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Delay, elasticity, and stability trade-offs in rate control

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Delay, Elasticity, and Stability Trade-Offs in Rate Control

Priya Ranjan, Eyad H. Abed, and Richard J. La

Abstract— We adopt the optimization framework for rate allocation problem proposed by Kelly and characterize the stability condition with arbitrary communication delay. We demonstrate that there is a fundamental trade-off between the price elasticity of demand of users and responsiveness of the resources through a choice of price function as well as between the system stability and resource utilization.

I. INTRODUCTION

With the unprecedented growth and popularity of the Internet the problem of rate/congestion control is emerging as a more crucial problem. Poor management of congestion can render one part of a network inaccessible to the rest and significantly degrade the performance of networking applications. Kelly has proposed an optimization framework for rate allocation in the Internet [4]. Using the proposed framework he has shown that the system optimum is achieved at the equilibrium between the end users and resources. Based on this observation researchers have proposed various rate-based algorithms that solve the system optimization problem or its relaxation [4], [8], [9]. However, the convergence of these algorithms has been established only in the absence of feedback delay, and the impact of feedback delay has been left open as well as any trade-off that may exist between stability and selected utility and cost functions.

In this work we establish a delay-independent global stability criterion for system optimization problem in the presence of arbitrary delay for simple one resource problem with one flow. This flow could be interpreted to represent aggregate behavior of the users in the system. This is derived using the invariance-based global stability results for nonlinear delay-differential equations [10], [3], [2]. This kind of global stability results are different from that based on Lyapunov or Razumikhin theorems in the sense that in our set up also hints the structure of emerging periodic orbits (like their periodicity and amplitude) in the case of loss of stability.

Generally speaking, our results tell us that if the user and resource curves have a stable market equilibrium, then corresponding dynamical equation for flow-optimization will converge to the optimal point in the presence of arbitrary delay. This result essentially shows that stability is related to utility and price curves in a fundamental way. In particular, for a given price curve, it is possible to design stable user utility functions such that the ensuing dynamical system converges to the optimal flow irrespective of communication delay. Conversely, if the underlying market equilibrium is unstable then it is possible to find a large enough delay for which the optimal point loses its stability and gives way to oscillations. These results provide an interesting perspective for designing end user algorithms and active queue management (AQM) mechanisms. It is also worth

noting that in general characterizing the exact conditions for stability with a delay is difficult. Hence, our result provides a simple way of dealing with the problem of widely varying feedback delay in communication networks through a clever choice of the user utility function and price functions.

This paper is organized as follows. Section II describes the optimization problem for rate control. Relevant previous work on the stability criterion of a system given by a delayed differential equation is given in Section III. Our main results are presented in Section IV, which is followed by numerical examples in Section V. We conclude the paper in Section VI.

II. BACKGROUND

In this section we briefly describe the rate control problem in the proposed optimization framework. Consider a flow traversing a single resource. The rate control problem can be formulated as the following net utility optimization problem from the end user's point of view [4]:

$$\begin{aligned} \max_x \quad & U(x) - x \cdot p(x) \\ \text{s. t.} \quad & x \leq C \end{aligned} \quad (1)$$

where x is the rate, $U(x)$ is the utility of the user when it receives a rate of x , $p(x)$ is the price per unit flow the user has to pay when the rate is x , and C is the capacity of the resource. The proposed end user algorithm in the absence of delay is given by the following differential equation [6].

$$\frac{d}{dt}x(t) = k(w(t) - x(t)\mu(t)) \quad (2)$$

where $w(t)$ is the price per unit time user is willing to pay, $\mu(t) = p(x(t))$, and $k, k > 0$, is a gain parameter.

