

ABSTRACT

Title of Dissertation: **ON THE GEOMETRY
OF CERTAIN NON-BASIC
AFFINE DELIGNE-LUSZTIG VARIETIES**
 Zhiming Li
 Doctor of Philosophy, 2025

Dissertation Directed by: **Professor Thomas J. Haines**
 Department of Mathematics

Let F be a non-Archimedean local field with ring of integers \mathcal{O}_F , uniformizer t , residue field $\kappa = \mathcal{O}_F/(t)$, and let $L = \check{F}$ be the completion of the maximal unramified extension of F , with ring of integers \mathcal{O}_L and Frobenius automorphism $\sigma = \sigma_{L/F}$. Let G be an unramified connected reductive group over F , fix a Borel subgroup $B \subset G$ and a maximal split torus $T \subset B$, and write $K = G(\mathcal{O}_L)$ for the hyperspecial subgroup. For $b \in G(L)$ and a dominant cocharacter $\mu \in X_\bullet(T)$, the (affine Grassmannian) affine Deligne–Lusztig variety is the locally closed subfunctor

$$X_\mu^G(b)(\bar{\kappa}) = \{ x \in G(L)/K : x^{-1}b\sigma(x) \in Kt^\mu K \},$$

viewed as a (perfect) κ -scheme locally of finite type. For a Levi subgroup $M \subset G$ and a basic element $b \in M(L)$, there is a natural “reduction to Levi” morphism (see [GHKR06, Prop. 5.6.1])

$$\beta : X_\mu^G(b) \longrightarrow \bigsqcup_{\mu_M \in S_M(\mu, v_b)} X_{\mu_M}^M(b),$$

where the disjoint union runs over those M -dominant cocharacters μ_M that are maximal with the properties $(\mu_M)_{\text{dom}} \preceq \mu$ and $X_{\mu_M}^M(b) \neq \emptyset$.

We already have a good understanding of the geometric structure of the affine Deligne–Lusztig variety when b is basic: Fox and Imai studied the irreducible components of the affine Deligne–Lusztig variety $X_{\mu}^G(b)$ over $G = \text{GU}(2, n - 2)$ when b is a basic element in G (see [FI22]). This thesis focuses primarily on the case $G = \text{GL}_n$ when b is not basic in G but basic in M , which is a *standard* (block diagonal) Levi, to study $X_{\mu}^G(b)$, using the reduction to Levi map β . A further emphasis is placed on the *Drinfeld case*, where the base $X_{\mu_M}^M(b)$ has dimension zero, and in this case we prove that β is a (globally) trivial algebraic bundle with constant affine space fiber. Moreover, when μ is the G -dominant representative of an M -minuscule cocharacter μ_M , we show that β is a Zariski locally trivial vector bundle over $X_{\mu_M}^M(b)$ under natural non-degeneracy conditions on (M, μ, b) . In this setting we compute the dimension of every geometric fiber of β and, under explicit slope-compatibility hypotheses, establish that β is a Zariski locally trivial algebraic vector bundle over $X_{\mu_M}^M(b)$.

Our approach exploits an Iwasawa-type factorization relative to $P = MN$ together with an explicit block-matrix/lattice analysis that converts the fiber condition into linear-algebraic data governed by valuations of matrix entries.

ON THE GEOMETRY OF CERTAIN NON-BASIC AFFINE
DELIGNE–LUSZTIG VARIETIES

by

Zhiming Li

Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2025

Advisory Committee:

Professor Thomas J. Haines, Chair
Professor Larry Washington, Committee Member
Professor Niranjan Ramachandran, Committee Member
Professor Patrick Brosnan, Committee Member
Professor William Gasarch, Dean's Representative

© Copyright by
Zhiming Li
2025

Acknowledgments

I owe my gratitude to all the people who have made this thesis possible and because of whom my graduate experience has been one that I will cherish forever.

First and foremost I'd like to thank my advisor, Thomas J. Haines for giving me this interesting topic and research project. He has given me plenty of help and advice on this project. It has been a pleasure to work with and learn from him, not only his profound knowledge, but also his admirable character.

I would also like to acknowledge help and support from my colleagues and some of the staff members, who always give me kind help and have enriched my graduate life in many ways.

Brothers and sisters in the church: I am deeply grateful for your faithful prayers, encouragement, and fellowship throughout my graduate years. Your love and practical help—meals, rides, and countless check-ins—and the peace we share in worship sustained me through long nights and setbacks. Thank you for reminding me to seek wisdom, to work with integrity, and to rest in God's grace.

I owe my deepest thanks to my family - my mother and father who have always stood by me and guided me through my career, and have pulled me through against impossible odds at times.

Lastly, thank you all and thank God!

Table of Contents

Acknowledgements	ii
Table of Contents	iii
Chapter 1: Introduction	1
Chapter 2: Background for the ADLV on GL_n	6
2.1 Iwasawa decomposition	7
2.2 Coset parametrization and admissibility	9
Chapter 3: Triviality of β for the Drinfeld Case in a 2-Block Levi M	17
3.1 The canonical Drinfeld case	17
3.2 The general Drinfeld case	30
Chapter 4: Triviality of β for the Drinfeld Case in a 3-Block Levi M	42
4.1 Analysis of E_1, E_2, E_3 for pairwise distinct $(k_1, d_1), (k_2, d_2), (k_3, d_3)$	46
4.2 The degenerate case where some (k_i, d_i) are equal	49
Chapter 5: Triviality of β for the Drinfeld Case in a N -Block Levi M	53
Chapter 6: The Non-Minuscule Case: Triviality of β Without Linearity	61
6.1 The two-block canonical Drinfeld case where $\mu = 2\omega_1^\vee$	63
Bibliography	71

Chapter 1: Introduction

Let F be any non-Archimedean local field, with ring of integers \mathcal{O}_F , a fixed uniformizer t , and residue field $\kappa = \mathcal{O}_F/(t)$. Denote by $L = \check{F}$ the completion of its maximal unramified extension, with ring of integers \mathcal{O}_L and Frobenius automorphism $\sigma = \sigma_{L/F}$. Let G be an unramified connected reductive group over F , fix a Borel subgroup $B \subset G$ and a maximal split torus $T \subset B$, and write $K = G(\mathcal{O}_L)$ for the hyperspecial subgroup. For $b \in G(L)$ and a dominant cocharacter $\mu \in X_\bullet(T)$, the affine Deligne–Lusztig variety $X_\mu^G(b)$ is defined by

$$X_\mu^G(b) = \{ x \in G(L)/K : x^{-1}b\sigma(x) \in Kt^\mu K \}.$$

Write $B(G)$ for the set of σ -conjugacy classes in $G(L)$ and let

$$\nu = \nu(\cdot) : B(G) \longrightarrow (X_\bullet(T)_\mathbb{Q})_{\text{dom}}^\Gamma$$

be the Newton map to dominant, Γ -invariant rational cocharacters. Here

$$\Gamma := \text{Gal}(L/F) = \text{Gal}(\check{F}/F) \cong \widehat{\mathbb{Z}},$$

topologically generated by the arithmetic Frobenius $\sigma = \sigma_{L/F}$. The group Γ acts on $X_\bullet(T)$ via

the natural Galois action on T_L , and $(\cdot)^\Gamma$ denotes the Γ -invariants; the subscript “dom” is taken with respect to B (equivalently B_L). For $b \in G(L)$ we write $\nu_b := \nu([b])$ for its Newton point and denote by

$$J_b(R) := \{ g \in G(L \otimes_F R) : g^{-1}b\sigma(g) = b \}$$

the σ -centralizer of b (an algebraic group over F), with *defect*

$$\text{def}_G(b) := \text{rank}_F G - \text{rank}_F J_b.$$

We say that b is *basic* if its Newton point is central, i.e. $\nu_b \in X_\bullet(Z(G))_{\mathbb{Q}}$; equivalently, $[b]$ is basic in $B(G)$. We say that b is *superbasic* if it is not σ -conjugate into any proper Levi subgroup of G defined over F ; equivalently, $[b]$ does not lie in the image of $B(M) \rightarrow B(G)$ for any proper Levi $M \subsetneq G$. (For $G = \text{GL}_n$, b is basic iff the associated isocrystal is isoclinic, and b is superbasic iff, in addition, it is not block-diagonalizable into a proper Levi.)

A simplest version of classical Deligne–Lusztig varieties. Let G be an unramified connected reductive group over the finite field \mathbb{F}_q , $B \subset G$ a Borel subgroup with flag variety G/B , and Fr the q -Frobenius. For a Weyl group element $w \in W$, the *classical Deligne–Lusztig variety* is

$$X_w(1) := \{ gB \in G(\overline{\mathbb{F}}_q)/B : g^{-1}\text{Fr}(g) \in BwB \}.$$

These varieties are smooth, quasi-projective, and satisfy $\dim X_w(1) = \ell(w)$. Deligne and Lusztig used $X_w(1)$ for w in the finite Weyl group to classify the irreducible representations of the finite group of Lie type $G(\mathbb{F}_q)$. Affine Deligne–Lusztig varieties (ADLVs) are the natural p -adic/affine

analogues obtained by replacing B with a parahoric (e.g. $K = G(\mathcal{O}_L)$) and allowing a general element b ; they arose from studying the geometry of Shimura varieties over finite fields.

Fact 1.0.1. *Rapoport’s dimension formula (see [GHKR06, Conjecture 1.0.1] and [Vie06]). Let G be an unramified connected reductive group, and $\mu \in X_\bullet(T)$ dominant. If $X_\mu^G(b) \neq \emptyset$, then*

$$\dim X_\mu^G(b) = \langle \rho, \mu - \nu_b \rangle - \frac{1}{2} \text{def}_G(b), \quad (1.0.1)$$

where ρ is the half-sum of the positive roots (with respect to B).

In particular, in situations where the right-hand side of (1.0.1) vanishes, the corresponding affine Deligne–Lusztig variety is zero-dimensional. For a Levi $M \subset G$, and $b \in M(L)$ basic in M , and a M -dominant minuscule cocharacter μ with $X_\mu^M(b) \neq \emptyset$, if $\dim X_\mu^M(b) = 0$, we refer to this zero-dimensional basic locus in a Levi M as the *Drinfeld case* for (M, μ, b) .

Let $G = \text{GL}_n$ and set a standard Levi subgroup M of G by

$$M = \prod_{i=1}^r \text{GL}_{n_i}, \quad \sum_{i=1}^r n_i = n,$$

so that M is “ r -block” standard. Note that in this case Γ acts on $X_\bullet(T)$ trivially. Let $b \in M(L)$ be M -basic. For a G -dominant $\mu \in X_\bullet(T)$ with $X_\mu^G(b) \neq \emptyset$, define

$$\Sigma(\mu)_M = \{ \mu_M \in X_\bullet(T) : \mu_M \text{ is } M\text{-dominant and } (\mu_M)_{\text{dom}} \preceq \mu \},$$

$$S_M(\mu) = \{ \mu_M \in \Sigma(\mu)_M : \mu_M \text{ is maximal under } \preceq_M \text{ in } \Sigma(\mu)_M \},$$

and

$$S_M(\mu, v_b) = \{\mu_M \in S_M(\mu) : X_{\mu_M}^M(b) \neq \emptyset\}.$$

Notice first that the projection of b to each factor GL_{n_i} , say b_i , is basic in GL_{n_i} . Let $\nu_{b_i} = \lambda_i \cdot (1, \dots, 1)$ be its Newton point, so $n_i \lambda_i = v(\det b_i)$. For any $\mu_M \in S_M(\mu, v_b)$ write $\mu_M = (\mu^{(1)}, \dots, \mu^{(r)})$ with $\mu^{(i)} \in X_{\bullet}(\mathrm{GL}_{n_i})$ dominant. The non-emptiness condition for $X_{\mu_M}^M(b)$ forces the blockwise sum constraint

$$\langle \mathbf{1}_{n_i}, \mu^{(i)} \rangle = \sum_{a=1}^{n_i} \mu_a^{(i)} = n_i \lambda_i = v(\det b_i) \quad (1 \leq i \leq r).$$

If further $\mu^{(i)}$ is minuscule in GL_{n_i} , one must have $\mu^{(i)} = \underbrace{(1 + q_i, \dots, 1 + q_i)}_{p_i}, q_i, \dots, q_i$, where $q_i = \lfloor \lambda_i \rfloor$, $p_i = n_i(\lambda_i - \lfloor \lambda_i \rfloor)$. In what follows we are primarily interested in the case where μ_M is M -minuscule and $(\mu_M)_{\mathrm{dom}} = \mu$. In this case, for any element $\mu' \in S_M(\mu, v_b)$ and any $1 \leq i \leq r$, its GL_{n_i} component has the same weight as the GL_{n_i} component of μ_M . The GL_{n_i} component of μ_M is minuscule for all i so one has $\mu_M \preceq_M \mu'$, and hence by maximality one has $\mu_M = \mu'$. Hence in this case the set $S_M(\mu, v_b)$ is a singleton: there is a unique μ_M compatible with (μ, b) . The reduction morphism

$$\beta : X_{\mu}^G(b) \longrightarrow X_{\mu_M}^M(b)$$

(on the unique non-empty summand) is induced by the canonical projection $G \rightarrow M$ coming from the Iwasawa decomposition relative to $P = MN$.

Our main goal is, in the Drinfeld case for (M, μ_M, b) ,

1. Compute $\dim \beta^{-1}(x)$ for every geometric point $x \in X_{\mu_M}^M(b)$.

2. Give conditions under which β is (Zariski) locally trivial, and in favorable cases a *vector bundle*, with explicitly computed rank.

The philosophy is that the geometry of $X_{\mu_M}^M(b)$ is basically “understood” when b is basic in M (see [FI22]), so the point is to relate the general ADLV to those which are basic. Since any F -Levi subgroup of G is $G(F)$ -conjugate to a standard one, the statements hold for a general Levi subgroup of G .

Chapter 2: Background for the ADLV on GL_n

Basic and superbasic elements in GL_n . For $G = GL_n$, an element $b \in GL_n(L)$ is *basic* iff the isocrystal $(L^n, b\sigma)$ is *isoclinic*, i.e. all its Newton slopes are equal: $\nu_b = \lambda \cdot (1, \dots, 1)$ with a single $\lambda \in \mathbb{Q}$. It is *superbasic* iff it is not σ -conjugate into any proper Levi of GL_n , equivalently, the isocrystal is simple (no nontrivial slope-stable decomposition).

