)$, we say $\lambda \leq \mu$ if and only if $\mu - \lambda \in \mathbb{Z}_{\geq 0}\Pi^\vee$. Let ρ be the half sum of positive roots of G and $\rho^\vee$ be the half sum of positive coroots.

Let W be the Weyl group $N_G(T)/T$. Then W acts on $X^*(T)$ and $X_*(T)$. For each $w \in W$, let $\lambda \leq_w \mu$ if and only if $w^{-1}(\lambda) \leq w^{-1}(\mu)$. Corresponding to the choice of simple roots Π, we have a set of simple reflections $S = \{s_\alpha\}_{\alpha \in \Pi}$ generating W, and (W, S) is a Coxeter system. Then there is a length function ℓ on elements of W; let $w_0 \in W$ be the longest element.

Fix a pinning of G compatible with T and B, i.e. a collection of root homomorphisms $x_\alpha : \mathbb{G}_a \to U$ for each simple root $\alpha \in \Pi$. Each root homomorphism x_α also uniquely determines an opposite root homomorphism $y_\alpha : \mathbb{G}_a \to w_0 U w_0^{-1}$. Fix also pinning $\{x_{\alpha^\vee}\}_{\alpha^\vee \in \Pi^\vee}$ of $\widehat{G}$ compatible with $\widehat{T}$ and $\widehat{B}$.

Let σ be an automorphism of G preserving the pinning $\{x_\alpha\}_{\alpha \in \Pi}$, meaning that σ preserves T and B, and that $\sigma \circ x_\alpha = x_{\sigma(\alpha)}$ for all $\alpha \in \Pi$. Let G^σ be the fixed point subgroup, and $G^{\sigma,\circ}$ the neutral component of that fixed point subgroup.

Since σ acts on the constituents of the root datum of $\widehat{G}$, there is a unique action of σ

on $\widehat{G}$ preserving its root datum and the pinning $\{x_{\alpha^\vee}\}$. Specifically, $\widehat{G}$ is generated by the images of cocharacters generating $X^*(T)$ and by the root and opposite root homomorphisms $x_{\alpha^\vee}$ and $y_{\alpha^\vee}$ for $\alpha^\vee \in \Pi^\vee$. It is thus sufficient to define σ on these images. Let R be a K-algebra, and suppose $g \in \widehat{G}(R)$. If $g = \lambda(t)$ for some $t \in \mathbb{G}_m(R)$ and $\lambda \in X^*(T)$, let $\sigma(g) = \sigma(\lambda)(t)$. And if $g = x_{\alpha^\vee}(u)$ for some $u \in \mathbb{G}_a(R)$, then let $\sigma(g) = x_{\sigma(\alpha^\vee)}(u)$; similarly, if $g = y_{\alpha^\vee}(u)$, let $\sigma(g) = y_{\sigma(\alpha^\vee)}(u)$.

For a complex, smooth, linear algebraic group H we have the loop group, positive loop group, negative loop group, and strictly negative loop group functors from $\mathbb{C}$-algebras to sets given by $LH : R \mapsto H(R(\!(\varpi)\!))$, $L^+H : R \mapsto H(R[\![\varpi]\!])$, $L^-H : R \mapsto H(R[\varpi^{-1}]) \subset LH(R)$, and $L^{--}H : R \mapsto \ker(L^-H(R) \to H(R))$, respectively. The étale sheafification of the quotient functor $LH/L^+H : R \mapsto H(R(\!(\varpi)\!))/H(R[\![\varpi]\!])$ is known as Gr_H, the affine Grassmannian of H, and is representable by an ind-finite type (strict) ind-scheme over $\mathbb{C}$. The ind-scheme Gr_H is ind-projective if and only if H is reductive (see, for instance, [3] Theorem 4.5.1(iv)). For this reason, it is essential to this proof to assume G is reductive.

For each cocharacter $\nu \in X_*(T)$ we have by definition a homomorphism $\nu : \mathbb{G}_m \to T \subset G$, as well as a homomorphism $\nu : L\mathbb{G}_m \to LT$. Let $\varpi^\nu \in LT(\mathbb{C})$ be the image of ϖ under this homomorphism, and let $\varpi^\nu x_0$ be the image of ϖ under the composition

$$L\mathbb{G}_m(\mathbb{C}) \xrightarrow{\nu} LT(\mathbb{C}) \to LG(\mathbb{C}) \to \mathrm{Gr}_G(\mathbb{C}),$$

where x_0 is the natural basepoint of $\mathrm{Gr}_G(\mathbb{C})$, corresponding to the trivial coset in $LG(\mathbb{C})/L^+G(\mathbb{C})$.

Given a locally closed, reduced sub-ind-scheme $Y \subset \mathrm{Gr}_G$, let $\overline{Y}$ be the reduced closure. If G is reduced, then $\overline{Y}$ is ind-projective. Gr_G has a Cartan stratification by L^+G-orbits.

Given a cocharacter μ, let Gr_G^μ be the L^+G-orbit $\mathrm{Gr}_G^\mu = L^+G \cdot \varpi^\mu x_0$. I refer to these orbits as Schubert cells, and their closures as Schubert varieties. Schubert cells and Schubert varieties are reduced, finite-type, complex schemes. Typically μ will be taken dominant, since $\mathrm{Gr}_G^\mu = \mathrm{Gr}_G^{w(\mu)}$ for all $w \in W$. If μ is dominant, we have the following closure relations from the Cartan stratification:

$$\overline{\mathrm{Gr}_G^\mu} = \coprod_{\substack{\lambda \in X_*(T)^+ \\ \lambda \leq \mu}} \mathrm{Gr}_G^\lambda.$$

We also have, for each $w \in W$, an Iwasawa stratification. The strata of the Iwasawa stratification are known as semi-infinite cells. In contrast with Schubert varieties, semi-infinite cells and their closures are not representable by schemes. Given w and a cocharacter ν, let S_w^ν be the orbit $S_w^\nu = wLUw^{-1} \cdot \varpi^\nu x_0$. Then we have the following closure relations (see eg [22] Proposition 3.1(a)):

$$\overline{S_w^\nu} = \coprod_{\substack{\eta \in X_*(T) \\ \eta \leq_w \nu}} S_w^\eta$$

From a geometric description of the complex points, we have an intersection criterion (see eg [22] equation (3.5) in the proof of Theorem 3.2). That geometric description is

$$S_w^\nu(\mathbb{C}) = \{x \in \mathrm{Gr}_G(\mathbb{C}) \mid \lim_{s \to 0} w(\rho^\vee)(s) \cdot x = \varpi^\nu x_0\}, \tag{2.2}$$

where $w(\rho^\vee) : \mathbb{G}_m(\mathbb{C}) \to T(\mathbb{C})$ is a homomorphism of complex groups. As a consequence of this description, $S_w^\eta \cap S_{w'}^\nu \neq \varnothing$ only if $\nu \leq_w \eta$ and $\eta \leq_{w'} \nu$. Indeed, if $p \in (S_w^\eta \cap S_{w'}^\nu)(\mathbb{C})$, then both $\varpi^\eta x_0$ and $\varpi^\nu x_0$ are in the closure of the $T(\mathbb{C})$-orbit of p. Since both S_w^η and $S_{w'}^\nu$

are T-invariant, that means in particular that $\varpi^\eta x_0 \in \overline{S^\nu_{w'}}(\mathbb{C})$ and $\varpi^\nu x_0 \in \overline{S^\eta_w}(\mathbb{C})$, implying the inequalities.

Given a reduced, irreducible, projective subvariety $X \subset \mathrm{Gr}_G$, the sheaf $IC_X = j_{!*}(\mathbb{C}[\dim X])$ is the unique perverse sheaf restricting to constant coefficients on the non-singular locus of X. In the case $X = \overline{\mathrm{Gr}^\mu_G}$ for a dominant cocharacter μ, this sheaf is L^+G-equivariant and known simply as IC_μ. The category of L^+G-equivariant perverse sheaves on closed subvarieties of Gr_G consists of only direct sums of IC_μ for dominant μ and is referred to as $P_{L^+G}(\mathrm{Gr}_G)$.

2.2 Outline of proof

Here I will summarize the proof, adapted from [15], of Theorem 2.0.1. I do this primarily to see that the hypothesis that G is reductive is sufficient. Suppose G is a complex, semisimple, simply connected group, and let σ be a pinning-preserving automorphism of G.

The proof is geometric in nature, relying on the geometry of the affine Grassmannian Gr_G. Of particular importance is the geometric Satake equivalence, which constructs an explicit and canonical bijection between certain varieties contained in Gr_G, called MV cycles, and basis vectors of highest-weight representations of $\widehat{G}$. For precise statements of the several theorems from [22] I am referring to when I say "the geometric Satake equivalence," see the Appendix A. The most important results, stated according to conventions from Section 2.1, are summarized here:

Theorem 2.2.1. *Let μ be a dominant cocharacter, and let $\lambda \in Wt(\mu)$.*

i. $S^\lambda_{w_0} \cap \overline{\mathrm{Gr}^\mu_G}$ *is equidimensional, and* $\dim(S^\lambda_{w_0} \cap \overline{\mathrm{Gr}^\mu_G}) = \langle \rho, \mu - \lambda \rangle$

ii. $S^\lambda_{w_0} \cap S^\mu_e$ *is equidimensional, and* $\dim(S^\lambda_{w_0} \cap S^\mu_e) = \langle \rho, \mu - \lambda \rangle$

iii. $\mathbb{H}^\bullet(\mathrm{Gr}_G, IC_\mu) = \displaystyle\bigoplus_{\lambda \in Wt(\mu)} H_c^{-2\langle \rho, \lambda \rangle}(S^\lambda_{w_0}, IC_\mu) = V_\mu$

iv. $H_c^{-2\langle \rho, \lambda \rangle}(S^\lambda_{w_0}, IC_\mu) = \displaystyle\bigoplus_{A \in \mathrm{Irr}\,(S^\lambda_{w_0} \cap \overline{\mathrm{Gr}^\mu_G})} K[A] = V_\mu(\lambda).$

v. $P_{L^+G}(\mathrm{Gr}_G, \mathbb{Z})$ *is isomorphic as a tensor category to* $\mathrm{Rep}_{\mathbb{Z}}(\widehat{G})$*.*

The direct sum in statement iv. is indexed by irreducible components A of the variety $S^\lambda_{w_0} \cap \overline{\mathrm{Gr}^\mu_G}$. These irreducible components are the MV cycles. The heart of the proof of Theorem 2.0.1 is to establish a bijective correspondence between those MV cycles in Gr_G which are invariant under the action of σ and all MV cycles in Gr_{G^σ}.

This is done in three steps. First, the collection of MV cycles is generalized to a larger collection of what are here called AMV cycles (see Section 2.3 for definition), after Anderson's polytope calculus [1]. Using Kamnitzer's indexing of AMV cycles by **i**-Lusztig data (see Section 2.4), a convenient criterion for σ-invariance of an AMV cycle is found, as well as a procedure for finding the datum of a corresponding AMV cycle in Gr_{G^σ}. Second, a criterion of [1] for when an AMV cycle intersects generically with an MV cycle is applied to show that the restriction of this correspondence to MV cycles is also bijective. Finally, the eigenvalues of the σ-action on the $\widehat{G}$-representation V_μ are examined to ensure that the twisted character of σ is exactly as expected, completing the proof of Theorem 2.0.1.

Based on the short description above, it is not obvious how the hypothesis that G is semisimple is used. However, much of the literature explicitly makes this and other assumptions.

The two complications introduced by relaxing the hypotheses from semisimple and simply connected to reductive come in the form of two different disconnected spaces. First, if G is not either semisimple or adjoint, then the fixed point subgroup G^σ is not necessarily connected. Using the classification of reductive groups, it is more convenient to work with $G^{\sigma,\circ}$ than G^σ whenever dealing with root data. However, it is easier, and in my opinion more natural, to relate the geometry of $(\mathrm{Gr}_G)^\sigma$ to Gr_{G^σ} than to $\mathrm{Gr}_{G^{\sigma,\circ}}$. Thankfully, the affine Grassmannians of G^σ and $G^{\sigma,\circ}$ are isomorphic for all reductive G (see Proposition 2.3.1). So results including Proposition 2.3.2 and its consequences still go through without issue, as they may be applied to Gr_{G^σ}. But the group whose category of representations is isomorphic to $P_{L^+G^\sigma}(\mathrm{Gr}_{G^\sigma})$ is $\widehat{G^{\sigma,\circ}}$, as needed for Theorem 2.0.1.

The second complication is that $\pi_0(\mathrm{Gr}_G) = \pi_1(G)$. So if G is not semisimple and simply connected, then Gr_G is not connected. However, in the framework of AMV cycles introduced by [1], this is not a complication at all. In fact, both Kamnitzer and Hong work primarily with stable AMV cycles, which are AMV cycles translated by $X_*(T)$ to be contained in $\overline{S_e^0}$, which is itself contained in the neutral component of Gr_G. This does not rule out possible sticking points in passing between MV cycles, AMV cycles, stable AMV cycles and back to MV cycles, but every result solely relating to stable AMV cycles holds automatically for all reductive groups.

And the result necessary to restrict the bijection on the level of AMV cycles to MV cycles is Theorem 2.3.4, which is proved again without a need for triviality of $\pi_1(G)$. More details are presented in Section 2.3.

In short, considering reductive groups, rather than semisimple groups—much less simply connected groups—is no more complicated for the proof of Theorem 2.0.1. Every potential obstacle is either immaterial or easily deflected. In particular, Proposition 2.3.1 and the preceding lemmas are the only results I had not previously found in the literature.

2.3 Action of σ on orbits

For the remainder of the paper, suppose G is a connected, reductive, complex group, and σ is a pinning-preserving automorphism on G.

Unlike in the case G is semisimple and simply connected, we cannot count on G^{σ} to be a connected group in general. This leads us to a choice: should we work with $G^{\sigma,\circ}$, to which the classification of connected, reductive groups applies, or should we work directly with G^{σ}, which has a simpler description relative to G? Thanks to the upcoming Proposition 2.3.1, it is immaterial whether we work with G^{σ} or $G^{\sigma,\circ}$. I work primarily with G^{σ} for simplicity, and the final result will hold for $G^{\sigma,\circ}$. I will avoid referring to root datum of G^{σ} where possible. Note, however, that $\operatorname{Hom}(\mathbb{G}_m, T^{\sigma}) = \operatorname{Hom}(\mathbb{G}_m, T^{\sigma,\circ})$, so $X_*(T^{\sigma}) = X_*(T^{\sigma,\circ})$.

First, we will need a few lemmas relating loop groups of algebraic groups and their quotients. While the following lemmas will be applied only to group schemes over $\mathbb{C}$, their proofs hold in greater generality. In this section, the field k is only assumed to have characteristic 0. In particular, no assumption is made about its topology or algebraic closure.

Lemma 2.1. Let G be an affine group scheme over a field k of characteristic 0, and suppose $H \subset G$ is a smooth normal subgroup with affine quotient G/H. There is a natural isomorphism of functors $L^+G/L^+H \to L^+(G/H)$, where L^+G/L^+H is the étale quotient.

In particular, if $H \subset G$ is a normal subgroup and both groups are reductive, then

L^+G/L^+H and $L^+(G/H)$ are canonically isomorphic group schemes.

Proof. Let the quotient map be denoted $q_0 : G \to G/H$. Note that $L^+G = \varprojlim G^{(n)}$, where $G^{(n)}$ is the nth jet group $R \mapsto G(R[\varpi]/(\varpi^{n+1}))$. Similarly, $L^+(G/H) = \varprojlim (G/H)^{(n)}$. Hence the map $q : L^+G \to L^+(G/H)$ corresponds to the inverse system of morphisms

$$\begin{tikzcd}
\cdots \arrow[r] & G^{(n)} \arrow[r, "i_n"] \arrow[d, "q_n"] & G^{(n-1)} \arrow[r, "i_{n-1}"] \arrow[d, "q_{n-1}"] & \cdots \arrow[r] & G^{(1)} \arrow[r, "i_1"] \arrow[d, "q_1"] & G \arrow[d, "q_0"] \\
\cdots \arrow[r] & (G/H)^{(n)} \arrow[r, "j_n"] & (G/H)^{(n-1)} \arrow[r, "j_{n-1}"] & \cdots \arrow[r] & (G/H)^{(1)} \arrow[r, "j_1"] & G/H
\end{tikzcd}$$

The first step of the proof of surjectivity will proceed inductively. For each $n \geq 1$, I will use surjectivity of q_{n-1} and formal smoothness of q_0 to prove that q_n is surjective. Then I will use the surjectivity of each q_n to show that q is surjective, with kernel L^+H.

Suppose q_{n-1} is surjective, and let $g_n \in (G/H)^{(n)}(R)$, for a k-algebra R, with image $g_{n-1} \in (G/H)^{(n-1)}(R)$. By surjectivity of q_{n-1}, there is a lift $\tilde{g}_{n-1} \in G^{(n-1)}(S)$ lying over g_{n-1}, where $R \to S$ is an étale k-algebra homomorphism. Then by formal smoothness of q_0, there is a simultaneous lift in S-points $\tilde{g}_n \in G^{(n)}(S)$ of both g_n and $\tilde{g}_{n-1}$. Indeed, g_n corresponds to a morphism $\operatorname{Spec}(R[\varpi]/(\varpi^{n+1})) \to G/H$ (and by precomposition, to a morphism $\operatorname{Spec}(S[\varpi]/(\varpi^{n+1})) \to G/H$), and $\tilde{g}_{n-1}$ corresponds to a morphism $\operatorname{Spec}(S[\varpi]/(\varpi^{n})) \to G$ such that $q_0 \circ \tilde{g}_{n-1}$ is equal to $j_n(g_n)$ as a morphism $\operatorname{Spec}(S[\varpi]/(\varpi^{n})) \to G/H$. Then by the infinitesimal lifting property of formally smooth morphisms, there is a lift $\tilde{g}_n$ in the diagram below. In particular, q_n is surjective.

$$\begin{tikzcd}
\operatorname{Spec}(S[\varpi]/(\varpi^{n})) \arrow[r, "\tilde{g}_{n-1}"] \arrow[d] & G \arrow[d, "q_0"] \\
\operatorname{Spec}(S[\varpi]/(\varpi^{n+1})) \arrow[r, "g_n"] \arrow[ur, dashed, "\tilde{g}_n"] & G/H
\end{tikzcd}$$

Now suppose $g \in L^+(G/H)(R)$. For each n there is a corresponding element $g_n \in (G/H)^{(n)}(R)$. In particular, there is an element $g_0 \in (G/H)(R)$ with a lift in S-points $\tilde{g}_0 \in G(S)$, for some étale $R \to S$. By above, for each n there is also a lift $\tilde{g}_n \in (G/H)^{(n)}(S)$ of g_n. These $\tilde{g}_n$ form an inverse system, and thus correspond to an element $\tilde{g} \in L^+G(S)$ lifting g. Thus q is surjective.

Now consider the kernel of q. Let $g \in L^+G(R)$ for a k-alebra R, and suppose $q(g) = e \in L^+(G/H)(R)$. Then $g \in \ker(q_0)(R[\![\varpi]\!]) = H(R[\![\varpi]\!]) = L^+H(R)$. Similarly, for $g \in L^+H(R) = H(R[\![\varpi]\!])$, we have $q(g)$ corresponds to the identity in $(G/H)(R[\![\varpi]\!]) = L^+(G/H)(R)$, and so $g \in \ker q(R)$. □

Lemma 2.2. Let R be a k-algebra over a field k of characteristic 0. If $I \subset R(\!(\varpi)\!)$ is a set of nonzero idempotents, then $I \subset R$.

As a result, given a decomposition of $R(\!(\varpi)\!)$ (respectively $R[\![\varpi]\!]$) into nontrivial Cartesian factors, there is a corresponding decomposition of R into nontrivial factors, such that each factor of $R(\!(\varpi)\!)$ (or $R[\![\varpi]\!]$) is a Laurent series ring (formal power series ring). Geometrically, we can say that the natural maps of k-schemes induced by inclusion of k-algebras

$$\operatorname{Spec} R(\!(\varpi)\!) \to \operatorname{Spec} R[\![\varpi]\!] \to \operatorname{Spec} R$$

are bijective on connected components.

Proof. First I will show that an idempotent $e \in R(\!(\varpi)\!)$ must be contained in $R[\![\varpi]\!]$. For an element $x \in R(\!(\varpi)\!)$, let x_n be the degree-n coefficient. That is, $x = \sum_{n\in\mathbb{Z}} x_n\varpi^n$. Let n_0 be the minimal nonzero degree of e, so that $e = \sum_{n\geq n_0} e_n\varpi^n$.

Suppose, for contradiction, that there is some idempotent element $e \notin R[\![\varpi]\!]$, so that $n_0 < 0$. I will show that $e_{n_0} = 0$, contradicting the claim that $e \notin R[\![\varpi]\!]$.

Comparing coefficients of e and e^2, we have

$$e_n = (e^2)_n = \sum_{i=n_0}^{n-n_0} e_i e_{n-i} \tag{2.3}$$

for all $n \geq n_0$. Then for each n, let

$$f_n := e_n - \sum_{i=n_0}^{n-n_0} e_i e_{n-i} \in R.$$

Note that, by equation (2.3), each $f_n = 0$ in R. If $n < n_0$, we have $e_n = 0$, so $f_n = -\sum e_i e_{n-i}$ *a priori*. If furthermore $n < 2n_0$, then every term of the sum is also 0 *a priori*. I will construct e_{n_0} as an R-linear combination of such f_n, to show that $e_{n_0} = 0$. I claim

$$e_{n_0} = f_{n_0} + \sum_{i=2n_0}^{0} (4 - \frac{6}{n_0} i) e_{n_0-i} f_i. \tag{2.4}$$

Substituting the f_n with e_n, we can expand the right-hand side of equation (2.4) as

$$\Big(e_{n_0} - \sum_{i=n_0}^{0} e_i e_{n_0-i}\Big) + \sum_{i=2n_0}^{0} (4 - \frac{6}{n_0} i) e_{n_0-i} \Big(e_i - \sum_{j=n_0}^{i-n_0} e_j e_{i-j}\Big). \tag{2.5}$$

The expression in equation (2.5) is a non-constant polynomial in the various e_n. We can rewrite, ordering by total degree in e_n:

$$e_{n_0} - \Big(\sum_{i=n_0}^{0} e_i e_{n_0-i} - \sum_{i=2n_0}^{0} (4 - \frac{6}{n_0} i) e_{n_0-i} e_i\Big) - \Big(\sum_{i=2n_0}^{0} (4 - \frac{6}{n_0} i) e_{n_0-i} \sum_{j=n_0}^{i-n_0} e_j e_{i-j}\Big) \qquad (2.6)$$

Gathering like terms, we can rewrite equation (2.6) as

$$e_{n_0} + \sum_{n_0 \le i \le \frac{n_0}{2}} c_i \, e_i e_{n_0-i} + \sum_{n_0 \le i \le 0} \sum_{i \le j \le \frac{n_0-i}{2}} c_{i,j} \, e_i e_j e_{n_0-i-j} \qquad (2.7)$$

for some finite collections of rational coefficients $\{c_i\}$ and $\{c_{i,j}\}$. Note that we can ignore the terms where $i < n_0$ (and thus also those where $j > 0$), as they are *a priori* equal to 0. I claim that in fact every coefficient c_i or $c_{i,j}$ of equation (2.7) is 0.

First consider the c_i. Suppose that $i < n_0 - i$. Then

$$c_i = -2 + (4 - \frac{6}{n_0} i) + (4 - \frac{6}{n_0}(n_0 - i)) = 0.$$

On the other hand, if $i = n_0 - i$, then $i = n_0/2$ (and n_0 is even), and

$$c_{n_0/2} = -1 + (4 - \frac{6}{n_0} \cdot \frac{n_0}{2}) = 0.$$

Now consider the $c_{i,j}$, with cases based on which of i, j, and $n_0 - i - j$ are equal. First suppose $i = j = n_0 - i - j = n_0/3$. Then

$$c_{n_0/3, n_0/3} = 4 - \frac{6}{n_0}(n_0 - \frac{n_0}{3}) = 0.$$

Now suppose $i = j < n_0 - 2i$. We can see that

$$c_{i,i} = 2(4 - \frac{6}{n_0}(n_0 - i)) + (4 - \frac{6}{n_0}2i) = 0.$$

Next suppose $i < j = n_0 - i - j$. Then $j = (n_0 - i)/2$. We can see that

$$c_{i,(n_0-i)/2} = (4 - \frac{6}{n_0}(n_0 - i)) + 2(4 - \frac{6}{n_0} \cdot \frac{n_0 + i}{2}) = 0.$$

Finally, suppose $i < j < n_0 - i - j$. Then we can see

$$c_{i,j} = 2(4 - \frac{6}{n_0}(n_0 - i)) + 2(4 - \frac{6}{n_0}(n_0 - j)) + 2(4 - \frac{6}{n_0}(i + j)) = 0.$$

We therefore have canceling of every term in equation (2.7) except for e_{n_0}. Thus equation (2.4) holds, and $e_{n_0} = 0$ in R. So if $e \in R(\!(\varpi)\!)$ is idempotent, then $e \in R[\![\varpi]\!]$.

Now suppose $e \in R[\![\varpi]\!]$ is idempotent. Then in particular e_0 is idempotent, as $e_0 = (e^2)_0 = e_0^2$. Suppose for contradiction that $e \notin R$, so there is some $n > 0$ such that $e_n \neq 0$. Let n_1 be the smallest such integer. Then

$$e_{n_1} = (e^2)_{n_1} = 2e_0e_{n_1},$$

but also

$$e_{n_1} = (e^3)_{n_1} = 3e_0^2e_{n_1} = 3e_0e_{n_1} = \frac{3}{2}e_{n_1},$$

which is only possible if $e_{n_1} = 0$. Therefore $e = e_0 \in R$. □

Lemma 2.3. Suppose G is an étale group scheme over a field k of characteristic 0. Then there are canonical isomorphisms $LG \cong L^+G \cong G$.

Proof. Let R be a k-algebra, and let $k[G] = \Gamma(G, \mathscr{O}_G)$; note $k[G]$ is a Cartesian product of finitely many fields, all finite and separable over k; write $k[G] = k_1 \times \cdots \times k_r$. The respective R-point sets are equal to $\operatorname{Hom}_{k\text{-Alg}}(k[G], R(\!(\varpi)\!))$, $\operatorname{Hom}_{k\text{-Alg}}(k[G], R[\![\varpi]\!])$, and $\operatorname{Hom}_{k\text{-Alg}}(k[G], R)$. It is sufficient to show that for all $f \in k[G]$ and all homomorphisms $\phi : k[G] \to R(\!(\varpi)\!)$, we have $\phi(f) \in R$. As a result, all $\phi : k[G] \to R[\![\varpi]\!]$ factor through R as well.

First I would like to reduce the domain to k_i, a factor of $k[G]$ and a finite separable extension of k. Note that in general, for two k-algebras $A \cong \prod_{i\in I} A_i$ and $B \cong \prod_{j \in J} B_j$, we have

$$\operatorname{Hom}_{k\text{-Alg}}(A, B) = \prod_{j\in J}\Big(\coprod_{i\in I}\operatorname{Hom}_{k\text{-Alg}}(A_i, B_j)\Big).$$

Thus it is sufficient to show, if $R(\!(\varpi)\!) \cong \prod_{j\in J} B_j$, that for all homomorphisms $\phi_{i,j} : k_i \to B_j$ we have $\phi(k_i) \subset R \cap B_j$. By Lemma 2.2, we know that $B_j \cong R_j(\!(\varpi)\!)$ for a k-algebra R_j. In particular, $R_j = e_j R$ for an idempotent $e_j \in R$, and thus $R \cap R_j(\!(\varpi)\!) = R_j$. Thus it is sufficient to assume $\phi : k_i \to R(\!(\varpi)\!)$ where k_i is a finite separable extension of k and $\operatorname{Spec} R$ is connected, and show that the image of ϕ is contained in R.

Suppose k_i is a factor of $k[G]$, in particular a finite separable extension of k, and let $f \in k_i$, to show that $\phi(f) \in R$, where $\operatorname{Spec} R$ is connected. Both f and $\phi(f)$ are invertible and satisfy a separable, irreducible polynomial P over k. For an element $x \in R(\!(\varpi)\!)$, write x_n for the degree-n coefficient of x, that is $x = \sum_{n\in\mathbb{Z}} x_n \varpi^n$. Then $\phi(f)_n = 0$ for all sufficiently small n; let n_0 be the smallest integer such that $\phi(f)_{n_0} \neq 0$. Invertibility of $\phi(f)$

implies that $\phi(f)_{n_0}$ is invertible in R, and in particular not nilpotent. Then algebraicity of $\phi(f)$ over k implies that $n_0 \geq 0$. Therefore $\phi(f) \in R[\![\varpi]\!]$.

Let $P(T) = \sum_{j=0}^{d} P_j T^j$, where $P(f) = 0$ in k_i and $P(\phi(f)) = \phi(P(f)) = 0$ in $R[\![\varpi]\!]$, and all $P_j \in k$. Assume, for the sake of contradiction, that $\phi(f) \notin R$, so there is some integer $n \geq 1$ such that $\phi(f)_n \neq 0$; let n_1 be the smallest. I claim that $\phi(f)_{n_1} = 0$, contradicting the assumption and implying $\phi(f) \in R$. Indeed, $\phi(f)_0$ must satisfy $P(\phi(f)_0) = P(\phi(f))_0 = 0$. Now consider $\phi(f)_{n_1}$. Since $P(\phi(f)) = 0$, we have $P(\phi(f))_{n_1} = 0$ in particular. We can expand this as

$$\begin{aligned}
P(\phi(f))_{n_1} &= \Big(\sum_{j=0}^{d} P_j \phi(f)^j\Big)_{n_1} \\
&= \sum_{j=0}^{d} P_j \left(\Big(\sum_{n\geq 0} \phi(f)_n \varpi^n\Big)^j\right)_{n_1} \\
&= \sum_{j=0}^{d} P_j \Big(\sum_{k_1+\cdots+k_{j-1}+k_j=n_1} \phi(f)_{k_1}\cdots\phi(f)_{k_j}\Big) \\
&= \sum_{j=0}^{d} P_j (j\phi(f)_0^{j-1}\phi(f)_{n_1}) \\
&= P'(\phi(f)_0)\phi(f)_{n_1} = 0,
\end{aligned}$$

where P' is the formal derivative of P. Since P is separable and irreducible, and since $P(\phi(f)_0) = 0$ and $\operatorname{Spec} R$ is connected, $P'(\phi(f)_0) \in k(\phi(f)_0) \subset R$ and is in particular neither 0 nor a zero divisor. Thus the only way $P'(\phi(f)_0)\phi(f)_{n_1} = 0$ is if $\phi(f)_{n_1} = 0$.

□

Note that Lemma 2.2 is essential in the proof of Lemma 2.3. If there were a k-algebra R such that $R(\!(\varpi)\!)$ had any idempotents not contained in R, then for any nontrivial étale group G, we would have $LG(R) \supsetneq G(R)$. For instance, the square roots of unity group $\boldsymbol{\mu}_2$ is étale over k and has $k[\boldsymbol{\mu}_2] \cong k \times k$. If there were a ring R with an idempotent $e \in R(\!(\varpi)\!) \setminus R$, then $1 - e$ would also be an idempotent in $R(\!(\varpi)\!) \setminus R$, and the map $\phi : k \times k \to R(\!(\varpi)\!)$ given by $\phi(1,0) = e$ and $\phi(0,1) = 1 - e$ would be a homomorphism such that $\phi(k[\boldsymbol{\mu}_2]) \not\subset R$. However, by Lemma 2.3, we know $L\boldsymbol{\mu}_2 \cong \boldsymbol{\mu}_2$.

Proposition 2.3.1. *Let G^σ be a possibly disconnected, split, reductive group over a field k of characteristic 0, and let $G^{\sigma,\circ}$ be the identity component. Then the natural map of functors*

$$\mathrm{Gr}_{G^{\sigma,\circ}} \xrightarrow{\eta} \mathrm{Gr}_{G^\sigma}$$

is an isomorphism $\mathrm{Gr}_{G^{\sigma,\circ}} \cong \mathrm{Gr}_{G^\sigma}$ of étale sheaves over k.

Proof. Recall the affine Grassmannian is the étale sheafification of a presheaf $P\mathrm{Gr}_{G^\sigma} : k\text{-Alg} \to \mathrm{Sets}$ defined by $R \mapsto LG^\sigma(R)/L^+G^\sigma(R)$. In order to prove that η is an isomorphism of sheaves, it is sufficient to prove η is both injective and surjective (as a map of sheaves).

Injectivity of η follows from injectivity of the the presheaf map $\eta^P : P\mathrm{Gr}_{G^{\sigma,\circ}} \to P\mathrm{Gr}_{G^\sigma}$. Let R be an arbitrary k-algebra. Then the component map η^P_R is injective. Indeed, an element $x \in P\mathrm{Gr}_{G^{\sigma,\circ}}(R)$ can be written as a coset $x = gL^+G^{\sigma,\circ}(R)$, where $g \in LG^{\sigma,\circ}(R)$, and $\eta^P_R(x) = gL^+G^\sigma(R)$. This definition of η^P_R makes sense because $LG^{\sigma,\circ} \subset LG^\sigma$ and is well-defined since $L^+G^{\sigma,\circ} \subset L^+G^\sigma$. The map η^P_R is also injective. Indeed, if g and g' are two elements of $LG^{\sigma,\circ}(R)$ such that $gL^+G^\sigma(R) = g'L^+G^\sigma(R)$, let $h \in L^+G^\sigma(R)$ be any element such that $gh = g'$. Then in fact $h \in L^+G^{\sigma,\circ}(R)$, otherwise $gh \notin LG^{\sigma,\circ}(R)$.

In order to show that η is surjective, I find it convenient to sheafify; the aim is to show that for all k-algebras R, and for all $x \in \mathrm{Gr}_{G^\sigma}(R)$, there is some étale k-algebra morphism $R \to S$ such that x, viewed by restriction as a point in $\mathrm{Gr}_{G^\sigma}(S)$, lifts to a point $\tilde{x} \in \mathrm{Gr}_{G^{\sigma,\circ}}(S)$. To do so, consider the following diagram of étale sheaves, for which the rows are exact (as sheaves in pointed sets):

$$\begin{tikzcd}
1 \arrow[r] & L^+G^{\sigma,\circ} \arrow[r] \arrow[d] & LG^{\sigma,\circ} \arrow[r] \arrow[d] & \mathrm{Gr}_{G^{\sigma,\circ}} \arrow[r] \arrow[d, "\eta"] & 1 \\
1 \arrow[r] & L^+G^{\sigma} \arrow[r] & LG^{\sigma} \arrow[r] & \mathrm{Gr}_{G^{\sigma}} \arrow[r] & 1
\end{tikzcd}$$

Let $x \in \mathrm{Gr}_{G^\sigma}(R)$. By surjectivity of $LG^\sigma \to \mathrm{Gr}_{G^\sigma}$, there is some lift $g \in LG^\sigma(S_1)$ of x, where $R \to S_1$ is étale. Then if we can find some $h \in L^+G^\sigma(S_2)$ such that $gh^{-1} \in LG^{\sigma,\circ}(S_2)$ (where $S_1 \to S_2$ is étale), then $\eta([gh^{-1}]) = x$. The reason we can find such h (and S_2) is that the étale quotient functors $L^+G^\sigma/L^+G^{\sigma,\circ}$ and $LG^\sigma/LG^{\sigma,\circ}$ are isomorphic—in fact, they are isomorphic to the algebraic group $G^\sigma/G^{\sigma,\circ}$, so we can take $h \in G^\sigma(S_2)$. In particular, if $h' \in LG^\sigma(S_1)$ is any point with $g(h')^{-1} \in LG^{\sigma,\circ}(S_1)$, then $[h'] \in (LG^\sigma/LG^{\sigma,\circ})(S_1) = (G^\sigma/G^{\sigma,\circ})(S_1)$ lifts to $h \in G^\sigma(S_2)$ for some S_2.

Use the previous two lemmas to see that the sheaves $L^+G^\sigma/L^+G^{\sigma,\circ}$ and $LG^\sigma/LG^{\sigma,\circ}$ really are isomorphic. The group $G^{\sigma,\circ}$ is reductive and a normal subgroup of the reductive group G^σ, and the quotient $G^\sigma/G^{\sigma,\circ}$ is étale. Thus by Lemma 2.1, $L^+G^\sigma/L^+G^{\sigma,\circ} \cong L^+(G^\sigma/G^{\sigma,\circ})$, and by Lemma 2.3, $G^\sigma/G^{\sigma,\circ} \cong L^+(G^\sigma/G^{\sigma,\circ})$.