The case where $w(t)$ is a fixed constant, *i.e.*, $U(x) = \log(x)$, is studied in [5]. In this paper we assume that $w(t) = x(t) \cdot U'(x(t))$ with a family of utility functions to be specified shortly [6]. Now, suppose that congestion signal generated at the resource, *i.e.*, $p(x(t))$, is returned to the user after a fixed round trip time T . In the presence of delay the interaction is given by the following delayed differential equation

$$\frac{d}{dt}x(t) = k(w(t) - x(t-T)\mu(t-T)) \quad (3)$$

$$= k(x(t)U'(x(t)) - x(t-T)p(x(t-T))) \quad (4)$$

After normalizing time by T and replacing $t = s \cdot T$, eq. 4 becomes:

$$\frac{d}{Tds}x(s) = k(x(s)U'(x(s)) - x(s-1)p(x(s-1))) \quad (5)$$

$$\frac{\nu d}{ds}x(s) = x(s)U'(x(s)) - x(s-1)p(x(s-1)) \quad (6)$$

where $\nu = \frac{1}{Tk}$. It is precisely eq. 6 we are interested in from stability point of view. For $T \gg 1$, this equation can be seen as a following singular perturbation

$$\frac{\nu d}{dt}x(t) = g(x(t)) - f(x(t-1)) \quad (7)$$

of general nonlinear difference equation with continuous argument given by

$$g(x(t)) = f(x(t-1)), \quad t \geq 0 \quad (8)$$

where $g(x) = xU'(x)$ and $f(y) = yp(y)$ in the context of eq. 6. Under certain natural invertibility conditions on $g(\cdot)$, it leads to much studied equation [12]

$$x(t) = F(x(t-1)), \quad t \geq 0 \quad (9)$$

where $F(\cdot) = g^{-1}(f(\cdot))$. For the solution of eq. 9 to be continuous for $t \geq -1$, along with the continuity of F and $\phi(\cdot)$, which is the initial function, a so-called consistency condition $\lim_{t \rightarrow -0} \phi(t) = F(\phi(-1))$ is required [3], [12].

It turns out that a great deal about the asymptotic stability of eq. 7 can be learned from the asymptotic behavior of following difference equation, with Z_+ denoting the set of positive integers:

$$x_{n+1} = F(x_n), \quad n \in Z_+ \quad (10)$$

Some of relevant previous work is presented in the following section.

III. PREVIOUS WORK

In this section we summarize some of relevant work presented in [2]. Consider a nonlinear delay differential equation of the following form:

$$\dot{x}(t) = f(x(t-\tau)) - g(x(t)) \quad (11)$$

where functions f and g are continuous for $\mathfrak{R}_+ = \{x : x \geq 0\}$ with the values in \mathfrak{R}_+ . we make following additional assumptions on these functions:

1. $g(x)$ is strictly increasing, $g(0) = 0$, and $\lim_{x \rightarrow \infty} g(x) = \infty$,

2. there is exactly one point $\bar{x} > 0$ such that $f(\bar{x}) = g(\bar{x})$; moreover, $f(x) > g(x)$ in $(0, \bar{x})$ and $f(x) < g(x)$ in $(\bar{x}, +\infty)$.

Eq. 11 can be written in a singular perturbation form by change of coordinates $t = \tau \cdot s$ and $\mu = \frac{1}{\tau}$.

$$\frac{\mu d}{dt}x(t) = f(x(t-1)) - g(x(t)) \quad (12)$$

Now define $F(x) := g^{-1}(f(x))$. Invariance and global stability of one dimensional map F can be translated to those of eq. 11 for arbitrary time delay τ as described here [3].

Let $I \subset \mathfrak{R}_+$ be a closed interval which is invariant under F . Also, let $X := C([-1, 0], \mathfrak{R}_+)$ and $X_I := \{\phi \in X : \phi(s) \in I \forall s \in [-1, 0]\}$.

Theorem 1: The set X_I is invariant under eq. 12. For all $\phi \in X_I$ the corresponding solution $x(t; \phi)$ belongs to I for all $\mu \geq 0$.

Now suppose that $I_0 = \bigcap_{n \geq 0} F^n(I)$ degenerates into a single point, *i.e.*, map F has an asymptotically stable fixed point. Then, the following theorem holds.

Theorem 2: If \bar{x} is the globally attracting fixed point of the map F , then for any initial function $\phi \in X$ and every $\mu > 0$ the corresponding solution $x(t)$ of eq. 12 approaches \bar{x} .