Let $\tau_n \in GL_n(L)$ be the cyclic shift matrix defined by

$$\tau_n e_1 = t e_n, \quad \tau_n e_i = e_{i-1} \quad (2 \leq i \leq n).$$

Then one checks

$$(\tau_n \sigma)^n = t \cdot \sigma^n, \quad (t^q \tau_n^p \sigma)^n = t^{qn+p} \cdot \sigma^n,$$

so $(L^n, t^q \tau_n^p \sigma)$ has single slope $\lambda = q + \frac{p}{n}$. By the classification (see [Kot97]), every isoclinic n -dimensional isocrystal of slope λ is isomorphic to one of the form $(L^n, t^q \tau_n^p \sigma)$ with $0 \leq p < n$ and $q = \lfloor \lambda \rfloor$; in particular:

- If b is *superbasic*, then b is σ -conjugate to $t^q \tau_n^p$ with $\gcd(p, n) = 1$ (the simple cyclic block of dimension n).
- In general, if b is *basic* of slope λ , then b is σ -conjugate into a standard Levi to a block diagonal $\bigoplus_i t^{q_i} \tau_{s_i}^{p_i}$ with all ratios $p_i/s_i = \lambda - \lfloor \lambda \rfloor$ equal; thus b is built from the same

cyclic shift/wrap pattern as powers of τ_\bullet .

Consequently, for purposes of our combinatorial/block computations on affine Deligne-Lusztig varieties, basic elements in GL_n may be treated as (block) powers of τ_n , with the central factor t^q merely shifting the (central) Newton slope and not affecting the cyclic index bookkeeping.

In the Drinfeld situation for GL_n (i.e. the minuscule case with zero-dimensional basic locus), the Euclidean decomposition $p \equiv 1$ or $n - 1 \pmod{n}$ is forced; in other words, modulo inversion of the cyclic direction, one is essentially in the two cases $b = \tau_n$ or $b = \tau_n^{n-1}$. Taking μ to be the corresponding minuscule coweight (so that the Hodge point matches the Newton slope in the basic setting), the condition $g^{-1}b\sigma(g) \in K t^\mu K = K b K$ is equivalent to $g^{-1}b\sigma(g) \in J_b \cdot K$, and the natural map

$$J_b \longrightarrow X_\mu^{\mathrm{GL}_n}(b), \quad j \longmapsto jK,$$

induces an identification $X_\mu^{\mathrm{GL}_n}(b) \cong J_b/(J_b \cap K)$, i.e. the basic locus in the Drinfeld case is a single J_b -orbit of hyperspecial cosets.

2.1 Iwasawa decomposition

Let $P = MU$ be the standard parabolic with Levi M and unipotent radical U . By Iwasawa,

$$G(L) = P(L)K = M(L)U(L)K,$$

and recall that the reduction to Levi map β is defined by the natural projection:

$$X^G = G(L)/K \longrightarrow M(L)/K_M = X^M.$$

We will simply denote K to be the maximal compact subgroups for all the subgroups of G in the following.

We first establish the argument for a “2-block” standard Levi M in G . Let $G = \mathrm{GL}_{n+m}$, $M = \mathrm{GL}_n \times \mathrm{GL}_m$, and $b \in M(L)$ a basic element in M . Write $b = \mathrm{diag}(b_1, b_2)$, and take μ_i to be the unique dominant minuscule cocharacter having the same total weight as b_i in the corresponding block, and finally take $\mu = (\mu_1 \oplus \mu_2)_{\mathrm{dom}}$. Fix a representative $A \oplus B$ such that $(A \oplus B) \bmod K \in X_{\mu_1}^{\mathrm{GL}_n}(b_1) \times X_{\mu_2}^{\mathrm{GL}_m}(b_2)$, then any coset $gK \in \beta^{-1}(A \oplus B)$ can be written as

$$g = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \quad C \in L^{n \times m}.$$

We try to use C above to parametrize the fiber $\beta^{-1}(A \oplus B)$, then restricting to $gK \in X_{\mu}^G(b)$ mapping to $(A \oplus B) \bmod K$ imposes two conditions on C :

(i) When will two parameters C_1, C_2 represent the same coset in $G(L)/K$,

(ii) the admissibility $g^{-1}b\sigma(g) \in K t^{\mu} K$.

2.2 Coset parametrization and admissibility

Write

$$g = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \quad gK \in \beta^{-1}(A \oplus B).$$

1. *Coset identification.* Two such blocks C_1, C_2 define the same coset iff

$$\begin{pmatrix} A & C_1 \\ 0 & B \end{pmatrix} K = \begin{pmatrix} A & C_2 \\ 0 & B \end{pmatrix} K \iff A^{-1}(C_1 - C_2) \in \mathcal{O}_L^{n \times m}.$$

2. *Admissibility.* Computing

$$g^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0 & B^{-1} \end{pmatrix}, \quad \sigma(g) = \begin{pmatrix} \sigma(A) & \sigma(C) \\ 0 & \sigma(B) \end{pmatrix},$$

one finds

$$g^{-1}b\sigma(g) = \begin{pmatrix} A^{-1}b_1\sigma(A) & A^{-1}(b_1\sigma(C) - CB^{-1}b_2\sigma(B)) \\ 0 & B^{-1}b_2\sigma(B) \end{pmatrix}.$$

Now we rewrite g as

$$g = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = (A \oplus B) \times C', \quad C' := A^{-1}C \in L^{n \times m}.$$

Set

$$M_1 = A^{-1} b_1 \sigma(A) \in K t^{\mu_1} K, \quad M_2 = B^{-1} b_2 \sigma(B) \in K t^{\mu_2} K.$$

Then the congruent condition for the new parameter C' is rewritten as:

$$C'_1 \sim C'_2 \iff C'_1 - C'_2 \in \mathcal{O}_L^{n \times m}.$$

and the admissibility condition $g^{-1} b \sigma(g) \in K t^\mu K$ is equivalent to force

$$\begin{pmatrix} M_1 & M_1 \sigma(C') - C' M_2 \\ 0 & M_2 \end{pmatrix} \in K t^\mu K.$$

This reformulation in terms of C' and the conjugates M_1, M_2 makes it clear that the fiber is cut out by a system of linear congruences in the entries of C' . We firstly recall the following lemma:

Lemma 2.2.1 (Smith normal form criterion for Cartan decomposition). *For any $g \in \mathrm{GL}_n(L)$, define the fractional \mathcal{O}_L -ideal by:*

$$\Delta_k(g) = \mathcal{O}_L(\text{determinant of all } k \times k \text{ minors of } g) \subset L, \quad k = 1, \dots, n,$$

which must be principal ideals. Given integers

$$a_1 \leq a_2 \leq \dots \leq a_n,$$

then

$$g \in K \operatorname{diag}(t^{a_1}, \dots, t^{a_n}) K$$

if and only if

$$v(\Delta_k(g)) = a_1 + a_2 + \cdots + a_k \quad \text{for all } k = 1, \dots, n.$$

Then to study what conditions should be imposed on C' , we need to apply the criterion of Lemma 2.1 for the given M_1, M_2 .

Corollary 2.2.2. *Let*

$$g = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \text{GL}_{n+m}(L),$$

where $A \in \text{GL}_n(L)$ and $B \in \text{GL}_m(L)$ are monomial matrices, i.e. each row of A and each column of B contains exactly one nonzero entry. For $1 \leq i \leq n$, let $\alpha_i = v(A_{i,\sigma(i)})$ be the valuation of the unique nonzero in row i of A , and for $1 \leq j \leq m$, let $\beta_j = v(B_{\tau(j),j})$ be the valuation of the unique nonzero in column j of B . Order the combined list $\{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m\}$ into $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_{n+m}$. Then

$$g \in K \text{diag}(t^{\gamma_1}, \dots, t^{\gamma_{n+m}}) K$$

if and only if for every entry c_{ij} of the off-diagonal block C , one has

$$v(c_{ij}) \geq \min\{\alpha_i, \beta_j\}.$$

Proof of Corollary 2.2.2. By left and right multiplication with permutation matrices in K (which

preserves the Cartan double coset and the determinantal ideals Δ_k , we may assume

$$A = \text{diag}(t^{\alpha_1}, \dots, t^{\alpha_n}), \quad B = \text{diag}(t^{\beta_1}, \dots, t^{\beta_m}),$$

so that

$$g = \begin{pmatrix} \text{diag}(t^{\alpha_1}, \dots, t^{\alpha_n}) & C \\ 0 & \text{diag}(t^{\beta_1}, \dots, t^{\beta_m}) \end{pmatrix}.$$

Let $\Gamma = \{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m\}$ and write its nondecreasing rearrangement as $\gamma_1 \leq \dots \leq$

γ_{n+m} .

Sufficiency. Assume that for all i, j one has

$$v(c_{ij}) \geq \min\{\alpha_i, \beta_j\}.$$

Fix k and consider any nonzero $k \times k$ minor of g . Such a minor necessarily selects some set $I \subset \{1, \dots, n\}$ of rows from the top block and some set $J \subset \{1, \dots, m\}$ of rows from the bottom block, with $|I| + |J| = k$, together with the same number of columns; because the lower-left block is zero, any chosen left (resp. right) column must be matched with a top (resp. bottom) row, except possibly when a top row is paired with a right column via an entry of C . Expanding the determinant as a sum over matchings, every monomial term is a product of factors, each of which is either t^{α_i} (if a top row i is matched to its left column i), or t^{β_j} (if a bottom row j is matched to its right column j), or c_{ij} (if a top row i is matched to a right column j through C). By the hypothesis $v(c_{ij}) \geq \min\{\alpha_i, \beta_j\}$, the valuation of any such product is bounded below by the sum of the corresponding α - and β -contributions in which each c_{ij} is replaced by $\min\{\alpha_i, \beta_j\}$.

In particular, the minimal possible valuation among all terms is achieved by the “block-diagonal” matching that avoids C altogether and pairs s top rows with s left columns and $k - s$ bottom rows with $k - s$ right columns, for some $0 \leq s \leq k$. Hence

$$v(\Delta_k(g)) = \min_{0 \leq s \leq k} \left(\alpha_{(1)} + \cdots + \alpha_{(s)} + \beta_{(1)} + \cdots + \beta_{(k-s)} \right),$$

where $\alpha_{(r)}$ (resp. $\beta_{(r)}$) denotes the r -th smallest α (resp. β). This minimum is exactly $\gamma_1 + \cdots + \gamma_k$.

By the Smith normal form criterion (Lemma 2.2.1), we conclude

$$g \in K \operatorname{diag}(t^{\gamma_1}, \dots, t^{\gamma_{n+m}}) K.$$

Necessity. Conversely, suppose there exist indices (i_0, j_0) with

$$v(c_{i_0 j_0}) < \min\{\alpha_{i_0}, \beta_{j_0}\}.$$

We will produce a $k \times k$ minor whose valuation is strictly smaller than $\gamma_1 + \cdots + \gamma_k$, contradicting the Smith criterion.

Let S be the multiset of the $k - 1$ smallest elements of $\Gamma \setminus \{\max(\alpha_{i_0}, \beta_{j_0})\}$; that is, from Γ remove the larger of $\alpha_{i_0}, \beta_{j_0}$ and take the $k - 1$ smallest remaining elements. Then

$$\gamma_1 + \cdots + \gamma_k = \min\{\alpha_{i_0}, \beta_{j_0}\} + \sum_{x \in S} x.$$

Form the $k \times k$ submatrix by choosing:

- the top row i_0 and the right column $n + j_0$ (thus contributing the entry $c_{i_0 j_0}$);

- for each $\alpha_i \in S$, the top row i and the left column i (contributing t^{α_i});
- for each $\beta_j \in S$, the bottom row $n + j$ and the right column $n + j$ (contributing t^{β_j}).

Because all other chosen entries lie on the block diagonal and the lower-left block is zero, this submatrix is upper triangular up to a single off-diagonal $c_{i_0 j_0}$, so its determinant is (up to a unit)

$$\det = c_{i_0 j_0} \cdot \prod_{\alpha_i \in S} t^{\alpha_i} \cdot \prod_{\beta_j \in S} t^{\beta_j},$$

and therefore

$$v(\det) = v(c_{i_0 j_0}) + \sum_{x \in S} x < \min\{\alpha_{i_0}, \beta_{j_0}\} + \sum_{x \in S} x = \gamma_1 + \cdots + \gamma_k.$$

Thus $v(\Delta_k(g)) < \gamma_1 + \cdots + \gamma_k$, contradicting the Cartan/Smith criterion. Hence necessarily $v(c_{ij}) \geq \min\{\alpha_i, \beta_j\}$ for all i, j .

Combining the two directions completes the proof. □

We give an example below to help readers better understand the above Corollary.

Example 2.2.3. ($A = \tau_2, B = \tau_3$). Recall

$$\tau_2 = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & 0 & 0 \end{pmatrix}.$$

These are monomial. By rows of A we get

$$\alpha_1 = v(1) = 0, \quad \alpha_2 = v(t) = 1,$$

and by columns of B we get

$$\beta_1 = v(t) = 1, \quad \beta_2 = v(1) = 0, \quad \beta_3 = v(1) = 0.$$

The multiset $\{\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3\} = \{0, 1, 1, 0, 0\}$ rearranges to

$$\gamma_\bullet = (0, 0, 0, 1, 1).$$

Hence the corollary says that

$$g = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in K \operatorname{diag}(1, 1, 1, t, t) K \iff v(c_{ij}) \geq \min\{\alpha_i, \beta_j\}.$$

Writing the 2×3 matrix $C = (c_{ij})$, the inequalities are

	$j = 1$	$j = 2$	$j = 3$
$i = 1$	$v(c_{11}) \geq 0$	$v(c_{12}) \geq 0$	$v(c_{13}) \geq 0$
$i = 2$	$v(c_{21}) \geq 1$	$v(c_{22}) \geq 0$	$v(c_{23}) \geq 0$

Thus *only* the lower-left entry must be divisible by t ; all other off-block entries need only be integral. If, for instance, $v(c_{21}) = 0$, one can choose a 4×4 minor using c_{21} whose valuation

drops below $\gamma_1 + \cdots + \gamma_4 = 1$, contradicting the Smith normal form criterion.

Chapter 3: Triviality of β for the Drinfeld Case in a 2-Block Levi M

Let $r > 1$ and recall that $\tau_r \in \mathrm{GL}_r(L)$ denotes the *canonical superbasic element*. By definition, one computes

$$\tau_r = \begin{pmatrix} 0 & 1 & & \\ \vdots & \ddots & \ddots & \\ 0 & & \ddots & 1 \\ t & 0 & \cdots & 0 \end{pmatrix},$$

i.e. the unique $r \times r$ matrix with t in the lower-left corner, 1 on the super-diagonal, and 0 elsewhere. Moreover notice that $\tau_1 = (t) \in \mathrm{GL}_1(L)$.