It remains to be seen that $LG^\sigma/LG^{\sigma,\circ} \cong L(G^\sigma/G^{\sigma,\circ})$. Note that we have a natural map $LG^\sigma \to L(G^\sigma/G^{\sigma,\circ})$ by applying the loop group functor L to the quotient map $q_0 : G^\sigma \to G^\sigma/G^{\sigma,\circ}$. By Lemma 2.3, $L(G^\sigma/G^{\sigma,\circ}) \cong G^\sigma/G^{\sigma,\circ} \cong L^+(G^\sigma/G^{\sigma,\circ})$. Then the map $LG^\sigma \to L(G^\sigma/G^{\sigma,\circ})$ is surjective, since the map $L^+G^\sigma \to L^+(G^\sigma/G^{\sigma,\circ})$ factors through it, and is itself surjective by Lemma 2.1. And the kernel is $LG^{\sigma,\circ}$, for reasons essentially identical to those in the proof of Lemma 2.1: if the quotient map kills $g \in LG^{\sigma}(R)$, then g corresponds to an element of $G^{\sigma}(R(\!(\varpi)\!))$ also killed by quotient, and thus $g \in G^{\sigma,\circ}(R(\!(\varpi)\!)) = LG^{\sigma,\circ}(R)$. And if $g \in LG^{\sigma,\circ}(R)$, then the quotient map kills g when viewed as an $R(\!(\varpi)\!)$-point of $G^{\sigma,\circ}$.

□

It follows from the same reasoning that the natural map $G^{\sigma,\circ} \hookrightarrow G^{\sigma}$ induces an isomorphism of categories

$$P_{L^+G^{\sigma,\circ}}(\mathrm{Gr}_{G^{\sigma,\circ}}) \xrightarrow{\sim} P_{L^+G^{\sigma}}(\mathrm{Gr}_{G^{\sigma}}). \tag{2.8}$$

Specifically, recall that the cocharacter lattices $X_*(T^{\sigma})$ and $X_*(T^{\sigma,\circ})$ are isomorophic. And for each cocharacter $\mu \in X_*(T^{\sigma})^+$, the map η restricts to an isomorphism $\mathrm{Gr}^{\mu}_{G^{\sigma}} \cong \mathrm{Gr}^{\mu}_{G^{\sigma,\circ}}$. I do not directly use this fact, but I think it is worth acknowledging.

For the remainder of the paper, the groups G, G^{σ}, and $G^{\sigma,\circ}$ will be assumed to lie over the field $k = \mathbb{C}$, and their dual groups will lie over the algebraically closed K field of characteristic 0.

I will prove equation (2.1) of Theorem 2.0.1 holds using G^{σ}. Then since the two categories in equation (2.8) are isomorphic, they share a Tannakian dual group, which by Theorem 2.2.1 v. has root datum dual to the connected reductive group $G^{\sigma,\circ}$, i.e. is isomorphic to the group $\widehat{G^{\sigma,\circ}}$.

Much of the proof follows from an understanding of the relationship between the σ-action on semi-infinite cells, the σ-fixed sub-ind-scheme of a σ-invariant semi-infinite cell, and the corresponding semi-infinite cell of the σ-fixed point affine Grassmannian.

Proposition 2.3.2. *i. There is a natural injective map of ind-schemes* $\mathrm{Gr}_{G^\sigma} \to \mathrm{Gr}_G$, *and the reduced ind-scheme* $(\mathrm{Gr}_{G^\sigma})_{red}$ *may be identified with* $(\mathrm{Gr}_G)^\sigma_{red}$.

ii. For $\mu \in X_*(T)^+$, $\sigma(\mathrm{Gr}_G^\mu) = \mathrm{Gr}_G^{\sigma(\mu)}$

iii. For $\mu \in X_*(T)^{+,\sigma}$, *we can identify* $(\mathrm{Gr}_G^\mu)^\sigma = \mathrm{Gr}_{G^\sigma}^\mu$.

iv. For $\nu \in X_*(T)$ *and* $w \in W$, $\sigma(S_w^\nu) = S_{\sigma(w)}^{\sigma(\nu)}$

v. For $\nu \in X_*(T)^\sigma$ *and* $w \in W^\sigma$, *we can identify* $(S_w^\nu)^\sigma = (S_\sigma)_w^\nu$, *where* $(S_\sigma)_w^\nu$ *is the semi-infinite cell* $wL(U^\sigma)w^{-1} \cdot \varpi^\nu x_0 \subset \mathrm{Gr}_{G^\sigma}$.

Note that, as perverse sheaves are defined using the étale topology, the identification of reduced ind-schemes in Statement i. implies an identification of Satake categories as well. Note also that Gr_{G^σ} may be reduced even if Gr_G is not.

Proof. Statements ii. and iv. are immediate.

Note that σ acts on LG, preserving L^+G. Thus we have, a priori, an action of σ on Gr_G and an injective map of functors $\mathrm{Gr}_{G^\sigma} \to (\mathrm{Gr}_G)^\sigma$.

Statements i. and iii. follow from statement v., along with the observation that for a sub-ind-scheme $X \subset \mathrm{Gr}_G$, $X^\sigma = X \cap (\mathrm{Gr}_G)^\sigma$. Note that statement i. is taken only on the level of reduced structures, since it follows from taking a union of locally closed sub-ind-schemes. The same is technically true for statement iii., but Schubert varieties are already reduced.

To see statement v., consider the action of $wLUw^{-1}$ on S_w^ν: there is a subgroup,

$J_G(w,\nu) \subset wLUw^{-1}$, with a simply transitive action on S_w^ν. Define $J_G(w,\nu)$ as follows:

$$J_G(w,\nu) := wLUw^{-1} \cap \varpi^\nu L^{--}G\varpi^{-\nu}.$$

By construction of $J_G(w,\nu)$, it is clear that for σ-invariant w and ν, $J_G(w,\nu)^\sigma = J_{G^\sigma}(w,\nu)$. Therefore the σ-fixed points of S_w^ν are exactly those in the orbit of the σ-fixed subgroup $J_{G^\sigma}(w,\nu)$.

So let us see that the action of the subgroup $J_G(w,\nu)$ is simply transitive on S_w^ν, implying Statement v.. It is well known that LU has a decomposition $L^{--}U \cdot L^+U$. Since $L^{--}G \supset L^{--}U$ acts freely on Gr_G and $LG \supset LU$ has stabilizer $L^+G \supset L^+U$ at the basepoint x_0, this decomposition implies that $J_G(e,0) = L^{--}U$ acts simply transitively on S_e^0. Similarly, we have $J_G(w,0)$ acting simply transitively on S_w^0 for all $w \in W$.

For more general $J_G(w,\nu)$, consider the decomposition

$$wLUw^{-1} = J_G(w,\nu) \cdot (wLUw^{-1} \cap \varpi^\nu L^+G\varpi^{-\nu}).$$

It is immediate both that this is a decomposition of $wLUw^{-1}$ (from normality of wUw^{-1} in wBw^{-1}), and also that $wLUw^{-1} \cap \varpi^\nu L^+G\varpi^{-\nu}$ is the stabilizer of $\varpi^\nu x_0$ in $wLUw^{-1}$. Therefore $J_G(w,\nu)$ acts simply transitively on S_w^ν, as needed.

□

Note also that closure relations hold as expected, simply by intersection. Specifically, given a σ-invariant dominant cocharacter μ,

$$\overline{\mathrm{Gr}^{\mu}_{G^\sigma}} = \overline{\mathrm{Gr}^{\mu}_{G}} \cap \mathrm{Gr}_{G^\sigma} = (\coprod_{\substack{\lambda\in X_*(T)^+\\ \lambda\leq\mu}} \mathrm{Gr}^{\lambda}_{G}) \cap \mathrm{Gr}_{G^\sigma} = \coprod_{\substack{\lambda\in X_*(T)^+\\ \lambda\leq\mu}} (\mathrm{Gr}^{\lambda}_{G} \cap \mathrm{Gr}_{G^\sigma}) = \coprod_{\substack{\lambda\in X_*(T)^{+,\sigma}\\ \lambda\leq\mu}} \mathrm{Gr}^{\lambda}_{G^\sigma},$$

and given σ-invariant $\nu \in X_*(T)$ and $w \in W^\sigma$,

$$\overline{(S_\sigma)^{\nu}_{w}} = \overline{S^{\nu}_{w}} \cap \mathrm{Gr}_{G^\sigma} = (\coprod_{\substack{\eta\in X_*(T)\\ \eta\leq_w\nu}} S^{\eta}_{w}) \cap \mathrm{Gr}_{G^\sigma} = \coprod_{\substack{\eta\in X_*(T)\\ \eta\leq_w\nu}} (S^{\eta}_{w} \cap \mathrm{Gr}_{G^\sigma}) = \coprod_{\substack{\eta\in X_*(T)^\sigma\\ \eta\leq_w\nu}} (S_\sigma)^{\eta}_{w}.$$

Either of these equalities implies that the cocharacter lattice $X_*(T^\sigma) \subset X_*(T)$ inherits the partial order $\leq$. This can also be seen combinatorially; see Section 2.7 for details.

The primary varieties in consideration in this proof are Mirković–Vilonen (MV) cycles and Anderson–Mirković–Vilonen (AMV) cycles. MV cycles of coweight (λ, μ) are irreducible components of the intersection $S^{\lambda}_{w_0} \cap \overline{\mathrm{Gr}^{\mu}_{G}}$, and according to the geometric Satake correspondence they index a basis for $V_\mu(\lambda)$. On the other hand, AMV cycles of coweight (λ, μ) are irreducible components of the variety $\overline{S^{\lambda}_{w_0} \cap S^{\mu}_{e}}$. Many authors refer to AMV cycles as simply "MV cycles." The following proposition will make clear how closely related they are, and why AMV cycles may be considered a generalization of MV cycles. Note that, for the purposes of the geometric Satake equivalence, it is not important whether we are dealing with an equi-dimensional variety or its closure. Indeed, we can use top-dimensional cohomology with compact support, which in this case depends only on dimension and number of components. However, it is (formally) convenient to require AMV cycles to be projective when defining their moment polytopes.

Proposition 2.3.3 ([1] Proposition 3)**.** *If A is an irreducible component of $\overline{S^{\lambda}_{w_0} \cap S^{\mu}_{e}}$ and*

$A \subset \overline{\mathrm{Gr}^{\mu}_{G}}$*, then* A *is the closure of an MV cycle of coweight* (λ, μ)*. If* A' *is an MV cycle of coweight* (λ, μ)*, then* $\overline{A'}$ *is an irreducible component of* $\overline{S^{\lambda}_{w_0} \cap S^{\mu}_{e}}$*. Thus the closures of MV cycles of coweight* (λ, μ) *are exactly the AMV cycles of coweight* (λ, μ) *contained in* $\overline{\mathrm{Gr}^{\mu}_{G}}$*.*

Proof. This result follows from dimension estimates in Theorem 2.2.1 i. and ii..

First suppose A is an irreducible component of $\overline{S^{\lambda}_{w_0} \cap S^{\mu}_{e}}$ and that $A \subset \overline{\mathrm{Gr}^{\mu}_{G}}$. Of course we have $A \subset \overline{S^{\lambda}_{w_0}} \cap \overline{\mathrm{Gr}^{\mu}_{G}}$. Now the Iwasawa stratification implies

$$\overline{S^{\lambda}_{w_0}} \cap \overline{\mathrm{Gr}^{\mu}_{G}} = \coprod_{\substack{\nu \in X_*(T) \\ \nu \geq \lambda}} (S^{\nu}_{w_0} \cap \overline{\mathrm{Gr}^{\mu}_{G}}) = (S^{\lambda}_{w_0} \cap \overline{\mathrm{Gr}^{\mu}_{G}}) \cup X,$$

where $\dim X < \dim A$. And so $A' := A \cap (S^{\lambda}_{w_0} \cap \overline{\mathrm{Gr}^{\mu}_{G}})$ is dense in A. Since A is irreducible, this implies A' is an MV cycle.

Now suppose $A = \overline{A'}$ where A' is an MV cycle of coweight (λ, μ). It is sufficient to see that $A' \subset \overline{S^{\lambda}_{w_0} \cap S^{\mu}_{e}}$. Note that by Theorem A.1(a),

$$\dim (S^{\mu}_{e} \cap \overline{\mathrm{Gr}^{\mu}_{G}}) = \dim (\overline{\mathrm{Gr}^{\mu}_{G}}) = 2\langle \rho, \mu \rangle,$$

implying that $S^{\mu}_{e} \cap \overline{\mathrm{Gr}^{\mu}_{G}}$ is dense in $\overline{\mathrm{Gr}^{\mu}_{G}}$, and in particular, $\overline{\mathrm{Gr}^{\mu}_{G}} \subset \overline{S^{\mu}_{e}}$. So $A' \subset S^{\lambda}_{w_0} \cap \overline{S^{\mu}_{e}}$. Again using the Iwasawa stratification, we have

$$S^{\lambda}_{w_0} \cap \overline{S^{\mu}_{e}} = \coprod_{\substack{\nu \in X_*(T) \\ \nu \leq \mu}} (S^{\lambda}_{w_0} \cap S^{\nu}_{e}) = (S^{\lambda}_{w_0} \cap S^{\mu}_{e}) \cup Y,$$

where again $\dim Y < \dim A'$. And so $A' \cap (S^{\lambda}_{w_0} \cap S^{\mu}_{e})$ is dense in A', and $A' \subset \overline{S^{\lambda}_{w_0} \cap S^{\mu}_{e}}$. □

Working with AMV cycles rather than MV cycles is convenient. The primary reason is that they are defined as components of the intersection of a pair of semi-infinite cells, rather than components of a semi-infinite cell and a Schubert variety. One useful consequence is that $X_*(T)$ acts on the set of AMV cycles by translation, where such a translation of an MV cycle is no longer necessarily an MV cycle.

Given an AMV cycle A and a cocharacter $\nu \in X_*(T)$, let $\nu \cdot A = \varpi^\nu A$. We have $\varpi^\nu S_w^\lambda = S_w^{\lambda+\nu}$ by normality of wUw^{-1} in wBw^{-1}, so if A has coweight (λ, μ), then $\nu \cdot A$ is an AMV cycle and has coweight $(\lambda + \nu, \mu + \nu)$. Given an AMV cycle A of coweight (λ, μ), the $X_*(T)$-orbit of A has one AMV cycle of coweight $(\lambda - \mu, 0)$. This AMV cycle is called the stable AMV cycle representing A, and denoted A_0. For many purposes, I will work with stable AMV cycles only, equivalent to assuming the second coweight is 0. In these cases I will use the subscript $_0$. This is especially convenient for the consideration of non-simply-connected groups G, since stable AMV cycles are contained in the neutral component of the affine Grassmannian.

The following theorem of Anderson is useful for determining which AMV cycles are MV cycles.

Theorem 2.3.4 ([1] Theorem 1 (1)). *Let G be a semisimple group over $\mathbb{C}$. There exists a family of polytopes $\mathcal{MV} = (P_A)_{A \in \mathbb{B}}$ in $X_*(T)_\mathbb{R}$ with parameter set $\mathbb{B}$ graded by Λ^- (i.e. $\mathbb{B} = \bigcup_{\nu \in \Lambda^-} \mathbb{B}_\nu$) such that weight multiplicities may be calculated according to the following rule: If V_μ is an irreducible representation of $\widehat{G}$ with highest weight μ, then the multiplicity of the weight λ in V_μ equals the number of $A \in \mathbb{B}_{\lambda-\mu}$ for which $P_A + \mu \subset \mathrm{Conv}(W \cdot \mu)$.*

Above, $X_*(T)_\mathbb{R} = X_*(T) \otimes_\mathbb{Z} \mathbb{R}$, $\mathbb{B}_\nu$ is the set of irreducible components of $\overline{X(\nu, 0)}$, and Λ^- is the set of negative coweights in $X_*(T^{sc})$, i.e. the negative coroot semilattice of

G. And P_A is the moment polytope of the MV cycle A, defined as follows:

Definition 2.3.5 (Moment polytope)**.** *Suppose X is an irreducible, projective, T-invariant subvariety $X \subset \mathrm{Gr}_G$. Then define the moment polytope of X as*

$$P_X := \mathrm{Conv}\,(\nu \in X_*(T) \mid \varpi^\nu x_0 \in X).$$

This definition is inspired by the image of the moment map $\Phi : \mathrm{Gr}_G \to X_*(T)_\mathbb{R}$ of the action of T on Gr_G. However, for our purposes, there is no need to define Φ, only the image of T-invariant subvarieties.

Note that Schubert varieties, semi-infinite cells, and AMV cycles are all T-invariant. It is helpful to note some properties of the moment map and polytopes it produces:

Proposition 2.3.6 ([1] Proposition 4 and proof)**.**

i. The T-fixed points of Gr_G are the $\varpi^\nu x_0$. Those in Gr_G^μ are the $\varpi^{W\cdot\mu} x_0$. Those in $\overline{\mathrm{Gr}_G^\mu}$ are the $\varpi^\nu x_0$ where

$$\nu \in \mathrm{Conv}\,(W \cdot \mu) \cap (\mu + X_*(T^{sc})).$$

The one in S_w^ν is $\varpi^\nu x_0$. Those in $\overline{S_w^\nu}$ are the $\varpi^\eta x_0$ where $\eta \leq_w \mu$.

ii. If X is a one-dimensional T-orbit, then $P_{\overline{X}}$ is a line segment in a coroot direction joining two coweights in a common coset modulo $X_*(T^{sc})$.

iii. If X is any projective, irreducible, T-invariant variety, then P_X is the convex hull of the images of its T-fixed points.

Proof. Statement i. is well known. Statement iii. is an immediate consequence of the definition of the moment polytope used here.

Suppose $X \subset \mathrm{Gr}_G$ is a one-dimensional T-orbit. Let x be a complex point $x \in X(\mathbb{C})$. By the Iwasawa stratification, for all $w \in W$ there is a unique ν_w for which $x \in S_w^{\nu_w}$. For each w, the set of T-fixed points in $\overline{S_w^{\nu_w}}$ is $\{\varpi^\eta x_0 \mid \ \eta \leq_w \nu_w\}$. So the set of T-fixed points in the intersection of $S_w^{\nu_w}$ for all $w \in W$ is contained in $\mathrm{Conv}\,(\nu_w \mid w \in W)$. In particular, $P_{\overline{X}} \subset \mathrm{Conv}\,(\nu_w \mid w \in W)$.

In fact, $P_{\overline{X}} = \mathrm{Conv}\,(\nu_w \mid w \in W)$. Recall the geometric description of the semi-infinite cells (2.2): $x \in S_w^\nu(\mathbb{C})$ if and only if

$$\lim_{t \to 0} w(\rho^\vee)(t) \cdot x = \varpi^\nu x_0.$$

The $\varpi^{\nu_w} x_0$ for $w \in W$ are therefore limit points for the torus action on X, and contained in $\overline{X}$.

Since X is a one-dimensional quotient of T, it must be isomorphic to $\mathbb{G}_m$, and so $X(\mathbb{C})$ has at most two distinct limit points in $\mathrm{Gr}_G(\mathbb{C})$. But if there is only one, call it ν, then $X(\mathbb{C}) \subset (S_{w_0}^\nu \cap S_e^\nu)(\mathbb{C})$, which consists of the single point $\varpi^\nu x_0$, violating the assumption that X is one-dimensional. So let the two distinct cocharacters in the set $\{\nu_w \mid w \in W\}$ be denoted ν and η. Suppose $\nu = \nu_e$, so that $\nu \geq \eta$. Then also we have $\eta = \nu_{w_0}$.

Find some $w \in W$ and simple reflection $s_i \in W$ such that $\nu \geq_w \eta$ but $\nu \leq_{ws_i} \eta$, and note that then $\ell(w) < \ell(ws_i)$. Such w and s_i can be found by choosing a reduced word $\mathbf{i} = (i_1, i_2, \ldots, i_{\ell(w_0)})$ for w_0 and comparing ν and η under the order $\leq_{s_{i_1} \cdots s_{i_k}}$ for each $0 \leq k \leq \ell(w_0)$. Then $w^{-1}(\nu) \leq w^{-1}(\eta)$, so $w^{-1}(\nu - \eta)$ is a sum of positive coroots; and

$s_i w^{-1}(\nu) \geq s_i w^{-1}(\eta)$, so $s_i w^{-1}(\nu - \eta)$ is a sum of negative coroots. This is only possible when $w^{-1}(\nu - \eta) = n\alpha_i^\vee$ for some integer $n > 0$. And so $\nu - \eta$ is a multiple of the positive coroot $w(\alpha_i^\vee)$. □

I will sketch Anderson's proof of Theorem 2.3.4 below, to see that it does not depend on the assumption G is semisimple.

Although T-invariance is the primary consideration for the moment map, it is convenient to also consider the "dilation" action of $\mathbb{G}_m$ on Gr_G defined by $\varpi \mapsto c\varpi \in R((\varpi))$ for $c \in \mathbb{G}_m(R)$, since we can usefully describe a fixed point in the closure of a $(\mathbb{G}_m \times T)$-orbit. Note also that Schubert varieties, semi-infinite cells, and AMV cycles are dilation-invariant as well as T-invariant. The following statement about the fixed points of torus actions is well-known. The proof in the current case is taken from Anderson.

Lemma 2.4 ([1] Lemma 6)**.** Every $(\mathbb{G}_m \times T)$-orbit $X \subset \mathrm{Gr}_G$ has a T-fixed point $\varpi^\eta x_0 \in \overline{X}$, such that $X \subset \mathrm{Gr}_G^\eta$.

Note that in the statement above, η is not required to be dominant.

Proof. Suppose X is a $(\mathbb{G}_m \times T)$-orbit in Gr_G. By the Cartan stratification, X must be contained in some Gr_G^μ. There is a point $x \in \overline{X}(\mathbb{C}) \cap (G(\mathbb{C}) \cdot \varpi^\mu x_0)$, found as a limit of the dilation action. Then there is some fixed point $\varpi^\eta x_0$ in the closure of the $T(\mathbb{C})$-orbit of x. Since $T \subset G$, we still have $\varpi^\eta x_0 \in G(\mathbb{C}) \cdot \varpi^\mu x_0$; in particular, η is in the Weyl orbit $W \cdot \mu$. And since X is both $\mathbb{G}_m$- and T-invariant, both x and $\varpi^\eta x_0$ are contained in $\overline{X}(\mathbb{C})$. □

Now we are ready to prove Theorem 2.3.4.

Proof of Theorem 2.3.4. First suppose A is an MV cycle of coweight (λ, μ) for some dominant cocharacter μ and some $\lambda \in Wt(\mu)$. Then $A \subset \overline{\mathrm{Gr}_G^\mu}$, and so $P_A \subset P_{\overline{\mathrm{Gr}_G^\mu}} = \mathrm{Conv}\,(W \cdot \mu)$.

Now suppose A is an AMV cycle of coweight (λ, μ), and suppose $P_A \subset \mathrm{Conv}\,(W \cdot \mu)$. Then for every vertex ν of P_A and each $w \in W$, we have $\nu \leq_w w(\mu)$. Since A is $(\mathbb{G}_m \times T)$-invariant, it is a union of $(\mathbb{G}_m \times T)$-orbits. So every complex point $a \in A(\mathbb{C})$ is contained in such an orbit $a \in X(\mathbb{C}) \subset A(\mathbb{C})$, and by Lemma 2.4 there is a cocharacter η such that $\varpi^\eta x_0 \in \overline{X}(\mathbb{C}) \subset A(\mathbb{C})$ and $X \subset \mathrm{Gr}_G^\eta$. In particular, $a \in \mathrm{Gr}_G^\eta(\mathbb{C})$. By the assumption that $P_A \subset \mathrm{Conv}\,(W \cdot \mu)$, the difference $\mu - w(\eta)$ must be a nonnegative real combination of simple roots for each $w \in W$. And since $\varpi^\eta x_0 \in A(\mathbb{C})$, and A is irreducible, it follows that all $w(\eta)$ are in the same coset modulo $X_*(T^{sc})$ as the vertices ν of P_A—so in fact $\eta \in Wt(\mu)$. Therefore $a \in \overline{\mathrm{Gr}_G^\mu}(\mathbb{C})$. □

The following observation summarizes the convenience of moment polytopes in studying the geometry of the affine Grassmannian via semi-infinite cells. It is a direct consequence of Anderson's work.

Lemma 2.5. Let X be a T-invariant, projective, irreducible subvariety of Gr_G, and let $\nu \in X_*(T)$. Then $X \cap S_w^\nu$ is dense in X if and only if ν is $\leq_w$-maximal among vertices of the moment polytope P_X.

Proof. First suppose $X \cap S_w^\nu$ is dense in X. Then $X = X \cap \overline{S_w^\nu}$. So if $\eta \not\leq_w \nu$, then $X \cap S_w^\eta = \varnothing$. In particular, $\varpi^\eta x_0 \notin X$. And so $\eta \notin P_X$, unless $\eta \notin \nu + X_*(T^{sc})$, in which case η cannot be a vertex of P_X.

Now suppose ν is $\leq_w$-maximal among vertices of P_X. By the Iwasawa stratification,

$X \subset \overline{S_w^\nu}$. In particular,

$$X = \coprod_{\eta \leq_w \nu} (X \cap S_w^\eta)$$

and so

$$X = \overline{\coprod_{\eta \leq_w \nu} (X \cap S_w^\eta)} = \bigcup_{\eta \leq_w \nu} (\overline{X \cap S_w^\eta}),$$

with the last equality holding since there are only finitely many semi-infinite cells intersecting X. By irreducibility and completeness of X, there is thus some $\eta \leq_w \nu$ such that $X = \overline{X \cap S_w^\eta}$. But the only $\eta \leq_w \nu$ with $\varpi^\nu x_0 \in S_w^\eta$ is $\eta = \nu$. Therefore $X = \overline{X \cap S_w^\nu}$. $\square$

These results suggest the construction of GGMS strata: small, T-invariant subvarieties stratifying Gr_G, whose closures are sometimes AMV cycles. A GGMS stratum, named for Gelfand, Goresky, MacPherson, and Serganova, is an intersection of a sequence of semi-infinite cells indexed by W. A sequence $\nu_\bullet = (\nu_w)_{w \in W}$ of cocharacters such that $\nu_w \geq_w \nu_{w'}$ for all $w, w' \in W$ is known as a GGMS datum, and specifies the GGMS stratum

$$A(\nu_\bullet) := \bigcap_{w \in W} S_w^{\nu_w}.$$

GGMS data are in bijection with pseudo-Weyl polytopes, or convex polytopes in $X_*(T)_\mathbb{R}$ whose edges are in root directions, and whose vertices are cocharacters in a common coset modulo $X_*(T^{sc})$. Combining the information in Proposition 2.3.6, we see that if A is an AMV cycle, then P_A is a pseudo-Weyl polytope, and the vertices of P_A form a GGMS datum $\nu_\bullet^A$. By Lemma 2.5, the GGMS stratum $GGMS(A) := A(\nu_\bullet^A)$ is dense in A, and is the minimal intersection of semi-infinite cells with this property.

As a consequence of their construction as the intersections of semi-infinite cells, the

σ-action on the set of GGMS strata adheres to the following dichotomy:

Lemma 2.6. i. A GGMS stratum $A(\nu_\bullet)$ is σ-invariant if and only if the sequence of cocharacters $\nu_\bullet$ is σ-invariant, meaning $\sigma(\nu_w) = \nu_{\sigma(w)}$ for all $w \in W$.

ii. If $A(\nu_\bullet)$ is not σ-invariant, then $A(\nu_\bullet)^\sigma = \varnothing$.

Proof. First, note that Gr_G is stratified by GGMS strata. Intersecting the Iwasawa stratifications of Gr_G for all $w \in W$, we have

$$\mathrm{Gr}_G = \coprod_{\nu_\bullet} A(\nu_\bullet).$$

where the disjoint union runs over all GGMS data $\nu_\bullet$.

Then let $\nu_\bullet = (\nu_w)_{w\in W}$. We have

$$\sigma(A(\nu_\bullet)) = \sigma\left(\bigcap_{w\in W} S_w^{\nu_w}\right) = \bigcap_{w\in W} S_{\sigma(w)}^{\sigma(\nu_w)} = A(\sigma(\nu_\bullet)),$$

where $\sigma(\nu_\bullet) = (\sigma(\nu_{\sigma^{-1}(w)}))_{w\in W}$. Then σ permutes GGMS strata, implying ii. And i. follows from observing that $\sigma(\nu_\bullet) = \nu_\bullet$ if and only if $\nu_w = \sigma(\nu_\bullet)_w$ for each $w \in W$. $\square$

2.4 Indexing using **i**-Lusztig data

Unfortunately, not all GGMS strata are dense in some AMV cycle. In order to work with an indexing set where we can be sure the result is an AMV cycle, we use **i**-Lusztig data.

Let A be an AMV cycle with GGMS datum $\nu_\bullet$. If we walk along the edges of the moment polytope P_A from ν_e to ν_{w_0} such that the vertices are indexed by Weyl elements of strictly increasing lengths, by recording the length of each edge we produce a sequence of $\ell(w_0)$ nonnegative integers $n_\bullet \in \mathbb{Z}_{\geq 0}^{\ell(w_0)}$ called the **i**-Lusztig datum of A. This **i**-Lusztig datum uniquely determines the stable AMV cycle A_0, where **i** is the corresponding reduced word for w_0. The **i**-Lusztig strata are in a useful bijection with stable AMV cycles:

Theorem 2.4.1 ([17] Theorem 4.2)**.** *Given fixed reduced word* **i** *for* w_0*, the set of* **i***-Lusztig data are in bijective correspondence with stable AMV cycles.*

$$\mathbb{Z}_{\geq 0}^{\ell(w_0)} \leftrightarrow \{\text{stable AMV cycles}\}$$

Note that since Theorem 2.4.1 concerns stable AMV cycles, there is no question that it applies to connected, reductive groups, and not just to simply connected, semisimple groups.

Proof. From the construction of the **i**-Lusztig datum of an AMV cycle, it is sufficient to find an inverse mapping of **i**-Lusztig data to stable AMV cycles. This can be done explicitly.

Note Conv $(W \cdot \rho^\vee)$ is a pseudo-Weyl polytope, and has **i**-Lusztig datum $(1, \ldots, 1)$ for every reduced word **i** for w_0; indeed, for any two neighboring vertices $w(\rho^\vee)$ and $ws_i(\rho^\vee)$, the difference is a single coroot $w(\rho^\vee - s_i(\rho^\vee)) = \pm w(\alpha_i^\vee)$. This polytope is sometimes known as the permutahedron.

One way to construct a subvariety of Gr_G whose GGMS datum has a given **i**-Lusztig datum $n_\bullet$ is to intersect semi-infinite cells correponding to the cocharacters encountered in the path from ν_e to ν_{w_0} corresponding to **i**, the directions of which are determined by

$\mathbf{i}$, and which can all be seen as the directions of edges of the permutahedron. It will turn out that this construction produces a collection of irreducible varieties in bijection with the $\mathbf{i}$-Lusztig data.

Fix $\mathbf{i} = (i_1, \ldots, i_{\ell(w_0)})$, a reduced word for w_0. Let $(w_k^{\mathbf{i}})_{0 \leq k \leq \ell(w_0)}$ be the sequence of Weyl group elements corresponding to the the initial (as in leftmost, assuming W acts from the left) k letters of $\mathbf{i}$: $w_0^{\mathbf{i}} = e$, $w_k^{\mathbf{i}} = w_{k-1}^{\mathbf{i}} s_{i_k}$, and $w_{\ell(w_0)}^{\mathbf{i}} = w_0$.

For $1 \leq k \leq \ell(w_0)$, let $\beta_k^{\mathbf{i},\vee}$ be the difference $w_k^{\mathbf{i}}(\rho^\vee) - w_{k-1}^{\mathbf{i}}(\rho^\vee)$, which is the direction of an edge of the permutahedron. Then $\beta_k^{\mathbf{i},\vee}$ is a negative coroot; specifically, $\beta_k^{\mathbf{i},\vee} = -w_{k-1}^{\mathbf{i}}(\alpha_{i_k}^\vee)$:

$$\begin{aligned} w_k^{\mathbf{i}}(\rho^\vee) - w_{k-1}^{\mathbf{i}}(\rho^\vee) &= w_{k-1}^{\mathbf{i}}(s_{i_k}(\rho^\vee) - \rho^\vee) \\ &= w_{k-1}^{\mathbf{i}}(s_{i_k}(\frac{1}{2}\sum_{\alpha^\vee \in \Phi^{\vee,+}} \alpha^\vee) - \frac{1}{2}\sum_{\alpha^\vee \in \Phi^{\vee,+}} \alpha^\vee) \\ &= w_{k-1}^{\mathbf{i}}(-\alpha_{i_k}^\vee), \end{aligned}$$

since s_{i_k} permutes all positive coroots except for $\alpha_{i_k}^\vee$, which it transposes with $-\alpha_{i_k}^\vee$. Let $\nu_0 = 0$, and for each $k \geq 1$ let $\nu_k = \nu_{k-1} + \beta_k^{\mathbf{i},\vee}$. Each ν_k is $\leq_{w_k^{\mathbf{i}}}$-maximal in the sequence (ν_k), since $w_k^{\mathbf{i}}(\rho^\vee)$ is $\leq_{w_k^{\mathbf{i}}}$-maximal in $W \cdot \rho^\vee$. Then let

$$A_0^{\mathbf{i}}(n_\bullet) := \bigcap_{0 \leq k \leq \ell(w_0)} S_{w_k^{\mathbf{i}}}^{\nu_k}.$$

Then $\overline{A_0^{\mathbf{i}}(n_\bullet)}$ is a projective subvariety of $\overline{X(\nu_{\ell(w_0)}, 0)} = \overline{S_{w_0}^{\nu_{\ell w_0}} \cap S_e^0}$. Note also that the map

$$\{n_\bullet \mid \mathbf{i}\text{-Lusztig data}\} \to \{A_0 \mid \text{stable AMV cycles}\}$$

given by $n_\bullet \mapsto \overline{A_0^{\mathbf{i}}(n_\bullet)}$ is injective.

Suppose that $A_0^{\mathbf{i}}(n_\bullet)$ is irreducible. Then it is not hard to see that $\overline{A_0^{\mathbf{i}}(n_\bullet)}$ is a stable AMV cycle, proving the theorem.

Indeed, if A_0 is a stable MV cycle with GGMS datum $\nu_\bullet$ and $\mathbf{i}$-Lusztig datum $n_\bullet$, we have containment

$$A_0(\nu_\bullet) \subset A_0^{\mathbf{i}}(n_\bullet) \subset \overline{X(\nu_{w_0}, 0)}, \qquad A_0(\nu_\bullet) \subset A_0 \subset \overline{X(\nu_{w_0}, 0)}.$$

Then by Lemma 2.5, $A_0^{\mathbf{i}}(n_\bullet) \cap A_0$ is dense in A_0. Since A_0 is projective, the result follows from irreducibility of $A_0^{\mathbf{i}}(n_\bullet)$.

And all stable AMV cycles are constructed in this way: the number of stable AMV cycles of a given coweight $(\nu, 0)$ is the Kostant partition function of ν, defined as the number of ways ν can be written as a combination of negative coroots. The reason for this is that the Kostant partition function is a sharp upper bound on the dimension of the weight space $V_\mu(\mu+\nu)$, and by Theorem 2.3.4 every stable AMV cycle A_0 represents an MV cycle $n\rho^\vee \cdot A_0$ for sufficiently large n. And since every negative coroot is the direction of exactly one edge in the paths defined by $\mathbf{i}$, the Kostant partition function of ν is exactly the number of distinct $\mathbf{i}$-Lusztig data of coweight $(\nu, 0)$.