IV. RATE CONTROL WITH FEEDBACK DELAY

We study the rate allocation problem in Kelly's optimization framework described in Section II [4] with the following class of price functions:

$$p(y) = \left(\frac{y}{C}\right)^b, \quad \text{where } b > 0 \quad (13)$$

This kind of marking function arises if the resource is modeled as $M/M/1$ queue with a service rate C packet per unit time and a packet receives a mark with a congestion indication signal if it arrives at the queue to find at least b packets in the queue.

The class of utility functions we consider has the form

$$U_a(x) = -\frac{1}{a} \frac{1}{x^a}, \quad a \geq 0. \quad (14)$$

In particular, $a = 1$ has been found useful for modeling the utility function of Transmission Control Protocol (TCP) algorithms [7]. We say that a user u_1 with utility function $U_{a_1}(x)$ is greedier than another user u_2 with utility function $U_{a_2}(x)$ if $a_2 > a_1$. One can interpret the notion of greed here using the notion of price elasticity of demand [13]. With the utility functions of the form in eq. 14 one can easily show that the price elasticity of demand decreases with a as follows. Given a price p , the optimal rate $x^*(p)$ of the user that maximizes the net utility $U_a(x) - x \cdot p$ is given by $p^{-\frac{1}{1+a}}$. The price elasticity of demand, which measures how responsive the demand is to a change in price, is defined to be the percent change in demand divided by the percent change in price [13]. In our case the price elasticity of demand is given by

$$\begin{aligned} \frac{p}{x^*(p)} \frac{dx^*(p)}{dp} &= \frac{p}{p^{-\frac{1}{1+a}}} \cdot \frac{-1}{1+a} p^{-\frac{1}{1+a}-1} \\ &= \frac{-1}{1+a}. \end{aligned} \quad (15)$$

Therefore, one can see that price elasticity of the demand decreases with a , *i.e.*, the larger a is, the less responsive the demand is.

In the presence of time delay T the end user algorithm with a utility function in eq. 14 is given by

$$\dot{x}(t) = k \left(\frac{1}{x(t)^a} - x(t-T) \left(\frac{x(t-T)}{C} \right)^b \right), \quad (16)$$

By substituting $t = T \cdot s$

$$\nu \dot{x}(s) = \frac{1}{x(s)^a} - x(s-1) \left(\frac{x(s-1)}{C} \right)^b \quad (17)$$

where $\nu = \frac{1}{T \cdot k}$. In order to apply the theorems in Section III, we can compare the forms in eq. 11 with that of eq. 16 where $g(x) = -k \frac{1}{x^a}$ and $f(x) = -kx \left(\frac{x}{C}\right)^b$. It is clear that although eq. 16 looks similar to eq. 11 it does not satisfy all the assumptions required to apply these theorems. In particular, these functions have their range in negative real numbers.

By making certain simple substitutions we can make eq. 17 resemble the well studied eq. 11. Consider the following substitution: $y(t) = x(t)U'(x(t)) := g(x(t))$ and $f(x(t)) =$

$x(t) \left(\frac{x(t)}{C} \right)^b$. We first make the following assumptions on the functions $g(x)$ and $f(x)$.

Assumption 1: (i) The function $g(x)$ is strictly decreasing with $-g'(x) > 0$ for all $x > 0$.

(ii) The function $f(x)$ is strictly increasing for all $x > 0$.

This allows us the following change of coordinate:

$$x(t) = g^{-1}(y(t)), \quad (18)$$

$$\dot{x}(t) = \frac{\dot{y}(t)}{g'(g^{-1}(y(t)))} \quad (19)$$

$$\nu \dot{y}(t) = g'(g^{-1}(y(t)))(y(t) - f(g^{-1}(y(t-1)))) \quad (20)$$

where the inverse $g^{-1}(\cdot)$ exists from assumption 1. Let $\kappa(y(t)) := -g'(g^{-1}(y(t)))$. Clearly, $\kappa(y(t)) > 0$ under assumption 1. Using this substitution in eq. 20 we get the following form which resembles eq. 11 closely, except for a multiplicative state-dependent gain $\kappa(y(t))$.