3.1 The canonical Drinfeld case

We firstly study the case where b is canonically superbasic in a 2-Block Levi M , say the component of b in each $r \times r$ block is τ_r . We retain all notation from above: $G = \mathrm{GL}_{n+m}$, $M = \mathrm{GL}_n \times \mathrm{GL}_m$, and for the canonical superbasic

$$b = \mathrm{diag}(\tau_n, \tau_m) \in M(L)$$

we write $\mu = (1, 1, 0, \dots, 0)$ and $\mu_M = (1, 0, \dots, 0) \oplus (1, 0, \dots, 0)$. Recall that for the superbasic element $\tau_r \in \mathrm{GL}_r(L)$ and the minuscule coweight $\omega_r = (1, 0, \dots, 0)$, the affine Deligne-Lusztig variety

$$X_{\omega_r}^{\mathrm{GL}_r}(\tau_r) = \{ gK \in \mathrm{GL}_r(L)/K \mid g^{-1}\tau_r\sigma(g) \in K t^{\omega_r} K \}$$

identifies canonically with the σ -centralizer quotient

$$X_{\omega_r}^{\mathrm{GL}_r}(\tau_r) \cong J_{\tau_r} / (J_{\tau_r} \cap K).$$

In particular, in our case we may fix

$$M_1 = \tau_n \in \mathrm{GL}_n(L), \quad M_2 = \tau_m \in \mathrm{GL}_m(L).$$

Notice that we are mainly interested in the case where b is NOT basic in G , in which case $m \neq n$, but now we also include the case $m = n$ for the study of our whole setting which can be seen later.

Writing

$$C' = A^{-1}C \in L^{n \times m}, \quad h = M_1 \sigma(C') - C' M_2 = (h_{ij})_{1 \leq i \leq n, 1 \leq j \leq m},$$

Applying Corollary 2.2.2, $g^{-1}b\sigma(g) \in K t^\mu K$ iff:

$$h_{n,1} = (\tau_n \sigma(C') - C' \tau_m)_{n,1} \in t \mathcal{O}_L, \quad h_{ij} \in \mathcal{O}_L \text{ for all other } (i, j).$$

Theorem 3.1.1. *Fix integers $n, m \geq 1$. Let $G = \mathrm{GL}_{n+m}$, $M = \mathrm{GL}_n \times \mathrm{GL}_m$, and set*

$$b = \mathrm{diag}(\tau_n, \tau_m), \quad \mu = (1, 1, 0, \dots, 0).$$

Then for every point $(A \oplus B) \bmod K \in X_{\mu_M}^M(b)$, the fiber of $\beta: X_{\mu}^G(b) \rightarrow X_{\mu_M}^M(b)$ has dimension

$$\dim_{\bar{k}} \beta^{-1}(A \oplus B) = \min\{n, m\} - 1,$$

in exact agreement with the Rapoport dimension formula

$$\dim(X_{\mu}^G(b)) = \langle \rho, \mu - \bar{v}_b \rangle - \frac{1}{2} \mathrm{def}_G(b).$$

Moreover, as varieties over \bar{k} , β is a trivial bundle with fiber

$$\begin{cases} \mathbb{A}^{\min\{n, m\} - 1} & \text{if } n \neq m, \\ \mathbb{A}^{k-1} \times \left(L^{\sigma^k} / \mathcal{O}_{L^{\sigma^k}} \right)^k & \text{if } n = m = k. \end{cases}$$

Before proving the main theorem, we introduce a convenient shorthand and establish an integrality lemma.

Definition 3.1.2. For elements $w, v \in L$, we write

$$w \xrightarrow{\sigma} v \iff \sigma(w) - v \in \mathcal{O}_L.$$

Since for any $w \in L$, σ has finite order on the image $\bar{w} \in L/\mathcal{O}_L$, one sees that the relation $\xrightarrow{\sigma}$ generates an equivalent relation \sim on L/\mathcal{O}_L , and each equivalence class of it has finitely many elements.

For elements $w, v \in L$, we write

$$\begin{aligned} w \xrightarrow{t\sigma} v &\iff t\sigma(w) - v \in \mathcal{O}_L, \\ w \xrightarrow{t^{-1}\sigma} v &\iff \sigma(w) - tv \in \mathcal{O}_L. \end{aligned}$$

It is easy to see that $\xrightarrow{t\sigma}, \xrightarrow{t^{-1}\sigma}$ induce a reverse relation pair on $(L/\mathcal{O}_L)/\sim$ which we denote by $\xrightarrow{t}, \xrightarrow{t^{-1}}$ respectively. Moreover, one has

$$(w \xrightarrow{t} v) \wedge (w \xrightarrow{t} u) \implies v = u,$$

as well as

$$w \xrightarrow{t} w \iff w = \mathcal{O}_L, \text{ the trivial class,}$$

that is, the relation \xrightarrow{t} extends to a cotree order (i.e. each element in the set has at most 1 successor) on $(L/\mathcal{O}_L)/\sim$ with the unique terminal element \mathcal{O}_L .

Lemma 3.1.3 (Integrality on a Cyclic Tree). *Let $R = \{r_1, \dots, r_N\} \subset L$ be a finite set whose*

images in L/\mathcal{O}_L form a cycle under the relations $\xrightarrow{\sigma}$, $\xrightarrow{t\sigma}$, or $\xrightarrow{t^{-1}\sigma}$. Suppose furthermore that R admits a piecewise partition

$$R = \left(\bigsqcup_{i=1}^r W_i \right) \sqcup \left(\bigsqcup_{i=1}^s S_i \right)$$

into disjoint subchains W_i and S_i satisfying:

- (1) Within each W_i or S_i , every arrow is of type $\xrightarrow{\sigma}$.
- (2) For each adjacent pair W_i, W_j , they are connected by an arrow of type $\xrightarrow{\sigma}$ or $\xrightarrow{t^{-1}\sigma}$ ($\xrightarrow{t\sigma}$ resp.).
- (3) In each $S_i = \{s_{i,1}, \dots, s_{i,r_i}\}$, the first element $s_{i,1}$ admits an incoming arrow of type $\xrightarrow{t^{-1}\sigma}$, and the last element s_{i,r_i} admits an outgoing arrow of type $\xrightarrow{t\sigma}$.
- (4) At least one adjacent pair (W_i, W_j) admits an arrow of type $\xrightarrow{t^{-1}\sigma}$ ($\xrightarrow{t\sigma}$ resp.).

Then:

- Every element of each W_i belongs to \mathcal{O}_L .
- Within each S_i , once one entry is fixed modulo \mathcal{O}_L , all others are uniquely determined modulo \mathcal{O}_L . Equivalently, the set of solutions in S_i is a torsor under the one-dimensional $\bar{\kappa}$ -vector space $t^{-1}\mathcal{O}_L/\mathcal{O}_L$.

Proof. By (3), each subchain S_i is "sandwiched" between a unique incoming arrow of type $\xrightarrow{t^{-1}\sigma}$ and a unique outgoing arrow of type $\xrightarrow{t\sigma}$. These two arrows necessarily connect S_i to (resp. from) the same σ -equivalence class among the W_j by the uniqueness of the immediate successor.

Now remove all the S_i from the original cyclic diagram in R , replacing each by a finite concatenation of pure $\xrightarrow{\sigma}$ arrows. The result is a smaller loop whose arrows are only of type

$\xrightarrow{\sigma}$ or $\xrightarrow{t^{-1}\sigma}$ ($\xrightarrow{t\sigma}$ resp.). Tracing this loop from any element necessarily returns to the starting point, and hence by (4), each such parameter must coincide with its own $t^{-1}\sigma$ -successor ($t\sigma$ -successor resp.), forcing it into \mathcal{O}_L .

Finally, re-inserting each S_i between its prescribed incoming and outgoing arrows shows that each S_i contributes exactly one free class in $t^{-1}\mathcal{O}_L/\mathcal{O}_L$. These yield the independent $\bar{\kappa}$ -parameters claimed, completing the proof. \square

Proof of Theorem 3.1 in the Case $n \neq m$

Proof. Write $C' = (c_{ij})$ for $1 \leq i \leq n, 1 \leq j \leq m$, and extend the indices periodically by

$$c_{i,j} = c_{i',j'} \quad \text{where } i' \equiv i \pmod{n}, j' \equiv j \pmod{m}.$$

Then the admissibility conditions

$$h_{ij} = (\tau_n \sigma(C') - C' \tau_m)_{ij} \in \mathcal{O}_L \quad (\forall (i,j) \neq (n,1)), \quad h_{n1} \in t\mathcal{O}_L$$

translate into the relations

$$\sigma(c_{i,j}) - c_{i-1,j-1} \in \mathcal{O}_L \quad (1 < i \leq n, 1 < j \leq m),$$

$$t\sigma(c_{1,j}) - c_{n,j-1} \in \mathcal{O}_L \quad (1 < j \leq m),$$

$$\sigma(c_{i,1}) - t c_{i-1,m} \in \mathcal{O}_L \quad (1 < i \leq n),$$

$$t\sigma(c_{1,1}) - t c_{n,m} \in t\mathcal{O}_L \iff \sigma(c_{1,1}) - c_{n,m} \in \mathcal{O}_L.$$

Starting from any c_{ij} and for each time we subtract both indexes by 1, we then get a cyclic loop:

$$c_{i,j} \rightarrow c_{i-1,j-1} \rightarrow \cdots c_{i+1,j+1} \rightarrow c_{i,j},$$

where the arrows are $\xrightarrow{\sigma}$, $\xrightarrow{t\sigma}$ or $\xrightarrow{t^{-1}\sigma}$. By certain symmetric argument of the following one, without loss of generality, we can assume $n < m$. For each $1 \leq i \leq n-1$, define the i -th diagonal block parameter set

$$S_i = \{ c_{j, m-i+j} \mid 1 \leq j \leq i \},$$

i.e., the set of the i -th “strict upper off-diagonal”, and for $n \leq i \leq m$, define the i -th diagonal block parameter set

$$W_i = \{ c_{j, m-i+j} \mid 1 \leq j \leq n \},$$

i.e., the set of the i -th “diagonal other than the strict upper off-diagonals”. Then the loop above consists of some specific W_i 's and S_i 's. A direct inspection shows that for any given loop above, the hypotheses (1)-(3) of Lemma 3.3 hold for the collection $\{W_i\}$ and $\{S_i\}$ that appear in the loop with the arrow in hypotheses (2) the $\xrightarrow{t\sigma}$ (the assumption $n < m$ plays a role here: if instead $n > m$, the arrow in hypotheses (2) is $\xrightarrow{t^{-1}\sigma}$). We have to check the hypothesis (4): There is at least one adjacent pair (W_i, W_j) that admits an arrow of type $\xrightarrow{t\sigma}$ in the loop. Such an arrow can only from $c_{1,j}$ to $c_{n,j-1}$ for some $1 < j \leq m$, and $c_{1,j}, c_{n,j-1}$ can both be contained in W_i 's iff $j \leq m-n+1$. Therefore it suffices to show that the loop contains $c_{1,j}$ for some $1 < j \leq m-n+1$. Write $d = \gcd(n, m)$, then it is easy to observe that two parameters $c_{1,j}, c_{1,j'}$ belong to the same loop iff $j \equiv j' \pmod{d}$. Since $m > n$, one has that the list $2, \dots, m-n+1$ is a full list of

representatives in $\mathbb{Z}/d\mathbb{Z}$, so it has nontrivial intersection with any loop above.

Therefore by Lemma 3.3, each $w \in W_i$ lies in \mathcal{O}_L , and within each S_i there is a single free class in $t^{-1}\mathcal{O}_L/\mathcal{O}_L$. Hence the total number of free parameters is $\sum_{i=1}^{n-1} 1 = n - 1$, as claimed.

Moreover, we can choose the coordinates $c_{1,m}, \dots, c_{n-1,m}$ uniformly as the free parameters so that we see β is trivial with fiber \mathbb{A}^{n-1} . □

Next we establish a discreteness lemma to prove the remaining part of Theorem 3.1.

Lemma 3.1.4 (Discreteness on a Cyclic Tree). *Let*

$$R = \{r_1, \dots, r_N\} \subset L$$

be a finite set whose images in L/\mathcal{O}_L form a single cycle under the relations $\xrightarrow{\sigma}, \xrightarrow{t\sigma}, \xrightarrow{t^{-1}\sigma}$.

Suppose furthermore that R admits a decomposition into disjoint subchains

$$R = \left(\bigsqcup_{i=1}^r W_i \right) \sqcup \left(\bigsqcup_{i=1}^s S_i \right)$$

satisfying the conditions (1)-(3) of Lemma 3.3, and in addition

(4') Every adjacent pair (W_i, W_{i+1}) in the cycle is connected by an arrow of type $\xrightarrow{\sigma}$.

Let $d = |R|$ be the length of the cycle. Then for any chosen base-point $w_0 \in W_i$, the iterated congruences around the loop give

$$\sigma^d(w_0) - w_0 \in \mathcal{O}_L.$$

Moreover, once w_0 is fixed modulo \mathcal{O}_L , all other elements of every W_i are uniquely determined, while each S_i carries a free parameter identified with $(t^{-1}w_0 + t^{-1}\mathcal{O}_L)/\mathcal{O}_L$. In particular, the solution space for w_0 is $L^{\sigma^d}/\mathcal{O}_{L^{\sigma^d}}$, and the total solution set of R is $\bar{\kappa}^{\{|S_i|\}} \times L^{\sigma^d}/\mathcal{O}_{L^{\sigma^d}}$, which is a finite dimensional affine $\bar{\kappa}$ -space times a discrete set.

