ABSTRACT

Title of dissertation: A NEW CONSTRUCTION OF THE TAME LOCAL LANGLANDS CORRESPONDENCE FOR GL(n,F), n A PRIME

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In this thesis, we give a new construction of the tame local Langlands correspondence for $GL(n, F)$, n a prime, where F is a non-archimedean local field of characteristic zero. The Local Langlands Correspondence for $GL(n, F)$ has been proven recently by Henniart, Harris/Taylor. In the tame case, supercuspidal representations correspond to characters of elliptic tori, but the local Langlands correspondence is unnatural because it involves a twist by some character of the torus. Taking the cue from the theory of real groups, supercuspidal representations should instead be parameterized by characters of covers of tori. Stephen DeBacker has calculated the distribution characters of supercuspidal representations for $GL(n, F)$, n prime, and they are written in terms of functions on elliptic tori. Over the reals, Harish-Chandra parameterized discrete series representations of real groups by describing their distribution characters restricted to compact tori. Those distribution characters are written down in terms of functions on a canonical double cover of real tori. We show that if one writes down a natural analogue of Harish-Chandra's

distribution character for p-adic groups, then it is the distribution character of a unique supercuspidal representation of $GL(n, F)$, where n is prime, away from the local character expansion. These results pave the way for a natural construction of the tame local Langlands correspondence for $GL(n, F)$, n a prime. In particular, there is no need to introduce any character twists.

A NEW CONSTRUCTION OF THE TAME LOCAL LANGLANDS CORRESPONDENCE FOR GL(n,F), n A PRIME

by

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Chapter 1

Introduction

In this thesis, we reexamine the local Langlands correspondence for $GL(\ell, F)$, where ℓ is prime and F is a p-adic field of characteristic 0, using character theory and ideas from the theory of real reductive groups. Our main result is a construction of the tame local Langlands correspondence which circumvents some of the difficulties of [5]. In particular, there are certain technical choices in the construction of the local Langlands correspondence which are explained from our point of view. As a result, the construction of the local Langlands correspondence can be made to appear more natural.

If $\ell = 2$, we assume that the residual characteristic of F is not 2. If $\ell > 2$, we assume that the residual characteristic of F is greater than 2ℓ. We make these assumptions for three reasons. Firstly, because our methods require the knowledge of the supercuspidal character formulas for $GL(\ell, F)$, and these have only been completely computed so far in the cases where the residual characteristic of F is greater than ℓ (see [9]). Secondly, it is unclear whether our methods will generalize to wildly ramified situations. Thirdly, we will use some results from [21], which assumes in the case of $\ell > 2$ that the residual characteristic of F is greater than 2ℓ. Once the supercuspidal characters for $GL(n, F)$ become available for arbitrary n, we expect that our methods will generalize to the case where the residual characteristic of

F is coprime to n. Our results illuminate some new ideas about character theory of p-adic groups and local Langlands for p-adic groups not known before. In particular, irreducible Weil group representations $W_F \to GL(\ell, \mathbb{C})$ and supercuspidal representations of $GL(\ell, F)$ are naturally parameterized not by certain characters of elliptic tori known as *admissible pairs*, but by genuine characters of double covers of elliptic tori, as is the case for admissible representations of real groups. We show that the supercuspidal representations of $GL(\ell, F)$ are naturally parameterized by genuine characters of double covers of elliptic tori using character theory. To do this we rewrite supercuspidal characters in terms of double covers of elliptic tori as in Harish-Chandra's discrete series character formula and as in the Weyl character formula. Rewriting the supercuspidal character formulas in this way paves the way for a more natural rendition of local Langlands for $GL(\ell, F)$. In particular, it eliminates the need for any finite order character twists in the local Langlands correspondence for $GL(\ell, F)$ that arise in the work of [5], [14]. As we shall see, our results and formulas also give justification and reason to the character formulas that first appeared in Sally/Shalika, which may look like they came out of nowhere.

Recall that the local Langlands correspondence for $GL(n, F)$ is a parametrization of representations of $GL(n, F)$ by representations of the Weil-Deligne group:

$$\{W'_F \to GL(n, \mathbb{C})\} \leftrightarrow \{\textit{irreducible admissible representations of } GL(n, F)\}$$

Definition 1.0.1. Let E/F be an extension of degree n, n relatively prime to p, and let χ be a character of E^*. The pair $(E/F, \chi)$ is an *admissible pair* if

(i) χ does not come via the norm from a proper subfield of E containing F.

(ii) If the restriction $\chi|_{1+\mathfrak{p}_E}$ comes via the norm from a proper subfield $E \supset L \supset F$, then E/L is unramified.

In the tame case, Howe constructs a map (see [11])

$$\{admissible\ pairs\ (E/F,\chi)\} \to \{supercuspidal\ representations\ of\ GL(n,F)\}$$

$$(E/F,\chi) \mapsto \pi_\chi$$

which associates supercuspidal representations π_χ of $GL(n,F)$ to admissible pairs $(E/F,\chi)$. This map is a bijection (see [14]). Moreover, the irreducible representations $W_F \to GL(n,\mathbb{C})$ are all of the form $\phi(\chi) := Ind_{W_E}^{W_F}(\chi)$ for some admissible pair $(E/F,\chi)$ (where via the Artin map $W_E^{ab} \cong E^*$, we may pull back the character χ of E^*, to a character, denoted χ, of W_E), and we obtain a bijection

$$\{admissible\ pairs\ (E/F,\chi)\} \to \{irreducible\ W_F \to GL(n,\mathbb{C})\}$$

$$(E/F,\chi) \mapsto Ind_{W_E}^{W_F}(\chi)$$

The problem is that the obvious map,

$$\phi(\chi) \mapsto \pi_\chi,$$

the so-called "naive correspondence", is not the local Langlands correspondence because π_χ has the wrong central character. Instead, the local Langlands correspondence is given by

$$\phi(\chi) \mapsto \pi_{\chi\Delta_\chi}$$

for some subtle finite order character Δ_χ which depends on the extension E/F. The presence of the character twist Δ_χ makes the correspondence look unnatural. We will show that if one considers genuine characters of a canonical double cover of elliptic tori rather than characters of elliptic tori, then one obtains a natural local Langlands correspondence. We do this in the following way.

Taking the cue from the theory of real groups, we use genuine characters $\tilde{\chi}$ of certain double covers of elliptic tori, denoted $T(F)_{\tau\circ\rho}$, instead of characters of elliptic tori $T(F)$, to parameterize both representations of W_F and supercuspidal representations of $GL(\ell, F)$ using character theory. We give a method for attaching a genuine character of a double cover of elliptic tori satisfying certain regularity conditions, to a supercuspidal Weil parameter of $GL(\ell, F)$:

$$\{irreducible\ W_F \to GL(\ell, \mathbb{C})\} \leftrightarrow \{regular\ genuine\ characters\ of\ T(F)_{\tau\circ\rho}\} \quad (1.1)$$

$$\phi \mapsto \tilde{\chi}$$

Moreover, as we shall show, supercuspidal characters of $GL(\ell, F)$ correspond naturally to genuine characters of $T(F)_{\tau\circ\rho}$ satisfying certain regularity conditions, rather than admissible pairs $(E/F, \chi)$. Given a genuine character $\tilde{\chi}$ of $T(F)_{\tau\circ\rho}$ satisfying certain regularity conditions, we write down a conjectural Harish-Chandra type character formula, denoted $F(\tilde{\chi})$. We show that this naturally gives a bijection

$$\{regular\ genuine\ characters\ of\ T(F)_{\tau\circ\rho}\} \leftrightarrow \left\{ \begin{array}{c} supercuspidal\ representations \\ of\ GL(\ell, F) \end{array} \right\}$$

$$\tilde{\chi} \mapsto \pi(\tilde{\chi}) \quad (1.2)$$

where $\pi(\tilde{\chi})$ is the unique supercuspidal representation of $GL(\ell, F)$, whose character, restricted to a certain natural subset of $T(F)$ (to be described later), is $F(\tilde{\chi})$.

Then the composition of bijections (1.1) and (1.2),

$$\phi \mapsto \tilde{\chi} \mapsto \pi(\tilde{\chi}),$$

is the local Langlands correspondence for $GL(\ell, F)$.

Let us explain why double covers of tori play a role. We start by considering the group $PGL(2, F)$. First recall that the representations of $PGL(2, F)$ are precisely the representations of $GL(2, F)$ with trivial central character. One of the conditions of the local Langlands correspondence for $GL(n, F)$ says that if $\phi : W_F \to GL(n, \mathbb{C})$ is irreducible, then $det(\phi) = \omega_{\pi(\phi)}$, where $\omega_{\pi(\phi)}$ denotes the central character of $\pi(\phi)$, and where $\pi(\phi)$ denotes the supercuspidal representation of $GL(n, F)$ that corresponds to ϕ under the local Langlands correspondence. Here we are viewing $det(\phi)$ as a character of F^* in the following way. As the image of $det(\phi)$ is in $\mathbb{C}^*$, $det(\phi)$ is trivial on $[W_F, W_F]$, and therefore factors to a character of $F^* \cong W_F^{ab}$ via the Artin map. Let ϕ be a supercuspidal Weil parameter for $PGL(2, F)$ (that is, an irreducible representation $W_F \to GL(2, \mathbb{C})$ that parameterizes a supercuspidal representation of $GL(2, F)$ with trivial central character). Then $\phi = Ind_{W_E}^{W_F}(\chi)$, for some admissible pair $(E/F, \chi)$. It is a fact that $det(Ind_{W_E}^{W_F}(\chi)) = \chi|_{F^*} \otimes \delta_{E/F}$, where $\delta_{E/F} = det(Ind_{W_E}^{W_F}(1))$. In the case that E/F is quadratic, $\delta_{E/F} = \aleph_{E/F}$, where $\aleph_{E/F}$ is the local class field theory character of F^* relative to E/F. Now, since $\pi(\phi)$ has trivial central character, the condition $det(\phi) = \omega_{\pi(\phi)}$ implies that $\chi|_{F^*} \otimes \aleph_{E/F} = 1$, so $\chi|_{F^*} = \aleph_{E/F}$. Therefore, the supercuspidal representations of

$PGL(2, F)$ are parameterized by the admissible pairs $(E/F, \chi)$ where $\chi|_{F^*} = \aleph_{E/F}$.

Now, recall again that supercuspidal representations of $GL(2, F)$ are parameterized by characters of elliptic tori. One might ask whether the supercuspidal representations of $PGL(2, F)$ are parameterized by characters of its elliptic tori[1] E^*/F^*. However, we have just seen that the supercuspidal representations of $PGL(2, F)$ are parameterized by characters χ of E^* whose restriction to F^* is $\aleph_{E/F}$. Such a χ is not a character of the elliptic torus E^*/F^* in $PGL(2, F)$. Rather, it is a genuine character of a double cover of E^*/F^* in the following way. There is a canonical exact sequence

$$1 \longrightarrow F^* \longrightarrow E^* \longrightarrow E^*/F^* \longrightarrow 1$$

$$w \mapsto [w]$$

Reducing this sequence mod $ker(\aleph_{E/F}) = N(E^*)$, we get an exact sequence

$$1 \longrightarrow F^*/N(E^*) \longrightarrow E^*/N(E^*) \longrightarrow E^*/F^* \longrightarrow 1$$

where N denotes the norm map from E to F. Since $F^*/N(E^*) \cong \mathbb{Z}/2\mathbb{Z}$ by Local Class Field Theory, we have that $E^*/N(E^*)$ is a double cover of E^*/F^*. Then the character χ of E^* naturally factors to a character, denoted $\tilde{\chi}$, of $E^*/N(E^*)$, given by $\tilde{\chi}([w]) := \chi(w)\ \forall [w] \in E^*/N(E^*)$ (since it is a character of E^* that is trivial on

[1] One can view E^1 as an elliptic torus in PGL(2,F), where $E^1 := \{w \in E^* : N(w) = 1\}$. While it's true that $E^1 \cong E^*/F^*$ by Hilbert's Theorem 90 if E/F is tame and quadratic, it is not necessarily true that $E^1 \cong E^*/F^*$ if E/F is degree ℓ, where ℓ is an odd prime (since E/F might not be Galois, and so Hilbert's Theorem 90 wouldn't hold). Therefore, in order to present a unified approach in this thesis, we view E^*/F^* naturally as the elliptic torus in $PGL(2, F)$ and $PGL(\ell, F)$, and no problems arise.

$N(E^*))$. Moreover, χ is not trivial on all of F^*, so $\tilde{\chi}$ doesn't factor to a character of E^*/F^*. This means that $\tilde{\chi}$ is a genuine character of $E^*/N(E^*)$. Therefore, we are getting naturally that the supercuspidal representations of $PGL(2, F)$ (i.e. the supercuspidal representations of $GL(2, F)$ with trivial central character) are parameterized by genuine characters of a double cover of the elliptic torus E^*/F^* inside $PGL(2, F)$. In fact, this double cover $E^*/N(E^*)$ is none other than an analogue of the ρ-cover that appears in the theory over the reals, which is a natural double cover of a real torus. We shall explain this shortly.

We can apply the same above reasoning to the case of $PGL(\ell, F)$ where ℓ is an arbitrary prime. In this case, the central character condition $det(\phi) = \omega_{\pi(\phi)}$ implies that if $(E/F, \chi)$ is an admissible pair corresponding to a supercuspidal representation of $PGL(\ell, F)$, then $\chi|_{F^*} \equiv \delta_{E/F}$, where $\delta_{E/F} = det(Ind_{W_E}^{W_F}(1))$. We can again reduce the sequence

$$1 \to F^* \to E^* \to E^*/F^* \to 1$$

mod $ker(\delta_{E/F})$ to obtain a sequence

$$1 \to F^*/ker(\delta_{E/F}) \to E^*/ker(\delta_{E/F}) \to E^*/F^* \to 1$$

Then χ is naturally a genuine character of $E^*/ker(\delta_{E/F})$. Sometimes $\delta_{E/F} = 1$, in which case χ factors to a character of E^*/F^*, but if $\delta_{E/F} \neq 1$, χ lifts to a genuine character $\tilde{\chi}$ of $E^*/ker(\delta_{E/F})$, which is a nontrivial double cover of E^*/F^*. In fact, these natural covers appear in general for $PGL(n, F)$ where n is general and $(n, p) = 1$, and so our results should generalize to this setting. We will show that these double covers that arise naturally in $PGL(\ell, F)$ will also arise in $GL(2, F)$ and

$GL(\ell, F)$, and studying these covers will lead to a natural description of the local Langlands correspondence for $GL(\ell, F)$. In fact, the double covers that arise from $PGL(\ell, F)$ and $GL(\ell, F)$ are a p-adic analogue of a natural double cover that arises in the theory of real groups.

In the theory of real groups, admissible homomorphisms $W_{\mathbb{R}} \to {}^L G$ naturally produce genuine characters of the *ρ-cover of* $T(\mathbb{R})$, denoted $T(\mathbb{R})_\rho$, a double cover of $T(\mathbb{R})$, which we now define. First, we need:

Definition 1.0.2. Let A, B, and C be groups, and suppose we have homomorphisms $\phi_1 : A \to C$, $\phi_2 : B \to C$. Then the *pullback* of these two homomorphisms is the group

$$A \times_C B := \{(a, b) \in A \times B \mid \phi_1(a) = \phi_2(b)\}$$

together with projections

$$\pi_1 : A \times_C B \to A \quad \pi_2 : A \times_C B \to B$$

$$(a, b) \mapsto a \quad (a, b) \mapsto b$$

Then the following diagram commutes:

$$\begin{array}{ccc} A \times_C B & \xrightarrow{\pi_1} & A \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \phi_1} \\ B & \xrightarrow{\phi_2} & C \end{array}$$

Definition 1.0.3. Let G be a connected reductive group over $\mathbb{R}$, and let $T \subset G$ a maximal torus over $\mathbb{R}$. Let $X^*(T)$ be the character group of T. Let Δ^+ be a set of positive roots of G with respect to T. Let $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$. Then $2\rho \in X^*(T)$. Viewing

2ρ as also a character of $T(\mathbb{R})$ by restriction, we define the *ρ-cover of* $T(\mathbb{R})$ as the pullback of the two homomorphisms

$$2\rho : T(\mathbb{R}) \to \mathbb{C}^* \qquad \Upsilon : \mathbb{C}^* \to \mathbb{C}^*$$

$$t \mapsto 2\rho(t) \qquad\qquad z \mapsto z^2$$

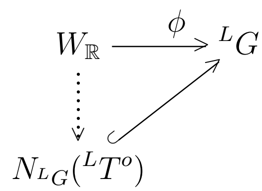
We denote the ρ-cover by $T(\mathbb{R})_\rho$, and so the following diagram commutes:

$$\begin{array}{ccc} T(\mathbb{R})_\rho & \xrightarrow{\rho} & \mathbb{C}^* \\ \downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle \Upsilon} \\ T(\mathbb{R}) & \xrightarrow{2\rho} & \mathbb{C}^* \end{array}$$

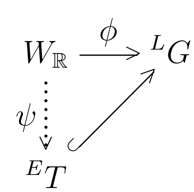
Note that because of the commutativity of the diagram, although ρ is not necessarily a character of $T(\mathbb{R})$, ρ is naturally a character of $T(\mathbb{R})_\rho$. Moreover, Π is the canonical projection $\Pi(t, \lambda) = t$.

The genuine characters of $T(\mathbb{R})_\rho$ that naturally arise from Weil parameters are used to form L-packets. In the case of $GL(n, \mathbb{R})$, L-packets are singletons, and we have that the irreducible admissible representations of $GL(n, \mathbb{R})$ and admissible homomorphisms $W_\mathbb{R} \to GL(n, \mathbb{C})$ are in natural bijection with genuine characters $\tilde{\chi}$ of $T(\mathbb{R})_\rho$. The composition of these two parameterizations is in fact the local Langlands correspondence for $GL(n, \mathbb{R})$.

More explicitly, the local Langlands correspondence for real groups can be roughly stated as follows: Let G be a connected reductive group over $\mathbb{R}$. Let $\phi : W_\mathbb{R} \to {}^L G$ be a parameter. Since $\phi(\mathbb{C}^*)$ is an abelian group consisting of semisimple elements, we can arrange for $\phi(\mathbb{C}^*) \subset {}^L T^o$, for some dual maximal torus ${}^L T^0$. However, it isn't necessarily true that ${}^L T \subset {}^L G$. Instead, $\phi(W_\mathbb{R})$ is contained in a slightly more general group, called an *E-group*. In fact, one can show easily that $\phi(W_{\mathbb{R}}) \subset N_{^LG}(^LT^o)$ for some dual maximal torus $^LT^o$. That is, any admissible homomorphism $\phi : W_{\mathbb{R}} \to {}^LG$ factors:



for some maximal torus $^LT^o$. This is the starting point for describing the local Langlands correspondence for real groups. As we mentioned, the image of $W_{\mathbb{R}}$ does not necessarily lie in the L-group of a maximal torus. In other words, if $\phi(\mathbb{C}^*) \subset {}^LT^0$, then the group generated by $^LT^0$ and $\phi(j)$ is not necessarily the L-group of a maximal torus. However, this group is an extension of $Gal(\mathbb{C}/\mathbb{R})$ by $^LT^0$, called an *E-group*. It will then be true that every quasi-admissible homomorphism $\phi : W_{\mathbb{R}} \to {}^LG$ will factor



where ^ET is an E-group, and therefore one is reduced to studying $\psi : W_{\mathbb{R}} \to {}^ET$. Then, there is a torus $T(\mathbb{R}) \subset G(\mathbb{R})$ such that ψ naturally gives rise to a genuine character $\tilde{\chi}$ of the group $T(\mathbb{R})_\rho$, where $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ for some choice of positive roots Δ^+ of G with respect to T. Then, from $\tilde{\chi}$, one constructs a set of representations $J(\Delta^+, \tilde{\chi})$ to form the L-packet of ϕ.

One can write down more explicitly the correspondence for relative discrete series representations, and this is the motivation for our work. Harish-Chandra proved that all relative discrete series character formulas are of a certain form when restricted to the compact (mod center) torus $T(\mathbb{R})$, written in terms of functions on $T(\mathbb{R})_\rho$. To state the theorem, we make a few preliminary remarks.

Definition 1.0.4. Let G be a connected reductive group over $\mathbb{R}$, $T \subset G$ a maximal torus over $\mathbb{R}$. Let Δ^+ be a set of positive roots of G with respect to T. Define

$$\Delta^0(h, \Delta^+) := \prod_{\alpha \in \Delta^+} (1 - \alpha^{-1}(h)),\ h \in T(\mathbb{R})$$

Recall that in general, ρ is not in $X^*(T)$. Therefore, $\rho(h)$ does not make sense if $h \in T(\mathbb{R})$. However, ρ is a well-defined character of $T(\mathbb{R})_\rho$. If $\tilde{h} \in T(\mathbb{R})_\rho$ is any element such that $\Pi(\tilde{h}) = h$, we may consider the function $\Delta^0(h, \Delta^+)\rho(\tilde{h})$. This function lives on $T(\mathbb{R})_\rho$, and we have the following theorem.

Theorem 1.0.5. *(Harish-Chandra) Let G be a connected reductive group, defined over $\mathbb{R}$. Suppose that G contains a Cartan subgroup T that is defined over $\mathbb{R}$ and that is compact mod center. Let Δ^+ be a set of positive roots of G with respect to T. Let $\rho := \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$. Let $\tilde{\chi}$ be a genuine character of $T(\mathbb{R})_\rho$ that is regular. Let $W := W(G(\mathbb{R}), T(\mathbb{R}))$ be the relative Weyl group of $G(\mathbb{R})$ with respect to $T(\mathbb{R})$. Let $\epsilon(s) := (-1)^{\ell(s)}$ where $\ell(s)$ is the length of the Weyl group element $s \in W$. Let $T(\mathbb{R})^{reg}$ denote the regular set of $T(\mathbb{R})$. Then there exists a unique constant $\epsilon(\tilde{\chi}, \Delta^+) = \pm 1$, depending only on $\tilde{\chi}$ and Δ^+, and a unique relative discrete series representation of $G(\mathbb{R})$, denoted $\pi(\tilde{\chi})$, such that*

$$\theta_{\pi(\tilde{\chi})}(h) = \epsilon(\tilde{\chi}, \Delta^+) \frac{\sum_{s \in W} \epsilon(s) \tilde{\chi}({}^s\tilde{h})}{\Delta^0(h, \Delta^+)\rho(\tilde{h})},\ h \in T(\mathbb{R})^{reg}$$

where $\tilde{h} \in T(\mathbb{R})_\rho$ *is any element such that* $\Pi(\tilde{h}) = h$. *Moreover, every relative discrete series character of* $G(\mathbb{R})$ *is of this form.*

We can be more specific about the constant $\epsilon(\tilde{\chi}, \Delta^+)$. In particular, $\epsilon(\tilde{\chi}, \Delta^+) = (-1)^{\ell(s)}$ where $s \in W$ makes $d\tilde{\chi}$ dominant for Δ^+, $d\tilde{\chi}$ denoting the differential of $\tilde{\chi}$.

For $GL(2,\mathbb{R})$ (which is the only general linear group besides $GL(1,\mathbb{R})$ that has relative discrete series), the local Langlands correspondence for relative discrete series representations is as follows. Fix a positive set of roots Δ^+ of G with respect to T. Let $\phi : W_\mathbb{R} \to GL(2,\mathbb{C})$ be a relative discrete series parameter, and let $T(\mathbb{R})$ be the compact mod center torus of $GL(2,\mathbb{R})$. Then ϕ naturally gives rise to a genuine character $\tilde{\chi}$ of $T(\mathbb{R})_\rho$. By Harish-Chandra's discrete series theorem, $\tilde{\chi}$ gives rise to a unique relative discrete series representation, denoted $\pi(\tilde{\chi})$, whose character, restricted to the regular elements of $T(\mathbb{R})$, is

$$F(\tilde{\chi}) := \epsilon(\tilde{\chi}, \Delta^+) \frac{\sum_{s\in W} \epsilon(s)\tilde{\chi}({}^s\tilde{h})}{\Delta^0(h, \Delta^+)\rho(\tilde{h})}, \ h \in T(\mathbb{R})^{reg}$$

where $\tilde{h} \in T(\mathbb{R})_\rho$ is any element such that $\Pi(\tilde{h}) = h$. The map

$$\phi \mapsto \pi(\tilde{\chi}) \tag{1.3}$$

is the local Langlands correspondence for relative discrete series representations of $GL(2,\mathbb{R})$. Thus, one can write down the correspondence for relative discrete series in terms of character theory. This is the approach we take in this paper, and we will show that the correspondence (1.3) carries over naturally to the p-adic setting.

One of the results that we will prove is an analogue of Harish-Chandra's theorem for $GL(\ell, F)$, where F is a p-adic field of characteristic zero. In doing this, we give a new realization of the tame local Langlands correspondence for $GL(\ell, F)$, and the character twists Δ_χ go away. Before we present our main results, we need to define the covers of tori that will be essential, which are an analogue of the ρ-cover that appears in the theory over the reals.

Let G be a connected reductive group defined over F, and T a maximal torus in G defined over F. Let Δ^+ be a choice of positive roots of G with respect to T. Let $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$. Let λ be a character of K^*, where K/F is the minimal splitting field of T.

Definition 1.0.6. We define the *$\lambda \circ \rho$-cover of $T(F)$*, denoted $T(F)_{\lambda\circ\rho}$, as the pullback of the two homomorphisms

$$\lambda \circ 2\rho : T(F) \to \mathbb{C}^* \qquad \Upsilon : \mathbb{C}^* \to \mathbb{C}^*$$

$$t \mapsto \lambda \circ 2\rho(t) \qquad z \mapsto z^2$$

$$\begin{array}{ccc} T(F)_{\lambda\circ\rho} & \xrightarrow{\lambda\circ\rho} & \mathbb{C}^* \\ \downarrow \Pi & & \downarrow \Upsilon \\ T(F) & \xrightarrow{\lambda\circ 2\rho} & \mathbb{C}^* \end{array}$$

That is, $T(F)_{\lambda\circ\rho} = \{(z, w) \in T(F) \times \mathbb{C}^* : \lambda(2\rho(z)) = w^2\}$

Note that the above map $T(F)_{\lambda\circ\rho} \to \mathbb{C}^*$ sends (z, w) to w. We have denoted this map by $\lambda\circ\rho$, even though this map is not literally λ composed with ρ. Moreover, Π is the canonical projection $\Pi(z, w) = z$.

Our main results will be the following theorems.

Theorem 1.0.7. *Let $G(F) = GL(\ell, F)$ where ℓ is prime, and let $T(F) = E^*$ be an elliptic torus in $GL(\ell, F)$, so $E = F(\sqrt[\ell]{\Delta})$ for some $\Delta \in F^*$. Let L be the unique* *unramified extension of F of degree $\ell - 1$. Let τ_o be any character of $(EL)^*$ whose restriction to L^* is $\aleph_{EL/L}$, where $\aleph_{EL/L}$ is the local class field theory character of L^* relative to EL/L. Let $\tau := \tau_o \mid\,\mid_{EL}$ where $\mid\,\mid_{EL}$ denotes the EL-adic absolute value. Let Δ^+ be a set of positive roots of G with respect to T. Let $\rho := \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. Let $T(F)_{\tau\circ\rho}$ be the $\tau\circ\rho$ cover of $T(F)$. Let $\tilde{\chi}$ be a genuine character of $T(F)_{\tau\circ\rho}$ that is regular. Let $W = W(G(F), T(F))$ denote the relative Weyl group of $G(F)$ with respect to $T(F)$. If $s \in W(G(F), T(F))$, let $\epsilon(s) := (-1, \Delta)^{\ell(s)(\ell+1)}$, where $(,)$ denotes the Hilbert symbol of F and $\ell(s)$ denotes the length of s. Let $T(F)^{reg}$ denote the regular elements of $T(F)$.*

Then there exists a unique constant $\epsilon(\tilde{\chi}, \Delta^+, \tau)$, depending only on $\tilde{\chi}, \Delta^+$, and τ, and a unique supercuspidal representation of $GL(\ell, F)$ denoted $\pi(\tilde{\chi})$, such that

$$\theta_{\pi(\tilde{\chi})}(z) = \epsilon(\tilde{\chi}, \Delta^+, \tau)\frac{\sum_{s\in W}\epsilon(s)\tilde{\chi}({}^s w)}{\tau(\Delta^0(z, \Delta^+))(\tau\circ\rho)(w)},\ \forall z \in T(F)^{reg}\ : 0 \le n(z) \le r/2$$

where $w \in T(F)_{\tau\circ\rho}$ is any element such that $\Pi(w) = z$ and r is the depth of $\pi(\tilde{\chi})$. Moreover, every supercuspidal character of $GL(\ell, F)$ is of this form.

We will define all of the notation in the above theorem in Chapters 5-8, including $n(z)$, $\epsilon(\tilde{\chi}, \Delta^+, \tau)$, and regularity. We remark that $n(z)$ comes from a canonical filtration on the torus $T(F)$, and is defined in [9]. Notice that when we treat the case of depth zero representations (i.e. $r = 0$), the range $\{z \in T(F)^{reg} : 0 \le n(z) \le r/2\}$ becomes $\{z \in T(F)^{reg} : n(z) = 0\}$. We wish to make a few comments about the constant $\epsilon(\tilde{\chi}, \Delta^+, \tau)$. Firstly, $\epsilon(\tilde{\chi}, \Delta^+, \tau)^4 \in \mathbb{R}^*$. Moreover, $|\epsilon(\tilde{\chi}, \Delta^+, \tau)|$ is a known real number in that it has to do with a canonical measure on the Lie algebra. The subtlety of $\epsilon(\tilde{\chi}, \Delta^+, \tau)$ is in the value of $\frac{\epsilon(\tilde{\chi}, \Delta^+, \tau)}{|\epsilon(\tilde{\chi}, \Delta^+, \tau)|} \in \{\pm 1, \pm i\}$.

Now let ϕ be a supercuspidal Weil parameter for $GL(\ell, F)$. We will show in Chapters 5 and 8 how to construct a regular genuine character, $\tilde{\chi}$, of $T(F)_{\tau\circ\rho}$, from ϕ. We will then prove the following theorem.

Theorem 1.0.8. *The assignment*

$$\phi \mapsto \tilde{\chi} \mapsto \pi(\tilde{\chi})$$

is the Local Langlands correspondence for $GL(\ell, F)$.

Let us be a bit more explicit about the representation $\pi(\tilde{\chi})$. In particular, if $\phi : W_F \to GL(\ell, \mathbb{C})$ is a supercuspidal Weil parameter for $GL(\ell, F)$, and $\phi = Ind_{W_E}^{W_F}(\chi)$ for some admissible pair $(E/F, \chi)$, then the $\pi(\tilde{\chi})$ that ϕ maps to under the previous theorem is $\pi_{\chi\Delta_\chi}$.

We expect these theorems to hold for $GL(n, F)$ where n coprime to the residual characteristic of F, as well as the analogous theorems for more general reductive groups. This will be the subject of future work.

The strategy in the thesis will be as follows: we will show that rather than using admissible pairs to construct the local Langlands correspondence, it is more natural to use genuine characters of 2-fold covers of elliptic tori, satisfying certain regularity type conditions. In particular, we show that Weil parameters are naturally in bijection with these genuine characters. We then naturally write down a Harish-Chandra type character formula $F(\tilde{\chi})$ for the supercuspidal representations of $GL(\ell, F)$ in terms of genuine characters $\tilde{\chi}$ of the double cover $T(F)_{\tau\circ\rho}$ of E^* as in Harish-Chandra's theorem. We prove that $F(\tilde{\chi})$ is indeed the character (on the range $\{z \in T(F)^{reg} : 0 \leq n(z) \leq r/2\}$) of a unique supercuspidal representation

$\pi(\tilde{\chi})$ of $GL(\ell, F)$, and that the map

$$\phi \mapsto \tilde{\chi} \mapsto \pi(\tilde{\chi})$$

is the local Langlands correspondence for $GL(\ell, F)$. Therefore there is no need to introduce the character twists Δ_χ that arise in the work of [5] and [14]. Therefore, as a byproduct of our work, we have given an explanation for the character twists Δ_χ that appear in the local Langlands correspondence for $GL(\ell, F)$.

We now briefly present an outline of the thesis. In section 2, we introduce some notation that will be used throughout. In section 3, we recall the necessary theory from real groups that we need. In particular, we describe some of the basic ingredients of the local Langlands correspondence for real reductive groups, following [2]. In section 4, we recall the necessary background to describe the local Langlands correspondence for $GL(2, F)$, following [5]. In particular, we introduce the notion of an admissible pair, and describe how such pairs parameterize both irreducible two-dimensional representations of the Weil group and supercuspidal representations of $GL(2, F)$. We then introduce the character twists Δ_χ that arise in the local Langlands correspondence, and then state the local Langlands correspondence as is stated in [5]. In section 5, we introduce the relevant double covers that play a role in our theory. We then define the relevant double covers of tori, and show how to incorporate them into the local Langlands theory for $PGL(2, F)$ and $GL(2, F)$. We then rewrite the supercuspidal characters of $GL(2, F)$ in terms of regular genuine characters of double covers of elliptic tori, and show that the distribution characters are determined by the values on the range $\{z \in T(F)^{reg} : 0 \leq n(z) \leq r/2\}$. Finally, we present a natural construction of the tame local Langlands correspondence for positive depth supercuspidal representations of $GL(2,F)$ using the above theory. In section 6, we treat the case of depth zero representations of $GL(2,F)$, and the theory is similar. The construction of the local Langlands correspondence we present in section 5 works equally well for the depth zero representations. In section 7, we recall the necessary background to describe the local Langlands correspondence for $GL(\ell,F)$ where ℓ is an odd prime, following [14]. In particular, we introduce the notion of an admissible pair, and describe how they parameterize both irreducible ℓ-dimensional representations of the Weil group and supercuspidal representations of $GL(\ell,F)$. We then introduce the character twists Δ_χ that arise in the local Langlands correspondence, and then state the local Langlands correspondence as is stated in [14]. In section 8, we develop our general theory for $GL(\ell,F)$, which carries over directly from the theory we developed for $GL(2,F)$ in section 5. In section 9, we treat the case of depth zero representations of $GL(\ell,F)$, and the theory is analogous.

Chapter 2

Notation and Definitions

Let F denote a local field of characteristic zero, $\mathfrak{o}_F$ its ring of integers, and $\mathfrak{p}_F$ the maximal ideal of $\mathfrak{o}_F$. We let p denote a uniformizer of F. Let k_F denote the residue field of F with cardinality q. We choose an element $\Phi \in Gal(\overline{F}/F)$ whose inverse induces on $\overline{k_F}$ the map $x \mapsto x^q$. Throughout, we fix once and for all a nontrivial additive character ψ of F of level one. If E/F is a separable extension, N will denote the norm map from E to F, $Tr_{E/F}$ will denote the trace map from E to F, and $Aut(E/F)$ will denote the group of automorphisms of E that fix F pointwise. When we write a decomposition $w = p^n u$ where $w \in F^*$, we mean that $u \in \mathfrak{o}_F^*$. If E/F is quadratic and $E = F(\delta)$, we will frequently decompose an element $w \in E$ as $w = p^n u + p^m v\delta$ where we are viewing E as a vector space over F with basis $1, \delta$, and $u, v \in \mathfrak{o}_F^*$. If E/F is quadratic, then we will write $\overline{w}$ instead of $\upsilon(w)$ where $1 \neq \upsilon \in Gal(E/F)$. Let $(,)_F$ denote the Hilbert symbol of F; most of the time we will write $(,)$ when there is no confusion about the field. We also set $U_F^n := 1 + \mathfrak{p}_F^n$ and $U_F = \mathfrak{o}_F^*$. If E/F is Galois, we let $\aleph_{E/F}$ denote the local class field theory character of F^* relative to the extension E/F. If K is a local non-archimedean field of characteristic zero, we let $|\ |_K$ denote the K-adic absolute value of K. In Chapters 5-6, τ_o will denote any character of E^* whose restriction to F^* is $\aleph_{E/F}$, where E/F is a tame quadratic extension, and we will set $\tau := \tau_o \,|\ |_E$. In Chapters

8-9, τ_o will denote any character of $(EL)^*$ whose restriction to L^* is $\aleph_{EL/L}$, where E/F is a tame degree ℓ extension and L/F is the degree $\ell - 1$ unramified extension, and we will set $\tau := \tau_o \mid \mid_{EL}$. We will generally write $\mid \mid$ when it is clear which field we are referring to.

If

$$1 \to \mathbb{Z}/2\mathbb{Z} \to A \to B \to 1$$

is an exact sequence of groups, then a character β of A is said to be *genuine* if $\beta|_{\mathbb{Z}/2\mathbb{Z}}$ is not trivial (that is, β does not arise from a character $A \to B$). When we say that a 2-fold cover of a group (as above) *splits*, we mean that the exact sequence splits. If G denotes any group, then G^{ab} denotes its abelianization. If G is a connected reductive group defined over F and T is a maximal torus in G defined over F, then we will frequently write the finite relative Weyl group as W instead of $W(G(F), T(F))$. We will write $T(F)^{reg}$ for the set of regular elements in $T(F)$. If B is a normal subgroup of A and $a \in A$, then we will write $[a]$ to denote the class of a in A/B.

Chapter 3

Background from real groups

In order to motivate the theory that we wish to develop for p-adic groups, we describe the corresponding theory over $\mathbb{R}$ since this is what our theory is based upon. We will briefly recall the relevant theory of the local Langlands correspondence over $\mathbb{R}$. A more detailed account is given in the appendix. More information about the general theory can be found in [2].

3.1 Covers of Tori

It will be important to describe a part of the local Langlands correspondence having to do with discrete series representations. Recall Definition (1.0.3). The Weyl group acts on $T(\mathbb{R})_\rho$ as follows: If $(t, \lambda) \in T(\mathbb{R})_\rho$, then define

$$s(t, \lambda) := (st, e^{s^{-1}\rho - \rho}(t)\lambda) \quad \forall s \in W(G(\mathbb{R}), T(\mathbb{R})) \tag{3.1}$$

Definition 3.1.1. A genuine character $\tilde{\chi}$ of $T(\mathbb{R})_\rho$ is called *regular* if ${}^s\tilde{\chi} \neq \tilde{\chi}$ $\forall s \in W(G(\mathbb{R}), T(\mathbb{R}))$ where ${}^s\tilde{\chi}(t, \lambda) := \tilde{\chi}(s^{-1}(t, \lambda))$.

3.2 Discrete series Langlands paramaters and character formulas

In this section we will briefly describe the local Langlands correspondence for discrete series representations of real groups. Let G be a connected reductive group over $\mathbb{R}$ that contains a compact torus. It is known that this is equivalent to $G(\mathbb{R})$ having discrete series representations.

Definition 3.2.1. Let t be an indeterminate and let k denote the rank of G. For $h \in G$, define the Weyl denominator $D_G(h)$ by

$$det(t+1-Ad(h)) = D_G(h)t^k + ...(terms\ of\ higher\ order)$$

Then if Δ is the set of roots of T in G,

$$D_G(h) = \prod_{\alpha \in \Delta} (1-\alpha(h)).$$

Definition 3.2.2. Let G be a connected reductive group over $\mathbb{R}$, $T \subset G$ a maximal torus over $\mathbb{R}$. Let Δ^+ be a set of positive roots of G with respect to T. Define

$$\Delta^0(h, \Delta^+) := \prod_{\alpha \in \Delta^+} (1-\alpha^{-1}(h)),\ h \in T(\mathbb{R})$$

Then if the cardinality of Δ^+ is n, we have

$$(-1)^n D_G(h) = \Delta^0(h, \Delta^+)^2 (2\rho)(h)$$

Then, if we define $|\rho(h)| := |2\rho(h)|^{\frac{1}{2}}$ we get that

$$|D_G(h)|^{\frac{1}{2}} = |\Delta^0(h, \Delta^+)||\rho(h)|$$

In general, ρ is not in $X^*(T)$. Therefore, ρ is not a well-defined character of $T(\mathbb{R})$. However, ρ is a well-defined character of $T(\mathbb{R})_\rho$. If $\tilde{h} \in T(\mathbb{R})_\rho$ maps to $h \in T(\mathbb{R})$ via the canonical projection, then

$$|D_G(h)|^{\frac{1}{2}} = |\Delta^0(h, \Delta^+)||\rho(h)| = |\Delta^0(h, \Delta^+)||\rho(\tilde{h})|$$

Recall Theorem (1.0.5). One can explicitly calculate the discrete series characters of $GL(2,\mathbb{R})$ using the theory from [2], but the details are tedious.

We conclude the section by describing the local Langlands correspondence for discrete series representations of $GL(2,\mathbb{R})$. Fix a positive set of roots Δ^+ of G with respect to T. Let $\phi : W_{\mathbb{R}} \to GL(2,\mathbb{C})$ be a discrete series Weil parameter. By the theory in [2], ϕ caonically gives rise to a genuine character $\tilde{\chi}$ of $T(\mathbb{R})_\rho$. By Harish-Chandra's discrete series theorem, $\tilde{\chi}$ canonically gives rise to a unique discrete series representation, denoted $\pi(\tilde{\chi})$, whose distribution character is

$$\theta_{\pi(\tilde{\chi})}(h) := \epsilon(\tilde{\chi},\Delta^+)\frac{\displaystyle\sum_{s\in W}\epsilon(s)\tilde{\chi}({}^s\tilde{h})}{\Delta^0(h,\Delta^+)\rho(\tilde{h})},\ h \in T(\mathbb{R})^{reg}$$

where $\tilde{h} \in T(\mathbb{R})_\rho$ is any element such that $\Pi(\tilde{h}) = h$. Then the map $\phi \mapsto \pi(\tilde{\chi})$ is the local Langlands correspondence for discrete series representations of $GL(2,\mathbb{R})$. The rest of the thesis will be devoted to proving the analogous result for $GL(\ell, F)$, where F is a local non-Archimedean field of characteristic zero, and ℓ is prime.

Chapter 4

Existing Description of Local Langlands for $GL(2, F)$

In this chapter, we describe the construction of the local Langlands correspondence for $GL(2, F)$ as explained in [5].

4.1 Admissible Pairs

Let E/F be a tamely ramified quadratic extension and χ a character of E^*. Recall that N denotes the norm map from E to F.

Definition 4.1.1. The pair $(E/F, \chi)$ is called an *admissible pair* if

(i) χ does not factor through N and

(ii) If $\chi|_{1+\mathfrak{p}_E}$ factors through N, then E/F is unramified.

Admissible pairs $(E/F, \chi)$, $(E'/F, \chi')$ are said to be F-isomorphic if there exists an F-isomorphism $j : E \to E'$ such that $\chi(w) = \chi'(j(w))\ \forall w \in E^*$. In the case $E = E'$, this amounts to saying that $1 \neq \upsilon \in Aut(E/F)$ and $\chi(w) = \chi'(\upsilon(w))\ \forall w \in E^*$.

We write $\mathbb{P}_2(F)$ for the set of F-isomorphism classes of admissible pairs. Note that if $(E/F, \chi)$ is an admissible pair and if ϕ is a character of F^*, then the pair $(E/F, \chi \otimes \phi_E)$ is an admissible pair where $\phi_E = \phi \circ N$.

Definition 4.1.2. Let $(E/F, \chi)$ be an admissible pair where χ is level n. We say that $(E/F, \chi)$ is *minimal* if $\chi|_{U_E^n}$ does not factor through N.

Note that any admissible pair $(E/F, \chi)$ is isomorphic to one of the form $(E/F, \chi' \otimes \phi_E)$ where ϕ is a character of F^* and $(E/F, \chi')$ is a minimal admissible pair.

Definition 4.1.3. Let v_E denote the E-adic valuation. An element $\alpha \in E \setminus F$ is called *minimal* over F if the algebra $E = F[\alpha]$ is a field and, setting $n = -v_E(\alpha)$, one of the following holds:

(1) E/F is totally ramified and n is odd;

(2) E/F is unramified and, for a prime element p of F, the coset $p^n\alpha + \mathfrak{p}_E$ generates the finite field extension k_E/k.

Proposition 4.1.4. *Let E/F be a tamely ramified quadratic extension, and let χ be a character of E^* of level $m \geq 1$. Let $\alpha(\chi) \in \mathfrak{p}_E^{-m}$ satisfy $\chi(1+x) = \psi_E(\alpha(\chi)x)$ $\forall x \in \mathfrak{p}_E^m$.*[1] *Then $(E/F, \chi)$ is a minimal (admissible) pair if and only if the element $\alpha(\chi)$ is minimal over F.*

Proof. See [2, Proposition 18.2] □

4.2 Depth zero supercuspidal representations of $GL(2, F)$

In this section we recall the parameterization of the depth zero supercuspidal represention[s] via a subclass of admissible pairs, following [2, Chapter 19].

Let $(E/F, \chi)$ be an admissible pair where χ has level 0. By definition of admissible pair, this implies that E/F is unramified.

[1] In [5], the notation α is used. We prefer to use the notation $\alpha(\chi)$ since this element depends on the character χ.