$$\nu \dot{y}(t) = \kappa(y(t))(f(g^{-1}(y(t-1))) - y(t)) \quad (21)$$

It is eq. 21 which we wish to study and show that there is a close correspondence between invariance and global stability properties of map $y_{n+1} = f(g^{-1}(y_n)) := F(y_n)$ and those of eq. 21. In particular, we wish to prove that if $y_{n+1} = F(y_n)$ has a fixed point then eq. 21 will have a uniformly constant solution for all possible time delays $T \geq 0$ if the initial function's range is contained in the immediate basin of attraction of this fixed point. The proofs are based on the invariance property of the underlying map $F(\cdot)$ and the monotonicity of function $g(\cdot)$. The map $F(y)$ is strictly decreasing because $g^{-1}(y)$ is strictly decreasing under assumption 1 and a composition of a strictly increasing and a strictly decreasing function (f is strictly increasing from assumption 1) is a strictly decreasing function.

Assumption 2: Suppose now that $I \subset \{x : \epsilon \leq x \leq \bar{X}\}$, where ϵ is some small positive constant and \bar{X} is an arbitrarily large constant, is a closed invariant interval under F . In particular let $I = [a \ b]$ be compact.

Let $X := C([-1, 0], \mathbb{R}_+)$, and $X_I := \{\phi \in X : \phi(s) \in I \ \forall s \in [-1, 0]\}$. Under this assumption, we have invariance for the solution of eq. 21 for all time $t \geq 0$ and for all $\nu \geq 0$.

Theorem 3: Invariance: If $\phi \in X_I$, the corresponding solution $y(t) = y(t; \phi)$ satisfies $y(t) \in I$ for all $t \geq 0$. It means that set I is invariant under eq. 21.

Proof: Let t_0 be the first time when solution $y(t; \phi)$ leaves I with $\phi \in X_I$. In particular, we can assume that $y(t_0) = b$ and every right hand neighborhood of t_0 will have a $t_1 > t_0$ such that $y(t_1) > b$. Then, we can find a point $t_2, t_0 < t_2 < t_0 + 1$, such that $y(t_2) > b$ and $\dot{y}(t_2) > 0$. Since $y(t_2 - 1) \leq b$, we have $\dot{y}(t_2) = \frac{\kappa(y(t_2))(f(g^{-1}(y(t_2-1))) - y(t_2))}{\nu} < 0$ from eq. 21 and assumption 2 that I is invariant under F , i.e., $f(g^{-1}(y(t_2 - 1))) \leq b$. This contradicts with the earlier assertion about the positivity of $\dot{y}(t_2)$.

Similarly, lets assume that $y(t_0) = a$ and the trajectory exits from left end of the interval. Then, every right hand neighborhood of t_0 will have a $t_1 > t_0$ such that $0 < y(t_1) < a$ due to the smoothness of solutions, and we can find $t_2, t_0 < t_2 < t_0 + 1$, such that $0 < y(t_2) < a$ and $\dot{y}(t_2) < 0$. From that $y(t_2 - 1) \geq a$, we have $\dot{y}(t_2) = \frac{\kappa(y(t_2))(f(g^{-1}(y(t_2-1))) - y(t_2))}{\nu} > 0$ from

eq. 21 and assumption 2. This contradicts with the negativity of $\dot{y}(t_2) < 0$. Hence, the theorem follows. ■

Here we note that uniform positivity of $\kappa(y(t))$ over positive real line is crucial to the proof. Next theorem considers the case when map F has an attracting fixed point y^* with immediate basin of attraction $J_0 : F^n y_0 \rightarrow y^*$ for any $y_0 \in J_0$. Let $X_{J_0} = C([-1, 0], J_0)$ then following theorem holds.

Theorem 4: For any $\nu > 0$ and $\phi \in X_{J_0}$, $\lim_{t \rightarrow \infty} y_\phi^\nu(t) = y^*$.

Before proving the theorem we will state a Lemma which is the key to the proof of above theorem.

Lemma 1: Suppose that an interval J is mapped by F into itself. If none of the endpoints of the interval $F(J)$ is fixed point then for every $\phi \in X_J = C([-1, 0], J)$ there exists a finite $t_0 = t_0(\phi, \nu, \kappa)$ such that $y_\phi^\nu(t) \in F(J)$ for all $t \geq t_0$.