Proof. First, replace each $s_i \in S_i$ by $\tilde{s}_i = t s_i$. Under this rescaling every $\xrightarrow{t\sigma}$ or $\xrightarrow{t^{-1}\sigma}$ arrow becomes a plain $\xrightarrow{\sigma}$ arrow, so the entire cycle on $\tilde{R} = (\bigsqcup W_i) \sqcup (\bigsqcup \tilde{S}_i)$ is now a σ -cycle of length d . Fix a base-point $w_0 \in W_i$; following the unique σ -arrows around the loop yields

$$\sigma^d(w_0) - w_0 \in \mathcal{O}_L.$$

Each W_i -element is then uniquely recovered from w_0 via successive σ -arrows, and each original s_i is determined by $(t^{-1}\tilde{s}_i + t^{-1}\mathcal{O}_L)/\mathcal{O}_L$, with \tilde{s}_i itself uniquely determined by $w_0 \pmod{\mathcal{O}_L}$. Again, inside one S_i , once a base-point s_0 is fixed, the other s_i 's are determined by the σ -arrows. Consequently the full solution set is a product of one copy of the discrete group $L^{\sigma^d}/\mathcal{O}_{L^{\sigma^d}}$ (parametrizing w_0) and one copy of $\bar{\kappa}$ for each S_i , as claimed. \square

Proof of Theorem 3.1 in the Case $n = m = k$

Proof. Retain the notation and periodic indexing from the $n \neq m$ case. Observe that the index range

$$1 < j \leq m - n + 1 = 1$$

is empty, so no $c_{1,j}$ ever appears and hence the condition (4') in Lemma 3.4 holds. Notice that each loop has length

$$\frac{nm}{\gcd(n, m)} = \frac{k^2}{k} = k.$$

There are exactly k loops: one “pure” diagonal loop $W_k = \{c_{1,1}, c_{2,2}, \dots, c_{k,k}\}$ and $k - 1$ “off-diagonal” loops each of which traverses a single W_i and a single S_j .

By Lemma 3.4 each off-diagonal loop yields a one-dimensional affine \bar{k} -parameter together with the discrete factor $L^{\sigma^k}/\mathcal{O}_{L^{\sigma^k}}$. The pure diagonal loop, having no S_i , contributes only the discrete quotient $L^{\sigma^k}/\mathcal{O}_{L^{\sigma^k}}$. Altogether there are $k - 1$ nontrivial loops, together with a discrete factor $(L^{\sigma^k}/\mathcal{O}_{L^{\sigma^k}})^k$, which can be parametrized uniformly by $c_{1,k}, \dots, c_{k,k}$, as required. \square

We compute some easy examples below to help readers understand the above argument better.

Example 3.1.5. Take $n = 1, m = 1$, so $G = \mathrm{GL}_2, M = \mathrm{GL}_1 \times \mathrm{GL}_1$. Fix

$$b_1 = t, \quad b_2 = 1, \quad \mu = (1, 0).$$

Then $A, B \in L^\times$ satisfy $A^{-1}t\sigma(A) \in K tK, B^{-1}\sigma(B) \in K$. Recall that $X_{(1)}^{\mathrm{GL}_1}(1) = F/\mathcal{O}_F \cong \mathbb{Z}$, thus we may take $A = t^{r_1}, B = t^{r_2}$. Then the parameter $C \in L$ is subject to

$$t\sigma(C) - C \in A\mathcal{O}_L.$$

Write $v(-)$ for the valuation on L . Since $v(t\sigma(C) - C) = \min\{v(C) + 1, v(C)\} = v(C)$, the

condition $t\sigma(C) - C \in A\mathcal{O}_L$ is equivalent to

$$v(C) \geq v(A).$$

Hence $C \in A\mathcal{O}_L$, and modulo the equivalence $C \sim C'$ when $C - C' \in A\mathcal{O}_L$, the fiber is

$$A\mathcal{O}_L/A\mathcal{O}_L \cong \{0\},$$

a single point. In particular $\dim \beta^{-1}(A \oplus B) = 0$.

Example 3.1.6. Take $G = \mathrm{GL}_3$, $n = 2$, $m = 1$. Let

$$b_1 = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix} \in \mathrm{GL}_2(L), \quad b_2 = t \in L^\times, \quad \mu = (1, 1, 0).$$

Any element of the fiber above $(A \oplus B) \bmod K \in X_{\mu_M}^M(b)$ may be written as

$$g = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = (A \oplus B) \times C', \quad C' := A^{-1}C \in L^{2 \times 1}.$$

Set

$$M_1 = A^{-1}b_1\sigma(A) \in K \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} K, \quad M_2 = B^{-1}b_2\sigma(B) \in KtK.$$

Notice that

$$X_{(1,0)}^{\mathrm{GL}_2}(b_1) = X_{b_1}(b_1) = J_{b_1}/(J_{b_1} \cap K)$$

is itself a basic locus, so we may take $M_1 = b_1$ and $M_2 = t$.

Then the congruence condition is rewritten as:

$$C'_1 \sim C'_2 \iff C'_1 - C'_2 \in \mathcal{O}_L^{2 \times 1}.$$

and the admissibility condition $g^{-1}b\sigma(g) \in K t^\mu K$ is equivalent to the single matrix-equation

$$M_1 \sigma(C') - C' M_2 = h = (h_{ij}) \in (\mathcal{O}_L)_{11} \oplus (t\mathcal{O}_L)_{21}.$$

Writing $C' = (c_1, c_2)^T$, the admissibility condition

$$M_1 \sigma(C') - C' M_2 = b_1 \sigma(C') - t C' = \begin{pmatrix} \sigma(c_2) - t c_1 \\ t(\sigma(c_1) - c_2) \end{pmatrix} \in (\mathcal{O}_L)_{11} \oplus (t\mathcal{O}_L)_{21}$$

implies $c_1 \in \mathcal{O}_L$ and $c_2 \in \mathcal{O}_L$. Therefore the dimension of $\beta^{-1}((A \oplus B) \bmod K)$ is 0.

Example 3.1.7 (recover a known basic example). Take $G = \mathrm{GL}_2$, $n = m = 1$. Let

$$b_1 = b_2 = 1, \quad \mu = (0, 0) = 1.$$

Here one sees b is in fact basic in G but we can still use the off-block analysis. With the same notations as before, it is obvious that $M_1 = 1$ and $M_2 = 1$, and the admissibility condition

$g^{-1}b\sigma(g) \in K t^\mu K = K$ is

$$\sigma(C') - C' \in \mathcal{O}_L.$$

We know that $\{c \in L : \sigma(c) - c \in \mathcal{O}_L\} = \mathcal{O}_L + F$, therefore we recover the fact:

$$X_1^{\text{GL}_2}(\text{Id}) = \text{GL}_2(F)/\text{GL}_2(\mathcal{O}_F),$$

hence β is trivial with fiber isomorphic to F/\mathcal{O}_F , which is a discrete set and has dimension 0.

Example 3.1.8 (a non-trivial example). Set $G = \text{GL}_5$, $n = 2$, $m = 3$, with

$$b_1 = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & 0 & 0 \end{pmatrix}, \quad \mu = (1, 1, 0, 0, 0),$$

Again we have

$$X_{(1,0)}^{\text{GL}_2}(b_1) = X_{b_1}(b_1) = J_{b_1}/(J_{b_1} \cap K) \quad X_{(1,0,0)}^{\text{GL}_3}(b_2) = X_{b_2}(b_2) = J_{b_2}/(J_{b_2} \cap K)$$

are themselves basic loci.

Thus we can again take $M_1 = b_1$, $M_2 = b_2$ and write

$$g = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = A \oplus B \times C', \quad C' = A^{-1}C \in L^{2 \times 3}.$$

The admissibility condition $g^{-1}b\sigma(g) \in K t^\mu K$ becomes

$$b_1 \sigma(C') - C' b_2 \in \mathcal{O}_L^5 \oplus (t\mathcal{O}_L)_{21}.$$

Write $C' = (c_{ij})$, one computes

$$b_1 \sigma(C') - C' b_2 = \begin{pmatrix} \sigma(c_{21}) - t c_{13} & \sigma(c_{22}) - c_{11} & \sigma(c_{23}) - c_{12} \\ t \sigma(c_{11}) - t c_{23} & t \sigma(c_{12}) - c_{21} & t \sigma(c_{13}) - c_{22} \end{pmatrix}.$$

The “loop tracing” argument implies that $c_{13} \in t^{-1}\mathcal{O}_L$, and the other c_{ij} ’s lie in \mathcal{O}_L . Therefore the solution space is one-dimensional over \bar{k} with free parameter c_{13} . In particular the map β is a trivial vector bundle over the basic loci $X_{(1,0)}^{\text{GL}_2}(b_1) \times X_{(1,0,0)}^{\text{GL}_3}(b_2)$.

3.2 The general Drinfeld case

Next we treat the general Drinfeld case in $M = \text{GL}_n \times \text{GL}_m$. Recall that for a superbasic b in $M(L)$, we can write

$$b = (\tau_n^{d_1}, \tau_m^{d_2}),$$

with $\gcd(d_1, n) = \gcd(d_2, m) = 1$. Since $\tau_r^r = t\text{Id}_r$, perform the Euclidean divisions

$$d_1 = q_1 n + p_1, \quad 0 < p_1 < n, \quad d_2 = q_2 m + p_2, \quad 0 < p_2 < m,$$

so that

$$b = (t^{q_1} \tau_n^{p_1}, t^{q_2} \tau_m^{p_2}),$$

still with $\gcd(p_1, n) = \gcd(p_2, m) = 1$.

The unique M -dominant minuscule μ_M is then

$$\mu_M = \underbrace{(1 + q_1, \dots, 1 + q_1, q_1, \dots, q_1)}_{p_1} \oplus \underbrace{(1 + q_2, \dots, 1 + q_2, q_2, \dots, q_2)}_{p_2},$$

and we take μ to be its G -dominant rearrangement. In the Drinfeld case, we have $\dim X_{\mu_M}^M(b) = 0$, which forces $p_1 \in \{1, n - 1\}$ and $p_2 \in \{1, m - 1\}$.

Theorem 3.2.1. *(layered case) In the setting of Drinfeld case above, assume $q_1 \neq q_2$. Then for every $(A, B) \bmod K \in X_{\mu_M}^M(b)$ the fiber*

$$\beta^{-1}(A, B) \subset X_{\mu}^G(b)$$

is zero-dimensional (indeed, a single K -coset over \bar{k}).

Proof. In the Drinfeld case setting of GL_n , we can write

$$M_1 = t^{q_1} \tau_n^{p_1}, \quad M_2 = t^{q_2} \tau_m^{p_2},$$

with $p_1 \in \{1, n - 1\}$ and $p_2 \in \{1, m - 1\}$. Without loss of generality assume $q_1 < q_2$ and set

$\delta := q_2 - q_1 \geq 1$. As in Section 2, put $C' = A^{-1}C$ and

$$h = M_1 \sigma(C') - C' M_2.$$

By Corollary 2.2.2 together with the Drinfeld (minuscule) shape of μ_M , the admissibility condition $g^{-1}b\sigma(g) \in K t^\mu K$ forces all entries of h to have valuation $\geq q_1$, and the entries with minimal possible valuation are exactly those lying in $t^{q_1} \mathcal{O}_L$. Factoring out t^{q_1} we obtain the normalized congruence system

$$\tilde{h} := \tau_n^{p_1} \sigma(C') - t^\delta C' \tau_m^{p_2}, \quad (3.2.1)$$

with some entries lying in $t\mathcal{O}_L$ and all others in \mathcal{O}_L .

Now recall the shape of τ_r^p : it is the permutation matrix for the shift by p with a factor t exactly at the wrap positions; more precisely, multiplication on the left by $\tau_n^{p_1}$ sends an entry of C' either to a *unit* multiple (coefficient 1) of a σ -shifted entry or to a *uniformizer* multiple (coefficient t) of a σ -shifted entry, depending on whether the row-shift wraps. Likewise, multiplication on the right by $\tau_m^{p_2}$ sends an entry either to a unit multiple or to a t -multiple, depending on whether the column-shift wraps.

Consequently, each scalar congruence extracted from (3.2.1) has the form

$$u \sigma(c_{i,j}) - t^\delta v c_{i',j'} \in t^{\text{val}(u)} \mathcal{O}_L, \quad u, v \in \{1, t\}, \quad (3.2.2)$$

linking a pair of entries $(i, j) \mapsto (i', j')$ determined by the shifts p_1 and p_2 . Rewriting (3.2.2)

gives

$$\sigma(c_{i,j}) \equiv t^k c_{i',j'} \pmod{\mathcal{O}_L}, \quad k = \delta + v_t - u_t \in \{\delta - 1, \delta, \delta + 1\},$$

where $u_t = 1$ if $u = t$ and 0 otherwise (similarly for v_t). Since $\delta \geq 1$, we always have $k \geq 0$; moreover $k = 0$ can occur only when $\delta = 1$ and simultaneously $u = t, v = 1$. In all other cases $k \geq 1$.

View the index set $\{(i, j)\}$ as vertices of a directed graph whose edges are the relations $(i, j) \rightarrow (i', j')$ furnished by (3.2.2). Because the loop advances by $(-p_1, -p_2)$ modulo (n, m) , and $p_1 = 1, n - 1$ and $p_2 = 1, m - 1$, we have $i' = i \pm 1, j' = j \pm 1$. Along any loop, compose the congruences to obtain

$$\sigma^r(c) \equiv t^K c \pmod{\mathcal{O}_L}, \quad (3.2.3)$$

for some length $r \geq 1$ and integer $K = \sum k_\ell$ equal to the sum of the exponents k along the loop. By the discussion above, every edge satisfies $k_\ell \geq 0$, and $k_\ell = 0$ can only happen when $\delta = 1$ and $\sigma(c_{i,j})$ is shifted by a uniformizer multiple while $c_{i',j'}$ is shifted by a unit multiple at that step. It is impossible for *every* step in a loop to satisfy this simultaneous wrap condition: indeed, the left wrap occurs precisely on a subset of p_1 consecutive rows, and the right wrap on a subset of p_2 consecutive columns; as the loop advances by either plus or minus one index modulo (n, m) , it must exit at least one of these wrap subsets. Hence every loop contains at least one edge with $k_\ell > 0$, so $K \geq 1$.

From (3.2.3) we deduce $t^K c - \sigma^r(c) \in \mathcal{O}_L$ with $K \geq 1$, which forces $c \in \mathcal{O}_L$ (apply valuations and use that σ preserves v). Since this holds for the representative c of each σ -orbit of entries in every loop, all entries of C' lie in \mathcal{O}_L . By the coset identification $C'_1 \sim C'_2 \iff C'_1 - C'_2 \in \mathcal{O}_L^{n \times m}$, it follows that all admissible C' represent the same coset as 0, and hence the

fiber $\beta^{-1}(A, B)$ consists of a single K -coset, and hence $\dim_{\bar{k}} \beta^{-1}(A, B) = 0$. \square

Therefore, the only potentially nontrivial situation for a 2-block Levi M arises when $q_1 = q_2$. In this case the pair (b, μ) differs from $(\tau_n^{p_1} \oplus \tau_m^{p_2}, \mu')$ by the central twist t^{q_1} , so by dividing out this scalar we reduce to the case $q_1 = q_2 = 0$ without changing the fiber structure. The canonical superbasic case $p_1 = p_2 = 1$ has already been treated; thus it remains to analyze the complementary Drinfeld configurations

$$(p_1, p_2) \in \{(n-1, m-1), (1, m-1), (n-1, 1)\}.$$

By symmetry between the two blocks, it suffices to handle the mixed cases $(1, m-1)$ and $(n-1, m-1)$; the remaining mixed case follows by swapping the roles of n and m . We now turn to these three residual cases.