Lemma 4.2.1. *Let E/F be an unramified quadratic extension, let χ be a character of E^* of level zero, and let $\upsilon \in Aut(E/F)$, $\upsilon \neq 1$. The following are equivalent:*

(i) the pair $(E/F, \chi)$ is admissible

(ii) $\chi \neq \chi^{\upsilon}$

(iii) $\chi|_{U_E} \neq \chi^{\upsilon}|_{U_E}$

where $\chi^{\upsilon}(w) := \chi(\upsilon(w))$.

Proof. See [5, Lemma 19.1] □

Returning to the admissible pair $(E/F, \chi)$ of level zero, write $k_E = \mathfrak{o}_E/\mathfrak{p}_E$. Then k_E/k_F is a quadratic extension, where $k_F = \mathfrak{o}_F/\mathfrak{p}_F$. Moreover, since $\chi|_{1+\mathfrak{p}_E} = 1$, $\chi|_{U_E}$ is the inflation of a character, call it again χ, of k_E^*. By the theory of finite groups of Lie type, the character χ then gives rise to an irreducible cuspidal representation λ' of $GL(2, k_F)$. Let λ be the inflation of λ' to $GL(2, \mathfrak{o}_F)$. We may extend λ to a representation Λ of $K := F^*GL(2, \mathfrak{o}_F)$ by setting $\Lambda|_{F^*} = \chi|_{F^*}$, and induce the resulting representation to all of G (see [5, Chapter 19]). Set

$$\pi_\chi = cInd_K^G \Lambda$$

where $cInd$ denotes compact induction.

Then these are all the depth zero representations of $GL(2, F)$. In particular, if $\mathbb{P}_2(F)_0$ denotes the set of admissible pairs of level zero and $\mathbb{A}_2^0(F)_0$ denotes the set of equivalence classes of depth zero supercuspidal representations of $GL(2, F)$, then we have the following proposition.

Proposition 4.2.2. *The map $(E/F, \chi) \mapsto \pi_\chi$ induces a bijection*

$$\mathbb{P}_2(F)_0 \to \mathbb{A}_2^0(F)_0$$

Furthermore, if (E/F, χ) $\in \mathbb{P}_2(F)_0$, then:

(i) if ϕ is a character of F^* of level zero, then $\pi_{\chi\phi_E} = \phi\pi_\chi$

(ii) if $\pi = \pi_\chi$, then $\omega_\pi = \chi|_{F^*}$

Proof. See [5, Section 19.1].

□

4.3 Positive depth supercuspidal representations of $GL(2, F)$

In this section we recall the parameterization of the positive depth supercuspidal represention via a subclass of admissible pairs, following [5, Chapter 19].

First we let $(E/F, \chi)$ be a minimal admissible pair such that χ has level $n \geq 1$. We set $\psi_E = \psi \circ Tr_{E/F}$.

Proposition 4.3.1. *Let m,n be integers, $0 \leq m < r \leq 2m+1$. Let ψ be a character of F of level one. Let $a \in F$. Define the character ψ_a by $\psi_a(x) = \psi(a(x-1))$. The map $a \mapsto \psi_a|_{U_F^{m+1}}$ induces an isomorphism*

$$\mathfrak{p}_F^{-r}/\mathfrak{p}_F^{-m} \to (U_F^{m+1}/U_F^{r+1})^\wedge$$

Proof. See [5, Proposition 1.8]

□

We apply this proposition to the character χ of E^* of an admissible pair $(E/F, \chi)$. Let $\llcorner\lrcorner$ denote the floor function. Then, the restriction of χ to $U_E^{\llcorner n/2 \lrcorner+1}$ defines a character of $U_E^{\llcorner n/2 \lrcorner+1}$ that is trivial on U_E^{n+1}. Therefore, it defines an element of $(U_E^{m+1}/U_E^{r+1})^\wedge$ where $m = \llcorner n/2 \lrcorner$ and $r = n$. Therefore, by the previous proposition, since ψ_E is a character of E of level one, there is an element $\alpha(\chi) \in \mathfrak{p}_E^{-n}$ such that $\chi(1+x) = \psi_E(\alpha(\chi)x) \;\; \forall x \in \mathfrak{p}_E^{\llcorner n/2 \lrcorner + 1}$.

To the pair of data $(E/F, \chi)$ and $\alpha(\chi)$, where $(E/F, \chi)$ is a minimal admissible pair, one can attach a supercuspidal representation π_χ of $GL(2, F)$ (see [5, Chapter 20]). We will not need the details of this construction, but will recall the relevant facts from [5] as we need them.

In general, let $(E/F, \chi)$ be an arbitrary admissible pair of level $n \geq 1$. As we mentioned before, there is a character ϕ of F^* and a character χ' of E^* such that $(E/F, \chi')$ is a minimal admissible pair and $\chi = \chi' \phi_E$. We define $\pi_\chi = \phi \pi_{\chi'}$. The result is independent of the choice of decomposition $\chi = \chi' \phi_E$.

In all cases, the equivalence class of the representation π_χ depends only on the isomorphism class of the admissible pair $(E/F, \chi)$. Let $\mathbb{A}_2^0(F)$ denote the set of equivalence classes of all irreducible supercuspidal representations of $GL(2, F)$. Then together with Proposition (4.2.2), we have a map

$$\mathbb{P}_2(F) \to \mathbb{A}_2^0(F)$$

$$(E/F, \chi) \mapsto \pi_\chi$$

defined independently of all choices.

Theorem 4.3.2. *The map* $(E/F, \chi) \mapsto \pi_\chi$ *induces a bijection*

$$\mathbb{P}_2(F) \to \mathbb{A}_2^0(F) \quad if \quad p \neq 2$$

Furthermore, if $(E/F, \chi) \in \mathbb{P}_2(F)$*, then:*

(i) if χ *has level* $l(\chi)$*, then* $l(\pi_\chi) = l(\chi)/e(E|F)$*.*

(ii) $\omega_{\pi_\chi} = \chi|_{F^*}$

(iii) the pair $(E/F, \chi^\vee)$ *is admissible and* $\pi_{\chi^\vee} = \pi_\chi^\vee$

(iv) if ϕ *is a character of* F^**, then* $\pi_{\chi\phi_E} = \phi\pi_\chi$.

Proof. See [5, Theorem 20.2]

□

4.4 Weil parameters

In this section, we recall the statement of the local Langlands correspondence for $GL(2,F)$. Most of what we say here is taken straight from [5].

Let $\mathbb{G}_2(F)$ denote the set of equivalence classes of 2-dimensional, semisimple, Weil-Deligne representations ([5, Section 31]). Again, let $\mathbb{A}_2(F)$ denote the set of equivalence classes of irreducible, smooth representations of $GL(2,F)$.

We first state the local Langlands correspondence, and then roughly describe the elements behind the statement.

Theorem 4.4.1. *There is a unique map*

$$\pi : \mathbb{G}_2(F) \to \mathbb{A}_2(F)$$

such that

$$L(\chi\pi(\phi), s) = L(\chi \otimes \phi, s), \tag{4.1}$$

$$\epsilon(\chi\pi(\phi), s, \psi) = \epsilon(\chi \otimes \phi, s, \psi), \tag{4.2}$$

for all $\phi \in \mathbb{G}_2(F)$ and all characters χ of F^*. The map π is a bijection.

The map π is the Langlands correspondence for $GL(2, F)$. We make some preliminary remarks. We have a decomposition

$$\mathbb{G}_2(F) = \mathbb{G}_2^1(F) \cup \mathbb{G}_2^0(F)$$

where $\mathbb{G}_2^0(F)$ is the set of equivalence classes of irreducible smooth representations of W_F of dimension two, and $\mathbb{G}_2^1(F)$ denotes the classes of Deligne representations $(\phi, V, \mathfrak{n}) \in \mathbb{G}_2(F)$ (see [5, Section 31]) for which the representation ϕ of W_F is reducible. Likewise, we write

$$\mathbb{A}_2(F) = \mathbb{A}_2^1(F) \cup \mathbb{A}_2^0(F)$$

where $\mathbb{A}_2^0(F)$ denotes the representations $\pi \in \mathbb{A}_2(F)$ that are supercuspidal, and $\mathbb{A}_2^1(F)$ denotes the representations $\pi \in \mathbb{A}_2(F)$ that are not supercuspidal.

Then it is a fact (cf [2, p. 221]) that the Langlands correspondence π must map $\mathbb{G}_2^1(F)$ to $\mathbb{A}_2^1(F)$ and $\mathbb{G}_2^0(F)$ to $\mathbb{A}_2^0(F)$. We are only concerned with the map on $\mathbb{G}_2^0(F)$, which is the heart of the matter. That is :

Theorem 4.4.2. *[5, Theorem 33.4] There is a unique map*

$$\pi : \mathbb{G}_2^0(F) \to \mathbb{A}_2^0(F)$$

with the property

$$\epsilon(\chi \otimes \phi, s, \psi) = \epsilon(\chi\pi(\phi), s, \psi) \tag{4.3}$$

for all $\phi \in \mathbb{G}_2^0(F)$, all characters χ of F^*. Moreover, the map π is a bijection.

A very important property of the map π is:

Proposition 4.4.3. *Let π be a map satisfying (4.3) of Theorem 4.4.2. Then:*

(i) If $\phi \in \mathbb{G}_2^0(F)$ and $\pi = \pi(\phi)$, then $\omega_\pi = det(\phi)$.

(ii) The map π satisfies (4.3).

Proof. See [5, Section 33]

□

We next turn to the question of parameterizing representations of W_F by admissible pairs. We have already parameterized the supercuspidal representations of $GL(2, F)$ by admissible pairs.

Let $\mathbb{P}_2(F)$ denote again the set of admissible pairs. Recall that there is a local Artin reciprocity isomorphism given by $W_E^{ab} \cong E^*$. Then, if $(E/F, \xi) \in \mathbb{P}_2(F)$, ξ gives rise to a character of W_E^{ab}, which we can pullback to a character, also denoted ξ, of W_E. We can then form the induced representation $\phi_\xi = Ind_{W_E}^{W_F}\xi$ of W_F. We sometimes denote this representation by $Ind_{E/F}\xi$.

Theorem 4.4.4. *Suppose the residual characteristic of F is not 2. If $(E/F, \xi)$ is an admissible pair, the representation ϕ_ξ of W_F is irreducible. The map $(E/F, \xi) \mapsto \phi_\xi$ induces a bijection*

$$\mathbb{P}_2(F) \to \mathbb{G}_2^0(F)$$

Proof. See [5, Chapter 33]. □

We therefore have canonical bijections

$$\mathbb{P}_2(F) \to \mathbb{A}_2^0(F), \qquad \mathbb{P}_2(F) \to \mathbb{G}_2^0(F) \tag{4.4}$$

$$(E/F, \xi) \mapsto \pi_\xi, \qquad (E/F, \xi) \mapsto \phi_\xi,$$

given by Theorem 4.4.4 and Theorem 4.3.2. Combining both of these bijections, we obtain a bijection

$$\mathbb{G}_2^0(F) \to \mathbb{A}_2^0(F) \tag{4.5}$$

$$\phi_\xi \mapsto \pi_\xi$$

However, this bijection is NOT the map π demanded in Theorem 4.4.2. The reason is as follows. If $(E/F, \xi) \in \mathbb{P}_2(F)$, then by [5, Proposition 29.2], representation ϕ_ξ has determinant $\aleph_{E/F} \otimes \xi|_{F^*}$, whereas π_ξ has central character $\xi|_{F^*}$, contrary to the requirement of Proposition 4.4.3. To obtain the map π of 4.4.2, we must therefore systematically modify the bijection (4.5), which we proceed to do now.

If K/F is a finite separable extension, let

$$\lambda_{K/F}(\psi) = \frac{\epsilon(R_{K/F}, s, \psi)}{\epsilon(1_K, s, \psi_K)}$$

denote the Langlands constant, as in [2, 34.3].

Proposition 4.4.5. *Let K/F be a tamely ramified quadratic extension.*

(i) If K/F is unramified, then $\aleph_{K/F}$ is unramified of order 2 and

$$\lambda_{K/F}(\psi) = -1$$

(ii) If K/F is totally ramified, then $\aleph_{K/F}$ is the non-trivial character of $F^*/N_{K/F}(K^*)$ and

$$\lambda = \tau(\aleph_{K/F}, \psi)/q^{\frac{1}{2}}.$$

In particular, $\lambda_{K/F}(\psi)^2 = \aleph_{K/F}(-1)$

Proof. See [5, p. 255] for notation and proof.

□

Now, let $(E/F, \xi) \in \mathbb{P}_2(F)$ be an admissible pair. We associate to this pair a character $\Delta = \Delta_\xi$ of E^* of level zero. First:

Definition 4.4.6. Let $(E/F, \xi)$ be an admissible pair in which E/F is unramified. Define Δ_ξ to be the unique quadratic unramified character of E^*.

The ramified case is more involved. We recall that μ_F denotes the group of roots of unity in F of order prime to the residual characteristic of F. Let E/F be a totally tamely ramified quadratic extension, let ϖ be a uniformizer of E, and let $\beta \in E^*$. Since $U_E = \mu_E U_E^1 = \mu_F U_E^1$, there is a unique root of unity $\zeta(\beta, \varpi) \in \mu_F$ such that

$$\beta \varpi^{-v_E(\beta)} = \zeta(\beta, \varpi) \ (mod\ U_E^1).$$

Definition 4.4.7. (i) Let $(E/F, \xi) \in \mathbb{P}_2(F)$ be a minimal admissible pair such that E/F is totally tamely ramified. Let n be the level of ξ and let $\alpha(\chi) \in \mathfrak{p}_E^{-n}$ satisfy $\xi(1+x) = \psi_E(\alpha(\chi)x), x \in \mathfrak{p}_E^n$. There is a unique character $\Delta = \Delta_\xi$ of E^* such that:

$$\Delta|_{U_E^1} = 1, \quad \Delta|_{F^*} = \aleph_{E/F},$$

$$\Delta(\varpi) = \aleph_{E/F}(\zeta(\alpha(\chi), \varpi))\lambda_{E/F}(\psi)^n,$$

for any prime element ϖ of E. The definition of Δ_ξ is independent of the choices of ψ and $\alpha(\chi)$.

(ii) Let $(E/F, \xi) \in \mathbb{P}_2(F)$ and suppose that E/F is totally tamely ramified. Write $\xi = \xi'\chi_E$ for a minimal admissible pair $(E/F, \xi')$ and a character χ of F^*. Define

$$\Delta_\xi = \Delta_{\xi'}.$$

The definition of Δ_ξ is independent of the choice of decomposition $\xi = \xi'\chi_E$.

Lemma 4.4.8. *(i) If $(E/F, \xi)$ is an admissible pair, the pair $(E/F, \xi\Delta_\xi)$ is admissible and the isomorphism class depends only on that of $(E/F, \xi)$. The character Δ_ξ satisfies $\Delta_\xi^2 = 1$, except when E/F is totally ramified and $q \equiv 3 \pmod 4$. In the exceptional case, Δ_ξ has order 4.*

(ii) The map

$$\mathbb{P}_2(F) \to \mathbb{P}_2(F)$$

$$(E/F, \xi) \mapsto (E/F, \xi\Delta_\xi)$$

is bijective.

Proof. Part (1) follows directly from the definitions of Δ_ξ. Part (2) follows from the observation that Δ_ξ is tamely ramified, depending only on E/F and $\xi|_{U_E^1}$. □

We can finally state the local Langlands correspondence for $GL(2, F)$, as described in [5].

Theorem 4.4.9. Tame Local Langlands Correspondence

Suppose the residual characteristic of F is not 2.

(i) For $\phi \in \mathbb{G}_2^0(F)$, define $\pi(\phi) = \pi_{\xi\Delta_\xi}$ in the notation of (4.4) for any $(E/F, \xi) \in \mathbb{P}_2(F)$ such that $\phi \cong \phi_\xi$. The map

$$\pi : \mathbb{G}_2^0(F) \to \mathbb{A}_2^0(F)$$

is a bijection satisfying

$$\epsilon(\chi \otimes \phi, s, \psi) = \epsilon(\chi\pi(\phi), s, \psi),$$

for all characters χ of F^*.

(ii) The map π satisfies

$$\pi(\chi \otimes \phi) = \chi\pi(\phi) \quad \textit{and} \quad \pi(\phi^\vee) = \pi(\phi)^\vee,$$

for all ϕ and all characters χ of F^*.

Proof. See [5, Chapter 34]. □

We make two concluding remarks: Because of the uniqueness properties, π is the Langlands correspondence when the residual characteristic of F is not 2. Moreover, it is important to note that the construction gives $\omega_{\pi(\phi)} = det(\phi)$.

Chapter 5

Our constructions in the positive depth case for $GL(2,F)$

5.1 Covers of Tori

In this section, we define a special cover of a torus that we will need throughout. This cover is an analogue of $T(\mathbb{R})_\rho$ in the setting of p-adic groups. Recall Definition (1.0.2). Let G be a connected reductive group defined over F, and T a maximal torus in G defined over F. Let Δ^+ be a choice of positive roots of G with respect to T. Let $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. Let λ be a character of K^*, where K/F is the minimal splitting field of T. Note that the image of 2ρ, restricted to $T(F)$, lies in K^*.

Definition 5.1.1. We define the *$\lambda\circ\rho$-cover of* $T(F)$, denoted $T(F)_{\lambda\circ\rho}$, as the pullback of the two homomorphisms

$$\lambda\circ 2\rho : T(F) \to \mathbb{C}^* \qquad \Upsilon : \mathbb{C}^* \to \mathbb{C}^*$$

$$t \mapsto \lambda\circ 2\rho(t) \qquad z \mapsto z^2$$

$$\begin{array}{ccc} T(F)_{\lambda\circ\rho} & \xrightarrow{\lambda\circ\rho} & \mathbb{C}^* \\ \big\downarrow \Pi & & \big\downarrow \Upsilon \\ T(F) & \xrightarrow{\lambda\circ 2\rho} & \mathbb{C}^* \end{array}$$

That is, $T(F)_{\lambda\circ\rho} = \{(z,w)\in T(F)\times\mathbb{C}^* : \lambda(2\rho(z)) = w^2\}$

Note that the above map $T(F)_{\lambda\circ\rho}\to\mathbb{C}^*$ sends (z,λ) to λ. We have denoted this map by $\lambda\circ\rho$, even though this map is not literally λ composed with ρ. Moreover,

Π is the canonical projection $\Pi(z, \lambda) = z$. We will use these maps repeatedly.

For example, if $T(F) = E^*/F^*$ is an elliptic torus in $PGL(2, F)$, then let $z \in T(F)$. Then let $w \in E^*$ such that $w \mapsto z = [w]$ under the map $E^* \to E^*/F^*$. Then, let ρ be half the standard positive root of $PGL(2, F)$ and let ρ' be half the standard positive root of $GL(2, F)$. Let α be the standard positive root of $PGL(2, F)$ and let α' be the standard positive root of $GL(2, F)$. Then $\alpha'(w) = \alpha'(xw)\ \forall x \in F^*$ since roots are trivial on the center F^* of $GL(2, F)$. Thus, α' factors to E^*/F^*, an elliptic torus in $PGL(2, F)$. In fact α' factors to α. Well, $\alpha'(w) = w/\overline{w}$, and we get that $w/\overline{w} = \alpha'(w) = \alpha([w]) = \alpha(z)$. Therefore, $\lambda \circ 2\rho(z) = \lambda(w/\overline{w})$.

5.2 Setup

In this chapter we first prove an analogue of Harish-Chandra's discrete series theorem (see Theorem (1.0.5)) for the positive depth supercuspidal representations of $GL(2, F)$.

Theorem 5.2.1. *Let $G(F) = GL(2, F)$, and let $T(F) = E^*$ be an elliptic torus in $GL(2, F)$, so $E = F(\sqrt{\Delta})$ for some $\Delta \in F^*$. Let τ_o be any character of E^* whose restriction to F^* is $\aleph_{E/F}$, where $\aleph_{E/F}$ is the local class field theory character of F^* relative to E/F. Let $\tau := \tau_o \mid\ \mid_E$ where $\mid\ \mid_E$ denotes the E-adic absolute value. Let Δ^+ be a set of positive roots of G with respect to T. Let $\rho := \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$. Let $T(F)_{\tau \circ \rho}$ be the $\tau \circ \rho$ cover of $T(F)$ as in Definition (5.1.1). Let $\tilde{\chi}$ be a genuine character of $T(F)_{\tau \circ \rho}$ that is regular. Let $W(G(F), T(F))$ be the relative Weyl group of $G(F)$ with respect to $T(F)$. If $s \in W(G(F), T(F))$, let $\epsilon(s) := (-1, \Delta)^{\ell(s)}$, where $(,)$ denotes* *the Hilbert symbol of F and $\ell(s)$ denotes the length of s. Let $T(F)^{reg}$ denote the regular elements of $T(F)$.*

Then there exists a unique constant $\epsilon(\tilde{\chi}, \Delta^+, \tau)$, depending only on $\tilde{\chi}, \Delta^+$, and τ, and a unique supercuspidal representation of $GL(2, F)$ denoted $\pi(\tilde{\chi})$, such that

$$\theta_{\pi(\tilde{\chi})}(z) = \epsilon(\tilde{\chi}, \Delta^+, \tau)\frac{\sum_{s \in W} \epsilon(s)\tilde{\chi}({}^s w)}{\tau(\Delta^0(z, \Delta^+))(\tau \circ \rho)(w)}, \ \forall z \in T(F)^{reg} \ : 0 \leq n(z) \leq r/2$$

where $w \in T(F)_{\tau \circ \rho}$ is any element such that $\Pi(w) = z$ and r is the depth of $\pi(\tilde{\chi})$. Moreover, every supercuspidal character of $GL(\ell, F)$ is of this form.

Definition 5.2.2. Let $\tilde{\chi}$ be a genuine character of $T(F)_{\tau \circ \rho}$. We define the function $F(\tilde{\chi}) : T(F)^{reg} \to \mathbb{C}$ by

$$F(\tilde{\chi})(z) = \epsilon(\tilde{\chi}, \Delta^+, \tau)\frac{\sum_{s \in W} \epsilon(s)\tilde{\chi}({}^s w)}{\tau(\Delta^0(z, \Delta^+))(\tau \circ \rho)(w)}, \ \forall z \in T(F)^{reg}$$

We will define all of the notation in the above theorem in the next several sections, including $n(z)$, $\epsilon(\tilde{\chi}, \Delta^+, \tau)$, and the definition of regular. We will also regularly use the fact (see Theorem (A.0.3)) that $W(G(F), T(F)) = Aut(E/F)$. Notice that there is only one main difference between this character formula and the character formula for discrete series of real reductive groups (see Theorem (1.0.5)). The difference is in the denominator. For real groups, the denominator is $\Delta^0(h, \Delta^+)\rho(\tilde{h})$, and in the above statement, it is $\tau(\Delta^0(h, \Delta^+))(\tau \circ \rho)(\tilde{h})$ for some character τ. If we were to literally transport the character formula of Theorem (1.0.5) to the p-adic case, then the denominator $\Delta^0(h, \Delta^+)\rho(\tilde{h})$ would take values in $\overline{F}^*$, which would be problematic since characters must take values in $\mathbb{C}^*$. Therefore, a natural thing to try is to introduce a $\mathbb{C}^*$-valued character τ into the denominator in order that the denominator takes values in $\mathbb{C}^*$.

We note that all of our calculations in the next two chapters will assume that we have chosen the standard positive set of roots of $GL(2, \overline{F})$ with respect to the standard split maximal torus. Our main results, however, will be seen to be independent of any choice of positive roots.

Let us remark also that in the course of proving the above statement for $GL(2, F)$, we must show that a supercuspidal representation of $GL(2, F)$ is determined by the character's values on elements of the torus E^* in the range $\{z \in E^* : 0 \leq n(z) \leq r/2\}$. Actually, we will even show that a supercuspidal representation of $GL(2, F)$ is determined by the character values on the range $\{w \in E^* : n(w) = 0\}$, which is stronger. This will also be true for $GL(\ell, F)$.

Now let ϕ be a supercuspidal Weil parameter for $GL(2, F)$. We will show later in this section how to construct a regular genuine character, $\tilde{\chi}$, of $T(F)_{\tau\circ\rho}$, from ϕ. We will then prove the following theorem.

Theorem 5.2.3. *The assignment*

$$\phi \mapsto \tilde{\chi} \mapsto \pi(\tilde{\chi})$$

is the Local Langlands correspondence for $GL(2, F)$*.*

We begin by making some definitions. Let F denote a non-Archimedean local field of characteristic zero with residual characteristic coprime to 2. Let E/F be a quadratic extension. Write $E = F(\sqrt{\Delta})$ for some square-free element $\Delta \in F^*$ and let $\delta := \sqrt{\Delta}$.

Then since E^* embeds as an elliptic torus in $GL(2,F)$ via the map

$$E^* \hookrightarrow GL(2,F)$$
$$a + b\delta \mapsto \begin{pmatrix} a & b \\ b\Delta & a \end{pmatrix}$$

we likewise have E^*/F^* embedded in $PGL(2,F)$ as an elliptic torus as well via the natural map $GL(2,F) \to PGL(2,F) = GL(2,F)/F^*$. It is useful sometimes to view E^*/F^* as sitting inside $SO(2,1) \cong PGL(2,F)$.

We now introduce a notion of regularity that we will need. Let E/F be a tamely ramified quadratic extension and χ a character of E^*. Recall that N denotes the norm map from E to F.

Definition 5.2.4. χ is called *regular* if χ does not factor through N. If χ is regular, we call the pair $(E/F, \chi)$ a *regular pair*.

All definitions we have made in the previous chapter for admissible pairs, we also make for regular pairs and regular characters. For example, as we defined the notion of minimal admissible pair, we make the same definition for minimal regular pair. In particular, we also define the character twists Δ_χ for a regular pair $(E/F, \chi)$ exactly the same way they were defined for admissible pairs. For example, if $(E/F, \chi)$ is a regular pair where E/F is unramified, then Δ_χ is the unique unramified quadratic character of E^*. Given a regular pair $(E/F, \chi)$, one may also construct a supercuspidal representation π_χ as in the previous chapter, but this construction is not one to one.

Our constructions and results do not require the stronger notion of admissible pair. We will sometimes say that χ is regular when the field E is understood.

We now explain why double covers of tori play a role. We start by considering the group $PGL(2,F)$. First recall that the representations of $PGL(2,F)$ are precisely the representations of $GL(2,F)$ with trivial central character. One of the conditions of the local Langlands correspondence for $GL(n,F)$ says that if $\phi : W_F \to GL(n,\mathbb{C})$ is irreducible, then $det(\phi) = \omega_{\pi(\phi)}$, where $\omega_{\pi(\phi)}$ denotes the central character of $\pi(\phi)$, and where $\pi(\phi)$ denotes the supercuspidal representation of $GL(n,F)$ that corresponds to ϕ under the local Langlands correspondence. Here we are viewing $det(\phi)$ as a character of F^* in the following way. As the image of $det(\phi)$ is in $\mathbb{C}^*$, $det(\phi)$ is trivial on $[W_F, W_F]$, and therefore factors to a character of $F^* \cong W_F^{ab}$ via the Artin map. Let ϕ be a supercuspidal Weil parameter for $PGL(2,F)$ (that is, an irreducible representation $W_F \to GL(2,\mathbb{C})$ that parameterizes a supercuspidal representation of $GL(2,F)$ with trivial central character). Then $\phi = Ind_{W_E}^{W_F}(\chi)$, for some regular pair $(E/F,\chi)$. Since we are using the notion of regular pair here rather than admissible pair, there may be a choice involved here. That is, there may be another regular pair $(E_1/F,\chi_1)$ such that $\phi = Ind_{W_{E_1}}^{W_F}(\chi_1)$ as well. However, this will not matter, and we will show that our results and constructions are independent of all choices. It is a fact (see [5, Proposition 29.2]) that $det(Ind_{W_E}^{W_F}(\chi)) = \chi|_{F^*} \otimes \delta_{E/F}$, where $\delta_{E/F} = det(Ind_{W_E}^{W_F}(1))$. In the case that E/F is quadratic, $\delta_{E/F} = \aleph_{E/F}$, where $\aleph_{E/F}$ is the local class field theory character of F^* relative to E/F. Therefore, in this case we will use $\delta_{E/F}$ and $\aleph_{E/F}$ interchangeably. Now, since $\pi(\phi)$ has trivial central character, the condition $det(\phi) = \omega_{\pi(\phi)}$ becomes

$\chi|_{F^*} \otimes \aleph_{E/F} = 1$, so $\chi|_{F^*} = \aleph_{E/F}$. Therefore, the supercuspidal representations of $PGL(2, F)$ naturally correspond to regular pairs $(E/F, \chi)$ where $\chi|_{F^*} = \aleph_{E/F}$.

One might ask whether the supercuspidal representations of $PGL(2, F)$ are parameterized by characters of its elliptic tori E^*/F^*, as is the case for $GL(2, F)$. However, we have just seen that the supercuspidal representations of $PGL(2, F)$ are parameterized by characters χ of E^* whose restriction to F^* is $\aleph_{E/F}$. Such a χ is not a character of the elliptic torus E^*/F^* in $PGL(2, F)$. Rather, it is a genuine character of a double cover of E^*/F^* in the following way: there is an exact sequence

$$1 \longrightarrow F^* \longrightarrow E^* \longrightarrow E^*/F^* \longrightarrow 1$$

$$w \mapsto [w]$$

Reducing this sequence by $ker(\aleph_{E/F}) = N(E^*)$, we get an exact sequence

$$1 \longrightarrow F^*/ker(\aleph_{E/F}) \longrightarrow E^*/ker(\aleph_{E/F}) \longrightarrow E^*/F^* \longrightarrow 1.$$

Then, this exact sequence becomes

$$1 \longrightarrow F^*/N(E^*) \longrightarrow E^*/N(E^*) \longrightarrow E^*/F^* \longrightarrow 1$$

where N denotes the norm map from E to F. Since $F^*/N(E^*) \cong \mathbb{Z}/2\mathbb{Z}$ by Local Class Field Theory, we have that $E^*/N(E^*)$ is a double cover of the elliptic torus E^*/F^* in $PGL(2, F)$. Then the character χ of E^* naturally factors to a character $\tilde{\chi}$ of $E^*/N(E^*)$, given by $\tilde{\chi}([w]) := \chi(w)\ \forall [w] \in E^*/N(E^*)$ (since it is a character of E^* that is trivial on $N(E^*)$). Moreover, χ is not trivial on all of F^*, so doesn't factor to a character of E^*/F^*. This means that $\tilde{\chi}$ is a genuine character of $E^*/N(E^*)$. Therefore, we are getting that the supercuspidal representations of $PGL(2, F)$ (i.e. the supercuspidal representations of $GL(2, F)$ with trivial central character) naturally correspond to genuine characters of a double cover of the torus E^*/F^* inside $PGL(2, F)$. We note that the double cover $E^*/N(E^*)$ splits if and only if $(-1, \Delta) = 1$ (see [3]). That is, $E^*/N(E^*) \cong E^*/F^* \times \mathbb{Z}/2\mathbb{Z}$ if and only if $(-1, \Delta) = 1$. In fact, this double cover $E^*/N(E^*)$ is none other than an analogue of the ρ-cover that appears in the theory over the reals (see Definition (1.0.3)), which is a natural double cover of a real torus inside of the real group considered. We explain this now.

Relative to the standard positive root of $PGL(2, F)$, let ρ be half the positive root. An elliptic torus in $PGL(2, F)$ is of the form $T(F) = E^*/F^*$. Then if $z \in T(F)$, $2\rho(z) = z$. Fix a character τ_o of E^* whose restriction to F^* is $\aleph_{E/F}$, and set $\tau := \tau_o| \, |_E$. Recall the denominator

$$\tau(\Delta^0(z, \Delta^+))(\tau \circ \rho)(w)$$

that was defined in Theorem (5.2.1). Although $(\tau \circ \rho)(w)$ is not naturally a function on E^*/F^* since in particular ρ is not naturally a function on E^*/F^*, it is by definition a function on the $\tau \circ \rho$-cover of E^*/F^*. Recall that our current situation, since $T(F) = E^*/F^*$, then $T(F)_{\tau\circ\rho} = \{(z, w) \in E^*/F^* \times \mathbb{C}^* : \tau(2\rho(z)) = w^2\}$

We can now identify the natural double cover that we are handed from the Local Langlands correspondence for $PGL(2, F)$, with this cover:

Lemma 5.2.5. $E^*/N(E^*) \cong T(F)_{\tau\circ\rho}$

Proof. Define the map

$$E^*/N(E^*) \xrightarrow{\kappa} T(F)_{\tau\circ\rho}$$

$$[w] \mapsto ([w], \tau_o(w)|2\rho([w])|^{1/2})$$

where we are taking the positive square root of the absolute value. To show injectivity, suppose $\kappa([w]) = ([1], 1)$, where $w \in E^*$. Since $[w] = [1]$, we get $w \in F^*$. But since $\tau_o(w)|2\rho([w])|^{1/2} = 1$, we get that $w \in N(E^*)$ since $\tau_o|_{F^*} = \aleph_{E/F}$ and since $|2\rho([w])| = 1$ since $[w] = 1$. To show surjectivity, suppose $([w], \lambda) \in T(F)_{\tau\circ\rho}$, where $w \in E^*$. Then, by definition of $T(F)_{\tau\circ\rho}$, we get that $\tau(2\rho([w])) = \lambda^2$. This means that $\tau_o(w/\overline{w})|2\rho([w])| = \lambda^2$. But τ_o is trivial on the norms, so we have that $\tau_o(w/\overline{w}) = \tau_o(w^2/N(w)) = \tau_o(w)^2$. Therefore, $\lambda = \pm\tau_o(w)|2\rho([w])|^{1/2}$. If $\lambda = \tau_o(w)|2\rho([w])|^{1/2}$, then we get that $\kappa([w]) = ([w], \lambda)$. If $\lambda = -\tau_o(w)|2\rho([w])|^{1/2}$, then let $x \in F^* \setminus N(E^*)$. Then $\kappa([xw]) = ([w], \lambda)$. Therefore, κ is surjective. Since κ is clearly a homomorphism, κ is an isomorphism. □

We note that the importance of the term $|2\rho([w])|^{1/2}$ comes from the fact that

$$|D([w])|^{\frac{1}{2}} = |\Delta^0([w], \Delta^+)||2\rho([w])|^{1/2},$$

an observation made in Chapter 3. The reason why this is important is that the term $|D(w)|^{1/2}$ appears in the supercuspidal characters (see Section (5.3)). We will need this fact in the character formulas for $PGL(2, F)$, $GL(2, F)$, $PGL(\ell, F)$, and $GL(\ell, F)$ where ℓ is an odd prime.

Now let's write down the character formula for a supercuspidal representation of $PGL(2, F)$. In order to do this, we need to move to the setting of $T(F)_{\tau\circ\rho}$. In particular, the proposed character formula involves genuine characters of $T(F)_{\tau\circ\rho}$.

Let $\phi : W_F \to GL(2, \mathbb{C})$ be a supercuspidal parameter for $PGL(2, F)$ so that $\phi = Ind_{W_E}^{W_F}(\chi)$ for some regular pair $(E/F, \chi)$. As discussed earlier, this gives us a genuine character $\tilde{\chi}$ of $E^*/N(E^*)$.

Definition 5.2.6. A genuine character $\tilde{\eta}$ of $E^*/N(E^*)$ is called *regular* if $(E/F, \eta)$ is regular, where η is the pullback of $\tilde{\eta}$ to E^*. A genuine character $\tilde{\lambda}$ of $T(F)_{\tau\circ\rho}$ is called *regular* if $\tilde{\lambda} \circ \kappa$ is regular.

Now recall from Theorem (5.2.1) the proposed character formula

$$F(\tilde{\chi})(z) := \epsilon(\tilde{\chi}, \Delta^+, \tau)\frac{\sum_{s\in W} \epsilon(s)\tilde{\chi}({}^s w)}{\tau(\Delta^0(z, \Delta^+))(\tau \circ \rho)(w)}, \quad z \in T(F)^{reg}$$

where $w \in T(F)_{\tau\circ\rho}$ is any element such that $\Pi(w) = z$. We naturally constructed a genuine character $\tilde{\chi}$ of $E^*/N(E^*)$. However, the functions in $F(\tilde{\chi})$ have domain $T(F)_{\tau\circ\rho}$. Recall that $T(F)_{\tau\circ\rho} \cong E^*/N(E^*)$ by Lemma (5.2.5), so we can pull the function $(\tau \circ \rho)(w)$ and the Weyl group action in $F(\tilde{\chi})$ back to $E^*/N(E^*)$ via this isomorphism, and leave our constructed $\tilde{\chi}$ as living on $E^*/N(E^*)$. That is, we consider

$$F(\tilde{\chi})(z) = \epsilon(\tilde{\chi}, \Delta^+, \tau)\frac{\sum_{s\in W} \epsilon(s)\tilde{\chi}({}^s[w])}{\tau(\Delta^0(z, \Delta^+))(\tau \circ \rho)(\kappa([w]))}, \quad z \in T(F)^{reg}$$

where $[w] \in E^*/N(E^*)$ such that $\Pi(\kappa([w])) = z$. Unwinding the definitions, we see that

$(\tau \circ \rho)(\kappa([w])) = \tau_o(w)|2\rho([w])|^{1/2}\ \forall [w] \in E^*/N(E^*)$, where we also write $[w]$ as the element in E^*/F^*.

We also need to define the Weyl group action. The Weyl group action on the $\tau \circ \rho$-cover is obtained as follows. If $([w], \lambda)$ is an element of $T(F)_{\tau\circ\rho}$, then analogously to the real case (recall equation (3.1) in Chapter 3), define $s([w], \lambda) = (s[w], \lambda\tau((s^{-1}\rho - \rho)([w])))$ for $s \in W = W(G(F), T(F)) = Aut(E/F)$, the relative Weyl group. Note that this is well-defined. Simplifying this expression, we get $s([w], \lambda) = ([\overline{w}], \lambda\tau(\overline{w}/w))$ when $s \in W$ is nontrivial. Then, since our character formula lives on $E^*/N(E^*)$, we must pull back this action from $T(F)_{\tau\circ\rho}$ to $E^*/N(E^*)$ via κ. Doing this, we see that we get

$$s[w] = \kappa^{-1}(s\kappa([w])) = \kappa^{-1}(s([w], \tau_o(w)|2\rho([w])|^{1/2})) =$$

$$\kappa^{-1}([\overline{w}], \tau_o(w)|2\rho([w])|^{1/2}\tau(\overline{w}/w)) = \kappa^{-1}([\overline{w}], \tau_o(\overline{w})|2\rho([\overline{w}])|^{1/2}) =$$

$$[\overline{w}] \ \forall [w] \in E^*/N(E^*)$$

when $s \in W = Aut(E/F)$ is nontrivial, since $|2\rho([w])| = |w/\overline{w}| = 1 \ \forall w \in E^*$.

We note that the definition of regularity for a genuine character of $T(F)_{\tau\circ\rho}$ is analogous to the definition of regularity for a genuine character $\tilde{\lambda}$ of $T(\mathbb{R})_\rho$ for real groups, since the notion in the setting of real groups is that $\tilde{\lambda}$ is not fixed by any element of the real Weyl group $W(G(\mathbb{R}), T(\mathbb{R}))$.

Finally we can write down the character formula. Recall again the formula (see Theorem (5.2.1))

$$F(\tilde{\chi})(z) := \epsilon(\tilde{\chi}, \Delta^+, \tau)\frac{\sum_{s\in W} \epsilon(s)\tilde{\chi}(^s w)}{\tau(\Delta^0(z, \Delta^+))(\tau\circ\rho)(w)} \quad z \in T(F)^{reg}$$

where $w \in T(F)_{\tau\circ\rho}$ such that $\Pi(w) = z$.

Then, pulling $(\tau\circ\rho)(w)$ and the Weyl group action back to $E^*/N(E^*)$ via κ, we get

$$F(\tilde{\chi})(z) = \epsilon(\tilde{\chi}, \Delta^+, \tau)\frac{\tilde{\chi}([w]) + (-1, \Delta)\tilde{\chi}([\overline{w}])}{\tau_o(1 - 1/z)|\Delta^0(z, \Delta^+)|\tau_o(w)|2\rho(z)|^{1/2}}$$

where $z \in E^*/F^*$ and $[w] \in E^*/N(E^*)$ is some element that maps to z under the map

$E^*/N(E^*) \to E^*/F^*$. We can also pull this character formula all the way back to E^*, and we get

$$F(\tilde{\chi})(z) = \epsilon(\tilde{\chi}, \Delta^+, \tau) \frac{\chi(w) + (-1, \Delta)\chi(\overline{w})}{\tau_o(1 - \overline{w}/w)|\Delta^0(w, \Delta^+)|\tau_o(w)|2\rho(w)|^{1/2}}$$

where $z \in E^*/F^*$ and $w \in E^*$ is some element that maps to z under the map $E^* \to E^*/F^*$. We will see that this proposed character formula for $PGL(2, F)$ is independent of the choice of τ.

Note that our formula simplifies:

$$F(\tilde{\chi})(z) = \epsilon(\tilde{\chi}, \Delta^+, \tau) \frac{\tilde{\chi}([w]) + (-1, \Delta)\tilde{\chi}({}^s[w])}{\tau_o(1 - 1/z)|\Delta^0(z, \Delta^+)|\tau_o(w)|2\rho(z)|^{1/2}} =$$
$$\epsilon(\tilde{\chi}, \Delta^+, \tau) \frac{\chi(w) + (-1, \Delta)\chi(\overline{w})}{\tau_o(w - \overline{w})|D(z)|^{1/2}}$$

The reason is that if $[w] \in E^*/N(E^*)$ maps to $z \in E^*/F^*$, then $2\rho(z) = w/\overline{w}$, since the positive root of $GL(2, F)$ factors to $PGL(2, F)$ since roots are trivial on the center. Moreover, it's clear that the positive root of $GL(2, F)$ sends w to $w/\overline{w}$, and therefore $\Delta^0(z, \Delta^+) = 1 - \overline{w}/w$. Finally, we recall that $|D(z)|^{1/2} = |\Delta^0(z, \Delta^+)||2\rho(z)|^{1/2}$ from Chapter 3.

We also note that if we had made the other choice of Δ^+, the denominator in our character formula would include the term $\tau_o(\overline{w} - w)$ instead of $\tau_o(w - \overline{w})$. However, because our definition of $\epsilon(\tilde{\chi}, \Delta^+, \tau)$ includes the term $\epsilon(\Delta^+)$ (see Section (5.3)), our overall character formula $F(\tilde{\chi})$ remains the same regardless of the choice of positive root. The same line of reasoning is true for the case of $GL(2, F)$, which we present next.

Summing up, noting that $T(F)_{\tau\circ\rho} \cong E^*/N(E^*)$, then we have given a method of assigning a conjectural character formula for a supercuspidal representation of $PGL(2,F)$, to a supercuspidal Weil parameter of $PGL(2,F)$, given by

$$\left\{ \begin{array}{l} irreducible\ \phi : W_F \to GL(2,\mathbb{C}) \\ \\ with\ det(\phi) = 1 \end{array} \right\} \mapsto \quad \tilde{\chi} \in \widehat{T(F)}_{\tau\circ\rho} \mapsto F(\tilde{\chi})$$

We wish to make an important comment here: In the above derivation of our character formula, we chose an isomorphism

$$E^*/N(E^*) \cong T(F)_{\tau\circ\rho}$$

$$[w] \mapsto ([w], \tau_o(w)|2\rho([w])|^{1/2})$$

which will be important for the proposed character formula. What if we chose a different isomorphism? Well, any other isomorphism is of the form

$$E^*/N(E^*) \cong T(F)_{\tau\circ\rho}$$

$$[w] \mapsto ([w], \tau_o(w)|2\rho([w])|^{1/2}\lambda(w))$$

for some character λ of $E^*/N(E^*)$. However, it is easy to see that in order for this map to be bijective, one is forced to take a λ that is a non-genuine character of $E^*/N(E^*)$ (which is a double cover of E^*/F^*). That is, λ factors to E^*/F^*. Moreover, for this to even be a morphism, it is easy to see that we need that $\lambda^2 = 1$. Therefore, λ is a quadratic character of E^* whose restriction to F^* is trivial. But since E/F has degree 2, $E^*/(E^*)^2F^* \cong \mathbb{Z}/2\mathbb{Z}$. Therefore, $\lambda = 1$ or the nontrivial quadratic character of E^* whose restriction to F^* is trivial. Therefore, there are two possible choices of isomorphism

$$E^*/N(E^*) \cong T(F)_{\tau\circ\rho}$$

and we chose one of them.

When we do the case of $PGL(\ell, F)$, we will see that there are no choices involved, and that any isomorphism we make is unique. In the $PGL(2, F)$ case, the cover $E^*/N(E^*)$ does not necessarily split. For $PGL(\ell, F)$, the cover $E^*/ker(\delta_{E/F})$ does split when $\delta_{E/F} \neq 1$, and the splitting is unique, so there are no choices that we can make, and the cover is isomorphic (with a unique choice of isomorphism) to the $\tau \circ \rho$-cover.