Irreducibility is somewhat difficult. The heart of the proof of irreducibility is the construction of surjective maps

$$\mathbf{y}_{\mathbf{i}} : B(n_\bullet) \to A_0^{\mathbf{i}}(n_\bullet),$$

where $B(n_\bullet)$ is the subfunctor of $L\mathbb{G}_a^{\ell(w_0)}$ defined on R-points as

$$B(n_\bullet)(R) = \{(x_1, \dots, x_{\ell(w_0)}) \in R(\!(\varpi)\!)^{\ell(w_0)} \mid \mathrm{val}_\varpi(x_k) = n_k \text{ for } 1 \leq k \leq \ell(w_0)\}.$$

Clearly $B(n_\bullet)$ is an irreducible ind-scheme, so the existence of a surjective map of sheaves $\mathbf{y_i}$ implies that $A_0^{\mathbf{i}}(n_\bullet)$ is irreducible as well. For details and construction of $\mathbf{y_i}$, see [17] Theorem 4.5. This construction is explicit, but somewhat complicated. Note that it does not depend in any way on simply-connectedness or semisimplicity of G. □

It is also the case that, fixing a reduced word $\mathbf{i}$, the variety $X(\nu, 0)$ is stratified by $\mathbf{i}$-Lusztig strata:

$$X(\nu, 0) = S_{w_0}^{\nu} \cap S_e^0 = \coprod_{\substack{n_\bullet \in \mathbb{Z}_{\geq 0}^{\ell(w_0)} \\ \nu_{\ell(w_0)} = \nu}} A_0^{\mathbf{i}}(n_\bullet). \tag{2.9}$$

Definition 2.4.2 (σ-compatibility)**.** *A reduced word* $\mathbf{i}$ *for the longest element* $w_0 \in W$ *is said to be* σ*-compatible if there is a (uniquely determined) reduced word* $\mathbf{i}_\sigma$ *for the longest element* $w_{0,\sigma} \in W^\sigma$ *such that* $\mathbf{i}$ *is an expansion of* $\mathbf{i}_\sigma$*.*

The group W^σ *is a Coxeter group whose simple reflections are the longest elements in the subgroup generated by the simple reflections in a single* σ*-orbit (see Proposition 2.7.2). If* $\mathbf{i}_\sigma$ *is a sequence of orbits* $(\eta_1, \dots, \eta_{\ell(w_{0,\sigma})})$*, we say* $\mathbf{i}$ *is an expansion of* $\mathbf{i}_\sigma$ *if it can be partitioned into consecutive subsequences of letters corresponding to the orbits of* $\mathbf{i}_\sigma$*. If this is the case, then each such consecutive subsequence will be the longest word in the Coxeter subgroup generated by the simple reflections in the corresponding orbit.*

Example 2.4.3. *To illustrate the concept of* σ*-compatibility, consider the involution on the standard pinning of* SL_5*, with fixed-point subgroup* $(SL_5)^\sigma \cong SO(5)$*. On the set of simple* *roots, σ acts by $(14)(23)$, with two orbits: $\eta_1 = \{1,4\}$ and $\eta_2 = \{2,3\}$. Then*

$$W = \langle s_1, s_2, s_3, s_4 \mid m_{13} = m_{14} = m_{24} = 2,\ m_{12} = m_{23} = m_{34} = 3\rangle$$
$$W^\sigma = \langle s_{\eta_1}, s_{\eta_2} \mid m_{\eta_1\eta_2} = 4\rangle,$$

so one (of two) possible reduced word for $w_{0,\sigma}$ is $\mathbf{i}_\sigma = (\eta_1, \eta_2, \eta_1, \eta_2)$. Since $m_{14} = 2$, the longest words on the letters in η_1 are $(1,4)$ and $(4,1)$. Since $m_{23} = 3$, the longest words on the letters in η_2 are $(2,3,2)$ and $(3,2,3)$. There are 16 reduced words for w_0 expanding $\mathbf{i}_\sigma$, including

$$\mathbf{i} = (1,4,2,3,2,1,4,2,3,2).$$

Proposition 2.4.4. *If $A_0^{\mathbf{i}}(n_\bullet)$ is an $\mathbf{i}$-Lusztig stratum, then $\sigma(A_0^{\mathbf{i}}(n_\bullet)) = A_0^{\sigma(\mathbf{i})}(n_\bullet)$.*

Proof. Let $\mathbf{i}$ be a reduced word for w_0, and $n_\bullet$ an $\mathbf{i}$-Lusztig datum. Consider

$$\sigma(A_0^{\mathbf{i}}(n_\bullet)) = \sigma\left(\bigcap_{0\le k\le \ell(w_0)} S^{\nu_k}_{w_k^{\mathbf{i}}}\right) = \bigcap_{0\le k\le \ell(w_0)} S^{\sigma(\nu_k)}_{\sigma(w_k^{\mathbf{i}})}.$$

Now $\sigma(w_k^{\mathbf{i}}) = w_k^{\sigma(\mathbf{i})}$, and each $\sigma(\nu_k) = \sigma(\nu_{k-1}) + n_k\sigma(\beta_k^{\mathbf{i},\vee})$. We have

$$\sigma(\beta_k^{\mathbf{i},\vee}) = \sigma(w_k^{\mathbf{i}}(\rho^\vee) - w_{k-1}^{\mathbf{i}}(\rho^\vee)) = w_k^{\sigma(\mathbf{i})}(\rho^\vee) - w_{k-1}^{\sigma(\mathbf{i})}(\rho^\vee) = \beta_k^{\sigma(\mathbf{i}),\vee}.$$

By induction the $\sigma(\mathbf{i})$-Lusztig datum for $\sigma(A_0^{\mathbf{i}}(n_\bullet))$ is thus also $n_\bullet$. □

Suppose $\mathbf{i}$ and $\mathbf{i}'$ are distinct reduced words for w_0 that both expand a common reduced word $\mathbf{i}_\sigma$ for $w_{0,\sigma}$. Let $n_\bullet$ be an $\mathbf{i}$-Lusztig datum. Then there is an explicit procedure for producing the $\mathbf{i}'$-Lusztig datum $n'_\bullet$ for $A_0^{\mathbf{i}}(n_\bullet)$.

Lemma 2.7 ([17] Proposition 5.2)**.** Let $\mathbf{i}$ and $\mathbf{i}'$ be two reduced words for w_0 related by a braid move corresponding to a pair of simple roots either disconnected or connected by an edge of weight 1. Specifically, let i_k and i_{k+1} be the indices of a pair of simple roots, and let $m_{i_k,i_{k+1}} \leq 3$ be the order of $s_{i_k}s_{i_{k+1}}$ in W. Then suppose

$$\mathbf{i} = (i_1, \ldots, i_{k-1}; i_k, i_{k+1}, i_k, \ldots; i_{k+m_{i_k,i_{k+1}}+1}, \ldots, i_{\ell(w_0)}),$$

$$\mathbf{i}' = (i_1, \ldots, i_{k-1}; i_{k+1}, i_k, i_{k+1}, \ldots; i_{k+m_{i_k,i_{k+1}}+1}, \ldots, i_{\ell(w_0)}).$$

Then define a function $R_{\mathbf{i}}^{\mathbf{i}'} : \mathbb{Z}_{\geq 0}^{\ell(w_0)} \to \mathbb{Z}_{\geq 0}^{\ell(w_0)}$ as follows:

1. For $k' < k$ or $k' > k + m_{i_k,i_{k+1}}$, let $R_{\mathbf{i}}^{\mathbf{i}'}(n_\bullet)_{k'} = n_{k'}$
2. Suppose $m_{i_k,i_{k+1}} = 2$. Then let $R_{\mathbf{i}}^{\mathbf{i}'}(n_\bullet)_k = n_{k+1}$ and let $R(n_\bullet)_{k+1} = n_k$.
3. Suppose $m_{i_k,i_{k+1}} = 3$, and let $p = \min\{n_k, n_{k+2}\}$. Then let $R_{\mathbf{i}}^{\mathbf{i}'}(n_\bullet)_k = n_{k+1}+n_{k+2}-p$, $R_{\mathbf{i}}^{\mathbf{i}'}(n_\bullet)_{k+1} = p$, and $R_{\mathbf{i}}^{\mathbf{i}'}(n_\bullet)_{k+2} = n_k + n_{k+1} - p$.

If $n_\bullet$ is the $\mathbf{i}$-Lusztig datum of a stable AMV cycle A_0, then $R_{\mathbf{i}}^{\mathbf{i}'}(n_\bullet)$ is the $\mathbf{i}'$-Lusztig datum of A_0.

Since any two simple roots in a σ-orbit are either disconnected or connected by an edge of weight 1 in the Dynkin diagram, we are only interested in the cases above. However, [17] proves the proposition in all cases. Since all reduced words for w_0 are related by sequences of braid moves, Kamnitzer's full proposition implies the existence of a function $R_{\mathbf{i}}^{\mathbf{i}'}$ for all pairs of reduced words $(\mathbf{i}, \mathbf{i}')$. As a consequence of the cases above, we can define $R_{\mathbf{i}}^{\mathbf{i}'}$ whenever $\mathbf{i}$ and $\mathbf{i}'$ expand a common word $\mathbf{i}_\sigma$. Thus we have justification for the following definition.

Definition 2.4.5 (σ-invariant $\mathbf{i}$-Lusztig datum)**.** *Fix a reduced word* $\mathbf{i}$ *for* w_0 *which expands a reduced word* $\mathbf{i}_\sigma$ *for* $w_{0,\sigma}$*. Then an* $\mathbf{i}$*-Lusztig datum* $n_\bullet$ *is* σ*-invariant if it is constant on each* σ*-orbit. If* $n_\bullet$ *is a* σ*-invariant* $\mathbf{i}$*-Lusztig datum, then let* $\bar{n}_\bullet$ *be the corresponding* $\mathbf{i}_\sigma$*-Lusztig datum.*

Consider Example 2.4.3. In this case, an $\mathbf{i}$-Lusztig datum $n_\bullet$ is σ-invariant if and only if $n_1 = n_2$, $n_3 = n_4 = n_5$, $n_6 = n_7$, and $n_8 = n_9 = n_{10}$. If this is the case, then $\bar{n}_\bullet = (n_1, n_3, n_6, n_8)$.

Note that $\mathbf{i}$ and $\sigma(\mathbf{i})$ are two different reduced words for w_0 expanding $\mathbf{i}_\sigma$, and $R_{\mathbf{i}}^{\sigma(\mathbf{i})}(n_\bullet) = n_\bullet$ if and only if $n_\bullet$ is σ-invariant. Thus, given a σ-compatible reduced word $\mathbf{i}$ for w_0, an $\mathbf{i}$-Lusztig datum $n_\bullet$ is σ-invariant if and only if the stable MV cycle $\overline{A_0^{\mathbf{i}}(n_\bullet)}$ is σ-invariant.

Lemma 2.8. Let $\mathbf{i}$ be a σ-compatible reduced word for w_0, and let $n_\bullet$ be a σ-invariant $\mathbf{i}$-Lusztig datum. Then we can identify the fixed-point subvariety

$$(A_0^{\mathbf{i}}(n_\bullet))^\sigma = A_0^{\mathbf{i}_\sigma}(\bar{n}_\bullet).$$

Proof. Recall the surjective map

$$\mathbf{y}_{\mathbf{i}} : B(n_\bullet) \to A_0^{\mathbf{i}}(n_\bullet)$$

from the proof of Theorem 2.4.1, which is defined explicitly in the proof of [17] Theorem 4.5. Consider also the map

$$\mathbf{y}_{\mathbf{i}_\sigma} : B_\sigma(\bar{n}_\bullet) \to A_0^{\mathbf{i}_\sigma}(\bar{n}_\bullet),$$

defined analogously for G^σ. It is clear from the explicit definition of $\mathbf{y}_{\mathbf{i}}$ that in the square

$$\begin{tikzcd}
B_\sigma(\bar{n}_\bullet) \arrow[r, "\mathrm{diag}"] \arrow[d, "\mathbf{y}_{\mathbf{i}_\sigma}"'] & B(n_\bullet) \arrow[d, "\mathbf{y}_{\mathbf{i}}"] \\
A_0^{\mathbf{i}_\sigma}(\bar{n}_\bullet) \arrow[r, "\iota"] & A_0^{\mathbf{i}}(n_\bullet)
\end{tikzcd}$$

the map ι is well-defined and injective, and the square commutes, for σ-invariant $n_\bullet$. In particular, $A_0^{\mathbf{i}_\sigma}(\bar{n}_\bullet) \subset (A_0^{\mathbf{i}}(n_\bullet))^\sigma = A_0^{\mathbf{i}}(n_\bullet) \cap \mathrm{Gr}_{G^\sigma}$.

The reverse inclusion follows from the $\mathbf{i}_\sigma$- and $\mathbf{i}$-Lusztig stratifications from equation (2.9) of the varieties

$$(X(\nu_{\ell(w_0)}, 0))^\sigma = (S_{w_0}^{\nu_{\ell(w_0)}} \cap S_e^0) \cap \mathrm{Gr}_{G^\sigma} = (S_\sigma)_{w_{0,\sigma}}^{\nu_{\ell(w_0)}} \cap (S_\sigma)_e^0$$

and

$$X(\nu_{\ell(w_0)}, 0) = S_{w_0}^{\nu_{\ell(w_0)}} \cap S_e^0.$$

Note also the semi-infinite cells $S_{w_0}^\nu$ and S_e^0 are σ-invariant, since σ-invariance of the $\mathbf{i}$-Lusztig datum $n_\bullet$ implies $\nu_{\ell(w_0)}$ is a σ-invariant cocharacter. □

Lemma 2.9. Let $n_\bullet$ be a σ-invariant $\mathbf{i}$-Lusztig datum, where $\mathbf{i}$ expands $\mathbf{i}_\sigma$. Suppose $A_0^{\mathbf{i}}(n_\bullet)$ is of coweight $(\nu, 0)$. Then $A_0^{\mathbf{i}_\sigma}(\bar{n}_\bullet)$ is also of coweight $(\nu, 0)$.

Proof. Let $\pi : \{1, \dots, \ell(w_0)\} \to \{1, \dots, \ell(w_{0,\sigma})\}$ be the surjection of indices of the words. It is sufficient to prove that, for each $1 \leq k' \leq \ell(w_{0,\sigma})$, we have

$$\bar{n}_{k'} \beta_{k'}^{\mathbf{i}_\sigma, \vee} = \sum_{k \in \pi^{-1}(k')} n_k \beta_k^{\mathbf{i}, \vee}. \tag{2.10}$$

Then since $\bar{n}_{k'} = n_k$ for all $k \in \pi^{-1}(k')$, equation (2.10) is equivalent to

$$\beta_{k'}^{\mathbf{i}_\sigma,\vee} = \sum_{k \in \pi^{-1}(k')} \beta_k^{\mathbf{i},\vee} \tag{2.11}$$

for all k'. For each k' there are two possibilities. Either $\pi^{-1}(k') = \{k, k+1, k+2\}$, where $\alpha_{i_k}^\vee = \alpha_{i_{k+2}}^\vee$ and $\alpha_{i_{k+1}}^\vee$ are two distinct simple coroots connected by an edge of weight 1 in the Dynkin diagram of G; or all simple coroots corresponding to $k \in \pi^{-1}(k')$ are pairwise disconnected. In the first case we can say the orbit is of type A_2, and in the second we can say it is of type $A_1 \times \cdots \times A_1$.

First suppose k' corresponds to an orbit of type A_2. Recall

$$\beta_{k+j}^{\mathbf{i},\vee} = w_{k+j}^{\mathbf{i}}(\rho^\vee) - w_{k+j-1}^{\mathbf{i}}(\rho^\vee) = w_{k+j-1}^{\mathbf{i}}(s_{i_{k+j}}(\rho^\vee) - \rho^\vee) \quad (\text{for } j = 0, 1, 2), \quad \text{and}$$

$$\beta_{k'}^{\mathbf{i}_\sigma,\vee} = w_{k'}^{\mathbf{i}_\sigma}(\rho_\sigma^\vee) - w_{k'-1}^{\mathbf{i}_\sigma}(\rho_\sigma^\vee) = w_{k'-1}^{\mathbf{i}_\sigma}(s_{i_{k'}}(\rho_\sigma^\vee) - \rho_\sigma^\vee).$$

Then the right hand side of (2.11) is

$$\begin{aligned} \beta_k^{\mathbf{i},\vee} + \beta_{k+1}^{\mathbf{i},\vee} + \beta_{k+2}^{\mathbf{i},\vee} &= w_{k+2}^{\mathbf{i}}(\rho^\vee) - w_{k-1}^{\mathbf{i}}(\rho^\vee) \\ &= w_{k-1}^{\mathbf{i}}(s_{i_k} s_{i_{k+1}} s_{i_{k+2}}(\rho^\vee) - \rho^\vee) \\ &= w_{k'-1}^{\mathbf{i}_\sigma}(s_{i_{k'}}(\rho_\sigma^\vee) - \rho_\sigma^\vee). \end{aligned} \tag{2.12}$$

The equality of line (2.12) follows from Propositions 2.7.2 and 2.7.3: $s_{i_{k'}} = s_{i_k} s_{i_{k+1}} s_{i_{k+2}}$ is the longest element of the Coxeter subgroup generated by letters in the orbit η, and so by induction $w_{k-1}^{\mathbf{i}} = w_{k'-1}^{\mathbf{i}_\sigma}$.

The case k' corresponds to an orbit of type $A_1 \times \cdots \times A_1$ is similar, except the orbit can be of any order, and each index in the orbit appears only once. Then in this case $s_{i_{k'}} = s_{i_k} s_{i_{k+1}} \cdots s_{i_{k+|\eta|-1}}$, and so equation (2.11) holds. □

2.5 The bijection for AMV cycles

Theorem 2.5.1. *Let (λ, μ) be a coweight with μ dominant, $\lambda \leq \mu$, and both fixed by σ. Then taking σ-fixed points induces a bijection between σ-invariant MV cycles in* Gr_G *of coweight and (λ, μ) and MV cycles in* Gr_{G^σ} *of coweight (λ, μ).*

Proof. First I will establish the bijection between AMV cycles. The action of $-\mu$ on the set of AMV cycles gives a bijection between AMV cycles of coweight (λ, μ) and stable AMV cycles of coweight $(\lambda - \mu, 0)$.

Fix a σ-compatible reduced word $\mathbf{i}$ for w_0 expanding the reduced word $\mathbf{i}_\sigma$ for $w_{0,\sigma}$. Theorem 2.4.1 gives a bijection between $\mathbf{i}$-Lusztig data and all stable AMV cycles, and restricts to a bijection between those of coweight $(\lambda - \mu, 0)$. And by Lemma 2.8, there is a bijection between σ-invariant $\mathbf{i}$-Lusztig strata of coweight $(\lambda - \mu, 0)$ and $\mathbf{i}_\sigma$-Lusztig strata of coweight $(\lambda - \mu, 0)$, given by taking fixed points.

Then by composing the corresponding bijection between $\mathbf{i}_\sigma$-Lusztig strata of coweight $(\lambda - \mu, 0)$ and AMV cycles of coweight (λ, μ), we have the desired bijection on the level of AMV cycles.

It remains to see that the bijection on AMV cycles restricts to one on MV cycles. Recall Proposition 2.3.3: MV cycles of coweight (λ, μ) correspond bijectively to those AMV cycles of coweight (λ, μ) contained in $\overline{\mathrm{Gr}^\mu_G}$. Clearly, if A is an AMV cycle and $A^\sigma \not\subset \overline{\mathrm{Gr}^\mu_{G^\sigma}}$, then $A \not\subset \overline{\mathrm{Gr}_G^\mu}$. However, the converse is more subtle.

Let A be a σ-invariant AMV cycle as in the hypotheses, and suppose $A \not\subset \overline{\mathrm{Gr}_G^\mu}$; i.e. A is not an MV cycle. Let $(\nu_w)_{w\in W}$ be the GGMS datum of A, the W-indexed sequence of vertices in the moment polytope P_A. Then the GGMS stratum of A is

$$GGMS(A) := \bigcap_{w\in W} S_w^{\nu_w}.$$

Note that $\overline{\mathrm{Gr}_G^\mu}$ is an AMV cycle of coweight $(-\mu, \mu)$, and thus the closure of its GGMS stratum:

$$\overline{\mathrm{Gr}_G^\mu} = \overline{\bigcap_{w\in W} S_w^{w(\mu)}}$$

Now if $A \not\subset \overline{\mathrm{Gr}_G^\mu}$, then its GGMS stratum does not intersect with $\overline{\mathrm{Gr}_G^\mu}$ at all. Indeed, suppose there exists a complex point

$$\begin{aligned} p \in \left(GGMS(A) \cap \overline{\mathrm{Gr}_G^\mu}\right)(\mathbb{C}) &= \left(\bigcap_{w\in W} S_w^{\nu_w}(\mathbb{C})\right) \cap \left(\overline{\bigcap_{w\in W} S_w^{w(\mu)}}(\mathbb{C})\right) \\ &\subset \bigcap_{w\in W} \left(S_w^{\nu_w}(\mathbb{C}) \cap \overline{S_w^{w(\mu)}}(\mathbb{C})\right). \end{aligned}$$

Then for each $w \in W$ the closure relations for semi-infinite cells imply $\nu_w \leq_w w(\mu)$, so $P_A \subset \mathrm{Conv}(W \cdot \mu)$. By Theorem 2.3.4, this implies A is in fact an MV cycle, contrary to assumption.

By Lemma 2.5, $GGMS(A)$ is dense in A. So if $(A \cap \overline{\mathrm{Gr}_G^\mu})^\sigma$ were dense in A^σ, there would have to be some complex point

$$p \in \left(GGMS(A)^{\sigma} \cap \overline{\mathrm{Gr}^{\mu}_{G^{\sigma}}}\right)(\mathbb{C}) \subset \left(GGMS(A) \cap \overline{\mathrm{Gr}^{\mu}_{G}}\right)(\mathbb{C}),$$

but the last intersection is empty.

As a result, the map

$$\left\{ A \subset \mathrm{Gr}_G \;\middle|\; \begin{array}{c} \sigma\text{-invariant MV cycles} \\ \text{of coweight } (\lambda,\mu) \end{array} \right\} \longrightarrow \left\{ A \subset \mathrm{Gr}_{G^{\sigma}} \;\middle|\; \begin{array}{c} \text{MV cycles of} \\ \text{coweight } (\lambda,\mu) \end{array} \right\}$$

$$A \longmapsto \overline{A^{\sigma}} \cap (S_{\sigma})^{\lambda}_{w_{0,\sigma}}$$

is a bijection. □

2.6 Eigenvalues

I would like to prove that equation (2.1) holds for all σ-invariant (λ,μ). It is known that the number of σ-invariant MV cycles of coweight (λ,μ) in Gr_G is the same as the number of MV cycles of coweight (λ,μ) in $\mathrm{Gr}_{G^{\sigma}}$, and that the bijection is given by taking the σ-fixed points. From a naive understanding of the geometric Satake equivalence, it is thus clear that there *is* a linear map $\sigma' : V_{\mu}(\lambda) \to V_{\mu}(\lambda)$ such that equation (2.1) holds, replacing σ with σ'. Indeed, one may simply choose a basis $\{e_A\}$ for $V_{\mu}(\lambda)$ indexed by MV cycles A, and let σ' be any map such that $\sigma'(e_A)$ is a scalar multiple if $e_{\sigma(A)}$ for all A and $\sigma'(e_A) = e_A$ for σ-invariant A. However, it is not clear at this point that the map constructed from the action of σ on $\widehat{G}$ shares these properties. We need to take a more careful look at the geometric Satake equivalence to see that there is indeed a basis $\{e_A\}$ for which σ satisfies these properties.

But first, we need to more carefully define the operator we are considering, as well as construct an alternative operator for comparison. In the end, we will need to make some (limited) choice to identify the operators on $V_\mu(\lambda)$, so from here I will start decorating them to keep them distinct.

On the one hand, we have constructed $\widehat{G}$ as the group dual to G using the root datum, and given it an arbitrary pinning (although there is a canonical choice one could make, to me it is arbitrary since I will make an identification that is not canonical in any case). This pinning uniquely determines an automorphism of $\widehat{G}$, which I will now call $\hat{\sigma}$, preserving $\widehat{B}$ and $\widehat{T}$, and compatible with the root homomorphisms $x_{\alpha^\vee}$. It is now straightforward to construct an action of $\hat{\sigma}$ on irreducible highest-weight representations V_μ of $\widehat{G}$ when μ is $\hat{\sigma}$-invariant. Since $\hat{\sigma}^*$ is a tensor auto-equivalence on $\mathrm{Rep}_{\mathbb{C}}(\widehat{G})$, we have $\hat{\sigma}^* V_\mu$ is an irreducible highest-weight representation. Furthermore, given a vector $v \in V_\mu$ of weight λ and $t \in \widehat{T}$, we have $\hat{\sigma}(t) \cdot v = \sigma^{-1}(\lambda)(t)v$, so as an element of $\hat{\sigma}^* V_\mu$, v has weight $\sigma^{-1}(\lambda)$. In particular, the highest weight of $\hat{\sigma}^* V_\mu$ is $\sigma^{-1}(\mu)$. In the case μ is σ-invariant, we thus have $\hat{\sigma}^* V_\mu \cong V_\mu$. By Schur's lemma, there is up to scalar a unique isomorphism of representations $\hat{\sigma}^* V_\mu \to V_\mu$. However, since $\hat{\sigma}^* V_\mu$ and V_μ share underlying vector spaces, we may canonically choose the isomorphism which fixes the highest-weight line $V_\mu(\mu)$ pointwise. This automorphism on the underlying vector space is the action of σ on V_μ, and will be known as $\hat{\sigma}_\mu : V_\mu \to V_\mu$. It is difficult, from this construction, to directly deduce precise eigenvalues of $\hat{\sigma}_\mu$ for vectors outside of the highest-weight line $V_\mu(\mu)$ (and other extreme-weight lines $V_\mu(w(\mu))$).

On the other hand, we have a construction of the dual group of G using the Tannakian formalism: $\widetilde{G}$ is the group of fiber functor automorphisms of $P_{L^+G}(\mathrm{Gr}_G)$. I will construct a linear isomorphism $\tilde{\sigma}_\mu : \mathbb{H}^\bullet(\mathrm{Gr}_G, IC_\mu) \to \mathbb{H}^\bullet(\mathrm{Gr}_G, IC_\mu)$ for σ-invariant cocharacters μ of G, and identify $\widetilde{G} \cong \widehat{G}$ in such a way that (using uniqueness from Schur's lemma) the automorphisms $\hat{\sigma}_\mu$ and $\tilde{\sigma}_\mu$ must be equal. However, the identification of $\widetilde{G}$ with $\widehat{G}$ is non-canonical, and in particular is sensitive to the pinning on G. In order to identify $\widetilde{G}$ with $\widehat{G}$, I will construct a pinning on $\widetilde{G}$ that is preserved by an automorphism $\tilde{\sigma}$ of $\widetilde{G}$.

The advantage of considering these "tilde" constructions is that, as an induced map on cohomology groups, it will be much more straightforward to prove that $\tilde{\sigma}_\mu$ fixes all basis vectors corresponding to σ-invariant MV cycles, and thus satisfies equation (2.1).

I will henceforth refer to the functor $\mathbb{H}^\bullet(\mathrm{Gr}_G, -)$ as F, as in "fiber." Similarly, for cocharacters μ and λ, let $F_\lambda IC_\mu$ be the group $H_c^{-2\langle\rho,\mu-\lambda\rangle}(S_{w_0}^\lambda \cap \mathrm{Gr}_G^\mu, \mathbb{C})$.

The map $\tilde{\sigma}_\mu$ will be constructed from an identification $IC_\mu \cong \sigma^! IC_\mu$. (Note that, since σ is an étale morphism of varieties, $\sigma^*\mathscr{A}$ is canonically isomorphic to $\sigma^!\mathscr{A}$ for all perverse sheaves $\mathscr{A}$ on Gr_G. Similarly, since σ is proper, $\sigma_!\mathscr{A}$ is canonically isomorphic to $\sigma_*\mathscr{A}$.) Suppose we have an isomorphism $\phi_\mu : IC_\mu \to \sigma^! IC_\mu$. Then, using the counit ϵ of the adjunction $(\sigma_!, \sigma^!)$ we have an isomorphism

$$\sigma_! IC_\mu \xrightarrow{\sigma_!\phi_\mu} \sigma_!\sigma^! IC_\mu \xrightarrow{\epsilon} IC_\mu$$

Since Gr_G is an ind-proper ind-scheme, F is canonically isomorphic to a direct sum of compactly supported cohomology functors, implying $F\, IC_\mu \cong F\sigma_! IC_\mu$. Thus, we can compose arrows in the following (commutative) diagram

$$\begin{tikzcd}
F\,IC_\mu \arrow[r, "F\phi_\mu"] \arrow[d, "can"] & F\sigma^! IC_\mu \arrow[d, "can"] & \\
F\sigma_! IC_\mu \arrow[r, "F\sigma_!\phi_\mu"] & F\sigma_!\sigma^! IC_\mu \arrow[r, "F\epsilon"] & F\,IC_\mu
\end{tikzcd}$$

to produce a map $\tilde{\sigma}_\mu := F\epsilon \circ can \circ F\phi_\mu$, given an isomorphism ϕ_μ.

Similarly to the construction of $\hat{\sigma}_\mu$, we can construct ϕ_μ canonically. For all x in the smooth locus Gr_G^μ of $\overline{\mathrm{Gr}_G^\mu}$, the stalks of both IC_μ and $\sigma^! IC_\mu$ at x are one-dimensional and concentrated in degree $-2\langle\rho,\mu\rangle$. Furthermore, the basepoint $\varpi^\mu x_0$ is preserved by σ if μ is. So let ϕ_μ be the unique isomorphism $IC_\mu \to \sigma^! IC_\mu$ that restricts to the identity on the stalk at $\varpi^\mu x_0$.

Lemma 2.10. Suppose μ is a σ-invariant dominant cocharacter. As constructed above, the map $\tilde{\sigma}_\mu$ is a direct sum of natural maps on top-dimensional cohomology groups with compact support, with constant coefficients, induced by the morphisms of varieties $\sigma : S_{w_0}^\lambda \cap \mathrm{Gr}_G^\mu \to S_{w_0}^{\sigma(\lambda)} \cap \mathrm{Gr}_G^\mu$. As a result, for all irreducible components $A \subset S_{w_0}^\lambda \cap \mathrm{Gr}_G^\mu$, the fundamental class

$$[A] \in H_c^{2\langle\rho,\mu-\lambda\rangle}(S_{w_0}^\lambda \cap \mathrm{Gr}_G^\mu, \mathbb{C})$$

satisfies $\tilde{\sigma}_\mu([A]) \in \mathbb{C}[\sigma(A)]$. And in particular, if $\sigma(A) = A$, then $\tilde{\sigma}_\mu([A]) = [A]$ exactly.

Proof. Recall by the geometric Satake correspondence, specifically 2.2.1 iii. and iv., that there is a natural, canonical isomorphism

$$GSE_\mu : F\,IC_\mu \xrightarrow{\sim} \bigoplus_{\lambda \in X_*(T)} H_c^{-2\langle\rho,\lambda\rangle}(S_{w_0}^\lambda, IC_\mu) \xrightarrow{\sim} \bigoplus_{\lambda \in Wt(\mu)} H_c^{2\langle\rho,\mu-\lambda\rangle}(S_{w_0}^\lambda \cap \mathrm{Gr}_G^\mu, \mathbb{C}).$$

So the goal will be to show that a natural map on cohomology groups induced by the morphism $\sigma : S^{\lambda}_{w_0} \cap \mathrm{Gr}^{\mu}_{G} \to S^{\sigma(\lambda)}_{w_0} \cap \mathrm{Gr}^{\mu}_{G}$ commutes with GSE_{μ} and fixes pointwise the cohomology groups corresponding to σ-invariant MV cycles.

Cohomology with compact support is a covariant functor. That is, given a map of sheaves $\phi : \mathscr{F} \to \mathscr{G}$ on a variety Y, we have for each i a map on cohomology with compact support

$$H^i_c(\phi) : H^i_c(Y, \mathscr{F}) \to H^i_c(Y, \mathscr{G})$$

Given a morphism of varieties $f : X \to Y$, we have naturally induced such a map on sheaves $\epsilon : f_! f^! \mathbb{C}_Y \to \mathbb{C}_Y$, where ϵ is the counit of the adjunction $(f_!, f^!)$. Furthermore, $H^i_c(Y, f_! f^! \mathbb{C}_Y)$ is canonically isomorphic to $H^i_c(X, f^! \mathbb{C}_Y)$. If f is an isomorphism of varieties, this is further isomorphic to $H^i_c(X, \mathbb{C}_X)$ as $f^! \mathbb{C}_Y \cong \mathbb{C}_X$. Thus, after choosing an appropriate isomorphism $\mathbb{C}_X \to f^! \mathbb{C}_Y$, we have a map

$$\mathrm{Tr}^i_f : H^i_c(X, \mathbb{C}) \to H^i_c(Y, \mathbb{C}).$$

This map is constructed in [31, 0GJY] and referred to as the trace map of f.

Let $i = 2\langle \rho, \mu - \lambda \rangle$, where μ is a σ-invariant cocharacter and $\lambda \in Wt(\mu)$. Then the cohomology group $H^i_c(S^{\lambda}_{w_0} \cap \mathrm{Gr}^{\mu}_{G}, \mathbb{C})$ is spanned by fundamental classes $[A]$, where A is an MV cycle of coweight (λ, μ), and $H^i_c(S^{\sigma(\lambda)}_{w_0} \cap \mathrm{Gr}^{\mu}_{G}, \mathbb{C})$ is spanned by fundamental classes $[\sigma(A)]$. Looking on stalks, we see that $\mathrm{Tr}^i_{\sigma}([A])$ is a scalar multiple of $[\sigma(A)]$. Supposing λ is σ-invariant and choosing the normalization $\mathbb{C} \to \sigma^! \mathbb{C}$ corresponding to that for $\tilde{\sigma}_{\mu}$, which is identity on stalks preserved by σ, we get that in the case $\sigma(A) = A$, $\mathrm{Tr}^i_{\sigma}([A]) = [A]$

exactly.

By the similarity in construction of the two horizontal maps, the following diagram commutes:

$$\begin{tikzcd}
F\,IC_\mu \arrow[r, "\tilde{\sigma}_\mu"] \arrow[d, "GSE_\mu"] & F\,IC_\mu \arrow[d, "GSE_\mu"] \\
\bigoplus_{\lambda \in Wt(\mu)} H_c^{2\langle \rho, \mu-\lambda \rangle}(S_{w_0}^{\lambda} \cap \mathrm{Gr}_G, \mathbb{C}) \arrow[r, "\mathrm{Tr}_\sigma^i"] & \bigoplus_{\lambda \in Wt(\mu)} H_c^{2\langle \rho, \mu-\lambda \rangle}(S_{w_0}^{\sigma(\lambda)} \cap \mathrm{Gr}_G, \mathbb{C})
\end{tikzcd}$$

Therefore $\tilde{\sigma}_\mu$ acts on fundamental classes as expected. □

Note that $\tilde{\sigma}_\mu$ thus satisfies (2.1).

In order to identify the actions $\tilde{\sigma}$ and $\hat{\sigma}$, we extend the construction of $\tilde{\sigma}_\mu$ to cases where μ is not σ-invariant, constructing an automorphism $\tilde{\sigma} : \widetilde{G} \to \widetilde{G}$. Given such an automorphism $\tilde{\sigma}$, we can verify that the linear map $\tilde{\sigma}^* V_\mu \to V_\mu$ on induced on σ-invariant highest-weight representations of $\widetilde{G}$ by the automorphism $\tilde{\sigma}$ is equal to the linear operator $\tilde{\sigma}_\mu$ defined above. Then, in order to prove that $\hat{\sigma}_\mu$ satisfies equation (2.1), it will be sufficient to verify that $\widetilde{G}$ may be identified with $\widehat{G}$ in such a way that the automorphisms $\tilde{\sigma}$ and $\hat{\sigma}$ commute with the identification $\widetilde{G} \to \widehat{G}$.

Consider the following proposition, an immediate consequence of definitions in the first chapter of [8]:

Proposition 2.6.1. *Let $\widehat{G}$ be a reductive group, and let $F : \mathrm{Rep}_{\mathbb{C}}(\widehat{G}) \to \mathrm{Vec}_{\mathbb{C}}$ be the natural fiber functor. Given a tensor auto-equivalence $T : (\mathrm{Rep}_{\mathbb{C}}(\widehat{G}), \otimes) \to (\mathrm{Rep}_{\mathbb{C}}(\widehat{G}), \otimes)$ and an isomorphism of fiber functors $\phi : FT \to F$, there is a corresponding automorphism τ of $\widehat{G}$, given by $\tau(g) = \phi \circ g \circ \phi^{-1}$. In the other direction, an automorphism τ can be* *used to construct such a pair, using the functor* $\tau^*(\rho, V) = (\rho \circ \tau, V)$*, and the isomorphism* $F\tau^* \to F$ *which takes identity on objects in* $\mathrm{Vec}_{\mathbb{C}}$*.*

$$\begin{array}{ccc}
\mathrm{Aut}\,(\widehat{G}) & \begin{array}{c}\longrightarrow \\ \\ \longleftarrow\end{array} & \left\{\begin{array}{c} \textit{tensor auto-equivalences with} \\ \textit{an isomorphism of fiber functors} \\ T : (\mathrm{Rep}_{\mathbb{C}}(\widehat{G}), \otimes) \to (\mathrm{Rep}_{\mathbb{C}}(\widehat{G}), \otimes) \\ \phi : FT \to F \end{array}\right\} \\
\tau & \longmapsto & (\tau^*, \mathrm{id}) \\
(g \mapsto \phi \circ g \circ \phi^{-1}) & \longleftarrow\!\shortmid & (T, \phi)
\end{array}$$

We could impose an equivalence relation on the right hand side above and tautologically the pairs (T, ϕ) modulo equivalence would be in bijection with automorphisms. All we need is a sufficient condition for construction of an automorphism $\tilde{\sigma} : \widetilde{G} \to \widetilde{G}$.