3.2.1 The case $p_1 = n-1, p_2 = m-1$

Write

$$\tau_n^{n-1} = t \cdot \tau_n^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ t & 0 & \cdots & 0 & 0 \\ 0 & t & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & t & 0 \end{pmatrix}, \quad \tau_m^{m-1} = t \cdot \tau_m^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ t & 0 & \cdots & 0 & 0 \\ 0 & t & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & t & 0 \end{pmatrix}.$$

Thus τ_r^{r-1} is the “reverse cyclic shift”: it shifts indices by -1 with a factor t on the subdiagonal and a unit at the $(1, r)$ -entry. The admissibility equation in this case reads

$$h = \tau_n^{n-1} \sigma(C') - C' \tau_m^{m-1},$$

and by Corollary 2.2.2 (applied with the monomial factors τ_n^{n-1} and τ_m^{m-1}) entrywise one has

$$h_{i,j} = t \sigma(c_{i-1,j}) - t c_{i,j+1} \in t\mathcal{O}_L \quad (1 < i \leq n, 1 \leq j < m),$$

$$h_{1,j} = \sigma(c_{n,j}) - t c_{1,j+1} \in \mathcal{O}_L \quad (1 \leq j < m),$$

$$h_{i,m} = t \sigma(c_{i-1,m}) - c_{i,1} \in \mathcal{O}_L \quad (1 < i \leq n),$$

$$h_{1,m} = \sigma(c_{n,m}) - c_{1,1} \in \mathcal{O}_L.$$

The integrality pattern is obtained from the canonical case $p_1 = p_2 = 1$ with the direction of the arrows reversed as well as the roles of the arrows of type $t\sigma / t^{-1}\sigma$ interchanged. Hence all loop-tracing and congruence arguments from the case $p_1 = p_2 = 1$ apply essentially the same way to C' here as before. In particular, the fiber description, the number of free parameters, and the (local) triviality of β in this $(n-1, m-1)$ configuration agree with the canonical case, up to the above interchange.

Theorem 3.2.2. *Fix integers $n, m \geq 1$. Let $G = \mathrm{GL}_{n+m}$, $M = \mathrm{GL}_n \times \mathrm{GL}_m$, and set*

$$b = \mathrm{diag}(\tau_n^{n-1}, \tau_m^{m-1}), \quad \mu = (1, 1, \dots, 1, 0, 0).$$

Then for every point $(A \oplus B) \bmod K \in X_{\mu_M}^M(b)$, the fiber of $\beta: X_\mu^G(b) \rightarrow X_{\mu_M}^M(b)$ has dimension

$$\dim_{\bar{k}} \beta^{-1}(A \oplus B) = \min\{n, m\} - 1,$$

in exact agreement with the Rapoport dimension formula

$$\dim(X_\mu^G(b)) = \langle \rho, \mu - \bar{v}_b \rangle - \frac{1}{2} \text{def}_G(b).$$

Moreover, as varieties over \bar{k} , β is a trivial bundle with fiber

$$\begin{cases} \mathbb{A}^{\min\{n, m\}-1} & \text{if } n \neq m, \\ \mathbb{A}^{k-1} \times \left(L^{\sigma^k} / \mathcal{O}_{L^{\sigma^k}} \right)^k & \text{if } n = m = k. \end{cases}$$

Sketch of proof. This is the “dual” version of Theorem 3.1.1. Write

$$h = \tau_n^{n-1} \sigma(C') - C' \tau_m^{m-1}.$$

As noted in the preceding subsection, the integrality pattern for h is obtained from the canonical case $p_1 = p_2 = 1$ with the direction of the arrows reversed as well as the roles of the arrows of type $t\sigma / t^{-1}\sigma$ interchanged. Consequently, all loop-tracing and congruence arguments from Theorem 3.1.1 apply verbatim to C' as well.

Define the diagonal “chains” S_i and W_i exactly as in the canonical case: S_i consists of the strictly upper off-diagonals of height i ($1 \leq i \leq \min\{n, m\} - 1$), while W_i consists of the remaining diagonals (including the possible main diagonal). The coset-congruence $C'_1 \sim$

$C'_2 \iff C'_1 - C'_2 \in \mathcal{O}_L^{n \times m}$ and the admissibility bounds for h imply that each W_i contributes no free $\bar{\kappa}$ -parameter, whereas each S_i contributes exactly one. Thus the number of free parameters is $\sum_{i=1}^{\min\{n,m\}-1} 1 = \min\{n, m\} - 1$, realized explicitly, e.g., by the entries on the last column of the strictly upper off-diagonals

$$c_{1,m}, c_{2,m}, \dots, c_{\min\{n,m\}-1, m}.$$

The fiber description then reduces exactly the same way as in Theorem 3.1.1. □

3.2.2 The case $p_1 = 1, p_2 = m - 1$

In this mixed Drinfeld configuration we have

$$h = \tau_n \sigma(C') - C' \tau_m^{m-1}.$$

Writing $C' = (c_{ij})_{1 \leq i \leq n, 1 \leq j \leq m}$, and using that τ_n shifts rows up by $+1$ with a factor t on wrap ($(n, 1)$ -entry), while τ_m^{m-1} shifts columns down by -1 with a factor t on the subdiagonal and a unit at $(1, m)$, one checks the entrywise formulas

$$h_{i,j} = \sigma(c_{i+1,j}) - t c_{i,j+1} \in \mathcal{O}_L \quad (1 \leq i < n, 1 \leq j < m),$$

$$h_{n,j} = t \sigma(c_{1,j}) - t c_{n,j+1} \in t\mathcal{O}_L \quad (1 \leq j < m),$$

$$h_{i,m} = \sigma(c_{i+1,m}) - c_{i,1} \in \mathcal{O}_L \quad (1 \leq i < n),$$

$$h_{n,m} = t \sigma(c_{1,m}) - c_{n,1} \in \mathcal{O}_L.$$

By inspecting the congruence relations above, the only $t\sigma$ -edge in the index diagram arises from

$h_{n,m} = t\sigma(c_{1,m}) - c_{n,1}$, that is,

$$c_{1,m} \xrightarrow{t\sigma} c_{n,1}.$$

All other edges are of type σ or $t^{-1}\sigma$. Consequently, except the degenerate cases that we list in the following theorem, loop-tracing forces integrality to propagate through every cycle, and the only free parameter occurs at the entry $c_{1,m}$ (modulo \mathcal{O}_L). Hence $c_{1,m}$ is the unique free variable, and the fiber $\beta^{-1}(A \oplus B)$ is an affine line over $\bar{\kappa}$; in particular, $\dim_{\bar{\kappa}} \beta^{-1}(A \oplus B) = 1$.

Theorem 3.2.3. *Let $n, m \geq 1$, $G = \mathrm{GL}_{n+m}$, $M = \mathrm{GL}_n \times \mathrm{GL}_m$, and consider the mixed Drinfeld configuration*

$$b = \mathrm{diag}(\tau_n, \tau_m^{m-1}), \quad \mu = (\underbrace{1, \dots, 1}_m, 0, \dots, 0),$$

with the analogous statement holding after swapping the two blocks (τ_n^{n-1}, τ_m) . Except for the degenerate case $n = m = 2$ (which reduces to the canonical case $p_1 = p_2 = 1$), the following holds: for every point $(A \oplus B) \bmod K \in X_{\mu_M}^M(b)$, the fiber of $\beta: X_{\mu}^G(b) \rightarrow X_{\mu_M}^M(b)$ is an affine line over $\bar{\kappa}$, hence

$$\dim_{\bar{\kappa}} \beta^{-1}(A \oplus B) = 1,$$

and β is (Zariski) a trivial \mathbb{A}^1 -bundle over its image.

Proof. Retain the mixed Drinfeld configuration and the entrywise relations

$$h_{i,j} = \sigma(c_{i+1,j}) - t c_{i,j+1} \in \mathcal{O}_L \quad (1 \leq i < n, 1 \leq j < m),$$

$$h_{n,j} = t \sigma(c_{1,j}) - t c_{n,j+1} \in t\mathcal{O}_L \quad (1 \leq j < m),$$

$$h_{i,m} = \sigma(c_{i+1,m}) - c_{i,1} \in \mathcal{O}_L \quad (1 \leq i < n),$$

$$h_{n,m} = t \sigma(c_{1,m}) - c_{n,1} \in \mathcal{O}_L.$$

Thus every edge in the index diagram is labelled by σ or $t^{-1}\sigma$, except for the *unique* $t\sigma$ -edge

$$c_{1,m} \xrightarrow{t\sigma} c_{n,1}.$$

We focus on the $t^{-1}\sigma$ -edges:

$$\boxed{c_{i+1,j} \xrightarrow{t^{-1}\sigma} c_{i,j+1}} \quad (1 \leq i < n, 1 \leq j < m). \quad (*)$$

Loops on the torus and their number. Indices live on the torus $\mathbb{Z}/n \times \mathbb{Z}/m$. Every edge moves by

$$(i, j) \mapsto (i - 1, j + 1),$$

so directed cycles are exactly the orbits of the translation by $(-1, +1)$. Hence the diagram decomposes into $d := \gcd(n, m)$ disjoint loops; an invariant of a loop is

$$s(i, j) := i + j \pmod{d}.$$

(Indeed s is unchanged by $(-1, +1)$.)

Covering all loops by $t^{-1}\sigma$ -targets. Consider the two strips of sources for (*):

$$V := \{(i+1, 1) : 1 \leq i < n\}, \quad H := \{(2, j) : 1 \leq j < m\}.$$

These give $t^{-1}\sigma$ -edges with targets

$$(i, 2) \quad (i = 1, \dots, n-1) \quad \text{and} \quad (1, j+1) \quad (j = 1, \dots, m-1),$$

respectively. When $n, m \geq 3$, the residues $s(i, 2) = i + 2$ (for $i = 1, \dots, n-1$) and $s(1, j+1) = j + 2$ (for $j = 1, \dots, m-1$) form $(n-1) + (m-1) - 1 = m + n - 3$ consecutive classes modulo d . We have

$$m + n - 3 \geq \min\{n, m\} \geq d,$$

so every residue class $s \in \mathbb{Z}/d\mathbb{Z}$ occurs among these targets. Equivalently, each loop contains at least one $t^{-1}\sigma$ -edge. Thus the fiber $\beta^{-1}(A \oplus B)$ is an affine line over $\bar{\kappa}$ with coordinate $c_{1,m}$:

$$\dim_{\bar{\kappa}} \beta^{-1}(A \oplus B) = 1.$$

Degenerate exceptions. By the trial and error method, the above counting fails only for the case $n = m = 2$. A direct inspection shows that this is exactly the degenerate canonical case singled out in the theorem. This completes the proof. \square

Notice that the above argument of Drinfeld cases also holds for the exceptional trivial cases when one or both of m, n equal to 1, which can be thought as the canonical case or the dual case or the layered case, and the dimension of the fiber in this case is always 0. For integers $n, m \geq 1$

and exponents $d_1 \in \{1, n - 1\} + n\mathbb{Z}$, $d_2 \in \{1, m - 1\} + m\mathbb{Z}$, we encode the fiber dimension in the Drinfeld case $(\tau_n^{d_1}, \tau_m^{d_2})$ above by

$$\mathbf{d}_{\text{Dr}}(n, d_1; m, d_2) := \dim_{\bar{k}} \beta^{-1}(A \oplus B), \quad (3.2.4)$$

which is independent of the choice of $(A \oplus B) \in X_{\mu_M}^M(b)$.

Chapter 4: Triviality of β for the Drinfeld Case in a 3-Block Levi M

In this chapter we generalize our two-block Levi analysis to the three-block Levi case and prove the triviality of β in this case. This serves as the idea of the general N -block Levi case. Let

$$G = \mathrm{GL}_{k_1+k_2+k_3}, \quad M = \mathrm{GL}_{k_1} \times \mathrm{GL}_{k_2} \times \mathrm{GL}_{k_3}, \quad k_1, k_2, k_3 \geq 1$$

and choose the superbasic elements $\tau_{k_i} \in \mathrm{GL}_{k_i}(L)$. Set $b_i = \tau_{k_i}^{d_i}$ such that $d_i \equiv \pm 1 \pmod{k_i}$ and μ_i to be the corresponding minuscule cocharacter of b_i in GL_{k_i} . Write

$$b = \mathrm{diag}(b_1, b_2, b_3), \quad \mu = (\mu_1 \oplus \mu_2 \oplus \mu_3)_{\mathrm{dom}}.$$

Consider next a general block-upper-triangular matrix

$$g = \begin{pmatrix} A & D_1 & D_2 \\ 0 & B & D_3 \\ 0 & 0 & C \end{pmatrix} \in G(L),$$

where $A \in \mathrm{GL}_{k_1}(L)$, $B \in \mathrm{GL}_{k_2}(L)$, $C \in \mathrm{GL}_{k_3}(L)$ and D_i are the off-diagonal blocks of appropriate sizes. Recall the Iwasawa decomposition in 2.1: $G(L) = M(L)U(L)K = P(L)K$,

hence every coset gK admits a representative (A, B, C, D_1, D_2, D_3) . Our goal is: given a base point $(A, B, C) \in X_{\mu_1}^{\text{GL}_{k_1}}(b_1) \times X_{\mu_2}^{\text{GL}_{k_2}}(b_2) \times X_{\mu_3}^{\text{GL}_{k_3}}(b_3)$, determine exactly which D_1, D_2, D_3 lie in the fiber $\beta^{-1}(A, B, C)$.