We should note that in the theory of real groups, via the theory from [2], a Langlands parameter naturally induces a genuine character of a double cover of $T(\mathbb{R})$. This double cover, as we have explained, is isomorphic to the ρ-cover of $T(\mathbb{R})$, and a choice of isomorphism is made. However, there is a canonical way to choose an isomorphism, and one uses the theory of E-groups to do this.

Let us now compute our proposed character formula for $GL(2, F)$. Let ρ be half the standard positive root of $GL(2, F)$. An elliptic torus in $GL(2, F)$ is of the form $T(F) = E^*$. Recall the denominator

$$\tau(\Delta^0(z, \Delta^+))(\tau \circ \rho)(w)$$

that was defined in Theorem (5.2.1). Although $(\tau \circ \rho)(w)$ is not naturally a function on E^* since in particular ρ is not naturally a function on E^*, it is naturally a function on the $\tau \circ \rho$-cover of E^*. We now introduce yet another cover which is isomorphic to $T(F)_{\tau\circ\rho}$. This cover is just the pullback of the cover $E^*/N(E^*) \to E^*/F^*$ in $PGL(2,F)$, to $GL(2,F)$.

Definition 5.2.7. Let $\Upsilon : E^*/N(E^*) \to E^*/F^*$ be the canonical projection map given by $\Upsilon([z]) := [z]$. We define $E^* \times_{E^*/F^*} E^*/N(E^*)$ as the group arising in the following pullback diagram:

$$\begin{array}{ccc} E^* \times_{E^*/F^*} E^*/N(E^*) & \longrightarrow & E^*/N(E^*) \\ \downarrow & & \downarrow \Upsilon \\ E^* & \xrightarrow{w \mapsto [w]} & E^*/F^* \end{array}$$

That is,

$$E^* \times_{E^*/F^*} E^*/N(E^*) = \{(w,z) \in E^* \times E^*/N(E^*) : [w] = [z] \in E^*/F^*\}$$

Then we have

Lemma 5.2.8. $E^* \times_{E^*/F^*} E^*/N(E^*) \cong T(F)_{\tau\circ\rho}$

Proof. An explicit isomorphism is given by

$$E^* \times_{E^*/F^*} E^*/N(E^*) \xrightarrow{\kappa} T(F)_{\tau\circ\rho}$$

$$(w,[z]) \mapsto (w, \aleph_{E/F}(z/w)\tau_o(w)|2\rho(w)|^{1/2})$$

To see that this is injective, note that if $w = 1$, then $z \in F^*$ by definition of pullback. But then $\aleph_{E/F}(z) = 1$ implies that $z \in N(E^*)$. To see surjectivity, suppose that $(w,\lambda) \in T(F)_{\tau\circ\rho}$. Then by definition of the pullback, we get $\tau_o(w/\overline{w})|2\rho(w)| = \lambda^2$. But $\tau_o(w/\overline{w}) = \tau_o(w^2/N(w)) = \tau_o(w)^2$. Thus, $\lambda = \pm\tau_o(w)|2\rho(w)|^{1/2}$. If $\lambda = \tau_o(w)|2\rho(w)|^{1/2}$, then $\kappa(w,[w]) = (w,\lambda)$. If $\lambda = -\tau_o(w)|2\rho(w)|^{1/2}$, then $\kappa(w,[xw]) = (w,\lambda)$, where $x \in F^* \setminus N(E^*)$. $\square$

Here we have again chosen an isomorphism $E^* \times_{E^*/F^*} E^*/N(E^*) \cong T(F)_{\tau\circ\rho}$. This is important for the character formula. We will explain why this choice of isomorphism is natural.

We note again that the importance of the term $|2\rho(w)|^{1/2}$ comes from the fact that

$$|D(w)|^{\frac{1}{2}} = |\Delta^0(w, \Delta^+)||2\rho(w)|^{1/2},$$

an observation made in Chapter 3. The reason why this is important is that the term $|D(w)|^{1/2}$ apears in the supercuspidal characters (see Section (5.3)). We will need this fact in the character formulas for $PGL(2, F)$, $GL(2, F)$, $PGL(\ell, F)$, and $GL(\ell, F)$ where ℓ is an odd prime.

Now let's write down the character formula for a supercuspidal representation of $GL(2, F)$. In order to do this, we need to move to the setting of $T(F)_{\tau\circ\rho}$. In particular, the proposed character formula involves genuine characters of $T(F)_{\tau\circ\rho}$.

Now let $\phi : W_F \to GL(2, \mathbb{C})$ be a supercuspidal parameter so that $\phi = Ind_{W_E}^{W_F}(\chi)$ for some regular pair $(E/F, \chi)$. Then this canonically gives a genuine character $\tilde{\chi}$ of $E^* \times_{E^*/F^*} E^*/N(E^*)$ as follows. Define $\tilde{\chi}(w, [z]) := \chi(w)\aleph_{E/F}(z/w)$.

Definition 5.2.9. A genuine character $\tilde{\eta}$ of $E^* \times_{E^*/F^*} E^*/N(E^*)$ is called *regular* if $(E/F, \eta)$ is regular, where $\eta(w) := \tilde{\eta}(w, [z])\aleph_{E/F}(z/w)$. A genuine character $\tilde{\lambda}$ of $T(F)_{\tau\circ\rho}$ is called *regular* if $\tilde{\lambda} \circ \kappa$ is regular.

We have therefore given a map $\widehat{E^*} \to (E^* \times_{E^*/F^*} E^*/N(E^*))^\wedge$ given by $\eta \mapsto \tilde{\eta}$, where $\tilde{\eta}(w, [z]) := \eta(w)\aleph_{E/F}(z/w)$. Note that we have a canonical map in the other direction, $(E^* \times_{E^*/F^*} E^*/N(E^*))^\wedge \to \widehat{E^*}$, given by $\tilde{\eta} \mapsto \eta$, where $\eta(w) :=$

$\tilde{\eta}(w,[z])\aleph_{E/F}(z/w)$. We will regularly go back and forth between characters of E^* and genuine characters of $E^* \times_{E^*/F^*} E^*/N(E^*)$. In particular, when we write $\tilde{\chi}$, a genuine character of $E^* \times_{E^*/F^*} E^*/N(E^*)$, we will sometimes keep in mind that there is a canonical character χ of E^* that $\tilde{\chi}$ comes from via the above maps.

Now recall the proposed character formula from Theorem (5.2.1):

$$F(\tilde{\chi})(z) := \epsilon(\tilde{\chi}, \Delta^+, \tau)\frac{\sum_{s \in W} \epsilon(s)\tilde{\chi}({}^s w)}{\tau(\Delta^0(z, \Delta^+))(\tau \circ \rho)(w)}, \quad z \in T(F)^{reg}$$

where $w \in T(F)_{\tau\circ\rho}$ such that $\Pi(w) = z$. We have naturally constructed a genuine character $\tilde{\chi}$ of $E^* \times_{E^*/F^*} E^*/N(E^*)$. However, the functions in $F(\tilde{\chi})$ have domain $T(F)_{\tau\circ\rho}$. Recall that $T(F)_{\tau\circ\rho} \cong E^* \times_{E^*/F^*} E^*/N(E^*)$, so we can pull the function $(\tau \circ \rho)(w)$ and the Weyl group action in $F(\tilde{\chi})$ back to $E^* \times_{E^*/F^*} E^*/N(E^*)$ via this isomorphism from Lemma (5.2.8), and leave our constructed $\tilde{\chi}$ as living on $E^* \times_{E^*/F^*} E^*/N(E^*)$. That is, we consider

$$F(\tilde{\chi})(w) := \epsilon(\tilde{\chi}, \Delta^+, \tau)\frac{\sum_{s \in W} \epsilon(s)\tilde{\chi}({}^s(w,[z]))}{\tau(\Delta^0(w, \Delta^+))(\tau \circ \rho)(\kappa(w,[z]))}, \quad w \in T(F)^{reg}$$

where $(w,[z]) \in E^* \times_{E^*/F^*} E^*/N(E^*)$ such that $\Pi(\kappa((w,[z]))) = w$. Unwinding the definitions, we see that $(\tau \circ \rho)(\kappa((w,[z]))) = \aleph_{E/F}(z/w)\tau_o(w)|2\rho(w)|^{1/2} \;\; \forall \; (w,[z]) \in E^* \times_{E^*/F^*} E^*/N(E^*)$.

We also need to define the Weyl group action. The Weyl group action on the $\tau \circ \rho$-cover is obtained as follows. If (w, λ) is an element of $T(F)_{\tau\circ\rho}$, then analogously to the real case (recall equation (3.1) in Chapter 3), define $s(w, \lambda) = (sw, \lambda\tau((s^{-1}\rho - \rho)(w)))$ for $s \in W = W(G(F), T(F)) = Aut(E/F)$, the relative Weyl group. Note that this is well-defined. Simplifying this expression, we get

$s(w,\lambda) = (\overline{w}, \lambda\tau(\overline{w}/w))$ when $s \in W = Aut(E/F)$ is nontrivial. Then, since our character formula lives on $E^* \times_{E^*/F^*} E^*/N(E^*)$, we must pull back this action from $T(F)_{\tau\circ\rho}$ to $E^* \times_{E^*/F^*} E^*/N(E^*)$ via κ. Doing this, we see that we get

$$s(w,[z]) = \kappa^{-1}(s\kappa(w,[z])) = \kappa^{-1}(s(w, \aleph_{E/F}(z/w)\tau_o(w)|2\rho(w)|^{1/2})) =$$

$$\kappa^{-1}(\overline{w}, \aleph_{E/F}(z/w)\tau_o(w)|2\rho(w)|^{1/2}\tau(\overline{w}/w))) =$$

$$\kappa^{-1}(\overline{w}, \aleph_{E/F}(\overline{z}/\overline{w})\tau_o(\overline{w})|2\rho(\overline{w})|^{1/2}) =$$

$$(\overline{w}, [\overline{z}])\ \forall (w,[z]) \in E^* \times_{E^*/F^*} E^*/N(E^*)$$

when $s \in W = Aut(E/F)$ is nontrivial. Note that in defining this Weyl group action, we implicitly used the isomorphism

$$E^* \times_{E^*/F^*} E^*/N(E^*) \cong T(F)_{\tau\circ\rho}$$

$$(w,z) \mapsto (w, \aleph_{E/F}(z/w)\tau(w)).$$

We note that the definition of regularity for a genuine character of $T(F)_{\tau\circ\rho}$ is analogous to the definition of regularity for a genuine character $\tilde{\lambda}$ of $T(\mathbb{R})_\rho$ for real groups, since the notion in the setting of real groups is that $\tilde{\lambda}$ is not fixed by any element of the real Weyl group $W(G(\mathbb{R}), T(\mathbb{R}))$.

Finally, we can write down the character formula. Pulling back $\tau\circ\rho$ and the Weyl group action back to $E^* \times_{E^*/F^*} E^*/N(E^*)$ via κ, the character formula is

$$F(\tilde{\chi})(w) := \epsilon(\tilde{\chi}, \Delta^+, \tau)\frac{\sum_{s\in W} \epsilon(s)\tilde{\chi}({}^s(w,[z]))}{\tau(\Delta^0(w,\Delta^+))(\tau\circ\rho(\kappa(w,[z])))} =$$

$$\epsilon(\tilde{\chi}, \Delta^+, \tau)\frac{\sum_{s\in W} \epsilon(s)\chi({}^s w)\aleph_{E/F}({}^s(z/w))}{\tau(\Delta^0(w,\Delta^+))\tau_o(w)\aleph_{E/F}(z/w)|2\rho(w)|^{1/2}} =$$

$$\epsilon(\tilde{\chi}, \Delta^+, \tau) \frac{\sum_{s \in W} \epsilon(s)\chi({}^s w)}{\tau(\Delta^0(w, \Delta^+))\tau_o(w)|2\rho(w)|^{1/2}} =$$

$$\epsilon(\tilde{\chi}, \Delta^+, \tau) \frac{\sum_{s \in W} \epsilon(s)\chi({}^s w)}{\tau_o(w - \overline{w})|\Delta^0(w, \Delta^+)||2\rho(w)|^{1/2}} =$$

$$\epsilon(\tilde{\chi}, \Delta^+, \tau) \frac{\chi(w) + (-1, \Delta)\chi(\overline{w})}{\tau_o(w - \overline{w})|D(w)|^{1/2}}, \quad w \in T(F)^{reg}$$

since $|D(w)|^{1/2} = |\Delta^0(w, \Delta^+)||2\rho(w)|^{1/2}$ from Chapter 3.

We also note that if we had made the other choice of Δ^+, the denominator in our character formula would include the term $\tau_o(\overline{w} - w)$ instead of $\tau_o(w - \overline{w})$. However, because our definition of $\epsilon(\tilde{\chi}, \Delta^+, \tau)$ includes the term $\epsilon(\Delta^+)$ (see Section (5.3)), our overall character formula $F(\tilde{\chi})$ remains the same regardless of the choice of positive root.

Summing up, noting that $T(F)_{\tau\circ\rho} \cong E^* \times_{E^*/F^*} E^*/N(E^*)$, then we have given a method of assigning a conjectural character formula for a supercuspidal representation of $GL(2, F)$ to a supercuspidal Weil parameter of $GL(2, F)$, given by

$$\left\{ \textit{irreducible } \phi : W_F \to GL(2, \mathbb{C}) \right\} \mapsto \tilde{\chi} \in \widehat{T(F)}_{\tau\circ\rho} \mapsto F(\tilde{\chi})$$

We wish to make the following important comment. In our formulation above, we chose an isomorphism

$$E^* \times_{E^*/F^*} E^*/N(E^*) \cong T(F)_{\tau\circ\rho}$$

$$(w, [z]) \mapsto (w, \aleph_{E/F}(z/w)\tau_o(w)|2\rho(w)|^{1/2})$$

What if we chose a different isomorphism? Well, it is easy to see that any other isomorphism is of the form

$$E^* \times_{E^*/F^*} E^*/N(E^*) \cong T(F)_{\tau\circ\rho}$$

$$(w,[z]) \mapsto (w, \aleph_{E/F}(z/w)\tau_o(w)|2\rho(w)|^{1/2}\lambda(w,z))$$

for some character λ of $E^* \times E^*/N(E^*)$. But to make this map bijective, it must be that λ is in fact a non-genuine character of $E^* \times_{E^*/F^*} E^*/N(E^*)$ (which is a double cover of E^*), and therefore, λ factors to E^*, so any isomorphism is of the form

$$E^* \times_{E^*/F^*} E^*/N(E^*) \cong T(F)_{\tau\circ\rho}$$

$$(w,[z]) \mapsto (w, \aleph_{E/F}(z/w)\tau_o(w)|2\rho(w)|^{1/2}\lambda(w))$$

for some character λ of E^*. For this to be even a morphism, we need that $\lambda^2 = 1$. Moreover, using this isomorphism for our character formula, we get

$$F(\tilde{\chi})(w) = \epsilon(\tilde{\chi}, \Delta^+, \tau)\frac{\sum_{s\in W}\epsilon(s)\chi({}^s w)\lambda({}^s w)}{\tau_o(w-\overline{w})|D(w)|^{1/2}} \quad w \in T(F)^{reg}$$

The key point now is that if we take a regular pair $(E/F, \chi)$ for $PGL(2,F)$ (i.e. such that $\chi|_{F^*} = \aleph_{E/F}$), and stick it in this character formula, we want to obtain a supercuspidal character of $PGL(2,F)$. This is the bare minimum that we would ask for in a character formula for $GL(2,F)$ if we wanted it to generalize a character formula for $PGL(2,F)$. Suppose we make this request. Well, we will show with a lot of work that the supercuspidal representation corresponding to

$$F(\tilde{\chi})(w) = \epsilon(\tilde{\chi}, \Delta^+, \tau)\frac{\sum_{s\in W}\epsilon(s)\chi({}^s w)\lambda({}^s w)}{\tau_o(w-\overline{w})|D(w)|^{1/2}} \quad w \in T(F)^{reg}$$

will be $\pi_{\chi\Delta_\chi\lambda}$. Then since representations of $PGL(2,F)$ have trivial central character, and since $(\chi\Delta_\chi)|_{F^*} = 1$, then this would force $\lambda|_{F^*} = 1$. Therefore, λ is a character of E^* such that $\lambda|_{F^*} = 1$ and $\lambda^2 = 1$. Thus, λ factors to a character of E^*/F^* whose square is 1. Since E/F has degree 2, $(E^*)^2F^* = \mathbb{Z}/2\mathbb{Z}$, and thus this forces $\lambda = 1$ or λ the unique nontrivial quadratic character of E^* whose restriction to F^* is trivial. What we conclude is that this bare minimum requirement on our character formula for $GL(2,F)$ forces us to only really consider two possible isomorphisms

$$E^* \times_{E^*/F^*} E^*/N(E^*) \cong T(F)_{\tau\circ\rho}$$

Another requirement we should have is the following: The character formula for $GL(2,F)$ must be compatible with the character formula for $PGL(2,F)$. That is, if we take a supercuspidal representation of $GL(2,F)$ with trivial central character, and feed it into the general character formula for $GL(2,F)$ and $PGL(2,F)$, we must obtain the same formula. It should not be the case that if one takes a representation of $GL(2,F)$ with trivial central character, and view it as a representation of $GL(2,F)$ or $PGL(2,F)$, one gets different character formulas. Let's say this a different way: We chose an isomorphism

$$E^* \times_{E^*/F^*} E^*/N(E^*) \cong T(F)_{\tau\circ\rho}$$

But it is important to note that if any other isomorphism was chosen, then the resulting character formula would not have been compatible with the formula for $PGL(2,F)$. That is, if we took a regular pair $(E/F,\chi)$ for $PGL(2,F)$ (i.e. such that $\chi|_{F^*} = \aleph_{E/F}$) and fed it through our $PGL(2,F)$ and $GL(2,F)$ character formulas described earlier in this section, we would get different supercuspidal representations. Thus, in a sense, the theory for $GL(2,F)$ is determined by the theory for $PGL(2,F)$.

We should note again that in the theory of real groups, via the theory from [2], a Langlands parameter naturally induces a genuine character of a double cover of $T(\mathbb{R})$. This double cover, as we have explained, is isomorphic to the ρ-cover of $T(\mathbb{R})$, and a choice of isomorphism is made. However, there is a canonical way to choose an isomorphism, and one uses the theory of E-groups to do this.

Again, as for $PGL(2, F)$, we will see that the proposed character formula for $GL(2, F)$ is independent of the choice of τ.

5.3 The constant $\epsilon(\tilde{\chi}, \Delta^+, \tau)$

We now turn to the question of defining the constant $\epsilon(\tilde{\chi}, \Delta^+, \tau)$. We recall the main theorem describing the distribution characters of positive depth supercuspidal representations of $GL(\ell, F)$, where ℓ is prime. We note that there is an analogous definition of regular pair $(E/F, \chi)$ when E/F has degree ℓ, and this is discussed further in Section (7.1).

Theorem 5.3.1. *[9, Theorem 5.3.2]* [1] *Let $(E/F, \chi)$ be a regular pair where E/F has degree ℓ and χ has positive level, and write $G' = E^*$. Let $\pi = \pi_\chi$ be the associated positive depth supercuspidal representation of $GL(\ell, F)$ given by Theorem (4.3.2). Then*

[1] In [9], X_π is the notation used instead of $\alpha(\chi)$ (recall the notation $\alpha(\chi)$ from Section 4.3). The notation X_π is a bit misleading, because the element X_π depends on χ, not just π. Since the notation in [9] and [5] differ, we need to choose a set of notation. We prefer to use the notation $\alpha(\chi)$.

$$\frac{\theta_\pi(\gamma)}{deg(\pi)} = \begin{cases} C\lambda(\sigma)\sum_{\overline{w}\in W}\chi({}^w t) & \textit{if } n(\gamma)=0 \textit{ and } \gamma = {}^g t \textit{ with } g\in G \textit{ and} \\ & t\in G' \\ C\sum_{\overline{w}\in W}\chi({}^w t)\gamma(\alpha(\chi),{}^w Y) & \textit{if } 0<n(\gamma)\le r/2 \textit{ and } \gamma={}^g t \textit{ where} \\ & t=z(1+Y) \textit{ with } z\in Z, g\in G, \\ & \textit{and } Y\in \mathfrak{g}'_{n(\gamma)}\setminus(\mathfrak{z}_{n(\gamma)}+\mathfrak{g}'_{n(\gamma)^+}) \\ \chi(z)\mu_{\alpha(\chi)}(Y) & \textit{if } n(\gamma)>r/2 \textit{ and } \gamma=z(1+{}^gY) \textit{ with} \\ & g\in G, z\in Z, \textit{ and } Y\in V_{n(\gamma)} \\ 0 & \textit{otherwise} \end{cases}$$

Here, $C := c_\psi(\mathfrak{g}')c_\psi^{-1}(\mathfrak{g})|D(\gamma)|^{-\frac{1}{2}}|\eta(\alpha(\chi))|^{-\frac{1}{2}}$ is defined in [9, Section 5.3].[2] The notation is explained in [9, Chapters 4-5]. In particular, D and η denote the Weyl discriminants of G and $\mathfrak{g}$, respectively.

To define $\epsilon(\tilde{\chi}, \Delta^+, \tau)$, we need to calculate the constant $\gamma(\alpha(\chi), Y)$ in the above theorem. We refer the reader to [9, Theorem 5.3.2] and [9, page 55] for various notation. We follow the same notation.

Recall $G = GL(2, F), \mathfrak{g} = \mathfrak{gl}(2, F), G' = E^*, \mathfrak{g}' = E$ where E/F is a quadratic extension. We have a direct sum decomposition $\mathfrak{g} = \mathfrak{g}' + \mathfrak{g}^\perp$ where the perpendicular is taken with respect to the trace form $< Z_1, Z_2 >:= Tr(Z_1Z_2)\ \forall Z_1, Z_2 \in \mathfrak{g}$. Recall our fixed additive character ψ of F. Let $V, W \in \mathfrak{g}^\perp$. Then we define

$$Q_{(\alpha(\chi),Y)}(V, W) := (trace([\alpha(\chi), W][V, Y])/2).$$

[2]In [9], Λ is used to denote a fixed additive character of F. In [5], ψ is used to denote a fixed additive character of F. We prefer to use the notation ψ.

$Q_{(\alpha(\chi),Y)}(V,W)$ is a non-degenerate, symmetric, bilinear form on $\mathfrak{g}^{\perp}$. Then, $\gamma_{(\alpha(\chi),Y)}$ is by definition the Weil Index of $\psi \circ Q_{(\alpha(\chi),Y)}$ (see [17]). Let us calculate $\gamma_{(\alpha(\chi),Y)}$. We will use various properties about the Weil index listed in the appendix.

We may embed G' in G as follows. If $E = F(\delta)$, where $\delta = \sqrt{\Delta}$, then we have

$$G' \hookrightarrow G$$
$$a + d\delta \mapsto \begin{pmatrix} a & d \\ d\Delta & a \end{pmatrix}, \quad a, d \in F, a, d \neq 0$$

Then $\mathfrak{g}'$ may be identified in the same way as $\left\{ \begin{pmatrix} a' & d' \\ d'\Delta & a' \end{pmatrix} : a, d \in F \right\}$.

Lemma 5.3.2.

$$\mathfrak{g}^{\perp} = \left\{ \begin{pmatrix} a & b \\ -b\Delta & a \end{pmatrix} : a, b \in F \right\}$$

Proof. Note that $\mathfrak{g}'$ is generated as a vector space by matrices of the form $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ and $\begin{pmatrix} 0 & y \\ y\Delta & 0 \end{pmatrix}$, for $x, y \in F^*$. With this into account, $\mathfrak{g}^{\perp}$ is determined as follows : Let $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{g}$. Then $C \in \mathfrak{g}^{\perp}$ if and only if the following two conditions hold:

(i) Trace $\left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = 0$

(ii) Trace $\left(\begin{pmatrix} 0 & y \\ y\Delta & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = 0$

Thus, $C \in \mathfrak{g}^\perp$ if and only if $a = -d$ and $c = -b\Delta$. Therefore,

$$\mathfrak{g}^\perp = \left\{ \begin{pmatrix} a & b \\ -b\Delta & a \end{pmatrix} : a, b \in F \right\}$$

□

Lemma 5.3.3. *If* $\alpha(\chi) = \begin{pmatrix} a & x \\ x\Delta & a \end{pmatrix}$ *and* $Y = \begin{pmatrix} t & y \\ y\Delta & t \end{pmatrix}$ *are arbitrary elements in* $\mathfrak{g}'$*, then the matrix of the quadratic form* $Q_{(\alpha(\chi),Y)}$ *is*

$$\begin{pmatrix} 4xy\Delta & 0 \\ 0 & -4xy\Delta^2 \end{pmatrix}$$

Proof. First note that the matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ -\Delta & 0 \end{pmatrix}$ form a basis for $\mathfrak{g}^\perp$. Let $V = \begin{pmatrix} a & b \\ -b\Delta & -a \end{pmatrix}, W = \begin{pmatrix} c & d \\ -d\Delta & -c \end{pmatrix} \in \mathfrak{g}^\perp$ be arbitrary. Then

$$Q_{(\alpha(\chi),Y)}(V, W) = -4bdxy\Delta^2 + 4acxy\Delta.$$

Now $Q_{(\alpha(\chi),Y)}(A, A) = 4xy\Delta$, $Q_{(\alpha(\chi),Y)}(B, B) = -4xy\Delta^2$, and $Q_{(\alpha(\chi),Y)}(A, B) = Q_{(\alpha(\chi),Y)}(B, A) = 0$. Therefore the matrix of the quadratic form $Q_{(\alpha(\chi),Y)}$ is

$$\begin{pmatrix} 4xy\Delta & 0 \\ 0 & -4xy\Delta^2 \end{pmatrix}$$

□

Lemma 5.3.4. *Let* $\alpha(\chi) = \begin{pmatrix} a & x \\ x\Delta & a \end{pmatrix}$ *and* $Y = \begin{pmatrix} t & y \\ y\Delta & t \end{pmatrix} \in \mathfrak{g}'$*. Then*

$$\gamma(\alpha(\chi), Y) = (x, \Delta)_F (y, \Delta)_F \; \gamma_F(\Delta, \psi)$$

Proof. By Definition (A.0.9), we have

$$\gamma(\psi \circ Q_{(\alpha(\chi),Y)}) = h_F(Q_{(\alpha(\chi),Y)})\gamma_F(\psi)^n\gamma_F(det(Q_{(\alpha(\chi),Y)}), \psi)$$

In our case, $n = 2$, and we also have that by Lemma (A.0.10),

$$h_F(Q_{(\alpha(\chi),Y)}) = (4xy\Delta, -4xy\Delta^2)_F = (\Delta, -xy)_F$$

Then, Lemmas (A.0.8) and (5.3.3) imply that

$$\gamma_F(\psi)^2 = \gamma_F(-1, \psi)^{-1} \; and$$

$$\gamma_F(det(Q_{(\alpha(\chi),Y)}), \psi) = \gamma_F(-16x^2y^2\Delta^3, \psi) = \gamma_F(-\Delta, \psi)$$

Thus, by Lemma (A.0.8),

$$\gamma(\psi \circ Q_{(\alpha(\chi),Y)}) = (\Delta, -xy)_F\gamma_F(-1, \psi)^{-1}\gamma_F(-\Delta, \psi) =$$

$$(\Delta, -xy)_F \; \gamma_F(\Delta, \psi)(-1, \Delta)_F = (x, \Delta)(y, \Delta)\gamma_F(\Delta, \psi)$$

□

Definition 5.3.5. Let $(E/F, \chi)$ be a regular pair such that χ has positive level. Associated to $(E/F, \chi)$ is an element $\alpha(\chi)$ (from Section (4.3)) and a supercuspidal representation $\pi := \pi_\chi$ via Theorem (4.3.2). Now, $\alpha(\chi) \in E^*$, so $\alpha(\chi) = a + x_\chi\delta$ for some $a, x_\chi \in F$. Let $deg(\pi)$ denote the formal degree of π. Let Δ^+ be a choice of a positive root of $GL(2, \overline{F})$ with respect to the diagonal maximal torus $T(\overline{F})$. Define $\epsilon(\Delta^+)$ to be 1 if Δ^+ is the standard positive root and define $\epsilon(\Delta^+)$

to be $\tau_o(-1)$ if Δ^+ is the opposite positive root. Then, we define $\epsilon(\tilde{\chi}, \Delta^+, \tau) := deg(\pi)(x_\chi, \Delta)\gamma_F(\Delta, \psi)\tau_o(2\delta)c_\psi(\mathfrak{g}')c_\psi^{-1}(\mathfrak{g})|\eta(\alpha(\chi))|^{-\frac{1}{2}}\epsilon(\Delta^+)$ where $c_\psi(\mathfrak{g}'), c_\psi(\mathfrak{g})$, and $\eta(\alpha(\chi))$ are defined in [9, Chapter 5].

In the calculations we will make throughout the rest of this chapter and the next, we will make a choice of Δ^+ to be the standard set of positive roots. Therefore, the term $\epsilon(\Delta^+)$ is just 1, and therefore this term will not appear in most of our calculations and formulas. We will show later that all of our results will be independent of the choice of Δ^+.

5.4 On certain decompositions associated to elements of E^*

Before moving on, we need to understand the sets $n(\gamma) = 0$ and $0 < n(\gamma) \leq r/2$ (cf Theorem (5.3.1)). We will determine explicitly how to write an element $w \in E^*$ such that $0 < n(w) \leq r/2$ in the form $z(1+Y)$, where z,Y are from Theorem (5.3.1). We will refer to $z(1+Y)$ as *the decomposition of* w.

We recall some relevant notions and definitions from [9, Section 5.3, Section 3.2]. We define a filtration on E^* by setting $G'_t := 1 + \mathfrak{p}_E^{\ulcorner te \urcorner}$ for $t > 0$, where e is the ramification index of E over F and $G' := E^*$. For example, when E/F is ramified, $e = 2$, thus $G'_1 = 1 + \mathfrak{p}_E^2$ and $G'_{1/4} = 1 + \mathfrak{p}_E$. We also define $G'_{t^+} := \bigcup_{s>t} G'_s$ for $t > 0$. For example, $G_{0^+} = 1 + \mathfrak{p}_E$. We set $G'_0 = \mathfrak{o}_E^*$. Recall that $Z(G)$ denotes the center of $G = GL(2, F)$, so $Z(G) = F^*$.

Definition 5.4.1. Let $w \in Z(G)G'_{0^+}$. Then $n(w)$ is defined by $w \in Z(G)G'_{n(w)} \setminus Z(G)G'_{n(w)^+}$. The *decomposition of* w by definition is the writing of w in the form

$w = ab$ where $a \in Z(G), b \in G'_{n(w)}$ and such that w is not in $Z(G)G'_{n(w)^+}$.

Definition 5.4.2. If w is not in $Z(G)G'_{0+} = F^*(1 + \mathfrak{p}_E)$, then define $n(w) = 0$.

We will now determine the decompositions of all elements of E^*. We separate this into various cases. We first deal with the situation where E/F is ramified. Then $E = F(\sqrt{p})$ or $E = F(\sqrt{dp})$, where $d \in \mathfrak{o}_F^*$ is not a square. Everything we prove will be for $E = F(\sqrt{p})$, but the analogous results hold for $E = F(\sqrt{dp})$ with the same proofs.

Lemma 5.4.3. *Let $w = p^n u + p^m v\delta \in E^*$, where $u, v \in \mathfrak{o}_F^*$, and $n, m \in \mathbb{Z}$ such that $n \leq m$. We can rewrite w as*

$$w = p^n u \left(1 + p^{m-n+\frac{1}{2}} \frac{v}{u}\right)$$

Thus, $w \in F^* U_E^{2m-2n+1}$. Moreover, w is not in $F^* U_E^{2m-2n+2}$. Therefore, the decomposition of w is

$$w = p^n u \left(1 + p^{m-n+\frac{1}{2}} \frac{v}{u}\right) = p^n u \left(1 + \sqrt{p}^{2m-2n+1} \frac{v}{u}\right)$$

Proof. Suppose by way of contradiction that $w = x(1 + s(\sqrt{p})^{2m-2n+2}), s \in U_E, x \in F^*$. So

$p^n u(1 + v' p^{m-n+\frac{1}{2}}) = x(1 + s p^{m-n+1})$, where $v' = \frac{v}{u}$. Then

$$x^{-1} p^n u(1 + v' p^{m-n+\frac{1}{2}}) = 1 + s p^{m-n+1}$$

Well, $x \in F^*$ is arbitrary, therefore $x^{-1} p^n u \in F^*$ is arbitrary, so the proof of the Lemma reduces to showing that there is no $y \in F^*$ such that $y(1 + v' p^{m-n+\frac{1}{2}}) = 1 + s p^{m-n+1}$ for $v' \in U_F, s \in U_E$. By way of contradiction, suppose such a y existed.

We consider power series expansions of various elements. Let $y = p^k(y_0 + y_1 p + ...)$, where $k \in \mathbb{Z}, y_0 \neq 0$, let $v' = v'_0 + v'_1 p + ...$, where $v'_0 \neq 0$, and let $s = s_0 + s_1\sqrt{p} + ...$, where $s_0 \neq 0$. Then we have

$$p^k(y_0 + y_1 p + ...)(1 + (v'_0 + v'_1 p + ...)p^{m-n+\frac{1}{2}}) = 1 + (s_0 + s_1\sqrt{p} + ...)p^{m-n+1}$$

Comparing leading coefficients, this implies that $k = 0$ and $y_0 = 1$. Therefore,

$$(1 + y_1 p + ...)(1 + v'_0 p^{m-n+\frac{1}{2}} + v'_1 p^{m-n+3/2} + ...) = 1 + (s_0 + s_1\sqrt{p} + ...)p^{m-n+1}$$

But expanding the left hand side, we see that

$$(1 + y_1 p + ...)(1 + v'_0 p^{m-n+\frac{1}{2}} + v'_1 p^{m-n+3/2} + ...) = 1 + v'_0 p^{m-n+\frac{1}{2}} + y_1 p + ...$$

On the other hand, $1 + (s_0 + s_1\sqrt{p} + ...)p^{m-n+1}$ does not have a $p^{m-n+\frac{1}{2}}$ term, and this implies that $v'_0 = 0$. But v' is a unit, so we have a contradiction. Therefore, the decomposition of w as $w = z(1 + Y)$ is

$$w = p^n u \left(1 + p^{m-n+\frac{1}{2}} \frac{v}{u}\right)$$

□

Lemma 5.4.4. *Let $w = p^n u + p^m v\delta \in E^*$ where $u = 0$ and $v \neq 0$. Then w is not in $F^* U_E^1$, and so $n(w) = 0$.*

Proof. Suppose by way of contradiction that $w = x(1 + s\sqrt{p})$ where $s \in U_E$, $x \in F^*$. Then $x^{-1} p^m v\delta \in U_E^1$. x is arbitrary, so the proof of the Lemma reduces to showing that there is no $y \in F^*$ such that $y\delta \in U_E^1$. Suppose such a y existed. Then this would mean that $y\delta$ has to have leading term 1 in its power series expansion, which is clearly impossible since $\delta = \sqrt{p}$. □

Lemma 5.4.5. *Let $w = p^n u + p^m v\delta \in E^*$, where $u, v \in F$ both non zero, $n, m \in \mathbb{Z}$ such that $n > m$. Then w is not in $F^*(1 + \mathfrak{p}_E)$. Thus, $n(w) = 0$.*

Proof. Suppose by way of contradiction that $w = x(1 + s\sqrt{p}), x \in F^*, s \in U_E$. $m < n$, so rewrite w as

$$w = p^m v \left(\sqrt{p} + p^{n-m}\frac{u}{v}\right)$$

Let $u' = \frac{u}{v}$. Then $x^{-1}p^m v(p^{\frac{1}{2}} + p^{n-m}u') = 1 + sp^{\frac{1}{2}}$. Since x is arbitrary in F^*, the proof of the Lemma reduces to showing that there is no $y \in F^*$ such that

$$y(p^{\frac{1}{2}} + p^{n-m}u') \in 1 + \mathfrak{p}_E$$

where $u' \in U_F, s \in U_E$. By way of contradiction, suppose such a y existed, with $y = p^k(y_0 + y_1 p + ...),\ y_0 \neq 0$, so that

$$p^k(y_0 + y_1 p + ...)(p^{\frac{1}{2}} + p^{n-m}u') = 1 + s_0 p^{\frac{1}{2}} + s_1 p + ...$$

for some $s = s_0 p^{\frac{1}{2}} + s_1 p + ... \in 1 + \mathfrak{p}_E$. Comparing leading terms, this implies that $k = 0$. But the leading term of $(y_0 + y_1 p + ...)(p^{\frac{1}{2}} + p^{n-m}u')$ is not 1, so we have a contradiction. Therefore, w is not in $F^*(1 + \mathfrak{p}_E)$, so $n(w) = 0$. ☐

We note that if $w = p^n u + p^m v\delta \in E^*$ where $v = 0$ and $u \neq 0$, then $w \in F^*$, and this case is irrelevant since the character is only valid on the regular set.

We now describe the decomposition of elements of E^* when E/F is unramified. Then $E = F(\sqrt{\Delta})$ where $\Delta \in F^*$ is a non-square unit. In this case, for example we have $G'_1 = 1 + \mathfrak{p}_E$ and $G'_{1/4} = 1 + \mathfrak{p}_E$.

Lemma 5.4.6. *Let $w = p^n u + p^m v\delta \in E^*$, where $u, v \in F$ both non zero, $n, m \in \mathbb{Z}$ such that $n < m$. We can rewrite w as*

$$w = p^n u \left(1 + p^{m-n} \frac{\delta v}{u}\right)$$

Thus, $w \in F^* U_E^{m-n}$, so $n(w) > 0$. Moreover, w is not in $F^* U_E^{m-n+1}$. Therefore, the decomposition of w is

$$w = p^n u \left(1 + p^{m-n} \frac{\delta v}{u}\right)$$

Proof. Suppose by way of contradiction that $w = x(1+sp^{m-n+1})$, for $s \in U_E, x \in F^*$. Then, we get $x^{-1}p^n u(1 + p^{m-n}\frac{v\delta}{u}) = 1 + sp^{m-n+1}$. x is arbitrary in F^*, so the proof of the Lemma reduces to showing that there is no $y \in F^*$ such that $y(1+p^{m-n}\frac{v\delta}{u}) \in 1+\mathfrak{p}_E^{m-n+1}$. Suppose such a y existed. We consider power series expansions of various elements. Let $y = p^k(y_0 + y_1 p + ...)$. So $p^k(y_0 + y_1 p + ...)(1 + p^{m-n}\frac{v\delta}{u}) \in 1 + \mathfrak{p}_E^{m-n+1}$. Comparing leading terms we get that $k = 0$ and $y_0 = 1$. Thus

$$(1 + y_1 p + ...) \left(1 + p^{m-n} \frac{v\delta}{u}\right) = 1 + sp^{m-n+1} \tag{5.1}$$

for some $s \in U_E$. Comparing powers of p on both sides of (5.1), we must have that $y_1 = y_2 = ... = y_{m-n-1} = 0$. But consider the y_{m-n} term. Write $v' = \frac{v}{u}$. Let $v' = v'_0 + v'_1 p + ...$, $\delta = \delta_0 + \delta_1 p + ...$ Then $(1 + y_1 p + ...)(1 + p^{m-n}\frac{v\delta}{u}) = 1 + y_{m-n}p^{m-n} + v'_0\delta_0 p^{m-n} + ...$ Again, comparing terms in (5.1), we must have that $y_{m-n} + v'_0\delta_0 = 0$. Since u, v are non-zero units, we have $v'_0 \neq 0$. Thus, $\delta_0 = \frac{-y_{m-n}}{v'_0}$. But this is a contradiction. The reason is that $\frac{-y_{m-n}}{v'_0} \in \mu_F$, but δ_0 can't be in μ_F. To see this, recall that $\delta^2 = \Delta$. If $\Delta = a_0 + a_1 p + ...$, then $\delta^2 = \Delta$ implies $\delta_0^2 = a_0$.

So if $\delta_0 \in \mu_F$, then this says that a_0 is a square in μ_F, so by Lemma (A.0.6), Δ would be a square, which is a contradiction. Therefore,

$$w = p^n u(1 + p^{m-n}\frac{v\delta}{u})$$

is the decomposition in this case. $\square$

Lemma 5.4.7. *Let $w = p^n u + p^m v\delta \in E^*$, where $u = 0$ and v is non-zero. Then, w is not in $F^*(1 + \mathfrak{p}_E)$, and thus $n(w) = 0$.*

Proof. Suppose by way of contradiction that $w \in F^*(1+\mathfrak{p}_E)$, so $p^m v\delta = x(1+sp)$, $s \in U_E, x \in F^*$. Thus $x^{-1}p^m v\delta = 1 + sp$. x is arbitrary in F^*, therefore $x^{-1}p^m v$ is arbitrary in F^*, so the proof of the Lemma reduces to showing that there is no $y \in F^*$ such that $y\delta \in 1 + \mathfrak{p}_E$. Well let $y = y_0 + y_1 p + ...$, $\delta = \delta_0 + \delta_1 p + ...$ be the power series expansions of y, δ. Then suppose

$$(y_0 + y_1 p + ...)(\delta_0 + \delta_1 p + ...) \in 1 + \mathfrak{p}_E$$

Comparing leading terms, this implies that $y_0\delta_0 = 1$. Let $\Delta = a_0 + a_1 p + ...$ Recall that a_0 is not a square in F by Lemma (A.0.6). Since $\delta^2 = \Delta$, we have that $\delta_0^2 = a_0$.

Then squaring both sides of the equation $y_0\delta_0 = 1$, we get that $a_0 y_0^2 = 1$, which implies that a_0 is a square in μ_F since $y_0 \in \mu_F$. This is a contradiction, since we mentioned before that a_0 can't be a square in F. Therefore, w is not in $F^*(1 + \mathfrak{p}_E)$, so $n(w) = 0$. $\square$

Lemma 5.4.8. *Let $w = p^n u + p^m v\delta \in E^*$, where u, v are both nonzero and $n = m$. Then $n(w) = 0$, i.e. w is not in $F^*(1 + \mathfrak{p}_E)$.*

Proof. Rewrite w as $w = p^n u(1 + \frac{v\delta}{u})$. Suppose by way of contradiction that $w = x(1 + sp)$, $s \in U_E, x \in F^*$. Then $x^{-1}p^n u(1 + \frac{v\delta}{u}) = 1 + sp$. But $x \in F^*$ is arbitrary, so $x^{-1}p^n u \in F^*$ is arbitrary. Therefore, the proof of the Lemma reduces to showing that there is no $y \in F^*$ such that $y(1 + \frac{v\delta}{u}) = 1 + sp \in U_E^1$. Suppose by way of contradiction that such a y existed. Well, let $y = p^k(y_0 + y_1 p + ...)$. Then we have

$$p^k(y_0 + y_1 p + ...)(1 + \frac{v\delta}{u}) \in U_E^1$$

Comparing leading terms we get that $k = 0$ and $y_0 = 1$. Thus,

$$(1 + y_1 p + y_2 p^2 + ...)(1 + \frac{v\delta}{u}) = 1 + sp$$

Let $y' = y_1 p + y_2 p^2 + ...$ Then $(1 + y_1 p + y_2 p^2 + ...)(1 + \frac{v\delta}{u}) = 1 + \frac{v\delta}{u} + y' + y'\frac{v\delta}{u}$. Now, since $v, u \in U_F, \delta \in U_E$ we have $y', y'\frac{v\delta}{u} \in \mathfrak{p}_E$. Recall that we are trying to show a contradiction in

$$1 + \frac{v\delta}{u} + y' + y'\frac{v\delta}{u} \in 1 + \mathfrak{p}_E$$

This is equivalent to showing a contradiction in

$$\frac{v\delta}{u} + y' + y'\frac{v\delta}{u} \in \mathfrak{p}_E$$

But since $y' + y'\frac{v\delta}{u} \in \mathfrak{p}_E$, this is equivalent to showing a contradiction in

$$\frac{v\delta}{u} \in \mathfrak{p}_E$$

by subtracting $y' + y'\frac{v\delta}{u}$ from both sides. But $\frac{v\delta}{u}$ is a unit, so we have a contradiction, and so w is not in $F^*(1 + \mathfrak{p}_E)$, so $n(w) = 0$. □

Lemma 5.4.9. *Let $w = p^n u + p^m v\delta \in E^*$, and where u, v are both nonzero, and $n > m$. Then $n(w) = 0$, i.e. w is not in $F^*(1 + \mathfrak{p}_E)$.*

Proof. Suppose by way of contradiction that $w = p^n u + p^m v\delta \in F^* U_E^1$. Rewrite w as $w = p^m v(\delta + p^{n-m}\frac{u}{v})$. So suppose by way of contradiction that $p^m v(\delta + p^{n-m}\frac{u}{v}) = x(1 + sp), s \in U_E, x \in F^*$. So $x^{-1} p^m v(\delta + p^{n-m}\frac{u}{v}) \in 1 + \mathfrak{p}_E$. Since $x \in F^*$ is arbitrary, the proof of the Lemma reduces to showing there is no $y \in F^*$ such that $y(\delta + p^{n-m}\frac{u}{v}) \in 1 + \mathfrak{p}_E$. Suppose such a y existed. Let $y = p^k(y_0 + y_1 p + ...)$, $\delta = \delta_0 + \delta_1 p + ...$ By comparing coefficients, get $k = 0$ and $y_0\delta_0 = 1$. But $y_0 \in \mu_F$, so this implies that $\delta_0 \in \mu_F$. But this is a contradiction since if $\Delta = \Delta_0 + \Delta_1 p + ...$, then since $\Delta_0 = \delta_0^2$, we'd have that Δ_0 is the square of an element of μ_F, which would imply by Lemma (A.0.6) that Δ is a square. Therefore, w is not in $F^*(1+\mathfrak{p}_E)$, so $n(w) = 0$. $\square$

We note that if $w = p^n u + p^m v\delta \in E^*$, where $v = 0$ and $u \neq 0$, then $w \in F^*$, and this case is irrelevant since the character is only valid on the regular set.