We already have a tensor auto-equivalence of functors $\sigma^! : P_{L^+G}(\mathrm{Gr}_G) \to P_{L^+G}(\mathrm{Gr}_G)$, so we want an isomorphism of fiber functors $\phi : F\sigma^! \to F$. As before, we can use the counit $\epsilon : \sigma_! \sigma^! IC_\mu \to IC_\mu$.

$$F\sigma^! IC_\mu \xrightarrow{can} F\sigma_!\sigma^! IC_\mu \xrightarrow{F\epsilon} F\, IC_\mu$$

So we let $\phi = F\epsilon \circ can$; then $\tilde{\sigma}_\mu(g) = \phi \circ g \circ \phi^{-1}$, as needed.

Finally, we need to compare $\tilde{\sigma}$ with $\hat{\sigma}$. Consider that, for σ-invariant μ, $\tilde{\sigma}$ satisfies a commutative diagram

$$\begin{tikzcd}
F\, IC_\mu \arrow[r, "\tilde{\sigma}_\mu"] \arrow[d, "g"'] & F\, IC_\mu \arrow[d, "\tilde{\sigma}(g)"] & F\sigma^! IC_\mu \arrow[l, "F\epsilon \circ can"'] \arrow[d, "g"] \\
F\, IC_\mu \arrow[r, "\tilde{\sigma}_\mu"] & F\, IC_\mu & F\sigma^! IC_\mu \arrow[l, "F\epsilon \circ can"']
\end{tikzcd} \tag{2.13}$$

In fact $\tilde{\sigma}_\mu$ factors through $F\sigma^! IC_\mu$, implying the linear operator on $F\, IC_\mu$ induced by the automorphism $\tilde{\sigma}$ is exactly $\tilde{\sigma}_\mu$.

Now we turn our attention to the question of identifying $\widetilde{G}$ and $\widehat{G}$. They are by GSE 2.2.1 v. abstractly isomorphic. Since a pinning-preserving automorphism is determined uniquely by its action on the Dynkin diagram and the pinning it preserves, it is sufficient to prove that there is *some* pinning preserved by $\tilde{\sigma}$; any choice of such a preserved pinning will imply $\tilde{\sigma} = \hat{\sigma}$ after identification $\widetilde{G} \cong \widehat{G}$. Note that the pinning preserved by $\tilde{\sigma}$ depends on the isomorphism $\sigma : \mathrm{Gr}_G \to \mathrm{Gr}_G$, which ultimately depends on the pinning on G. It is for this reason that the identification $\widetilde{G} \cong \widehat{G}$ is not canonical.

To begin with, consider that $\widetilde{G}$ has a natural choice of maximal torus and Borel $\widetilde{T} \subset \widetilde{B} \subset \widetilde{G}$, implying that identification with $\widehat{G}$ may only vary by conjugation by $\widetilde{T}$. Indeed, let $\widetilde{T} \subset \widetilde{G}$ consist of those fiber functor automorphisms that preserve weight spaces of all representations, and $\widetilde{B} \subset \widetilde{G}$ consist of those fiber functor automorphisms that preserve the positive cone of weight spaces. In particular, $\widetilde{B}$ is generated by $\widetilde{T}$ and $\widetilde{U}_{\alpha^\vee}$ for $\alpha^\vee \in \Phi^{\vee,+}$, where we have $\widetilde{U}_{\alpha^\vee}$ defined by the property

$$g(F_\lambda IC_\mu) \subset F_{\lambda+\alpha^\vee} IC_\mu. \tag{2.14}$$

Since the action of $\tilde{\sigma}$ on V_μ permutes weight spaces according to the action of σ on $X_*(T)$, the group automorphism $\tilde{\sigma}$ preserves $\widetilde{T}$ and $\widetilde{B}$. And $\widetilde{T}$ can be identified with $\widehat{T}$, such that $X_*(\widetilde{T}) = X_*(\widehat{T})$. In particular, $\tilde{\sigma}|_{\widetilde{T}} = \hat{\sigma}|_{\widehat{T}}$. Similarly, $\tilde{\sigma}$ permutes root subgroups according to the action of σ on $\Phi^\vee$. Indeed, equations (2.13) and (2.14) imply that $g \in \widetilde{U}_{\alpha^\vee}$ maps to $\tilde{\sigma}(g) \in \widetilde{U}_{\sigma(\alpha^\vee)}$.

It remains to be seen that there exists a pinning $\{x_{\alpha_i^\vee}\}_{\alpha_i^\vee \in \Pi^\vee}$ preserved by $\tilde{\sigma}$. However, if such a pinning exists, in each σ-orbit $\eta \subset \Pi^\vee$, one root homomorphism $x_{\alpha_i^\vee} : \mathbb{G}_a \to \widetilde{U}_{\alpha_i^\vee}$ may be determined arbitrarily, with $x_{\sigma(\alpha_i^\vee)} := \tilde{\sigma} \circ x_{\alpha_i^\vee}$. In the case σ acts freely on a simple root, i.e. $|\eta|$ is equal to the order of σ, there is no further obstruction: an arbitrary choice of pinning for one $\alpha_i^\vee \in \eta$ will determine a root homomorphism respected by $\tilde{\sigma}$ for all $\alpha_j^\vee \in \eta$. However, if $\sigma^n(\alpha_i^\vee) = \alpha_i^\vee$ for some n less than the order of σ, we need to know σ^n preserves $\widetilde{U}_{\alpha_i^\vee}$ pointwise. It is sufficient to consider the case $n = 1$, as in the following lemma of Hong.

Lemma 2.11 ([15] Lemma 4.3)**.** Suppose some simple $\alpha_i^\vee \in \Pi^\vee$ is σ-invariant, i.e. $\sigma(\alpha_i^\vee) = \alpha_i^\vee$. Then $\tilde{\sigma}$ fixes $\widetilde{U}_{\alpha_i^\vee}$ pointwise.

Proof. We may assume $\widetilde{G}$ is semisimple and almost simple. Indeed, $\widetilde{G}^{der} = \widetilde{G}_1 \times \cdots \times \widetilde{G}_m$ where each $\widetilde{G}_j$ is semisimple and almost simple, and the inclusion $\widetilde{U}_{\alpha_i^\vee} \hookrightarrow \widetilde{G}$ factors through some $\widetilde{G}_j \to \widetilde{G}^{der} \to \widetilde{G}$. If σ preserves $\alpha_i^\vee$, this inclusion commutes with $\tilde{\sigma}$. So suppose $\widetilde{G} = \widetilde{G}_j$.

We compare two different actions of σ on $\tilde{\mathfrak{g}}$, the Lie algebra of $\widetilde{G}$. The first, $d\tilde{\sigma}$, comes from differentiating the automorphism $\tilde{\sigma}$ at the identity. The second, $\tilde{\sigma}_{\gamma^\vee}$, comes from viewing $\tilde{\mathfrak{g}}$ as the representation $V_{\gamma^\vee}$, where $\gamma^\vee \in \Phi^\vee$ is the highest coroot. As noted earlier, $\tilde{\sigma}_{\gamma^\vee}$ fixes the weight space $F_{\alpha_i^\vee} IC_{\gamma^\vee} = \tilde{\mathfrak{g}}_{\alpha_i^\vee}$ pointwise, so it is sufficient to prove that $d\tilde{\sigma} = \tilde{\sigma}_{\gamma^\vee}$, as tangent space isomorphisms, implying the automorphism $\tilde{\sigma}$ fixes $\widetilde{U}_{\alpha_i^\vee}$ pointwise.

For each $\alpha^\vee \in \Phi^\vee$, let $e_{\alpha^\vee}$ be the fundamental class of the (unique) MV cycle of coweight $(\alpha^\vee, \gamma^\vee)$ in Gr_G. Note that for σ-invariant $\alpha^\vee$, we have $\tilde{\sigma}_{\gamma^\vee}(e_{\alpha^\vee}) = e_{\alpha^\vee}$. We can also identify $\tilde{\mathfrak{h}} = Lie(\widetilde{T})$ with $X_*(\widetilde{T}) \otimes_{\mathbb{Z}} \mathbb{C}$, to understand the action of $d\tilde{\sigma}$ on $\tilde{\mathfrak{h}}$.

Schur's lemma implies that the two maps may only differ by a constant scalar: let $d\tilde{\sigma} = c \cdot \tilde{\sigma}_{\gamma^\vee}$. Furthermore, by commutativity of diagram (2.13), differentiating the adjoint action of $\widetilde{G}$ on $\tilde{\mathfrak{g}}$, we have $\tilde{\sigma}_{\gamma^\vee}([a, b]) = [d\tilde{\sigma}(a), \tilde{\sigma}_{\gamma^\vee}(b)]$.

The highest root $\gamma \in \Phi$ is σ-invariant, so $\tilde{\sigma}_{\gamma^\vee}$ fixes the image of $\gamma : \mathbb{G}_m \to \widetilde{T}$ pointwise and $d\tilde{\sigma}$ fixes $\mathbb{C} \cdot \gamma \subset \tilde{\mathfrak{h}}$ as well. So $d\tilde{\sigma}([e_{\gamma^\vee}, e_{-\gamma^\vee}]) = [e_{\gamma^\vee}, e_{-\gamma^\vee}] \in \mathbb{C} \cdot \gamma$. Since $d\tilde{\sigma}$ is a Lie algebra homomorphism, we also have

$$d\tilde{\sigma}([e_{\gamma^\vee}, e_{-\gamma^\vee}]) = [d\tilde{\sigma}(e_{\gamma^\vee}), d\tilde{\sigma}(e_{-\gamma^\vee})] = c^2 \cdot [\tilde{\sigma}_{\gamma^\vee}(e_{\gamma^\vee}), \tilde{\sigma}_{\gamma^\vee}(e_{-\gamma^\vee})] = c^2 \cdot [e_{\gamma^\vee}, e_{-\gamma^\vee}].$$

The last equality holds by σ-invariance of $\gamma^\vee$ and $-\gamma^\vee$. And so $c^2 = 1$, and we must have $c = \pm 1$.

Now by comparing trace of $\tilde{\sigma}_{\gamma^\vee}$ and $d\tilde{\sigma}$ on $\tilde{\mathfrak{h}}$, we see $c \neq -1$. In particular, $\tilde{\sigma}_{\gamma^\vee}$ preserves $\tilde{\mathfrak{h}} = F_0 IC_{\gamma^\vee}$, so equation (2.1) implies that $\operatorname{tr}(\tilde{\sigma} \mid \tilde{\mathfrak{h}}) \geq 0$. Similarly, since σ acts on $X^*(T) = X_*(\widetilde{T})$ by permutation of characters forming a basis, we have $\operatorname{tr}(d\tilde{\sigma} \mid X_*(\widetilde{T}) \otimes_{\mathbb{Z}} K) \geq 0$ as well. But by assumption, there is a σ-invariant simple coroot $\alpha_i^\vee$, and so $\operatorname{tr}(d\tilde{\sigma} \mid X_*(\widetilde{T}) \otimes_{\mathbb{Z}} K) > 0$. So the ratio of those two traces, $c = \operatorname{tr}(d\tilde{\sigma}) / \operatorname{tr}(\tilde{\sigma}_{\gamma^\vee})$, must be nonnegative, hence $c = 1$. □

2.7 Root data and an application

Here I will be explicit about the root datum and pinning of $\widehat{G}$ and $\widehat{G^{\sigma,\circ}}$, both in terms of the pinned root datum tuple, and more simply the root system. I will also prove Theorem 7.7 of [11], which was originally stated without proof.

Suppose G is a quasi-split, connected, reductive group over a non-Archimedean local field F. Then let Σ be the root system of G and let $\breve{\Sigma}$ be the échelonnage root system of $G_{\breve{F}}$, as defined in [11]. By Corollary 5.3 in [11], $\breve{\Sigma}^\vee$ is the root system for $\widehat{G}^{I,\circ}$, where I is the inertia group of F. Then, in light of Theorem 2.0.1, to prove Theorem 7.7 of [11] it is sufficient to prove that $N'_\tau(\breve{\Sigma}^\vee)$ is equal to the set of roots of $\widehat{\widehat{G}^{I,\circ}}^{\tau,\circ}$.

We will work with pinned root data. According to the classification of connected reductive groups (see, for instance, [30]), a group H over an algebraically closed field is determined up to isomorphism by its root datum: a quadruple $(X, \Phi, X^\vee, \Phi^\vee)$, where X and $X^\vee$ are dual, finitely generated, free abelian groups; and $\Phi \subset X$ and $\Phi^\vee \subset X^\vee$ are dual reduced root systems. The group H is determined up to inner automorphism if the root datum is associated to a particular choice of maximal torus $T \subset H$. If the root datum is based, or endowed with a system of simple roots and coroots $\Pi \subset \Phi$ and $\Pi^\vee \subset \Phi^\vee$ corresponding to a choice of Borel $T \subset B \subset H$, then the datum determines H up to inner automorphism by an element of T. Since the systems of roots and coroots can be constructed from the quadruple $(X, \Pi, X^\vee, \Pi^\vee)$, there is no need to give the entire 6-tuple. Finally, H is determined up to unique automorphism by a pinning: a collection of root homomorphisms $x_{\alpha_i} : \mathbb{G}_a \to SL_2 \to H$ for $\alpha_i \in \Pi$.

When we say σ preserves a pinning of a connected reductive group H over an algebraically closed field K, we mean that if H has pinned root datum $(X, \Pi, X^\vee, \Pi^\vee, \{x_{\alpha_i}\})$, then σ preserves $T \subset B \subset H$, and $\sigma \circ x_{\alpha_i} = x_{\sigma(\alpha_i)}$ for each $\alpha_i \in \Pi$. Then σ also acts on X, Π, $X^\vee$, and $\Pi^\vee$, preserving Dynkin diagram edges and abelian group structure.

Proposition 2.7.1. *Let G be a complex, connected, reductive group with pinning $T \subset B \subset$*

G and $\{x_{\alpha_i}\}_{\alpha_i\in\Pi}$. Let the corresponding pinned root datum be denoted $(X, \Pi, X^\vee, \Pi^\vee, \{x_{\alpha_i}\})$. Let σ be an automorphism of G preserving its pinning. Then the fixed point subgroup $G^\sigma \subset G$ is a closed subgroup, and is reductive. The neutral component $G^{\sigma,\circ} \subset G^\sigma \subset G$ is also a closed subgroup and a connected, reductive group. The root datum of $G^{\sigma,\circ}$ is $(X_\sigma/tor, \operatorname{res}_\sigma(\Pi), (X^\vee)^\sigma, N'_\sigma(\Pi^\vee), \{x_{\alpha_\eta}\})$, where

- *i. X_σ is the group of σ-coinvariants $X/\langle x - \sigma(x)\rangle$. X_σ/tor is the quotient by all torsion elements.*
- *ii. For every orbit of simple roots $\eta \subset \Pi$, there is a single root α_η equal to the image of any $\alpha_i \in \eta$ under the quotient map $X \to X_\sigma/tor$. Then $\operatorname{res}_\sigma(\Pi) = \{\alpha_\eta \mid \eta \in \Pi/\sigma\}$.*
- *iii. $(X^\vee)^\sigma \subset X^\vee$ is the subgroup of σ-invariant cocharacters.*
- *iv. For every σ-orbit $\eta \subset \Pi^\vee$, we have $\alpha_\eta^\vee = \sum_{\alpha_i^\vee\in\eta} \alpha_i^\vee$ in the case η consists of pairwise disconnected simple roots, and $\alpha_\eta^\vee = 2(\sum_{\alpha_i^\vee\in\eta}\alpha_i^\vee)$ in the case η consists of a pair of simple roots connected by an edge. Then $N'_\sigma(\Pi^\vee) = \{\alpha_\eta^\vee \mid \eta \in \Pi^\vee/\sigma\}$.*

Proof. See [32] chapters 7–8. See also [12] and [11]. □

Note that if $\pi : X \to X_\sigma/tor$ and we have a character $\lambda \in X_\sigma/tor$ and cocharacter $\mu \in (X^\vee)^\sigma$, then $\langle\lambda, \mu\rangle$ is given by $\langle\tilde{\lambda}, \mu\rangle$, where $\tilde{\lambda}$ is any lift $\pi(\tilde{\lambda}) = \lambda$.

Proposition 2.7.2. *The group of σ-invariants W^σ is the Coxeter group generated by s_η, where for each orbit η of simple roots, s_η is the longest element of the group generated by simple reflections in η.*

Furthermore, W^σ is the Weyl group for $G^{\sigma,\circ}$, i.e. $W^\sigma = N_G(T^{\sigma,\circ})/T^{\sigma,\circ}$.

Proof. See [32] and [12]. ☐

We define $\operatorname{res}_\sigma(\Phi)$ and $N'_\sigma(\Phi^\vee)$ for the root system and coroot system of $G^{\sigma,\circ}$ as follows:

$$\operatorname{res}_\sigma(\Phi) := W^\sigma \cdot \operatorname{res}_\sigma(\Pi), \qquad \text{and} \qquad N'_\sigma(\Phi^\vee) := W^\sigma \cdot N'_\sigma(\Pi^\vee).$$

Proposition 2.7.3. *The half sum of positive coroots is equal for G and $G^{\sigma,\circ}$: $\rho^\vee = \rho^\vee_\sigma$.*

Proof. Recall $\rho^\vee$ is the sum of fundamental coweights, or dual vectors to the simple roots under the natural perfect pairing $\langle\cdot,\cdot\rangle$. For $\alpha_i \in \Pi$, let λ^i be the corresponding fundamental coweight. Similarly, for $\alpha_\eta \in \operatorname{res}_\sigma(\Pi)$, let λ^η be the corresponding fundamental coweight. It is sufficient to show that for each η,

$$\lambda^\eta = \sum_{i\in\eta} \lambda^i.$$

For any two orbits ζ and η,

$$\langle \alpha_\zeta, \sum_{i\in\eta} \lambda^i\rangle = \sum_{i\in\eta} \langle \alpha_j, \lambda^i\rangle = \delta_{\zeta,\eta},$$

where α_j is any of the simple roots mapping to α_ζ. This works regardless of type of orbits, since $\operatorname{res}_\sigma$ treats all simple roots uniformly. ☐

Note that $\rho_\sigma \neq \rho$. In particular, if λ is a σ-invariant cocharacter, then in general $|\langle \rho_\sigma, \lambda\rangle| \leq |\langle \rho, \lambda\rangle|$. As a result, by Theorem 2.2.1 i., $\dim(A^\sigma) \leq \dim A$ for a σ-invariant AMV cycle A.

Now we can prove the following theorem, which appears in [11] as Theorem 7.7, although it is not proved in full generality there.

Theorem 2.7.4. *Let G be a quasi-split, connected reductive group over a non-Archimedean local field F with inertia group I and geometric Frobenius τ. Let $\widehat{G}$ be the complex dual of G, and let $\widehat{G}^I$ be the fixed point subgroup of $\widehat{G}$, and $\widehat{G}^{I,\circ}$ the neutral component. Let the root system of $\widehat{G}^{I,\circ}$ be denoted $\breve{\Sigma}^\vee$. Then τ is an outer automorphism of $\widehat{G}^{I,\circ}$ preserving the natural pinning. Let $V_{\lambda,\xi}$ be the highest-weight representation of $\widehat{G}^{I,\circ} \rtimes \langle \tau \rangle$ where τ acts by the scalar $\xi \in \mathbb{C}^\times$ on weight spaces associated to weights $\nu \in W^\tau \cdot \lambda$.*

Let $\lambda \in X^*(\widehat{T}^{I,\circ})^{+,\tau}$ be a dominant, τ-invariant character of $\widehat{T}^{I,\circ}$. There is an equality

$$\sum_{\nu \in Wt(\lambda)^\tau} \operatorname{tr}(\tau \mid V_{\lambda,1}(\nu)) e^\nu = \sum_{w \in W^\tau} w \left(\prod_{\alpha \in N'_\tau(\breve{\Sigma}^\vee)^+} \frac{1}{1 - e^{-\alpha}} \right) e^{w(\lambda)}.$$

Proof. The group $\widehat{G}^{I,\circ}$ is connected and reductive. And since G is quasi-split over F, τ acts on $\widehat{G}^{I,\circ}$, preserving the natural pinning. Then the theorem follows from Theorem 2.0.1 by two observations. First, $V_{\lambda,1} = V_\lambda$ as a vector space and carries the same normalized τ-action as that described in Section 2.6.

And second, $N'_\tau(\breve{\Sigma}^\vee)$ is the set of roots of $\widehat{\widehat{\widehat{G}^{I,\circ}}^{\tau,\circ}}$. Indeed, $\breve{\Sigma}^\vee$ is the set of roots of $\widehat{G}^{I,\circ}$, so it is the set of coroots of the complex dual group $\widehat{\widehat{G}^{I,\circ}}$. Then by Proposition 2.7.1 iv., $N'_\tau(\breve{\Sigma}^\vee)$ is the set of coroots of the neutral component of the τ-fixed subgroup $\widehat{\widehat{G}^{I,\circ}}^{\tau,\circ} \subset \widehat{\widehat{G}^{I,\circ}}$. And finally, by again taking complex dual, $N'_\tau(\breve{\Sigma}^\vee)$ is the set of roots of $\widehat{\widehat{\widehat{G}^{I,\circ}}^{\tau,\circ}}$. □

Note that in Theorem 2.7.4, G is assumed to be quasi-split over a non-Archimedean local field. In particular, G may be ramified. Then the root system $\breve{\Sigma}^\vee$ for $\widehat{G}^{I,\circ}$ may be determined combinatorially from the absolute root datum of G, along with the action of the Galois group. In particular, as shown in [11] Theorem 6.8, $N'_\tau(\breve{\Sigma}^\vee)$ is equal to the root system $\widetilde{\Sigma}_0^\vee$ appearing in the Lusztig character formula of [18]. This is a necessary ingredient in the proof of Theorem D in [11].

Chapter 3: Cellular Paving of Convolution Fibers for Quasi-Split Groups

Let G be a quasi-split, connected, reductive group over a Laurent series field $K = k(\!(t)\!)$. Let W be the Iwahori–Weyl group of $LG(k) = G(k(\!(t)\!))$, and for each p-tuple $w_\bullet = (w_1, \ldots, w_p) \in W^p$ and choice of standard parahoric subgroup $\mathscr{P}_{\mathbf{f}} \subset LG(k)$ corresponding to a facet $\mathbf{f}$ in an apartment of G consider the convolution morphism

$$m_{w_\bullet,\mathbf{f}} : X_{w_\bullet,\mathscr{P}_{\mathbf{f}}} := X_{w_1,\mathscr{P}_{\mathbf{f}}} \tilde{\times} \cdots \tilde{\times} X_{w_p,\mathscr{P}_{\mathbf{f}}} \to X_{w_*,\mathbf{f}}$$

defined on the twisted product of Schubert varieties $X_{w_i,\mathbf{f}} \subset \mathscr{F}_{\mathbf{f}}$ (see §3.2). Such morphisms have long played an important role in the geometric Langlands program and in the study of the geometry of Schubert varieties. For example, if $w_\bullet = (s_1, ..., s_p)$ is a sequence of simple affine reflections, $w = s_1 \cdots s_p$ is a reduced word, and $\mathbf{f}$ is the base alcove $\mathbf{a}$, then $X_{s_\bullet,\mathbf{a}} \to X_{w,\mathbf{a}}$ is the Demazure resolution of singularities of $X_{w,\mathbf{a}}$. If $\mathbf{f}$ is a (minimal) facet containing an absolutely special vertex with corresponding special maximal parahoric group scheme $\mathscr{G}$ and $\mathscr{P}_{\mathbf{f}} =: L^+\mathscr{G}$ is a (twisted) positive loop group and $w_\bullet = \mu_\bullet = (\mu_1, \ldots, \mu_r)$ is a sequence of cocharacters of G, the corresponding convolution morphism is used to define the convolution of $L^+\mathscr{G}$-equivariant perverse sheaves on the affine Grassmannian $\mathrm{Gr}_G = LG/L^+\mathscr{G}$, and hence it plays a key role in the twisted geometric Satake correspondence.

It is common to study these morphisms and other aspects of the geometry of partial affine flag varieties in the context of split groups. However, for many purposes, including the study of Shimura varieties, local models, and the geometric Langlands conjectures, it is essential to understand the geometry of these spaces in a more general context. The context adopted in this paper will be that of a connected, tamely ramified, quasi-split, reductive group G defined over a Laurent series field $K = k(\!(t)\!)$ where the residue field k is either algebraically closed or finite. If k is finite of order q and $G_{ad,\tilde{K}}$ contains a factor of type D_4, we also assume either $3 \mid q$ or $3 \mid q-1$.

As previously stated, there are many applications of the geometric study of the fibers of these convolution morphisms. The purpose of this paper is to establish a suitable paving of these fibers by products of affine spaces and punctured affine spaces. This generalizes a result of Haines [10], proved in the case G is a split group defined over k.

Fix a be a quasi-split, tamely ramified, connected, reductive group G over a field $K = k(\!(t)\!)$, where k is either algebraically closed or finite. If k is finite of order q and the split adjoint form of G has a factor of type D_4, we also assume that either $3 \mid q$ or $3 \mid q-1$. Let $\mathscr{O} := k[\![t]\!] \subset K$. Let $S \subset G$ be a maximal K-split torus and let T be the centralizer of S in G, which is a maximal torus as G is quasi-split over K. Let $\mathscr{I} \subset G(K)$ be an Iwahori subgroup containing $\mathscr{T}(\mathscr{O}) \subset T(K)$, where $\mathscr{T}$ is the neutral component of the lft Néron model for T. In the Euclidean building $\mathscr{B}(G, K)$ let $\mathscr{A}$ be the fundamental apartment preserved by $N_G(S)(K)$ and let $\mathbf{a} \subset \mathscr{A}$ be the fundamental alcove preserved by $\mathscr{I}$, with absolutely special origin $\mathbf{0} \in \mathscr{A}$ in the closure of $\mathbf{a}$.

For each facet $\mathbf{f}$ contained in the fundamental alcove $\mathbf{a}$ we have an associated parahoric subgroup $\mathscr{P}_{\mathbf{f}} \subset G(K)$ preserving $\mathbf{f}$, and $\mathscr{I} \subset \mathscr{P}_{\mathbf{f}}$. Then $\mathscr{P}_{\mathbf{f}}$ defines an algebraic group scheme over $\mathscr{O}$, also referred to as $\mathscr{P}_{\mathbf{f}}$. The schemes G and $\mathscr{P}_{\mathbf{f}}$ also define functors in groups over k: let $LG(R) = G(R((t)))$ and let $L^+\mathscr{P}_{\mathbf{f}}(R) = \mathscr{P}_{\mathbf{f}}(R[\![t]\!])$ for a k-algebra R. The functor LG is represented by an ind-affine ind-scheme over k and the functor $L^+\mathscr{P}_{\mathbf{f}}$ is represented by an affine scheme over k; thus both are étale sheaves over Aff_k. Let the partial affine flag variety $\mathscr{F}_{\mathbf{f}}$ be the étale sheafification of the functor $R \mapsto LG(R)/L^+\mathscr{P}_{\mathbf{f}}(R)$, which is represented by an ind-projective ind-scheme over k. This is shown explicitly in [27, Theorem A] for the case k is algebraically closed, though it is pointed out there that general case is proved by the exhaustion of the reduced structure of $\mathscr{F}_{\mathbf{f}}$ by Schubert varieties, which is shown in [24, §8].

We have a stratification of the (reduced) partial affine flag variety $\mathscr{F}_{\mathbf{f}}$ by $L^+\mathscr{P}_{\mathbf{f}}$-orbits as follows. Let W be the Iwahori–Weyl group for G over K, which has a natural action on the fundamental apartment $\mathscr{A}$. Then W is a quasi-Coxeter group with a Bruhat partial order $\leq$ inherited from the Coxeter subgroup W_{af} generated by the set of simple reflections S_{af} (see §3.2 for a defintion). For an alcove $\mathbf{f}$ in the closure of $\mathbf{a}$ let $W_{\mathbf{f}} \subset W_{\mathrm{af}}$ be the subgroup fixing $\mathbf{f}$ pointwise. Then the $\mathscr{P}_{\mathbf{f}}$-orbits are referred to as Schubert cells and are in one-to-one correspondence with the double coset space $W_{\mathbf{f}}\backslash W/W_{\mathbf{f}}$, which also inherits the Bruhat order $\leq$. Given $w \in W_{\mathbf{f}}\backslash W/W_{\mathbf{f}}$, let $\dot{w} \in LG(k)$ be a lift. Then let $Y_w = L^+\mathscr{P}_{\mathbf{f}} \cdot \dot{w}e_{\mathbf{f}}$ where $e_{\mathbf{f}}$ is the natural basepoint corresponding to the identity coset $L^+\mathscr{P}_{\mathbf{f}} \in \mathscr{F}_{\mathbf{f}}(k)$. The reduced closure of Y_w is the Schubert variety X_w corresponding to w, and provides the following stratification:

$$X_w := \overline{Y_w} = \coprod_{\substack{v \in W_{\mathbf{f}}\backslash W/W_{\mathbf{f}} \\ v \leq w}} Y_v$$

Then we have the convolution morphism $m_{w_\bullet}$ defined on sequences $w_\bullet = (w_1, \ldots, w_p) \in$

$(W_{\mathbf{f}}\backslash W/W_{\mathbf{f}})^p$:

$$m_{w_\bullet} : X_{w_1} \tilde{\times} \cdots \tilde{\times} X_{w_p} \to X_{{}^{\mathbf{f}}w_*^{\mathbf{f}}}$$

where ${}^{\mathbf{f}}w_*^{\mathbf{f}}$ is the $\mathbf{f}$-Demazure product ${}^{\mathbf{f}}w_1^{\mathbf{f}} * \cdots * {}^{\mathbf{f}}w_p^{\mathbf{f}}$, where each ${}^{\mathbf{f}}w_i^{\mathbf{f}}$ is the maximal representative in W of w_i. See [7, §4.2] for a discussion of this operation, and proof that it is well-defined on $W_{\mathbf{f}}\backslash W/W_{\mathbf{f}}$. In the case $\mathbf{f} = \mathbf{a}$ this is equal to the Demazure product w_*. The $\bar{k}$-points of the twisted product in the domain of $m_{w_\bullet}$ are p-tuples $(g_1 e_{\mathbf{f}}, \dots, g_p e_{\mathbf{f}}) \in \mathscr{F}_{\mathbf{f}}(\bar{k})$ such that $g_{i-1}^{-1} g_i e_{\mathbf{f}} \in X_{w_i}$ for each $1 \leq i \leq p$, taking g_0 to be the identity. Then $m_{w_\bullet}$ is simply the projection onto the last component:

$$m_{w_\bullet}(g_1 e_{\mathbf{f}}, \dots, g_p e_{\mathbf{f}}) = g_p e_{\mathbf{f}}.$$

There is a similar morphism to the above, but with restriction of the domain to twisted products of Schubert cells, which I will use to make a more general statement. On $\bar{k}$-points, we have

$$Y_{w_1} \tilde{\times} \cdots \tilde{\times} Y_{w_p} = \{(g_1 e_{\mathbf{f}}, \dots, g_p e_{\mathbf{f}}) \mid g_{i-1}^{-1} g_i e_{\mathbf{f}} \in Y_{w_i} \text{ for each } i\}.$$

Then let $m_{w_\bullet}^{\circ}$ be the restriction of $m_{w_\bullet}$ to $Y_{w_1} \tilde{\times} \cdots \tilde{\times} Y_{w_p}$.

Let us discuss what is meant by paving. Let $\mathscr{C}$ be a class of schemes over k. Then a k-scheme X is paved by varieties in $\mathscr{C}$, or simply paved by $\mathscr{C}$, if there is an exhaustion of X by closed subschemes $X_0 \subset \cdots \subset X_p$ such that each $X_p - X_{p-1}$ is contained in $\mathscr{C}$.

Now we can state the main theorems of the paper:

Theorem 3.0.1. *Let G be a tamely ramified, quasi-split, connected, reductive group over*

K. Let $s_\bullet = (s_1, \ldots, s_p)$ be a tuple of simple reflections $s_i \in S_{\mathrm{af}}$. Then for a point $x \in X_{s_*}$ in the image of the convolution morphism of Schubert varieties in $\mathscr{F}_{\mathbf{a}}$,

$$m_{s_\bullet} : X_{s_1} \tilde{\times} \cdots \tilde{\times} X_{s_p} \to X_{s_*},$$

the fiber $m_{s_\bullet}^{-1}(x)$ is paved by finite-dimensional affine spaces.

More generally, we will not always be able to say a convolution fiber is paved by affine spaces, but rather by spaces that are finite products of affine spaces $\mathbb{A}^n$ as well as punctured affine spaces $\mathbb{A}^n - \mathbb{A}^0$, where $n \leq 3$. Let $\mathscr{C}$ be the class of such spaces.

Theorem 3.0.2. *Let G be a group as in Theorem 3.0.1, and fix a facet $\mathbf{f} \subset \mathscr{A}$. Let $w_\bullet = (w_1, \ldots, w_p)$ be a tuple of elements of $W_{\mathbf{f}} \backslash W / W_{\mathbf{f}}$. Then for a point $x \in X_{\mathbf{f} w_*^{\mathbf{f}}, \mathbf{f}}$ in the image of the convolution morphism of Schubert varieties*

$$m_{w_\bullet} : X_{w_1, \mathbf{f}} \tilde{\times} \cdots \tilde{\times} X_{w_p, \mathbf{f}} \to X_{\mathbf{f} w_*^{\mathbf{f}}, \mathbf{f}},$$

the fiber $m_{w_\bullet}^{-1}(x)$ is paved by $\mathscr{C}$.

As a result, in the case $k = \mathbb{F}_q$ we will find formulas for structure constants of parahoric Hecke algebras. In particular it will be immediately clear that they are polynomials in q and $q - 1$ with nonnegative integer coefficients.

3.1 Summary of approach

The paper proceeds as follows. First, we will make necessary definitions, constructions, and conventions. This includes references for construction of various group schemes associated to G, especially the root subgroups and affine root subgroups; the Iwahori–Weyl group W, and two semidirect product decompositions; the Bruhat–Tits building and fundamental apartment; parahoric subgroups and parahoric loop group functors; and partial affine flag varieties with stratifications by Schubert cells. This is followed by a detailed description of the isomorphism classes of root groups and affine root groups. Then the branching of the Bruhat–Tits building at codimension 1 facets is described in terms of k-points of affine root groups. Since the k-points of fibers of convolution morphisms can be modeled by certain galleries in the building, understanding these branching points is essential to the proof of Theorems. 3.0.1 and 3.0.2.

Then an essential lemma for the inductive step, Lemma 3.3.6 is proved. First an Iwahori-type decomposition of the pro-unipotent Iwahori functor is proved, so it is a direct product of affine root groups and a certain other functor in groups over k. Then that decomposition is used to prove a certain triviality property called "stratified triviality."

Next the main theorems are proved. First Theorem 3.0.1 is proved inductively, with the help of stratified triviality. Given a sequence $s_\bullet = (s_1, \dots, s_p)$, we fix a point $ve_{\mathbf{a}}$ in the image of $m_{s_\bullet}$ and show inductively that the fiber over $ve_{\mathbf{a}}$ is paved by affine spaces. If we let $m_{s'_\bullet}$ be the convolution morphism defined by the sequence $s'_\bullet = (s_1, \dots, s_{p-1})$. We know that the $(p-1)$-component of any point in the fiber lying over $ve_{\mathbf{a}}$ must be contained in $Y_{vs_p,\mathbf{a}}$ or $Y_{v,\mathbf{a}}$. Then we fix a point x lying in the image of $m_{s'_\bullet}$ that may also be the $(p-1)$th component of the fiber lying over $ve_{\mathbf{a}}$. Analyzing cases, we see that the set of possible points in the image of $m_{s_\bullet}$ for given x is describe by one of the following possibilities: it is in correspondence with an affine root subgroup $\mathsf{U}_{\alpha+r}$ for some affine root $\alpha + r$, or consists of a single point, or is empty. By triviality of $m_{s_\bullet}$ over the intersection of its image with $Y_{v,\mathbf{a}}$, we can conclude that the fiber of $m_{s_\bullet}$ lying over $ve_{\mathbf{a}}$ is stratified by products of one of the above spaces and the fiber over x in the image of $m_{s'_\bullet}$, and thus paved by affine spaces as desired.

Next Theorem 3.0.2 is proved in a special case similar to the context of 3.0.1—we again assume that $s_\bullet$ is a length-p sequence of simple reflections, and again work with the facet $\mathbf{f} = \mathbf{a}$. The only difference between this case and the one before is that we work with the convolution morphism $m^\circ_{s_\bullet}$ for a twisted product of Schubert cells $Y_{s_\bullet}$ rather Schubert varieties, using the stratification of $X_{s_\bullet}$ by various twisted products of Schubert cells. This is necessary for the upcoming generalizations. The use of Schubert cells introduces some slight complication, but the case structure is very similar. Again we fix $ve_{\mathbf{a}}$ in the image of $m^\circ_{s_\bullet}$ and look at various cases of points in the image of $m^\circ_{s'_\bullet}$. However, in this case it is possible that for a given point in the image of $m^\circ_{s'_\bullet}$, the intersection of $Y_{v,\mathbf{a}}$ and the possible points in the image of $m^\circ_{s_\bullet}$ with given $(p-1)$th component are in correspondence with $\mathsf{U}_{\alpha+r}$, or a single point, or empty, or with $\mathsf{U}_{\alpha+r} - \{1\}$. We similarly conclude the paving result holds for $m^\circ_{s_\bullet}$ using induction and stratified triviality, but we need a larger class of varieties to describe the paving.