Proof: From last invariance theorem it is clear that $y_\phi^\nu \in J$ for all $t \geq 0$. The claim here is that after certain time t_0 it will be limited by $F(J) \subset J$.

First, assume that $\phi(0) \in \overline{F(J)}$. Then it can be shown that $y_\phi^\nu(t) \in F(J)$ for all $t \geq t_0$ by contradiction. Suppose that this is not true and let t_0 be the first time when $y_\phi^\nu(t)$ leaves the interval $\overline{F(J)}$. In particular assume that it leaves from the right end, i.e., every right-sided neighborhood of t_0 contains a point t_1 such that $y_\phi^\nu(t_1) > \sup \overline{F(J)}$. Then, the same neighborhood also contains a point t_2 such that $y_\phi^\nu(t_2) > \sup \overline{F(J)}$ and $\dot{y}_\phi^\nu(t_2) > 0$. As $y_\phi^\nu(t) \in J$ for all $t \in [t_0 - 1, t_0]$ and also $t_0 < t_2 < t_0 + 1$ can be assumed, we have $\dot{y}(t_2) = \frac{\kappa(y(t_2))(f(g^{-1}(y(t_2-1))) - y(t_2))}{\nu} < 0$ from eq. 21. This contradicts the earlier assumption that $\dot{y}(t_2) > 0$. The other case where $y_\phi^\nu(t)$ leaves the interval from the left end can be handled similarly.

Now assume that $\phi(0) \notin \overline{F(J)}$. Particularly, let $\phi(0) > \sup \overline{F(J)}$. Claim here is that $y_\phi^\nu(t)$ is decreasing for all $t \in [0, t_0]$, where $t_0 \leq \infty$ is the first point with $y_\phi^\nu(t_0) = \sup \overline{F(J)}$. We first argue that $t_0 < \infty$ by contradiction. Suppose $t_0 = \infty$ and, hence, $y_\phi^\nu(t) > \sup \overline{F(J)}$ for all $t \geq 0$. From eq. 21 we have $\dot{y}(t_2) = \frac{\kappa(y(t_2))(f(g^{-1}(y(t_2-1))) - y(t_2))}{\nu} < 0$ because $f(g^{-1}(y(t_2 - 1))) \leq \sup \overline{F(J)}$. Then there exists a limit $\bar{y} = \lim_{t \rightarrow \infty} y_\phi^\nu(t) > \sup \overline{F(J)}$ due to Bolzano-Weierstrass theorem argument, which states that every strictly decreasing sequence which is bounded from below has a limit [1]. As \bar{y} is not a fixed point of map F , $\kappa(\bar{y})(\bar{y} - f(g^{-1}(\bar{y}))) := \delta > 0$. This tells us from eq. 21 that $\dot{y}(t) = \frac{\kappa(y(t))(f(g^{-1}(y(t-1))) - y(t))}{\nu} < -\frac{\delta}{2\nu}$ for large enough t . This implies that $y_\phi^\nu(t) \rightarrow -\infty$ as $t \rightarrow \infty$ which is a contradiction, for this means that $y_\phi^\nu(t)$ crosses $\sup \overline{F(J)}$ for some finite t . Hence, $t_0 < \infty$. Now, we invoke the first part of proof where system is restarted at time t_0 with $y(t_0) = \sup \overline{F(J)}$ and $y(t) \in J \ \forall t \in [t_0 - 1, t_0]$. Using that argument $y(t) \in F(J)$ for all $t \geq t_0$. The other case can be handled similarly. ■

Now we provide the proof for above theorem using this lemma.