5.5 The proof that our conjectural character formulas agree with positive depth supercuspidal characters

Here we prove that on the range $\{z \in T(F)^{reg} : 0 \leq n(z) \leq r/2\}$, our conjectured character formula agrees with the character of a specific positive depth supercuspidal representation of $GL(2, F)$, and this supercuspidal is the one given by the local Langlands correspondence. In Sections (5.6) and (5.7), we prove that there are no other supercuspidal representations whose character agrees with ours on this range. Therefore, we have a canonical way of attaching a supercuspidal representation to our conjectured character formula $F(\tilde{\chi})$. Note again that when we study the ranges $\{w \in E^* : 0 \leq n(w) \leq r/2\}$, we do not consider elements $w \in F^*$ since distribution characters are not defined on F^*.

In the remainder of the chapter and the next, we will deal exclusively with $GL(2, F)$, and so we set $T(F) = E^*$. Recall from Section (5.2) that our proposed character formula simplifies to

$$F(\tilde{\chi})(w) = \epsilon(\tilde{\chi}, \Delta^+, \tau)\ \frac{\chi(w) + (-1, \Delta)\chi(\overline{w})}{\tau_o(w - \overline{w})|D(w)|^{1/2}}, \quad w \in T(F)^{reg}$$

It will be useful for computational purposes to rewrite this formula as

$$F(\tilde{\chi})(w) = \epsilon(\tilde{\chi}, \Delta^+, w)\ \frac{\chi(w) + (-1, \Delta)\chi(\overline{w})}{\tau_o(\frac{w-\overline{w}}{2\delta})}, \quad w \in T(F)^{reg}$$

where

$$\epsilon(\tilde{\chi}, \Delta^+, w) := deg(\pi)(x_\chi, \Delta)_F \gamma(\Delta, \psi) C$$

and

$$C := c_\psi(\mathfrak{g}')c_\psi^{-1}(\mathfrak{g})|D(\gamma)|^{-1/2}|\eta(\alpha(\chi))|^{-1/2}$$

is as in Section (5.3). We will use this rewritten version for the rest of Chapter 5. Note that $F(\tilde{\chi})$ is independent of the choice of τ because $\frac{w-\overline{w}}{2\delta} \in F^*$, and we have required only that $\tau_o|_{F^*} = \aleph_{E/F}$. We will start by assuming that all of our regular pairs $(E/F, \chi)$ are *minimal* (cf Section (4.1)), and then we will show that there is no harm in assuming this, and that all of our results are true for arbitrary regular pairs.

First we consider the case that E/F is ramified, so we may take $E = F(\delta)$ where $\delta = \sqrt{p}$ or $\delta = \sqrt{dp}$ where $d \in \mathfrak{o}_F^*$ is not a square. Without loss of generality we let $\delta = \sqrt{p}$. The same proofs work in the case $\delta = \sqrt{dp}$.

We must first conduct a careful analysis of the supercuspidal characters in the $0 < n(w) \leq r/2$ range. Recall that $n(w) > 0$ if and only if either

i) $w = p^n u + p^m v\delta$ where $n \leq m$ or

ii) $w = p^n u \in F^*$ with $u \neq 0$

We ignore case (ii) because characters are only defined on the regular set.

Lemma 5.5.1. $\gamma(\alpha(\chi), Y) = (x_\chi, \Delta)_F \gamma(\Delta, \psi)\mu(\frac{w-\overline{w}}{2\delta})\mu(w) \;\; \forall w \in E^* : n(w) > 0$ *for any character μ of E^* whose restriction to F^* is $\aleph_{E/F}$ and whose order is a power of 2. Similarly,* $\gamma(\alpha(\chi), {}^sY) = (x_\chi, \Delta)_F \gamma(\Delta, \psi)\mu(\frac{\overline{w}-w}{2\delta})\mu(\overline{w}) \;\; \forall w \in E^* : n(w) > 0$ *for any character μ of E^* and whose restriction to F^* is $\aleph_{E/F}$ and whose order is a power of 2, where $1 \neq s \in W = W(G(F), T(F))$.*

Proof. Let us recall one of the results of Section (5.3) involving $\gamma(\alpha(\chi), Y)$: If $\alpha(\chi), Y$ embed in $\mathfrak{g}'$ as $\alpha(\chi) = \begin{pmatrix} a & x_\chi \\ x_\chi\Delta & a \end{pmatrix}$ and $Y = \begin{pmatrix} t & y \\ y\Delta & t \end{pmatrix}$, then

$$\gamma(\alpha(\chi), Y) = (x_\chi, \Delta)_F (y, \Delta)_F \;\gamma_F(\Delta, \psi)$$

If $w = m + n\delta \in E^*$ where $m, n \in F^*$, and $n(w) > 0$, then as we have seen in the previous section, the decomposition of w is $w = m(1 + n/m\delta)$. Thus, since $(n/m, \Delta) = (n, \Delta)(m, \Delta)$, and since $m = \frac{w+\overline{w}}{2}$ and $n = \frac{w-\overline{w}}{2\delta}$, we have that

$$\gamma(\alpha(\chi), Y) = (x_\chi, \Delta)_F \gamma(\Delta, \psi)\mu\left(\frac{w-\overline{w}}{2\delta}\right)\mu\left(\frac{w+\overline{w}}{2}\right)$$

where $\alpha(\chi) = a + x_\chi\delta$.

Now, $\frac{w+\overline{w}}{2} \in F^*$, so $\mu(\frac{w+\overline{w}}{2}) = \mu(w)\mu(\frac{1}{2}(1 + \overline{w}/w))$, but since $n(w) > 0$, we have $w \in F^*U_E^1$, so $\overline{w}/w \subset U_E^1$ since if $w = xu$ where $x \in F^*$, $u \in U_E^1$, then $\overline{w}/w = \overline{u}/u$, which is clearly in U_E^1. So $\frac{1}{2}(1+\overline{w}/w) \in U_E^1$, and therefore $\mu(\frac{1}{2}(1+\overline{w}/w)) = 1$ by Lemma (A.0.5). $\square$

Therefore, we can simplify the supercuspidal characters in the $0 < n(w) \leq r/2$ range to

$$\theta_\pi(w) =$$

$$deg(\pi)C(x_\chi, \Delta)_F\gamma(\Delta,\psi)\mu(\frac{w-\overline{w}}{2\delta})\ (\phi(w)\mu(w) + (-1,\Delta)\phi(\overline{w})\mu(\overline{w}))$$

$\forall w \in E^* : 0 < n(w) \leq r/2$ and for any character μ of E^* whose restriction to F^* is $\aleph_{E/F}$ whose order is a power of 2 (we use also that $\mu(\frac{\overline{w}-w}{2\delta}) = (-1,\Delta)\mu(\frac{w-\overline{w}}{2\delta})$). We can rewrite this formula as

$$\theta_\pi(w) = \epsilon(\tilde{\chi}, \Delta^+, w)\ \frac{\phi(w)\mu(w) + (-1,\Delta)\phi(\overline{w})\mu(\overline{w})}{\mu(\frac{w-\overline{w}}{2\delta})}\ \forall w \in E^* : 0 < n(w) \leq r/2.$$

Recall that our conjectured character formula is

$$F(\tilde{\chi})(z) = \epsilon(\tilde{\chi}, \Delta^+, w)\ \frac{\chi(w) + (-1,\Delta)\chi(\overline{w})}{\tau_o(\frac{w-\overline{w}}{2\delta})}\quad w \in T(F)^{reg}$$

where $\epsilon(\tilde{\chi}, \Delta^+, w) = deg(\pi)(x_\chi, \Delta)_F\gamma(\Delta,\psi)C$. We have therefore proven the following proposition.

Proposition 5.5.2. *$F(\tilde{\chi})$ agrees with the character of the supercuspidal representation $\pi_{\chi\mu^{-1}}$ in the $0 < n(w) \leq r/2$ range, where μ is any character of E^* whose restriction to F^* is $\aleph_{E/F}$ and whose order is a power of 2.*

So the above analysis shows $F(\tilde{\chi})(w) = \theta_\pi(w)\ \forall w \in E^* : 0 < n(w) \leq r/2$ where $\phi = \chi\mu^{-1}$ for any character μ of E^* whose restriction to F^* is $\aleph_{E/F}$ and whose order is a power of 2. The reason is because for such a μ, $\tau_o(\frac{w-\overline{w}}{2\delta}) = \mu(\frac{w-\overline{w}}{2\delta})$

since $\frac{w-\overline{w}}{2\delta} \in F^*$ and $\tau_o|_{F^*} = \mu|_{F^*} = \aleph_{E/F}$. Equivalently, the above analysis shows $F(\tilde{\chi})(w) = \theta_\pi(w)\ \forall w \in E^* : 0 < n(w) \leq r/2$ where $\phi = \chi\mu$ for any character μ of E^* whose restriction to F^* is $\aleph_{E/F}$ and whose order is a power of 2, since if μ is a character of E^* whose restriction to F^* is $\aleph_{E/F}$ and whose order is a power of 2, then so is μ^{-1}. We will use the $\phi = \chi\mu$ version in what follows for ease of notation, although it makes no difference which version we use. We have therefore proven the following proposition.

We will need to investigate the $n(w) = 0$ range to see which such characters μ can arise, if any. We will show that our conjectured formula agrees with a supercuspidal character in the $n(w) = 0$ range, for a unique μ. We will also show that $\mu = \Delta_\chi$, the twist coming from the local Langlands correspondence. A priori there is no reason for our proposed formula to agree with any supercuspidal character on the $n(w) = 0$ level. We now investigate the $n(w) = 0$ range.

Lemma 5.5.3.

$$F(\tilde{\chi})(w) = \epsilon(\tilde{\chi}, \Delta^+, w)\ (\chi(w)\Omega(\frac{w}{\delta}) + \chi(\overline{w})\Omega(\frac{\overline{w}}{\delta}))\ \ \forall w \in E^* : n(w) = 0$$

for any character Ω of E^* whose restriction to F^* is $\aleph_{E/F}$ and whose order is a power of 2.

Proof. Recall that $n(w) = 0$ if and only if either

i) $w = p^n u + p^m v\delta$, where u, v are both nonzero, $n > m$, or

ii) $w = p^m v\delta$, where v is nonzero.

In case (i), $\frac{w}{\delta} = p^m v + p^n \frac{u}{\delta} = p^m v(1 + p^{n-m}\frac{u}{v\delta})$. Note that $1 + p^{n-m}\frac{u}{v\delta} \in U_E^1$ since $n > m$ (and thus $val(\frac{p^{n-m}}{\delta}) > 0$). Recall that $\tau_o(\frac{w-\overline{w}}{2\delta}) = \Omega(\frac{w-\overline{w}}{2\delta})$ for any character Ω of E^* whose restriction to F^* is $\aleph_{E/F}$. So suppose Ω has order a power of 2. Then Ω is trivial on U_E^1 by Lemma A.0.5, and thus we have that $\Omega(\frac{w}{\delta}) = \Omega(p^m v(1+p^{n-m}\frac{u}{v\delta})) = \Omega(p^m v) = \Omega(\frac{w-\overline{w}}{2\delta}) = \tau_o(\frac{w-\overline{w}}{2\delta})$. In case (ii), it's clear that $\Omega(\frac{w}{\delta}) = \Omega(\frac{w-\overline{w}}{2\delta}) = \tau_o(\frac{w-\overline{w}}{2\delta})$. Therefore, we get that

$$\tau_o(\frac{w-\overline{w}}{2\delta}) = \Omega(\frac{w}{\delta})\ \forall w \in E^* : n(w) = 0$$

where Ω is any character of E^* whose order is a power of 2.

Thus, since $\tau_o(-1) = (-1,\Delta)$, $F(\tilde{\chi})$ simplifies in the $n(w) = 0$ range.

$$F(\tilde{\chi})(w) = \epsilon(\tilde{\chi},\Delta^+,w)\ \frac{\chi(w) + (-1,\Delta)\chi(\overline{w})}{\tau_o(\frac{w-\overline{w}}{2\delta})} =$$

$$\epsilon(\tilde{\chi},\Delta^+,w)\ (\chi(w)\tau_o(\frac{w-\overline{w}}{2\delta}) + (-1,\Delta)^2\chi(\overline{w})\tau_o(\frac{\overline{w}-w}{2\delta})) =$$

$$\epsilon(\tilde{\chi},\Delta^+,w)\ (\chi(w)\Omega(\frac{w}{\delta}) + \chi(\overline{w})\Omega(\frac{\overline{w}}{\delta}))\ \ \forall w \in E^* : n(w) = 0$$

□

Now, since E/F is ramified, $\lambda(\sigma) = 1$ [9, page 65]. Therefore, we would like to show that

$$\epsilon(\tilde{\chi},\Delta^+,w)\ (\chi(w)\Omega(\frac{w}{\delta}) + \chi(\overline{w})\Omega(\frac{\overline{w}}{\delta})) =$$

$$deg(\pi)C(\chi(w)\mu(w) + \chi(\overline{w})\mu(\overline{w}))\ \forall w \in E^* : n(w) = 0$$

for some character μ of E^* whose restriction to F^* is $\aleph_{E/F}$ and whose order is a power of 2.

Note that we proved earlier that $F(\tilde{\chi})(w) = \theta_\pi(w)\ \ \forall w \in E^* : 0 < n(w) \leq r/2$ for any such μ. So what we will now prove is that there exists a μ such that $F(\tilde{\chi})(w) = \theta_\pi(w)\ \ \forall w \in E^* : 0 \leq n(w) \leq r/2$. This μ will be Δ_χ. We will later show that this μ is unique.

Lemma 5.5.4.

$$\epsilon(\tilde{\chi}, \Delta^+, w)\ (\chi(w)\Omega(\frac{w}{\delta}) + \chi(\overline{w})\Omega(\frac{\overline{w}}{\delta})) =$$

$$deg(\pi)C(\chi(w)\mu(w) + \chi(\overline{w})\mu(\overline{w}))\ \forall w \in E^* : n(w) = 0$$

for some character μ of E^* whose restriction to F^* is $\aleph_{E/F}$ and whose order is a power of 2.

Proof. Unwinding the definitions, one can see that to prove the lemma, it suffices to show that $(x_\chi, \Delta)\gamma(\Delta, \psi)\chi(w)\Omega(\frac{w}{\delta}) = \chi(w)\mu(w)$, for some character μ of E^* whose restriction to F^* is $\aleph_{E/F}$, and whose order is a power of 2, which simplifies to $\Omega(\delta) = (x_\chi, \Delta)_F\gamma(\Delta, \psi)\frac{\Omega(w)}{\mu(w)}$. But recall that Ω was any arbitrary character of E^* whose restriction to F^* was $\aleph_{E/F}$, and whose order was a power of 2. If we let $\Omega = \mu$, then we are reduced to showing that there is some character μ of E^* whose restriction to F^* was $\aleph_{E/F}$, and whose order is a power of 2, such that $\mu(\delta) = (x_\chi, \Delta)_F\gamma(\Delta, \psi)$. We claim that $\mu := \Delta_\chi$ is such a character.

So we want to show that $\Delta_\chi(\delta) = (x_\chi, \Delta)_F\gamma(\Delta, \psi)$. We need to investigate the term $\alpha(\phi)$. Note that $\alpha(\phi) = \alpha(\chi)$ since $\mu|_{1+\mathfrak{p}_E} \equiv 1$. We prefer to work with $\alpha(\chi)$. Firstly, since E/F is ramified, χ has odd level $n = 2m + 1$ [5, Chapter 19]. We also have $\alpha(\chi) \in \mathfrak{p}_E^{-n}$ (cf Section (4.3)). So let $\alpha(\chi) = p^k u + p^\ell v\delta$. The fact that n is odd implies that $\ell < k$. Therefore, rewriting $\alpha(\chi)$ as $\alpha(\chi) = p^\ell\sqrt{p}(v + p^{k-\ell-\frac{1}{2}}u)$, we get $\aleph_{E/F}(\zeta(\alpha(\chi), \varpi)) = (v_0, \Delta)$ where v_0 is the leading term of the power series expansion of v. But $(v_0, \Delta) = (v, \Delta)$. Moreover, $\alpha(\chi) \in \mathfrak{p}_E^{-n}$ implies that $-n = 2\ell + 1$. Now, by definition of x_χ, we get that $x_\chi = p^\ell v$.

Therefore, by definition of Δ_χ, we have $\Delta_\chi(\delta) = (v, \Delta)\lambda_{E/F}(\psi)^n$ and we wish to show that this is equal to $(x_\chi, \Delta)\gamma(\Delta, \psi) = (p^\ell v, \Delta)\gamma(\Delta, \psi)$. Cancelling out terms, we want to show that $(p^\ell, \Delta)\gamma(\Delta, \psi) = \lambda_{E/F}(\psi)^{-2\ell-1}$ since $-n = 2\ell + 1$. Well, we know that

$$\lambda_{E/F}(\psi)^{-2\ell} = (-1)^\ell$$

(cf [5, page 217]) Thus we are reduced to showing that $\lambda_{E/F}(\psi) = \gamma(\Delta, \psi)^{-1}$ since $(p, \Delta) = -1$. But we prove this in Section (5.8). □

Therefore, we have proven the following, when E/F is ramified.

Theorem 5.5.5. *$F(\tilde{\chi})$ agrees with the character of the supercuspidal representation $\pi_{\chi\Delta_\chi}$ on the range $\{w \in E^* : 0 \leq n(w) \leq r/2\}$.*

What we have actually proven is that if $(E/F, \chi)$ is a *minimal* regular pair with E/F ramified and χ having positive level, then $F(\tilde{\chi})$ agrees with the character of the supercuspidal representation $\pi_{\chi\Delta_\chi}$ on the range $\{w \in E^* : 0 \leq n(w) \leq r/2\}$. To prove this for an arbitrary regular pair follows from this. For if $(E/F, \chi)$ is an arbitrary regular pair, then there exists a minimal regular pair $(E/F, \chi')$ such that $\chi = \chi'\phi_E$ where $\phi_E = \phi \circ N_{E/F}$ for some $\phi \in \widehat{F^*}$. Moreover, $\pi_\chi = \phi\pi_{\chi'}$ by definition. We proved above that $F(\widetilde{\chi'}) = \theta_{\pi_{\chi'\Delta_{\chi'}}}$ on the range $\{w \in E^* : 0 \leq n(w) \leq r/2\}$. Therefore, $\theta_{\pi_{\chi\Delta_\chi}}(w) = \theta_{\pi_{\chi'\phi_E\Delta_{\chi'\phi_E}}}(w) = \theta_{\pi_{\chi'\phi_E\Delta_{\chi'}}}(w) = \phi_E(w)\theta_{\pi_{\chi'\Delta_{\chi'}}}(w) = \phi_E(w)F(\widetilde{\chi'})(w) = F(\widetilde{\chi'\phi_E})(w) = F(\tilde{\chi})(w)$ on the range $\{w \in E^* : 0 \leq n(w) \leq r/2\}$.

Now we consider the case E/F is unramified, so $\delta = \sqrt{\Delta}$, where $\Delta \in \mathfrak{o}_F^*$ is not a square.

Recall that the conjectured formula is

$$F(\tilde{\chi})(z) = \epsilon(\tilde{\chi}, \Delta^+, w)\ \frac{\chi(w) + (-1,\Delta)\chi(\overline{w})}{\tau_o(\frac{w-\overline{w}}{2\delta})} \quad w \in T(F)^{reg}$$

where $\epsilon(\tilde{\chi}, \Delta^+, w) = deg(\pi)(x_\chi, \Delta)_F\gamma(\Delta, \psi)C$. Note that $(-1, \Delta) = 1$ since E/F is unramified. This term will therefore sometimes be removed from formulas.

We again first conduct a careful analysis of the supercuspidal characters evaluated on the range $\{w \in E^* : 0 < n(w) \leq r/2\}$. Recall from Section (5.4) that when E/F is unramified, $n(w) > 0$ if and only if either

i) $w = p^n u + p^m v\delta,\ u, v \in \mathfrak{o}_F^*$ where $n < m$ or

ii) $w = p^n u \in F^*,\ u \in \mathfrak{o}_F^*$

Again, we ignore case (ii) since characters are only valid on the regular set.

Proposition 5.5.6. *$F(\tilde{\chi})$ agrees with the character of the supercuspidal representation $\pi_{\chi\mu^{-1}}$ in the $0 < n(w) \leq r/2$ range, where μ is any character of E^* whose restriction to F^* is $\aleph_{E/F}$ and whose order is a power of 2.*

Proof. Consider elements from case (i). Then again we claim that $\gamma(\alpha(\chi), Y) = (x_\chi, \Delta)_F\gamma(\Delta, \psi)\mu(\frac{w-\overline{w}}{2\delta})\mu(w)$ and $\gamma(\alpha(\chi), {}^sY) = (x_\chi, \Delta)_F\gamma(\Delta, \psi)\mu(\frac{\overline{w}-w}{2\delta})\mu(\overline{w})$ where $1 \neq s \in W = Aut(E/F)$ and $\alpha(\chi) = a + x_\chi\delta$, for $a, x_\chi \in F$. The reasoning is as follows. We have $\mu(\frac{w+\overline{w}}{2}) = \mu(w)\mu(\frac{1}{2}(1 + \frac{\overline{w}}{w}))$. Now, $w = p^n u + p^m v\delta$ with $n < m$, so we may rewrite $\frac{\overline{w}}{w} = \frac{p^n u - p^m v\delta}{p^n u + p^m v\delta} = \frac{1 - p^{m-n}\frac{v\delta}{u}}{1 + p^{m-n}\frac{v\delta}{u}} = \frac{1 - p^{m-n}\frac{v\delta}{u}}{1 - (-p^{m-n}\frac{v\delta}{u})}$. Now rewriting $\frac{1}{1-(-p^{m-n}\frac{v\delta}{u})}$ as a power series, we get $\frac{1}{2}(1 + \frac{\overline{w}}{w}) \in U_E^1$, and so $\mu(\frac{1}{2}(1 + \frac{\overline{w}}{w})) = 1$ by Lemma (A.0.5). Therefore, $\mu(\frac{w+\overline{w}}{2}) = \mu(w)$.

Thus, the supercuspidal character on the $0 < n(w) \leq r/2$ range simplifies to

$$\theta_\pi(w) =$$

$$deg(\pi)C(x_\chi,\Delta)_F\gamma(\Delta,\psi)\mu(\frac{w-\overline{w}}{2\delta})\ (\phi(w)\mu(w)+(-1,\Delta)\phi(\overline{w})\mu(\overline{w}))$$

$$\forall w \in E^* : 0 < n(w) \leq r/2$$

for any character μ of E^* whose restriction to F^* is $\aleph_{E/F}$ and whose order is a power of 2, since $\mu(\frac{\overline{w}-w}{2\delta}) = (-1,\Delta)\mu(\frac{w-\overline{w}}{2\delta})$. We can rewrite this formula as

$$\theta_\pi(w) = \epsilon(\tilde{\chi},\Delta^+,w)\ \frac{\phi(w)\mu(w)+(-1,\Delta)\phi(\overline{w})\mu(\overline{w})}{\mu(\frac{w-\overline{w}}{2\delta})}\quad \forall w \in E^* : 0 < n(w) \leq r/2$$

□

The above analysis shows $F(\tilde{\chi})(w) = \theta_\pi(w)\ \forall w \in E^* : 0 < n(w) < r/2$ where $\phi = \chi\mu^{-1}$ for any character μ of E^* whose restriction to F^* is $\aleph_{E/F}$ and whose order is a power of 2. The reason is that for such a μ, $\tau_o(\frac{w-\overline{w}}{2\delta}) = \mu(\frac{w-\overline{w}}{2\delta})$ since $\frac{w-\overline{w}}{2\delta} \in F^*$ and $\tau_o|_{F^*} = \mu|_{F^*} = \aleph_{E/F}$. Equivalently, the above analysis shows $F(\tilde{\chi})(w) = \theta_\pi(w)\ \forall w \in E^* : 0 < n(w) < r/2$ where $\phi = \chi\mu$, since if μ is a character of E^* whose restriction to F^* is $\aleph_{E/F}$ and whose order is a power of 2, then so is μ^{-1}. We will use the $\phi = \chi\mu$ version in what follows for ease of notation, although it makes no difference which version we use.

We will need to investigate the $n(w) = 0$ range to see which such characters μ can arise, if any. We will show that our conjectured formula agrees with a supercuspidal character in the $n(w) = 0$ range, for a unique μ. We will also show that $\mu = \Delta_\chi$, the twist coming from the local Langlands correspondence. A priori there is no reason for our proposed formula to agree with any supercuspidal character on the $n(w) = 0$ level. We now investigate the $n(w) = 0$ range.

Lemma 5.5.7.

$$F(\tilde{\chi})(w) = \epsilon(\tilde{\chi}, \Delta^+, w)\ (\chi(w)\Omega(\frac{w}{\delta}) + \chi(\overline{w})\Omega(\frac{\overline{w}}{\delta}))\ \forall w \in E^* : n(w) = 0$$

for the unique unramified character Ω of E^* whose restriction to F^* is $\aleph_{E/F}$.

Proof. Recall that $n(w) = 0$ if and only if either

i) $w = p^n u + p^m v\delta$, where u, v are both nonzero, $n > m$, or

ii) $w = p^m v\delta$, where v is nonzero.

iii) $w = p^n u + p^m v\delta$, where $n = m$ and u, v are both non-zero.

In case (i), $\frac{w}{\delta} = p^m v + p^n \frac{u}{\delta} = p^m v(1 + p^{n-m}\frac{u}{v\delta})$. Note that $1 + p^{n-m}\frac{u}{v\delta} \in U_E^1$. Recall that $\tau_o(\frac{w-\overline{w}}{2\delta}) = \Omega(\frac{w-\overline{w}}{2\delta})$ since $\tau_o|_{F^*} = \aleph_{E/F}$. But Ω is trivial on U_E^1 by Lemma (A.0.5). Therefore, we have $\Omega(\frac{w}{\delta}) = \Omega(p^m v(1 + p^{n-m}\frac{u}{v\delta})) = \Omega(p^m v) = \Omega(\frac{w-\overline{w}}{2\delta}) = \tau_o(\frac{w-\overline{w}}{2\delta})$. In case (ii), it's clear that $\Omega(\frac{w}{\delta}) = \Omega(\frac{w-\overline{w}}{2\delta}) = \tau_o(\frac{w-\overline{w}}{2\delta})$. In case (iii), $w = p^n u + p^m v\delta$ with $n = m$, so $w = p^n(u + v\delta)$. But then, $u + v\delta \in \mathfrak{o}_E^*$ since if the power series expansions of $u, v,\ \delta$ are $u = u_0 + ..., v = v_0 + ..., \delta = \delta_0 + ...$, then if $u + v\delta \in \mathfrak{o}_E - \mathfrak{o}_E^*$, then comparing leading coefficients gives $u_0 + v_0\delta_0 = 0$, so $\delta_0 = -u_0/v_0 \in \mu_F$, which as we have seen, implies that Δ is a square. Thus, $\Omega(w) = \Omega(p^n)\Omega(u + v\delta) = \Omega(p^n) = \Omega(p^n v)$ since $\Omega(v) = 1$ since Ω is unramified. In particular, since $\Omega(\delta) = 1$, we get $\Omega(\frac{w}{\delta}) = \tau_o(\frac{w-\overline{w}}{2\delta})$.

Therefore, in the $n(w) = 0$ case, $\tau_o(\frac{w-\overline{w}}{2\delta}) = \Omega(\frac{w}{\delta})$.

Since $\tau_o(-1) = (-1, \Delta) = 1$, $F(\tilde{\chi})$ simplifies in the $n(w) = 0$ range.

$$F(\tilde{\chi})(z) = \epsilon(\tilde{\chi}, \Delta^+, w)\ \frac{\chi(w) + \chi(\overline{w})}{\tau_o(\frac{w-\overline{w}}{2\delta})} =$$

$$\epsilon(\tilde{\chi}, \Delta^+, w)\ (\chi(w)\tau_o(\frac{w-\overline{w}}{2\delta}) + \chi(\overline{w})\tau_o(\frac{\overline{w}-w}{2\delta})) =$$

$$\epsilon(\tilde{\chi}, \Delta^+, w)\ (\chi(w)\Omega(\frac{w}{\delta}) + \chi(\overline{w})\Omega(\frac{\overline{w}}{\delta}))\ \forall w \in E^* : n(w) = 0.$$

□

We now want to show that $F(\tilde{\chi})(w)$ is equal to

$$\theta_\pi(w) = deg(\pi) C\lambda(\sigma)(\phi(w) + \phi(\overline{w}))\ \forall w \in E^* : n(w) = 0$$

for $\phi = \chi\mu$ for some character μ of E^* whose restriction to F^* is $\aleph_{E/F}$ and whose order is a power of 2. Note that $\Omega(\delta) = 1$ since Ω is unramified, and therefore, we are reduced to showing that

$$\epsilon(\tilde{\chi}, \Delta^+, w)\ (\chi(w)\Omega(w) + \chi(\overline{w})\Omega(\overline{w})) =$$

$$deg(\pi) C\lambda(\sigma)(\chi(w)\mu(w) + \chi(\overline{w})\mu(\overline{w}))\ \forall w \in E^* : n(w) = 0$$

for some character μ. We will show that $\mu = \Delta_\chi$ works.

Note that we proved earlier that $F(\tilde{\chi})(w) = \theta_\pi(w)\ \ \forall w \in E^* : 0 < n(w) \leq r/2$ for any such μ. So what we will prove now is that there exists a μ such that $F(\tilde{\chi})(w) = \theta_\pi(w)\ \ \forall w \in E^* : 0 \leq n(w) \leq r/2$.

Unwinding the definitions, one can see that to prove $F(\tilde{\chi})(w) = \theta_\pi(w)\ \forall w \in E^* : n(w) = 0$ for $\phi = \chi\mu$, it suffices to show that

$$(x_\chi, \Delta)\gamma(\Delta, \psi)\chi(w)\Omega(w) = \chi(w)\mu(w)\lambda(\sigma)\ \forall w \in E^* : n(w) = 0.$$

Lemma 5.5.8. $\lambda(\sigma) = (x_\chi, \Delta)_F \gamma(\Delta, \psi)$.

Proof. $\lambda(\sigma)$ is defined such that if $G = GL(\ell, F)$, and if the inducing subgroup of the representation is $G' G_{x,r/2}$ (see [9, page 7,34-35,65]), then we have the $\lambda(\sigma) = 1$ if

$\mathfrak{g}_{x,r/2} = \mathfrak{g}_{x,r/2^+}$, and $\lambda(\sigma) = (-1)^{\ell-1}$ otherwise, as long as E/F is unramified. Here $G = GL(2, F)$, so $\ell = 2$. We don't know which point in the building x is. However, in our case, since E/F is unramified, the point in the building always refers to the maximal compact subgroup $G(\mathfrak{o}_F)$ [5, Lemma 20.3]. Therefore, $\mathfrak{g}_{x,r/2} \neq \mathfrak{g}_{x,r/2^+}$ if and only if $r/2 \in \mathbb{Z}$ (see [9, page 34-35]). That is, $\lambda(\sigma) = 1$ if r is odd, and $\lambda(\sigma) = -1$ if r is even, where r is the depth of the representation π_ϕ via Theorem (4.3.2). Therefore, $\lambda(\sigma) = (-1)^{r+1}$. Note that the depth of π_ϕ equals the depth of π_χ since $\mu|_{1+\mathfrak{p}_E} \equiv 1$, i.e. μ has level zero.

Now, consider the term (x_χ, Δ). We need to investigate the term $\alpha(\phi)$. Note that $\alpha(\phi) = \alpha(\chi)$ since again, $\mu|_{1+\mathfrak{p}_E} \equiv 1$. We prefer to work with $\alpha(\chi)$. Recall that $\alpha(\chi) \in \mathfrak{p}_E^{-n}$. Here, $n = r$, since according to [9, page 34], $\alpha(\chi) \in \mathfrak{g}'_{-r} \setminus \mathfrak{g}'_{-r^+}$. Now let $\alpha(\chi) = p^k u + p^l v\delta$. We need a lemma.

Lemma 5.5.9. *$l = -n$.*

Proof. Well, since $(E/F, \chi)$ is a minimal regular pair (cf [5, Section 19.2 line 1]), we have that $\alpha(\chi)$ is a *minimal element* over F (see [5, Proposition 18.2]). By [5, Section 13.4], $\alpha(\chi)$ is minimal over F if and only if $(\alpha(\chi) + \mathfrak{p}_E^{-n+1}) \bigcap F = \emptyset$, where $n = -v_E(\alpha(\chi))$, where v_E denotes valuation. Now, it must be the case that either $l = -n$ or $k = -n$ (or both). The reason is that $v_E(\alpha(\chi)) = -n$. So suppose by way of contradiction that $k = -n$ but $l \neq -n$. Now let $w = p^a u' + p^b v'\delta \in \mathfrak{p}_E^{-n+1}$. Then $\alpha(\chi) + w = p^k u + p^l v\delta + p^a u' + p^b v'\delta$. Then $\alpha(\chi) + w \in F$ if and only if $p^l v\delta + p^b v'\delta = 0$. Then since $\alpha(\chi) \in \mathfrak{p}_E^{-n} \setminus \mathfrak{p}_E^{-n+1}$ and since $k = -n$, we must have that $l \geq -n$. But we have assumed $l \neq -n$, so that means $l \geq -n + 1$. Thus,

$b \geq -n+1$. Therefore, just pick any $a \in \mathbb{Z}$ such that $a \geq -n+1$ and any u' and set $p^b v' = -p^l v$. Then we get that $p^a u' + p^b v' \delta \in \mathfrak{p}_E^{-n+1}$ and that $\alpha(\chi) + w \in F$. Thus, $(\alpha(\chi) + \mathfrak{p}_E^{-n+1}) \bigcap F \neq \emptyset$, a contradiction, and the lemma is proven. □

Therefore, $\alpha(\chi) = p^k u + p^l v \delta$, and $l = -n$. Returning to the proof of the proposition, recall that $x_\chi = p^l v$. Thus, $(x_\chi, \Delta) = (p^{-n} v, \Delta)$. Also, since $r = n$, we have $\lambda(\sigma) = (-1)^{r+1} = (-1)^{n+1}$. Therefore, we are reduced to showing that

$$(p^{-n} v, \Delta)\gamma(\Delta, \psi) = (-1)^{n+1}$$

so equivalently, $\gamma(\Delta, \psi) = -1$. But since ψ has level 1, it is a fact that $\gamma(\Delta, \psi) = -1$ (we will show this in Section (5.8)). □

Proposition 5.5.10. *$F(\tilde{\chi})(w) = \theta_\pi(w)$ $\forall w \in E^* : n(w) = 0$, for $\phi = \chi\mu$, for some character μ of E^* whose restriction to F^* is $\aleph_{E/F}$ and whose order is a power of 2. In particular, $F(\tilde{\chi})(w) = \theta_\pi(w)$ $\forall w \in E^* : n(w) = 0$ for $\phi = \chi\Delta_\chi$.*

Proof. By the previous lemma, the conjectured equation

$$(x_\chi, \Delta)\gamma(\Delta, \psi)\chi(w)\Omega(w) = \chi(w)\mu(w)\lambda(\sigma) \ \forall w \in E^* : n(w) = 0$$

simplifies to $\Omega(w) = \mu(w)$ $\forall w \in E^* : n(w) = 0$. But recall that μ is a character of E^* whose restriction to F^* is $\aleph_{E/F}$ and whose order is a power of 2. There is only one character μ of E^* whose restriction to F^* is $\aleph_{E/F}$, whose order is a power of 2, and that equals Ω on the $n(w) = 0$ range. Indeed, this forces μ to be Δ_χ. □

Therefore, we have proven the following, when E/F is unramified.

Theorem 5.5.11. *$F(\tilde{\chi})$ agrees with the character of the supercuspidal representation $\pi_{\chi\Delta_\chi}$ on the range $\{w \in E^* : 0 \leq n(w) \leq r/2\}$.*

What we have actually proven is that if $(E/F, \chi)$ is a *minimal* regular pair with E/F unramified and χ having positive level, then $F(\tilde{\chi})$ agrees with the character of the supercuspidal representation $\pi_{\chi\Delta_\chi}$ on the range $\{w \in E^* : 0 \leq n(w) \leq r/2\}$. To prove this for an arbitrary regular pair follows. For if $(E/F, \chi)$ is an arbitrary regular pair, then there exists a minimal regular pair $(E/F, \chi')$ such that $\chi = \chi'\phi_E$ where $\phi_E = \phi \circ N_{E/F}$ for some $\phi \in \widehat{F^*}$. Moreover, $\pi_\chi = \phi\pi_{\chi'}$ by definition. We proved above that $F(\widetilde{\chi'}) = \theta_{\pi_{\chi'\Delta_{\chi'}}}$ on the range $\{w \in E^* : 0 \leq n(w) \leq r/2\}$. Therefore, $\theta_{\pi_{\chi\Delta_\chi}}(w) = \theta_{\pi_{\chi'\phi_E\Delta_{\chi'\phi_E}}}(w) = \theta_{\pi_{\chi'\phi_E\Delta_{\chi'}}}(w) = \phi_E(w)\theta_{\pi_{\chi'\Delta_{\chi'}}}(w) = \phi_E(w)F(\widetilde{\chi'})(w) = F(\widetilde{\chi'\phi_E})(w) = F(\tilde{\chi})(w)$ on the range $\{w \in E^* : 0 \leq n(w) \leq r/2\}$.

5.6 On whether there are two positive depth character formulas coming from the same Cartan

Given a supercuspidal parameter for $GL(2, F)$, we constructed a conjectural character formula for a positive depth supercuspidal representation of $GL(2, F)$. We showed that it agrees with the character formula of the positive depth supercuspidal representation $\pi_{\chi\Delta_\chi}$ on the set $\{w \in E^* : 0 \leq n(w) \leq \frac{r}{2}\}$, where r is the depth of the representation. In the next two sections we show that there are no other supercuspidal representations whose character agrees with $F(\tilde{\chi})$ on the set $\{w \in E^* : 0 \leq n(w) \leq r/2\}$, thereby solving the uniqueness question in Theorem (5.2.1). In fact, we show something stronger. In the next two sections, we show that a supercuspidal representation of $GL(2, F)$ is uniquely determined by the restriction of its distribution character to the $n(w) = 0$ range. In this section, we show that if the distribution characters of two positive depth supercuspidal representations, both coming from the same Cartan, agree on the $n(w) = 0$ range, then the supercuspidal representations are isomorphic. That is, we prove the following theorem.

Theorem 5.6.1. *Suppose* $(E/F, \chi_1)$ *and* $(E/F, \chi_2)$ *are admissible pairs such that* $F(\tilde{\chi}_1)(w) = F(\tilde{\chi}_2)(w)\ \forall w \in E^* : n(w) = 0$. *Then,* $\chi_1 = \chi_2^{\upsilon}$ *for some* $\upsilon \in Aut(E/F)$.

We wish to make the following important note. Recall that in the previous sections, we constructed a character formula $F(\tilde{\chi})$ from a regular pair $(E/F, \chi)$. The above theorem and Theorem (5.7.1) will together prove that a positive depth supercuspidal representation of $GL(2, F)$ is uniquely determined by its restriction of its distribution character to the $n(w) = 0$ range. We are claiming in the above theorem and in Theorem (5.7.1) that it is sufficient to consider admissible pairs rather than regular pairs in order to prove that a positive depth supercuspidal representation of $GL(2, F)$ is uniquely determined by its restriction of its distribution character to the $n(w) = 0$ range. This is because the positive depth supercuspidal representations of $GL(2, F)$ are parameterized by admissible pairs, and so it is sufficient to consider just admissible pairs.

Note that we are using our formulation $F(\tilde{\chi})$ of the supercuspidal characters of $GL(2, F)$, and we are implicitly using the fact which we proved that every supercuspidal character of $GL(2, F)$ is of the form $F(\tilde{\chi})$ for some $\tilde{\chi}$ and some Cartan

E^*.

Lemma 5.6.2. *Let E/F be ramified. Suppose $(E/F,\chi_1)$ and $(E/F,\chi_2)$ are admissible pairs such that $F(\tilde{\chi}_1)(w) = F(\tilde{\chi}_2)(w)\ \forall w \in E^* : n(w) = 0$. Then, $\chi_1 = \chi_2^{\upsilon}$ for some $\upsilon \in Aut(E/F)$.*

Proof. We may assume without loss of generality that $E = F(\sqrt{p})$. Recall that $n(w) = 0$ if and only if

(i) $w = p^n u + p^m v\sqrt{p}$ for $n > m$ or

(ii) $w = p^m v\sqrt{p}$ for $v \neq 0$.

Then, assume $F(\tilde{\chi}_1)(w) = F(\tilde{\chi}_2)(w)$ on the range $\{w \in E^* : n(w) = 0\}$. Recall that this means that

$$\epsilon(\tilde{\chi}_1, \Delta^+, w)\frac{\chi_1(w) + (-1,\Delta)\chi_1(\overline{w})}{\tau_o(\frac{w-\overline{w}}{2\delta})} =$$

$$\epsilon(\tilde{\chi}_2, \Delta^+, w)\frac{\chi_2(w) + (-1,\Delta)\chi_2(\overline{w})}{\tau_o(\frac{w-\overline{w}}{2\delta})} \quad \forall w \in E^* : n(w) = 0$$

in the notation of Section 4.4 first page. Recall that

$$\epsilon(\tilde{\chi}, \Delta^+, w) = (x_\chi, \Delta)\gamma(\Delta,\psi)deg(\pi)C$$

, where $\alpha(\chi) = a + x_\chi\delta$. Let us write $deg(\pi_1), deg(\pi_2)$ for the $deg(\pi)$ that are associated to the pairs $(E/F,\chi_1)$ and $(E/F,\chi_2)$, respectively. Recall also that

$$C := c_\psi(\mathfrak{g}')c_\psi^{-1}(\mathfrak{g})|D(w)|^{-\frac{1}{2}}|\eta(\alpha(\chi))|^{-\frac{1}{2}}$$

Define $C_i := c_\psi(\mathfrak{g}')c_\psi^{-1}(\mathfrak{g})|D(w)|^{-\frac{1}{2}}|\eta(\alpha(\chi_i))|^{-\frac{1}{2}}$ for $i = 1, 2$. Therefore we get that

$$deg(\pi_1)(x_{\chi_1}, \Delta)\gamma(\Delta,\psi)C_1\frac{\chi_1(w) + (-1,\Delta)\chi_1(\overline{w})}{\tau_o(\frac{w-\overline{w}}{2\delta})} =$$

$$deg(\pi_2)(x_{\chi_2},\Delta)\gamma(\Delta,\psi)C_2\frac{\chi_2(w)+(-1,\Delta)\chi_2(\overline{w})}{\tau_o(\frac{w-\overline{w}}{2\delta})} \quad \forall w\in E^*: n(w)=0$$

Cancelling out like terms, we get

$$deg(\pi_1)(x_{\chi_1},\Delta)|\eta(\alpha(\chi_1))|^{-\frac{1}{2}}(\chi_1(w)+(-1,\Delta)\chi_1(\overline{w}))=$$

$$deg(\pi_2)(x_{\chi_2},\Delta)|\eta(\alpha(\chi_2))|^{-\frac{1}{2}}(\chi_2(w)+(-1,\Delta)\chi_2(\overline{w})) \quad \forall w\in E^*: n(w)=0$$

Now, let

$$c_1 = deg(\pi_1)(x_{\chi_1},\Delta)|\eta(\alpha(\chi_1))|^{-\frac{1}{2}}$$

and

$$c_2 = deg(\pi_2)(x_{\chi_2},\Delta)|\eta(\alpha(\chi_2))|^{-\frac{1}{2}}.$$

Then we have that

$$c_1(\chi_1(w)+(-1,\Delta)\chi_1(\overline{w})) = c_2(\chi_2(w)+(-1,\Delta)\chi_2(\overline{w})) \; \forall w\in E^*: n(w)=0$$

Then the same proof of [21, Lemma 5.1], but adjusted to our situation here, shows that $\chi_1|_{F^*(1+\mathfrak{p}_E)} = \chi_2^{\upsilon}|_{F^*(1+\mathfrak{p}_E)}$ for some $\upsilon\in Aut(E/F)$. For the following arguments, it suffices without loss of generality to assume $\upsilon=1$.