At this point, we can prove Theorem 3.0.2 in its general context in two separate deductions. First, we remove the assumption that $w_\bullet$ is a length-p sequence of simple reflections, using reduced words for the constituent Iwahori–Weyl group elements to view $Y_{w_\bullet}$ as a twisted product of a longer sequence of simple reflections $Y_{s_{\bullet,\bullet}}$. Second, we remove the assumption that $\mathbf{f} = \mathbf{a}$ and deal with partial affine flag varieties. This requires a lemma that relates Schubert cells in $\mathscr{F}_{\mathbf{f}}$ to those in $\mathscr{F}$. Given this, we can decompose the image of $m^\circ_{s_\bullet}$, into intersections with left $L^+\mathscr{I}$-orbits, where stratified triviality still holds. Thus we can apply our previous work, completing the proof of Theorem 3.0.2.

Finally, it is shown how this result can be applied to structure constants of parahoric Hecke algebras.

3.1.1 Leitfaden

The remaining sections are as follows: Section 3.2 establishes notations and conventions, as well as references to nontrivial constructions and detailed descriptions of root groups and affine root groups. Section 3.3 proves Lemma 3.3.6 as well as Proposition 3.3.5, which is a consequence of a result in [7]. Section 3.4 contains the inductive proofs of Theorems 3.0.1 and 3.0.2, as well as the case reductions used. Section 3.5 applies Theorem 3.0.2 to a result on structure constants for parahoric Hecke algebras.

3.2 Notation and basic constructions

3.2.1 General notation

Fix a field $K := k(\!(t)\!)$ where k is either algebraically closed or finite. Let the ring of integers of K be denoted $\mathscr{O} := k[\![t]\!]$. Normalize a valuation v_K on K such that $v_K(t) = 1$. Fix a connected, quasi-split reductive group scheme G over K and fix a minimal Galois field extension $\tilde{K}/K$ over which G splits. If k is finite of order q and the split adjoint form $G_{ad,\tilde{K}}$ contains a factor of type D_4, then we also assume either $3 \mid q$ or $3 \mid q-1$. Let the residue field of $\tilde{K}$ be denoted $\tilde{k}$, so that $\tilde{K} = \tilde{k}(\!(u)\!)$ for some uniformizer u. Let d be the index of $\tilde{K}$ over $\tilde{k}(\!(t)\!)$, so that we have a valuation $v_{\tilde{K}}$ on $\tilde{K}$ extending the valutaion on K, for which $v_{\tilde{K}}(u) = \frac{1}{d}$.

If k is finite, let $\breve{K} = \bar{k}(\!(t)\!)$ and $\breve{\mathscr{O}} = \bar{k}[\![t]\!]$ where $\bar{k}$ is the algebraic closure of k, and let $\tau \in \mathrm{Gal}(\breve{K}/K)$ be the Frobenius. If k is algebraically closed, then let $\breve{K} = K$, let $\breve{\mathscr{O}} = \mathscr{O}$, and let τ be the identity on K.

Where it reasonable to do so, the text style of the name of a group scheme will denote the base ring over which it is initially defined. That is, groups defined over K or algebraic extensions will have names in math italics, e.g. G, those defined over $\mathscr{O}$ or integral extensions will have script letters, e.g. $\mathscr{G}$, and those defined over k or algebraic extensions will have sans-serif letters, e.g. G.

Fix a maximal K-split torus $S \subset G$. Let T be the centralizer of S in G. Since G is quasi-split over K, T is a maximal torus. Choose a Borel $B \subset G$ defined over K and containing T. We also have an opposite borel B^- such that $B \cap B^- = T$, as well as maximal unipotent subgroups $U \lhd B$ and $U^- \lhd B^-$ such that $B = TU$ and $B^- = U^-T$.

Let $\Phi = \Phi(G, S)$ be the relative root system of G and let $\tilde{\Phi} = \Phi(G_{\tilde{K}}, T_{\tilde{K}})$ be the absolute root system. Corresponding to the choice of B we have sets of positive roots $\Phi^+ \subset \Phi$ and $\tilde{\Phi}^+ \subset \tilde{\Phi}$. We have, respectively, relative and absolute character lattices $X^*(S) \supset \Phi$ and $X^*(T_{\tilde{K}}) \supset \tilde{\Phi}$ as well as relative and absolute cocharacter lattices $X_*(S)$ and $X_*(T_{\tilde{K}})$. Completing the relative and absolute root data, we have relative and absolute coroot systems $\Phi^\vee \subset X_*(S)$ and $\tilde{\Phi}^\vee \subset X_*(T_{\tilde{K}})$.

The action of $\Gamma_G := \mathrm{Gal}(\tilde{K}/K)$ on $\tilde{K}$ induces an action on $G_{\tilde{K}}$. Recall we assume that if k is not algebraically closed and the split adjoint form of G has a factor of type D_4 k is not algebraically closed then either $3 \mid q$ or $3 \mid q - 1$. The primary reason for this is to ensure that Γ_G has a nice description. More specifically, for each component $\Phi_i \subset \Phi$ let Γ_i be the corresponding Galois group of the splitting subfield $K_i \subset \tilde{K}$. Then since G is quasi-split and tamely ramified, the assumption on q implies the extension K_i/K is cyclic of index at most 3. In particular Φ_i is not of type ${}^6D_{4,2}$ and $\mathrm{Gal}(K_i/K) \neq S_3$; see §3.2.2. Let $I \subset \Gamma_G$ be the inertia subgroup $I = \mathrm{Gal}(\tilde{K}/\tilde{k}(\!(t)\!))$. Then $I = \prod_i \langle \sigma_i \rangle$ where each σ_i has order at most 3, and $\Gamma_G = I \times \langle \bar{\tau} \rangle$ where $\bar{\tau}$ generates $\mathrm{Gal}(\tilde{k}/k)$ and has order dividing 6. If G is almost simple, then Γ_G is cyclic of order at most 3.

We have an action of Γ_G on the absolute root datum $(\tilde{\Phi}, X^*(T_{\tilde{K}}), \tilde{\Phi}^\vee, X_*(T_{\tilde{K}}))$ and on the ring $\tilde{k}[\![u]\!]$. See for instance [10] for a detailed description of the relation between the absolute and relative root data of G given the action of Γ_G. In particular, $X_*(S) = X_*(T_{\tilde{K}})^{\Gamma_G}$.

We have an isomorphism of K-groups [13] which we exploit:

$$G \cong \mathrm{Res}_{\tilde{K}/K}\left(G_{\tilde{K}}\right)^{\Gamma_G}. \tag{3.1}$$

Fix a Chevalley group H over $\mathbb{Z}$ with root datum equal to the absolute root datum of G, with maximal torus and Borel $T_H \subset B_H \subset H$ over $\mathbb{Z}$. Then fix an isomorphism of $\tilde{K}$-group schemes $\phi : G_{\tilde{K}} \xrightarrow{\sim} H_{\tilde{K}}$ such that the pullbacks along inclusion $\phi_B : B_{\tilde{K}} \to B_{H,\tilde{K}}$ and $\phi_T : T_{\tilde{K}} \to T_{H,\tilde{K}}$ are also isomorphisms. Use ϕ to transport the action of the group Γ_G to $H_{\tilde{K}}$. By (3.1) we have $G \cong (\mathrm{Res}_{\tilde{K}/K}(H_{\tilde{K}}))^{\Gamma_G}$. Define an affine group scheme $\breve{\mathscr{G}}$ over $\breve{\mathscr{O}}$ as

$$\breve{\mathscr{G}} := \mathrm{Res}_{\breve{k}[\![u]\!]/\breve{\mathscr{O}}}\left(H_{\breve{k}[\![u]\!]}\right)^{I,\circ}.$$

Then $\breve{\mathscr{G}}$ is a parahoric group scheme for $G_{\breve{K}}$ over $\breve{\mathscr{O}}$. Let $\mathscr{G} = \mathrm{Res}_{\breve{\mathscr{O}}/\mathscr{O}}(\breve{\mathscr{G}})^{\tau,\circ}$, and $\mathscr{G}$ is also a parahoric group scheme over $\mathscr{O}$; see [24, §2.a.1.] and note that $\mathscr{G}$ is smooth with geometrically connected fibers over $\mathscr{O}$, with generic fiber G. We have an injective homomorphism $\mathscr{G}(\mathscr{O}) \hookrightarrow G(K)$ of abstract groups determined by the Chevalley basis of H. Let $\mathscr{T}$ be the neutral component of the lft Néron model for T, an $\mathscr{O}$-group scheme with an injection on $\mathscr{O}$-points $\mathscr{T}(\mathscr{O}) \subset T(K)$, as constructed in [5] 4.4.

The fundamental apartment $\mathscr{A}$ is canonically endowed with the structure of the affine linear space underlying $(X_*(T)_I)^\tau \otimes_{\mathbb{Z}} \mathbb{R}$. The fundamental apartment carries a natural action of $N_G(S)(K)$ with stabilizer $\mathscr{T}(\mathscr{O})$ (see for instance [33] §§1.1–1.3 for an abstract description). Let the origin $\mathbf{0}$ be a point in $\mathscr{A}$ fixed by $N_G(S)(K) \cap \mathscr{G}(\mathscr{O})$; in the correspondence between parahoric group schemes and facets in the building, $\mathscr{G}$ is the parahoric group scheme corresponding to the minimal facet containing $\mathbf{0}$. We identify any point $x \in \mathscr{A}$ with the real vector $x - \mathbf{0} \in (X_*(T)_I)^\tau \otimes_{\mathbb{Z}} \mathbb{R}$. From now on we use this identification to treat $\mathscr{A}$ as a real vector space.

Let the group $N_G(S)(K)/\mathscr{T}(\mathscr{O})$ be denoted W. Since G is quasi-split over K, W is the Iwahori–Weyl group for G over K. For each $w \in W$ we fix a lift $\dot{w} \in N_G(S)(K)$ as follows. For w preserving $\mathbf{0}$, fix a lift $\dot{w} \in N_G(S)(K) \cap \mathscr{G}(\mathscr{O})$. We also have a subgroup of W acting by translation on $\mathscr{A}$, which is naturally indexed by the Frobenius invariants of the inertia coinvariants of the cocharacter lattice $X_*(T)$. For an element $\mu \in (X_*(T)_I)^\tau$ let the corresponding translation element be denoted $t_\mu \in W$. The Kottwitz homomorphism $\kappa_T : T(K) \to (X_*(T)_I)^\tau$ is surjective [20], so we may choose a lift $\dot{t}_\mu \in \kappa_T^{-1}(\mu) \subset T(K)$. We establish the convention that the action of t_μ on $\mathscr{A}$ is translation by $-\mu$ as in [33]. There is a semidirect product decomposition $W = W_{\mathbf{0}} \ltimes (X_*(T)_I)^\tau$, so we can lift each element $wt_\mu \in W$ as a product of two lifts in $G(K)$ in this way.

We say a root $\alpha \in \Phi$ is singular if the intersection $\mathbb{R}_{>0}\alpha \cap \Phi$ is equal to the singleton

$\{\alpha\}$, and we say α is plural otherwise. If all roots in (a component of) Φ are singular, we say (that component of) Φ is reduced. If α is plural, there are exactly two roots in the intersection $\mathbb{R}_{>0}\alpha \cap \Phi$: either α is divisible and $\mathbb{R}_{>0}\alpha \cap \Phi = \{\frac{1}{2}\alpha, \alpha\}$, or else α is non-divisible, and $\mathbb{R}_{>0}\alpha \cap \Phi = \{\alpha, 2\alpha\}$.

Corresponding to each root $\alpha \in \Phi$ we have a root subgroup $U_\alpha \subset G$ defined as the minimal closed subgroup normalized by S and such that $\mathrm{Lie}(U_\alpha)$ contains $\mathfrak{g}_\alpha$, the Lie subalgebra of $\mathrm{Lie}(G)$ consisting of elements such that $\mathrm{Lie}(S)$ acts via α. For $U_\alpha \subset G$, there are a few possible isomorphism classes for U_α depending on α, detailed in Section 3.2.2. For each $\alpha \in \Phi$ we have a filtration on U_α by valuation, where the valuation maps $v_\alpha : U_\alpha(K) \to \mathbb{R}$ are those defined in [33] §1.4 and described concretely in Section 3.2.2. Then for $r \in \mathbb{R}$ let $\mathscr{U}_{\alpha+r} \subset U(K)$ be the subgroup consisting of those elements such that $v_\alpha(x) \geq r$, and so for $s \geq r$ we have $\mathscr{U}_{\alpha+s} \subset \mathscr{U}_{\alpha+r}$. In particular, the valuation is normalized such that $\mathscr{U}_{\alpha+0} = U_\alpha(K) \cap \mathscr{U}(\mathscr{O})$. Then for each $\mathscr{U}_{\alpha+r}$ we may also construct a smooth $\mathscr{O}$-group scheme, also denoted $\mathscr{U}_{\alpha+r}$, with generic fiber equal to U_α and an embedding of $\mathscr{O}$-points $\mathscr{U}_{\alpha+r}(\mathscr{O}) \subset U_\alpha(K)$ [5, §4.3.2].

For a root α and real number r we also have an affine linear functional on the fundamental apartment $\mathscr{A} \cong X_*(S)_{\mathbb{R}}$ given by $(\alpha+r)(x) = \alpha(x)+r$. Then the set of affine roots $\Phi_{\mathrm{af}} = \Phi_{\mathrm{af}}(G, S)$ consists of those functionals $\alpha + r$ that are "break points" in the filtration of $U_\alpha(K)$ by $\mathscr{U}_{\alpha+r}(\mathscr{O})$ as in [33]. That is, if α is singular or divisible, then a functional $\alpha + r$ is an affine root if $\mathscr{U}_{\alpha+r+\epsilon}(\mathscr{O}) \subsetneq \mathscr{U}_{\alpha+r}(\mathscr{O})$ for all small $\epsilon > 0$. If α is plural and non-divisible, then $\alpha + r$ is an affine root if the group

$$\frac{\mathscr{U}_{\alpha+r}(\mathscr{O})/\mathscr{U}_{\alpha+r+\epsilon}(\mathscr{O})}{\mathscr{U}_{2\alpha+2r}(\mathscr{O})/\mathscr{U}_{2\alpha+2r+2\epsilon}(\mathscr{O})}$$

is nontrivial for all small ϵ. If $\alpha + r$ is an affine root, then for a k-algebra R we define an affine root group scheme $\mathsf{U}_{\alpha+r}$ on R-points as

$$\mathsf{U}_{\alpha+r}(R) := \mathscr{U}_{\alpha+r}(R[\![t]\!])/\mathscr{U}_{\alpha+r+\epsilon}(R[\![t]\!])$$

for any sufficiently small ϵ. It is always sufficient to let $\epsilon < \frac{1}{2[\breve{K}:K]}$ (see Proposition 3.2.5). Then $\mathsf{U}_{\alpha+r}$ is represented by a finite-type unipotent group scheme over k [33] §3.5.1, and since K is a Laurent series field, there is a natural injective map $\mathsf{U}_{\alpha+r}(k) \hookrightarrow \mathscr{U}_{\alpha+r}(\mathscr{O}) \subset U_\alpha(K)$. The existence of this map can be seen from the classification of the groups $\mathsf{U}_{\alpha+r}$ as in Proposition 3.2.6. The dimension of the unipotent group $\mathsf{U}_{\alpha+r}$ is an important constant in the description of the Bruhat–Tits building. Let it be denoted $d(\alpha + r) = \dim_k(\mathsf{U}_{\alpha+r})$. A detailed description of the affine root groups is given Section 3.2.2.

The Iwahori–Weyl group $W = N_G(S)(K)/\mathscr{T}(\mathscr{O})$ acts on the fundamental apartment such that for a lift $\dot{w} \in G(K)$ as described above we have

$$\dot{w}\mathscr{U}_{\alpha+r}(\mathscr{O})\dot{w}^{-1} = \mathscr{U}_{w(\alpha+r)}(\mathscr{O})$$

where $w(\alpha + r)(x) = (\alpha + r)(w^{-1}(x))$. For each affine root $\alpha + r \in \Phi_{\text{af}}$ the zero set of the functional $\alpha + r : \mathscr{A} \to \mathbb{R}$ is denoted $\mathbf{H}_{\alpha+r}$ and referred to as a wall in the fundamental apartment. The connected components of

$$\mathscr{A} - \left(\bigcup_{\alpha+r\in\Phi_{\text{af}}} \mathbf{H}_{\alpha+r} \right)$$

are the alcoves of the fundamental apartment. The unique alcove **a** such that **0** is contained in the closure of $\mathbf{a}$ and such that $\alpha(x) > 0$ for all $\alpha \in \Phi^+$ is referred to as the fundamental alcove and is denoted $\mathbf{a}$. The alcoves define a triangulation of $\mathscr{A}$ by polysimplices, where the alcoves are the facets of maximal dimension. The polysimplices of this triangulation are referred to as the facets of the fundamental apartment, usually referred to with $\mathbf{f}$.

A facet $\mathbf{f}$ in the fundamental apartment is the intersection of the closure of an alcove and some (finite, possibly empty) set of walls. Then $\mathscr{A}$ is the union of facets. For an affine root $\alpha + r \in \Phi_{\text{af}}$ and a facet $\mathbf{f}$ in $\mathscr{A}$, we say $\alpha + r \overset{\mathbf{f}}{>} 0$ if $\alpha + r$ takes positive values on all points $x \in \mathbf{f}$. For $\mathbf{f} = \mathbf{a}$, then we say $\alpha + r > 0$ if and only if $\alpha + r \overset{\mathbf{a}}{>} 0$.

For each affine root $\alpha + r \in \Phi_{\text{af}}$ there is a reflection $s_{\alpha+r} \in W$ defined by

$$s_{\alpha+r}(x) := x - (\alpha + r)(x)\alpha^{\vee},$$

i.e. the reflection of $\mathscr{A}$ across the wall $\mathbf{H}_{\alpha+r}$. Note that if $\alpha + r$, $\lambda\alpha + \lambda r \in \Phi_{\text{af}}$ for some $\lambda \in \mathbb{R}^{\times}$ then $(\lambda\alpha)^{\vee} = \frac{1}{\lambda}\alpha^{\vee}$ and so $s_{\alpha+r} = s_{\lambda\alpha+\lambda r}$. A reflection $s_{\alpha+r}$ is called simple if there is a codimension 1 facet $\mathbf{f}$ contained in the closure of $\mathbf{a}$ and the closure of $s_{\alpha+r}(\mathbf{a})$. The set of all simple reflections is denoted S_{af}. The subgroup of $W_{\text{af}} \subset W$ generated by S_{af} is a Coxeter group, called the affine Weyl group, and acts simply transitively on alcoves in $\mathscr{A}$.

Let the subgroup of W preserving $\mathbf{a}$ be denoted Ω. The group Ω is abelian, and $W = \Omega \ltimes W_{\text{af}}$. Thus W is a quasi-Coxeter group, and inherits a length function ℓ and Bruhat order $\leq$ from W_{af}. Indeed, let $\tau w, \tau' w' \in \Omega \ltimes W_{\text{af}}$. Then $\ell(\tau w) = \ell(w)$ and $\tau w \leq \tau' w'$ if and only if $\tau = \tau'$ and $w \leq w'$.

Given a K-group scheme X, let the loop group LX be the functor on Aff_k whose R-points are defined to be $LX(R) = X(R(\!(t)\!))$ for a k-algebra R. Given an $\mathscr{O}$-group scheme

$\mathscr{X}$, let the positive loop group $L^+\mathscr{X}$ be the functor on Aff_k whose R-points are defined to be $L^+\mathscr{X}(R) = \mathscr{X}(R[\![t]\!])$ for a k-algebra R.

The Bruhat–Tits building $\mathscr{B}(G, K)$ is a polysimplicial complex constructed in [4]. Then $\mathscr{B}(G, K)$ carries an action of $G(K)$ such that

$$\mathscr{B}(G, K) = \bigcup_{g \in G(K)} g\mathscr{A}$$

and such that, for any two alcoves $\mathbf{a}_1$, $\mathbf{a}_2$ there is some apartment $A = g\mathscr{A}$ such that $\mathbf{a}_1$ and $\mathbf{a}_2$ are contained in A. For a detailed description of the set of alcoves containing a codimension 1 facet $\mathbf{f}$ in the building, see Section 3.2.4.

Given a facet $\mathbf{f} \subset \mathscr{A}$ let $W_{\mathbf{f}} \subset W_{\mathrm{af}}$ be the subgroup preserving $\mathbf{f}$. If $\mathbf{f}$ is contained in the closure of $\mathbf{a}$, then $(W_{\mathbf{f}}, S_{\mathbf{f}})$ is the parabolic subgroup of $(W_{\mathrm{af}}, S_{\mathrm{af}})$ generated by the set $S_{\mathbf{f}} \subset S_{\mathrm{af}}$ of simple reflections preserving $\mathbf{f}$, which all fix $\mathbf{f}$ pointwise. Given a pair of facets $\mathbf{f}', \mathbf{f}$ in the closure of $\mathbf{a}$, the quotient spaces $W/W_{\mathbf{f}}$ and $W_{\mathbf{f}'}\backslash W/W_{\mathbf{f}}$ inherit the Bruhat order $\leq$. In the case $\mathbf{f_0}$ is the facet containing $\mathbf{0}$ we have $W_{\mathbf{0}} := W_{\mathbf{f_0}}$ is isomorphic the finite Weyl group described earlier, such that $W = W_{\mathbf{0}} \ltimes (X_*(T)_I)^\tau$.

For each facet $\mathbf{f} \subset \mathscr{A}$ we also have a parahoric subgroup $\mathscr{P}_{\mathbf{f}} \subset G(K)$ fixing $\mathbf{f}$, as well as a corresponding smooth $\mathscr{O}$-group scheme with geometrically connected fibers and generic fiber isomorphic to G, also denoted $\mathscr{P}_{\mathbf{f}}$; the parahoric subgroup is isomorphic to the group of $\mathscr{O}$-points of $\mathscr{P}_{\mathbf{f}}$. Note that the $\mathscr{O}$-group scheme $\mathscr{G}$ constructed above is the parahoric group scheme corresponding to $\mathbf{0}$. In fact, in general, $\mathscr{P}_{\mathbf{f}}$ is constructed as a fixed-point subgroup of a parahoric group scheme corresponding to H. As is shown in [14], the $\mathscr{O}$-points $\mathscr{P}_{\mathbf{f}}$ are exactly the points in $G(K)$ that fix $\mathbf{f}$ pointwise and are contained in the kernel of the Kottwitz homomorphism κ_G. The group scheme $\mathscr{P}_{\mathbf{f}}$ over $\mathscr{O}$ is constructed in general in [4]. Note that $\mathscr{B}(G, K) = \mathscr{B}(H_{\tilde{K}}, \tilde{K})^{\Gamma_G}$ [25] and for a facet $\mathbf{f} \subset \mathscr{B}(G, K)$ let $\mathbf{f}_H \subset \mathscr{B}(H_{\tilde{K}}, \tilde{K})$ be the facet containing $\mathbf{f}$; then $\mathbf{f} = \mathbf{f}_H^{\Gamma_G}$. Let $\mathscr{H}_{\mathbf{f}_H}$ be the parahoric group scheme over $\tilde{k}[\![u]\!]$ corresponding to $\mathbf{f}_H$. Then we have

$$\mathscr{P}_{\mathbf{f}} = (\mathrm{Res}_{\tilde{k}[\![u]\!]/\mathscr{O}}(\mathscr{H}_{\mathbf{f}_H}))^{\Gamma_G,\circ}. \tag{3.2}$$

Indeed, let $\breve{\mathscr{P}}_{\breve{\mathbf{f}}}$ be the parahoric group scheme associated to the facet $\breve{\mathbf{f}} \subset \mathscr{B}(G_{\breve{K}}, \breve{K})$ containing $\mathbf{f}$. Then $\breve{\mathscr{P}}_{\breve{\mathbf{f}}} = (\mathrm{Res}_{\bar{k}[\![u]\!]/\breve{\mathscr{O}}}((\mathscr{H}_{\mathbf{f}_H})_{\bar{k}[\![u]\!]}))^{I,\circ}$ by [24, Equation (7.1)], noting that in general it is necessary to take the neutral component if G is not simply connected. If k is algebraically closed, then $\breve{\mathscr{P}}_{\breve{\mathbf{f}}} = \mathscr{P}_{\mathbf{f}}$ and $\Gamma_G = I$ so (3.2) follows. If not, then $(\mathrm{Res}_{\breve{\mathscr{O}}/\mathscr{O}}(\breve{\mathscr{P}}_{\breve{\mathbf{f}}}))^{\tau}$ is how $\mathscr{P}_{\mathbf{f}}$ is originally constructed [5, §5.1.8]. Furthermore, if $d = [\tilde{k} : k]$ then the action of τ^d is trivial on G and on H and so taking I-fixed points of $(\mathscr{H}_{\mathbf{f}_H})_{\bar{k}[\![u]\!]}$ commutes with taking τ^d-fixed points.

Then we have as well the positive loop group functor $L^+\mathscr{P}_{\mathbf{f}}$ on Aff_k, which is represented by an affine group scheme over k, though it is not of finite type. Furthermore, from (3.2) we have

$$L^+\mathscr{P}_{\mathbf{f}} = (\mathrm{Res}_{\tilde{k}/k}(L^+\mathscr{H}_{\mathbf{f}_H}))^{\Gamma_G,\circ}. \tag{3.3}$$

Associated to each parahoric positive loop group $L^+\mathscr{P}_{\mathbf{f}}$ there is a partial affine flag variety functor $\mathscr{F}_{\mathbf{f}}$, defined as the étale-sheafification of the presheaf functor on Aff_k

$$R \mapsto LG(R)/L^+\mathscr{P}_{\mathbf{f}}(R).$$

Since $\mathscr{P}_{\mathbf{f}}$ is a parahoric group scheme, the partial affine flag variety $\mathscr{F}_{\mathbf{f}}$ is an ind-projective ind-scheme over k [27], [24]. The natural basepoint in $\mathscr{F}_{\mathbf{f}}$ corresponding to the trivial coset $L^+\mathscr{P}_{\mathbf{f}}(k)$ is denoted $e_{\mathbf{f}} \in \mathscr{F}_{\mathbf{f}}(k)$.

The loop group LG naturally acts by left multiplication on $\mathscr{F}_{\mathbf{f}}$. Given a pair of facets $\mathbf{f}, \mathbf{f}'$ in the closure of $\mathbf{a}$, the $L^+\mathscr{P}_{\mathbf{f}'}$-orbits of $\mathscr{F}_{\mathbf{f}}$ are referred to as Schubert cells, and naturally indexed by elements of the double coset space $W_{\mathbf{f}'}\backslash W/W_{\mathbf{f}}$ [14, Proposition 8 and Remark 9]. Let $w \in W_{\mathbf{f}'}\backslash W/W_{\mathbf{f}}$ be a double coset, and let $\dot{w} \in G(K)$ be a lift of the minimal representative in W of w as described above. We abuse notation, suppressing the dot and simply referring to the point $\dot{w}e_{\mathbf{f}} \in \mathscr{F}_{\mathbf{f}}(k)$ as $we_{\mathbf{f}}$. The Schubert cell corresponding to $W_{\mathbf{f}'}wW_{\mathbf{f}}$ is defined as the orbit space

$$Y_{w,\mathbf{f}',\mathbf{f}} := L^+\mathscr{P}_{\mathbf{f}} \cdot we_{\mathbf{f}}.$$

If $\mathbf{f}' = \mathbf{f}$ (this is usually the case) then we denote a Schubert cell $Y_{w,\mathbf{f}} := Y_{w,\mathbf{f},\mathbf{f}}$. If the context is clear, we may also shorten to simply Y_w.

For a pair of facets $\mathbf{f}, \mathbf{f}'$ in the closure of $\mathbf{a}$, the affine flag variety $\mathscr{F}_{\mathbf{f}}$ is stratified by Schubert cells according to the Bruhat order $\leq$ on $W_{\mathbf{f}'}\backslash W/W_{\mathbf{f}}$. In particular, the Schubert variety $X_{w,\mathbf{f}',\mathbf{f}} := \overline{Y}_{w,\mathbf{f}',\mathbf{f}}$ is the reduced closure in $\mathscr{F}_{\mathbf{f}}$ of the corresponding Schubert cell [28]. Then we have for a pair $(v, w) \in W_{\mathbf{f}'}\backslash W/W_{\mathbf{f}}$ that $Y_{v,\mathbf{f}',\mathbf{f}} \subset X_{w,\mathbf{f}',\mathbf{f}}$ if and only if $v \leq w \in W_{\mathbf{f}'}\backslash W/W_{\mathbf{f}}$. Thus the Schubert cells for $v \leq w$ form the Iwahori stratification of the Schubert variety $X_{w,\mathbf{f}',\mathbf{f}}$:

$$X_{w,\mathbf{f}',\mathbf{f}} = \coprod_{\substack{v \in W_{\mathbf{f}'} \backslash W / W_{\mathbf{f}} \\ v \leq w}} Y_{v,\mathbf{f}',\mathbf{f}}.$$

In the case $\mathbf{f}' = \mathbf{f} = \mathbf{a}$, the corresponding parahoric is referred to as the Iwahori $\mathscr{I} := \mathscr{P}_{\mathbf{a}}$. In this case the the partial affine flag variety $\mathscr{F}_{\mathbf{a}}$ is referred to simply as the (full) affine flag variety $\mathscr{F}$. In the case $\mathbf{f}' = \mathbf{f}$ is the facet containing $\mathbf{0}$, we have $\mathscr{P}_{\mathbf{f}} = \mathscr{P}_{\mathbf{0}} = \mathscr{G}$ and the corresponding partial affine flag variety is the affine Grassmannian $\mathrm{Gr}_{\mathbf{0}} := \mathscr{F}_{\mathbf{f}} = \mathscr{F}_{\mathbf{0}}$.

In order to define twisted products of Schubert varieties, let us define the twisted product of a partial affine flag variety $\mathscr{F}_{\mathbf{f}}$. For a facet $\mathbf{f}$ in the closure of $\mathbf{a}$ we have a presheaf functor on Aff_k

$$\begin{aligned} \mathrm{Presh}(\tilde{\mathscr{F}}^p_{\mathbf{f}}) : R \mapsto & (LG(R))^p)/(L^+\mathscr{P}_{\mathbf{f}}(R))^p \\ & = LG(R) \times^{L^+\mathscr{P}_{\mathbf{f}}(R)} \cdots \times^{L^+\mathscr{P}_{\mathbf{f}}(R)} LG(R)/L^+\mathscr{P}_{\mathbf{f}}(R), \end{aligned}$$

where the right action of $(L^+\mathscr{P}_{\mathbf{f}}(R))^p$ on $(LG(R))^p$ is given by

$$(g_1, \ldots, g_p) \cdot (h_1, \ldots, h_p) = (g_1 h_1, h_1^{-1} g_2 h_2, \ldots, h_{p-1}^{-1} g_p h_p). \tag{3.4}$$

Then the étale sheafification of $\mathrm{Presh}(\tilde{\mathscr{F}}^p_{\mathbf{f}})$ is denoted $\tilde{\mathscr{F}}^p_{\mathbf{f}}$ or $\mathscr{F}_{\mathbf{f}} \tilde{\times} \cdots \tilde{\times} \mathscr{F}_{\mathbf{f}}$, and called the ($p$-fold) twisted product of $\mathscr{F}_{\mathbf{f}}$.

Denote the quotient morphism by $q : LG \to \mathscr{F}_{\mathbf{f}}$. Given an element $w \in W_{\mathbf{f}} \backslash W / W_{\mathbf{f}}$, let the sheaf $q^{-1}(Y_{w,\mathbf{f}})$ be denoted $\mathscr{P}_{\mathbf{f}} w \mathscr{P}_{\mathbf{f}}$, and let the sheaf $q^{-1}(X_{w,\mathbf{f}})$ be denoted $\overline{\mathscr{P}_{\mathbf{f}} w \mathscr{P}_{\mathbf{f}}}$. Both are étale subsheaves of LG. Then, given a sequence $w_\bullet := (w_1, \ldots, w_p)$ of elements of $W_{\mathbf{f}} \backslash W / W_{\mathbf{f}}$, we have

$$Y_{w_\bullet,\mathbf{f}} := Y_{w_1,\mathbf{f}} \mathbin{\tilde{\times}} \cdots \mathbin{\tilde{\times}} Y_{w_p,\mathbf{f}} := \mathscr{P}_{\mathbf{f}} w_1 \mathscr{P}_{\mathbf{f}} \times^{L^+\mathscr{P}_{\mathbf{f}}} \cdots \times^{L^+\mathscr{P}_{\mathbf{f}}} \mathscr{P}_{\mathbf{f}} w_p \mathscr{P}_{\mathbf{f}} / L^+\mathscr{P}_{\mathbf{f}} \subset \tilde{\mathscr{F}}_{\mathbf{f}}^p$$

$$X_{w_\bullet,\mathbf{f}} := X_{w_1,\mathbf{f}} \mathbin{\tilde{\times}} \cdots \mathbin{\tilde{\times}} X_{w_p,\mathbf{f}} := \overline{\mathscr{P}_{\mathbf{f}} w_1 \mathscr{P}_{\mathbf{f}}} \times^{L^+\mathscr{P}_{\mathbf{f}}} \cdots \times^{L^+\mathscr{P}_{\mathbf{f}}} \overline{\mathscr{P}_{\mathbf{f}} w_p \mathscr{P}_{\mathbf{f}}} / L^+\mathscr{P}_{\mathbf{f}} \subset \tilde{\mathscr{F}}_{\mathbf{f}}^p,$$

where the right action of $L^+\mathscr{P}_{\mathbf{f}}^p$ is that of (3.4). On $\bar{k}$-points we have

$$Y_{w_\bullet,\mathbf{f}}(\bar{k}) = \{(g_1 e_{\mathbf{f}}, \ldots, g_p e_{\mathbf{f}}) \mid g_{i-1}^{-1} g_i e_{\mathbf{f}} \in Y_{w_i}(\bar{k}) \text{ for each } 1 \leq i \leq p\}$$

$$X_{w_\bullet,\mathbf{f}}(\bar{k}) = \{(g_1 e_{\mathbf{f}}, \ldots, g_p e_{\mathbf{f}}) \mid g_{i-1}^{-1} g_i e_{\mathbf{f}} \in X_{w_i}(\bar{k}) \text{ for each } 1 \leq i \leq p\}.$$

If k is a finite field, then we can work with k-points by taking the Frobenius fixed points.

Let ${}^{\mathbf{f}}w_*^{\mathbf{f}}$ be the Demazure product of maximal representatives of $w_\bullet$. That is, for each $w_i \in W_{\mathbf{f}} \backslash W / W_{\mathbf{f}}$ let ${}^{\mathbf{f}}w_i^{\mathbf{f}} \in W$ be the maximal representative of the double coset. Then ${}^{\mathbf{f}}w_*^{\mathbf{f}}$ is the double coset corresponding to the Demazure product

$${}^{\mathbf{f}}w_*^{\mathbf{f}} = W_{\mathbf{f}} \, {}^{\mathbf{f}}w_1^{\mathbf{f}} * \cdots * {}^{\mathbf{f}}w_p^{\mathbf{f}} W_{\mathbf{f}}.$$

See [7, §4, especially Lemma 4.3.4] for more details on this product. We have a convolution morphism given by projection onto the last factor:

$$m_{w_\bullet} : X_{w_\bullet} \to X_{\mathbf{f}w_*^{\mathbf{f}}}$$

$$(g_1 e_{\mathbf{f}}, \ldots, g_p e_{\mathbf{f}}) \mapsto g_p e_{\mathbf{f}}.$$

In the case $\mathbf{f}$ is the minimal facet containing $\mathbf{0}$, we have $W_{\mathbf{f}} = W_{\mathbf{0}}$, and $W/W_{\mathbf{0}} \cong (X_*(T)_I)^\tau$, and $W_{\mathbf{0}}\backslash W/W_{\mathbf{0}}$ is represented by $(X_*(T)_I)^{\tau,+}$, which contains the set of dominant cocharacters. Then for a sequence $\mu_\bullet$ of dominant cocharacters we have ${}^{\mathbf{0}}\mu_*^{\mathbf{0}} = |\mu_\bullet| w_0$, where w_0 is the longest element of $W_{\mathbf{0}}$, and note that $|\mu_\bullet|$ represents the same double coset in $W_{\mathbf{0}}\backslash W/W_{\mathbf{0}}$.