Proof: For any $\phi \in X_{J_0}$, define $m = \inf\{\phi(s), s \in [-1, 0]\}$ and $M = \sup\{\phi(s), s \in [-1, 0]\}$. Clearly, $[m \ M] \subset J_0$. Let J' be the smallest closed invariant interval containing $[m \ M]$ which is a subset of J_0 . Then, from existence of fixed

point for the map $F, J' \supset F(J') \supset F(F(J')) \supset \dots$ and $\bigcap_{i \geq 0} F^i(J') = x^*$. Using invariance and Lemma 1 repeatedly one can find arbitrarily small estimates for the range of trajectories with large enough time. Hence, the proof. ■

These theorems can be directly applied to study the dynamical behavior of eq. 16 which is essentially described by the underlying discrete time difference equation

$$y_{n+1} = F(y_n) \quad (22)$$

$$\frac{1}{x_{n+1}^a} = x_n \left(\frac{x_n}{C} \right)^b, \quad x_n > 0 \quad (23)$$

$$x_{n+1} = \left(\frac{C^b}{x_n^{b+1}} \right)^{\frac{1}{a}} \quad (24)$$

Here the invariance set I is a subset of the interval $[\epsilon C]$, where ϵ is a positive constant greater than or equal to $F(C)$. Lets look into the dynamical behavior of map given by eq. 24. It has a fixed point

$$x^* = C^{\frac{b}{a+b+1}}, \quad (25)$$

and the market equilibrium price is given by $p^* = C^{-\frac{b(1+a)}{1+a+b}}$. The market equilibrium price can be obtained from that $x^* = p^{*-\frac{1}{1+a}}$. This expression of equilibrium flow shows that x^* increases with decreasing a and that's why we characterize the user with eq. 14 utility functions with smaller a greedier. The eigenvalue at this fixed point which is interestingly independent of the fixed point is

$$\lambda(x^*) = -\frac{b+1}{a}. \quad (26)$$

Here x^* will be a locally stable fixed point if $a > b+1$. According to the Sharkovsky cycle coexistence ordering [11] the most general condition for the fixed point x^* to be globally attracting is that the second iteration $F^2(\cdot)$ of the map $F(\cdot)$ does not have a fixed point in the relevant invariance set other than x^* , which is locally stable. These conditions hold in our example, and hence the fixed point x^* is globally stable in the invariance set. In addition, in practice since any initial user rate in the interval $(0, \epsilon]$ will be upper bounded by the physical link capacity C , after one iteration the user rate will lie in the invariance set. The above in turn implies the global delay independent stability of eq. 17. It is interesting to note that when the utility function of user is given by $U_1(x) = \frac{1}{x}$ as has been suggested for TCP algorithms, the delay independent stability of the system cannot be ensured by a price function of the form in eq. 13.

Our results have the following interpretation. If the functions $x \cdot U'(x)$ and $x \cdot p(x)$ have an intersecting point that is a stable fixed point, then the communication delays are irrelevant for system stability, and user rates and resource price converge to the system optimum. Furthermore, our results tell us that the stability of system depends critically on the user utility functions, more specifically on the parameter a , for a given price function. This can be seen from the eigenvalue $\lambda(x^*) = -\frac{b+1}{a}$. Larger values of b mean that the slope of the price function is steeper, which in turn implies that the price varies more widely in response to a change in rate x . Hence, in order to maintain the

stability of system, user demand should be less elastic, *i.e.*, the response of user to a change in price should be less dramatic. Thus, this presents a fundamental trade-off between the elasticity of user demand and responsiveness of price function. In other words, in order to keep the stability of system, if one wants to increase the responsiveness of one, then the responsiveness of the other must be sacrificed.

The above results have the following practical implications. Characterizing the exact stability conditions of the system with a given choice of utility and price functions is not easy. In addition, the round-trip delays of connections tend to vary widely. Therefore, one approach to designing a stable system is to select a pair of user utility and price functions in such a way the communication delay does not affect the stability of the system. This is, however, not to say that the dynamics of the system do not depend on the delay.

Our results also provide us with the following design guideline for the AQM mechanism and end user algorithms for efficient use of network resources. Note that from eq. 25 the fixed point x^* is strictly increasing in b and is strictly decreasing in a . Therefore, in order to increase the utilization at the fixed point, we should increase the ratio $\frac{b}{a}$. However, this ratio cannot be increased arbitrarily without losing the stability from eq. 26. Therefore, in order to achieve high utilization of the resource and maintain the stability of the system, the parameter b should be selected as large as possible and the parameter a should be selected just large enough so that $|\lambda(x^*)|$ is smaller than one. However, having the eigenvalue close to -1 comes at the price of a larger settling time. In order to reduce the settling time, the ratio of $\frac{b}{a}$ should be lowered. Therefore, the selection of parameters a and b presents a fundamental trade-off between stability, settling time, and utilization of the system. This is numerically demonstrated in the following section.