Now let $c := \frac{c_1}{c_2}$. Let $[\chi](w) := \chi(w)+(-1,\Delta)\chi(\overline{w})$. Then we have $c[\chi_1](w) = [\chi_2](w) \; \forall w \in E^*\setminus F^*(1+\mathfrak{p}_E)$ and we also have that $\chi_1|_{F^*(1+\mathfrak{p}_E)} = \chi_2|_{F^*(1+\mathfrak{p}_E)}$. Now let p_E be a uniformizer of E and recall that p is a uniformizer of F. We may take p_E so that $p_E^2 = p$. Since $\chi_1(p)=\chi_2(p)$, we have that $\chi_1(p_E)^2=\chi_2(p_E)^2$, and so $\chi_2(p_E)=\xi_2\chi_1(p_E)$ where ξ_2 could be plus or minus 1. Therefore, since $\chi_1|_{F^*(1+\mathfrak{p}_E)} = \chi_2|_{F^*(1+\mathfrak{p}_E)}$ and since $\chi_2(p_E)=\xi_2\chi_1(p_E)$, notice that this implies that $\chi_2(w)=\chi_1(w)\xi_2^{val(w)} \; \forall w\in E^*$, where $val(w)$ denotes the E-adic valuation of

$w \in E^*$. Therefore, plugging this into the formula

$$c[\chi_1](w) = [\chi_2](w)\ \forall w \in E^* \setminus F^*(1+\mathfrak{p}_E)$$

we obtain

$$c[\chi_1](w) = [\chi_2](w) = \chi_1(w)\xi_2^{val(w)} + \chi_1^{\upsilon}(w)\xi_2^{val(\upsilon(w))}\quad \forall w \in E^* \setminus F^*(1+\mathfrak{p}_E)$$

but since $val(w) = val(\upsilon(w))\ \forall w \in E^* \setminus F^*(1+\mathfrak{p}_E)$ and $\forall i$, this equality turns into

$$c[\chi_1](w) = \xi_2^{val(w)}[\chi_1](w)\ \forall w \in E^* \setminus F^*(1+\mathfrak{p}_E)$$

We will prove in the next section that $\exists w' \in E^* \setminus F^*(1+\mathfrak{p}_E)$ such that $[\chi_1](w') \neq 0$. Therefore, we can cancel $[\chi_1](w')$ from both sides to get

$$c = \xi_2^{val(w')}$$

Therefore, c is a square root of 1, i.e. c is plus or minus 1.

Now, suppose that $c = 1$. Then this says that $[\chi_1](w) = [\chi_2](w)\ \forall w \in E^* \setminus F^*(1+\mathfrak{p}_E)$ and $[\chi_1](w) = [\chi_2](w)\ \forall w \in F^*(1+\mathfrak{p}_E)$ (since $\chi_1|_{F^*(1+\mathfrak{p}_E)} = \chi_2|_{F^*(1+\mathfrak{p}_E)}$) and so $[\chi_1](w) = [\chi_2](w)\ \forall w \in E^*$, and so by linear independence of characters, $\chi_1 = \chi_2$, and we're done.

Suppose $c = -1$. Then this says that $\xi_2 = -1$. Therefore, this says that $\chi_2(w) = \chi_1(w)(-1)^{val(w)}$. But notice that this says that $\chi_2 = \chi_1 \otimes \phi_E$, where $\phi_E := \phi \circ N_{E/F}$ where $\phi = \aleph_{L/F}$ where L/F is the unique unramified degree 2 extension of F. We now wish to determine the relationship between $\alpha(\chi_1)$ and $\alpha(\chi_2)$. Recall that that $\chi_1(1+x) = \psi_E(\alpha(\chi_1)x)$ and $\chi_2(1+x) = \psi_E(\alpha(\chi_2)x)$. Then, it is clear that $\chi_1(1+x) = (\chi_1 \otimes \phi_E)(1+x)\ \ \forall x \in \mathfrak{p}_E$. Therefore, we can take

$\alpha(\chi_1) = \alpha(\chi_2)$. Therefore, since $c = \frac{c_1}{c_2} = \frac{deg(\pi_1)(x_{\chi_1},\Delta)|\eta(\alpha(\chi_1))|^{-\frac{1}{2}}}{deg(\pi_2)(x_{\chi_1},\Delta)|\eta(\alpha(\chi_2))|^{-\frac{1}{2}}}$, we get $c = \frac{deg(\pi_1)}{deg(\pi_2)}$. But these are just formal degrees, which are positive real numbers, and so we get a contradiction to the supposition that $c = -1$.

Therefore, $\chi_1 = \chi_2$ or $\chi_1 = \chi_2^v$, and so the admissible pairs are isomorphic, and so the supercuspidal representations are the same. □

Lemma 5.6.3. *Let E/F be unramified. Suppose $(E/F, \chi_1)$ and $(E/F, \chi_2)$ are admissible pairs such that $F(\tilde{\chi}_1)(w) = F(\tilde{\chi}_2)(w)$ $\forall w \in E^* : n(w) = 0$. Then, $\chi_1 = \chi_2^v$ for some $v \in Aut(E/F)$.*

Proof. See [21, page 16]. □

Hence, we are now finished with both cases.

5.7 On whether there are two positive depth character formulas coming from different Cartans

In this section we answer the question of uniqueness with respect to positive depth supercuspidal representations coming from distinct Cartans. We show that if the character formula $F(\tilde{\chi})$ comes from the Cartan E, then there is no supercuspidal representation coming from a different Cartan $E_1 \not\cong E$ whose character agrees with $F(\tilde{\chi})$ on the set $\{w \in E^* : 0 \leq n(w) \leq r/2\}$. We know that there is a supercupsidal representation whose character formula agrees with ours on this range, and there are no other supercuspidal characters coming from E^* that agree with $F(\tilde{\chi})$ on the range. In this section we show that the distribution characters of two supercuspidal representations, coming from different Cartans, can't agree on the $n(w) = 0$ range. This, together with the results from the previous section, shows that if $(E/F, \chi)$ is an admissible pair, then there is a unique positive depth supercuspidal representation whose distribution character agrees with $F(\tilde{\chi})$ on the range $\{w \in E^* : n(w) = 0\ \}$.

Note that we are using our formulation $F(\tilde{\chi})$ of the supercuspidal characters of $GL(2, F)$, and we are implicitly using the fact which we proved that every supercuspidal character on a Cartan E^* of $GL(2, F)$ is of the form $F(\tilde{\chi})$ for some $\tilde{\chi}$. We prove the following theorem.

Theorem 5.7.1. *Suppose* $(E/F, \chi)$ *and* $(E_1/F, \chi_1)$ *are admissible pairs with* $E \ncong E_1$. *Then* $\exists w \in E^* : n(w) = 0$ *such that* $F(\tilde{\chi})(w) \neq \theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w)$.

Well, there are many cases to check, and we split them up in a sequence of propositions.

Proposition 5.7.2. *Suppose* $(E/F, \chi)$ *and* $(E_1/F, \chi_1)$ *are admissible pairs with* E *ramified and* E_1 *unramified. Then* $\exists w \in E^* : n(w) = 0$ *such that* $F(\tilde{\chi})(w) \neq \theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w)$.

Proof. We claim that if the inducing representation of the representation coming from E_1 is $E_1^* G_{x,r/2}$ (see [9, page 34-35]), then one can't conjugate w into $E_1^* G_{x,r/2}$ for all $w \in E^* : n(w) = 0$. Therefore, we'd have that $\theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w) = 0\ \forall w \in E^* : n(w) = 0$, since supercuspidal character formulas are zero on elements that can't be conjugated into the inducing subgroup of the representation.

The strategy is as follows: If two elements have different determinants, then they certainly aren't conjugate. So let $w'g \in E_1^* G_{x,r/2}$, $w' \in E_1^*, g \in G_{x,r/2}$. Since

$r > 0$, $det(g) \in 1+\mathfrak{p}_F$, no matter whether the point in the building corresponds to the Iwahori or maximal compact. (This is easy to check, just look at the filtrations). In particular, $valuation(det(g)) = 0$. And if $w' \in E_1^*, w' = a + b\delta$, then $w' = \begin{pmatrix} a & b \\ b\Delta_{E_1} & a \end{pmatrix}$, so $det(w') = N_{E_1/F}(w)$. But E_1 is unramified, so the uniformizer of E_1 is p. So if $w' = p^s v \in E_1^*, v \in \mathfrak{o}_E^*$, then $valuation(det(w')) \in 2\mathbb{Z}$. In particular, $valuation(det(w'g)) \in 2\mathbb{Z}$. But if $w \in E^*$ such that $n(w) = 0$, then it is clear that $valuation(det(w)) \in 2\mathbb{Z} + 1$. Thus, $\theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w) = 0\ \forall w \in E^* : n(w) = 0$. If we can find a single element $w \in \{u \in E^* : n(u) = 0\}$ such that $F(\tilde{\chi})(w) \neq 0$, then we'd have a contradiction since we assumed that both character formulas agreed on the set $\{w \in E^* : n(w) = 0\}$. We now need to consider two subcases.

Subcase (a): Suppose $(-1, \Delta_E) = -1$. We suppose by way of contradiction that $F(\tilde{\chi})(w) = 0\ \forall w \in E^* : n(w) = 0$. Then this implies that $\chi(w) + (-1, \Delta_E)\chi(\overline{w}) = 0\ \forall w \in E^* : n(w) = 0$. Since $(-1, \Delta_E) = -1$, we have $\chi(w) = \chi(\overline{w})\ \forall w \in E^* : n(w) = 0$. Now, since any $w \in E^* : n(w) > 0$ is the product of elements from the set $\{w \in E^* : n(w) = 0\}$, we have clearly that also $\chi(w) = \chi(\overline{w})\ \forall w \in E^* : n(w) > 0$. Thus, $\chi(w) = \chi(\overline{w})\ \forall w \in E^*$, which contradicts the first condition in the definition of $(E/F, \chi)$ being an admissible pair.

Subcase (b): Suppose $(-1, \Delta_E) = 1$, and suppose again by way of contradiction that $F(\tilde{\chi})(w) = 0\ \forall w \in E^* : n(w) = 0$. Then $\chi(w) + \chi(\overline{w}) = 0\ \forall w \in E^* : n(w) = 0$. Thus, $\chi(w/\overline{w}) = -1\ \forall w \in E^* : n(w) = 0$, so $\chi(w^2/\overline{w}^2) = 1$. This says that $\chi(w/\overline{w})^2 = 1$ for all $w \in E^* : n(w) = 0$. But since the set $\{w \in E^* : n(w) = 0\}$ generates all of E^* as a group (i.e. any element w' with $n(w') > 0$ can be written as the product of elements coming from the set $\{w \in E^* : n(w) = 0\}$), we get that since χ is a multiplicative character, then $\chi(w/\overline{w})^2 = 1 \ \forall w \in E^*$. Thus, we get that $\chi|_{(E^*)^2} = \chi^{\upsilon}|_{(E^*)^2}$. But since $U_E^1 \subset (E^*)^2$ by Lemma (A.0.6), we get $\chi|_{U_E^1} = \chi^{\upsilon}|_{U_E^1}$, which contradicts the second condition of the definition of $(E/F, \chi)$ being an admissible pair. □

Proposition 5.7.3. *Suppose $(E/F, \chi)$ and $(E_1/F, \chi_1)$ are admissible pairs with E unramified and E_1 ramified. Then $\exists w \in E^* : n(w) = 0$ such that $F(\tilde{\chi})(w) \neq \theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w)$.*

We will need to split this proposition into two cases: $(-1, p) = 1$ and $(-1, p) = -1$. We have $E = F(\sqrt{\Delta})$, where $\Delta \in \mathfrak{o}_F^*$ is not a square, and without loss of generality $E_1 = F(\sqrt{p})$.

Lemma 5.7.4. *Suppose $(-1, p) = 1$. Suppose $(E/F, \chi)$ and $(E_1/F, \chi_1)$ are admissible pairs with $E \not\cong E_1$, with E unramified and E_1 ramified. Then $\exists w \in E^* : n(w) = 0$ such that $F(\tilde{\chi})(w) \neq \theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w)$.*

Proof. We use the same strategy as in Proposition (5.7.2). Recall that $w \in E^* : n(w) = 0$ if and only if

$$w = p^m v\delta \tag{5.2}$$

$$w = p^n u + p^m v\delta, \ n = m \tag{5.3}$$

or

$$w = p^n u + p^m v\delta,\ n > m \tag{5.4}$$

We wish to show that any element of $\{w \in E^* : n(w) = 0\}$ can't be conjugated into $E_1^* G_{x,r/2}$ (see [9, page 34-35]), and then we will have that $\theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w) = 0\ \forall w \in E^* : n(w) = 0$. We will then show that there exists an element $w \in E^* : n(w) = 0$ such that $F(\tilde{\chi})(w) \neq 0$. This will prove the Lemma.

We first show that if w is of the form $w = p^m v\delta$, then w can't be conjugated into $E_1^* G_{x,r/2}$ (see [9, page 34-35]), as follows: As in Proposition (5.7.2), we calculate determinants : Just as in Proposition (5.7.2), we have that if $zg \in E_1^* G_{x,r/2}$, then $det(zg) \in N_{E_1/F}(E_1^*)$. Now, if $w = p^m v\delta$, then $det(w) = N(w) = -p^{2m} v^2 \Delta$. To check whether this is a norm from E_1, we check Hilbert symbol :

$$(det(w), p) = (-p^{2m} v^2 \Delta, p) = (-\Delta, p) = -(-1, p) = -1$$

Therefore, this shows that $\theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w) = 0$ for all w of the form $w = p^m v\delta$.

Now we show that if $w = p^n u + p^m v\delta,\ n > m$, then w can't be conjugated into $E_1^* G_{x,r/2}$. So let $w = p^n u + p^m v\delta,\ n > m$. If $w' = p^k s + p^l t\sqrt{p} \in E_1^*$, then $N_{E_1/F}(w') = p^{2k} s^2 - p^{2l+1} t^2$. We wish to show that we can't have an equality of the form $N_{E/F}(w) = N_{E_1/F}(w')$ for any $w' \in E_1^*$. Thus, we want to show that we can't have an equality

$$p^{2n} u^2 - p^{2m} v^2 \Delta = p^{2k} s^2 - p^{2l+1} t^2$$

Since $m < n$, we can rewrite the left hand side of the above equality as $p^{2m}(-v^2\Delta + p^{2n-2m} u^2)$

Subcase (i): Suppose $k \leq l$. Then we can rewrite the above as

$$p^{2m}(-v^2\Delta + p^{2n-2m} u^2) = p^{2k}(s^2 - p^{2l-2k+1} t^2)$$

so that

$$p^{2m-2k}(-v^2\Delta + p^{2n-2m}u^2) = s^2 - p^{2l-2k+1}t^2$$

If $s \neq 0$, then this forces $m = k$ since $-v^2\Delta + p^{2n-2m}u^2$ and $s^2 - p^{2l-2k+1}t^2$ are both units. But then we'd have $-v^2\Delta \equiv s^2 \ mod \ p$, which would imply that if Δ_0, v_0, s_0 are the leading terms of the power series expansions, then $\Delta_0 = -\frac{s_0^2}{v_0^2}$. But since $(-1, p) = 1$, we have that -1 is a square mod p, and so Δ_0 is a square, which is a contradiction to the fact that Δ is a non-square unit. Now suppose $s = 0$. Then it is obvious that we cannot achieve the equality

$$p^{2m-2k}(-v^2\Delta + p^{2n-2m}u^2) = -p^{2l-2k+1}t^2$$

since $2m - 2k$ is even, and $2l - 2k + 1$ is odd, and since $-v^2\Delta + p^{2n-2m}u^2$ and t^2 are units.

Subcase (ii): Suppose $k > l$. Then the equality

$$p^{2m}(-v^2\Delta + p^{2n-2m}u^2) = p^{2k}s^2 - p^{2l+1}t^2$$

reduces to

$$p^{2m}(-v^2\Delta + p^{2n-2m}u^2) = p^{2l+1}(-t^2 + p^{2k-2l-1}s^2)$$

If $t \neq 0$, then this equality clearly can't happen because $2l + 1$ is odd and $2m$ is even, and since $-v^2\Delta + p^{2n-2m}u^2$ and $-t^2 + p^{2k-2l-1}s^2$ are units. Suppose $t = 0$. Then we need to answer whether it is possible that

$$p^{2m}(-v^2\Delta + p^{2n-2m}u^2) = p^{2k}s^2$$

Well, this equality would force $m = k$, and then we'd have

$$-v^2\Delta + p^{2n-2m}u^2 = s^2$$

Thus,

$$p^{2n-2m}u^2 = s^2 + v^2\Delta$$

Now, $s^2 + v^2\Delta$ is a unit if $(-1, p) = 1$, since saying $s^2 + v^2\Delta \equiv 0 \ mod \ p$ would say that Δ is a square, as in subcase (i). Therefore, the fact that $s^2 + v^2\Delta$ is a unit forces that $n = m$, which is a contradiction since we assumed that $w = p^n u + p^m v\delta$ with $n > m$.

Therefore, we have shown that if $w = p^n u + p^m v\delta,\ n > m$, then w can't be conjugated into $E_1^* G_{x,r/2}$. This is more tedious, and we will not show it here.

Summing up, we have shown that elements of the form (5.2) and (5.4) cannot be conjugated into $E_1^* G_{x,r/2}$, and so $\theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w) = 0$ for all w of the form (5.2) and (5.4). Now, if there exists a w of the form (5.2) or (5.4) such that $F(\tilde{\chi})(w) \neq 0$, then we are done, because the character formulas from the unramified torus and the ramified torus have different values on the same element. So we assume that $F(\tilde{\chi})(w) = 0$ for all w of the form (5.2) and (5.4), and then seek a contradiction. Well, this assumption clearly implies that $\chi(w) + (-1, \Delta)\chi(\overline{w}) = 0$ for all w of the form (5.2) and (5.4). $(-1, \Delta) = 1$, so we have $\chi(w) + \chi(\overline{w}) = 0$ for all w of the form (5.2) and (5.4). We will soon show that this implies that $F(\tilde{\chi})(w) = 0$ for all w of the form (5.3). This will show that $F(\tilde{\chi})$ vanishes on all elements $w \in E^* : n(w) = 0$. We will then use this to show a contradiction later. We now need the following lemma.

Lemma 5.7.5. $\chi|_{F^*U_E^1} = \chi^{\upsilon}|_{F^*U_E^1}$ *where* υ *is the generator of* $Aut(E/F)$.

Proof. One can check that any element of $1+\mathfrak{p}_E$ can be written as the product of an element of the form (5.2) and the Galois conjugate of an element of the form (5.4). Let $z \in U_E^1$. Therefore, we may write $z = w_1\overline{w_2}$, where w_1 is of the form (5.2) and w_2 is of the form (5.4). Now, since $F(\tilde{\chi})(w) = 0$ on all w of the form (5.2) and (5.4), we have

$$\chi(w_1) + \chi(\overline{w_1}) = 0$$

$$\chi(w_2) + \chi(\overline{w_2}) = 0$$

Multiplying the first equation by $\chi(w_2)$ and the second equation by $\chi(w_1)$, we conclude that

$$\chi(w_1 w_2) + \chi(w_2\overline{w_1}) = 0$$

$$\chi(w_1 w_2) + \chi(w_1\overline{w_2}) = 0$$

Therefore,

$$\chi(w_2\overline{w_1}) = \chi(w_1\overline{w_2})$$

Therefore we have proven that $\chi(z) = \chi(\overline{z})\ \forall z \in U_E^1$. Since $\chi(x) = \chi(\overline{x})\ \forall x \in F^*$, we get $\chi|_{F^*U_E^1} = \chi^{\upsilon}|_{F^*U_E^1}$ where $1 \neq \upsilon \in Aut(E/F)$. □

We want to eventually contradict the first condition of admissibility of the pair $(E/F, \chi)$. We need the following lemma, which we shall not prove, as it is not difficult.

Lemma 5.7.6. *Every element* b *of the form (5.3) can be written as a product* ac, *where* a *is either an element of the form (5.2) or (5.4), and* c *is an element of* $F^*U_E^1$.

Because of the above lemma, we have that if b is of the form (5.3), then if we write $b = ac$ as in Lemma (5.7.6), then $\chi(b) + \chi(\bar{b}) = \chi(ac) + \chi(\overline{ac}) = \chi(a)\chi(c) + \chi(\overline{a})\chi(\overline{c}) = \chi(a)\chi(c) + \chi(\overline{a})\chi(c)$ since we showed that $\chi|_{F^*U_E^1} = \chi^{\upsilon}|_{F^*U_E^1}$. Thus, $\chi(b) + \chi(\bar{b}) = \chi(c)(\chi(a) + \chi(\overline{a}))$. But recall that we assumed that $\chi(a) + \chi(\overline{a}) = 0$ for all elements a of the form (5.2) and of the form (5.4). Thus, we would have that $F(\tilde{\chi})(b) = 0$ for all w in of the form (5.3). Thus, given the Lemma, we'd have that $F(\tilde{\chi})(w) = 0$ for all w in cases (12), (13), and (14), so $F(\tilde{\chi})(w) = 0$ $\forall w \in E^* : n(w) = 0$.

So we have that $F(\tilde{\chi})(w) = 0$ $\forall w \in E^* : n(w) = 0$. Recall that we also have that $\chi|_{F^*U_E^1} = \chi^{\upsilon}|_{F^*U_E^1}$ where $1 \neq \upsilon \in Aut(E/F)$. We want to show that these two conditions will contradict the fact that $(E/F, \chi)$ is an admissible pair. To show a contradiction, note that by a similar argument as in the proof of Lemma (5.7.5), we can show that if $w_1, w_2 \in E^* : n(w_1) = n(w_2) = 0$, then $\chi(w_1\overline{w_2}) = \chi(\overline{w_1}w_2)$. (When we made the argument before, we only allowed w_1, w_2 to be in case (5.2) or of the form (5.4). But now that we know that $F(\tilde{\chi})$ vanishes also on elements of the form (5.3), we can include these elements as well to say that if w_1, w_2 are of the form (5.2), (5.3), or (5.4), then $\chi(w_1\overline{w_2}) = \chi(\overline{w_1}w_2)$. The same proof holds clearly). Now, if w_1 is in of the form (5.2) and w_2 is in of the form (5.3), then one can see that $w_1\overline{w_2}$ is of the form (5.3). Thus, $\chi(w_1\overline{w_2}) = \chi(\overline{w_1}w_2)$. This shows that $\chi(z) = \chi(\overline{z})$ for some element z of the form (5.3). But notice that the group generated by U_E^1, F^*, and any single element in of the form (5.3) is all of E^*. Thus, we get that $\chi = \chi^{\upsilon}$ on all of E^*, and therefore χ factors through N, so $(E/F, \chi)$ is not an admissible pair (violates the first condition of admissible pair). Finally, we are done with proving

Lemma (5.7.4). ☐

Lemma 5.7.7. *Suppose $(-1,p) = -1$. Suppose $(E/F,\chi)$ and $(E_1/F,\chi_1)$ are admissible pairs with E unramified and E_1 ramified. Then $\exists w \in E^* : n(w) = 0$ such that $F(\tilde{\chi})(w) \neq \theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w)$.*

Proof. Recall again that $w \in E^* : n(w) = 0$ if and only if

(i) $w = p^m v\delta$

(ii) $w = p^n u + p^m v\delta,\ n = m$, or

(iii) $w = p^n u + p^m v\delta,\ n > m$.

Moreover, $w \in E^* : n(w) > 0$ if and only if

(i') $w = p^n u + p^m v\delta,\ n < m$

(ii') $w = p^n u \in F^*$.

We want to show like in Lemma (5.7.4) that any character formula $\theta_{\pi_{\chi_1 \Delta_{\chi_1}}}$ vanishes on elements of the form (i) and (iii) from E^*. If we can do this, then the rest of the proof of Lemma (5.7.7) goes exactly the same way as in Lemma (5.7.4). Thus, all we need to show is that any character formula $\theta_{\pi_{\chi_1 \Delta_{\chi_1}}}$ vanishes on elements of case (i) and case (iii) from E^*. In Lemma (5.7.4), we were able to use arguments that rely heavily on the fact that $(-1,p)$ was 1 in that case. However, we can't use these same arguments in Lemma (5.7.7). Rather than finding a different proof, we can rely on a result of [20, Proposition 2, p. 101]. Let E_1^1 denote the norm 1 elements of E_1. First note that if ω is a character of E_1^*, then saying that $\omega|_{E_1^1} \equiv 1$ is the same thing as saying that ω factors through $N_{E_1/F}$ because of the exact sequence

$$1 \to E_1^1 \to E_1^* \to N_{E_1/F}(E_1^*) \to 1$$

where the third map is $w \mapsto N_{E_1/F}(w)$ and where E_1^1 denotes the norm 1 elements of E_1. i.e. this exact sequence says that any character ω of E_1^* that is trivial on E_1^1 arises as the pullback of a character of $N_{E_1/F}(E_1^*)$, which means that $\omega = \nu \circ N_{E_1/F}$ for some character ν of $N_{E_1/F}(E_1^*)$. Thus, saying that $\omega|_{E_1^1} \neq 1$ is the same as the first condition in the definition of admissible pair. Similarly, the above exact sequence induces an exact sequence

$$1 \to E_1 \bigcap U_{E_1}^1 \to E_1^* \bigcap U_{E_1}^1 \to N_{E_1/F}(U_{E_1}^1) \to 1$$

where all the maps are same. So this exact sequence says precisely that if $\omega|_{U_{E_1}^1}$ is trivial on $U_{E_1}^1 \bigcap E_1^1$, then $\omega|_{U_{E_1}^1}$ factors through the norms. All of this information together shows that the values on the ramified torus E_1 of the character formula of a supercuspidal representation coming from an admissible pair $(E_1/F, \chi_1)$ can be computed using [20, Proposition 2, Proposition (5.7.2), $\ell > 0$, p. 101]. This formula shows that $\theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w) = 0$ for all $w \in E^* : d(w) > |p|^{2\ell}$ where $d(w) = \frac{|(w-\overline{w})^2|}{|N(w)|}$ and $||$ denotes norm. But we clearly have that $d(w) = 1$ for all w of the form (i) and (iii), so since $\ell > 0$, we have that $\theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w) = 0$ for all $w \in E^*$ such that w is in case (i) and case (iii). Thus, we have shown what we need, and the same proofs in Lemma (5.7.4) will show that $\theta_{\pi_{\chi_1 \Delta_{\chi_1}}}$ vanishes on the set $\{w \in E^* : n(w) = 0\}$, and then we can proceed with the same proofs as in Lemma (5.7.4). Thus, we are done with Lemma (5.7.7). $\square$

Therefore, we have finished the proof of Proposition (5.7.3).

Proposition 5.7.8. *Suppose $(E/F,\chi)$ and $(E_1/F,\chi_1)$ are admissible pairs with $E \ncong E_1$, with E ramified and E_1 ramified. Then $\exists w \in E^* : n(w) = 0$ such that $F(\tilde{\chi})(w) \neq \theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w)$.*

Proof. Suppose without loss of generality that $E = F(\sqrt{p})$ and $E_1 = F(\sqrt{\Delta p})$, $\Delta \in \mathfrak{o}_F^*$ not a square. The proof for the case where $E = F(\sqrt{\Delta p})$ and $E = F(\sqrt{p})$ is similar. We first claim that elements $w \in E^*$ such that $n(w) = 0$ can't be conjugated into $E_1^* G_{x,r/2}$ (see [9, page 34-35]). This would then say that $\theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w) = 0$ on these elements. Then we will show that there is such an element w where $F(\tilde{\chi})(w) \neq 0$, and therefore we will be done.

Let's show that the elements $w \in E^*$ such that $n(w) = 0$ can't be conjugated into $E_1^* G_{x,r/2}$. Recall that $w \in E^*$ such that $n(w) = 0$ if and only if either $w = p^n u + p^m v\sqrt{p}$ with $n > m$, $u, v \neq 0$, or $w = p^m v\sqrt{p}$ with $v \neq 0$.

Suppose $w = p^m v\sqrt{p}$. Then $det(w) = N(w) = -p^{2m+1}v^2$. Recall that $det(E_1^* G_{x,r/2}) \subset N_{E_1/F}(E_1^*)$. If $w' = p^k s + p^l t\sqrt{\Delta p} \in E_1^*$, then $N_{E_1/F}(w') = p^{2k}s^2 - p^{2l+1}t^2\Delta$. So we want to show that $det(w)$ can't be of the form $N_{E_1/F}(w')$ for any $w' \in E_1^*$. This can be done by checking the Hilbert symbol. Well, $(det(w), \Delta p) = (-p^{2m+1}v^2, \Delta p) = (-p, \Delta p) = (-1, \Delta p)(p, \Delta p) = (\Delta p, \Delta p)(p, \Delta p) = (\Delta p^2, \Delta p) = (\Delta, \Delta p) = (\Delta, \Delta)(\Delta, p) = (\Delta, p) = -1$. Therefore, $det(w)$ can't be a norm from E_1.

We now need the following sublemma:

Lemma 5.7.9. *Suppose $w = p^n u + p^m v\sqrt{p}$ with $n > m$, $u, v \neq 0$. Then w can't be* *conjugated into* $E_1^* G_{x,r/2}$.

Proof. Recall as usual, $det(E_1^* G_{x,r/2}) \subset N_{E_1/F}(E_1^*)$. If $w' = p^k s + p^l t\sqrt{\Delta p} \in E_1^*$, then $N_{E_1/F}(w') = p^{2k}s^2 - p^{2l+1}t^2\Delta$. So we want to show that $det(w)$ can't be of the form $N_{E_1/F}(w')$ for any $w' \in E_1^*$. Thus, we want to investigate whether we can have an equality of the form $p^{2n}u^2 - p^{2m+1}v^2 = p^{2k}s^2 - p^{2l+1}t^2\Delta$. Since $n > m$, this equality reduces to $p^{2m+1}(-v^2 + p^{2n-2m-1}u^2) = p^{2k}s^2 - p^{2l+1}t^2\Delta$.

First suppose $k \leq l$. Then

$$p^{2m+1}(-v^2 + p^{2n-2m-1}u^2) = p^{2k}(s^2 - p^{2l-2k+1}t^2\Delta)$$

If $s \neq 0$, this equality is clearly impossible since both elements inside the parenthesis are units. Suppose $s = 0$. Then we the equation above reduces to

$$p^{2m+1}(-v^2 + p^{2n-2m-1}u^2) = -p^{2l+1}t^2\Delta$$

This forces $m = l$, and then we get

$$-v^2 + p^{2n-2m-1}u^2 = -t^2\Delta$$

If v, t, Δ have power series expansions $v = v_0 + v_1 p + ...$, $t = t_0 + t_1 p + ...$, $\Delta = \Delta_0 + \Delta_1 p + ...$, then comparing leading coefficients of the equation we get

$$-v_0^2 = -t_0^2\Delta_0$$

Thus, $\Delta_0 = (\frac{v_0}{t_0})^2$, which is a contradiction since Δ is a non-square unit so its leading term is a non-square mod p.

Now suppose $k > l$. So

$$p^{2m+1}(-v^2 + p^{2n-2m-1}u^2) = p^{2l+1}(-t^2\Delta + p^{2k-2l-1}s^2)$$

Now suppose $t \neq 0$. The since both elements within the parenthesis are units, we get $m = l$. So

$$-v^2 + p^{2n-2m-1}u^2 = -t^2\Delta + p^{2k-2l-1}s^2$$

Comparing leading coefficients of power series, with the obvious notation, we get

$$-v_0^2 = -t_0^2\Delta_0$$

so that $\Delta_0 = (\frac{v_0}{t_0})^2$, again a contradiction since Δ is a non-square. Now suppose $t = 0$. Then we get

$$p^{2m+1}(-v^2 + p^{2n-2m-1}u^2) = p^{2k}s^2$$

This is clearly a contradiction since the element within the parenthesis is a unit, and s^2 is also a unit, so the valuations of both sides are different. This proves Lemma (5.7.9). □

Now that we've shown that the elements $w \in E^*$ such that $n(w) = 0$ can't be conjugated into $E_1^* G_{x,r/2}$, we can finish the proof of Proposition (5.7.8) by using the same proof from subcases (a) and (b) from Proposition (5.7.2). □

We have now finished the proof of Theorem (5.7.1). Summing up, we have altogether shown that if $(E/F, \chi)$ is a regular pair such that χ has positive level, then there is a unique positive depth supercuspidal representation, $\pi_{\chi\Delta_\chi}$, whose character, on the range $\{z \in T(F)^{reg} : 0 \leq n(z) \leq r/2\}$, agrees with $F(\tilde{\chi})$. There is one minor point here to resolve. Is there possibly a depth zero supercuspidal representation whose character, on the range $\{z \in T(F)^{reg} : 0 \leq n(z) \leq r/2\}$, also equals $F(\tilde{\chi})$? We will prove in the next chapter that if $(E_1/F, \chi_1)$ is a regular pair corresponding to a depth zero supercuspidal representation π, then its character formula, on the range $\{z \in T(F)^{reg} : 0 \leq n(z) \leq r/2\}$, is

$$F(\tilde{\chi}_1)(w) = -\epsilon(\Delta^+ \frac{deg(\pi)}{deg(\sigma)} \left(\frac{\chi_1(w) + (-1, \Delta)\chi_1(\overline{w})}{\tau_o(\frac{w - \overline{w}}{2\delta})} \right), \quad w \in E_1^* \setminus F^*(1 + \mathfrak{p}_{E_1})$$

Then, the same arguments as in Theorems (5.6.1) and (5.7.1) show that the character of π cannot equal $F(\tilde{\chi})$, on the range $\{z \in T(F)^{reg} : 0 \leq n(z) \leq r/2\}$, unless $\pi \cong \pi(\tilde{\chi})$.

Therefore, combining Theorems (5.6.1), (5.7.1), and (5.5.11), we obtain the following result.

Theorem 5.7.10. *The assignment*

$$\left\{ \text{irreducible } \phi : W_F \to GL(2, \mathbb{C}) \right\} \mapsto \tilde{\chi} \in \widehat{T(F)}_{\tau \circ \rho} \mapsto \pi(\tilde{\chi})$$

from Section (5.2) is the Local Langlands correspondence for positive depth supercuspidal representations of $GL(2, F)$, where $\pi(\tilde{\chi})$ is the unique supercuspidal representation whose character, on the range $\{z \in T(F)^{reg} : 0 \leq n(z) \leq r/2\}$, is $F(\tilde{\chi})$.

5.8 Calculation of various needed constants

In this section we calculate various constants needed throughout the previous proofs. We calculate $\gamma(\Delta, \psi)$ where $\Delta \in \mathfrak{o}_F^*$ is not a square, and we calculate

$\lambda_{E/F}(\psi)$. Let $\Delta \in F^*$ be a non-square unit, and let ψ be an additive character of F. Let $\ell(\psi)$ denote the level of ψ.

Lemma 5.8.1. $\gamma_F(\Delta, \psi) = (-1)^{\ell(\psi)}$.

Proof. The notation in [17] is $\gamma_F(\Delta, \psi)$ instead of $\gamma(\Delta, \psi)$. Let $a \in F^*$.

By Definition (A.0.7), $\gamma_F(a, \psi) = \frac{\gamma_F(a\psi)}{\gamma_F(\psi)}$. Lemma (A.0.12) implies that

$$\frac{\gamma_F(a\psi)}{\gamma_F(\psi)} = \frac{\gamma_{\overline{F}}(\overline{a\psi})^{\ell(a\psi)}}{\gamma_{\overline{F}}(\overline{\psi})^{\ell(\psi)}}$$

where $\ell(\psi)$ is the level of ψ.

Now,

$$\gamma_{\overline{F}}(\overline{a\psi})^{\ell(a\psi)} = \gamma_{\overline{F}}(\overline{a}, \overline{\psi})^{\ell(a\psi)} \gamma_{\overline{F}}(\overline{\psi})^{\ell(a\psi)}$$

since again, by defintion of γ_F, we have that $\gamma_F(a, \psi) = \frac{\gamma_F(a\psi)}{\gamma_F(\psi)}$ (this is true for F a local field or finite field with characteristic not 2. cf Definition (A.0.7)). We therefore have

$$\frac{\gamma_{\overline{F}}(\overline{a\psi})^{\ell(a\psi)}}{\gamma_{\overline{F}}(\overline{\psi})^{\ell(\psi)}} = \frac{\gamma_{\overline{F}}(\overline{a}, \overline{\psi})^{\ell(a\psi)} \gamma_{\overline{F}}(\overline{\psi})^{\ell(a\psi)}}{\gamma_{\overline{F}}(\overline{\psi})^{\ell(\psi)}}$$

Now, $a \in F^*$ and ψ is a character of F. Let $a = p^m u$. Note that $\ell(a\psi) = \ell(\psi)$ if m is even and $\ell(a\psi) = \ell(\psi) + 1$ if m is odd.

Moreover,

$$\gamma_{\overline{F}}(\overline{a}, \overline{\psi}) = \left(\frac{\overline{a}}{\overline{F}}\right)$$

by Lemma (A.0.11), where $\left(\frac{\overline{a}}{\overline{F}}\right)$ denotes the Legendre symbol.

So altogether, we get $\gamma_F(a, \psi) = \left(\frac{\overline{a}}{\overline{F}}\right)^{\ell(a\psi)}$ if m is even (including $m = 0$) and $\gamma_F(a, \psi) = \left(\frac{\overline{a}}{\overline{F}}\right)^{\ell(a\psi)} \gamma_{\overline{F}}(\overline{\psi})$ if m is odd.

For our setting, $a = \Delta$ is a non-square unit. Thus,

$$\gamma_F(\Delta, \psi) = \left(\frac{\overline{\Delta}}{\overline{\overline{F}}}\right)^{\ell(\Delta\psi)} = (-1)^{\ell(\psi)}$$

□

Lemma 5.8.2. $\lambda_{E/F}(\psi) = \gamma_F(\Delta, \psi)(-1, \Delta)_F$

Proof. Recall that $\lambda_{E/F}(\psi)$ is the Langlands constant (cf [5, pages 216, 217, 241]), where ψ is an additive character of F. By [5, page 240-241], $\lambda_{E/F}$ is defined to be the Weil index of

$$q : E^* \to \mathbb{C}^*$$

where $q(z) := \psi(N(z))$. Therefore, the quadratic form in question is $N : E^* \to F^*$. The associated symmetric bilinear form associated is

$$(z, w)_q := \frac{Tr(z\overline{w})}{2}$$

We wish to therefore calculate the Weil index $\gamma(\psi \circ N)$, which will be the Langlands constant $\lambda_{E/F}(\psi)$.

Well, a basis for E/F is $1, \delta$, where $E = F(\delta), \delta = \sqrt{\Delta}$. Then, $(1, 1)_q = 1$, $(1, \delta)_q = (\delta, 1)_q = 0$, and $(\delta, \delta)_q = -\Delta$. Thus, the matrix of the quadratic form is

$$\begin{pmatrix} 1 & 0 \\ 0 & -\Delta \end{pmatrix}$$

Then, by Lemma (A.0.9),

$$\gamma(\psi \circ N) = h_F(N)\gamma_F(\psi)^n \gamma_F(det(N), \psi)$$

In our setting, $n = 2$, and by Lemmas (A.0.8) and (A.0.9)

$$\gamma(\psi \circ N) = (1, -\Delta)_F \gamma_F(\psi)^2 \gamma_F(-\Delta, \psi) = \gamma_F(-1, \psi)^{-1} \gamma_F(-\Delta, \psi) =$$

$$\gamma_F(-1, \psi) \gamma_F(-1, \psi)^{-2} \gamma_F(-\Delta, \psi) = \gamma_F(-1, \psi)(-1, -1)_F \gamma_F(-\Delta, \psi) =$$

$$\gamma_F(\Delta, \psi)(-1, -\Delta)_F = \gamma_F(\Delta, \psi)(-1, \Delta)_F$$

$\square$

Chapter 6

Our constructions in the depth zero case for $GL(2, F)$

6.1 On the proof that our conjectural character formulas agree with depth zero supercuspidal characters

In the following two sections, we prove Theorems (5.2.1) and (5.2.3) for the case of depth zero supercuspidal representations of $GL(2, F)$ in an analogous way as in the positive depth case.

Let us recall from the previous chapter that the proposed character formula

$$F(\tilde{\chi})(z) := \epsilon(\tilde{\chi}, \Delta^+, \tau) \frac{\sum_{s \in W} \epsilon(s) \tilde{\chi}({}^s w)}{\tau(\Delta^0(z, \Delta^+))(\tau \circ \rho)(w)}, \ w \in T(F)^{reg}$$

simplifies to

$$F(\tilde{\chi})(w) = \epsilon(\tilde{\chi}, \Delta^+, \tau) \left(\frac{\chi(w) + (-1, \Delta)\chi(\overline{w})}{\tau_o(w - \overline{w})|D(w)|^{1/2}} \right), \quad w \in T(F)^{reg}$$

where W denotes the relative Weyl group $W(G(F), T(F))$. For depth zero representations we define $\epsilon(\tilde{\chi}, \Delta^+, \tau) := -\frac{deg(\pi)\tau(2\delta)}{deg(\sigma)} \epsilon(\Delta^+)$, where $\epsilon(\Delta^+)$ is as in Section (5.3) and $deg(\pi), deg(\sigma)$ are as in Theorem (6.1.1). Therefore, the formula simplifies to

$$F(\tilde{\chi})(w) = -\frac{deg(\pi)}{deg(\sigma)} \left(\frac{\chi(w) + (-1, \Delta)\chi(\overline{w})}{\tau_o(\frac{w - \overline{w}}{2\delta})|D(w)|^{1/2}} \right) \quad w \in T(F)^{reg}$$

Let us recall the following theorem from [9].

Theorem 6.1.1. [1] *[9, Theorem 5.4.1]*

Suppose $\gamma \in F^* K_0^{reg}$.

$$\frac{\theta_\pi(\gamma)}{deg(\pi)} = \begin{cases} \chi_\pi(z)\frac{\chi_\sigma(\gamma)}{deg(\sigma)} & \text{if } \gamma = zw \text{ is unramified elliptic and } \gamma \\ & \text{is not in } F^*K_1, z \in Z, w \in K_0 \\ \chi_\pi(z) L.C.E. & \text{if } \gamma = z(1 + {}^g X) \text{ with } X \in \mathfrak{b}_1, g \in G, \text{ and } z \in Z \\ 0 & \text{otherwise} \end{cases}$$

Here, χ_π is the central character of the representation π.

We will compare our conjectured character formula to the supercuspidal characters of Theorem (6.1.1) on the range $\{z \in T(F)^{reg} : 0 \leq n(z) \leq r/2\}$. Note that for for depth zero representations, $r = 0$. We will show that this means that we are only interested in character values on the elements $\gamma = zw$ where $z \in Z, w \in K_0$ where γ is unramified elliptic and γ is not in F^*K_1.

We now compute the supercuspidal characters of Theorem (6.1.1). They are written in terms of regular pairs, so we recall a few notions regarding the regular pairs for depth zero supercuspidal representations. We will then compare the formula of Theorem (6.1.1) with our proposed character formula. Let $(E/F, \chi)$ be a regular pair corresponding to a depth zero supercuspidal representation via Theorem (4.3.2). This means that E/F is unramified and χ has level zero, so $\chi|_{U_E}$ gives rise to a character θ of the multiplicative group of the residue field $\mathbb{F}_{q^2}$ of E. Note that when E/F is unramified, $(E/F, \chi)$ is regular if and only if $(E/F, \chi)$ is admissible.

[1] Notice that this theorem is slightly different than the one from [9]. It is because there are a few typos in [9].

Let $\mathbb{G} := GL(2, \overline{\mathbb{F}_q})$. Let $\mathbb{T}$ be the maximal torus of $\mathbb{G}$ defined over $\mathbb{F}_q$ such that $\mathbb{T}^\Phi = \mathbb{F}_{q^2}^*$ is the elliptic torus in $GL(2, \mathbb{F}_q)$. Then, by Deligne-Lusztig theory, the pair $(\mathbb{T}, \theta)$ yields a generalized character $R_{\mathbb{T},\theta}$ of $\mathbb{G}(\mathbb{F}_q) = GL(2, \mathbb{F}_q)$. We need the values of this character on the semisimple elements.