Similarly, we have a convolution morphism of twisted products of Schubert cells, which stratify the twisted products of Schubert varieties:

$$m_{w_\bullet}^\circ : Y_{w_\bullet} \to X_{\mathbf{f}w_*^{\mathbf{f}}}$$

$$(g_1 e_{\mathbf{f}}, \ldots, g_p e_{\mathbf{f}}) \mapsto g_p e_{\mathbf{f}}.$$

The goal of this paper is to construct suitable pavings of the fibers of these convolution morphisms.

3.2.2 Description of root groups and valuations

For a root α let L_α, let C_α be the subset of $\tilde{\Phi}$ consisting of those absolute roots $\tilde{\alpha} \in \tilde{\Phi}$ such that $\tilde{\alpha}|_S = \alpha$, also known as the set of lifts of α. Then the group Γ_G permutes the root subgroups $U_{H,\tilde{\alpha}} \subset H$. Given any $\tilde{\alpha} \in C_\alpha$, we have a subgroup $\Gamma_{\tilde{\alpha}} \subset \Gamma_G$ stabilizing

$U_{H,\tilde{\alpha}}$. Let the field of definition of $\tilde{\alpha}$ be denoted $L_{\tilde{\alpha}} := \tilde{K}^{\Gamma_{\tilde{\alpha}}}$.

Proposition 3.2.1. *Let G be a quasi-split, connected, tamely ramified, reductive group over K, where $K = k((t))$, and suppose that k is either algebraically closed or finite of order q such that $3 \mid q$ or $3 \mid q-1$. Furthermore suppose that G is almost simple. Then G is not of type ${}^6D_{4,2}$.*

Proof. Suppose that G is of type ${}^6D_{4,2}$. Then by [5, §4.1.16(e)], we have a non-Galois intermediate extension $K \subset L \subset \tilde{K}$ of degree 3. Since $\tilde{K}/K$ is Galois by definition and the Galois group of ${}^6D_{4,2}$ is $\Gamma_G = S_3$, this means L must be equal to a totally ramified cubic extension $k((u'))/K$. Since we assumed G is tamely ramified over K, we must have $\operatorname{char} K \neq 3$.

If k has cubic roots of unity then we may choose a uniformizer u' for L such that $(u')^3 = t$, as in [24, §7(a)]. But then we see that $k((u'))/K$ is Galois. If k is algebraically closed or $k = \mathbb{F}_q$ where $3 \mid q-1$, then k has cubic roots of unity.

Thus if G is of type ${}^6D_{4,2}$ then $k = \mathbb{F}_q$, where $3 \nmid q$ and $3 \nmid q-1$. $\square$

Note that the assumption on the order of k is fairly weak: if q is any prime power not divisible by 3, then $3 \mid q^2 - 1$.

As a consequence of Proposition 3.2.1 and our assumptions on k, every component of Φ splits over a cyclic extension of K of order ≤ 3 [5, A.6.]. Thus for every absolute root $\tilde{\alpha} \in \tilde{\Phi}$, the field $L_{\tilde{\alpha}}$ is either equal to K or is a splitting field for its component of Φ. In particular, for a relative root $\alpha \in \Phi$, every lift $\tilde{\alpha} \in C_\alpha$ has the same field of definition; we thus define L_α to be the field of definition $L_{\tilde{\alpha}}$ for any lift $\tilde{\alpha} \in C_\alpha$.

Write $L_\alpha = \ell_\alpha((u_\alpha))$, where ℓ_α is the residue field and u_α is a uniformizer. We always have $[L_\alpha : K] \leq 3$ and L_α/K is either unramified or totally ramified. If L_α/K is unramified, then we say α is unramified, and we have $u_\alpha := t$, and if L_α/K is totally ramified, we say α is ramified, and we have $\ell_\alpha = k$.

The following well-known fact is frequently useful in understanding the structure of root groups and affine root groups.

Proposition 3.2.2. *Let F'/F be a finite field extension. Then $\mathrm{Res}_{F'/F}(\mathbb{G}_{a,F'})$ is isomorphic as a variety over F to $\mathbb{A}_F^{[F':F]}$.*

Proof. Let $d = [F' : F]$ and let $\{a_1, \ldots, a_d\} \subset F'$ be a fixed basis for F'/F. Then for an F-algebra R, the map

$$\mathbb{A}_F^d(R) \to \mathrm{Res}_{F'/F}(\mathbb{G}_{a,F'})(R) = R \otimes_F F'$$

$$(x_1, \ldots, x_d) \mapsto x_1 a_1 + \cdots + x_d a_d$$

is an isomorphism of F-schemes. □

For each $\alpha \in \Phi$ the root subgroup $U_\alpha \subset G$ is isomorphic to one of three possible unipotent groups. Let V_α be the group scheme over K representing that isomorphism class. If α is singular, then $V_\alpha = \mathrm{Res}_{L_\alpha/K}(\mathbb{G}_{a,L_\alpha})$, which is an $[L_\alpha : K]$-dimensional unipotent group scheme over K, and isomorphic as a variety to $\mathbb{A}_K^{[L_\alpha:K]}$. If α is plural and non-divisible, then U_α is isomorphic to the K-group scheme $H(L_\alpha, K)$ which is a maximal unipotent subgroup of the quasi-split form of $\mathrm{SU}_{3,L_\alpha/K}$ and defined in general in Definition 3.2.3. Points of $H(L_\alpha, K)(R)$ are always referred to by pairs $(x, y) \in R_{L_\alpha}^2$. If 2α is plural and divisible, then note that L_α/K is a quadratic Galois extension. We have

$V_{2\alpha} \subset \operatorname{Res}_{L_\alpha/K}(\mathbb{G}_{a,L_\alpha})$ is the subgroup consisting of points such that $\sigma(x) = -x$ where σ is the nontrivial automorphism of L_α/K. Thus we have a natural embedding $V_{2\alpha} \hookrightarrow V_\alpha$ given by $x \mapsto (0, x)$, and $V_{2\alpha}$ is a central subgroup of V_α. Furthermore, $V_{2\alpha}$ is isomorphic as a group scheme to $\mathbb{G}_{a,K}$ and in particular isomorphic as a variety to $\mathbb{A}^1_K$.

Definition 3.2.3. Let F be a field with a quadratic, Galois, tamely ramified extension F'/F. For any F-algebra R, the points of $H(F', F)(R)$ are are given by

$$H(F', F)(R) = \{(x, y) \in R^2_{F'} \mid x\sigma(x) + y + \sigma(y) = 0\},$$

and multiplication is given by

$$(x, y)(x', y') := (x + x', -x\sigma(x') + y + y').$$

Note that $H(F', F)$ is a 3-dimensional, unipotent, solvable, non-abelian affine group scheme over F. As an F-variety, $H(F', F)$ is 3-dimensional, and in particular isomorphic to $\mathbb{A}^3_F$. Let $F' = F(b)$. Assume, if $\operatorname{char} F \neq 2$, that $b + \sigma(b) = 0$. Such a b exists; for any $b' \in F' - F$ take $b := \frac{b' - \sigma(b')}{2}$. If $\operatorname{char} F = 2$, then instead assume $b + \sigma(b) = 1$. Again, such a b exists, since $x \mapsto x + \sigma(x)$ is an F-linear map $F' \to F$ whose kernel is F. Since $[F' : F] = 2$, the map must be surjective. If F has a valuation then also assume b is such that $v_{F'}(b)$ is the minimal nonnegative valuation for elements of $F' - F$. Then we have an isomorphism

$$\mathbb{A}^3_F(R) \to H(F',F)(R)$$

$$(x,x',y) \mapsto \begin{cases} \left(x+x'b, yb+\frac{1}{2}(x^2-x'^2b^2)\right), & \operatorname{char} F \neq 2 \\ \Big(x+x'b, y+(x^2+xx'+x'^2b\sigma(b))b\Big), & \operatorname{char} F = 2. \end{cases} \tag{3.5}$$

The map above is clearly injective.

Note that the point in the right-hand side is an element of $H(F',F)(R)$. Indeed, consider that the action of σ on an element $(x+yb, z+wb) \in H(F',F)(R)$ is to send $b \mapsto \sigma(b)$ and leave x, y, z, w unchanged. First suppose that $\operatorname{char} F \neq 2$, so $\sigma(b) = -b$. We have $yb + \sigma(yb) = y(b+\sigma(b)) = 0$, and $\sigma(b^2) = b$ so

$$\tfrac{1}{2}(x^2-x'^2b^2) + \sigma(\tfrac{1}{2}(x^2-x'^2b^2) + (x+x'b)\sigma(x+x'b) = x^2-x'^2b^2+x^2+x'^2b^2 = 0,$$

as needed. Now suppose that $\operatorname{char} F = 2$, so $b+\sigma(b) = 1$. Then $y+y=0$ and

$$\begin{aligned} (x+x'b)\sigma(x+x'b) &= x^2+xx'\sigma(b)+xx'b+x'^2b\sigma(b) \\ &= x^2+xx'+x'^2b\sigma(b), \end{aligned}$$

and so

$$(x+x'b)\sigma(x+x'b) + (x^2+xx'+x'^2b\sigma(b))b + \sigma((x^2+xx'+x'^2b\sigma(b))b) = 0.$$

To see that the map is injective, let $(z,w) \in H(F',F))(R)$. Then clearly $z = x+x'b$ for some $x, x' \in R$. If $\operatorname{char} F \neq 2$, then write $w' = w - \frac{1}{2}z\sigma(z) = w - \frac{1}{2}(x^2 - x'^2b^2)$. Then the equation $w + \sigma(w) + z\sigma(z)$ implies $w' + \sigma(w') = 0$, so $w = yb$ for some $y = w/b \in R$. Similarly, if $\operatorname{char} F = 2$, then write $w' = w - (x^2 + xx' + x'^2b\sigma(b))b$. Then by the same reasoning we must have $w' = \sigma(w')$.

Define a valuation v_α on $V_\alpha(K)$ as follows. We give a concrete description that matches the abstract definition in [33, Chapter 1]. In all cases let the valuation v_{L_α} be normalized such that $v_{L_\alpha}|_K = v_K$. If α is singular, then $V_\alpha(K) = L_\alpha$ and v_α is equal to v_{L_α}. If α is plural and non-divisible, then $V_\alpha(K) \subset L_\alpha^2$ and $v_\alpha(x, y) := \frac{1}{2}v_{L_\alpha}(y)$. If 2α is plural and divisible, then $V_{2\alpha}(K) \subset L_\alpha$ and $v_{2\alpha} := v_{L_\alpha}$. Note that this creates an inconsistency between valuations such that $v_{2\alpha}(y) \neq v_\alpha(0, y)$.

In order to define a valuation v_α on $U_\alpha(K)$, let us compare root subgroups of G to those of H. Since for each absolute root $\tilde{\alpha} \in \tilde{\Phi}$ we have the corresponding root subgroup $U_{H,\tilde{\alpha}} \subset H$ is isomorphic to $\mathbb{G}_{a,\mathbb{Z}}$, we have $U_{H,\tilde{\alpha}}(\tilde{K}) = \tilde{K}$. Then for an element $x \in U_{H,\tilde{\alpha}}(\tilde{K})$ we define $v_{\tilde{\alpha}}(x)$ to be $v_{\tilde{K}}(x)$. Note using the Chevalley basis for H we have a natural injective group homomorphism $U_{H,\tilde{\alpha}}(\tilde{k}[\![u]\!]) \hookrightarrow U_{H,\tilde{\alpha}}(\tilde{K})$ such that x is contained in the image of $U_{H,\tilde{\alpha}}(k[\![u]\!])$ if and only if $v_{\tilde{\alpha}}(x) \geq 0$.

For a relative root $\alpha \in \Phi$ let $C_\alpha \subset \tilde{\Phi}$ be the set of lifts of α in $\tilde{\Phi}$.

Lemma 3.2.4. *If $\alpha \in \Phi$ is a singular or divisible root, then we have a product decomposition of varieties over $\tilde{K}$*

$$(U_\alpha)_{\tilde{K}} \cong \prod_{\tilde{\alpha} \in C_\alpha} (U_{H,\tilde{\alpha}})_{\tilde{K}},$$

and as varieties over K we have

$$U_\alpha \cong \mathrm{Res}_{\tilde{K}/K}\,\Big(\prod_{\tilde{\alpha}\in C_\alpha} (U_{H,\tilde{\alpha}})_{\tilde{K}}\Big)^{\Gamma_G}.$$

If $\alpha \in \Phi$ is a plural and non-divisible root, then we have a product decomposition of varieties over $\tilde{K}$

$$(U_\alpha)_{\tilde{K}} \cong (U_{H,\tilde{\alpha}} \times U_{H,\tilde{\beta}} \times U_{H,\tilde{\alpha}+\tilde{\beta}})_{\tilde{K}},$$

and as varieties over K we have

$$U_\alpha \cong \mathrm{Res}_{\tilde{K}/K}\,((U_{H,\tilde{\alpha}} \times U_{H,\tilde{\beta}} \times U_{H,\tilde{\alpha}+\tilde{\beta}})_{\tilde{K}})^{\Gamma_G}.$$

Proof. From the isomorphism classes V_α (see [5, §A.3] for an explicit description) we know that if α is singular or divisible, then $(U_\alpha)_{\tilde{K}} \cong \mathbb{A}_{\tilde{K}}^{[L_\alpha:K]}$, and if α is plural and non-divisible then $(U_\alpha)_{\tilde{K}} \cong \mathbb{A}^3_{\tilde{K}}$. All that remains is to identify the root groups appearing in the product decomposition. It is clear that for all $\tilde{\alpha} \in C_\alpha$ the group $(U_{H,\tilde{\alpha}})_{\tilde{K}} \subset (U_\alpha)_{\tilde{K}}$. In the case α is singular or divisible, then by dimension we are done. In the case α is plural and non-divisible, then $C_\alpha = \{\tilde{\alpha}, \tilde{\beta}\}$ and $\tilde{\alpha}$ and $\tilde{\beta}$ are not orthogonal. (See [5, §4.1.4 Cas II], and note that there is some choice of simple roots for Φ such that α is a simple root.) Then since $U_{H,\tilde{\alpha}}$ and $U_{H,\tilde{\beta}}$ do not commute, subgroup closure of $(U_\alpha)_{\tilde{K}}$ thus requires that $(U_{H,\tilde{\alpha}+\tilde{\beta}})_{\tilde{K}} \subset (U_\alpha)_{\tilde{K}}$. Again, by dimension, we are done. □

Then for an element $x \in U_\alpha(K)$, we have a corresponding Γ_G-invariant tuple $(x_{\tilde{\alpha}}) \in \prod_{\tilde{\alpha}\in C'_\alpha} U_{H,\tilde{\alpha}}(\tilde{K})$, taking the product in some fixed order. Let $v_\alpha(x)$ be the minimum of $v_{\tilde{\alpha}}(x_{\tilde{\alpha}})$.

3.2.3 Description of affine root groups

There are a few cases for the set of $r \in \mathbb{R}$ such that $\alpha + r$ is an affine root, depending on whether α is singular, divisible, or non-divisible, as well as the degree and ramification of L_α/K. We have

Proposition 3.2.5. *Let $\alpha \in \Phi(G, S)$ be a relative root. Let the set of $r \in \mathbb{R}$ such that $\alpha + r \in \Phi_{\mathrm{af}}(G, S)$ be denoted R_α. If α is singular (respectively plural and non-divisible), let the smallest positive integer r such that $rR_\alpha = \mathbb{Z}$ (respectively such that $rR_\alpha + \frac{r}{2}R_{2\alpha} = \mathbb{Z}$) be denoted r_α. If 2α is plural and divisible, then let $r_{2\alpha} = \frac{r_\alpha}{2}$.*

- *i. If α is ramified and singular, then $R_\alpha = \frac{1}{[L_\alpha : K]}\mathbb{Z}$, and $r_\alpha = [L_\alpha : K]$.*
- *ii. If α is ramified, plural, and non-divisible, then $R_\alpha = \frac{1}{2}\mathbb{Z}$, and $r_\alpha = 4$.*
- *iii. If 2α is ramified, plural, and divisible, then $R_{2\alpha} = \frac{1}{2} + \mathbb{Z}$, and $r_{2\alpha} = 2$.*
- *iv. If α is unramified and singular, then $R_\alpha = \mathbb{Z}$, and $r_\alpha = 1$.*
- *v. If α is unramified, plural, and non-divisible, then $R_\alpha = \mathbb{Z}$, and $r_\alpha = 2$.*
- *vi. If 2α is unramified, plural, and divisible, then $R_{2\alpha} = \mathbb{Z}$, and $r_{2\alpha} = 1$.*

Proof. For singular α the result follows immediately from $V_\alpha(K) = L_\alpha$. If α is ramified then the result is in [23] Proposition 2.2.

Suppose 2α is plural, divisible, and unramified. Then $V_{2\alpha}(K) \subset L_\alpha$ consists of those $y \in L_\alpha$ such that $y + \sigma(y) = 0$. Recall we have $v_{2\alpha}(y) = v_{L_\alpha}(y) \in \mathbb{Z}$, so $R_{2\alpha} \subset \mathbb{Z}$. Write

$L_\alpha = \ell_\alpha(\!(t)\!)$, with $\ell_\alpha = k(b)$. If $r \in \mathbb{Z}$, we have an isomorphism of $\mathscr{O}$-modules

$$\phi_r : \mathscr{O} \to \mathscr{U}_{2\alpha+r}(\mathscr{O})$$

$$x \mapsto x(b - \sigma(b))t^r.$$

Then $\phi_r(x) \in \mathscr{U}_{2\alpha+r+\epsilon}(\mathscr{O})$ if and only if $v_K(x) \geq \epsilon$. So for all positive $\epsilon < 1$ we have an isomorphism of $\mathscr{O}$-modules $\mathscr{U}_{2\alpha+r}(\mathscr{O})/\mathscr{U}_{2\alpha+r+\epsilon}(\mathscr{O}) \cong \mathscr{O}/t\mathscr{O} \cong k$, and $2\alpha + r \in \Phi_{\mathrm{af}}$. Therefore $R_{2\alpha} = \mathbb{Z}$.

Suppose α is plural, non-divisible, and unramified. Then $L_\alpha = \ell_\alpha(\!(t)\!)$. Since the image of $v_{L_\alpha}(L_\alpha) \subset \mathbb{R}$ is equal to $\mathbb{Z}$ and for $(x, y) \in U_\alpha(K)$ we have $v_\alpha(x, y) = \frac{1}{2}v_{L_\alpha}(y)$, we have $R_\alpha \subset \frac{1}{2}\mathbb{Z}$. Let $r \in \mathbb{Z}$ and write $\ell_\alpha = k(b)$, where $b + \sigma(b) = 0$ if $\operatorname{char} k \neq 2$ and $b + \sigma(b) = 1$ if $\operatorname{char} k = 2$. Then there is an isomorphism of $\mathscr{O}$-modules

$$\phi_r : \mathscr{O}^3 \to \mathscr{U}_{\alpha+r}(\mathscr{O})$$

$$(x, x', y) \mapsto \begin{cases} ((x + x'b)t^r, (yb + \frac{1}{2}(x^2 - x'^2b^2))t^{2r}), & \operatorname{char} k \neq 2 \\ ((x + x'b)t^r, (v + (x^2 + xx' + x'^2 b\sigma(b))b)t^{2r}), & \operatorname{char} k = 2. \end{cases}$$

We have $\phi_r(x, x', y) \in \mathscr{U}_{\alpha+r+\epsilon}(\mathscr{O})$ for some $\epsilon > 0$ if and only if $\min(v_K(x), v_K(x')) \geq \epsilon$ and $v_K(y) \geq 2\epsilon$. Thus for all $\epsilon < \frac{1}{2}$ we have a quotient of $\mathscr{O}$-modules

$$\mathscr{U}_{\alpha+r}(\mathscr{O})/\mathscr{U}_{\alpha+r+\epsilon}(\mathscr{O}) \cong \mathscr{O}^3/(t\mathscr{O})^3 \cong k^3.$$

As seen in the case of 2α, for all $\epsilon < \frac{1}{2}$ we have $\mathscr{U}_{2\alpha+2r}(\mathscr{O})/\mathscr{U}_{2\alpha+2r+2\epsilon}(\mathscr{O}) \cong k \subsetneq k^3$.

Therefore $\alpha + r \in \Phi_{\mathrm{af}}$ and we have $\mathbb{Z} \subset R_\alpha$.

Now suppose $r \in \frac{1}{2} + \mathbb{Z}$. Then, using the notation as above, we have an isomorphism of $\mathscr{O}$-modules given by

$$\phi_r : \mathscr{O}^3 \to \mathscr{U}_{\alpha+r}(\mathscr{O})$$
$$(u, u', v) \mapsto \begin{cases} ((u + u'b)t^{r+\frac{1}{2}}, vbt^{2r} + \dfrac{u^2 - u'^2b^2}{2}t^{2r+1}), & \operatorname{char} k \neq 2 \\ ((u + u'b)t^{r+\frac{1}{2}}, vt^{2r} + (u^2 + uu' + u'^2b\sigma(b))bt^{2r+1}), & \operatorname{char} k = 2. \end{cases}$$

For all positive $\epsilon < \frac{1}{2}$, we have $\phi_r(u, u', v) \in \mathscr{U}_{\alpha+r+\epsilon}(\mathscr{O})$ if and only if $v_K(v) \geq 2\epsilon$. Then we have $\mathscr{U}_{\alpha+r}(\mathscr{O})/\mathscr{U}_{\alpha+r+\epsilon}(\mathscr{O}) \cong \mathscr{O}/t\mathscr{O} \cong k$. But we've seen that $\mathscr{U}_{2\alpha+2r}(\mathscr{O})/\mathscr{U}_{2\alpha+2r+2\epsilon}(\mathscr{O}) \cong k$ as well, and so $\alpha + r \notin \Phi_{\mathrm{af}}$. Therefore $R_\alpha = \mathbb{Z}$. This completes the proof. $\square$

There are a few possible isomorphism classes of the group scheme $\mathsf{U}_{\alpha+r}$ over k, depending on ramification of α as well as whether α is singular, divisible, or non-divisible. Note that the isomorphism class of $\mathsf{U}_{\alpha+r}$ does not depend on r, only on the finite root α. Recall that $d(\alpha + r)$ is defined as the dimension $\dim_k(\mathsf{U}_{\alpha+r})$. We have

Proposition 3.2.6. *Let $\alpha + r \in \Phi_{\mathrm{af}}$. Then as a k-variety $\mathsf{U}_{\alpha+r} \cong \mathbb{A}_k^{d(\alpha+r)}$. More specifically, we have*

i. If α is ramified, then as a k-group scheme $\mathsf{U}_{\alpha+r} \cong \mathbb{G}_{a,k}$ and $d(\alpha + r) = 1$.

ii. If α is unramified and singular, then as a k-group scheme $\mathsf{U}_{\alpha+r} \cong \operatorname{Res}_{\ell_\alpha/k}(\mathbb{G}_{a,\ell_\alpha})$ and $d(\alpha + r) = [\ell_\alpha : k] \leq 3$.

iii. If α is unramified, plural, and non-divisible, then as a k-group scheme $\mathsf{U}_{\alpha+r} \cong H(\ell_\alpha, k)$ and $d(\alpha + r) = 3$.

iv. If 2α is unramified, plural, and divisible, then as a k-group scheme $\mathsf{U}_{2\alpha+r} \cong \mathbb{G}_{a,k}$ and $d(\alpha+r) = 1$. If $2r \in 2\mathbb{Z}$, then for all k-algebras R there is a natural injection on R-points $\mathsf{U}_{2\alpha+2r}(R) \subset \mathsf{U}_{\alpha+r}(R)$.

One possible problem with decomposing subgroups of G into products of of affine root groups is that affine root groups may intersect. Specifically, it is possible for both $\alpha+r$ and $2\alpha+2r$ to be affine roots, so that $\mathsf{U}_{2\alpha+2r} \subset \mathsf{U}_{\alpha+r}$. Thus the natural map $\mathsf{U}_{\alpha+r} \times \mathsf{U}_{2\alpha+2r} \to G$ is not injective. We define two spaces related to $\mathsf{U}_{\alpha+r}$ as follows.

Definition 3.2.7. If α is ramified, singular, or divisible, then for $\alpha + r \in \Phi_{\mathrm{af}}$ let $\mathsf{U}'_{\alpha+r} = \mathsf{U}''_{\alpha+r} = \mathsf{U}_{\alpha+r}$.

If α is plural, non-divisible, and unramified, then recall $\mathsf{U}_{\alpha+r} \cong H(\ell_\alpha, k)$. Let $\mathsf{U}'_{\alpha+r} = \mathsf{U}_{\alpha+r}/\mathsf{U}_{2\alpha+2r}$. In this case we have $\mathsf{U}'_{\alpha+r} \cong \mathrm{Res}_{\ell_\alpha/k}(\mathbb{G}_{a,\ell_\alpha})$.

Let $\mathsf{U}''_{\alpha+r}$ be a subvariety of $\mathsf{U}_{\alpha+r}$ isomorphic to $\mathsf{U}'_{\alpha+r}$ as a variety. Specifically, if α is plural, non-divisible, and unramified, let σ be the nontrivial Galois automorphism $\ell_\alpha \to \ell_\alpha$ over k. Then write $\ell_\alpha = k(b)$ where $b+\sigma(b) = 0$ if $\operatorname{char} k \neq 2$ and $b+\sigma(b) = 1$ if $\operatorname{char} k = 2$. Then there is an embedding

$$\mathsf{U}'_{\alpha+r}(R) = R \otimes_k \ell_\alpha \to \mathsf{U}_{\alpha+r}(R) = H(\ell_\alpha, k)(R) \tag{3.6}$$

$$x \mapsto \begin{cases} (x, -\dfrac{x\sigma(x)}{2}) & \operatorname{char} k \neq 2 \\ (x, x\sigma(x)b) & \operatorname{char} k = 2. \end{cases}$$

Then $\mathsf{U}''_{\alpha+r}(R) \subset \mathsf{U}_{\alpha+r}(R)$ is defined to be the image of (3.6).

3.2.4 Branching in the building

The building $\mathscr{B}(G,K)$ is a union of apartments A all isomorphic to the fundamental apartment, and such that any two alcoves are contained in at least one apartment. Then $LG(k)$ acts transitively on the set of apartments, so for each A there is some $g \in LG(k)$ such that $A = g\mathscr{A}$. The intersection of any two apartments is a (possibly empty) convex polysimplicial subcomplex, i.e. a closed union of facets. We say a wall $\mathbf{H}$ in an apartment A is of type $\alpha+r$ if there is an element $g \in LG(k)$ in the kernel of the Kottwitz homomorphism κ_G, defined in [20], such that $A = g\mathscr{A}$ and $\mathbf{H} = g\mathbf{H}_{\alpha+r}$.

Lemma 3.2.8. *Let $\mathbf{a}'$ be an alcove in the building $\mathscr{B}(G,K)$ and let $\mathbf{f}$ be a codimension 1 facet contained in the closure of $\mathbf{a}'$. There is a unique wall $\mathbf{H}$ containing $\mathbf{f}$; suppose $\mathbf{H}$ is of type $\alpha + r \in \Phi_{\mathrm{af}}$. The set of alcoves $\mathbf{a}'' \neq \mathbf{a}'$ in $\mathscr{B}(G,K)$ such that $\mathbf{f}$ is contained in the closure of $\mathbf{a}''$ is indexed by the k-points of the affine root group $\mathsf{U}_{\alpha+r}$.*

Recall that $\mathsf{U}_{\alpha+r} \cong \mathbb{A}_k^{d(\alpha+r)}$ as a variety over k (see Propositon 3.2.6, Proposition 3.2.2, and Definition 3.2.3) and in particular that $\mathsf{U}_{\alpha+r}(k) = k^{d(\alpha+r)}$.

Proof. This is the content of §3.5 in [33], especially §§3.5.4 and 3.5.1.

In particular, the link in $\mathscr{B}(G,K)$ of a point $x \in \mathscr{A}$ is canonically isomorphic to the spherical building of a reductive group G_x defined over k—specifically, if x is in an arbitrary facet $\mathbf{f}$ then $\mathsf{G}_x = \mathsf{G}_{\mathbf{f}}$ is the reductive group formed by reducing the parahoric functor $L^+\mathscr{G}_{\mathbf{f}}$ modulo t and taking the maximal reductive quotient of the resulting k-group. The group G_x is quasi-split, of type $\Phi_x \subset \Phi$, where Φ_x consists only of roots α such that $x \in \mathbf{H}_{\alpha+r}$ for some affine root $\alpha + r$. For a codimension 1 facet $\mathbf{f}$, we have $\mathsf{G}_{\mathbf{f}} := \mathsf{G}_x$ for any $x \in \mathbf{f}$ is of

k-rank 1, so its relative root system is either type A_1 or BC_1.

Let $\mathbf{f}$ be a fixed codimension 1 facet in $\mathscr{A}$. Then the set of alcoves containing $\mathbf{f}$ is in correspondence with the set of chambers in the spherical building of $\mathsf{G}_{\mathbf{f}}$, which indexes the set of Borels in the neutral component $\mathsf{G}_{\mathbf{f}}^{\circ} \subset \mathsf{G}_{\mathbf{f}}$ containing a maximal k-split torus $\mathsf{S}_{\mathbf{f}} \subset \mathsf{G}_{\mathbf{f}}^{\circ}$. In particular, if $\mathsf{B}_{\mathbf{f}}$ is one such Borel, then

$$\{\mathbf{a}' \subset \mathscr{B}(G,K) \mid \mathbf{f} \subset \overline{\mathbf{a}'}\} = \mathsf{G}_{\mathbf{f}}^{\circ}(k)/\mathsf{B}_{\mathbf{f}}(k).$$

Since $\mathsf{G}_{\mathbf{f}}$ has k-rank 1, the subset of alcoves not equal to $\mathbf{a}$ is the big cell of $\mathsf{G}_{\mathbf{f}}^{\circ}(k)/\mathsf{B}_{\mathbf{f}}(k)$ and is indexed by the maximal unipotent subgroup $\mathsf{U}_{\mathbf{f}}(k) \subset \mathsf{G}_{\mathbf{f}}^{\circ}(k)$. In particular, $\mathsf{U}_{\mathbf{f}} \cong \mathsf{U}_{\alpha+r}$, where $\mathbf{H}_{\alpha+r}$ is the unique wall in $\mathscr{A}$ containing $\mathbf{f}$.

If $\mathbf{a}'$ is not contained in $\mathscr{A}$, then $\mathbf{a} = g\mathbf{a}''$ for some $\mathbf{a}'' \subset \mathscr{A}$ and some $g \in LG(k) \cap \ker \kappa_G$, and we also have $\mathbf{f} = g\mathbf{f}'$ for some $\mathbf{f}' \subset \overline{\mathbf{a}''}$. Then the set of alcoves containing $\mathbf{f}'$ other than $\mathbf{a}''$ is indexed by $\mathsf{U}_{\alpha+r}(k)$ for some $\alpha + r$ where $\mathbf{f}' \subset \mathbf{H}_{\alpha+r}$. Then the wall of $g\mathscr{A}$ containing $\mathbf{f}$ is of the same type, and the set of alcoves containing $\mathbf{f}$ in their closure other than $\mathbf{a}$ is also indexed by $\mathsf{U}_{\alpha+r}(k)$. $\square$

3.3 Stratified triviality

3.3.1 Descent of affine root subgroups

For a root $\alpha \in \Phi$, recall $C_\alpha \subset \tilde{\tilde{\Phi}}$ is the set of lifts of α in $\tilde{\tilde{\Phi}}$. Recall that as a consequence of the assumption that either $3 \mid q$ or $3 \mid q-1$, for all $\alpha \in \Phi$, the set C_α has order $[L_\alpha : K]$.

Recall that for a root $\alpha \in \Phi$ we refer to the lifts in $\tilde{\Phi}$ by C_α, and C_α is described in Lemma 3.2.4. We define an analogous subset of $\tilde{\Phi}_{\mathrm{af}}$ as follows, for the purpose of understanding the relationship between affine root subgroups of G over k and those of H over $\tilde{k}$.

Definition 3.3.1. Let $\alpha + r \in \Phi_{\mathrm{af}}$. If α is singular or divisible, then let

$$C_{\alpha+r} := \{\tilde{\alpha} + r \mid \tilde{\alpha} \in C_\alpha\}.$$

If α is plural and non-divisble, then write $C_\alpha = \{\tilde{\alpha}, \tilde{\beta}\}$ and let

$$C_{\alpha+r} := \{\tilde{\alpha} + r, \tilde{\beta} + r, \tilde{\alpha} + \tilde{\beta} + 2r\}.$$

Lemma 3.3.2. *Let $\alpha + r \in \Phi_{\mathrm{af}}$. We an have isomorphism of varieties over k*

$$\mathsf{U}_{\alpha+r} \cong \mathrm{Res}_{\tilde{k}/k} \Big(\prod_{\tilde{\alpha}+\tilde{r} \in C_{\alpha+r}} (\mathsf{U}_{H,\tilde{\alpha}+\tilde{r}})_{\tilde{k}} \Big)^{\Gamma_G}.$$

Note that in the case k is algebraically closed, this is proved in [24, §9].

Proof. First suppose α is singular or divisible. Then use Lemma 3.2.4 to write an element $x \in \mathsf{U}_{\alpha+r}(k)$ as a Γ_G-invariant sequence $x = (x_{\tilde{\alpha}}) \in \prod_{\tilde{\alpha} \in C_\alpha} U_{H,\tilde{\alpha}}(\tilde{K})$. We must have $v_{\tilde{\alpha}}(x_{\tilde{\alpha}}) = r$. In fact since the valuation on $U_\alpha(K)$ is defined by restriction of that on $\prod_{\tilde{\alpha} \in C_\alpha} U_{H,\tilde{\alpha}}(\tilde{K})$, we have $x \in \mathsf{U}_{\alpha+r}(k)$ if and only if each $x_{\tilde{\alpha}} \in \mathsf{U}_{H,\tilde{\alpha}+r}(\tilde{k})$.

Now suppose $(x, y) \in \mathsf{U}_{\alpha+r}(k) \subset U_\alpha(K)$ and as in Lemma 3.2.4 write $x = (x_{\tilde{\alpha}}, x_{\tilde{\beta}}) \in U_{H,\tilde{\alpha}}(\tilde{K}) \times U_{H,\tilde{\beta}}(\tilde{K})$, with $y \in U_{\tilde{\alpha}+\tilde{\beta}}(\tilde{K})$. Then as before, for $(x, y) \in \mathsf{U}_{\alpha+r}(k)$ it is necessary that $x_{\tilde{\alpha}} \in \mathsf{U}_{\tilde{\alpha}+r}(\tilde{k})$, that $x_{\tilde{\beta}} \in \mathsf{U}_{\tilde{\beta}+r}(\tilde{k})$, and that $y \in \mathsf{U}_{\tilde{\alpha}+\tilde{\beta}+2r}$. □

Note that if $\alpha + r \in \Phi_{\mathrm{af}}$ where α is a plural, non-divisible, and unramified root, then as a variety over k we have $\mathsf{U}_{\alpha+r} \cong \mathsf{U}''_{\alpha+r} \times \mathsf{U}_{2\alpha+2r}$, and in all other cases we have $\mathsf{U}''_{\alpha+r} = \mathsf{U}_{\alpha+r}$. For this reason we use $\mathsf{U}''_{\alpha+r}$ in product decompositions. Taking this into account and applying Lemma 3.3.2, in all cases we have

$$\mathsf{U}''_{\alpha+r} \cong \mathrm{Res}_{\tilde{k}/k}\Big(\prod_{\tilde{\alpha}\in C_\alpha} (\mathsf{U}_{H,\tilde{\alpha}+r})_{\tilde{k}}\Big)^{\Gamma_G}. \tag{3.7}$$

Note that the product in (3.7) is taken over absolute roots C_α, rather than the possibly larger set of affine roots in $C_{\alpha+r}$ as in Lemma 3.3.2.

3.3.2 An Iwahori-type decomposition

Let us define some more functors in groups over k. The goal of this section is to prove a decomposition of the pro-unipotent Iwahori group functor $L^{++}\mathscr{I}$, which will be defined shortly, generalizing the functor defined in [7, §3.7]. We will also make use of twisted strictly negative loop functors for the purpose of describing big cells of partial affine flag varieties.