The coset parametrization of g

Set

$$N = \begin{pmatrix} 0 & A^{-1}D_1 & A^{-1}D_2 \\ 0 & 0 & B^{-1}D_3 \\ 0 & 0 & 0 \end{pmatrix}, \quad U = \text{diag}(A, B, C).$$

Then $g = U(I + N)$ and $(I + N)^{-1} = I - N + N^2$. We want to use N to parametrize $\beta^{-1}(A, B, C)$. Define $E_1 = A^{-1}D_1, E_2 = A^{-1}D_2, E_3 = B^{-1}D_3$ and write $N = N(E_1, E_2, E_3)$. Assume $N' = N(E'_1, E'_2, E'_3)$ and $g' = U(I + N')$ is another representative of g . A direct calculation gives

$$g^{-1}g' = (I - N + N^2)(I + N') = I + N' - N - NN' + N^2.$$

Since

$$N = \begin{pmatrix} 0 & E_1 & E_2 \\ 0 & 0 & E_3 \\ 0 & 0 & 0 \end{pmatrix}, \quad N' = \begin{pmatrix} 0 & E'_1 & E'_2 \\ 0 & 0 & E'_3 \\ 0 & 0 & 0 \end{pmatrix},$$

one computes

$$N N' = \begin{pmatrix} 0 & 0 & E_1 E'_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad N^2 = \begin{pmatrix} 0 & 0 & E_1 E_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$g^{-1}g' = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} + \begin{pmatrix} 0 & E'_1 - E_1 & E'_2 - E_2 \\ 0 & 0 & E'_3 - E_3 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & E_1 E'_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & E_1 E_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Collecting terms gives the final block-matrix expression

$$g^{-1}g' = \begin{pmatrix} I & E'_1 - E_1 & E'_2 - E_2 & -E_1 E'_3 + E_1 E_3 \\ 0 & I & & E'_3 - E_3 \\ 0 & 0 & & I \end{pmatrix} \in K.$$

Therefore we get the congruent conditions:

$$E'_1 - E_1 \in \mathcal{O}_L^{k_1 \times k_2}, \quad E'_3 - E_3 \in \mathcal{O}_L^{k_2 \times k_3}, \quad E'_2 - E_2 - E_1 E'_3 + E_1 E_3 \in \mathcal{O}_L^{k_1 \times k_3}$$

The admissibility condition

Next we compute

$$g^{-1}b\sigma(g) = (g^{-1}b)\sigma(g).$$

First,

$$g^{-1}b = \begin{pmatrix} A^{-1}b_1 & -A^{-1}D_1B^{-1}b_2 & A^{-1}(D_1B^{-1}D_3 - D_2)C^{-1}b_3 \\ 0 & B^{-1}b_2 & -B^{-1}D_3C^{-1}b_3 \\ 0 & 0 & C^{-1}b_3 \end{pmatrix}.$$

Then multiplying on the right by $\sigma(g)$,

$$\sigma(g) = \begin{pmatrix} \sigma(A) & \sigma(D_1) & \sigma(D_2) \\ 0 & \sigma(B) & \sigma(D_3) \\ 0 & 0 & \sigma(C) \end{pmatrix},$$

yields the full block matrix

$$g^{-1}b\sigma(g) = \begin{pmatrix} M_1 & H_1 & H_2 \\ 0 & M_2 & H_3 \\ 0 & 0 & M_3 \end{pmatrix},$$

where

$$M_1 = A^{-1} b_1 \sigma(A),$$

$$H_1 = A^{-1} \left(b_1 \sigma(D_1) - D_1 B^{-1} b_2 \sigma(B) \right) = M_1 \sigma(E_1) - E_1 M_2,$$

$$\begin{aligned} H_2 &= A^{-1} \left(b_1 \sigma(D_2) - D_1 B^{-1} b_2 \sigma(D_3) + (D_1 B^{-1} D_3 - D_2) C^{-1} b_3 \sigma(C) \right) \\ &= M_1 \sigma(E_2) - E_2 M_3 - E_1 H_3 \end{aligned}$$

$$M_2 = B^{-1} b_2 \sigma(B),$$

$$H_3 = B^{-1} \left(b_2 \sigma(D_3) - D_3 C^{-1} b_3 \sigma(C) \right) = M_2 \sigma(E_3) - E_3 M_3,$$

$$M_3 = C^{-1} b_3 \sigma(C).$$

These explicit formulas are the starting point for checking admissibility $g^{-1} b \sigma(g) \in K t^\mu K$ and analyzing the resulting congruences on the blocks E_1, E_2, E_3 .

4.1 Analysis of E_1, E_2, E_3 for pairwise distinct $(k_1, d_1), (k_2, d_2), (k_3, d_3)$

We first study the case in which the pairs $(k_1, d_1), (k_2, d_2), (k_3, d_3)$ are pairwise distinct. Recall that for each block GL_{k_i} , one has

$$X_{\mu_i}^{\text{GL}_{k_i}}(b_i) \cong J_{b_i} / (J_{b_i} \cap K),$$

so we may henceforth fix

$$M_1 = b_1, \quad M_2 = b_2, \quad M_3 = b_3.$$

In the block-matrix computation

$$g^{-1}b\sigma(g) = \begin{pmatrix} M_1 & H_1 & H_2 \\ 0 & M_2 & H_3 \\ 0 & 0 & M_3 \end{pmatrix},$$

one finds

$$H_1 = b_1\sigma(E_1) - E_1b_2, \quad H_3 = b_2\sigma(E_3) - E_3b_3,$$

and

$$H_2 = b_1\sigma(E_2) - E_2b_3 - E_1H_3.$$

Applying Corollary 2.2.2 to the block matrix $g^{-1}b\sigma(g) = \begin{pmatrix} b_1 & H_1 & H_2 \\ 0 & b_2 & H_3 \\ 0 & 0 & b_3 \end{pmatrix}$, we see that admis-

sibility $g^{-1}b\sigma(g) \in K t^\mu K$ is equivalent to requiring:

$$(H_1)_{ij} = (b_1\sigma(E_1) - E_1b_2)_{ij} \in t^{\min(\text{row}_i(b_1), \text{col}_j(b_2))} \mathcal{O}_L \quad (4.1.1)$$

and

$$(H_3)_{ij} = (b_2\sigma(E_3) - E_3b_3)_{ij} \in t^{\min(\text{row}_i(b_2), \text{col}_j(b_3))} \mathcal{O}_L \quad (4.1.2)$$

and

$$(H_2)_{ij} = (b_2 \sigma(E_2) - E_2 b_3 - E_1 H_3)_{ij} \in t^{\min(\text{row}_i(b_1), \text{col}_j(b_3))} \mathcal{O}_L. \quad (4.1.3)$$

Conditions (4.1.1), (4.1.2), together with the coset congruences $E'_1 - E_1 \in \mathcal{O}_L^{k_1 \times k_2}$, $E'_3 - E_3 \in \mathcal{O}_L^{k_2 \times k_3}$, reduce the behaviour of E_1, E_3 exactly to the two-block analyses of Chapter 3. We want to control the “extra terms” in the congruent conditions and admissibility related to E_2 , say in the equations

$$E'_2 - E_2 - E_1 E'_3 + E_1 E_3 \in \mathcal{O}_L^{k_1 \times k_3}$$

and (4.1.3), we want to make $-E_1 E'_3 + E_1 E_3$ and $E_1 H_3$ as 0, so that we can reduce E_2 to the two-block analyses of Chapter 3 as well. Firstly we fix a full set of representatives (which we will determine later) in $\mathcal{O}_L^{k_2 \times k_3}$ for all possible choices of E_3 , and use them to parametrize E_3 , therefore we can make $-E_1 E'_3 + E_1 E_3 = E_1(E_3 - E'_3) = 0$.

In chapter 3, we know that in any Drinfeld setting there are at most $\min(k_1, k_2) - 1$ free parameters for E_1 , and in any case they are distributed among the entries of the strict upper off-diagonals. Therefore we can fix our choices of E_1 so that any entry of it other than the off-diagonal ones is 0. Therefore, to make $E_1 H_3 = 0$, it suffices to make the entries of H_3 other than its first row to be 0.

Write $E_3 = (c_{ij})_{1 \leq i \leq k_2, 1 \leq j \leq k_3}$. Then since $H_3 = b_2 \sigma(E_3) - E_3 b_3$, keeping the notations in Chapter 3, given free parameters in E_3 , any “loop” of E_3 can be traced so that at most one entry of H_3 that the loop corresponds is nonzero, and we can make those entries to be inside the first row of H_3 . So we get our desired set of representatives of E_3 and desired H_3 . We then have the following theorem:

Theorem 4.1.1. *Let $G = \mathrm{GL}_{k_1+k_2+k_3}$, $M = \mathrm{GL}_{k_1} \times \mathrm{GL}_{k_2} \times \mathrm{GL}_{k_3}$, and set $b_i = \tau_{k_i}^{d_i}$ such that $d_i \equiv \pm 1 \pmod{k_i}$ with $(k_1, d_1), (k_2, d_2), (k_3, d_3)$ pairwise distinct. Set μ_i to be the corresponding minuscule cocharacter of b_i and set*

$$b = \mathrm{diag}(b_1, b_2, b_3), \quad \mu = (\mu_1 \oplus \mu_2 \oplus \mu_3)_{\mathrm{dom}}.$$

Then for every point $(A, B, C) \bmod K \in X_{\mu_M}^M(b)$, the fiber of $\beta: X_{\mu}^G(b) \rightarrow X_{\mu_M}^M(b)$ has dimension

$$\dim_{\bar{k}} \beta^{-1}(A, B, C) = \sum_{i < j} \mathbf{d}_{\mathrm{Dr}}(k_i, d_i; k_j, d_j),$$

in exact agreement with the Rapoport dimension formula

$$\dim(X_{\mu}^G(b)) = \langle \rho, \mu - \bar{v}_b \rangle - \frac{1}{2} \mathrm{def}_G(b).$$

Moreover, as varieties over \bar{k} , β is a trivial bundle with fiber $\mathbb{A}^{\sum_{i < j} \mathbf{d}_{\mathrm{Dr}}(k_i, d_i; k_j, d_j)}$.

4.2 The degenerate case where some (k_i, d_i) are equal

Now suppose that

$$(k_2, d_2) = (k_3, d_3).$$

Then the two Levi-blocks of size k_2 share the same basic slope:

$$M_1 = b_2, \quad M_2 = M_3 = b_2.$$

As before, write

$$g = \begin{pmatrix} A & D_1 & D_2 \\ 0 & B & D_3 \\ 0 & 0 & C \end{pmatrix}, \quad U = \text{diag}(A, B, C), \quad N = \begin{pmatrix} 0 & E_1 & E_2 \\ 0 & 0 & E_3 \\ 0 & 0 & 0 \end{pmatrix},$$

so that $g = U(I + N)$. For another representative $g' = U(I + N')$ with $N' = (E'_1, E'_2, E'_3)$, one checks

$$g^{-1}g' = I + (N' - N) - NN' + N^2.$$

As in the distinct-slope case, this gives the congruences

$$E'_1 - E_1 \in \mathcal{O}_L^{k_1 \times k_1}, \quad E'_3 - E_3 \in \mathcal{O}_L^{k_1 \times k_3}, \quad E'_2 - E_2 - E_1 E'_3 + E_1 E_3 \in \mathcal{O}_L^{k_1 \times k_3}.$$

Next, computing

$$g^{-1}b\sigma(g) = \begin{pmatrix} b_1 & H_1 & H_2 \\ 0 & b_2 & H_3 \\ 0 & 0 & b_2 \end{pmatrix},$$

we find

$$H_1 = b_1 \sigma(E_1) - E_1 b_2, \quad H_3 = b_2 \sigma(E_3) - E_3 b_2,$$

$$H_2 = b_1 \sigma(E_2) - E_2 b_2 - E_1 H_3.$$

Since one may have $(k_1, d_1) = (k_2, d_2)$, we cannot zeroing out the first column of E_1 , but we claim that H_3 can be made identically zero. Indeed, writing $E_3 = (c_{ij})$ and tracing the cyclic

relations

$$H_3 = b_2 \sigma(E_3) - E_3 b_2$$

one finds (by the same “loop-tracing” argument of Chapter 3) a choice of base-point $w_0 \in L^{\sigma^{k_2}}$ such that each congruence becomes a genuine equality. Indeed, the last entry before we return to w_0 is either $t^{-1}\sigma^{k_2-1}w_0$ or $\sigma^{k_2-1}w_0$, and by either the relation $\xrightarrow{t\sigma}$ or $\xrightarrow{\sigma}$, we get $\sigma^{k_2}w_0 = w_0$. This forces $H_3 = 0$. Once E_3 is thus normalized, the remaining congruences on E_1 and then on E_2 each reduce to the two-block analyses of Chapter 3.

Notice that our assumption “ $(k_2, d_2) = (k_3, d_3)$ ” does not lose generality since in any cases we can reorder k_1, k_2, k_3 in the blocks by “change of basis”, and thus recovers the same dimension count and coordinates’ behaviour. In all cases, β remains a trivial bundle of the predicted rank.

We have the following theorem:

Theorem 4.2.1. *Let $G = \mathrm{GL}_{k_1+2k_2}$, $M = \mathrm{GL}_{k_1} \times \mathrm{GL}_{k_2} \times \mathrm{GL}_{k_2}$, and set $b_i = \tau_{k_i}^{d_i}$ such that $d_i \equiv \pm 1 \pmod{k_i}$. Set μ_i to be the corresponding minuscule cocharacter of b_i and set*

$$b = \mathrm{diag}(b_1, b_2, b_2), \quad \mu = (\mu_1 \oplus \mu_2 \oplus \mu_2)_{\mathrm{dom}}.$$

Then for every point $(A, B, C) \bmod K \in X_{\mu_M}^M(b)$, the fiber of $\beta: X_{\mu}^G(b) \rightarrow X_{\mu_M}^M(b)$ has dimension

$$\dim_{\bar{k}} \beta^{-1}(A, B, C) = 2\mathbf{d}_{\mathrm{Dr}}(k_1, d_1; k_2, d_2) + k_2 - 1.$$

Moreover, as varieties over \bar{k} , β is a trivial bundle with fiber

$$\begin{cases} \mathbb{A}^{2\mathbf{d}_D(k_1, d_1; k_2, d_2) + k_2 - 1} \times \left(L^{\sigma^{k_2}} / \mathcal{O}_{L^{\sigma^{k_2}}} \right)^{k_2} & \text{if } (k_1, d_1) \neq (k_2, d_2), \\ \mathbb{A}^{3k-3} \times \left(L^{\sigma^k} / \mathcal{O}_{L^{\sigma^k}} \right)^{3k} & \text{if } (k_1, d_1) = (k_2, d_2) = (k, d). \end{cases}$$

Chapter 5: Triviality of β for the Drinfeld Case in a N -Block Levi M

In this chapter we extend the block-matrix analysis to the general case

$$G = \mathrm{GL}_{k_1+k_2+\dots+k_N}, \quad M = \mathrm{GL}_{k_1} \times \mathrm{GL}_{k_2} \times \dots \times \mathrm{GL}_{k_N}.$$

Set $b_i = \tau_{k_i}^{d_i}$ such that $d_i \equiv \pm 1 \pmod{k_i}$ and μ_i to be the corresponding minuscule cocharacter of b_i in GL_{k_i} . Set the superbasic $b = \mathrm{diag}(b_1, b_2, \dots, b_N)$ and minuscule cocharacter $\mu = (\mu_1 \oplus \mu_2 \oplus \dots \oplus \mu_N)_{\mathrm{dom}}$. Write $\{(h_1, f_1), \dots, (h_m, f_m)\}$ for the distinct values among the (k_i, d_i) , each with multiplicities r_i . We shall show that for each point $(A_1, \dots, A_N) \bmod K \in X_{\mu_M}^M(b)$, the fiber $\beta^{-1}(A_1, \dots, A_N)$ is canonically isomorphic to

$$\mathbb{A}^{\sum_{1 \leq i < j \leq m} \mathbf{d}_{\mathrm{Dr}}(h_i, f_i; h_j, f_j) r_i r_j + \sum_{1 \leq i \leq m} (h_i - 1) \frac{r_i(r_i - 1)}{2}} \times \prod_{1 \leq i \leq m} \left(L^{\sigma^{h_i}} / \mathcal{O}_{L^{\sigma^{h_i}}} \right)^{h_i \frac{r_i(r_i - 1)}{2}},$$

of dimension $\sum_{1 \leq i < j \leq N} \mathbf{d}_{\mathrm{Dr}}(k_i, d_i; k_j, d_j)$, and hence β is a trivial bundle of that rank over the target.