Proposition 6.1.2. *If $s \in \mathbb{G}^\Phi$ is semisimple, then*

$$R_{\mathbb{T},\theta}(s) = \frac{\epsilon_{\mathbb{T}}\epsilon_{C^0(s)}}{|\mathbb{T}^\Phi||C^0(s)^\Phi|_p} \sum_{g \in \mathbb{G}^\Phi : g^{-1}sg \in \mathbb{T}^\Phi} \theta(g^{-1}sg)$$

Proof. [8, Proposition 7.5.3] □

We will define the various terms in the formula throughout the section. Let us now calculate the values of $R_{\mathbb{T},\theta}$ on the elliptic torus. Let $\mathbb{T}_s$ denote the split torus in $\mathbb{G}$. Then $\mathbb{T}$ is obtained from $\mathbb{T}_s$ by twisting by the canonical generator of the Weyl group, the permutation $w = (12)$ (cf [10, Definition 3.24]). That is,

$$t \in \mathbb{T}^\Phi \Longleftrightarrow g^{-1}tg \in \mathbb{T}_s^{w\Phi}$$

where $g^{-1}\Phi(g) \in N_{\mathbb{G}}(\mathbb{T}_s)$ maps to w in $W(\mathbb{G}, \mathbb{T}_s)$. Then, $\mathbb{T}_s^{w\Phi}$ is the group of fixed points of the composite morphism

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto \begin{pmatrix} a^q & 0 \\ 0 & b^q \end{pmatrix} \mapsto \begin{pmatrix} b^q & 0 \\ 0 & a^q \end{pmatrix}$$

This says that

$$a = b^q, b = a^q$$

which says that $a = a^{q^2}$ so that $a \in \mathbb{F}_{q^2}^*$. Therefore, the group of fixed points is

$$\left(\begin{array}{cc} a & 0 \\ 0 & a^q \end{array} \right) : a \in \mathbb{F}_{q^2}^*$$

Proposition 6.1.3.

$$R_{\mathbb{T},\theta}(s) = -\sum_{i=0}^{1} \theta(\upsilon^i(s))$$

for all regular semisimple s in $\mathbb{T}^\Phi$*, where* υ *is the generator of* $Gal(\mathbb{F}_{q^2}/\mathbb{F}_q)$

Proof. We first need a lemma:

Lemma 6.1.4. *Suppose* S *is a maximal torus in a connected reductive group* G *and* $s \in S$ *is regular. Suppose* $g \in G$ *satisfies* $g^{-1}sg \in S$*. Then* $g \in N_G(S)$*.*

Proof. Let $C^0(s)$ be the connected centralizer of s in G. Let $h \in C^0(s)$. Then defining $s' := g^{-1}sg$, we get that $hs' = s'h$, since $C^0(s) = S$. Therefore, $g \in N_G(C^0(s)) = N_G(S)$.

□

Let $s \in \mathbb{T}^\Phi$. Since we are assuming that s is regular semisimple, we have by the above Lemma that

$$\sum_{g \in \mathbb{G}^\Phi \ : \ g^{-1}sg \in \mathbb{T}^\Phi} \theta(g^{-1}sg) = \sum_{g \in N_{\mathbb{G}^\Phi}(\mathbb{T}^\Phi)} \theta(g^{-1}sg) =$$
$$|\mathbb{T}^\Phi| \sum_{w \in N_{\mathbb{G}^\Phi}(\mathbb{T}^\Phi)/\mathbb{T}^\Phi} \theta({}^w s)$$

Therefore, the values of $R_{\mathbb{T},\theta}$ on regular semisimple elements simplify to

$$R_{\mathbb{T},\theta}(s) = \frac{\epsilon_\mathbb{T}\epsilon_{C^0(s)}}{|\mathbb{T}^\Phi||C^0(s)^\Phi|_p}|\mathbb{T}^\Phi| \sum_{w \in N_{\mathbb{G}^\Phi}(\mathbb{T}^\Phi)/\mathbb{T}^\Phi} \theta({}^w s) = \frac{\epsilon_\mathbb{T}\epsilon_{C^0(s)}}{|C^0(s)^\Phi|_p} \sum_{i=0}^{1} \theta(\upsilon^i(s))$$

since the relative Weyl group is $W(\mathbb{G}(\mathbb{F}_q), \mathbb{T}(\mathbb{F}_q)) = Aut(\mathbb{F}_{q^2}/\mathbb{F}_q)$ (see Theorem (A.0.3)). It remains to calculate the constants in front.

Now, since $s \in \mathbb{T}^\Phi$ is regular semisimple, then $C^0(s) = \mathbb{T}$. Therefore, $|C^0(s)^\Phi| = q^2 - 1$, so $|C^0(s)^\Phi|_p = 1$, where $| \ |_p$ denotes the p-part of $| \ |$. Now, ϵ_H is defined to be $(-1)^{\mathbb{F}_q-rank\ of\ H}$, for any algebraic group H (see [10, page 66]). Therefore, $\epsilon_{\mathbb{T}} = -1$ and $\epsilon_{C^0(s)} = 1$. Therefore, the values of $R_{\mathbb{T},\theta}$ on regular semisimple elements of the elliptic torus are

$$R_{\mathbb{T},\theta}(s) = \epsilon_{\mathbb{T}}\epsilon_{C^0(s)} \sum_{i=0}^{1} \theta(v^i(s)) = -(\theta(s) + \theta(\overline{s}))$$

□

Our character formula is defined on the unramified elliptic torus E^*. We wish to show that our character formula agrees with a depth zero supercuspidal character on the sets where they are both defined. i.e. the set $(F^*K_0 \setminus F^*K_1) \bigcap E^*$.

We need the following Lemma:

Lemma 6.1.5. *$(F^*K_0 \setminus F^*K_1) \bigcap E^* = F^*A = E^* \setminus F^*(1 + \mathfrak{p}_E = \{z \in T(F)^{reg} : n(z) = 0\}$, where $A := \{p^n u + v\delta : n \geq 0\}$*

Proof. First note that $\mathfrak{o}_E^*$ consists of the union of the following three sets:

$A := \{p^n u + v\delta : n \geq 0\}$

$B := \{u + p^m v\delta : m > 0, u \ mod \ p \neq 1\}$

$C := \{u + p^m v\delta : m > 0, u \ mod \ p = 1\}$

Note that $(F^*K_0 \setminus F^*K_1) \bigcap E^*$ is equal to the set $(F^*K_0 \bigcap E^*) \setminus (F^*K_1 \bigcap E^*)$. Then we claim that the set $F^*K_0 \bigcap E^* \setminus F^*K_1 \bigcap E^*$ is precisely the set F^*A: It's clear that $F^*K_0 \bigcap E^* = F^*(K_0 \bigcap E^*)$. Moreover, one can check that $K_0 \bigcap E^* = \mathfrak{o}_E^*$. Therefore, $F^*K_0 \bigcap E^* = F^*\mathfrak{o}_E^* = E^*$. Likewise, one can see that $K_1 \bigcap E^* = 1 + \mathfrak{p}_E$ (cf [9, Lemma 3.2.1 (2)]), and therefore $F^*K_1 \bigcap E^* = F^*(K_1 \bigcap E^*) = F^*(1 + \mathfrak{p}_E)$. Now, since $\mathfrak{o}_E^* = A \bigcup B \bigcup C$, it's clear that $F^*\mathfrak{o}_E^* = F^*A \bigcup F^*B \bigcup F^*C$. Moreover, while it is true that C is strictly contained in B, it is clear that $F^*C = F^*B$. Moreover, we have that $K_1 \bigcap E^* = C$. Therefore, $F^*K_1 = F^*C$. Now, the following set identities are clear :

$$F^*K_0 \setminus F^*K_1 = F^*(A \bigcup B \bigcup C) \setminus F^*C = F^*A \bigcup F^*B \bigcup F^*C \setminus F^*C =$$

$$(F^*A \setminus (F^*A \bigcap F^*C)) \bigcup (F^*B \setminus (F^*B \bigcap F^*C))$$

We need the following sublemma:

Lemma 6.1.6. $F^*A \bigcap F^*C = \emptyset$

Proof. Suppose $F^*A \bigcap F^*C \neq \emptyset$, so that there is an $x, y, u, v, u', v' \in F^*$ such that $y(u + p^m v\delta) = x(p^n u' + v'\delta)$ where $u \ mod \ p = 1$. Then this says $x^{-1}y(u + p^m v\delta) = p^n u' + v'\delta$, and setting $x' := x^{-1}y$, we get that there is an $x' \in F^*$ such that $x'(u + p^m v\delta) = p^n u' + v'\delta$. But since $m > 0$, and $n \geq 0$, this is clearly impossible, because x' would have to have valuation $-m$ in order for this equality to work out. But then this would be a contradiction. $\square$

Back to the proof of Lemma 6.1.5: Note that we also have that $F^*B = F^*C$, so that $F^*B \bigcap F^*C = F^*B$. Thus, we finally have that $(F^*A \setminus (F^*A \bigcap F^*C)) \bigcup (F^*B \setminus (F^*B \bigcap F^*C)) = F^*A$. Therefore, indeed we have that $F^*K_0 \setminus F^*K_1 = F^*A$, and therefore $(F^*K_0 \setminus F^*K_1) \bigcap E^* = F^*A$ $\square$

We now can prove

Theorem 6.1.7. *$F(\tilde{\chi})$ agrees with the supercuspidal character of $\pi_{\chi\Delta_\chi}$ on $F^*A = \{z \in T(F)^{reg} : n(z) = 0\}$.*

Proof. Recall that Δ_χ is the unique quadratic unramified character of E^*. Therefore, we need to show that

$$-\frac{deg(\pi)}{deg(\sigma)}\left(\frac{\chi(w) + (-1,\Delta)\chi(\overline{w})}{\tau_o(\frac{w-\overline{w}}{2\delta})|D(w)|^{1/2}}\right) = -\frac{deg(\pi)}{deg(\sigma)}(\chi(w)\Delta_\chi(w)+\chi(\overline{w})\Delta_\chi(\overline{w}))\ \forall w \in F^*A$$

We first show that they are equal on the set A. Let $w \in A$, so $w = p^n u + v\delta, n \geq 0$. Then $D(w) = \frac{-(w-\overline{w})^2}{N(w)}$. Then $|D(w)| = |v^2\Delta|_F|N(w)|_F^{-1} = |w|_E^{-2} = 1$. Moreover, $\tau_o(\frac{w-\overline{w}}{2\delta}) = \tau_o(v) = 1$. Therefore, we are reduced to showing that

$$\chi(w) + \chi(\overline{w}) = \chi(w)\Delta_\chi(w) + \chi(\overline{w})\Delta_\chi(\overline{w})\ w \in A$$

But this is true since $\Delta_\chi(w) = 1\ \forall w \in A$ since Δ_χ is unramified. Therefore, both sides agree on A. Finally, since $\tau_o(x) = \Delta_\chi(x)\ \forall x \in F^*$, we have that both sides agree on F^*A. □

Note that in constructing the character formula $F(\tilde{\chi})$, we have chosen Δ^+ to be the standard positive root. If we had made the other choice of Δ^+, the denominator in our character formula would include the term $\tau_o(\overline{w} - w)$ instead of $\tau_o(w - \overline{w})$. However, because our definition of $\epsilon(\tilde{\chi}, \Delta^+, \tau)$ includes the term $\epsilon(\Delta^+)$, our overall character formula $F(\tilde{\chi})$ remains the same regardless of the choice of positive root.

6.2 On whether there are two character formulas coming from the same Cartan

In this section, we show that if the distribution characters of two depth zero supercuspidal representations, both coming from the unramified Cartan, agree on the $n(w) = 0$ range, then the supercuspidal representations are isomorphic.

Theorem 6.2.1. *Suppose $(E/F, \chi_1)$, $(E/F, \chi_2)$ are admissible pairs such that $F(\tilde{\chi}_1)(w) = F(\tilde{\chi}_2)(w)$ on the set $E^* \setminus F^*(1+\mathfrak{p}_E)$. Then $\chi_1 = \chi_2^{\upsilon}$ for some $\upsilon \in Aut(E/F)$.*

Proof. Assume $F(\tilde{\chi}_1)(w) = F(\tilde{\chi}_2)(w)$ on the set $E^* \setminus F^*(1+\mathfrak{p}_E)$. Then this implies (by [21, page 16]) that $\chi_1 = \chi_2^{\upsilon}$ for some $\upsilon \in Aut(E/F)$. □

Summing up, we have altogether shown that if $(E/F, \chi)$ is a regular pair such that χ has level zero, then there is a unique depth zero supercuspidal representation, $\pi_{\chi\Delta_\chi}$, whose character, on the range $\{z \in T(F)^{reg} : 0 \leq n(z) \leq r/2\} = \{z \in T(F)^{reg} : n(z) = 0\}$, agrees with $F(\tilde{\chi})$. There is one minor point here to resolve. Is there possibly a positive depth supercuspidal representation whose character, on the range $\{z \in T(F)^{reg} : n(z) = 0\}$, also equals $F(\tilde{\chi})$? Suppose $(E_1/F, \chi_1)$ is a regular pair corresponding to a positive depth supercuspidal representation π via Theorem (4.3.2). Then, the same arguments as in Theorems (5.6.1) and (5.7.1) show that the character of π cannot equal $F(\tilde{\chi})$, on the range $\{z \in T(F)^{reg} : n(z) = 0\}$, unless $\pi \cong \pi(\tilde{\chi})$.

Therefore, combining Theorems (6.2.1) and (6.1.7), we obtain the following result.

Theorem 6.2.2. *The assignment*

$$\left\{ \textit{irreducible } \phi : W_F \to GL(2,\mathbb{C}) \right\} \mapsto \tilde{\chi} \in \widehat{T(F)}_{\tau\circ\rho} \mapsto \pi(\tilde{\chi})$$

from Section (5.2) is the Local Langlands correspondence for depth zero supercuspidal representations of $GL(2,F)$*, where* $\pi(\tilde{\chi})$ *is the unique supercuspidal representation whose character, on the range* $\{z \in T(F)^{reg} : 0 \leq n(z) \leq r/2\} = \{z \in T(F)^{reg} : n(z) = 0\}$*, is* $F(\tilde{\chi})$*.*

Chapter 7

Existing Description of Local Langlands Correspondence for $GL(\ell, F)$, ℓ an odd prime

In this chapter, we describe the statement of the local Langlands correspondence for $GL(\ell, F)$ as explained in [14].

7.1 Admissible Pairs

Let E/F be a tamely ramified degree ℓ extension and χ a character of E^*, where ℓ is an odd prime. Recall that N denotes the norm map from E to F.

Definition 7.1.1. The pair $(E/F, \chi)$ is called an *admissible pair* if

(i) χ does not factor through N and

(ii) If $\chi|_{1+\mathfrak{p}_E}$ factors through N, then E/F is unramified.

Admissible pairs $(E/F, \chi)$, $(E'/F, \chi')$ are said to be F-isomorphic if there exists an F-isomorphism $j : E \to E'$ such that $\chi(w) = \chi'(j(w))\ \forall w \in E^*$. If $(E/F, \chi)$ is an admissible pair, then there is a Howe factorization $\chi = \chi'\phi_E$, where $\phi_E = \phi \circ N_{E/F}$, for some $\phi \in \widehat{F^*}$ and where $\chi' \in \widehat{E^*}$ is of minimal conductor. We write $\mathbb{P}_\ell(F)$ for the set of F-isomorphism classes of admissible pairs. Note that if $(E/F, \chi)$ is an admissible pair and if ϕ is a character of F^*, then the pair $(E/F, \chi \otimes \phi_E)$ is an admissible pair where $\phi_E = \phi \circ N$.

7.2 Depth zero supercuspidal representations of $GL(\ell, F)$

In this section we recall the parameterization of the depth zero supercuspidal representions via a subclass of admissible pairs.

Let $(E/F, \chi)$ be an admissible pair where χ has level 0. By definition of admissible pair, this implies that E/F is unramified. Write $k_E = \mathfrak{o}_E/\mathfrak{p}_E$. Then k_E/k_F is a degree ℓ extension, where $k_F = \mathfrak{o}_F/\mathfrak{p}_F$. Moreover, since $\chi|_{1+\mathfrak{p}_E} = 1$, $\chi|_{U_E}$ is the inflation of a character, call it again χ of k_E^*. By the theory of finite groups of Lie type, the character χ then gives rise to an irreducible cuspidal representation λ' of $GL(\ell, k_F)$. Let λ be the inflation of λ' to $GL(\ell, \mathfrak{o}_F)$. We may extend λ to a representation Λ of $K := F^*GL(\ell, \mathfrak{o}_F)$ by setting $\Lambda|_{F^*} = \chi|_{F^*}$, and induce the resulting representation to all of G. Set

$$\pi_\chi = cInd_K^G \Lambda$$

where $cInd$ denotes compact induction.

Then these are all the depth zero representations of $GL(\ell, F)$. In particular, if $\mathbb{P}_\ell(F)_0$ denotes the set of admissible pairs of level zero, and $\mathbb{A}_\ell^0(F)_0$ denotes the set of equivalence classes of depth zero supercuspidal representations of $GL(\ell, F)$, then we have

Proposition 7.2.1. *The map $(E/F, \chi) \mapsto \pi_\chi$ induces a bijection*

$$\mathbb{P}_\ell(F)_0 \to \mathbb{A}_\ell^0(F)_0$$

Furthermore, if $(E/F, \chi) \in \mathbb{P}_\ell(F)_0$, then:

(i) if ϕ is a character of F^* of level zero, then $\pi_{\chi\phi_E} = \phi\pi_\chi$

(ii) if $\pi = \pi_\chi$*, then* $\omega_\pi = \chi|_{F^*}$

Proof. See [14]. $\square$

7.3 Positive depth supercuspidal representations of $GL(\ell, F)$

In this section we recall the parametrization of the positive depth supercuspidal represention via a subclass of admissible pairs, following [14]. Let $\mathbb{A}^0_\ell(F)$ denote the set of all irreducible supercuspidal representations of $GL(\ell, F)$.

Theorem 7.3.1. *Suppose the residual characteristic of* F *is not equal to* ℓ*. There is a map* $(E/F, \chi) \mapsto \pi_\chi$ *that induces a bijection*

$$\mathbb{P}_\ell(F) \to \mathbb{A}^0_\ell(F)$$

If $(E/F, \chi) \in \mathbb{P}_\ell(F)$*, then:*

(i) if χ *has level* $l(\chi)$*, then* $l(\pi_\chi) = l(\chi)/e(E|F)$*.*

(ii) $\omega_{\pi_\chi} = \chi|_{F^*}$

(iii) the pair (E/F, $\chi^\vee$*) is admissible and* $\pi_{\chi^\vee} = \pi_\chi^\vee$

(iv) if ϕ *is a character of* F^**, then* $\pi_{\chi\phi_E} = \phi\pi_\chi$*.*

Proof. See [14] $\square$

7.4 Weil parameters

In this section, we recall the classical Langlands parameterization for $GL(\ell, F)$. Most of what we say here is taken straight from [14].

Let $\mathbb{G}_\ell(F)$ denote the set of equivalence classes of ℓ-dimensional, semisimple, Weil-Deligne representations of the Weil group W_F. Let $\mathbb{A}_\ell(F)$ denote the set of equivalence classes of irreducible, smooth representations of $GL(\ell, F)$.

We first state the local Langlands correspondence, and then roughly describe the elements behind the statement.

Theorem 7.4.1. *There is a unique map*

$$\pi : \mathbb{G}_\ell(F) \to \mathbb{A}_\ell(F)$$

such that

$$L(\chi\pi(\phi), s) = L(\chi \otimes \phi, s), \tag{7.1}$$

$$\epsilon(\chi\pi(\phi), s, \psi) = \epsilon(\chi \otimes \phi, s, \psi), \tag{7.2}$$

for all $\phi \in \mathbb{G}_\ell(F)$ and all characters χ of F^*. The map π is a bijection.

Proof. See [14]. □

The map π is the Langlands correspondence for $GL(\ell, F)$. We make some preliminary remarks. We have a decomposition

$$\mathbb{G}_\ell(F) = \mathbb{G}_\ell^1(F) \cup \mathbb{G}_\ell^0(F)$$

where $\mathbb{G}_\ell^0(F)$ is the set of equivalence classes of irreducible smooth representations of W_F of dimension ℓ, and $\mathbb{G}_\ell^1(F)$ denotes the classes of Weil-Deligne representations $(\phi, V, \mathfrak{n}) \in \mathbb{G}_\ell(F)$ for which the representation ϕ of W_F is reducible. Likewise, we write

$$\mathbb{A}_\ell(F) = \mathbb{A}_\ell^1(F) \cup \mathbb{A}_\ell^0(F)$$

where $\mathbb{A}_\ell^0(F)$ denotes the representations $\pi \in \mathbb{A}_\ell(F)$ that are supercuspidal, and $\mathbb{A}_\ell^1(F)$ denotes the representations $\pi \in \mathbb{A}_\ell(F)$ that are not supercuspidal.

Then it is a fact that the Langlands correspondence π must map $\mathbb{G}_\ell^1(F)$ to $\mathbb{A}_\ell^1(F)$ and $\mathbb{G}_\ell^0(F)$ to $\mathbb{A}_\ell^0(F)$. We are only concerned with the map on $\mathbb{G}_\ell^0(F)$, which is the heart of the matter. That is:

Theorem 7.4.2. *There is a unique map*

$$\pi : \mathbb{G}_\ell^0(F) \to \mathbb{A}_\ell^0(F)$$

with the property

$$\epsilon(\chi \otimes \phi, s, \psi) = \epsilon(\chi\pi(\phi), s, \psi) \tag{7.3}$$

for all $\phi \in \mathbb{G}_\ell^0(F)$, all characters χ of F^*. Moreover, the map π is a bijection.

Proof. See [14]. □

A very important property of the map π is:

Proposition 7.4.3. *Let π be a map satisfying (7.3) of Theorem (7.4.2). Then:*

(i) If $\phi \in \mathbb{G}_\ell^0(F)$ and $\pi = \pi(\phi)$, then $\omega_\pi = det(\phi)$.

(ii) The map π satisfies (7.3).

Proof. See [14]. □

We next turn to the question of parameterizing representations of W_F by admissible pairs. We have already parameterized the supercuspidal representations of

$GL(\ell, F)$ by admissible pairs. We assume throughout that the residual characteristic of F is not equal to ℓ.

Let $\mathbb{P}_\ell(F)$ denote again the set of admissible pairs. Recall that there is a local Artin reciprocity isomorphism given by $W_E^{ab} \cong E^*$. Then, if $(E/F, \xi) \in \mathbb{P}_\ell(F)$, ξ gives rise to a character of W_E^{ab}, which we can pullback to a character, also denoted ξ, of W_E. We can then form the induced representation $\phi_\xi = Ind_{W_E}^{W_F}\xi$ of W_F. We sometimes denote this representation by $Ind_{E/F}\xi$.

Theorem 7.4.4. *Suppose the residual characteristic of F is not equal to ℓ. If $(E/F, \xi)$ is an admissible pair, the representation ϕ_ξ of W_F is irreducible. The map $(E/F, \xi) \mapsto \phi_\xi$ induces a bijection*

$$\mathbb{P}_\ell(F) \to \mathbb{G}_\ell^0(F)$$

Proof. See [14]. □

We therefore have canonical bijections

$$\mathbb{P}_\ell(F) \to \mathbb{A}_\ell^0(F), \qquad \mathbb{P}_\ell(F) \to \mathbb{G}_\ell^0(F) \tag{7.4}$$

$$(E/F, \xi) \mapsto \pi_\xi, \qquad (E/F, \xi) \mapsto \phi_\xi,$$

given by Theorem (7.4.4) and Theorem (7.3.1). Combining both of these bijections, we obtain a bijection

$$\mathbb{G}_\ell^0(F) \to \mathbb{A}_\ell^0(F),$$

$$\phi_\xi \mapsto \pi_\xi \tag{7.5}$$

However, this bijection is NOT the map π demanded in Theorem (7.4.2). The reason is as follows. If $(E/F, \xi) \in \mathbb{P}_\ell(F)$, then by [5, Proposition 29.2], representation ϕ_ξ has determinant $\delta_{E/F} \otimes \xi|_{F^*}$ where $\delta_{E/F} := det(Ind_{W_E}^{W_F}(1))$, whereas π_ξ has central character $\xi|_{F^*}$, contrary to the requirement of Proposition (7.4.3). To obtain the map π of Theorem (7.4.2), we must systematically modify the bijection (7.5), which we proceed to do now.

Let $(E/F, \xi) \in \mathbb{P}_\ell(F)$ be an admissible pair. We associate to this pair a character $\Delta = \Delta_\xi$ of E^* of level zero.

Definition 7.4.5. Let $(E/F, \xi)$ be an admissible pair in which $\delta_{E/F} = 1$. Define Δ_ξ to be the trivial character of E^*.

Definition 7.4.6. Let $(E/F, \xi)$ be an admissible pair in which $\delta_{E/F} \neq 1$. Define Δ_ξ to be the unique quadratic unramified character of E^*.

Lemma 7.4.7. *(i) If $(E/F, \xi)$ is an admissible pair, the pair $(E/F, \xi\Delta_\xi)$ is admissible and the isomorphism class depends only on that of $(E/F, \xi)$. The character Δ_ξ satisfies $\Delta_\xi^2 = 1$.*

(ii) The map

$$\mathbb{P}_\ell(F) \to \mathbb{P}_\ell(F)$$

$$(E/F, \xi) \mapsto (E/F, \xi\Delta_\xi)$$

is bijective.

Proof. See [14]. □

We can finally state the local Langlands correspondence for $GL(\ell, F)$, as described in [14].

Theorem 7.4.8. Tame Local Langlands Correspondence

Suppose the residual characteristic of F is not equal to ℓ.

(i) For $\phi \in \mathbb{G}_\ell^0(F)$, define $\pi(\phi) = \pi_{\xi\Delta_\xi}$ in the notation of (7.4) for any $(E/F, \xi) \in \mathbb{P}_\ell(F)$ such that $\phi \cong \phi_\xi$. The map

$$\pi : \mathbb{G}_\ell^0(F) \to \mathbb{A}_\ell^0(F)$$

is a bijection satisfying

$$\epsilon(\chi \otimes \phi, s, \psi) = \epsilon(\chi\pi(\phi), s, \psi),$$

for all characters χ of F^*.

(ii) The map π satisfies

$$\pi(\chi \otimes \phi) = \chi\pi(\phi) \quad \textit{and} \quad \pi(\phi^\vee) = \pi(\phi)^\vee,$$

for all ϕ and all characters χ of F^*.

Proof. See [14]. □

We make two concluding remarks: Because of the uniqueness properties, π is the local Langlands correspondence when the residual characteristic of F is not equal to ℓ. Moreover, it is important to note that the construction gives $\omega_{\pi(\phi)} = det(\phi)$.

Chapter 8

Our constructions in the positive depth case for $GL(\ell, F)$, ℓ an odd prime

8.1 Preliminaries

In this Chapter we first prove an analogue of Harish-Chandra's discrete series theorem (see Theorem (1.0.5)) for the positive depth supercuspidal representations of $GL(\ell, F)$, where ℓ is an odd prime.

Theorem 8.1.1. *Let $G(F) = GL(\ell, F)$, and let $T(F) = E^*$ be an elliptic torus in $GL(\ell, F)$, so $E = F(\sqrt[\ell]{\Delta})$ for some $\Delta \in F^*$. Let L be the unique unramified extension of F of degree $\ell - 1$. Let τ_o be any character of $(EL)^*$ whose restriction to L^* is $\aleph_{EL/L}$, where $\aleph_{EL/L}$ is the local class field theory character of L^* relative to EL/L. Let $\tau := \tau_o \mid \mid_{EL}$, where $\mid \mid_{EL}$ denotes the EL-adic absolute value. Let Δ^+ be a set of positive roots of G with respect to T. Let $\rho := \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$. Let $T(F)_{\tau \circ \rho}$ be the $\tau \circ \rho$ cover of $T(F)$ as in Definition (5.1.1). Let $\tilde{\chi}$ be a genuine character of $T(F)_{\tau \circ \rho}$ that is regular. Let $W = W(G(F), T(F))$ be the relative Weyl group of $G(F)$ with respect to $T(F)$. If $s \in W(G(F), T(F))$, let $\epsilon(s) := (-1, \Delta)^{l(s)(\ell+1)}$, where $(,)$ denotes the Hilbert symbol of F and $\ell(s)$ denotes the length of s. Let $T(F)^{reg}$ denote the regular elements of $T(F)$.*

Then there exists a unique constant $\epsilon(\tilde{\chi}, \Delta^+, \tau)$, depending only on $\tilde{\chi}, \Delta^+$, and

τ, and a unique supercuspidal representation of $GL(\ell, F)$ denoted $\pi(\tilde{\chi})$, such that

$$\theta_{\pi(\tilde{\chi})}(z) = \epsilon(\tilde{\chi}, \Delta^+, \tau)\frac{\sum_{s\in W} \epsilon(s)\tilde{\chi}({}^s w)}{\tau(\Delta^0(z, \Delta^+))(\tau \circ \rho)(w)}, \ \forall z \in T(F)^{reg} \ : 0 \leq n(z) \leq r/2$$

where $w \in T(F)_{\tau\circ\rho}$ is any element such that $\Pi(w) = z$ and r is the depth of $\pi(\tilde{\chi})$. Moreover, every supercuspidal character of $GL(\ell, F)$ is of this form.

We will soon define $\epsilon(\tilde{\chi}, \Delta^+, \tau)$ and the notion of regularity. We will also regularly use the fact (see Theorem (A.0.3)) that $W(G(F), T(F)) = Aut(E/F)$.

We note that all of our calculations in the next two chapters will assume that we have chosen the standard positive set of roots of $GL(\ell, \overline{F})$ with respect to the standard split maximal torus. Our main results, however, will be seen to be independent of any choice of positive roots.

Let us remark also that in the course of proving the above theorem, we must show that a supercuspidal representation of $GL(\ell, F)$ is determined by the character values on elements of the torus E^* on the range $\{z \in T(F)^{reg} : 0 \leq n(z) \leq r/2\}$. Actually, we will even show that a supercuspidal representation of $GL(\ell, F)$ is determined by the character values on elements $w \in E^*$ satisfying $n(w) = 0$, which is stronger.

Now let ϕ be a supercuspidal Weil parameter for $GL(\ell, F)$. We will show later in this section how to construct a regular genuine character, $\tilde{\chi}$, of $T(F)_{\tau\circ\rho}$, from ϕ. We will then prove the following theorem.

Theorem 8.1.2. *The assignment*

$$\phi \mapsto \tilde{\chi} \mapsto \pi(\tilde{\chi})$$

is the Local Langlands correspondence for $GL(\ell, F)$.

We now introduce a notion of regularity that we will need. Let E/F be a tamely ramified degree ℓ extension and χ a character of E^*. Recall that N denotes the norm map from E to F.

Definition 8.1.3. χ is called *regular* if χ does not factor through N. If χ is regular, we call the pair $(E/F, \chi)$ a *regular pair*.

All definitions we have made in the previous chapter for admissible pairs, we also make for regular pairs and regular characters. In particular, we also define the character twists Δ_χ for a regular pair $(E/F, \chi)$ exactly the same way they were defined for admissible pairs. For example, if $(E/F, \chi)$ is a regular pair where $\delta_{E/F} \neq 1$, then Δ_χ is the unique unramified quadratic character of E^*. Given a regular pair $(E/F, \chi)$, one may also construct a supercuspidal representation π_χ as in the previous chapter, but this construction is not one to one.

Our constructions and results do not require the stronger notion of admissible pair. We will sometimes say that χ is regular when the field E is understood.

We first explain why double covers of tori play a role. We start by considering the group $PGL(\ell, F)$. First recall that the representations of $PGL(\ell, F)$ are precisely the representations of $GL(\ell, F)$ with trivial central character. One of the conditions of the local Langlands correspondence for $GL(n, F)$ says that if $\phi : W_F \to GL(n, \mathbb{C})$ is irreducible, then $det(\phi) = \omega_{\pi(\phi)}$, where $\omega_{\pi(\phi)}$ denotes the central character of $\pi(\phi)$, and where $\pi(\phi)$ denotes the supercuspidal representation of $GL(n, F)$ that corresponds to ϕ under the local Langlands correspondence. Here we are viewing $det(\phi)$ as a character of F^* in the following way. As the image of $det(\phi)$ is in $\mathbb{C}^*$, $det(\phi)$ is trivial on $[W_F, W_F]$, and therefore factors to a character of $F^* \cong W_F^{ab}$ via the Artin map. Let ϕ be a supercuspidal parameter for $PGL(\ell, F)$ (that is, an irreducible representation $W_F \to GL(\ell, \mathbb{C})$ that parameterizes a supercuspidal representation of $GL(\ell, F)$ with trivial central character). Then $\phi = Ind_{W_E}^{W_F}(\chi)$ for some regular pair $(E/F, \chi)$. Since we are using the notion of regular pair here rather than admissible pair, there may be a choice involved here. That is, there may be another regular pair $(E_1/F, \chi_1)$ such that $\phi = Ind_{W_{E_1}}^{W_F}(\chi_1)$ as well. However, this will not matter, and we will show that our results and constructions are independent of all choices. It is a fact (see [5, Proposition 29.2]) that $det(Ind_{W_E}^{W_F}(\chi)) = \chi|_{F^*} \otimes \delta_{E/F}$, where $\delta_{E/F} = det(Ind_{W_E}^{W_F}(1))$. In the case that E/F is degree ℓ, $\delta_{E/F}^2 = 1$ (see [14, Corollary 2.5.15]). Now, since $\pi(\phi)$ has trivial central character, the condition $det(\phi) = \omega_{\pi(\phi)}$ becomes $\chi|_{F^*} \otimes \delta_{E/F} = 1$, so $\chi|_{F^*} = \delta_{E/F}$. Therefore, the supercuspidal representations of $PGL(\ell, F)$ naturally correspond to regular pairs $(E/F, \chi)$ where $\chi|_{F^*} = \delta_{E/F}$.

One might ask whether the supercuspidal representations of $PGL(\ell, F)$ are parameterized by characters of its elliptic tori E^*/F^*, as is the case for $GL(\ell, F)$. However, we have just seen that the supercuspidal representations of $PGL(\ell, F)$ are parameterized by characters χ of E^* whose restriction to F^* is $\delta_{E/F}$. Such a χ is not necessarily a character of the elliptic torus E^*/F^* in $PGL(\ell, F)$. Rather, it is a genuine character of a double cover of E^*/F^* in the following way. There is an exact sequence

$$1 \longrightarrow F^* \longrightarrow E^* \longrightarrow E^*/F^* \longrightarrow 1$$

$$w \mapsto [w]$$

Reducing this sequence by $ker(\delta_{E/F})$, we get an exact sequence

$$1 \longrightarrow F^*/ker(\delta_{E/F}) \longrightarrow E^*/ker(\delta_{E/F}) \longrightarrow E^*/F^* \longrightarrow 1.$$

We first consider the case where $\delta_{E/F} \neq 1$. Then since $F^*/ker(\delta_{E/F}) \cong \mathbb{Z}/2\mathbb{Z}$, we have that $E^*/ker(\delta_{E/F})$ is a double cover of E^*/F^*. Then the character χ of E^* naturally factors to a character $\tilde{\chi}$ of $E^*/ker(\delta_{E/F})$, given by $\tilde{\chi}([w]) := \chi(w)\ \forall [w] \in E^*/ker(\delta_{E/F})$ (since it is a character of E^* that is trivial on $ker(\delta_{E/F})$). Moreover, it is not trivial on all of F^*, so doesn't factor to a character of E^*/F^*. This means that $\tilde{\chi}$ is a genuine character of $E^*/ker(\delta_{E/F})$. Therefore, we are getting that the supercuspidal representations of $PGL(\ell, F)$ (i.e. the supercuspidal representations of $GL(\ell, F)$ with trivial central character) naturally correspond to genuine characters of the double cover $E^*/ker(\delta_{E/F})$ of the torus E^*/F^* inside $PGL(\ell, F)$. In fact, this double cover $E^*/ker(\delta_{E/F})$ is none other than an analogue of the ρ-cover that appears in the theory over the reals (see Definition (1.0.3)), which is a natural double cover of a real torus inside of the real group considered. We will explain this momentarily. In the case of $\delta_{E/F} = 1$, there is no cover, since $E^*/ker(\delta_{E/F}) = E^*/F^*$. Therefore, in this case, supercuspidal representations of $PGL(\ell, F)$ naturally correspond to characters χ of E^* that are trivial on F^*, i.e. characters χ of the elliptic torus E^*/F^* in $PGL(\ell, F)$. There is still a small subtlety in this case, as the proposed character formula in Theorem (8.1.1) is still written in terms of a double cover of the elliptic tori. We can't just say that since the cover splits, then there is nothing to do, so we just put a sum of characters of E^*/F^* in the numerator and let the denominator be $\tau(\Delta^0(z,\Delta^+))(\tau\circ\rho)(w)$ because then the denominator is a function on a double cover of E^*/F^* and the numerator is a function on E^*/F^*. Since the denominator of the proposed character formula lives on a double cover, we must have that the numerator lives on a double cover as well, so we will need a method of going from the character χ of E^*/F^* to a genuine character $\tilde{\chi}$ of a double cover. We will see how to do this shortly. We will also explain shortly why double covers play a role in the general setting of $GL(\ell, F)$.

We first consider the case where $\delta_{E/F} \neq 1$. Relative to the standard positive system of roots of $PGL(\ell, F)$, let ρ be half the sum of the positive roots. An elliptic torus in $PGL(\ell, F)$ is of the form $T(F) = E^*/F^*$. Then if $[z] \in T(F) = E^*/F^*$, $2\rho([z]) = [z]$. Fix a character τ_o of $(EL)^*$ whose restriction to L^* is $\aleph_{EL/L}$, where L is the unramified extension of F of degree $\ell - 1$ and set $\tau := \tau_o \mid\,\mid_{EL}$. Recall the denominator

$$\tau(\Delta^0(z,\Delta^+))(\tau\circ\rho)(w)$$

that was defined in Theorem (8.1.1). As in the cases of $PGL(2, F)$ and $GL(2, F)$, we will incorporate the cover $T(F)_{\tau\circ\rho}$. Recall that our current situation, since $T(F) = E^*/F^*$, then $T(F)_{\tau\circ\rho} = \{([z], w) \in E^*/F^* \times \mathbb{C}^* : \tau(2\rho([z])) = w^2\}$.

We make an important note here. Note that since ℓ is odd, not only do we have $2\rho \in X^*(T)$, but we also have $\rho \in X^*(T)$. Therefore, if we consider ρ as a function on T as such, we may apply τ to the element $\rho(w)$ where $w \in E^*$. We will denote this resulting function $\rho_\tau(w)$, as this is a different function than the function $(\tau\circ\rho)(w)$ which naturally lives on $T(F)_{\tau\circ\rho}$. We make this a formal definition.

Definition 8.1.4. We define

$$\rho_\tau(w) := \tau(\rho(w)),\ w \in E^*$$

Here, we are viewing ρ as an element of $X^*(T)$, which we may do since ℓ is odd.

We see that we can now identify the natural double cover that we are handed from the Local Langlands correspondence for $PGL(\ell, F)$, with this cover.

Lemma 8.1.5. *$E^*/ker(\delta_{E/F}) \cong T(F)_{\tau\circ\rho}$, and this isomorphism is unique as an isomorphism of covering groups (i.e. as covers of $T(F)$).*

Proof. The key is that both groups split, and there is a unique splitting. Recall that an extension

$$1 \to \mathbb{Z}/2\mathbb{Z} \to B \to C \to 1$$

splits if and only if there exists a genuine $\mathbb{Z}/2\mathbb{Z}$-valued character $\alpha : B \to \mathbb{Z}/2\mathbb{Z}$. Well, a genuine $\mathbb{Z}/2\mathbb{Z}$-valued character of $E^*/ker(\delta_{E/F})$ means a quadratic character of E^* whose restriction to F^* is given by the unique unramified character of F^* of order 2. Equivalently, this is the unique unramified quadratic character of E^*. This is the character twist occurring in the local Langlands correspondence, so we call it Δ_χ, even though it doesn't depend on χ at all. Anyway, the splitting is given by

$$E^*/ker(\delta_{E/F}) \to E^*/F^* \times \mathbb{Z}/2\mathbb{Z}$$

$$w \mapsto (w, \Delta_\chi(w))$$

Moreover, this splitting is unique. To see this, note that there is a unique genuine quadratic character of $E^*/ker(\delta_{E/F})$, since $E^* = (E^*)^2F^*$ (here we are using that ℓ

is an odd prime). Now, it's also true that $T(F)_{\tau\circ\rho}$ also splits. This is because ρ lifts to E^*/F^*. In the end, we have that there is a unique covering isomorphism

$$E^*/ker(\delta_{E/F}) \xrightarrow{\kappa} T(F)_{\tau\circ\rho}$$

given by

$$[w] \mapsto ([w], \Delta_\chi(w)\rho_\tau(w))$$

□

Now let's write down the character formula for a supercuspidal representation of $PGL(\ell, F)$ in the case where $\delta_{E/F} \neq 1$. In order to do this, we need to move to the setting of $T(F)_{\tau\circ\rho}$. In particular, the proposed character formula involves genuine characters of $T(F)_{\tau\circ\rho}$.

Let $\phi : W_F \to GL(\ell, \mathbb{C})$ be a supercuspidal parameter for $PGL(\ell, F)$ so that $\phi = Ind_{W_E}^{W_F}(\chi)$ for some regular pair $(E/F, \chi)$. As discussed earlier, this gives us a genuine character $\tilde{\chi}$ of $E^*/ker(\delta_{E/F})$.

Definition 8.1.6. A genuine character $\tilde{\eta}$ of $E^*/ker(\delta_{E/F})$ is called *regular* if $(E/F, \eta)$ is regular, where η is the pullback of $\tilde{\eta}$ to E^*. A genuine character $\tilde{\lambda}$ of $T(F)_{\tau\circ\rho}$ is called *regular* if $\tilde{\lambda} \circ \kappa$ is regular.

Now recall from Theorem (8.1.1) the proposed formula

$$F(\tilde{\chi})(z) := \epsilon(\tilde{\chi}, \Delta^+, \tau)\frac{\sum_{s\in W} \epsilon(s)\tilde{\chi}({}^s w)}{\tau(\Delta^0(z, \Delta^+))(\tau\circ\rho)(w)}, \quad z \in T(F)^{reg}$$

where $w \in T(F)_{\tau\circ\rho}$ is any element such that $\Pi(w) = z$. We naturally have constructed a character $\tilde{\chi}$ of $E^*/ker(\delta_{E/F})$. However, the functions in $F(\tilde{\chi})$ have domain

$T(F)_{\tau\circ\rho}$. Recall that $T(F)_{\tau\circ\rho} \cong E^*/ker(\delta_{E/F})$, so we can pull the function $(\tau\circ\rho)(w)$ and the Weyl group action in $F(\tilde{\chi})$ back to $E^*/ker(\delta_{E/F})$ via this isomorphism from Lemma (8.1.5), and leave our constructed $\tilde{\chi}$ as living on $E^*/ker(\delta_{E/F})$. That is, we consider

$$F(\tilde{\chi})(z) = \epsilon(\tilde{\chi}, \Delta^+, \tau) \frac{\sum_{s\in W} \epsilon(s)\tilde{\chi}({}^s[w])}{\tau(\Delta^0(z, \Delta^+))(\tau \circ \rho)(\kappa([w]))}, \quad z \in T(F)^{reg}$$

where $[w] \in E^*/ker(\delta_{E/F})$ such that $\Pi(\kappa([w])) = z$. Unwinding the definitions, we see that $(\tau \circ \rho)(\kappa([w])) = \Delta_\chi(w)\rho_\tau(w)\ \forall [w] \in E^*/ker(\delta_{E/F})$.

We need to define the Weyl group action. The Weyl group action on the $\tau\circ\rho$-cover is obtained as follows. If $([w], \lambda)$ is an element of $T(F)_{\tau\circ\rho}$, then analogously to the real case (recall equation (3.1) in Chapter 3), define $s([w], \lambda) = (s[w], \lambda\tau((s^{-1}\rho - \rho)([w])))$ for $s \in W = W(G(F), T(F)) = Aut(E/F)$ which is the relative Weyl group. Note that this is well-defined. Simplifying this expression, we get $s([w], \lambda) = ([sw], \lambda\frac{\rho_\tau(sw)}{\rho_\tau(w)})$ since ρ lifts to the torus. Then, since our character formula lives on $E^*/ker(\delta_{E/F})$, we must pull back this action from $T(F)_{\tau\circ\rho}$ to $E^*/ker(\delta_{E/F})$ via κ. Doing this, we see that we get

$$s[w] = \kappa^{-1}(s\kappa([w])) = \kappa^{-1}(s([w], \Delta_\chi(w)\rho_\tau(w))) =$$

$$\kappa^{-1}([sw], \Delta_\chi(w)\rho_\tau(w)\frac{\rho_\tau(sw)}{\rho_\tau(w)}) = \kappa^{-1}([sw], \Delta_\chi(w)\rho_\tau(sw)) =$$

$$\kappa^{-1}([sw], \Delta_\chi(sw)\rho_\tau(sw)) = [sw]\ \forall [w] \in E^*/ker(\delta_{E/F})$$

for $s \in W = Aut(E/F)$.