Let $\mathscr{I}_H$ be the Iwahori group scheme for H corresponding to the alcove $\mathbf{a}_H \subset \mathscr{B}(H, \tilde{K})$ containing $\mathbf{a}$. Let R be a $\tilde{k}$-algebra and consider the functor $L^{++}\mathscr{I}_H$ on $\mathrm{Aff}_{\tilde{k}}$ that sends R to the preimage of $U_H(R)$ in $L^{+}\mathscr{I}_H(R)$ under the composition

$$\mathscr{I}_H(R[\![u]\!]) \hookrightarrow H(R[\![u]\!]) \to H(R[\![u]\!]/u).$$

Then $L^{++}\mathscr{I}_H$ is the pro-unipotent Iwahori functor for H over $\tilde{k}$. Then the pro-unipotent Iwahori functor for G over k is defined to be

$$L^{++}\mathscr{I} = (\mathrm{Res}_{\tilde{k}/k}\,(L^{++}\mathscr{I}_H))^{\Gamma_G,\circ}.$$

Given a facet $\mathbf{f}$ in the closure of $\mathbf{a}$, we also define a twisted strictly negative loop group functor $L^{--}\mathscr{P}_{\mathbf{f}}$ over k as in [13, §3]. Specifically, let $LH_{\tilde{K}}$ be the functor on $\mathrm{Aff}_{\tilde{k}}$ that sends a $\tilde{k}$-algebra R to $H(R(\!(u)\!))$. Let $L^{-}H_{\tilde{K}}$ be the functor on $\mathrm{Aff}_{\tilde{k}}$ that sends a $\tilde{k}$-algebra R to $H(R[u^{-1}])$, and let $L^{--}H_{\tilde{k}}$ be the kernel of the natural map $L^{-}H_{\tilde{K}} \to H_{\tilde{k}}$ defined by $u^{-1} \mapsto 0$. Then the negative H-Iwahori loop functor $L^{--}\mathscr{I}_H$ is the closed sub-functor-in-groups of $LH_{\tilde{K}}$ generated by $L^{--}H_{\tilde{K}}$ and the opposite unipotent group $(U_H^{-})_{\tilde{k}} \subset H_{\tilde{k}}$.

Now let $\mathbf{f}$ be a facet in the closure of $\mathbf{a}$, and let $\mathbf{f}_H$ be the facet of the building $\mathscr{B}(H, \tilde{K})$ containing $\mathbf{f}$. Then we construct the negative loop group $L^{--}\mathscr{H}_{\mathbf{f}_H}$ as in [13, Equation (3.8)]. Let W_H be the Iwahori–Weyl group for H over $\tilde{K}$ with Coxeter subgroup $(W_H)_{\mathrm{af}}$ generated by simple reflections. Then $L^{--}\mathscr{H}_{\mathbf{f}_H}$ is the intersection of the w-conjugates of $L^{--}\mathscr{I}_H \subset LH_{\tilde{K}}$ taken over the $w \in (W_H)_{\mathrm{af}}$ preserving f_H. Then we have

$$L^{--}\mathscr{P}_{\mathbf{f}} := (\mathrm{Res}_{\tilde{k}/k}\,(L^{--}\mathscr{H}_{\mathbf{f}_H}))^{\Gamma_G,\circ}, \tag{3.8}$$

as in [13, Equation (3.12)], where $(-)^{\circ}$ denotes the fiberwise neutral component. Then $L^{--}\mathscr{P}_{\mathbf{f}}$ is exactly the Frobenius invariants of the $\bar{k}$-functor $L^{--}\breve{\mathscr{G}}_{\mathbf{f}}$

Our current goal is to extend the following statement to the generality used in this paper.

Proposition 3.3.3 ([7], Proposition 3.7.4)**.** *Suppose G is split over K. Then for each facet $\mathbf{f}$ in the closure of $\mathbf{a}$ and each w in the Iwahori–Weyl group W there is a decomposition of the pro-unipotent Iwahori functor $L^{++}\mathscr{I}$ as a product of functors as follows:*

$$L^{++}\mathscr{I} = (L^{++}\mathscr{I} \cap wL^{--}\mathscr{P}_{\mathbf{f}}w^{-1}) \cdot (L^{++}\mathscr{I} \cap wL^{+}\mathscr{P}_{\mathbf{f}}w^{-1}),$$

and furthermore the first factor is isomorphic as a k-scheme to an affine space

$$L^{++}\mathscr{I} \cap wL^{--}\mathscr{P}_{\mathbf{f}}w^{-1} \cong \prod_{\alpha+r} \mathsf{U}_{\alpha+r} \cong \prod_{\alpha+r} \mathbb{A}_k^{d(\alpha+r)} \tag{3.9}$$

where the product is taken in any order and runs over those $\alpha + r \in \Phi_{\mathrm{af}}$ such that $\alpha + r > 0$ and $w^{-1}(\alpha + r) \overset{\mathbf{f}}{<} 0$.

In order to make a statement like Proposition 3.3.3 in greater generality, we need a few lemmas relating the affine root groups of G to those of H. In particular, since the product (3.9) may be taken in any order, it is possible to arrange the factors, affine root subgroups of H, in such a way that they imply an analogous result for G.

Lemma 3.3.4. *The map $L^{--}\mathscr{P}_{\mathbf{f}} \to \mathscr{F}_{\mathbf{f}}$ defined by left multiplication on the base point $e_{\mathbf{f}}$ is representable by a quasi-compact open immersion.*

Proof. In the case k is algebraically closed, this is [13, Corollary 3.9].

Suppose k is finite. Then by [13, Corollary 3.9] for any facet $\breve{\mathbf{f}} \subset \mathscr{B}(G_{\breve{K}}, \breve{K})$ the corresponding map

$$L^{--}\breve{\mathscr{P}}_{\breve{\mathbf{f}}} \to \breve{\mathscr{F}}_{\breve{\mathbf{f}}} \tag{3.10}$$

of ind-schemes on $\mathrm{Aff}_{\bar{k}}$ is a quasi-compact open immersion. Suppose $\breve{\mathbf{f}}$ is the unique facet of $\mathscr{B}(G_{\breve{K}}, \breve{K})$ containing $\mathbf{f}$. Recall from [5, §5.1.8] that $\mathscr{P}_{\mathbf{f}}$ is the unique descent over $\mathscr{O}$ of the group scheme $\breve{\mathscr{P}}_{\breve{\mathbf{f}}}$ over $\breve{\mathscr{O}}$. Then $L^+\mathscr{P}_{\mathbf{f}}$ is the unique descent over k of the group scheme $L^+\breve{\mathscr{P}}_{\breve{\mathbf{f}}}$ over $\bar{k}$. Similarly, $L^{--}\mathscr{P}_{\mathbf{f}}$ is the unique descent over k of the functor in groups $L^{--}\breve{\mathscr{P}}_{\breve{\mathbf{f}}}$. And since $(LG)_{\bar{k}} = L(G_{\breve{K}})$, by construction as an étale quotient we have $(\mathscr{F}_{\mathbf{f}})_{\bar{k}} = \breve{\mathscr{F}}_{\breve{\mathbf{f}}}$. Thus the base change of the map

$$L^{--}\mathscr{P}_{\mathbf{f}} \to \mathscr{F}_{\mathbf{f}} \tag{3.11}$$

to $\bar{k}$ is equal to (3.10). Since being a quasi-compact open immersion is fpqc-local on the base [31, 02YJ] and the map $\operatorname{Spec}\bar{k} \to \operatorname{Spec} k$ is fpqc, the map (3.11) is thus also a quasi-compact immersion by descent. $\square$

Proposition 3.3.5. *Let G be a connected, quasi-split, reductive group over K that splits over a tamely ramified extension $\tilde{K}$, and let $\mathbf{f}$ be a facet in the closure of $\mathbf{a}$. Then for each $w \in W$ we have a decomposition of $L^{++}\mathscr{I}$ as a product as follows:*

$$L^{++}\mathscr{I} = (L^{++}\mathscr{I} \cap wL^{--}\mathscr{P}_{\mathbf{f}}w^{-1}) \cdot (L^{++}\mathscr{I} \cap wL^{+}\mathscr{P}_{\mathbf{f}}w^{-1})$$

and furthermore the first factor is isomorphic as a k-scheme to an affine space:

$$L^{++}\mathscr{I} \cap wL^{--}\mathscr{P}_{\mathbf{f}}w^{-1} \cong \prod_{\alpha+r} \mathsf{U}''_{\alpha+r},$$

where the product is taken in any order and runs over those $\alpha + r \in \Phi_{\mathrm{af}}$ such that $\alpha + r > 0$

and $w^{-1}(\alpha + r) \overset{\mathbf{f}}{<} 0$.

Proof. Let $\mathbf{f}$ be a facet in the closure of $\mathbf{a}$, and let $\mathbf{f}_H \subset \mathscr{B}(H, \tilde{K})$ be the facet containing $\mathbf{f}$. Similarly, let $w \in W$, and note that $W \subset W_H^{\Gamma_G}$. Apply Theorem 3.3.3 to H with the parahoric group scheme $\mathscr{H}_{\mathbf{f}_H}$ and Iwahori–Weyl group element w. Then we have a decomposition

$$L^{++}\mathscr{I}_H = (L^{++}\mathscr{I}_H \cap wL^{--}\mathscr{H}_{\mathbf{f}_H}w^{-1}) \cdot (L^{++}\mathscr{I} \cap wL^{+}\mathscr{H}_{\mathbf{f}_H}w^{-1}).$$

Note that Γ_G preserves the factor $L^{++}\mathscr{I}_H \cap wL^{--}\mathscr{H}_{\mathbf{f}_H}w^{-1}$. Indeed, Γ_G permutes the affine root subgroups $\mathsf{U}_{\tilde{\alpha}+r}$ of (3.9), which are those such that $\tilde{\alpha} + r > 0$ and $w^{-1}(\tilde{\alpha} + r) \overset{\mathbf{f}_H}{<} 0$. Now Γ_G must preserve r, and for any $\tilde{\alpha} \in \tilde{\Phi}$ lying over a relative root α, we must have $\tilde{\alpha} + r \overset{\mathbf{a}_H}{>} 0$ if and only if $\alpha + r \overset{\mathbf{f}}{>} 0$, and similarly for $w^{-1}(\alpha + r) \overset{\mathbf{f}_H}{<} 0$. Thus the defining property for affine root groups to be contained in the first factor is invariant under action of Γ_G, and in particular the first factor is invariant under Γ_G. Thus we have

$$\begin{aligned} L^{++}\mathscr{I} &= (L^{++}\mathscr{I}_H)^{\Gamma_G,\circ} \\ &= (L^{++}\mathscr{I}_H \cap wL^{--}\mathscr{H}_{\mathbf{f}_H}w^{-1})^{\Gamma_G,\circ} \cdot (L^{++}\mathscr{I} \cap wL^{+}\mathscr{H}_{\mathbf{f}_H}w^{-1})^{\Gamma_G,\circ}. \end{aligned}$$

Furthermore, since $L^{++}\mathscr{I} = (L^{++}\mathscr{I}_H)^{\Gamma_G,\circ}$, $L^{+}\mathscr{P}_{\mathbf{f}} = (L^{+}\mathscr{H}_{\mathbf{f}_H})^{\Gamma_G,\circ}$ by (3.3), and $L^{--}\mathscr{P}_{\mathbf{f}} = (L^{--}\mathscr{H}_{\mathbf{f}_H})^{\Gamma_G,\circ}$, we have

$$L^{++}\mathscr{I} = (L^{++}\mathscr{I} \cap wL^{--}\mathscr{P}_{\mathbf{f}}w^{-1}) \cdot (L^{++}\mathscr{I} \cap wL^{+}\mathscr{P}_{\mathbf{f}}w^{-1}),$$

as needed.

Now consider the product decomposition of $L^{++}\mathscr{I} \cap wL^{--}\mathscr{P}_{\mathbf{f}}w^{-1}$. Write

$$L^{++}\mathscr{I}_H \cap wL^{--}\mathscr{H}_{\mathbf{f}_H}w^{-1} \cong \prod_{\tilde{\alpha}+r} \mathsf{U}_{H,\tilde{\alpha}+r},$$

where the product is taken over the absolute roots $\tilde{\alpha}+r \in \tilde{\Phi}_{\mathrm{af}}$ such that $\tilde{\alpha}+r > 0$ and $w^{-1}(\tilde{\alpha}+r) \overset{\mathbf{f}_H}{<} 0$. Since the product can be taken in any order, let us ensure that the absolute roots are grouped into Γ_G-orbits. Then by Lemma 3.3.2 and Equation (3.7), upon taking Γ_G-fixed points we have the desired decomposition. $\square$

Lemma 3.3.6 (Stratified triviality)**.** *Let* $\mathbf{f}$ *be a facet in the closure of* $\mathbf{a}$*, let* $w_\bullet$ *be a sequence of elements of* $W_{\mathbf{f}}\backslash W/W_{\mathbf{f}}$*, and let* $v \leq {}^{\mathbf{f}}w_*^{\mathbf{f}}$ *have* $ve_{\mathbf{f}}$ *in the image of the convolution morphism*

$$m : X_{w_\bullet,\mathbf{f}} \to X_{{}^{\mathbf{f}}w_*^{\mathbf{f}},\mathbf{f}}.$$

Then we have an isomorphism $m^{-1}(Y_{v,\mathbf{a},\mathbf{f}}) \cong m^{-1}(ve_{\mathbf{f}}) \times Y_{v,\mathbf{a},\mathbf{f}}$*.*

Proof. This result is proved for G essentially as in [10, Proposition 5.3]. $\square$

As an immediate consequence, to study the fiber over any point $ge_{\mathbf{f}} \in X_{w_*}$ it is sufficient to study the fibers over the base points of Schubert cells $ve_{\mathbf{f}}$.

Proposition 3.3.7. *Let* $\mathbf{f} \subset \mathscr{B}(G,K)$ *be a facet in the closure of* $\mathbf{a}$ *and let* $w_\bullet = (w_1,\ldots,w_p) \in (W_{\mathbf{f}}\backslash W/W_{\mathbf{f}})^p$*. Then the twisted product of Schubert cells* $Y_{w_\bullet,\mathbf{f}} = Y_{w_1,\mathbf{f}} \tilde{\times} \cdots \tilde{\times} Y_{w_p,\mathbf{f}}$ *and the twisted product of Schubert varieties* $X_{w_\bullet,\mathbf{f}} = X_{w_1,\mathbf{f}} \tilde{\times} \cdots \tilde{\times} X_{w_p,\mathbf{f}}$ *are representable by finite-type schemes over* k*. Moreover,* $X_{w_\bullet,\mathbf{f}}$ *is proper over* k*.*

Proof. The proof is inductive on p and in exactly the same manner as [10, Lemmas 11.11 and 11.13]. In particular by Lemma 3.3.4 the quotient map $\pi : LG \to \mathscr{F}_{\mathbf{f}}$ has a section on a Zariski-open neighborhood of the base point $e_{\mathbf{f}}$. Thus, after passing to an étale cover, π becomes Zariski-locally trivial. $\square$

3.4 Proof of main theorems

The proof of the main theorems proceeds in three stages. First Theorem 3.0.1 is proved in the case $\mathbf{f} = \mathbf{a}$ is the fundamental alcove. Then we drop the assumption that $s_1 \cdots s_p = s_*$, maintaining the assumptions that all w_i are simple reflections and that $\mathbf{f} = \mathbf{a}$. Then the assumption that w_i are simple reflections is dropped, so Theorem 3.0.2 is proved in the case $\mathbf{f} = \mathbf{a}$. Finally, it is explained how general parahoric groups follow from the case of the Iwahori.

Before we begin, let us establish some notation. For the time being we work with the affine flag variety $\mathscr{F} = \mathscr{F}_{\mathbf{a}}$. In this section we work on $\bar{k}$-points where convenient to specify morphisms of reduced schemes. Let us introduce relations $\overset{w}{\text{---}}$ and $\overset{\leq w}{\text{---}}$ on points in $\mathscr{F}(\bar{k})$. We say $ge_{\mathbf{a}} \overset{w}{\text{---}} g'e_{\mathbf{a}}$ if $g^{-1}g'e_{\mathbf{a}} \in Y_w(\bar{k})$, and $ge_{\mathbf{a}} \overset{\leq w}{\text{---}} g'e_{\mathbf{a}}$ if $g^{-1}g'e_{\mathbf{a}} \in X_w(\bar{k})$. So we can rephrase the set of $\bar{k}$-points of $Y_{w_\bullet}$ (and also $X_{w_\bullet}$):

$$Y_{w_1} \tilde{\times} \cdots \tilde{\times} Y_{w_p}(\bar{k}) = \{(g_1 e_{\mathbf{a}}, \dots, g_p e_{\mathbf{a}}) \in \mathscr{F}^p(\bar{k}) \mid g_{i-1}e_{\mathbf{a}} \overset{w_i}{\text{---}} g_i e_{\mathbf{a}} \text{ for all } 1 \leq i \leq p\}.$$

We always have $we_{\mathbf{a}} \overset{v}{\text{---}} wve_{\mathbf{a}}$, and for simple reflections we always have $we_{\mathbf{a}} \overset{s}{\text{---}} wse_{\mathbf{a}}$. In general, $ge_{\mathbf{a}} \overset{w}{\text{---}} g'e_{\mathbf{a}}$ if and only if $g'e_{\mathbf{a}} \overset{w^{-1}}{\text{---}} ge_{\mathbf{a}}$. In particular, if s is a reflection then $\overset{s}{\text{---}}$ is a symmetric relation and if s is a simple reflection then $\overset{\leq s}{\text{---}}$ is a symmetric relation.

Recall that for a simple reflection $s \in S_{\mathrm{af}}$ and an element $w \in W$ we always have $wY_s \subset Y_w \cup Y_{ws}$ and $wX_s \subset Y_w \cup Y_{ws}$. Recall from Lemma 3.2.8 that $wY_s \cong \mathsf{U}_{\alpha+r}$, where $s = s_{\alpha+r}$, and recall from Proposition 3.2.6 that $\mathsf{U}_{\alpha+r}(k) = k^{d(\alpha+r)}$. We also get that $wX_s = \mathsf{G}^{\circ}_{\mathbf{f}_s}/\mathsf{B}_{\mathbf{f}_s}$ where $\mathsf{G}^{\circ}_{\mathbf{f}_s}$ is the connected reductive group over k determined by the codimension 1 facet $\mathbf{f}_s$ between $w\mathbf{a}$ and $ws\mathbf{a}$ and $\mathsf{B}_{\mathbf{f}_s} \subset \mathsf{G}^{\circ}_{\mathbf{f}_s}$ is a Borel subgroup. Then wX_s is a one-point compactification of $\mathsf{U}_{\alpha+r}$.

Note that one of the following cases holds:

- $wX_s \cong \mathrm{Res}_{\ell/k}(\mathbb{P}^1_\ell)$ for a finite extension ℓ/k
- $wX_s \cong \mathrm{SU}_3 / B_{\mathrm{SU}_3}$

In each case, we have

$$wX_s - \{we_{\mathbf{a}}\} = wY_s \cong wX_s - \{wse_{\mathbf{a}}\} \cong \mathsf{U}_{\alpha+r} \cong \mathbb{A}^{d(\alpha+r}.$$

Note that by the stratification of Schubert varieties we have $X_s = Y_e \amalg Y_s$ in general. By Proposition 3.2.6, the disjoint union holds on k-points as well for finite k.

Recall also some basic facts about Coxeter groups and Bruhat–Tits theory. Given an element $w \in W$ and a simple reflection $s \in S_{\mathrm{af}}$, we have the BN-pair relations (see for example [29, §2.9]):

$$\mathscr{I}w\mathscr{I}s\mathscr{I} = \begin{cases} \mathscr{I}ws\mathscr{I}, & w < ws \\ \mathscr{I}ws\mathscr{I} \cup \mathscr{I}w\mathscr{I}, & ws < w \end{cases}$$

Lemma 3.4.1. *Suppose $s = s_{\alpha+r} \in S_{\mathrm{af}}$ and $w \in W$.*

1. *If $v < vs$, then $vX_s \cap Y_v = \{ve_{\mathbf{a}}\}$.*

Also $vY_s \cap Y_v = \varnothing$.

2. *If* $v < vs$, *then* $vX_s \cap Y_{vs} = vY_s \cong \mathsf{U}_{\alpha+r}$.

   Also $vY_s \cap Y_{vs} \cong \mathsf{U}_{\alpha+r}$.

3. *If* $vs < v$, *then* $vX_s \cap Y_v = vsX_s - \{vse_{\mathbf{a}}\} \cong \mathsf{U}_{\alpha+r}$.

   Also $vY_s \cap Y_v \cong \mathsf{U}_{\alpha+r} - \{1\}$.

4. *If* $vs < v$, *then* $vX_s \cap Y_{vs} = \{vse_{\mathbf{a}}\}$.

   Also $vY_s \cap Y_{vs} = \{vse_{\mathbf{a}}\}$.

3.4.1 Special case: paving by affines

Proof of Theorem 3.0.1. Induct on p to show that if $s_\bullet = (s_1, \ldots, s_p) \in S_{\mathrm{af}}^p$ is a sequence of simple reflections, then the fiber of the convolution morphism (shown with the map as defined on $\bar{k}$-points)

$$m : X_{s_\bullet} \to X_{s_*}$$

$$(g_1 e_{\mathbf{a}}, \ldots, g_p e_{\mathbf{a}}) \mapsto g_p e_{\mathbf{a}}$$

is paved by affine spaces.

Assume $p > 1$; if $p = 0$ or 1 then m is an isomorphism, and fibers are isomorphic to $\mathbb{A}^0$. Let $s'_\bullet = (s_1, \ldots, s_{p-1})$ and let $s'_* = s_1 * \cdots * s_{p-1}$. Then assume for induction that the fiber of $m' : X_{s'_\bullet} \to X_{s'_*}$ is paved by affine spaces.

Let $v \in W$ be an element $v \leq s_*$. Then $ve_{\mathbf{a}}$ is in the image of m. Indeed, there is some sequence $(\epsilon_i) \in \{0,1\}^p$ such that $v = s_1^{\epsilon_1} \cdots s_p^{\epsilon_p}$. Then we have

$$e_{\mathbf{a}} \overset{\leq s_1}{—} s_1^{\epsilon_1} e_{\mathbf{a}} \overset{\leq s_2}{—} \cdots \overset{\leq s_{p-1}}{—} \left(s_1^{\epsilon_1} \cdots s_{p-1}^{\epsilon_{p-1}}\right) e_{\mathbf{a}} \overset{\leq s_p}{—} v e_{\mathbf{a}}.$$

where $v = s_1^{\epsilon_1} \cdots s_p^{\epsilon_p} \leq s_*$, where each $\epsilon_i \in \{0,1\}$.

Let us now study the fiber over $ve_{\mathbf{a}}$.

For any $(g_1 e_{\mathbf{a}}, \ldots, g_{p-1} e_{\mathbf{a}}, v e_{\mathbf{a}}) \in vX_{s_p}(\bar{k})$, we have

$$e_{\mathbf{a}} \overset{v}{—} v e_{\mathbf{a}} \overset{\leq s_p}{—} g_{p-1} e_{\mathbf{a}}.$$

Since $g_{p-1} e_{\mathbf{a}} \in vX_{s_p}$, we have $g_{p-1} e_{\mathbf{a}} \in Y_v \cup Y_{vs_p}(\bar{k})$. We will understand the possible $g_{p-1} e_{\mathbf{a}}$ by decomposing into those lying over $ve_{\mathbf{a}}$ and those lying over $vs_p e_{\mathbf{a}}$. Define μ as the morphism that sends $\bar{k}$-points

$$\mu : m^{-1}(v e_{\mathbf{a}}) \to Y_v \cup Y_{vs_p}$$

$$(g_1 e_{\mathbf{a}}, \ldots g_{p-1} e_{\mathbf{a}}, v e_{\mathbf{a}}) \mapsto g_{p-1} e_{\mathbf{a}}.$$

Then the following facts hold about the spaces $\operatorname{im}(\mu)_v := \operatorname{im}(\mu) \cap Y_v$ and $\operatorname{im}(\mu)_{vs_p} := \operatorname{im}(\mu) \cap Y_{vs_p}$:

(i) These subsets are either empty, consist of a single point, or are isomorphic to $\mathsf{U}_{\alpha+r}$ where $s_p = s_{\alpha+r}$. In all cases they are each isomorphic to an affine space, whether $\varnothing$, $\mathbb{A}^0$, or $\mathbb{A}^{d(\alpha+r)}$.

(ii) One is closed, possibly empty in $\operatorname{im}(\mu)$, and the other is open, nonempty, and dense in $\mathrm{im}\,(\mu)$.

(iii) The spaces $\mu^{-1}(\mathrm{im}\,(\mu)_v)$ and $\mu^{-1}(\mathrm{im}\,(\mu)_{vs_p})$ are isomorphic to the respective spaces $m'^{-1}(\mathrm{im}\,(\mu)_v)$ and $m'^{-1}(\mathrm{im}\,(\mu)_{vs_p})$ under the obvious projection; in particular, μ is isomorphic to a domain restriction of m'.

Note that (iii) is apparent. The statements (i) and (ii) can be examined by cases, depending on the Bruhat order relationship between v, vs_p, s_*, and s'_*. We have 8 total cases, but not all are possible.

I. $s'_* < s'_* s_p = s_*$

II. $s'_* s_p < s'_* = s_*$

One of the cases I. or II. holds. For the purposes of the current proof, which one holds does not determine the shape of $\mathrm{im}\,(\mu)_v$ or $\mathrm{im}\,(\mu)_{vs_p}$.

1. $v < vs_p$
2. $vs_p < v$

One of these cases 1. or 2. holds, and both are possible regardless of whether I. or II. holds. In case 1. we always have $\mathrm{im}\,(\mu)_v = \{ve_{\mathbf{a}}\}$, while in 2. we always have $\mathrm{im}\,(\mu)_{vs_p} = \{vs_p e_{\mathbf{a}}\}$.

a. Both $v \leq s'_*$ and $vs_p \leq s'_*$.

b. Either $v \not\leq s'_*$ or $vs_p \not\leq s'_*$.

Since by assumption $v \leq s_*$, at least one of $v \leq s_* s_p$ or $vs_p \leq s_* s_p$ must hold by a standard fact about (quasi-) Coxeter groups (see for instance [10, Lemma 6.3]). Indeed, in case

I. that is exactly the statement either $v \leq s'_*$ or $vs_p \leq s'_*$. And in II. we already have $v \leq s_* = s'_*$. Note also that in case II.2. in particular we have $vs_p < v \leq s_* = s'_*$ and so a. holds automatically.

In case a. we always have $\mathrm{im}\,(\mu) = vX_{s_p}$. In case b. we always have $\mathrm{im}\,(\mu)$ consists of one point, either $\{ve_{\mathbf{a}}\}$ if $v \leq s'_*$, or $\{vs_pe_{\mathbf{a}}\}$ if $vs_p \leq s'_*$.

We have:

1.a. $v < vs_p \leq s'_*$

In this case we have $\mathrm{im}\,(\mu)_v = \{ve_{\mathbf{a}}\}$, and we have $\mathrm{im}\,(\mu)_{vs_p} = vY_{s_p}$. The latter is open and dense.

1.b. $v < vs_p$, and $v \leq s'_*$ but $vs_p \not\leq s'_*$

In this case, we have $\mathrm{im}\,(\mu)_v = \{ve_{\mathbf{a}}\}$, and we have $\mathrm{im}\,(\mu)_{vs_p} = \varnothing$. The former is open and dense.

2.a. $vs_p < v \leq s'_*$

In this case, we have $\mathrm{im}\,(\mu)_v = vs_pX_{s_p} - \{vs_pe_{\mathbf{a}}\}$, and we have $\mathrm{im}\,(\mu)_{vs_p} = \{vs_pe_{\mathbf{a}}\}$. The former is open and dense.

2.b. $vs_p < v$, and $vs_p \leq s'_*$ but $v \not\leq s'_*$

In this case, we have $\mathrm{im}\,(\mu)_v = \varnothing$ and $\mathrm{im}\,(\mu)_{vs_p} = \{vs_pe_{\mathbf{a}}\}$. The latter is open and dense.

Thus in every case we have facts (i) and (ii) hold. Then by fact (iii) we observe

$$m^{-1}(ve_{\mathbf{a}}) = \mu^{-1}(\mathrm{im}\,(\mu)_v) \cup (\mathrm{im}\,(\mu)_{vs_p}) \cong m'^{-1}(\mathrm{im}\,(\mu)_v) \cup m'^{-1}(\mathrm{im}\,(\mu)_{vs_p}).$$

Since both $\mathrm{im}\,(\mu)_v$ and $\mathrm{im}\,(\mu)_{vs_p}$ are contained in $\mathscr{I}$-orbits, we can apply stratified triviality of m: in particular,

$$m'^{-1}(\mathrm{im}\,(\mu)_v) \cong m'^{-1}(ve_{\mathbf{a}}) \times \mathrm{im}\,(\mu)_v,$$

$$m'^{-1}(\mathrm{im}\,(\mu)_{vs_p}) \cong m'^{-1}(vs_pe_{\mathbf{a}}) \times \mathrm{im}\,(\mu)_{vs_p},$$

and so the paving result for $m^{-1}(ve_{\mathbf{a}})$ holds by induction. □

3.4.2 The case of the Iwahori

Let us now prove Theorem 3.0.2 in the case $\mathbf{f} = \mathbf{a}$, and all w_i are simple reflections s_i. We will deduce the full generality from this case.

Proof of Theorem 3.0.2, simple reflections. Let $\mathbf{f} = \mathbf{a}$, and let $(s_1, \ldots, s_p)$ be a sequence of simple reflections. Consider the Schubert *cell* convolution morphisms, defined in terms of $\bar{k}$-points as

$$m^\circ : Y_{s_\bullet} \to X_{s_*}, \qquad (g_1e_{\mathbf{a}}, \ldots, g_pe_{\mathbf{a}}) \mapsto g_pe_{\mathbf{a}}$$

$$m^{\circ\prime} : Y_{s'_\bullet} \to X_{s'_*}, \qquad (g_1e_{\mathbf{a}}, \ldots, g_{p-1}e_{\mathbf{a}}) \mapsto g_{p-1}e_{\mathbf{a}}.$$

As before, fix $v \leq s_*$, and assume that $ve_{\mathbf{a}} \in \mathrm{im}\,(m^\circ)$, and consider the morphism, defined below in terms of $\bar{k}$-points

$$\mu^\circ : m^{\circ -1}(ve_{\mathbf{a}}) \to Y_v \cup Y_{vs_p}$$

$$(g_1 e_{\mathbf{a}}, \ldots, g_{p-1}e_{\mathbf{a}}, ve_{\mathbf{a}}) \mapsto g_{p-1}e_{\mathbf{a}}.$$

Again we have property (iii) holds immediately. Let us examine cases to see that property (ii) holds and that a modification of (i) holds:

(i′) Both $\mathrm{im}\,(\mu^\circ)_v$ and $\mathrm{im}\,(\mu^\circ)_{vs_p}$ are either empty, consist of a single point, or are indexed either by $\mathsf{U}_{\alpha+r}$ or $\mathsf{U}_{\alpha+r} - \{1\}$ where $s_p = s_{\alpha+r}$. In all cases they are each isomorphic as varieties to either $\varnothing$, $\mathbb{A}^0$, $\mathbb{A}^{d(\alpha+r)}$, or $\mathbb{A}^{d(\alpha+r)} - \mathbb{A}^0$.

Let us examine the possible cases, after a few remarks.

First note that, just as we cannot guarantee $ve_{\mathbf{a}} \in \mathrm{im}\,(m^\circ)$ when we are dealing with Schubert cells, we cannot guarantee $Y_v \subset \mathrm{im}\,(m^{\circ\prime})$ or $Y_{vs_p} \subset \mathrm{im}\,(m^{\circ\prime})$. However, by Lemma 3.3.6 we have either $Y_v \subset \mathrm{im}\,(m^{\circ\prime})$ or else $Y_v \cap \mathrm{im}\,(m^{\circ\prime}) = \varnothing$, and similarly for Y_{vs_p}.

Note that now we never have $ve_{\mathbf{a}} \in \mathrm{im}\,(\mu^\circ)$. As a result, in case 1. ($v < vs_p$) we always have $\mathrm{im}\,(\mu^\circ)_v = \varnothing$. Case 2. is not much different, so long as $Y_{vs_p} \subset \mathrm{im}\,(m^{\circ\prime})$. And cases a. and b. no longer have uniform interpretations.

1.a. $v < vs_p \leq s'_*$

In this case we have $\mathrm{im}\,(\mu^\circ)_v = \varnothing$, and we have $\mathrm{im}\,(\mu)_{vs_p} = vY_{s_p} \cap \mathrm{im}\,(m^{\circ\prime})$.

1.b. $v < vs_p$, and $v \leq s'_*$ but $vs_p \not\leq s'_*$

In this case $\mathrm{im}\,(\mu^\circ) = \varnothing$.

2.a. $vs_p < v \leq s'_*$

In this case we have $\operatorname{im}(\mu^\circ)_v = (vY_{s_p} - \{vs_pe_{\mathbf{a}}\}) \cap \operatorname{im}(m^{\circ\prime})$, and $\operatorname{im}(\mu^\circ)_{vs_p} = \{vs_pe_{\mathbf{a}}\} \cap \operatorname{im}(m^{\circ\prime})$. If $\operatorname{im}(\mu^\circ)_v$ is nonempty, it is open and dense; and if it is empty then $\operatorname{im}(\mu^\circ)_{vs_p}$ is either empty or open and dense.

2.b. $vs_p < v$, and $vs_p \leq s'_*$ but $v \not\leq s'_*$

In this case we have $\operatorname{im}(\mu^\circ)_v = \varnothing$, and we have $\operatorname{im}(\mu^\circ)_{vs_p} = \{vs_pe_{\mathbf{a}}\} \cap \operatorname{im}(m^{\circ\prime})$.

The induction follows exactly the same structure as in the case of Schubert varieties.

□

With some careful setup, we can see the more general case, still assuming $\mathbf{f} = \mathbf{a}$, as an immediate consequence.

Lemma 3.4.2. *If $w \in W_{\text{af}}$, then let $w = s_1 \cdots s_\ell$ be a reduced expression. Then the convolution morphism is an isomorphism of schemes over k*

$$m^\circ_{s_\bullet} : Y_{s_\bullet} \xrightarrow{\sim} Y_w.$$

Proof. This is a well known fact. We show this inductively on $\ell = \ell(w)$. It is true trivially for $\ell \leq 1$. Then suppose it is true for all words of length $< \ell$ and let $w \in W_{\text{af}}$ be an element of length ℓ and suppose $s_{\alpha+r} \in S_{\text{af}}$ is such that $ws_{\alpha+r} < w$. Let $s_\bullet = (s_1, \ldots, s_{\ell-1}, s_\ell = s_{\alpha+r})$ be a reduced word for w and $s'_\bullet = (s_1, \ldots, s_{\ell-1})$ be the corresponding reduced word for $ws_{\alpha+r}$. Let $ge_{\mathbf{a}} \in Y_w(\bar{k})$. As seen above, the intersection $wY_{s_{\alpha+r}}(\bar{k}) \cap Y_{ws_{\alpha+r}}(\bar{k})$ consists of the unique point $ws_{\alpha+r}e_{\mathbf{a}}$, and so the set

$$\{g'e_{\mathbf{a}} \mid g'e_{\mathbf{a}} \in Y_{ws_{\alpha+r}}(\bar{k}) \text{ and } g'e_{\mathbf{a}} \overset{s_{\alpha+r}}{—} ge_{\mathbf{a}}\}$$

also consists of a single point, call it $g'e_{\mathbf{a}}$. In fact, this is a surjective map to $Y_{ws_{\alpha+r}}(\bar{k})$ as for each $g'e_{\mathbf{a}} \in Y_{ws_{\alpha+r}}(\bar{k})$ the set

$$\{g''e_{\mathbf{a}} \mid g''e_{\mathbf{a}} \in Y_w(\bar{k}) \text{ and } g'e_{\mathbf{a}} \overset{s_{\alpha+r}}{—} g''e_{\mathbf{a}}\}$$

is in correspondence with $\mathsf{U}_{\alpha+r}(\bar{k}) = \bar{k}^{d(\alpha+r)} \neq \varnothing$.