Fix a point $(A_1, \dots, A_N) \bmod K \in X_{\mu_M}^M(b)$. By Iwasawa decomposition we may write any lift $g \in G(L)$ of this coset in the form

$$g = U(I + E), \quad U = \mathrm{diag}(A_1, \dots, A_N), \quad E = (E_{ij})_{1 \leq i, j \leq N},$$

where $E_{ij} = 0$ unless $i < j$. For another representative $g' = U(I + E')$, one checks

$$g^{-1}g' = (I - E + E^2 - \cdots + (-1)^{N-1}E^{N-1})(I + E') = I + \Delta,$$

and for each $i < j$, we compute

$$\Delta_{ij} = E'_{ij} - E_{ij} - \sum_{i < k < j} E_{ik} \Delta_{kj}. \quad (5.0.1)$$

The coset-congruences $g^{-1}g' \in K$ yield, for each $i < j$, $\Delta_{ij} \in \mathcal{O}_L^{k_i \times k_j}$. In other words, one solves the (i, j) -entry of Δ by first taking the naive difference $E'_{ij} - E_{ij}$ and then subtracting all the “earlier” corrections coming from $E_{ik} \Delta_{kj}$. This triangular recursion uniquely determines every Δ_{ij} .

Next, write

$$g^{-1}b\sigma(g) = \begin{pmatrix} M_1 & H_{12} & \cdots & H_{1N} \\ & M_2 & \ddots & \vdots \\ & & \ddots & H_{N-1,N} \\ & & & M_N \end{pmatrix},$$

where

$$M_i = A_i^{-1} b_i \sigma(A_i),$$

and for each $i < j$, we compute

$$H_{ij} = M_i \sigma(E_{ij}) - E_{ij} M_j - \sum_{i < \ell < j} E_{i\ell} H_{\ell j}. \quad (5.0.2)$$

Using induction on (i, j) in lexicographic order in (5.0.1) and the two-block base case, we see that we can assume $\Delta_{1N} = E'_{1N} - E_{1N}$ and $\Delta_{ij} = 0$ for other (i, j) (i.e. $E'_{ij} = E_{ij}$ for other (i, j)), which means that the $1N$ -th block is the only non-trivial block.

As before, in the Drinfeld setting we can take $M_i = b_i$. We want to eliminate all the terms $E_{i\ell} H_{\ell j}$ in (5.0.2) so that the $1N$ -th block is completely reduced to the two-block base case. In the case that all the (k_i, d_i) 's are pairwise distinct, from Chapter 4 we see that we can canonically make all $E_{i\ell}$ have zeros on the first column while making all $H_{\ell j}$ have zeros except on the first row, so that $E_{i\ell} H_{\ell j} = 0$ and we finish the argument.

For the degenerate case where some (k_i, d_i) 's are the same, we use another approach. Notice that this reduces to the case where b is basic in G , which we would not be interested in itself. But here we need to study the arithmetic computation in this case to get the desired computation in our general situation. Firstly we introduce the following lemma:

Lemma 5.0.1. *Suppose*

$$G = \mathrm{GL}_{Nk}, \quad M = \underbrace{\mathrm{GL}_k \times \cdots \times \mathrm{GL}_k}_{N \text{ factors}}, \quad d \in \{1, k-1\} + k\mathbb{Z}, \quad b = \mathrm{diag}(\tau_k^d, \dots, \tau_k^d).$$

Set μ_0 to be the corresponding minuscule cocharacter of τ_k^d and set $\mu = \underbrace{(\mu_0 \oplus \cdots \oplus \mu_0)}_{N \text{ factors}}_{\mathrm{dom}}$. Then b is basic in G , and the ADLV $X_\mu^G(b)$ coincides with the basic locus $X_b(b)$, and there is a canonical identification

$$X_\mu^G(b) = X_b(b) \cong J_b / (J_b \cap K).$$

In particular, $X_\mu^G(b)$ is a single J_b -orbit of hyperspecial cosets, and the reduction to Levi map

$$\beta: X_\mu^G(b) \rightarrow X_{\mu_M}^M(b) \text{ is trivial with fiber } \left(\mathbb{A}^{k-1} \times \left(L^{\sigma^k} / \mathcal{O}_{L^{\sigma^k}} \right)^k \right)^{\frac{N(N-1)}{2}}.$$

Proof. Since all block-slopes of b agree, the Newton point $\bar{\nu}_b$ factorizes through the center of G , so b is basic in G . Keeping the notations above, by Chapter 4, we know that we can make all H_{ij} in the expression of $g^{-1}b\sigma(g)$ as zero, so that $g \in J_b$ as required, and it follows that $X_\mu^G(b) = X_b(b) \cong J_b/(J_b \cap K)$. The fact that $\beta: X_\mu^G(b) \rightarrow X_{\mu_M}^M(b)$ is trivial with fiber $\left(\mathbb{A}^{k-1} \times \left(L^{\sigma^k}/\mathcal{O}_{L^{\sigma^k}}\right)^k\right)^{\frac{N(N-1)}{2}}$ follows from that for each $i < j$, E_{ij} is parametrized by $\mathbb{A}^{k-1} \times \left(L^{\sigma^k}/\mathcal{O}_{L^{\sigma^k}}\right)^k$, which is derived from the previous arguments. \square

Next we state the first main theorem in this chapter:

Theorem 5.0.2. *Let*

$$G = \mathrm{GL}_n, \quad M = \mathrm{GL}_{k_1} \times \cdots \times \mathrm{GL}_{k_N}.$$

For $d_i \in \{1, k_i - 1\} + k_i\mathbb{Z}$, set μ_i to be the corresponding minuscule cocharacter of $\tau_{k_i}^{d_i}$ and

$$b = \mathrm{diag}(\tau_{k_1}^{d_1}, \dots, \tau_{k_N}^{d_N}), \quad \mu = (\mu_1 \oplus \mu_2 \oplus \cdots \oplus \mu_N)_{\mathrm{dom}}.$$

Write $\{(h_1, f_1), \dots, (h_m, f_m)\}$ for the distinct values among the (k_i, d_i) , each with multiplicities r_i . For each $i = 1, \dots, m$, let

$$b_i = \mathrm{diag}(\underbrace{\tau_{h_i}^{f_i}, \dots, \tau_{h_i}^{f_i}}_{r_i}) \in \mathrm{GL}_{h_i r_i}(L),$$

and let $\mu^{(i)}$ be the $\mathrm{GL}_{h_i r_i}$ -dominant rearrangement of the direct sum of r_i -copies of “the minuscule dominant cocharacter corresponding to $\tau_{h_i}^{f_i}$ ” in $\mathrm{GL}_{h_i r_i}$. Then the partial reduction to Levi

map

$$\beta_1: X_\mu^G(b) \longrightarrow \prod_{i=1}^m X_{\mu^{(i)}}^{\text{GL}_{h_i r_i}}(b_i)$$

is isomorphic, over $\bar{\kappa}$, to a trivial vector bundle of rank

$$\sum_{1 \leq i < j \leq m} \mathbf{d}_{\text{Dr}}(h_i, f_i; h_j, f_j) r_i r_j.$$

Consequently, by Lemma 5.0.1, the full map $\beta: X_\mu^G(b) \rightarrow X_{\mu_M}^M(b)$ is a trivial bundle with fiber:

$$\mathbb{A}^{\sum_{1 \leq i < j \leq m} \mathbf{d}_{\text{Dr}}(h_i, f_i; h_j, f_j) r_i r_j + \sum_{1 \leq i \leq m} (h_i - 1) \frac{r_i(r_i - 1)}{2}} \times \prod_{1 \leq i \leq m} \left(L^{\sigma^{h_i}} / \mathcal{O}_{L^{\sigma^{h_i}}} \right)^{h_i \frac{r_i(r_i - 1)}{2}}$$

.

Proof. By a proper change of basis, we may simply assume that $b = \text{diag}(b_1, \dots, b_m)$. Keep all the notations as before, and write

$$g^{-1} b \sigma(g) = \text{diag}(M_1, \dots, M_m) (I + (H_{ij})_{1 \leq i < j \leq m}),$$

then as (5.2), one has the recursion formula

$$H_{ij} = M_i \sigma(E_{ij}) - E_{ij} M_j - \sum_{i < \ell < j} E_{i\ell} H_{\ell j},$$

and by Lemma 5.0.1 we may take $M_i = b_i$. We proceed by induction on the pair (i, j) in lex

order. For the minimal pairs $j = i + 1$, the sum is empty and

$$H_{i,i+1} = b_i \sigma(E_{i,i+1}) - E_{i,i+1} b_{i+1}. \quad (5.0.3)$$

We see that every original $h_i \times h_{i+1}$ sub-block of $H_{i,i+1}$ in (5.0.3) exactly satisfies the two-block condition since b_i, b_{i+1} are diagonal block matrices with Drinfeld setting superbasic sub-blocks. Suppose we have arranged parameters so that $\sum_{i < \ell < j} E_{i\ell} H_{\ell j} = 0$, then the formula of H_{ij} reduces to

$$H_{ij} = b_i \sigma(E_{ij}) - E_{ij} b_j,$$

and one applies the two-block case on each original $h_i \times h_j$ sub-block of H_{ij} to solve for E_{ij} . Since h_i 's are pairwise distinct, we can indeed make each $E_{i\ell} H_{\ell j}$ as 0, by applying the same method in Chapter 4 to each original $h_i \times h_\ell$ sub-block of $E_{i\ell}$ and each $h_\ell \times h_j$ sub-block of $H_{\ell j}$. For each H_{ij} , there are $r_i r_j$ many original $h_i \times h_j$ sub-blocks so by Theorem 3.1.1, the theorem follows. □

Indeed, the map β_1 in Theorem 5.0.2 is the essential map that we want to look at, since it is a “vector bundle”, while the fiber map in Lemma 5.0.1 serves as the “degenerate case” where the element b is already basic in G . We have the following main theorem:

Theorem 5.0.3. *Let*

$$G = \mathrm{GL}_n, \quad b \in G(L),$$

and suppose $M \subset G$ is a Levi subgroup such that b is basic in M , and that M is maximal with this property. Writing μ_M for the unique M -dominant and minuscule cocharacter corresponding

to b and let $\mu = (\mu_M)_{\text{dom}}$. Assume that for some Levi $M' \subset M$ with $b \in M'(L)$ and $\mu_{M'}$ the unique M' -dominant cocharacter corresponding to b , the tuple $(M', \mu_{M'}, b)$ satisfies the Drinfeld setting, then the reduction to Levi map

$$\beta: X_{\mu}^G(b) \longrightarrow X_{\mu_{M'}}^M(b)$$

is isomorphic to a trivial vector bundle (over \bar{k}).

Proof. Since b is basic in a Levi $M \subset G$, under a “change of basis” and “re-identification of a uniformizer” and up to “ σ -conjugation isomorphism” we may assume

$$M = \prod_{i=1}^m \text{GL}_{n_i}, \quad b = \text{diag}(b_1, \dots, b_m),$$

where each $b_i \in \text{GL}_{n_i}(L)$ is of the form $\tau_{n_i}^{d_i}$. By a further “change of basis” within each GL_{n_i} -block, we can diagonalize b_i into r_i copies of a superbasic element of size $h_i \times h_i$:

$$b_i \sim \underbrace{\text{diag}(\tau_{h_i}^{f_i}, \dots, \tau_{h_i}^{f_i})}_{r_i \text{ times}}$$

with $\gcd(h_i, f_i) = 1$. Since $(M', \mu_{M'}, b)$ satisfies the Drinfeld setting for some Levi $M' \subset M$, we have $f_i \in \{1, h_i - 1\} + h_i\mathbb{Z}$. Maximality of M forces the pairs $(h_1, f_1), \dots, (h_m, f_m)$ to be pairwise distinct. Denote μ_i as the projection of μ_M onto the GL_{n_i} component, then Theorem 5.0.2 applied to M shows that the reduction to Levi map

$$\beta: X_{\mu}^G(b) \longrightarrow X_{\mu_M}^M(b) = \prod_{i=1}^m X_{\mu_i}^{\text{GL}_{h_i r_i}}(b_i)$$

is a trivial vector bundle, as claimed.

□

Chapter 6: The Non-Minuscule Case: Triviality of β Without Linearity

In the reduction morphism

$$\beta : X_{\mu}^G(b) \longrightarrow \bigsqcup_{\mu_M \in S_M(\mu, v_b)} X_{\mu_M}^M(b)$$

the geometry of the fibers over $X_{\mu_M}^M(b)$ is especially transparent when μ_M is the unique M -*minuscule and dominant* cocharacter corresponding to b and $\mu = (\mu_M)_{\text{dom}}$: all congruence constraints are linear modulo t , and the fibers assemble into Zariski locally trivial *vector bundles*. When $\mu \succ (\mu_M)_{\text{dom}}$, or μ_M itself is *not* minuscule for some $\mu_M \in S_M(\mu, v_b)$, the situation will be different. We will study some special examples in this case where the same Iwasawa/block analysis still apply but the fiber will be defined by some higher-level congruences, and the mod- t relations are no longer linear; in particular, the fiber need not carry a canonical vector space structure. Nevertheless, the reduction to Levi remains Zariski locally trivial as a principal bundle under an explicitly described set of parameters at higher congruence level.