We now study what effects the load of the system, *i.e.*, the number of users in the system, has on the stability of the system. Since the load on a resource is beyond the control of a network manager, ideally the stability of the system should not depend on the load. Suppose that there are $N, N \geq 1$, homogeneous users in the system. Since users are assumed to be homogeneous, we denote the rate of a user by $x(t)$. We assume that utility function of the users is of the form in eq. 14 and the price function used at the resource is that of eq. 13. Then, the end user algorithm is given by

$$\begin{aligned} \dot{x}^{(N)}(t) &= k \left(x^{(N)}(t) U'_a(t) - x^{(N)}(t-T) \cdot p(N \cdot x^{(N)}(t-T)) \right) \quad (27) \end{aligned}$$

$$= k \left(\frac{1}{x^{(N)}(t)^a} - x^{(N)}(t-T) \left(\frac{N \cdot x^{(N)}(t-T)}{C} \right)^b \right) \quad (28)$$

where a superscript (N) is used to denote the dependence on N . Following similar steps as in the single flow case above, the discrete time difference equation corresponding to eq. 22 - 24 of single flow case is given by

$$y_{n+1}^{(N)} = F^{(N)}(y_n^{(N)}) \quad (29)$$

$$\frac{1}{\left(x_{n+1}^{(N)}\right)^a} = x_n^{(N)} \left(\frac{N \cdot x_n^{(N)}}{C} \right)^b, \quad x_n^{(N)} > 0 \quad (30)$$

$$x_{n+1}^{(N)} = \left(\frac{(C/N)^b}{x_n^{(N)b+1}} \right)^{\frac{1}{a}} \quad (31)$$

Then, from eq. 31 the fixed point $x^{(N)*}$ is $\left(\frac{C}{N}\right)^{\frac{b}{a+b+1}}$ ¹ and the eigenvalue is given by $\lambda^{(N)}(x^{(N)*}) = -\frac{b+1}{a}$ and is independent of N . Therefore, the stability of the system does not depend on the number of users in the system. This can also be explained using the price elasticity of demand. Since, given a utility function of the form in eq. 14 for some $a > 0$, the price elasticity of the demand is constant for all $x > 0$ from eq. 15, one would expect the stability of the system to be independent of the operating point, *i.e.*, the fixed point, and capacity, but only on the choices of the utility and price functions that determine the responsiveness of the users and resource, respectively.

Clearly, the network designer can rescale the price function by a scalar, *i.e.*,

$$p(y) = \gamma \cdot \left(\frac{y}{C}\right)^b, \quad (32)$$

where $\gamma > 0$. When the price function is of the form in eq. 32, the fixed point of the system with N flows is given by $x^* = \gamma^{-\frac{1}{1+a+b}} \left(\frac{C}{N}\right)^{\frac{b}{1+a+b}}$. Furthermore, the value of γ does not change the eigenvalue at the fixed point, *i.e.*, the stability condition does not depend on γ . Hence, if the number of flows traversing the resource is known, then the resource can select an appropriate value of γ so that the fixed point of the system achieves high utilization. However, the problem with this is that smaller values of γ reduces the responsiveness of the price function.

V. NUMERICAL EXAMPLES

In this section we present numerical examples to validate our results presented in Section IV.