We note that the definition of regularity for a genuine character of $T(F)_{\tau\circ\rho}$ is analogous to the definition of regularity for a genuine character $\tilde{\lambda}$ of $T(\mathbb{R})_\rho$ for real groups when E/F is Galois, since the notion in the setting of real groups is that $\tilde{\lambda}$ is not fixed by any element of the real Weyl group $W(G(\mathbb{R}), T(\mathbb{R}))$.

Finally we can write down the character formula. Recall again the proposed formula

$$F(\tilde{\chi})(z) := \epsilon(\tilde{\chi}, \Delta^+, \tau) \frac{\sum_{s \in W} \epsilon(s) \tilde{\chi}({}^s w)}{\tau(\Delta^0(z, \Delta^+))(\tau \circ \rho)(w)} \quad z \in T(F)^{reg}$$

where $w \in T(F)_{\tau \circ \rho}$ is any element such that $\Pi(w) = z$.

Pulling back $(\tau \circ \rho)(w)$ and the Weyl group action to $E^*/ker(\delta_{E/F})$ via κ and incorporating $\tilde{\chi}$, we get

$$F(\tilde{\chi})(z) = \epsilon(\tilde{\chi}, \Delta^+, \tau) \frac{\sum_{s \in W} \epsilon(s) \tilde{\chi}({}^s w)}{\tau(\Delta^0(z, \Delta^+))\Delta_\chi(w)\rho_\tau(w)} =$$

$$\epsilon(\tilde{\chi}, \Delta^+, \tau) \frac{\sum_{s \in W} \epsilon(s) \tilde{\chi}({}^s[w])\Delta_\chi({}^s[w])}{\tau(\Delta^0(z, \Delta^+))\rho_\tau(w)} =$$

$$\epsilon(\tilde{\chi}, \Delta^+, \tau) \frac{\sum_{s \in W} \tilde{\chi}({}^s[w])\Delta_\chi({}^s[w])}{\tau(\Delta^0(z, \Delta^+))\rho_\tau(w)}$$

where $z \in E^*/F^*$ and $[w] \in E^*/ker(\delta_{E/F})$ is any element such that $[w]$ maps to z under the canonical map $E^*/ker(\delta_{E/F}) \to E^*/F^*$ (note that we've used here that $\Delta_\chi({}^s w) = \Delta_\chi(w)\ \forall s, w$). We can pull this character formula all the way back to E^* as well, and we get

$$F(\tilde{\chi})(z) = \epsilon(\tilde{\chi}, \Delta^+, \tau) \frac{\sum_{s \in W} \chi({}^s w)\Delta_\chi({}^s w)}{\tau(\Delta^0(w, \Delta^+))\rho_\tau(w)}$$

where $z \in E^*/F^*$ and $w \in E^*$ is any element such that w maps to z under the projection map $E^* \to E^*/F^*$. We will later see that this proposed character formula for $PGL(\ell, F)$ is independent of the choice of τ and the choice of positive roots Δ^+.

Summing up, noting that $T(F)_{\tau\circ\rho} \cong E^*/ker(\delta_{E/F})$, we have given a method of assigning a conjectural character formula for a supercuspidal representation of $PGL(\ell, F)$ when $\delta_{E/F} \neq 1$, to a supercuspidal Weil parameter of $PGL(\ell, F)$, given by

$$\left\{ \begin{array}{l} irreducible\ \phi : W_F \to GL(\ell, \mathbb{C}) \\ \\ with\ det(\phi) = 1 \end{array} \right\} \mapsto \quad \tilde{\chi} \in \widehat{T(F)}_{\tau\circ\rho} \mapsto F(\tilde{\chi})$$

We wish to make an important comment here: In the above derivation of our character formula, we implicitly chose an isomorphism

$$E^*/ker(\delta_{E/F}) \xrightarrow{\kappa} T(F)_{\tau\circ\rho}$$

$$[w] \mapsto ([w], \Delta_\chi(w)\rho_\tau(w))$$

However, we showed that there is only one isomorphism $E^*/ker(\delta_{E/F}) \cong T(F)_{\tau\circ\rho}$, so in the end there were no choices involved. This is in contrast with the case of $PGL(2, F)$, where there were two choices of isomorphism $E^*/N(E^*) \cong T(F)_{\tau\circ\rho}$, and we made a choice of isomorphism $E^*/N(E^*) \cong T(F)_{\tau\circ\rho}$.

Let us now compute the proposed character formula for $GL(\ell, F)$ when $\delta_{E/F} \neq 1$. An elliptic torus in $GL(\ell, F)$ is of the form $T(F) = E^*$. Let ρ be half the sum of the standard positive system of roots of $GL(\ell, F)$. Recall the denominator

$$\tau(\Delta^0(z, \Delta^+))(\tau \circ \rho)(w)$$

that was defined in Theorem (8.1.1). As in the cases of $PGL(2, F)$ and $GL(2, F)$, we will incorporate the cover $T(F)_{\tau\circ\rho}$. We now introduce yet another cover which is isomorphic to $T(F)_{\tau\circ\rho}$. This cover is just the pullback of the cover $E^*/ker(\delta_{E/F}) \to E^*/F^*$ in $PGL(\ell, F)$, to $GL(\ell, F)$.

Definition 8.1.7. Let $\Upsilon : E^*/ker(\delta_{E/F}) \to E^*/F^*$ be the canonical projection map given by $\Upsilon([z]) := [z]$. We define $E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F})$ as the group arising in the following pullback diagram:

$$\begin{array}{ccc} E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F}) & \longrightarrow & E^*/ker(\delta_{E/F}) \\ \big\downarrow & & \big\downarrow \Upsilon \\ E^* & \xrightarrow{w \mapsto [w]} & E^*/F^* \end{array}$$

That is, $E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F}) = \{(w, [z]) \in E^* \times E^*/ker(\delta_{E/F}) : [w] = [z] \in E^*/F^*\}$

Then we have

Lemma 8.1.8. *$E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F}) \cong T(F)_{\tau\circ\rho}$*

Proof. $T(F)_{\tau\circ\rho}$ splits since ρ is in the character lattice of T. Note that the cover $E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F})$ splits, as follows. Recall from Lemma (8.1.5) that the two-fold cover

$$E^*/ker(\delta_{E/F}) \to E^*/F^*$$

has a unique splitting given by

$$E^*/ker(\delta_{E/F}) \to E^*/F^* \times \mathbb{Z}/2\mathbb{Z}$$

$$[w] \mapsto ([w], \Delta_\chi(w)).$$

Then, since $E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F})$ is defined via a pullback diagram, we get that the canonical splitting of $E^*/ker(\delta_{E/F}) \to E^*/F^*$ induces, via pulling back, a canonical splitting of $E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F}) \to E^*$. One can see that it is given by

$$E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F}) \cong E^* \times \mathbb{Z}/2\mathbb{Z}$$

$$(w, [z]) \mapsto (w, \Delta_\chi(z))$$

(It is important to note that if any other splitting was chosen for the double cover $E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F})$, then the resulting character formula will not have been compatible with the formula for $PGL(\ell, F)$. More importantly, if we took a regular pair $(E/F, \chi)$ such that $\chi|_{F^*} = \delta_{E/F}$ and fed it through our $GL(\ell, F)$ character formula, we would not even get a supercuspidal representation of $PGL(\ell, F)$, as we will show later (if we want to develop any sort of general theory, we should have the theory for $GL(\ell, F)$ and $PGL(\ell, F)$ be compatible). Note that in the $GL(2, F)$ situation, we sometimes did get a supercuspidal representation of $PGL(2, F)$ this way, but then we required additionally that the $GL(2, F)$ and $PGL(2, F)$ formulas be compatible (that is, plugging a regular pair $(E/F, \chi)$ for $PGL(2, F)$ into either formula should give you the same formula). Thus, in a sense, the theory for $GL(\ell, F)$ is determined by the theory for $PGL(\ell, F)$. We will discuss this in more detail shortly.)

Therefore, since both covers split, we have naturally an isomorphism

$$E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F}) \xrightarrow{\kappa} T(F)_{\tau\circ\rho}$$

$$(w, [z]) \mapsto (w, \Delta_\chi(z)\rho_\tau(w))$$

The proof that this is an isomorphism follows as in the previous cases. $\square$

Now let's write down the character formula for a supercuspidal representation of $GL(\ell, F)$. In order to do this, we need to move to the setting of $T(F)_{\tau\circ\rho}$. In particular, the proposed character formula involves genuine characters of $T(F)_{\tau\circ\rho}$.

Now let $\phi : W_F \to GL(\ell, \mathbb{C})$ be a supercuspidal parameter so that $\phi = Ind_{W_E}^{W_F}(\chi)$ for some regular pair $(E/F, \chi)$. Then this canonically gives a genuine character $\tilde{\chi}$ of $E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F})$ as follows. Define $\tilde{\chi}(w, [z]) := \chi(w)\delta_{E/F}(z/w)$.

Definition 8.1.9. A genuine character $\tilde{\eta}$ of $E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F})$ is called *regular* if $(E/F, \eta)$ is regular, where $\eta(w) := \tilde{\eta}(w, [z])\delta_{E/F}(z/w)$. A genuine character $\tilde{\lambda}$ of $T(F)_{\tau\circ\rho}$ is called *regular* if $\tilde{\lambda} \circ \kappa$ is regular.

We have therefore given a map $\widehat{E^*} \to (E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F}))^\wedge$ given by $\eta \mapsto \tilde{\eta}$, where $\tilde{\eta}(w, [z]) := \eta(w)\delta_{E/F}(z/w)$. Note that we have a canonical map in the other direction, $(E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F}))^\wedge \to \widehat{E^*}$, given by $\tilde{\eta} \mapsto \eta$, where $\eta(w) := \tilde{\eta}(w, [z])\delta_{E/F}(z/w)$. We will regularly go back and forth between characters of E^* and genuine characters of $E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F})$. In particular, when we write $\tilde{\chi}$, a genuine character of $E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F})$, we will sometimes keep in mind that there is a canonical character χ of E^* that $\tilde{\chi}$ comes from via the above maps.

Now recall from Theorem (8.1.1) the proposed formula

$$F(\tilde{\chi})(z) := \epsilon(\tilde{\chi}, \Delta^+, \tau)\frac{\sum_{s\in W} \epsilon(s)\tilde{\chi}({}^s w)}{\tau(\Delta^0(z, \Delta^+))(\tau\circ\rho)(w)}, \quad z \in T(F)^{reg}$$

where $w \in T(F)_{\tau\circ\rho}$ is any element such that $\Pi(w) = z$. The functions in $F(\tilde{\chi})$ have domain $T(F)_{\tau\circ\rho}$. We have exhibited a natural isomorphism

$$E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F}) \cong T(F)_{\tau\circ\rho}$$

given by κ from Lemma (8.1.8), so we can pull the function $(\tau\circ\rho)(w)$ and the Weyl group action in $F(\tilde{\chi})$ back to $E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F})$ via this isomorphism, and leave our constructed $\tilde{\chi}$ as living on $E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F})$. That is, we consider

$$F(\tilde{\chi})(w) := \epsilon(\tilde{\chi}, \Delta^+, \tau)\frac{\sum_{s\in W} \epsilon(s)\tilde{\chi}(^s(w,[z]))}{\tau(\Delta^0(w,\Delta^+))(\tau\circ\rho)(\kappa(w,[z]))} \quad w \in T(F)^{reg}$$

where $(w,[z]) \in E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F})$ such that $\Pi(\kappa((w,[z]))) = w$. Unwinding the definitions, we see that $(\tau\circ\rho)(\kappa((w,[z]))) = \Delta_\chi(z)\rho_\tau(w)$ $\forall (w,[z]) \in E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F})$.

We also need to define the Weyl group action. The Weyl group action on the $\tau\circ\rho$-cover is obtained as follows. If (w,λ) is an element of $T(F)_{\tau\circ\rho}$, then analogously to the real case (recall equation (3.1) in Chapter 3), define $s(w,\lambda) = (sw, \lambda\tau((s^{-1}\rho - \rho)(w)))$, for $s \in W = W(G(F),T(F)) = Aut(E/F)$, the relative Weyl group. Note that this is well-defined. Simplifying this expression, we get $s(w,\lambda) = (sw, \lambda\frac{\rho_\tau(sw)}{\rho_\tau(w)})$ since ρ lifts to the torus. Then, since our character formula lives on $E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F})$, we must pull back this action from $T(F)_{\tau\circ\rho}$ to $E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F})$ via κ. Doing this, we see that we get

$$s(w,[z]) = \kappa^{-1}(s\kappa(w,[z])) = \kappa^{-1}(s(w,\Delta_\chi(z)\rho_\tau(w))) =$$

$$\kappa^{-1}(sw, \Delta_\chi(z)\rho_\tau(w)\frac{\rho_\tau(sw)}{\rho_\tau(w)}) = \kappa^{-1}(sw,\Delta_\chi(z)\rho_\tau(sw)) =$$

$$\kappa^{-1}(sw,\Delta_\chi(sz)\rho_\tau(sw)) = (sw,[sz]) \ \forall (w,[z]) \in E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F})$$

for $s \in W = W(G(F), T(F)) = Aut(E/F)$.

We note that the definition of regularity for a genuine character of $T(F)_{\tau\circ\rho}$ is analogous to the definition of regularity for a genuine character $\tilde{\lambda}$ of $T(\mathbb{R})_\rho$ for real groups when E/F is Galois, since the notion in the setting of real groups is that $\tilde{\lambda}$ is not fixed by any element of the real Weyl group $W(G(\mathbb{R}), T(\mathbb{R}))$.

Pulling back $\tau \circ \rho$ and the Weyl group action back to $E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F})$ via κ, our character formula becomes

$$F(\tilde{\chi})(w) = \epsilon(\tilde{\chi}, \Delta^+, \tau) \frac{\sum_{s\in W} \epsilon(s)\tilde{\chi}({}^s(w, [z]))}{\tau(\Delta^0(w, \Delta^+))(\tau\circ\rho)(\kappa(w, [z]))} =$$

$$\epsilon(\tilde{\chi}, \Delta^+, \tau) \frac{\sum_{s\in W} \epsilon(s)\chi({}^s w)\delta_{E/F}(z/w)}{\tau(\Delta^0(w, \Delta^+))\rho_\tau(w)\Delta_\chi(z)}, \quad w \in T(F)^{reg}$$

where $(w, [z]) \in E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F})$ is any element that maps to w under the canonical projection $E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F}) \to E^*$. This formula reduces to something simpler, which we show now. First note that $\Delta_\chi|_{F^*} = \delta_{E/F}$. Therefore, we get that $\delta_{E/F}(z/w) = \Delta_\chi(z/w)$. Therefore, the character formula simplifies to :

$$F(\tilde{\chi})(w) = \epsilon(\tilde{\chi}, \Delta^+, \tau) \frac{\sum_{s\in W} \epsilon(s)\chi({}^s w)\Delta_\chi(z/w)}{\tau(\Delta^0(w, \Delta^+))\rho_\tau(w)\Delta_\chi(z)} =$$

$$\epsilon(\tilde{\chi}, \Delta^+, \tau) \frac{\sum_{s\in W} \epsilon(s)\chi({}^s w)\Delta_\chi(z/w)}{\tau(\Delta^0(w, \Delta^+))\rho_\tau(w)\Delta_\chi(z)}, \quad w \in T(F)^{reg}$$

where $(w, [z]) \in E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F})$ is any element that maps to w under the canonical projection $E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F}) \to E^*$. Then, since $\Delta_\chi^2 = 1$, this formula reduces to

$$F(\tilde{\chi})(w) = \epsilon(\tilde{\chi}, \Delta^+, \tau) \frac{\sum_{s\in W} \epsilon(s)\chi({}^s w)\Delta_\chi({}^s w)}{\tau(\Delta^0(w, \Delta^+))\rho_\tau(w)} =$$

$$\epsilon(\tilde{\chi}, \Delta^+, \tau)\frac{\sum_{s\in W} \chi(^s w)\Delta_\chi(^s w)}{\tau(\Delta^0(w, \Delta^+))\rho_\tau(w)}, \quad w \in T(F)^{reg}$$

since $\Delta_\chi(^s w) = \Delta_\chi(w)\ \forall s \in W$. We will later see that our proposed character formula for $GL(\ell, F)$ is independent of the choice of τ and the choice of positive roots Δ^+.

Summing up, noting that $T(F)_{\tau\circ\rho} \cong E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F})$, then we have given a method of assigning a conjectural character formula for a supercuspidal representation of $GL(\ell, F)$ when $\delta_{E/F} \neq 1$, to a supercuspidal Weil parameter of $GL(\ell, F)$, given by

$$\left\{ \ irreducible\ \phi : W_F \to GL(\ell, \mathbb{C}) \ \right\} \ \mapsto \ \tilde{\chi} \in \widehat{T(F)}_{\tau\circ\rho} \mapsto F(\tilde{\chi})$$

We note the following important point: Recall that we chose a splitting

$$E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F}) \cong E^* \times \mathbb{Z}/2\mathbb{Z}$$

$$(w, [z]) \mapsto (w, \Delta_\chi(z))$$

This splitting is both canonical and natural, given how the cover was constructed, and given that the splitting of

$$E^*/ker(\delta_{E/F}) \to E^*/F^*$$

was unique. One might ask what if we choose a different splitting? Well, any other splitting is of the form

$$E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F}) \cong E^* \times \mathbb{Z}/2\mathbb{Z}$$

$$(w,z) \mapsto (w, \Delta_\chi(z)\lambda(w,z))$$

for some character $\lambda(w,z)$ of $E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F})$. One can check that for this to be a splitting (i.e. for the morphism to be bijective), $\lambda(w,z)$ must be a non-genuine character of $E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F})$, which means it factors to E^*. That is, any splitting is really of the form

$$E^* \times_{E^*/F^*} E^*/ker(\delta_{E/F}) \cong E^* \times \mathbb{Z}/2\mathbb{Z}$$

$$(w,[z]) \mapsto (w, \Delta_\chi(z)\lambda(w))$$

for some character λ of E^*. But since the second factor of this splitting is in $\mathbb{Z}/2\mathbb{Z}$, we must have that $\lambda^2 = 1$. Moreover, using this splitting for our character formula, we get

$$F(\tilde{\chi})(w) = \epsilon(\tilde{\chi}, \Delta^+, \tau) \frac{\sum_{s \in W} \chi({}^s w)\Delta_\chi({}^s w)\lambda({}^s w)}{\tau(\Delta^0(w, \Delta^+))\rho_\tau(w)} \quad w \in T(F)^{reg}$$

The key point now is that if we take a regular pair $(E/F, \chi)$ such that $\chi|_{F^*} = \delta_{E/F}$ (i.e. a regular pair for $PGL(\ell, F)$), and stick it in this character formula, we want to obtain a supercuspidal character of $PGL(\ell, F)$. This is the bare minimum that we would ask for in a character formula for $GL(\ell, F)$ if we wanted it to generalize a character formula for $PGL(\ell, F)$. It should not be the case that if you take a representation of $GL(\ell, F)$ with trivial central character, and view it as a representation of $GL(\ell, F)$ or $PGL(\ell, F)$, one gets a different local Langlands parameterization or character formula. If we make this request that we take a regular pair $(E/F, \chi)$ such that $\chi|_{F^*} = \delta_{E/F}$, and stick it in this character formula, we want to obtain a supercuspidal character of $PGL(\ell, F)$, then we get the following: since we will show that $\tau(\Delta^0(w,\Delta^+))\rho_\tau(w) = 1 \ \forall w \in E^*$ and since representations of $PGL(\ell, F)$ have trivial central character, and since $(\chi\Delta_\chi)|_{F^*} = 1$, then this would force $\lambda|_{F^*} = 1$. Therefore, λ is a character of E^* such that $\lambda|_{F^*} = 1$ and $\lambda^2 = 1$. Thus, λ factors to a character of E^*/F^* whose square is 1. However, since ℓ is odd, $E^* = (E^*)^2F^*$, and thus this forces $\lambda = 1$. What we conclude is that the splitting we used, which was natural to begin with, was the only splitting we could have even chosen in order to generalize our formula from $PGL(\ell, F)$.

We should note that in the theory of real groups, via the theory from [2], a Langlands parameter naturally induces a genuine character of a double cover of $T(\mathbb{R})$. This double cover, as we have explained, is isomorphic to the ρ-cover of $T(\mathbb{R})$, and a choice of isomorphism is made. However, there is a canonical way to choose an isomorphism, and one uses the theory of E-groups to do this.

Let us note that we will later see that our proposed character formula for $GL(\ell, F)$ is independent of the choice of τ.

We now consider the case where $\delta_{E/F} = 1$. Relative to the standard positive system of roots of $PGL(\ell, F)$, let ρ be half the sum of the positive roots. An elliptic torus in $PGL(\ell, F)$ is of the form $T(F) = E^*/F^*$. We still want to study the group $E^*/ker(\delta_{E/F})$, as in the other cases. However, recall that in this case, a supercuspidal parameter for $PGL(\ell, F)$ does not naturally yield a genuine character of a double cover of E^*/F^*. Rather, we are naturally handed a character of $E^*/ker(\delta_{E/F}) = E^*/F^*$, since $\delta_{E/F} = 1$. Therefore, the only natural double cover to consider in this case is the canonical split cover $E^*/F^* \times \mathbb{Z}/2\mathbb{Z}$. We will show that this setting still naturally fits into our theory. Recall the denominator

$$\tau(\Delta^0(z, \Delta^+))(\tau \circ \rho)(w)$$

that was defined in Theorem (8.1.1). As in the cases of $PGL(2, F)$ and $GL(2, F)$, we will incorporate the cover $T(F)_{\tau\circ\rho}$.

Lemma 8.1.10. *$E^*/F^* \times \mathbb{Z}/2\mathbb{Z} \cong T(F)_{\tau\circ\rho}$, and this isomorphism is unique as an isomorphism of covering groups (i.e. as covers of $T(F)$)..*

Proof. An explicit isomorphism is given by

$$E^*/F^* \times \mathbb{Z}/2\mathbb{Z} \xrightarrow{\kappa} T(F)_{\tau\circ\rho}$$

$$(z, \epsilon) \mapsto (z, \epsilon\rho_\tau(z))$$

To show that this covering isomorphism is unique, suppose that

$$E^*/F^* \times \mathbb{Z}/2\mathbb{Z} \xrightarrow{\kappa} T(F)_{\tau\circ\rho}$$

$$(z, \epsilon) \mapsto (z, \lambda(z, \epsilon))$$

was another isomorphism for some character λ of $E^*/F^* \times \mathbb{Z}/2\mathbb{Z}$. By commutativity of the pullback diagram that defines $T(F)_{\tau\circ\rho}$, we have that $\lambda(z, \epsilon)^2 = \tau(2\rho(z)) = \rho_\tau(z)^2$. Let $\phi(z, \epsilon) := \frac{\lambda(z,\epsilon)}{\rho_\tau(z)} = 1$. Then $\phi(z, \epsilon)^2 = 1$. So we ask what are the quadratic characters of $E^*/F^* \times \mathbb{Z}/2\mathbb{Z}$? Let $\alpha(z, \epsilon)$ be such a character. Then $\alpha(z, \epsilon) = \alpha_1(z)\alpha_2(\epsilon)$ for some character α_1 of E^*/F^* and some character α_2 of $\mathbb{Z}/2\mathbb{Z}$. Suppose $\alpha(z, \epsilon)$ is quadratic. So $1 = \alpha(z, \epsilon)^2 = \alpha_1(z)^2\alpha_2(\epsilon)^2$. But $\epsilon \in \mathbb{Z}/2\mathbb{Z}$, so $\alpha_2(\epsilon)^2 = 1$, so $\alpha_1(z)^2 = 1$. It is not difficult to see that since E/F has degree ℓ where ℓ is an odd prime, we have that $E^* = (E^*)^2F^*$. Therefore, the only quadratic character of E^*/F^* is the trivial character, so $\alpha_1 \equiv 1$. Therefore, we have that $\phi(z, \epsilon) = \phi_2(\epsilon)$ for some character ϕ_2 of $\mathbb{Z}/2\mathbb{Z}$. So ϕ_2 is either trivial, or the unique nontrivial character of $\mathbb{Z}/2\mathbb{Z}$. If $\phi_2 \equiv 1$, then $\lambda(z, \epsilon) = \rho_\tau(z)$, and it is easy to see that the map

$$E^*/F^* \times \mathbb{Z}/2\mathbb{Z} \xrightarrow{\kappa} T(F)_{\tau\circ\rho}$$

$$(z, \epsilon) \mapsto (z, \lambda(z, \epsilon))$$

is not bijective. Therefore, the only possibility is that ϕ_2 is the nontrivial character of $\mathbb{Z}/2\mathbb{Z}$. Therefore, $\lambda(z, \epsilon) = \epsilon\rho_\tau(z)$, so the covering isomorphism

$$E^*/F^* \times \mathbb{Z}/2\mathbb{Z} \xrightarrow{\kappa} T(F)_{\tau\circ\rho}$$

$$(z, \epsilon) \mapsto (z, \epsilon\rho_\tau(z))$$

is unique. □

Now let's write down the character formula for a supercuspidal representation of $PGL(\ell, F)$ in the case where $\delta_{E/F} = 1$. In order to do this, we need to move to the setting of $T(F)_{\tau\circ\rho}$. In particular, the proposed character formula involves genuine characters of $T(F)_{\tau\circ\rho}$.

Let $\phi : W_F \to GL(\ell, \mathbb{C})$ be a supercuspidal parameter for $PGL(\ell, F)$ so that $\phi = Ind_{W_E}^{W_F}(\chi)$ for some regular pair $(E/F, \chi)$. Now recall that we are trying to make sense of the proposed character formula

$$F(\tilde{\chi})(z) := \epsilon(\tilde{\chi}, \Delta^+, \tau)\frac{\sum_{s\in W} \epsilon(s)\tilde{\chi}({}^s w)}{\tau(\Delta^0(z, \Delta^+))(\tau\circ\rho)(w)}, \quad z \in T(F)^{reg}$$

where $w \in T(F)_{\tau\circ\rho}$ is any element such that $\Pi(w) = z$. We have that χ factors to a character of E^*/F^*, which we will also denote χ. However, the functions in $F(\tilde{\chi})$

have domain $T(F)_{\tau\circ\rho}$. We canonically have that

$$E^*/F^* \times \mathbb{Z}/2\mathbb{Z} \cong T(F)_{\tau\circ\rho}$$

$$(z,\epsilon) \mapsto (z, \epsilon\rho_\tau(z))$$

so we can pull the function $(\tau \circ \rho)(w)$ and the Weyl group action in $F(\tilde{\chi})$ back to $E^*/F^* \times \mathbb{Z}/2\mathbb{Z}$ via this isomorphism. That is, we consider

$$F(\tilde{\chi})(z) = \epsilon(\tilde{\chi}, \Delta^+, \tau) \frac{\sum_{s\in W} \epsilon(s)\tilde{\chi}({}^s(z,\epsilon))}{\tau(\Delta^0(z,\Delta^+))(\tau\circ\rho)(\kappa((z,\epsilon)))}, \quad z \in T(F)^{reg}$$

where $(z,\epsilon) \in E^*/F^* \times \mathbb{Z}/2\mathbb{Z}$ such that $\Pi(\kappa((z,\epsilon))) = z$. Unwinding the definitions, we see that $(\tau\circ\rho)(\kappa((z,\epsilon))) = \epsilon\rho_\tau(z)$ $\forall (z,\epsilon) \in E^*/F^* \times \mathbb{Z}/2\mathbb{Z}$. We can then canonically assign a genuine character $\tilde{\chi}$ of $E^*/F^* \times \mathbb{Z}/2\mathbb{Z}$ from the regular character χ by setting

$$\tilde{\chi} := \chi \otimes sgn$$

Definition 8.1.11. A genuine character $\tilde{\eta}$ of $E^* \times \mathbb{Z}/2\mathbb{Z}$ is called *regular* if $(E/F, \eta)$ is regular, where $\eta := \tilde{\eta} \otimes sgn$. A genuine character $\tilde{\lambda}$ of $T(F)_{\tau\circ\rho}$ is called *regular* if $\tilde{\lambda} \circ \kappa$ is regular.

We also need to define the Weyl group action. The Weyl group action on the $\tau\circ\rho$-cover is obtained as follows. If $([w], \lambda)$ is an element of $T(F)_{\tau\circ\rho}$, then analogously to the real case (recall equation (3.1) in Chapter 3), define $s([w], \lambda) = ([sw], \lambda\tau((s^{-1}\rho - \rho)(w)))$ for $s \in W = W(G(F), T(F)) = Aut(E/F)$, the relative Weyl group. Note that this is well-defined. Simplifying this expression, we get $s([w], \lambda) = ([sw], \lambda\frac{\rho_\tau(sw)}{\rho_\tau(w)})$ since ρ lifts to the torus. Then, since our character formula lives on $E^* \times_{E^*/F^*} E^*/N(E^*)$, we must pull back this action from $T(F)_{\tau\circ\rho}$ to $E^*/F^* \times \mathbb{Z}/2\mathbb{Z}$ via κ. Doing this, we see that we get

$$s([z],\epsilon) = \kappa^{-1}(s\kappa([z],\epsilon)) = \kappa^{-1}(s([z],\epsilon\rho_\tau(z))) =$$

$$\kappa^{-1}([sz],\epsilon\rho_\tau(z)\frac{\rho_\tau(sz)}{\rho_\tau(z)}) = \kappa^{-1}([sz],\epsilon\rho_\tau(sz)) = ([sz],\epsilon)$$

$$\forall([z],\epsilon) \in E^*/F^* \times \mathbb{Z}/2\mathbb{Z}$$

for $s \in W = Aut(E/F)$.

We note that the definition of regularity for a genuine character of $T(F)_{\tau\circ\rho}$ is analogous to the definition of regularity for a genuine character $\tilde{\lambda}$ of $T(\mathbb{R})_\rho$ for real groups when E/F is Galois, since the notion in the setting of real groups is that $\tilde{\lambda}$ is not fixed by any element of the real Weyl group $W(G(\mathbb{R}), T(\mathbb{R}))$.

Finally we can write down the character formula. Pulling back $(\tau \circ \rho)(w)$ and the Weyl group action to $E^*/F^* \times \mathbb{Z}/2\mathbb{Z}$ via κ and incorporating $\tilde{\chi}$, we get

$$F(\tilde{\chi})(z) = \epsilon(\tilde{\chi},\Delta^+,\tau)\frac{\sum_{s\in W}\epsilon(s)\tilde{\chi}({}^s(z,\epsilon))}{\tau(\Delta^0(z,\Delta^+))(\tau\circ\rho)(\kappa(z,\epsilon))} =$$

$$\epsilon(\tilde{\chi},\Delta^+,\tau)\frac{\sum_{s\in W}\epsilon(s)\epsilon\chi({}^s z)}{\tau(\Delta^0(z,\Delta^+))\epsilon\rho_\tau(z)} =$$

$$\epsilon(\tilde{\chi},\Delta^+,\tau)\frac{\sum_{s\in W}\epsilon(s)\chi({}^s z)}{\tau(\Delta^0(z,\Delta^+))\rho_\tau(z)} = \epsilon(\tilde{\chi},\Delta^+,\tau)\frac{\sum_{s\in W}\chi({}^s z)}{\tau(\Delta^0(z,\Delta^+))\rho_\tau(z)} \quad z \in T(F)^{reg}$$

We will later see that the proposed character formula for $PGL(\ell, F)$ is independent of the choice of τ and the choice of positive roots Δ^+.

Summing up, noting that $T(F)_{\tau\circ\rho} \cong E^*/F^* \times \mathbb{Z}/2\mathbb{Z}$, then we have given a method of assigning a conjectural character formula for a supercuspidal representation of $PGL(\ell, F)$ when $\delta_{E/F} = 1$, to a supercuspidal Weil parameter of $PGL(\ell, F)$, given by

$$\left\{ \begin{array}{l} irreducible\ \phi : W_F \to GL(\ell, \mathbb{C}) \\ \\ with\ det(\phi) = 1 \end{array} \right\} \mapsto \quad \tilde{\chi} \in \widehat{T(F)}_{\tau\circ\rho} \mapsto F(\tilde{\chi})$$

We wish to make an important comment here: In the above derivation of our character formula, we implicitly chose an isomorphism

$$E^*/F^* \times \mathbb{Z}/2\mathbb{Z} \xrightarrow{\kappa} T(F)_{\tau\circ\rho}$$

$$([z], \epsilon) \mapsto ([z], \epsilon\rho_\tau(z))$$

However, we showed that there is only one isomorphism $E^*/F^* \times \mathbb{Z}/2\mathbb{Z} \cong T(F)_{\tau\circ\rho}$, so in the end there were no choices involved. This is in contrast with the case of $PGL(2, F)$, where there were two choices of isomorphism $E^*/N(E^*) \cong T(F)_{\tau\circ\rho}$, and we made a choice of isomorphism $E^*/N(E^*) \cong T(F)_{\tau\circ\rho}$.

Let us now compute the proposed character formula for $GL(\ell, F)$ in the case that $\delta_{E/F} = 1$. Let ρ be half the sum of the standard positive system of roots of $GL(\ell, F)$. An elliptic torus in $GL(\ell, F)$ is of the form $T(F) = E^*$. Recall the denominator

$$\tau(\Delta^0(z, \Delta^+))(\tau \circ \rho)(w)$$

that was defined in Theorem (8.1.1). As in the cases of $PGL(2, F)$ and $GL(2, F)$, we will incorporate the cover $T(F)_{\tau\circ\rho}$. Note that $T(F)_{\tau\circ\rho}$ is isomorphic to $E^* \times \mathbb{Z}/2\mathbb{Z}$.

We now introduce yet another cover which is isomorphic to $T(F)_{\tau\circ\rho}$, completely analogously to the case of $\delta_{E/F} \neq 1$. This cover is just the pullback of the cover $E^*/F^* \times \mathbb{Z}/2\mathbb{Z} \to E^*/F^*$ in $PGL(\ell, F)$, to $GL(\ell, F)$.

Definition 8.1.12. Let $\Upsilon : E^*/F^* \times \mathbb{Z}/2\mathbb{Z} \to E^*/F^*$ be the canonical projection map given by $\Upsilon([z], \epsilon) := [z]$. We define $E^* \times_{E^*/F^*} (E^*/F^* \times \mathbb{Z}/2\mathbb{Z})$ as the group arising in the following pullback diagram:

$$\begin{array}{ccc} E^* \times_{E^*/F^*} (E^*/F^* \times \mathbb{Z}/2\mathbb{Z}) & \longrightarrow & E^*/F^* \times \mathbb{Z}/2\mathbb{Z} \\ \Big\downarrow & & \Big\downarrow \Upsilon \\ E^* & \xrightarrow{w \mapsto [w]} & E^*/F^* \end{array}$$

That is, $E^* \times_{E^*/F^*} (E^*/F^* \times \mathbb{Z}/2\mathbb{Z}) = \{(w, ([z], \epsilon)) : [w] = [z] \in E^*/F^*\}$

Then we have

Lemma 8.1.13. $E^* \times_{E^*/F^*} (E^*/F^* \times \mathbb{Z}/2\mathbb{Z}) \cong T(F)_{\tau\circ\rho}$

Proof. An explicit isomorphism is given by

$$E^* \times_{E^*/F^*} (E^*/F^* \times \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\kappa} T(F)_{\tau\circ\rho}$$

$$(w, ([z], \epsilon)) \mapsto (w, \epsilon\rho_\tau(w))$$

□

Now let's write down the character formula for a supercuspidal representation of $GL(\ell, F)$. In order to do this, we need to move to the setting of $T(F)_{\tau\circ\rho}$. In particular, the proposed character formula involves genuine characters of $T(F)_{\tau\circ\rho}$.

Now let $\phi : W_F \to GL(\ell, \mathbb{C})$ be a supercuspidal parameter so that $\phi = Ind_{W_E}^{W_F}(\chi)$ for some regular pair $(E/F, \chi)$. Then this gives a genuine character $\tilde{\chi}$ of $E^* \times_{E^*/F^*} (E^*/F^* \times \mathbb{Z}/2\mathbb{Z})$ as follows. Define $\tilde{\chi}(w, ([z], \epsilon)) := \chi(w)\epsilon$.

Definition 8.1.14. A genuine character $\tilde{\eta}$ of $E^* \times_{E^*/F^*} (E^*/F^* \times \mathbb{Z}/2\mathbb{Z})$ is called *regular* if $(E/F, \eta)$ is regular, where $\eta(w) := \tilde{\eta}(w, ([z], \epsilon))\epsilon$. A genuine character $\tilde{\lambda}$ of $T(F)_{\tau\circ\rho}$ is called *regular* if $\tilde{\lambda} \circ \kappa$ is regular.

We have therefore given a map $\widehat{E^*} \to (E^* \times_{E^*/F^*} (E^*/F^* \times \mathbb{Z}/2\mathbb{Z}))^\wedge$ given by $\eta \mapsto \tilde{\eta}$, where $\tilde{\eta}(w, ([z], \epsilon)) := \eta(w)\epsilon$. Note that we have a canonical map in the other direction, $(E^* \times_{E^*/F^*} (E^*/F^* \times \mathbb{Z}/2\mathbb{Z}))^\wedge \to \widehat{E^*}$, given by $\tilde{\eta} \mapsto \eta$, where $\eta(w) := \tilde{\eta}(w, ([z], \epsilon))\epsilon$. We will regularly go back and forth between characters of E^* and genuine characters of $E^* \times_{E^*/F^*} (E^*/F^* \times \mathbb{Z}/2\mathbb{Z})$. In particular, when we write $\tilde{\chi}$, a genuine character of $E^* \times_{E^*/F^*} (E^*/F^* \times \mathbb{Z}/2\mathbb{Z})$, we will sometimes keep in mind that there is a canonical character χ of E^* that $\tilde{\chi}$ comes from via the above maps.

Recall that the functions in $F(\tilde{\chi})$ have domain $T(F)_{\tau\circ\rho}$. We have exhibited a natural isomorphism $E^* \times_{E^*/F^*} (E^*/F^* \times \mathbb{Z}/2\mathbb{Z}) \cong T(F)_{\tau\circ\rho}$ given by κ, so we can pull the function $(\tau \circ \rho)(w)$ and the Weyl group action in $F(\tilde{\chi})$ back to $E^* \times_{E^*/F^*} (E^*/F^* \times \mathbb{Z}/2\mathbb{Z})$ via this isomorphism, and leave our constructed $\tilde{\chi}$ as living on $E^* \times_{E^*/F^*} (E^*/F^* \times \mathbb{Z}/2\mathbb{Z})$. That is, we consider

$$F(\tilde{\chi})(w) = \epsilon(\tilde{\chi}, \Delta^+, \tau) \frac{\sum_{s\in W} \epsilon(s)\tilde{\chi}({}^s(w, ([z], \epsilon)))}{\tau(\Delta^0(w, \Delta^+))(\tau \circ \rho)(\kappa(w, ([z], \epsilon)))} \quad w \in T(F)^{reg}$$

where $(w, ([z], \epsilon)) \in E^* \times_{E^*/F^*} (E^*/F^* \times \mathbb{Z}/2\mathbb{Z})$ such that $\Pi(\kappa((w, ([z], \epsilon)))) = w$. Unwinding the definitions, we see that $(\tau \circ \rho)(\kappa((w, (z, \epsilon)))) = \epsilon\rho_\tau(w)\ \forall (w, (z, \epsilon)) \in E^* \times_{E^*/F^*} (E^*/F^* \times \mathbb{Z}/2\mathbb{Z})$.

We also need to define the Weyl group action. The Weyl group action on the $\tau \circ \rho$-cover is obtained as follows. If (w, λ) is an element of $T(F)_{\tau\circ\rho}$, then analogously to the real case (recall equation (3.1) in Chapter 3), define $s(w,\lambda) = (sw, \lambda\tau((s^{-1}\rho - \rho)(w)))$ for $s \in W = W(G(F),T(F)) = Aut(E/F)$, the relative Weyl group. Note that this is well-defined. Simplifying this expression, we get $s(w,\lambda) = (sw, \lambda\frac{\rho_\tau(sw)}{\rho_\tau(w)})$ since ρ lifts to the torus. Then, since our character formula lives on $E^* \times_{E^*/F^*} (E^*/F^* \times \mathbb{Z}/2\mathbb{Z})$, we must pull back this action from $T(F)_{\tau\circ\rho}$ to $E^* \times_{E^*/F^*} (E^*/F^* \times \mathbb{Z}/2\mathbb{Z})$ via κ. Doing this, we see that we get

$$s(w,([z],\epsilon)) = \kappa^{-1}(s\kappa(w,([z],\epsilon))) = \kappa^{-1}(s(w,\epsilon\rho_\tau(w))) =$$

$$\kappa^{-1}(sw,\epsilon\rho_\tau(w)\frac{\rho_\tau(sw)}{\rho_\tau(w)}) = \kappa^{-1}(sw,\epsilon\rho_\tau(sw)) = (sw,([sz],\epsilon))$$

$$\forall (w,([z],\epsilon)) \in E^* \times_{E^*/F^*} (E^*/F^* \times \mathbb{Z}/2\mathbb{Z})$$

for $s \in W = Aut(E/F)$.

We note that the definition of regularity for a genuine character of $T(F)_{\tau\circ\rho}$ is analogous to the definition of regularity for a genuine character $\tilde{\lambda}$ of $T(\mathbb{R})_\rho$ for real groups when E/F is Galois, since the notion in the setting of real groups is that $\tilde{\lambda}$ is not fixed by any element of the real Weyl group $W(G(\mathbb{R}),T(\mathbb{R}))$.

Pulling back $\tau\circ\rho$ and the Weyl group action to $E^* \times_{E^*/F^*} (E^*/F^* \times \mathbb{Z}/2\mathbb{Z})$ via κ and incorporating $\tilde{\chi}$, our character formula becomes

$$F(\tilde{\chi})(w) = \epsilon(\tilde{\chi},\Delta^+,\tau)\frac{\sum_{s\in W}\epsilon(s)\tilde{\chi}({}^s(w,(z,\epsilon)))}{\tau(\Delta^0(w,\Delta^+))(\tau\circ\rho)(\kappa(w,(z,\epsilon)))} =$$

$$\epsilon(\tilde{\chi},\Delta^+,\tau)\frac{\sum_{s\in W}\epsilon(s)\chi({}^s w)\epsilon}{\tau(\Delta^0(w,\Delta^+))\epsilon\rho_\tau(w)} =$$

$$\epsilon(\tilde{\chi},\Delta^+,\tau)\frac{\sum_{s\in W}\epsilon(s)\chi({}^s w)}{\tau(\Delta^0(w,\Delta^+))\rho_\tau(w)} = \epsilon(\tilde{\chi},\Delta^+,\tau)\frac{\sum_{s\in W}\chi({}^s w)}{\tau(\Delta^0(w,\Delta^+))\rho_\tau(w)} \quad w \in T(F)^{reg}$$

where $(w, (z, \epsilon)) \in E^* \times_{E^*/F^*} (E^*/F^* \times \mathbb{Z}/2\mathbb{Z})$ is any element that maps to w under the canonical projection $E^* \times_{E^*/F^*} (E^*/F^* \times \mathbb{Z}/2\mathbb{Z}) \to E^*$. This is our character formula for $GL(\ell, F)$ in the case that $\delta_{E/F} = 1$. Again, we will see that the proposed character formula for $GL(\ell, F)$ is independent of the choice of τ and the choice of positive roots Δ^+.

Summing up, noting that $T(F)_{\tau\circ\rho} \cong E^* \times_{E^*/F^*} (E^* \times \mathbb{Z}/2\mathbb{Z})$, then we have given a method of assigning a conjectural character formula for a supercuspidal representation of $GL(\ell, F)$ when $\delta_{E/F} = 1$, to a supercuspidal Weil parameter of $GL(\ell, F)$, given by

$$\left\{ \textit{irreducible } \phi : W_F \to GL(\ell, \mathbb{C}) \right\} \mapsto \tilde{\chi} \in \widehat{T(F)}_{\tau\circ\rho} \mapsto F(\tilde{\chi})$$

Note that again, we chose various splittings in order to define the character formula for $GL(\ell, F)$ when $\delta_{E/F} = 1$. However, just like in the case $\delta_{E/F} \neq 1$, the splitting that we use is the only splitting we could have chosen in order to generalize the character formula for $PGL(\ell, F)$.

We should note again that in the theory of real groups, via the theory from [2], a Langlands parameter naturally induces a genuine character of a double cover of $T(\mathbb{R})$. This double cover, as we have explained, is isomorphic to the ρ-cover of $T(\mathbb{R})$, and a choice of isomorphism is made. However, there is a canonical way to choose an isomorphism, and one uses the theory of E-groups to do this.

We wish to make another important note. The case $\delta_{E/F} = 1$ is the only case where the "naive correspondence" of [5], [14], is the actual local Langlands correspondence. This is precisely because there is nothing interesting to see in the double cover. It is clear that in this special case, we didn't need to use double covers in order to obtain the local Langlands correspondence, and this is the only case where we could avoid using double covers of tori. What we are showing in this thesis, however, is that if we move to the setting of double covers of elliptic tori, then we can obtain the local Langlands correspondence in all cases by a "naive correspondence".