A first example: GL_2 with $\mu = (2, 0)$ and $b = \text{diag}(t, t)$

Here $M = \text{GL}_1 \times \text{GL}_1$, b is basic and central, and the set $S_M(\mu, v_b)$ consists of $\mu_M = (1, 1)$.

We show that the fiber of β is Zariski trivial with fiber $F/\mathcal{O}_F \times (\bar{\kappa} \setminus \kappa)$ (hence *not* a vector space

times a discrete set).

Choose representatives so that $M_1 = M_2 = t$ (block-monomial normalization). A point in the fiber over $(A \oplus B)$ is represented by

$$g = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix} K, \quad C \in L,$$

and a direct computation gives

$$g^{-1}b\sigma(g) = \begin{pmatrix} t & t(\sigma(C) - C) \\ 0 & t \end{pmatrix}.$$

By Corollary 2.2.2, the condition

$$g^{-1}b\sigma(g) \in K \operatorname{diag}(t^2, 1) K$$

reduces to the requirement that the reduction modulo t of the $(1, 2)$ -entry be a *unit*. Equivalently,

$$u := t(\sigma(C) - C) \in \mathcal{O}_L^\times. \tag{6.0.1}$$

Thus the fiber over $(A \oplus B)$ identifies *canonically* with

$$\{C \in L : t(\sigma(C) - C) \in \mathcal{O}_L^\times\} / \mathcal{O}_L.$$

We use the t -adic Witt expansion for C :

$$C = \sum_{i \geq i_0} a_i t^i, \quad a_i \in \bar{\kappa}$$

which preserves the addition, but not the multiplication when F is of mixed characteristics. Taking C modulo \mathcal{O}_L , we see $a_i = 0$ for $i \geq 0$. For the principal part of C , one has $a_{-1} \in \bar{\kappa} \setminus \kappa$, and $a_i \in \kappa$ for $i < -1$. The fiber has dimension 1 provided by a_{-1} , and it is *not* a vector space.

6.1 The two-block canonical Drinfeld case where $\mu = 2\omega_1^\vee$

Let $G = \mathrm{GL}_{n+m}$, $M = \mathrm{GL}_n \times \mathrm{GL}_m$, and consider the canonical Drinfeld configuration

$$b = \mathrm{diag}(\tau_n, \tau_m), \quad \mu = (2, 0, \dots, 0).$$

One can check that $S(\mu, v_b) = \{(1, 0, \dots, 0) \oplus (1, 0, \dots, 0)\}$ and we pick μ_M to be the unique element in it. As before, after block-monomial normalization we may assume

$$M_1 := A^{-1}b_1\sigma(A) = \tau_n, \quad M_2 := B^{-1}b_2\sigma(B) = \tau_m,$$

and a point of the fiber of β over $(A \oplus B) \bmod K$ is represented by

$$g = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I_n & C \\ 0 & I_m \end{pmatrix} K, \quad h := (1,2)\text{-block of } g^{-1}b\sigma(g) = \tau_n \sigma(C) - C \tau_m.$$

Claim. For the admissibility condition $g^{-1}b\sigma(g) \in Kt^\mu K$ with $t^\mu = \text{diag}(t^2, 1, \dots, 1)$, the off-diagonal block h satisfies

$$\boxed{h_{n,1} \in \mathcal{O}_L^\times, \quad h_{i,j} \in \mathcal{O}_L \text{ for all } (i,j) \neq (n,1).} \quad (6.1.1)$$

Proof via Smith normal form. Recall the Smith normal form criterion (Lemma 2.2.1): with $t^\mu = \text{diag}(t^2, 1, \dots, 1)$, the nondecreasing list of exponents is $(0, \dots, 0, 2)$; hence

$$v(\Delta_k(g^{-1}b\sigma(g))) = 0 \quad (1 \leq k \leq n+m-1), \quad v(\Delta_{n+m}(g^{-1}b\sigma(g))) = 2.$$

Step 1: Integrality of all entries of h . By the SNF requirement that $v(\Delta_1) = 0$ we already see $h_{i,j} \in \mathcal{O}_L$ for all (i,j) .

Step 2: The lower-left entry $h_{n,1}$ is a unit. Consider the $(n+m-1) \times (n+m-1)$ minors obtained by deleting one row and one column. There are four types:

(a) Delete a row and a column both in the τ_n block: the resulting minor is block upper-triangular with an $(n-1)$ -minor of τ_n and the full determinant of τ_m . Since $v(\det \tau_m) = 1$ and $v(\Delta_{n-1}(\tau_n)) \geq 0$, its valuation is ≥ 1 .

Delete a row and a column both in the τ_m block: similarly the valuation is $\geq v(\det \tau_n) = 1$.

(b) Delete a row in the τ_n block and a column in the τ_m block: a zero entry from the lower-left block will be picked up and therefore the determinant of the minor is 0.

(c) Delete a row in the τ_m block and a column in the τ_n block: either the row is not the last row of τ_m or the column is not the first column of τ_n , then any mixed minor of this

type necessarily picks up the unique t from τ_n (row n is present) or from τ_m (column 1 is present), and there is a unique corresponding $h_{i,j}$ such that the minor valuation $\geq 1 + v(h_{i,j}) \geq 1$.

- (d) The *special* mixed minor: delete the last row and the first column. Choose all remaining top columns and bottom columns; the only way to connect the blocks is through the entry $h_{n,1}$, while the $(n-1)$ -minor of τ_n avoiding the wrap t and the $(m-1)$ -minor of τ_m avoiding the wrap t both have valuation 0. Thus this minor has valuation exactly $v(h_{n,1})$.

By the SNF lemma, $v(\Delta_{n+m-1}(g^{-1}b\sigma(g))) = 0$. Among the above candidates, all minors in (a)-(c) have valuation ≥ 1 , whereas the special minor in (d) has valuation $v(h_{n,1})$. Therefore necessarily $v(h_{n,1}) = 0$, i.e. $h_{n,1} \in \mathcal{O}_L^\times$.

Combining Steps 1 and 2 yields

$$h_{n,1} \in \mathcal{O}_L^\times, \quad h_{i,j} \in \mathcal{O}_L \text{ for all } (i,j) \neq (n,1),$$

as claimed. □

Theorem 6.1.1. *Let $G = \mathrm{GL}_{n+m}$ and $M = \mathrm{GL}_n \times \mathrm{GL}_m$. Consider*

$$b = \mathrm{diag}(\tau_n, \tau_m), \quad \mu = (2, 0, \dots, 0).$$

Then the reduction to Levi morphism

$$\beta : X_\mu^G(b) \longrightarrow X_{\mu_M}^M(b)$$

is a Zariski trivial bundle. Moreover, each fiber has dimension

$$\dim_{\bar{k}} \beta^{-1}(A \oplus B) = \min\{n, m\} \quad \text{for every } (A \oplus B) \in X_{\mu_M}^M(b).$$

When $n \neq m$, we have

$$\beta^{-1}(A \oplus B) \cong \mathbb{G}_m \times \mathbb{A}^{\min\{n, m\}-1}$$

canonically over \bar{k} . Heuristically: the unique t -adic unit pivot forced by the Smith-normal-form condition contributes the \mathbb{G}_m -factor, while the remaining $\min\{n, m\} - 1$ free principal-part coefficients contribute the affine factors.

Proof. Keeping all the notations above, a point of the fiber of β over $(A \oplus B) \bmod K$ is represented by

$$g = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I_n & C \\ 0 & I_m \end{pmatrix} K, \quad h = (1,2)\text{-block of } g^{-1}b\sigma(g) = \tau_n \sigma(C) - C \tau_m.$$

Congruence equations and the arrow diagram. Write $C = (c_{i,j})_{1 \leq i \leq n, 1 \leq j \leq m}$, then the Smith normal form criterion for $Kt^\mu K$ with $t^\mu = \text{diag}(t^2, 1, \dots, 1)$ forces one extra *negative* valuation among the off-block relations; this appears only in the wrap relation linking the two corners:

$$\sigma(c_{1,1}) - c_{n,m} \in t^{-1} \mathcal{O}_L^\times. \quad (\star)$$

Loop tracing and the special loop. As in Chapter 3, we do the loop tracing on C , so directed cycles are orbits of the translation $(i, j) \mapsto (i - 1, j - 1)$ on the torus $\mathbb{Z}/n \times \mathbb{Z}/m$; there are

$d = \gcd(n, m)$ disjoint loops. The loop containing the special edge $c_{1,1} \rightarrow c_{n,m}$ closes after $\ell = \text{lcm}(n, m)$ steps. Chasing along that loop (composing the edge labels) gives

$$c_{1,1} = t^K \sigma^{\ell-1}(c_{n,m}) \quad \text{for some integer } K.$$

When $n \neq m$, we have seen in Chapter 3 that $K(m - n) > 0$. Without loss of generality, assume $n < m$ and hence $K > 0$. Applying σ and comparing with (\star) ,

$$\sigma(c_{1,1}) - c_{n,m} = t^K \sigma^\ell(c_{n,m}) - c_{n,m} \in t^{-1} \mathcal{O}_L^\times,$$

forces

$$c_{n,m} \in t^{-1} \mathcal{O}_L^\times.$$

Thus the *principal part* of $c_{n,m}$ is a unit of valuation -1 . This unit determines the entire chain

$$c_{n,m}, c_{n-1,m-1}, \dots, c_{1,m-n+1}$$

along the long off-diagonal in the $n \times m$ block (assuming $n < m$; the other case is symmetric),

but contributes only *one* free parameter: the class of $c_{n,m}$ modulo \mathcal{O}_L , i.e.

$$t^{-1} \mathcal{O}_L^\times / \mathcal{O}_L \cong \bar{\kappa}^\times \cong \mathbb{G}_m.$$

The long off-diagonal and the other strict off-diagonals contained in this loop serve as free parameters for this loop, while the remaining entries on this loop lie in \mathcal{O}_L .

Free parameters on strict off-diagonals. Every strict off-diagonal in C behaves exactly as in Chapter 3. The loop tracing there shows that each such strict off-diagonal contributes one free parameter in $t^{-1}\mathcal{O}_L/\mathcal{O}_L \cong \bar{\kappa}$, and different strict off-diagonals contribute *independently*. The number of these strict off-diagonals is $\min\{n, m\} - 1$; hence they contribute an affine factor $\mathbb{A}^{\min\{n, m\}-1}$.

Dimension count and triviality. Collecting the parameters:

(i) one multiplicative parameter) $\bar{\kappa}^\times \cong \mathbb{G}_m$, (ii) $\min\{n, m\}-1$ additive parameters) $\mathbb{A}^{\min\{n, m\}-1}$.

Therefore

$$\dim_{\bar{\kappa}} \beta^{-1}(A \oplus B) = 1 + (\min\{n, m\} - 1) = \min\{n, m\},$$

and when $n \neq m$ we obtain a canonical identification

$$\beta^{-1}(A \oplus B) \cong \mathbb{G}_m \times \mathbb{A}^{\min\{n, m\}-1}.$$

The same loop-solving shows that the coordinates of the fiber depend *algebraically* and *trivially* on the base point $(A \oplus B)$ (the base is zero-dimensional in this Drinfeld case), hence β is a Zariski trivial bundle.

□

Example 6.1.2. Take $n = 2, m = 3$, so

$$G = \mathrm{GL}_5, \quad M = \mathrm{GL}_2 \times \mathrm{GL}_3, \quad b = \mathrm{diag}(\tau_2, \tau_3), \quad \mu = (2, 0, 0, 0, 0).$$

Keeping notations above, entrywise one has

$$h_{1,1} = \sigma(c_{21}) - t c_{13}, \quad h_{2,1} = t \sigma(c_{11}) - t c_{23},$$

$$h_{1,2} = \sigma(c_{22}) - c_{11}, \quad h_{2,2} = t \sigma(c_{12}) - c_{21},$$

$$h_{1,3} = \sigma(c_{23}) - c_{12}, \quad h_{2,3} = t \sigma(c_{13}) - c_{22}.$$

By the Smith normal form criterion for $Kt^\mu K$ with $t^\mu = \text{diag}(t^2, 1, 1, 1, 1)$, one has

$$\sigma(c_{11}) - c_{23} \in t^{-1}\mathcal{O}_L^\times, \quad \text{and all other entries of } h \text{ are integral.} \quad (*)$$

Loop tracing. As in §3, encode the relations by arrows of type $t^{-1}\sigma$ and σ . There are $\gcd(2, 3) = 1$ loops on the 2×3 torus, and the unique loop contains the edge determined by (*). Chasing once around the loop (length $\text{lcm}(2, 3) = 6$) shows that

$$c_{11} = t \sigma^5(c_{23}),$$

hence from (*) we obtain

$$t\sigma^6(c_{23}) - c_{23} \in t^{-1}\mathcal{O}_L^\times \implies c_{23} \in t^{-1}\mathcal{O}_L^\times.$$

Thus the class of

$$u := \overline{c_{23}} \in t^{-1}\mathcal{O}_L^\times / \mathcal{O}_L \cong \bar{k}^\times$$

is a *multiplicative* free parameter (the \mathbb{G}_m -factor). The remaining strict off-diagonals contribute

exactly one *additive* parameter; equivalently, we may take, for instance,

$$v := \overline{c_{13}} \in t^{-1}\mathcal{O}_L/\mathcal{O}_L \cong \bar{\kappa},$$

while all other entries of C are uniquely determined by (u, v) via the congruences. Consequently,

$$\beta^{-1}(A \oplus B) \cong \mathbb{G}_m \times \mathbb{A}^1 \quad \text{and} \quad \dim_{\bar{\kappa}} \beta^{-1}(A \oplus B) = 2 = \min\{2, 3\}.$$

This realizes concretely the theorem in the case $G = \mathrm{GL}_5$, $M = \mathrm{GL}_2 \times \mathrm{GL}_3$.

Bibliography

- [GHKR06] Ulrich Görtz, Thomas J. Haines, Robert E. Kottwitz, and Daniel C. Reuman, *Dimensions of some affine Deligne–Lusztig varieties*, Ann. Sci. Éc. Norm. Supér. (4) **39** (2006), no. 3, 467–511.
- [Kot97] Robert E. Kottwitz, *Isocrystals with additional structure. II*, Compositio Mathematica **109** (1997), no. 3, 255–339.
- [FI22] Maria Fox and Naoki Imai, *The supersingular locus of the Shimura variety of $\mathrm{GU}(2, n - 2)$* (2022).
- [Vie06] Eva Viehmann, *The dimension of some affine Deligne–Lusztig varieties*, Ann. Sci. Éc. Norm. Supér. (4) **39** (2006), no. 3, 513–526. DOI:10.1016/j.ansens.2006.04.001. arXiv:math/0510385.