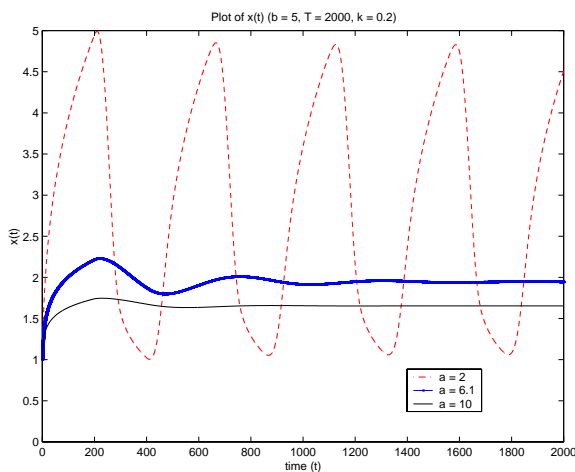


Fig. 1. Plot of $x(t)$ for $a = 2, 6.1, \text{ and } 10$.

Fig. 1 plots $x(t)$ for $C = 5, T = 200, k = 0.2, b = 5$, and various values of a . The value of parameter a is set to 2, 6.1, and 10, respectively. Note that $a = 2$ yields an eigenvalue $-\frac{b+1}{a} = -3$, which violates the stability condition. This is illustrated in

Fig. 1. As one can see the system does not converge to the optimal value of 2.73. On the other hand, the value of $a = 6.1$ leads to a stable system and the rate converges to the optimal value of 1.945 as demonstrated in the figure. When we further increase the value of a to 10 one can see that the utilization at the fixed point decreases with a larger value of a . However, the settling time improves with increasing a . Thus, this presents another trade-off between settling time and resource utilization as mentioned in Section IV.

VI. CONCLUSION

We have shown that dynamical stability of rate control problem for a simple one resource case is determined by the interaction of underlying utility and price functions. In particular, we have demonstrated a fundamental trade-off between users' price elasticity of demand and the responsiveness of the resource. Furthermore, there is another trade-off between the global stability of system and the utilization of the resource. These results offer some guidelines for jointly designing the end users algorithms and AQM mechanisms at the routers.

REFERENCES

- [1] P. M. Fitzpatrick, editor. *Advanced Calculus*. PWS Publishing Company, 1996.
- [2] A. F. Ivanov, M. A. Pinto, and S. I. Trofimchuk. Global behavior in nonlinear systems with delayed feedback. *Proc. Conference of Decision and Control, Sydney*, 2000.
- [3] A. F. Ivanov and A. N. Sharkovsky. Oscillations in singularly perturbed delay equations. *Dynamics Reported*, 1:165–224, 1991.
- [4] F. Kelly. Charging and rate control for elastic traffic. *European Transactions on telecommunications*, 8:33–37, January 1997.
- [5] F. Kelly. Mathematical modelling of the internet. In *Bjorn Engquist and Wilfried Schmid (Eds.), Mathematics Unlimited – 2001 and Beyond*. Springer-Verlag, Berlin, 2001.
- [6] F. Kelly, A. Maulloo, and D. Tan. Rate control for communication networks: shadow prices, proportional fairness and stability. *Journal of the Operational Research Society*, 49(3):237–52, March 1998.
- [7] S. Kunniyur and R. Srikant. End-to-end congestion control schemes: Utility functions, Random losses and ECN marks. In *Proc. of IEEE INFOCOM*, Tel-Aviv, Israel, 2000.
- [8] S. Kunniyur and R. Srikant. A time scale decomposition approach to adaptive ecn marking. In *Proc. of IEEE INFOCOM*, Anchorage, AK, 2001.
- [9] S. Low and D. Lapsley. Optimization flow control. *IEEE/ACM Transactions on Networking*, 7(6):861–874, June 1997.
- [10] J. Mallet-Paret and R. D. Nussbaum. Global continuation and asymptotic behaviour for periodic solutions of a differential-delay equation. *Annali di Matematica Pura ed Applicata*, 145(4):33–128, 1986.
- [11] A. N. Sharkovsky, S. F. Kolyada, A. G. Sivak, and V. V. Fedorenko. *Dynamics of one dimensional maps*. Kluwer Academic, 1997.
- [12] A N Sharkovsky, Y. L. Maistrenko, and E. Y. Romaneko. *Difference Equations and their applications: (Trans. from Russian)*. Kluwer Academic, 1995.
- [13] Hal R. Varian. *Intermediate Microeconomics*. Norton, New York, 1996.

¹Here we assume that the fixed point is smaller than C .