Furthermore, by stratified triviality of $m^{\circ}_{s_\bullet}$, we have

$$m^{\circ-1}_{s_\bullet}(ws_{\alpha+r}Y_{s_{\alpha+r}}) \cong m^{\circ-1}_{s'_\bullet}(ws_{\alpha+r}e_{\mathbf{a}}) \times ws_{\alpha+r}Y_{s_{\alpha+r}},$$

and in particular for any $ge_{\mathbf{a}} \in Y_w(\bar{k})$ we have

$$m^{\circ-1}_{s_\bullet}(ge_{\mathbf{a}}) \cong m^{\circ-1}_{s'_\bullet}(g'e_{\mathbf{a}}) \times wY_{s_p}.$$

Thus $Y_{s_\bullet} \cong Y_w \cong \mathsf{U}_{\alpha+r} \times Y_{ws_{\alpha+r}}$. □

Lemma 3.4.3. *Let $w \in W$, and let $\tau \in \Omega$. Then there is an isomorphism*

$$Y_w \cong \tau Y_w$$

and on $\bar{k}$-points we have

$$\tau Y_w(\bar{k}) = Y_{\tau w}(\bar{k}).$$

Proof. For all $y \in LG(\bar{k})$ we have an isomorphism $yY_w \cong Y_w$ following from group action of LG on $\mathscr{F}$. Then since $L^+\mathscr{I}(\bar{k})$ is normalized by Ω, we have for all $\tau g w e_{\mathbf{a}} \in \tau Y_w(\bar{k})$ that $\tau g w e_{\mathbf{a}} = (\tau g \tau^{-1})\tau w e_{\mathbf{a}} \in Y_{\tau w}(\bar{k})$ as well. □

As consequence of Lemmas 3.4.2 and 3.4.3, if $w = \tau s_1 \cdots s_q \in W$ is a reduced expression then let $s_\bullet = (s_1, \ldots, s_q)$, and we have

$$Y_w(\bar{k}) = \tau Y_{\tau^{-1}w}(\bar{k}) \cong Y_{s_\bullet}(\bar{k}).$$

Lemma 3.4.4. *Let $w_1, \tau w_2 \in W$ where $\tau \in \Omega$ and $w_i \in W_{\mathrm{af}}$. Then we have*

$$Y_{w_1} \tilde{\times} Y_{\tau w_2}(\bar{k}) \cong Y_{w_1\tau} \tilde{\times} Y_{w_2}(\bar{k})$$

Proof. Let $(g_1 e_{\mathbf{a}}, g_2 e_{\mathbf{a}}) \in Y_{w_1} \tilde{\times} Y_{\tau w_2}(\bar{k})$. Then write $g_1 e_{\mathbf{a}} = g_1' w_1 e_{\mathbf{a}}$ for some $g_1' \in L^+\mathscr{I}(\bar{k})$. By Lemma 3.4.3, we have $Y_{\tau w_2} = \tau Y_{w_2}$, so in particular $\tau^{-1} w_1^{-1} g_1'^{-1} g_2 e_{\mathbf{a}} \in Y_{w_2}$. Then, if it is well-defined, the following map is an isomorphism:

$$\begin{aligned} Y_{w_1} \tilde{\times} Y_{\tau w_2} &\to Y_{w_1\tau} \tilde{\times} Y_{w_2} \\ (g_1' w_1 e_{\mathbf{a}}, g_2 e_{\mathbf{a}}) &\mapsto (g_1' w_1 \tau e_{\mathbf{a}}, g_2 e_{\mathbf{a}}). \end{aligned} \tag{3.12}$$

Then (3.12) is well-defined if $g_1' w_1 e_{\mathbf{a}} = w_1 e_{\mathbf{a}}$ if and only if $g_1' w_1 \tau e_{\mathbf{a}} = w_1 \tau e_{\mathbf{a}}$. We have

$g_1' w_1 e_{\mathbf{a}} = w_1 e_{\mathbf{a}}$ if and only if

$$g_1' \in L^+\mathscr{I}(\bar{k}) \cap w_1 L^+\mathscr{I}(\bar{k}) w_1^{-1},$$

and $g_1' w_1 \tau e_{\mathbf{a}} = w_1 \tau e_{\mathbf{a}}$ if and only if

$$g_1' \in L^+\mathscr{I}(\bar{k}) \cap w_1 \tau L^+\mathscr{I}(\bar{k}) \tau^{-1} w_1^{-1} = L^+\mathscr{I}(\bar{k}) \cap w_1 L^+\mathscr{I}(\bar{k}) w_1^{-1},$$

as desired. This concludes the proof in the case $\mathbf{f} = \mathbf{a}$. ☐

Proof of Theorem 3.0.2, Iwahori. Let $w_\bullet = (w_1, \dots, w_p)$ be a sequence of elements of W and for each $1 \leq i \leq p$ let $w_i = \tau_i s_{i,1} \cdots s_{i,q_i}$ be a reduced expression. Then by inductive application of Lemma 3.4.2 we have an isomorphism

$$\begin{aligned}
Y_{w_\bullet} &= Y_{w_1} \tilde{\times} \cdots \tilde{\times} Y_{w_{p-2}} \tilde{\times} Y_{w_{p-1}} \tilde{\times} Y_{w_p} \\
&\cong Y_{w_1} \tilde{\times} \cdots \tilde{\times} Y_{w_{p-2}} \tilde{\times} Y_{w_{p-1}\tau_p} \tilde{\times} Y_{\tau_p^{-1} w_p} \\
&= Y_{w_1} \tilde{\times} \cdots \tilde{\times} Y_{w_{p-2}} \tilde{\times} Y_{\tau_{p-1}\tau_p w'_{p-1}} \tilde{\times} Y_{w'_p} \\
&\cong Y_{\tau_1 \cdots \tau_p w'_1} \tilde{\times} \cdots \tilde{\times} Y_{w'_{p-2}} \tilde{\times} Y_{w'_{p-1}} \tilde{\times} Y_{w'_p} \\
&= \tau_1 \cdots \tau_p (Y_{w'_1} \tilde{\times} \cdots \tilde{\times} Y_{w'_{p-2}} \tilde{\times} Y_{w'_{p-1}} \tilde{\times} Y_{w'_p}) \\
&\cong Y_{w'_1} \tilde{\times} \cdots \tilde{\times} Y_{w'_{p-2}} \tilde{\times} Y_{w'_{p-1}} \tilde{\times} Y_{w'_p},
\end{aligned}$$

where each $w'_i \in W_{\mathrm{af}}$ is an Ω-conjugate of $\tau_i^{-1} w_i$. Note that Ω permutes S_{af} and preserves W_{af}. For each i let $s'_{i,\bullet} = (s'_{i,1}, \dots, s'_{i,q_i})$ be the (reduced) word for w'_i obtained by conjugation by $\tau_{i+1} \cdots \tau_p$, and let $s'_{\bullet,\bullet} = (s'_{1,1}, \dots, s'_{1,q_1}, s'_{2,1}, \dots, s'_{p,q_p})$, with Demazure product $s'_{*,*}$.

Then the paving result for

$$m^{\circ}_{s'_{\bullet,\bullet}} : Y_{s'_{\bullet,\bullet}} \to X_{s'_{*,*}}$$

implies that

$$m^{\circ}_{w_\bullet} : Y_{w_\bullet} \to X_{w_*}$$

also has fibers paved by finite products of spaces of the form $\varnothing$, $\mathsf{U}_{\alpha+r}$, or $\mathsf{U}_{\alpha+r} - \{1\}$. $\square$

3.4.3 Generalization to other parahoric groups

Proposition 3.4.5. *Fix a facet* $\mathbf{f}$ *in the closure of* $\mathbf{a}$ *and let* $w \in W$ *be right-*$\mathbf{f}$*-minimal, meaning it is the minimal representative of its coset* $wW_{\mathbf{f}}$. *Then we have isomorphisms*

$$Y_{w,\mathbf{a},\mathbf{f}} \cong L^{++}\mathscr{I} \cap wL^{--}\mathscr{P}_{\mathbf{f}}w^{-1} \cong L^{++}\mathscr{I} \cap wL^{--}\mathscr{I}w^{-1} \cong Y_{w,\mathbf{a}}.$$

Let $\mathbf{f}$ be a facet in the closure of $\mathbf{a}$. Let $\mathscr{C}$ be the class of varieties that are finite products of the varieties $\mathsf{U}_{\alpha+r}$ or $\mathsf{U}_{\alpha+r} - \{1\}$ for $\alpha + r \in \Phi_{\mathrm{af}}$. Recall that for all $\alpha + r$, the group $\mathsf{U}_{\alpha+r}$ is isomorphic as a variety to $\mathbb{A}^1_k$, $\mathbb{A}^2_k$, or $\mathbb{A}^3_k$, and if k is algebraically closed then $\mathsf{U}_{\alpha+r} \cong \mathbb{A}^1_k$ for all $\alpha + r$. Then for a sequence $w_\bullet = (w_1, \ldots, w_p) \in (W_{\mathbf{f}}\backslash W/W_{\mathbf{f}})^p$, we show the reduced fibers of the convolution map

$$m^{\circ}_{w_\bullet} : Y_{w_\bullet,\mathbf{f}} \to X_{{}^{\mathbf{f}}w^{\mathbf{f}}_*,\mathbf{f}}$$

are paved by $\mathscr{C}$. With $w_\bullet$ fixed, let us refer to $m_{w_\bullet}$ and $m^{\circ}_{w_\bullet}$ simply as m and m°.

Similar to the proof in the case $\mathbf{f} = \mathbf{a}$, let us fix a point $ve_{\mathbf{f}} \in X_{{}^{\mathbf{f}}w^{\mathbf{f}}_*,\mathbf{f}}(\bar{k})$ where $v \leq {}^{\mathbf{f}}w^{\mathbf{f}}_*$

(and v is a right-minimal representative of $vW_{\mathbf{f}}$), assuming $ve_{\mathbf{f}}$ is in the image of $m^{\circ}_{w_{\bullet}}$, and consider the map defined on $\bar{k}$- points

$$\mu^{\circ} : m^{\circ -1}(ve_{\mathbf{f}}) \to X_{\mathbf{f} w^{\mathbf{f}}_{*}, \mathbf{f}}$$

$$(g_1 e_{\mathbf{f}}, \ldots, g_{p-1} e_{\mathbf{f}}, ve_{\mathbf{f}}) \mapsto g_{p-1} e_{\mathbf{f}}.$$

We have $\operatorname{im}(\mu^{\circ}) \subset \operatorname{im}(m'^{\circ}) \cap vY_{w_p^{-1}, \mathbf{f}}$, where $m'^{\circ} := m^{\circ}_{w'_{\bullet}}$ is the convolution morphism corresponding to the sequence $w'_{\bullet} := (w_1, \ldots, w_{p-1})$. We want to pass to $L^+\mathscr{I}$-orbits to apply the results above, particularly Lemma 3.3.6. To that end, consider that $L^+\mathscr{I} \subset L^+\mathscr{P}_{\mathbf{f}}$ and so we have

$$vY_{\eta, \mathbf{a}, \mathbf{f}} \subset vY_{w_p^{-1}, \mathbf{f}},$$

where $\eta \in W$ is any right-minimal representative of the double coset $w_p^{-1} \in W_{\mathbf{f}} \backslash W / W_{\mathbf{f}}$. In particular,

$$W_{\mathbf{f}} w_p^{-1} W_{\mathbf{f}} = \coprod_{\eta W_{\mathbf{f}} \subset W_{\mathbf{f}} w_p^{-1} W_{\mathbf{f}}} \eta W_{\mathbf{f}},$$

and so we have a locally closed stratification

$$vY_{w_p^{-1}, \mathbf{f}}(\bar{k}) = \coprod_{\eta} vY_{\eta, \mathbf{a}, \mathbf{f}}(\bar{k}).$$

Since we have $vY_{\eta, \mathbf{a}, \mathbf{f}} \subset \operatorname{im}(m'^{\circ})$ if and only if $vY_{\eta, \mathbf{a}, \mathbf{f}} \cap \operatorname{im}(m'^{\circ})$ is nonempty, to show that the reduced fiber of m° is paved by $\mathscr{C}$ it is sufficient to show that the nonempty reduced intersections

$$\operatorname{im}(m'^{\circ}) \cap vY_{\eta, \mathbf{a}, \mathbf{f}}$$

are paved by affines. Since η are assumed right-minimal, we have $vY_{\eta,\mathbf{a},\mathbf{f}} = vY_{\eta,\mathbf{a}}$. Then we have the following:

Lemma 3.4.6. *Let $\pi : \mathscr{F} \to \mathscr{F}_{\mathbf{f}}$ be the natural projection morphism. Let $x, y \in W$ be right-$\mathbf{f}$-minimal, and let $w \in W$ be any element. Then by restriction of π we have an isomorphism of* ***reduced*** *k-schemes*

$$\pi_{x,yW_{\mathbf{f}}} : Y_{x,\mathbf{a}} \cap \left(\coprod_{v \in W_{\mathbf{f}}} wY_{yv,\mathbf{a}} \right) \xrightarrow{\sim} Y_{x,\mathbf{a},\mathbf{f}} \cap wY_{y,\mathbf{a},\mathbf{f}}$$

Proof. Begin with the proper surjective morphism

$$\pi_{xW_{\mathbf{f}},yW_{\mathbf{f}}} : \left(\coprod_{v' \in W_{\mathbf{f}}} Y_{xv',\mathbf{a}} \right) \cap \left(\coprod_{v \in W_{\mathbf{f}}} wY_{yv,\mathbf{a}} \right) \longrightarrow Y_{x,\mathbf{a},\mathbf{f}} \cap wY_{y,\mathbf{a},\mathbf{f}} \ .$$

Then since x is right-$\mathbf{f}$-minimal, $Y_{x,\mathbf{a}}$ is closed in $\coprod_{v'} Y_{xv',\mathbf{a}}$, and $\pi_{x,yW_{\mathbf{f}}}$ is also proper. Then $\pi_{x,yW_{\mathbf{f}}}$ is also a monomorphism by Proposition 3.4.5, and so induces an isomorphism of reduced k-schemes. ☐

The general case follows from Lemma 3.4.6 applied to intersections of the form $Y_w \cap vY_{\eta,\mathbf{a},\mathbf{f}}$, which stratify $\operatorname{im}(m'^{\circ}) \cap vY_{w_p^{-1},\mathbf{f}}$ when taken over $w \leq w_p$. And this case follows from the work of Section 3.4.2.

3.5 Parahoric Hecke algebra structure constants

Let us now see how the results above can be applied to parahoric Hecke algebras. We have

Lemma 3.5.1. *Suppose $k = \mathbb{F}_q$ and suppose either $3 \mid q$ or $3 \mid q-1$. Let $\mathbf{f}$ be a facet contained in the closure of $\mathbf{a}$ and let $w_\bullet = (w_1, \dots, w_p)$ be a sequence of elements of $W_{\mathbf{f}}\backslash W/W_{\mathbf{f}}$. Then the fibers of the convolution morphism*

$$m_{w_\bullet} : X_{w_\bullet,\mathbf{f}} \to X_{{}^{\mathbf{f}}w_*^{\mathbf{f}},\mathbf{f}}$$

have the property that the number of $\mathbb{F}_q$-points of $m_{w_\bullet}(we_{\mathbf{f}})(\mathbb{F}_q)$ for a point $we_{\mathbf{f}} \in m_{w_\bullet}(X_{w_\bullet,\mathbf{f}})$ can be expressed as a polynomial

$$\# m_{w_\bullet}(we_{\mathbf{f}})(\mathbb{F}_q) = \sum_{a,b \in \mathbb{Z}_{\geq 0}} m_{a,b} q^a (q-1)^b$$

where $m_{a,b}$ are nonnegative integers, and almost all are 0.

Proof. By Theorem 3.0.2 as well as Proposition 3.2.6, it is sufficient to observe that the number of points in $\mathsf{U}_{\alpha+r}(\mathbb{F}_q)$ and $(\mathsf{U}_{\alpha+r} - \{1\})(\mathbb{F}_q)$ are nonnegative polynomials in q and $q-1$ for all affine roots $\alpha + r \in \Phi_{\mathrm{af}}$. Recall that $\mathsf{U}_{\alpha+r}$ is isomorphic as a variety to $\mathbb{A}_k^{d(\alpha+r)}$. Then we $\#\mathsf{U}_{\alpha+r}(\mathbb{F}_q) = q^{d(\alpha+r)}$, and also $\#(\mathsf{U}_{\alpha+r} - \{1\})(\mathbb{F}_q) = q^{d(\alpha+r)} - 1 = (q-1)(q^{d(\alpha+r)-1} + \cdots + 1)$. $\square$

Now we can deduce a result for parahoric Hecke algebras. Let F be a non-archimedean local field and let $k_F = \mathbb{F}_q$ be the residue field, with ring of integers $\mathscr{O}_F$, and suppose q is either a power of 3 or that $3 \mid q-1$. Let G be a quasi-split, connected, reductive group over F that splits over a tamely ramified extension, and let $T \subset B \subset G$ be a maximal torus and Borel subgroup of G, defined over F. For any facet $\mathbf{f}$ in the apartment $\mathscr{A}$, let $\mathscr{P}_{\mathbf{f}} \subset G(F)$ be the corresponding parahoric subgroup. Consider the parahoric Hecke algebra $\mathscr{H}(G(F)//\mathscr{P}_{\mathbf{f}}) = C_c(\mathscr{P}_{\mathbf{f}}\backslash G(F)/\mathscr{P}_{\mathbf{f}}, \mathbb{C})$ with convolution $*$ defined using the Haar measure on $G(F)$ such that $\mathscr{P}_{\mathbf{f}}$ has volume 1. Consider the basis of characteristic functions $f_w := 1_{\mathscr{P}_{\mathbf{f}} w \mathscr{P}_{\mathbf{f}}}$ indexed by elements $w \in W_{\mathbf{f}}\backslash W/W_{\mathbf{f}}$. Represent such a double coset by an element of maximal length $w \in {}^{\mathbf{f}}W^{\mathbf{f}}$.

Proposition 3.5.2. *For any $w_1, w_2 \in {}^{\mathbf{f}}W^{\mathbf{f}}$, we have*

$$f_{w_1} * f_{w_2} = \sum_{v \in {}^{\mathbf{f}}W^{\mathbf{f}}} c^v_{w_1,w_2}(q) f_v$$

where the structure constant is a nonnegative integer of the form

$$c^v_{w_1,w_2}(q) = \sum_{a,b \in \mathbb{Z}_{\geq 0}} m_{a,b} q^a (q-1)^b$$

for some collection of $m_{a,b} \in \mathbb{Z}_{\geq 0}$, which all vanish except for finitely many.

Proof. If F has residue field $\mathbb{F}_q$ then the structure constants are the same as those for $F = \mathbb{F}_q((t))$, so we may assume $F = \mathbb{F}_q((t))$. Then note that $c^v_{w_1,w_2}(q)$ is exactly the number of $\mathbb{F}_q$-points of the fiber over $ve_{\mathbf{f}}$ of the convolution morphism $Y_{w_1,\mathbf{f}} \tilde{\times} Y_{w_2,\mathbf{f}} \to X_{w_1 * w_2,\mathbf{f}}$. Thus we may apply Theorem 3.0.2 and Lemma 3.5.1. $\square$

Appendix A: Geometric Satake equivalence

I use several theorems of [22], along with one from [2], summarized in Theorem 2.2.1. Collectively, I refer to these statements as the geometric Satake equivalence. In wording more similar to that used by the original sources, I have the following:

Theorem A.1 ([22] Theorem 3.2)**.** a) The intersection $S_e^\nu \cap \mathrm{Gr}_G^\mu$ is nonempty precisely when $\varpi^\nu \in \overline{\mathrm{Gr}_G^\mu}$ and then $S_e^\nu \cap \overline{\mathrm{Gr}_G^\mu}$ is of pure dimension $\langle \rho, \mu + \nu \rangle$, if μ is chosen dominant.

b) The intersection $S_{w_0}^\nu \cap \mathrm{Gr}_G^\mu$ is nonempty precisely when $\varpi^\nu \in \overline{\mathrm{Gr}_G^\mu}$ and then $S_{w_0}^\nu \cap \overline{\mathrm{Gr}_G^\mu}$ is of pure dimension $-\langle \rho, \mu + \nu \rangle$, if μ is chosen anti-dominant.

Theorem A.2 ([22] Theroem 3.5)**.** For all $\mathscr{A} \in P_{L^+G}(\mathrm{Gr}_G, K)$ there is a canonical isomorphism

$$H_c^k(S_e^\nu, \mathscr{A}) \xrightarrow{\sim} H^k_{S_{w_0}^\nu}(\mathrm{Gr}_G, \mathscr{A})$$

and both sides vanish for $k \neq 2\langle \rho, \nu \rangle$.

In particular, the functors $F_\nu : P_{L^+G}(\mathrm{Gr}_G, K) \to \mathrm{Mod}_K$, defined by

$$F_\nu := H_c^{2\langle \rho, \nu \rangle}(S_e^\nu, -) = H^{2\langle \rho, \nu \rangle}_{S_{w_0}^\nu}(\mathrm{Gr}_G, -),$$

are exact.

Theorem A.3 ([22] Theorem 3.6)**.**

$$\mathbb{H}^\bullet \cong \bigoplus_{\nu \in X_*(T)} F_\nu = \bigoplus_{\nu \in X_*(T)} H_c^{2\langle \rho, \nu \rangle}(S_e^\nu, -) : P_{L^+G}(\mathrm{Gr}_G, K) \to \mathrm{Vec}_K$$

Proposition A.0.1 ([22] Proposition 3.10)**.** *Let R be a Noetherian ring of finite global dimension. There is a canonical identification*

$$H_c^{2\langle \rho, \lambda \rangle}(S_e^\lambda, IC_\mu(R)) \cong H_c^{2\langle \rho, \mu - \lambda \rangle}(S_e^\lambda \cap \mathrm{Gr}_G^\mu, R) \cong R[\mathrm{Irr}\,(S_e^\lambda \cap \overline{\mathrm{Gr}_G^\mu})],$$

here $R[\mathrm{Irr}\,(S_e^\lambda \cap \overline{\mathrm{Gr}_G^\mu})]$ stands for the free R-module generated by the irreducible components of $S_e^\lambda \cap \overline{\mathrm{Gr}_G^\mu}$.

Note that [22] Theorem 3.10 is proved using a constant sheaf on the smooth variety $S_e^\lambda \cap \mathrm{Gr}_G^\mu$, as written above. The second isomorphism thus follows from the bijection between $\mathrm{Irr}\,(S_e^\lambda \cap \mathrm{Gr}_G^\mu)$ and $\mathrm{Irr}\,(S_e^\lambda \cap \overline{\mathrm{Gr}_G^\mu})$.

Theorem A.4 ([22] Theorem 12.1)**.** The group scheme $\widetilde{G}_\mathbb{Z}$ is the split reductive group scheme over $\mathbb{Z}$ whose root datum is dual to that of G.

Here $\widetilde{G}_\mathbb{Z}$ is the group over $\mathbb{Z}$ whose category of representations is isomorphic as a tensor category to $P_{L^+G}(\mathrm{Gr}_G, \mathbb{Z})$. Mirković and Vilonen construct $\widetilde{G}_\mathbb{Z}$ as a $\mathbb{Z}$-scheme so a result analogous to Theorem A.4 will hold, by base change, for coefficients in any Noetherian ring R of finite global dimension. In particular, the complex group with dual root datum is $\widehat{G}$, and so $\widehat{G} \cong \widetilde{G} := \mathrm{Spec}\,\mathbb{C} \times_\mathbb{Z} \widetilde{G}_\mathbb{Z}$ has a representation category isomorphic to $P_{L^+G}(\mathrm{Gr}_G, \mathbb{C})$. Furthermore, $\widehat{G^{\sigma,\circ}} \cong \widetilde{G^{\sigma,\circ}} := \mathrm{Spec}\,\mathbb{C} \times_\mathbb{Z} \widetilde{G^{\sigma,\circ}}_\mathbb{Z}$.

Note that $\widetilde{G}$ is naturally endowed with a maximal torus and Borel $\widetilde{T} \subset \widetilde{B} \subset \widetilde{G}$, identifiable using representations of $\widetilde{G}$. In particular, consider the representation

$$\tilde{\mathfrak{g}}^{ss} := Lie([\widetilde{G}, \widetilde{G}]) = \bigoplus_i \mathbb{H}^\bullet(\mathrm{Gr}_G, IC_{\gamma_i^\vee}),$$

where i runs through the components of the Dynkin diagram of G, and $\gamma_i^\vee$ is the highest coroot in $\Phi_i^\vee$. Then we have a decomposition into weight spaces

$$\tilde{\mathfrak{g}}^{ss} = \bigoplus_i \left(H_c^0(S_{w_0}^0, IC_{\gamma_i^\vee}) \oplus \bigoplus_{\alpha^\vee \in \Phi_i^\vee} H_c^{-2\langle \rho, \alpha^\vee \rangle}(S_{w_0}^{\alpha^\vee}, IC_{\gamma_i^\vee}) \right) = \tilde{\mathfrak{g}}^{ss}(0) \oplus \bigoplus_{\alpha^\vee \in \Phi^\vee} \tilde{\mathfrak{g}}^{ss}(\alpha^\vee).$$

For $g \in \widetilde{G}$, we can say $g \in \widetilde{T}$ if g preserves all weight spaces of $\tilde{\mathfrak{g}}^{ss}$, and $g \in \widetilde{B}$ if g preserves the vector subspace

$$\tilde{\mathfrak{g}}^{ss}(0) \oplus \bigoplus_{\alpha^\vee \in \Phi^{\vee,+}} \tilde{\mathfrak{g}}^{ss}(\alpha^\vee).$$

Thus if we fix pinnings of $\widetilde{G}$ and $\widehat{G}$, we can identify the two groups uniquely.

Theorem A.5 ([22] Corollary 13.2)**.** Let R be a Noetherian ring of finite global dimension. The λ-weight spaces $S_\mu(\lambda)$ and $W_\mu(\lambda)$ of S_μ and W_μ, respectively, can both be canonically identified with the free R-module spanned by the irreducible components of $S_e^\lambda \cap \overline{\mathrm{Gr}_G^\mu}$. In particular, the ranks of these modules can be given by the number of irreducible components of $S_e^\lambda \cap \overline{\mathrm{Gr}_G^\mu}$.

In the corollary above, S_μ and W_μ are canonical R-representations of $\widehat{G}$. In particular, taking coefficients in $R = \mathbb{C}$ (or any other field), we have a natural map $S_\mu \to W_\mu$, bijective on underlying vector spaces, factoring through the irreducible highest weight representation V_μ. As a result, the underlying vector space of V_μ has a basis indexed by $\mathrm{Irr}\,(S_e^\lambda \cap \overline{\mathrm{Gr}_G^\lambda})$.

Lemma A.6 ([2] Proposition 5 (iii))**.** Let $\nu \in X_*(T)$ be such that $\nu \geq 0$. If $\mu \in X_*(T)^+$ is sufficiently dominant, then $S_{w_0}^{\mu-\nu} \cap S_e^{\mu} = S_{w_0}^{\mu-\nu} \cap \mathrm{Gr}_G^{\mu}$.

Here a dominant cocharacter μ may be considered "sufficiently dominant" if, for all simple roots $\alpha \in \Pi$, we have $\langle \alpha, \mu \rangle \geq N$, where N is some positive integer depending on the group G and the cocharacter ν.

These are all the statements necessary to state and prove Theorem 2.2.1:

Theorem A.7. Let μ be a dominant cocharacter, and let $\lambda \in Wt(\mu)$.

i. $S_{w_0}^{\lambda} \cap \overline{\mathrm{Gr}_G^{\mu}}$ is equidimensional, and $\dim(S_{w_0}^{\lambda} \cap \overline{\mathrm{Gr}_G^{\mu}}) = \langle \rho, \mu - \lambda \rangle$

ii. $S_{w_0}^{\lambda} \cap S_e^{\mu}$ is equidimensional, and $\dim(S_{w_0}^{\lambda} \cap S_e^{\mu}) = \langle \rho, \mu - \lambda \rangle$

iii. $\mathbb{H}^{\bullet}(\mathrm{Gr}_G, IC_\mu) = \bigoplus_{\lambda \in Wt(\mu)} H_c^{-2\langle \rho, \lambda \rangle}(S_{w_0}^{\lambda}, IC_\mu) = V_\mu$

iv. $H_c^{-2\langle \rho, \lambda \rangle}(S_{w_0}^{\lambda}, IC_\mu) = \bigoplus_{A \in \mathrm{Irr}\,(S_{w_0}^{\lambda} \cap \overline{\mathrm{Gr}_G^{\mu}})} \mathbb{C}[A] = V_\mu(\lambda)$.

v. $P_{L^+G}(\mathrm{Gr}_G, \mathbb{Z})$ is isomorphic as a tensor category to $\mathrm{Rep}_{\mathbb{Z}}(\widehat{G})$.

Proof. Statement i. follows immediately from A.1 (b). Note that $-\mu$ is anti-dominant exactly when μ is dominant.

Statement ii. is not a direct consequence of any statement in [22], but it is well-known. One way to see it follows from Lemma A.6 and statement i.. Indeed, note that for $\nu \in X_*(T)$, the translation of semi-infinite cells $\nu : S_w^{\eta} \to S_w^{\eta+\nu}$ is an isomorphism of ind-schemes. In particular,

$$\dim(S_{w_0}^{\lambda} \cap S_e^{\mu}) = \dim(S_{w_0}^{\lambda+\nu} \cap S_e^{\mu+\nu}).$$

So $\dim(S_{w_0}^{\lambda} \cap S_e^{\mu}) = \dim(S_{w_0}^{\lambda+n\rho^\vee} \cap S_e^{\mu+n\rho^\vee})$ for all integers n. For sufficiently large n, the character $\mu + n\rho^\vee$ is sufficiently dominant, with respect to G and $\mu - \lambda$, to satisfy the hypotheses of Lemma A.6. Therefore

$$\dim(S_{w_0}^{\lambda} \cap S_e^{\mu}) = \dim(S_{w_0}^{\lambda+n\rho^\vee} \cap S_e^{\mu+n\rho^\vee}) = \dim(S_{w_0}^{\lambda+n\rho^\vee} \cap \mathrm{Gr}_G^{\mu+n\rho^\vee}) = \langle \rho, \mu - \lambda \rangle.$$

Statements iii. and iv. are closely tied together. Theorem A.2 tells us that the global cohomology of L^+G-equivariant perverse sheaves decomposes canonically as a direct sum of cohomology groups with compact support, taken on semi-infinite cells. Specifically, for $\mathscr{A} \in P_{L^+G}(\mathrm{Gr}_G)$,

$$\mathbb{H}^\bullet(\mathrm{Gr}_G, \mathscr{A}) = \bigoplus_{\nu \in X_*(T)} H_c^{2\langle \rho, \nu \rangle}(S_e^{\nu}, \mathscr{A}).$$

By symmetry between our choice of Borel and its opposite, we have

$$H_c^{2\langle \rho, \nu \rangle}(S_e^{\nu}, IC_\mu) = H_c^{-2\langle \rho, \nu \rangle}(S_{w_0}^{\nu}, IC_\mu). \tag{A.1}$$

And of course by emptiness of the intersection of $S_{w_0}^{\lambda} \cap \mathrm{Gr}_G^{\mu}$ for $\lambda \notin Wt(\mu)$, the only λ appearing in the direct sum for $\mathbb{H}^\bullet$ are those contained in $Wt(\mu)$. Thus the first equality of iii. follows. And the first equality of iv. follows from Proposition A.0.1.

The second equality of iv. follows from Theorem A.5 and equation (A.1), which in turn implies the second equality of iii..

Statement v. is a restatement of Theorem A.4. $\square$

Bibliography

[1] Jared E. Anderson. “A polytope calculus for semisimple groups”. In: *Duke Mathematical Journal* 116.3 (2003), pp. 567–588. ISSN: 00127094. DOI: `10.1215/S0012-7094-03-11636-1`.

[2] Pierre Baumann and Stéphane Gaussent. “On Mirković-Vilonen cycles and crystal combinatorics”. In: *Representation Theory of the American Mathematical Society* 12.4 (2008), pp. 83–130. ISSN: 1088-4165. DOI: `10.1090/s1088-4165-08-00322-1`.

[3] Alexander A. Beilinson and Vladimir G. Drinfeld. “Quantization of Hitchin’s integrable system and Hecke eigensheaves”. 1991.

[4] François Bruhat and Jacques Tits. “Groupes réductifs sur un corps local: I. Données radicielles valuées”. In: *Publications Mathématiques de l’IHÉS* 41 (1972), pp. 5–251.

[5] François Bruhat and Jacques Tits. “Groupes réductifs sur un corps local: II. Schémas en groupes. Existence d’une donnée radicielle valuée”. In: *Publications Mathématiques de l’IHÉS* 60 (1984), pp. 5–184.

[6] William Casselman, Jorge Cely, and Thomas Hales. “The spherical Hecke algebra, partition functions, and motivic integration”. In: *Transactions of the American Mathematical Society* 371.9 (2019), pp. 6169–6212.

[7] Mark Andrea de Cataldo, Thomas J Haines, and Li Li. “Frobenius semisimplicity for convolution morphisms”. In: *Mathematische Zeitschrift* 289.1 (2018), pp. 119–169.

[8] Pierre Deligne and James S. Milne. “Tannakian Categories”. In: *Hodge Cycles, Motives, and Shimura Varieties*. Ed. by A. Dold and B. Eckmann. 2nd. Berlin: Springer-Verlag, 1989. Chap. II, pp. 101–228. ISBN: 3-540-11174-3. DOI: `10.1007/bfb0091627`.

[9] Thomas Haines. “Dualities for root systems with automorphisms and applications to non-split groups”. In: *Representation Theory of the American Mathematical Society* 22.1 (2018), pp. 1–26.

[10] Thomas J Haines. “Cellular pavings of fibers of convolution morphisms”. In: *arXiv preprint arXiv:2309.14218* (2023).

[11] Thomas J. Haines. “Dualities for root systems with automorphisms and applications to non-split groups”. In: *Representation Theory of the American Mathematical Society* 22.1 (2018), pp. 1–26. ISSN: 1088-4165. DOI: `10.1090/ert/512`.

[12] Thomas J. Haines. “On satake parameters for representations with parahoric fixed vectors”. In: *International Mathematics Research Notices* 2015.20 (2015), pp. 10367–10398. ISSN: 16870247. DOI: `10.1093/imrn/rnu254`.

[13] Thomas J Haines, João Lourenço, and Timo Richarz. "On the normality of Schubert varieties: remaining cases in positive characteristic". In: *arXiv preprint arXiv:1806.11001* 2.3 (2018), p. 5.

[14] Thomas Haines and Michael Rapoport. "On parahoric subgroups". In: *Advances in Mathematics* 219.1 (2008), pp. 188–198.

[15] Jiuzu Hong. "Mirković–Vilonen cycles and polytopes for a symmetric pair". In: *Representation Theory of the American Mathematical Society* 13.3 (2009), pp. 19–32.

[16] Jens C. Jantzen. "Darstellungen halbeinfacher algebraischer Gruppen und zugeordnete kontravariante Formen". PhD thesis. Mathematisches Institut der Universität Bonn, 1973.

[17] Joel Kamnitzer. "Mirković–Vilonen cycles and polytopes". In: *Annals of Mathematics* 171.1 (2010), pp. 245–294. ISSN: 0003486X. DOI: `10.4007/annals.2010.171.245`.

[18] Friedrich Knop. "On the Kazhdan-Lusztig basis of a spherical Hecke algebra". In: *Representation Theory of the American Mathematical Society* 9.14 (2005), pp. 417–425.

[19] Bertram Kostant. "Lie algebra cohomology and the generalized Borel–Weil theorem". In: *The Annals of Mathematics* 74.2 (1961), p. 329. DOI: `10.2307/1970237`.

[20] Robert E Kottwitz. "Isocrystals with additional structure. II". In: *Compositio Mathematica* 109.3 (1997), pp. 255–339.

[21] Shrawan Kumar, George Lusztig, and Dipendra Prasad. "Characters of simplylaced nonconnected groups versus characters of nonsimplylaced connected groups". In: (2009), pp. 99–101. DOI: `10.1090/conm/478/09321`. arXiv: `0701615 [math]`.

[22] Ivan Mirković and Kari Vilonen. "Geometric Langlands duality and representations of algebraic groups over commutative rings". In: *Annals of Mathematics* 166.1 (2007), pp. 95–143. ISSN: 0003486X. DOI: `10.4007/annals.2007.166.95`. arXiv: `0401222 [math]`.

[23] Georgios Pappas and Michael Rapoport. "Local models in the ramified case. III Unitary groups". In: *Journal of the Institute of Mathematics of Jussieu* 8.3 (2009), pp. 507–564.

[24] Georgios Pappas and Michael Rapoport. "Twisted loop groups and their affine flag varieties". In: *Advances in Mathematics* 219.1 (2008), pp. 118–198.

[25] Gopal Prasad. "Galois-fixed points in the Bruhat-Tits building of a reductive group". In: *Bulletin de la Société Mathématique de France* 129.2 (2001), pp. 169–174.

[26] Timo Richarz. "A new approach to the geometric Satake equivalence". In: *Documenta Mathematica* 19 (2014), pp. 209–246.

[27] Timo Richarz. "Affine Grassmannians and geometric Satake equivalences". In: *International mathematics research notices* 2016.12 (2016), pp. 3717–3767.

[28] Timo Richarz. "Schubert varieties in twisted affine flag varieties and local models". In: *Journal of Algebra* 375 (2013), pp. 121–147.

[29] Sean Rostami. “The Bernstein presentation for general connected reductive groups”. In: *Journal of the London Mathematical Society* 91.2 (2015), pp. 514–536.

[30] Tonny A. Springer. “Reductive Groups”. In: *Automorphic Forms, Representations, and L-functions*. Ed. by Armand Borel and William Casselman. 1979, pp. 3–28. ISBN: 0-8218-2437.

[31] The Stacks project authors. *The Stacks project*. `https://stacks.math.columbia.edu`. 2024.

[32] Robert Steinberg. *Endomorphisms of Linear Algebraic Groups*. 80. 1968, pp. 1–112. DOI: `10.1090/memo/0080`.

[33] Jacques Tits. “Reductive groups over local fields”. In: *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part*. Vol. 1. 1979, pp. 29–69.