8.2 The constant $\epsilon(\tilde{\chi}, \Delta^+, \tau)$

In this section we define the constant $\epsilon(\tilde{\chi}, \Delta^+, \tau)$. Again we recall the following theorem.

Theorem 8.2.1. *[9, Theorem 5.3.2] Let $(E/F, \chi)$ be a regular pair such that χ has positive level, and write $G' = E^*$. Let $\pi = \pi_\chi$ be the associated supercuspidal representation given by Theorem (7.3.1). Then*

$$\frac{\theta_\pi(\gamma)}{deg(\pi)} = \begin{cases} C\lambda(\sigma)\sum_{\overline{w}\in W}\chi({}^{w}t) & \textit{if } n(\gamma)=0 \textit{ and } \gamma={}^{g}t \textit{ with } g\in G \textit{ and} \\ & t\in G' \\ C\sum_{\overline{w}\in W}\chi({}^{w}t)\gamma(\alpha(\chi),{}^{w}Y) & \textit{if } 0<n(\gamma)\leq r/2 \textit{ and } \gamma={}^{g}t \textit{ where} \\ & t=z(1+Y) \textit{ with } z\in Z, g\in G, \\ & \textit{and } Y\in\mathfrak{g}'_{n(\gamma)}\setminus(\mathfrak{z}_{n(\gamma)}+\mathfrak{g}'_{n(\gamma)^+}) \\ \chi(z)\mu_{\alpha(\chi)}(Y) & \textit{if } n(\gamma)>r/2 \textit{ and } \gamma=z(1+{}^{g}Y) \textit{ with} \\ & g\in G, z\in Z, \textit{ and } Y\in V_{n(\gamma)} \\ 0 & \textit{otherwise} \end{cases}$$

The constant C is in Theorem (8.2.1) is defined as

$$C := c_\psi(\mathfrak{g}')c_\psi^{-1}(\mathfrak{g})|D(\gamma)|^{-1/2}|\eta(\alpha(\chi))|^{-1/2}$$

(see [9, Section 5.3.2]).

Definition 8.2.2. Define $\epsilon(\Delta^+)$ to be 1 if Δ^+ is the standard choice of positive roots of $GL(\ell,\overline{F})$ with respect to the diagonal maximal torus $T(\overline{F})$. Any other set of positive roots is of the form $s\Delta^+$ for some $s\in W(G(\overline{F}),T(\overline{F}))$. We then define $\epsilon(s\Delta^+)=\tau_o((-1)^{\ell(s)})$ where ℓ is the length of s. Let Δ^+ be any set of positive roots. We set

$$\epsilon(\tilde{\chi},\Delta^+,\tau) := deg(\pi)\lambda(\sigma)c_\psi(\mathfrak{g}')c_\psi^{-1}(\mathfrak{g})|\eta(\alpha(\chi))|^{-1/2}\tau_o((-1)^{\sum_{k=1}^{\frac{\ell-1}{2}}k})\epsilon(\Delta^+)$$

.

It will be useful to define another constant. We set

$$\epsilon(\tilde{\chi},\Delta^+,\tau)' := deg(\pi)\lambda(\sigma)c_\psi(\mathfrak{g}')c_\psi^{-1}(\mathfrak{g})|\eta(\alpha(\chi))|^{-1/2}\epsilon(\Delta^+).$$

In the calculations we will make throughout the rest of this chapter and the next, we will make a choice of Δ^+ to be the standard set of positive roots. Therefore, the term $\epsilon(\Delta^+)$ is just 1, and therefore this term will not appear in most of our calculations and formulas. We will show later that all of our results will be independent of the choice of Δ^+.

8.3 The case $\delta_{E/F} \neq 1$

In the next two sections, we show that the proposed character formula $F(\tilde{\chi})$ agrees with the character of the positive depth supercuspidal representation $\pi_{\chi\Delta_\chi}$ occuring in the local Langlands correspondence, on the range $\{z \in T(F)^{reg} : 0 \leq n(z) \leq r/2\}$. We will assume again without loss of generality, as in Chapter 5, that our regular pairs $(E/F, \chi)$ are such that χ has minimal conductor (In [21], the terminology conductor is used instead of the terms "minimal regular pair". We follow the terminology in [21] since we will use results from there). The same argument as in the end of Chapter 5 shows that this doesn't matter, and that all of our results are true for arbitrary regular pairs. We need a result from [21]. Unwinding all the definitions, it is shown in [21, Sections 6 and 7][1] that if $(E/F, \chi)$

[1] Actually, what is shown in [21, Sections 6 and 7] implies that the character values on the range $\{z \in T(F)^{reg} : 0 \leq n(z) \leq r/2\}$ are of a particular form, with some constants $c_1|D_G(z)|^{-\frac{1}{2}}$ and $c_\ell|D_G(z)|^{-\frac{1}{2}}$ in front of the character formulas, depending on whether the elliptic torus is unramified or ramified, respectively. But in Theorem (8.2.1), it is shown that the character values on the range $\{z \in T(F)^{reg} : 0 \leq n(z) \leq r/2\}$ are of the same form with a single constant in front of the formulas, namely, $deg(\pi)C\lambda(\sigma)$, regardless of whether the torus is ramified or unramfied. Therefore, because of this, and because of the rewritten version of the character formulas in [21,

is a regular pair where χ has positive level, then the character, θ, of the associated supercuspidal representation π_χ via Theorem (7.3.1), satisfies[2]

$$\theta(w) = deg(\pi)\lambda(\sigma)c_\psi(\mathfrak{g}')c_\psi^{-1}(\mathfrak{g})|\eta(\alpha(\chi))|^{-1/2} \frac{\sum_{s \in Aut(E/F)} \chi({}^s w)}{|D(w)|^{1/2}}$$

$$\forall w \in E^* : 0 \leq n(w) \leq r/2$$

We wish to show that $F(\tilde{\chi})$ agrees (on the $0 \leq n(w) \leq r/2$ range) with the character formula of the supercuspidal representation $\pi_{\chi\Delta_\chi}$. We will also show that a supercuspidal character is completely determined by its values on the $n(w) = 0$ range.

We first need some preliminary calculations:

Let Δ be the standard set of roots for $GL(\ell, F)$ with respect to the diagonal torus T, where ℓ is an odd prime, $\Delta = \{e_i - e_j : i \neq j\}$, and let Δ^+ be the standard set of positive roots. Then $\Delta^+ = \{e_i - e_j \mid i < j\}$.

Sections 6 and 7], it must be that what we write here as the formula for $\theta(w)$ is correct. Moreover, in a private communication, Loren mentioned to me that $c_1 = c_\ell = 1$

[2]In a private communication with Loren Spice, Loren told me that the notation r in [21] is the same as the notation r in [9]. In [9], there is an explicit formula for the supercuspidal characters on the range $0 \leq n(w) \leq r/2$. In [21], there is an explicit formula for the supercuspidal characters on the range $0 \leq n(w) \leq r$. Therefore, [21] has more information about the supercuspidal characters than [9]. In particular, the expression that we write above for $\theta(w)$ holds for supercuspidal representations coming from unramified and ramified tori definitely on the range $0 \leq n(w) < r$ (and for unramified tori, it also holds on $n(w) = r$, but for ramified tori, there is a problem at $n(w) = r$, which is called "at the level"), and therefore the expression that we write above for $\theta(w)$ holds for all supercuspidal representations of $GL(\ell, F)$ on on the range $0 \leq n(w) \leq r/2$, which is the only range on which there is an explicit formula for the supercuspidal characters in [9].

Recall that the definition of $\rho_\tau(w)$ was as follows. Since ℓ is odd, ρ is a legitimate character of T. Therefore, we may apply τ to the element $\rho(w)$ where $w \in E^*$, and we denote the resulting element $\rho_\tau(w)$ (where we view E^* as embedded in $T = T(\overline{F})$). In this section, we will view ρ as a character of T, so that we may eventually compute ρ_τ.

Lemma 8.3.1. *If w is the diagonal matrix*

$$\begin{pmatrix} w_1 & 0 & 0 & 0 & 0 \\ 0 & w_2 & 0 & 0 & 0 \\ 0 & 0 & w_3 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & w_\ell \end{pmatrix}$$

then

$$\Delta^0(w, \Delta^+)\rho(w) = \frac{\prod_{i<j}(w_i - w_j)}{(w_1 w_2 ... w_\ell)^{\frac{\ell-1}{2}}}$$

Proof. Note that $\sum_{\alpha \in \Delta^+} \alpha = (\ell-1)e_1 + (\ell-3)e_2 + ... + 2e_{\frac{\ell+1}{2}-1} + \hat{e}_{\frac{\ell+1}{2}} - 2e_{\frac{\ell+1}{2}+1} - ... - (\ell-3)e_{\ell-1} - (\ell-1)e_\ell$, where a hat over a character means that character doesn't exist in the sum with a non-zero coefficient. Thus, since the coefficients of this sum are all even integers, we have that $\rho = \frac{(\ell-1)}{2}e_1 + \frac{(\ell-3)}{2}e_2 + ... + e_{\frac{\ell+1}{2}-1} + \hat{e}_{\frac{\ell+1}{2}} - e_{\frac{\ell+1}{2}+1} - ... - \frac{(\ell-3)}{2}e_{\ell-1} - \frac{(\ell-1)}{2}e_\ell$ is a character of the torus. Then

$$\Delta^0(w, \Delta^+) = \prod_{\alpha \in \Delta^+}(1 - \alpha^{-1}(w)) = (1 - \frac{w_2}{w_1})(1 - \frac{w_3}{w_1})...(1 - \frac{w_3}{w_2})...(1 - \frac{w_\ell}{w_{\ell-1}})$$

Clearing denominators, we can rewrite this as

$$\Delta^0(w, \Delta^+) = \frac{(w_1 - w_2)(w_1 - w_3)...(w_2 - w_3)...(w_{\ell-1} - w_\ell)}{w_1^{\ell-1} w_2^{\ell-2} w_3^{\ell-3} ... w_{\ell-2}^2 w_{\ell-1}}$$

Thus,

$$\Delta^0(w,\Delta^+)\rho(w) = \frac{w_1^{\frac{\ell-1}{2}} w_2^{\frac{\ell-3}{2}} ... w_{\frac{\ell+1}{2}-1}(w_1-w_2)(w_1-w_3)...(w_2-w_3)...(w_{\ell-1}-w_\ell)}{w_{\frac{\ell+1}{2}+1}...w_{\ell-1}^{\frac{\ell-3}{2}} w_\ell^{\frac{\ell-1}{2}} w_1^{\ell-1} w_2^{\ell-2} w_3^{\ell-3}...w_{\ell-2}^2 w_{\ell-1}} =$$

$$\frac{(w_1-w_2)(w_1-w_3)...(w_2-w_3)...(w_{\ell-1}-w_\ell)}{(w_1w_2...w_\ell)^{\frac{\ell-1}{2}}} = \frac{\prod_{i<j}(w_i-w_j)}{(w_1w_2...w_\ell)^{\frac{\ell-1}{2}}}$$

□

Corollary 8.3.2. *If $w \in E^*$, then*

$$\Delta^0(w,\Delta^+)\rho(w) =$$

$$(-1)^{\sum_{k=1}^{\frac{\ell-1}{2}} k} \frac{N_{EL/L}((w-\upsilon(w))(w-\upsilon^2(w))(w-\upsilon^3(w))...(w-\upsilon^{\frac{\ell-1}{2}}(w)))}{N_{E/F}(w)^{\frac{\ell-1}{2}}}$$

where L is the unramified extension of F of degree $\ell-1$, and υ is an embedding of E into $\overline{F}$.

Proof. E^* naturally embeds as a torus in GL(ℓ,F). Over $\overline{F}$, E becomes a product of ℓ copies of $\overline{F}^*$, and via this isomorphism, an element $w \in E^*$ maps to

$$\begin{pmatrix} w & 0 & 0 & 0 & 0 \\ 0 & \upsilon(w) & 0 & 0 & 0 \\ 0 & 0 & \upsilon^2(w) & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \upsilon^{\ell-1}(w) \end{pmatrix}$$

(note that in the case that $\delta_{E/F} \neq 1$, E/F is never Galois. Even in the case $\delta_{E/F} = 1$, E/F is sometimes not Galois). Using this description of w in $\overline{F}^* \times \overline{F}^* \times ... \times \overline{F}^*$, let's calculate $\Delta^0(w,\Delta^+)\rho(w)$. From the calculations above,

$$\Delta^0(w,\Delta^+)\rho(w) = \frac{\prod_{0\le i<j\le \ell-1}(\upsilon^i(w)-\upsilon^j(w))}{(w\upsilon(w)...\upsilon^{\ell-1}(w))^{\frac{\ell-1}{2}}} =$$

$$\frac{\prod_{0\leq i<j\leq \ell-1}(\upsilon^i(w)-\upsilon^j(w))}{N(w)^{\frac{\ell-1}{2}}}$$

Elementary calculation shows that

$$\prod_{0\leq i<j\leq \ell-1}(\upsilon^i(w)-\upsilon^j(w)) =$$

$$(-1)^{\sum_{k=1}^{\frac{\ell-1}{2}} k}\ N_{EL/L}(w-\upsilon(w))N_{EL/L}(w-\upsilon^2(w))...N_{EL/L}(w-\upsilon^{\frac{\ell-1}{2}}(w))$$

Thus, we have that

$$\Delta^0(w,\Delta^+)\rho(w) =$$

$$(-1)^{\sum_{k=1}^{\frac{\ell-1}{2}} k}\ \frac{N_{EL/L}((w-\upsilon(w))(w-\upsilon^2(w))(w-\upsilon^3(w))...(w-\upsilon^{\frac{\ell-1}{2}}(w)))}{N_{E/F}(w)^{\frac{\ell-1}{2}}}$$

□

Recall that our proposed character formula reduces in the case $\delta_{E/F}\neq 1$ to

$$F(\tilde{\chi})(w) = \epsilon(\tilde{\chi},\Delta^+,\tau)\frac{\sum_{s\in W}\chi({}^s w)\Delta_\chi({}^s w)}{\tau(\Delta^0(w,\Delta^+))\rho_\tau(w)},\quad w\in T(F)^{reg}$$

Theorem 8.3.3. $F(\tilde{\chi})$ *agrees with the character of the supercuspidal representation* $\pi_{\chi\Delta_\chi}$ *on the range* $\{w\in E^* : 0\leq n(w)\leq r/2\}$.

Proof. Recall that τ_o is a character of $(EL)^*$ whose restriction to L^* is a local class field theory character $\aleph_{EL/L}$. Then, since τ_o is trivial on $N_{EL/L}((EL)^*)$, and since $N_{E/F}(w) = N_{EL/L}(w)$, we have that $\tau_o(\Delta^0(w,\Delta^+))\rho_{\tau_o}(w) = \tau_o((-1)^{\sum_{k=1}^{\frac{\ell-1}{2}} k})$. Thus, our character formula

$$F(\tilde{\chi})(w) = \epsilon(\tilde{\chi},\Delta^+,\tau)\frac{\sum_{s\in W}\chi({}^s(w)\Delta_\chi({}^s w)}{\tau(\Delta^0(w,\Delta^+))\rho_\tau(w))},\quad w\in E^*$$

becomes merely

$$F(\tilde{\chi})(w) = \epsilon(\tilde{\chi},\Delta^+,\tau)'\frac{\sum_{s\in W}\chi({}^s w)\Delta_\chi({}^s w)}{|\Delta^0(w,\Delta^+)\rho(w)|} =$$

$$\epsilon(\tilde{\chi}, \Delta^+, \tau)' \frac{\sum_{s \in W} \chi({}^s w) \Delta_\chi({}^s w)}{|D(w)|^{1/2}}, \ w \in E^*$$

since recall that $|\Delta^0(w, \Delta^+)\rho(w)| = |D(w)|^{1/2}$ from Chapter 3. $F(\tilde{\chi})$ is the character of a supercuspidal representation (on the range $\{w \in E^* : 0 \leq n(w) \leq r/2\}$, which we will show is enough to determine any supercuspidal representation), which we denote $\pi(\tilde{\chi})$. It is the character of the supercuspidal representation $\pi_{\chi\Delta_\chi}$ occurring in the local Langlands correspondence. $\square$

Note that $F(\tilde{\chi})$ is independent of the choice of τ. That is, all that matters is $\tau_o|_{L^*}$, which we have required from the outset is $\aleph_{EL/L}$.

Note that in the above, we have chosen Δ^+ to be the standard set of positive roots, which implies that $\epsilon(\Delta^+) = 1$. We wish to make the following observation. Suppose we made another choice of positive roots. Any other choice is of the form $s\Delta^+$ where Δ^+ is the standard choice of positive roots and $s \in W(G(\overline{F}), T(\overline{F}))$. Let ρ be half the sum of positive roots in Δ^+ and let ρ_s be half the sum of positive roots in $s\Delta^+$. Then $\Delta^0(w, s\Delta^+)\rho_s(w) = (-1)^{\ell(s)}\Delta^0(w, \Delta^+)\rho(w)$, where $\ell(s)$ is the length of s. Therefore, the denominator in our character formula for the choice $s\Delta^+$ would include the term $\tau_o((-1)^{\ell(s)})$. However, because our definition of $\epsilon(\tilde{\chi}, \Delta^+, \tau)$ includes the term $\epsilon(\Delta^+)$, our overall character formula $F(\tilde{\chi})$ remains the same regardless of the choice of positive roots. The same line of reasoning is true for the case of $\delta_{E/F} = 1$ and $PGL(\ell, F)$.

8.4 The case $\delta_{E/F} = 1$

Let E/F now be a degree ℓ extension such that $\delta_{E/F} = 1$. Recall that $\delta_{E/F} = 1 \Leftrightarrow \Delta_\chi = 1$. Recall that our proposed character formula reduces in the case $\delta_{E/F} = 1$ to

$$F(\tilde{\chi})(w) = \epsilon(\tilde{\chi}, \Delta^+, \tau)\frac{\sum_{s\in W}\chi({}^s w)}{\tau(\Delta^0(w,\Delta^+))\rho_\tau(w)}, \quad w \in T(F)^{reg}$$

Theorem 8.4.1. *$F(\tilde{\chi})$ agrees with the character of the supercuspidal representation $\pi_{\chi\Delta_\chi}$ on the range $\{w \in E^* : 0 \leq n(w) \leq r/2\}$.*

Proof. Since τ_o is trivial on $N_{EL/L}((EL)^*)$, and since $N_{E/F}(w) = N_{EL/L}(w)$, we have that

$\tau_o(\Delta^0(w,\Delta^+))\rho_{\tau_o}(w) = \tau_o((-1)^{\sum_{k=1}^{\frac{\ell-1}{2}} k})$. Thus, our character formula reduces to

$$F(\tilde{\chi})(w) = \epsilon(\tilde{\chi}, \Delta^+, \tau)\frac{\sum_{s\in W}\chi({}^s(w)}{\tau(\Delta^0(w,\Delta^+))\rho_\tau(w)} =$$

$$\epsilon(\tilde{\chi}, \Delta^+, \tau)'\frac{\sum_{s\in W}\chi({}^s w)}{|\Delta^0(w,\Delta^+)\rho(w)|} =$$

$$\epsilon(\tilde{\chi}, \Delta^+, \tau)'\frac{\sum_{s\in W}\chi({}^s w)}{|D(w)^{1/2}|}, \quad w \in E^*$$

on the range $\{w \in E^* : 0 \leq n(w) \leq r/2\}$ (which we will show is enough to completely determine the representation), this is the character of a supercuspidal representation. It is the character of the supercuspidal representation $\pi_{\chi\Delta_\chi}$ occurring in the local Langlands correspondence (in our current case $\delta_{E/F} = 1$, so it's true that $\Delta_\chi = 1$). □

Note again that $F(\tilde{\chi})$ is independent of the choice of τ. That is, all that matters is $\tau_o|_{L^*}$, which we have required from the outset is $\aleph_{EL/L}$. Also note again that we have chosen Δ^+ to be the standard set of positive roots, which implies that $\epsilon(\Delta^+) = 1$. Moreover, $F(\tilde{\chi})$ is independent of the choice of Δ^+ for the same reasoning as in the previous section.

8.5 On whether there are two positive depth character formulas coming from the same Cartan

Given a supercuspidal parameter for $GL(\ell, F)$, we constructed a conjectural character formula for a supercuspidal representation of $GL(\ell, F)$. We showed that it agrees with the character formula of the positive depth supercuspidal representation $\pi_{\chi\Delta_\chi}$ on the set $\{w \in E^* : 0 \leq n(w) \leq \frac{r}{2}\}$, where r is the depth of the representation. In the next two sections we show that there are no other positive depth supercuspidal representations whose character agree with $F(\tilde{\chi})$ on the set $\{w \in E^* : 0 \leq n(w) \leq r/2\}$, thereby solving the uniqueness question in Theorem (8.1.1). In fact, we show something stronger. In the next two sections, we show that a positive depth supercuspidal representation of $GL(\ell, F)$ is uniquely determined by the restriction of its distribution character to the $n(w) = 0$ range. In this section, we show that if the distribution characters of two positive depth supercuspidal representations, both coming from the same Cartan, agree on the $n(w) = 0$ range, then the supercuspidal representations are isomorphic. That is, we prove the following theorem.

Theorem 8.5.1. *Suppose* $(E/F, \chi_1)$ *and* $(E/F, \chi_2)$ *are admissible pairs such that* $F(\tilde{\chi}_1)(w) = F(\tilde{\chi}_2)(w)\ \forall w \in E^* : n(w) = 0$. *Then,* $\chi_1 = \chi_2^{\upsilon}$ *for some* $\upsilon \in Aut(E/F)$.

We will split the proof of this theorem into several cases.

We wish to make the following important note. Recall that in the previous sections, we constructed a character formula $F(\tilde{\chi})$ from a regular pair $(E/F, \chi)$. The above theorem and Theorem (8.6.1) will together prove that a positive depth supercuspidal representation of $GL(\ell, F)$ is uniquely determined by its restriction of its distribution character to the $n(w) = 0$ range. We are claiming in the above theorem and in Theorem (8.6.1) that it is sufficient to consider admissible pairs rather than regular pairs in order to prove that a positive depth supercuspidal representation of $GL(\ell, F)$ is uniquely determined by its restriction of its distribution character to the $n(w) = 0$ range. This is because the positive depth supercuspidal representations of $GL(\ell, F)$ are parameterized by admissible pairs, and so it is sufficient to consider just admissible pairs.

Proposition 8.5.2. *Let* E/F *be ramified Galois. Suppose* $(E/F, \chi_1), (E/F, \chi_2)$ *are admissible pairs such that* $F(\tilde{\chi}_1)(w) = F(\tilde{\chi}_2)(w)\ \forall w \in E^* : n(w) = 0$. *Then* $\chi_1 = \chi_2^{\upsilon}$ *for some* $\upsilon \in Aut(E/F)$.

Proof. Assume $F(\tilde{\chi}_1)(w) = F(\tilde{\chi}_2)(w)$ on the set $\{w \in E^* : n(w) = 0\}$. We partially follow a proof in [21, page 16]. We have

$$c_1|D(w)|^{-\frac{1}{2}}[\chi_1](w) = c_2|D(w)|^{-\frac{1}{2}}[\chi_2](w)\ \forall w \in E^* \setminus F^*(1 + \mathfrak{p}_E)$$

where $c_1, c_2 \in \mathbb{R}$ are some non-zero constants appearing in front of the ramified character formulas in the $n(w) = 0$ range (cf [21, Corollary 7.15]), and where $[\chi](w) := \sum_{\upsilon \in Aut(E/F)} \chi^{\upsilon}(w)$ and D is the Weyl denominator for $GL(n, F)$.

Therefore, $c_1[\chi_1] = c_2[\chi_2]$ on $E^* \setminus F^*(1 + \mathfrak{p}_E)$. Now, we will prove in the next section that $[\chi_1]$ doesn't vanish completely on $E^* \setminus F^*(1 + \mathfrak{p}_E)$. Assuming this fact, we then have that by [21, Lemma 5.1], $\chi_1|_{F^*(1+\mathfrak{p}_E)} = \chi_2^{\upsilon'}|_{F^*(1+\mathfrak{p}_E)}$ for some $\upsilon' \in Aut(E/F)$. Since replacing χ_2 with $(\chi_2)^{\upsilon'}$ doesn't change anything in the calculations, we may assume that $\upsilon' = 1$.

Now let $c := \frac{c_1}{c_2}$. Then we have $c[\chi_1](w) = [\chi_2](w)\ \forall w \in E^* \setminus F^*(1 + \mathfrak{p}_E)$ and we also have that $\chi_1|_{F^*(1+\mathfrak{p}_E)} = \chi_2|_{F^*(1+\mathfrak{p}_E)}$. Now let π_E be the uniformizer of E and let π_F be the uniformizer of F. Then since $\pi_E^{\ell} = \pi_F$, and since $\chi_1(\pi_F) = \chi_2(\pi_F)$, we have that $\chi_1(\pi_E)^{\ell} = \chi_2(\pi_E)^{\ell}$, and so $\chi_2(\pi_E) = \xi_{\ell}\chi_1(\pi_E)$ for some ℓth root of unity ξ_{ℓ}. Therefore, since $\chi_1|_{F^*(1+\mathfrak{p}_E)} = \chi_2|_{F^*(1+\mathfrak{p}_E)}$ and since $\chi_2(\pi_E) = \xi_{\ell}\chi_1(\pi_E)$, notice that this implies that $\chi_2(w) = \chi_1(w)\xi_{\ell}^{val(w)}\ \forall w \in E^*$, where $val(w)$ denotes the E-adic valuation of $w \in E^*$. Therefore, plugging this into the formula

$$c[\chi_1](w) = [\chi_2](w)\ \forall w \in E^* \setminus F^*(1 + \mathfrak{p}_E)$$

we obtain

$$c[\chi_1](w) = [\chi_2](w) =$$

$$\chi_1(w)\xi_{\ell}^{val(w)} + \chi_1^{\upsilon}(w)\xi_{\ell}^{val(\upsilon(w))} + \chi_1^{\upsilon^2}(w)\xi_{\ell}^{val(\upsilon^2(w))} + \cdots + \chi_1^{\upsilon^{\ell-1}}(w)\xi_{\ell}^{val(\upsilon^{\ell-1}(w))}$$

$$\forall w \in E^* \setminus F^*(1 + \mathfrak{p}_E)$$

but since $val(w) = val(\upsilon^i(w))\ \forall w \in E^* \setminus F^*(1 + \mathfrak{p}_E)$ and $\forall i$, this equality turns into

$$c[\chi_1](w) = \xi_\ell^{val(w)}[\chi_1](w) \ \forall w \in E^* \setminus F^*(1+\mathfrak{p}_E) \tag{8.1}$$

Now, again, since we proved that $[\chi_1]$ doesn't vanish completely on $E^* \setminus F^*(1+\mathfrak{p}_E)$, there exists a $w' \in E^* \setminus F^*(1+\mathfrak{p}_E)$ such that $[\chi_1](w') \neq 0$, and thus plugging that into Equation (5.6), we can cancel $[\chi_1](w')$ from both sides to get

$$c = \xi_\ell^{val(w')}$$

Therefore, c is an ℓth root of unity.

Now, suppose that $c = 1$. Then this implies that $[\chi_1](w) = [\chi_2](w) \ \forall w \in E^* \setminus F^*(1+\mathfrak{p}_E)$. We already know that $[\chi_1](w) = [\chi_2](w) \ \forall w \in F^*(1+\mathfrak{p}_E)$ (since $\chi_1|_{F^*(1+\mathfrak{p}_E)} = \chi_2|_{F^*(1+\mathfrak{p}_E)}$) and so we get $[\chi_1](w) = [\chi_2](w) \ \forall w \in E^*$. Therefore, by linear independence of characters, $\chi_1 = \chi_2$, and we're done.

Suppose $c \neq 1$. Then since ℓ is odd and since we concluded above that c is an ℓth root of unity, $c \in \mathbb{C} \setminus \mathbb{R}$. But the original definition of c was that $c = \frac{c_1}{c_2}$, and these constants c_1 and c_2 are from [21, Corollary 7.15]. In particular, they are both real numbers. Since the quotient of two real numbers can't be a non-real complex number, we have a contradiction, and therefore we are done. □

Proposition 8.5.3. *Let E/F be ramified non-Galois. Suppose $(E/F, \chi_1), (E/F, \chi_2)$ are admissible pairs such that $F(\tilde{\chi}_1)(w) = F(\tilde{\chi}_2)(w) \ \forall w \in E^* : n(w) = 0$. Then $\chi_1 = \chi_2$.*

Proof. Assume $F(\tilde{\chi}_1)(w) = F(\tilde{\chi}_2)(w)$ on the set $\{w \in E^* : n(w) = 0\}$. Then since $Aut(E/F) = 1$, by inspecting the formula for $F(\tilde{\chi}_1), F(\tilde{\chi}_2)$, we see that there are constants c_1, c_2 such that $c_1\chi_1(w) = c_2\chi_2(w)\ \forall w : n(w) = 0$. So we have that $c\chi_1(w) = \chi_2(w)\ \forall w : n(w) = 0$ where $c = \frac{c_1}{c_2}$. By [21, Lemma 5.1], this says that $\chi_1(w) = \chi_2(w)\ \forall w \in F^*(1 + \mathfrak{p}_E)$. But then, we can use the exact same proof as in Proposition (8.5.2) adapted to the fact that we don't have a sum of Galois conjugates of characters but just the character itself. □

Proposition 8.5.4. *Let E/F be unramified. Suppose $(E/F, \chi_1), (E/F, \chi_2)$ are admissible pairs such that $F(\tilde{\chi}_1)(w) = F(\tilde{\chi}_2)(w)\ \forall w \in E^* : n(w) = 0$. Then $\chi_1 = \chi_2^v$ for some $v \in Aut(E/F)$.*

Proof. See [21, page 16]). □

Hence, we are now done with all cases.

8.6 On whether there are two positive depth character formulas coming from different Cartans

In this section we answer the question of uniqueness with respect to positive depth supercuspidal representations coming from distinct Cartans. We show that if the character formula $F(\tilde{\chi})$ comes from the Cartan E, then there is no positive depth supercuspidal representation coming from a different Cartan $E_1 \ncong E$ whose character agrees with $F(\tilde{\chi})$ on the set $\{w \in E^* : 0 \leq n(w) \leq r/2\}$. We know that there is a positive depth supercuspidal representation whose character formula agrees with ours on this range, and there are no other positive depth supercuspidal characters coming from E^* that agree with $F(\tilde{\chi})$ on the range. In this section we show that the distribution characters of two positive depth supercuspidal representations, coming from different Cartans, can't agree on the $n(w) = 0$ range. This, together with the results from the previous section, shows that if $(E/F, \chi)$ is a regular pair such that χ has positive level, then there is a unique positive depth supercuspidal representation whose distribution character agrees with $F(\tilde{\chi})$ on the range $w \in E^* : n(w) = 0$.

Note that we are using our formulation $F(\tilde{\chi})$ of the supercuspidal characters of $GL(\ell, F)$, and we are implicitly using the fact which we proved that every supercuspidal character on a Cartan E^* of $GL(\ell, F)$ is of the form $F(\tilde{\chi})$ for some $\tilde{\chi}$.

Theorem 8.6.1. *Suppose $(E/F, \chi)$ and $(E_1/F, \chi_1)$ are admissible pairs with $E \ncong E_1$. Then*

$\exists w \in E^* : n(w) = 0$ such that $F(\tilde{\chi})(w) \neq \theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w)$.

There are many cases to check, and we split them up in a sequence of propositions.

Proposition 8.6.2. *Suppose $(E/F, \chi)$ and $(E_1/F, \chi_1)$ are admissible pairs with E ramified Galois and E_1 unramified. Then $\exists w \in E^* : n(w) = 0$ such that $F(\tilde{\chi})(w) \neq \theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w)$.*

Proof. Suppose E is ramified Galois, and E_1 is unramified. Then it is shown in [22] that $\theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w) = 0$ $\forall w \in E^* : n(w) = 0$. Thus, if we can find a single element of $\{w \in E^* : n(w) = 0\}$ such that $F(\tilde{\chi})(w) \neq 0$, then we'd be done.

By way of contradiction, suppose $F(\tilde{\chi})(w) = 0 \ \forall w \in E^* : n(w) = 0$. This says that

$$\chi(w) + \chi(\upsilon(w)) + \chi(\upsilon^2(w)) + ... + \chi(\upsilon^{n-1}(w)) = 0 \ \forall w \in E^* : n(w) = 0,$$

where υ generates $Aut(E/F)$, since $\chi(w)+\chi(\upsilon(w))+\chi(\upsilon^2(w))+...+\chi(\upsilon^{n-1}(w))$ is the numerator of these character formulas in the $n(w) = 0$ range. Let $\chi^{\upsilon}, \chi^{\upsilon^2}, ..., \chi^{\upsilon^{n-1}}$ denote the characters

$$\chi^{\upsilon}(w) = \chi(\upsilon(w)), \chi^{\upsilon^2}(w) = \chi(\upsilon^2(w)), ..., \chi^{\upsilon^{n-1}}(w) = \chi(\upsilon^{n-1}(w)).$$

Then we are supposing by way of contradiction that $\chi(w) + \chi^{\upsilon}(w) + \chi^{\upsilon^2}(w) + ... + \chi^{\upsilon^{n-1}}(w) = 0 \ \forall w \in E^* : n(w) = 0$.

Recall the decomposition $E^* = \pi_E^{\mathbb{Z}} \times \mu_F \times U_E^1$, where μ_F are roots of unity and $U_E^1 = 1 + \mathfrak{p}_E$, and π_E is the uniformizer of E. This decomposes further into

$$E^* = (\pi_E^{\ell\mathbb{Z}} \times \mu_F \times U_E^1) \bigcup (\pi_E^{\ell\mathbb{Z}+1} \times \mu_F \times U_E^1) \bigcup \cdots \bigcup (\pi_E^{\ell\mathbb{Z}+\ell-1} \times \mu_F \times U_E^1).$$

Let $B := E^* \setminus F^*(1 + \mathfrak{p}_E) = \{w \in E^* : n(w) = 0\}$. Let $B_1 := \pi_E^{\ell\mathbb{Z}+1} \times \mu_F \times U_E^1$. Let $w, z_1, ..., z_\ell \in B_1$. Then $wz_1, wz_2, ..., wz_\ell \in B$. Therefore, because we assumed by way of contradiction that $\theta(w) = 0 \ \forall w \in E^* : n(w) = 0$, we have that

$$\chi(wz_1) + \chi^{\upsilon}(wz_1) + \chi^{\upsilon^2}(wz_1) + ... + \chi^{\upsilon^{\ell-1}}(wz_1) = 0$$

$$\chi(wz_2) + \chi^{\upsilon}(wz_2) + \chi^{\upsilon^2}(wz_2) + ... + \chi^{\upsilon^{\ell-1}}(wz_2) = 0$$

$$\cdots$$

$$\chi(wz_\ell) + \chi^{\upsilon}(wz_\ell) + \chi^{\upsilon^2}(wz_\ell) + ... + \chi^{\upsilon^{\ell-1}}(wz_\ell) = 0$$

Rewriting this, we have

$$\chi(z_1)\chi(w) + \chi^{\upsilon}(z_1)\chi^{\upsilon}(w) + ... + \chi^{\upsilon^{\ell-1}}(z_1)\chi^{\upsilon^{\ell-1}}(w) = 0$$

$$\chi(z_2)\chi(w) + \chi^{\upsilon}(z_2)\chi^{\upsilon}(w) + ... + \chi^{\upsilon^{\ell-1}}(z_2)\chi^{\upsilon^{\ell-1}}(w) = 0$$

$$\cdots$$

$$\chi(z_\ell)\chi(w) + \chi^{\upsilon}(z_\ell)\chi^{\upsilon}(w) + ... + \chi^{\upsilon^{\ell-1}}(z_\ell)\chi^{\upsilon^{\ell-1}}(w) = 0$$

Letting $\chi_1 = \chi$, $\chi_2 = \chi^{\upsilon}$, $\chi_3 = \chi^{\upsilon^2}$, ..., $\chi_\ell = \chi^{\upsilon^{\ell-1}}$, then in matrix form, this becomes

$$\begin{pmatrix} \chi_1(z_1) & \chi_2(z_1) & \chi_3(z_1) & \cdots & \chi_\ell(z_1) \\ \chi_1(z_2) & \chi_2(z_2) & \chi_3(z_2) & \cdots & \chi_\ell(z_2) \\ \chi_1(z_3) & \chi_2(z_3) & \chi_3(z_3) & \cdots & \chi_\ell(z_3) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \chi_1(z_\ell) & \chi_2(z_\ell) & \chi_3(z_\ell) & \cdots & \chi_\ell(z_\ell) \end{pmatrix} \begin{pmatrix} \chi_1(w) \\ \chi_2(w) \\ \chi_3(w) \\ \vdots \\ \chi_\ell(w) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

This says that the determinant of the matrix

$$\begin{pmatrix} \chi_1(z_1) & \chi_2(z_1) & \chi_3(z_1) & \cdots & \chi_\ell(z_1) \\ \chi_1(z_2) & \chi_2(z_2) & \chi_3(z_2) & \cdots & \chi_\ell(z_2) \\ \chi_1(z_3) & \chi_2(z_3) & \chi_3(z_3) & \cdots & \chi_\ell(z_3) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \chi_1(z_\ell) & \chi_2(z_\ell) & \chi_3(z_\ell) & \cdots & \chi_\ell(z_\ell) \end{pmatrix}$$

is zero for all $z_1, z_2, ..., z_\ell \in B_1$. Now fix some $z_2, ..., z_\ell \in B_1$. Then, expanding the determinant by the first row, we get the important equality that

$$a_1\chi_1(z_1) + a_2\chi_2(z_1) + ... + a_\ell\chi_\ell(z_1) = 0 \ \forall z_1 \in B_1$$

where a_i are the determinants of the obvious $(\ell-1)\times(\ell-1)$ subminors. Now we are about to replace z_1 with other elements in B_1 to our advantage. We split into 2 subcases:

SubCase 1: Suppose that none of the a_i vanished. First note that if $\alpha_1, \alpha_2, ..., \alpha_\ell \in F^*(1+\mathfrak{p}_E)$, then $\alpha_i z_1 \in B_1 \ \forall i$. Therefore, the equality

$$a_1\chi_1(z_1) + a_2\chi_2(z_1) + ... + a_\ell\chi_\ell(z_1) = 0 \ \forall z_1 \in B_1$$

implies that

$$a_1\chi_1(\alpha_1 z_1) + a_2\chi_2(\alpha_1 z_1) + ... + a_\ell\chi_\ell(\alpha_1 z_1) = 0$$

$$a_1\chi_1(\alpha_2 z_1) + a_2\chi_2(\alpha_2 z_1) + ... + a_\ell\chi_\ell(\alpha_2 z_1) = 0$$

$$\vdots$$

$$a_1\chi_1(\alpha_\ell z_1) + a_2\chi_2(\alpha_\ell z_1) + ... + a_\ell\chi_\ell(\alpha_\ell z_1) = 0$$

which, written differently, is

$$\chi_1(\alpha_1)a_1\chi_1(z_1) + \chi_2(\alpha_1)a_2\chi_2(z_1) + ... + \chi_\ell(\alpha_1)a_\ell\chi_\ell(z_1) = 0$$

$$\chi_1(\alpha_2)a_1\chi_1(z_1) + \chi_2(\alpha_2)a_2\chi_2(z_1) + ... + \chi_\ell(\alpha_2)a_\ell\chi_\ell(z_1) = 0$$

$$\vdots$$

$$\chi_1(\alpha_\ell)a_1\chi_1(z_1) + \chi_2(\alpha_\ell)a_2\chi_2(z_1) + ... + \chi_\ell(\alpha_\ell)a_\ell\chi_\ell(z_1) = 0$$

Therefore, we have that

$$\begin{pmatrix} \chi_1(\alpha_1) & \chi_2(\alpha_1) & \chi_3(\alpha_1) & \cdots & \chi_\ell(\alpha_1) \\ \chi_1(\alpha_2) & \chi_2(\alpha_2) & \chi_3(\alpha_2) & \cdots & \chi_\ell(\alpha_2) \\ \chi_1(\alpha_3) & \chi_2(\alpha_3) & \chi_3(\alpha_3) & \cdots & \chi_\ell(\alpha_3) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \chi_1(\alpha_\ell) & \chi_2(\alpha_\ell) & \chi_3(\alpha_\ell) & \cdots & \chi_\ell(\alpha_\ell) \end{pmatrix} \begin{pmatrix} a_1\chi_1(z_1) \\ a_2\chi_2(z_1) \\ a_3\chi_3(z_1) \\ \vdots \\ a_\ell\chi_\ell(z_1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Thus the determinant of

$$\begin{pmatrix} \chi_1(\alpha_1) & \chi_2(\alpha_1) & \chi_3(\alpha_1) & \cdots & \chi_\ell(\alpha_1) \\ \chi_1(\alpha_2) & \chi_2(\alpha_2) & \chi_3(\alpha_2) & \cdots & \chi_\ell(\alpha_2) \\ \chi_1(\alpha_3) & \chi_2(\alpha_3) & \chi_3(\alpha_3) & \cdots & \chi_\ell(\alpha_3) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \chi_1(\alpha_\ell) & \chi_2(\alpha_\ell) & \chi_3(\alpha_\ell) & \cdots & \chi_\ell(\alpha_\ell) \end{pmatrix}$$

is zero for all $\alpha_1, \alpha_2, ..., \alpha_\ell \in F^*(1 + \mathfrak{p}_E)$. Thus, again, fixing $\alpha_2, ..., \alpha_\ell$, and expanding this determinant along the top row, we get that

$$b_1\chi_1(\alpha_1) + b_2\chi_2(\alpha_1) + ... + b_n\chi_\ell(\alpha_1) = 0 \ \ \forall \alpha_1 \in F^*(1 + \mathfrak{p}_E)$$

where b_i are the determinants of the obvious $(n-1)\times(n-1)$ subminors. Now we split again into three more subcases of this particular subcase.

Subcase 1(i): Suppose none of the b_i vanish. Then, we have that the character

$$b_1\chi_1 + ... + b_n\chi_\ell$$

vanishes on the group $F^*(1+\mathfrak{p}_E)$. Then by orthogonality of characters, we get that $\chi_1 = \chi_2 = ... = \chi_\ell$ on the group $F^*(1+\mathfrak{p}_E)$. More specifically, assume by way of contradiction that all the characters χ_i are mutually distinct. Then, by using an inner product on characters, we get that

$$(b_1\chi_1 + ... + b_n\chi_\ell, \chi_1) = b_1$$

on the one hand, but on the other hand, since $b_1\chi_1 + ... + b_n\chi_\ell = 0$, we get that

$$(b_1\chi_1 + ... + b_n\chi_\ell, \chi_1) = 0$$

which says that $b_1 = 0$, a contradiction. Thus we must have that $\chi_i = \chi_j$ for some $i \neq j$. But then this says that $\chi^{\upsilon^r}(\alpha_1) = \chi^{\upsilon^s}(\alpha_1)$ where $r = i-1, s = j-1$. But notice that $\upsilon(\alpha) \in F^*(1+\mathfrak{p}_E)$ $\forall \alpha \in F^*(1+\mathfrak{p}_E)$. Therefore, we have that $\chi^{\upsilon^r}(\upsilon(\alpha_1)) = \chi^{\upsilon^s}(\upsilon(\alpha_1))$, which says that $\chi^{\upsilon^{r+1}}(\alpha_1) = \chi^{\upsilon^{s+1}}(\alpha_1)$ $\forall \alpha_1 \in F^*(1+\mathfrak{p}_E)$, $\chi^{\upsilon^{r+2}}(\alpha_1) = \chi^{\upsilon^{s+2}}(\alpha_1)$ $\forall \alpha_1 \in F^*(1+\mathfrak{p}_E)$, etc. A quick analysis shows that this implies that $\chi_1 = \chi_2 = ... = \chi_\ell$ on $F^*(1+\mathfrak{p}_E)$, contradicting admissibility of the pair $(E/F, \chi)$ since E/F is ramified.

Subcase 1(ii): Suppose there was some b_i that was zero. Without loss of generality, suppose $b_1 = 0$. Then

$$b_2\chi_2 + ... + b_n\chi_\ell = 0$$

on $F^*(1 + \mathfrak{p}_E)$. If none of $b_2, b_3, ..., b_n$ vanish, then you use the orthogonality of characters argument as in Subcase 1(i). On the other hand, suppose without loss of generality that $b_2 = 0$. Then $b_3\chi_3 + ... + b_n\chi_\ell = 0$ on $F^*(1+\mathfrak{p}_E)$. If none of the $b_3, ..., b_n$ vanish, then again you use the orthogonality of characters argument. So we are eventually led to consider the case that $b_1 = b_2 = b_3 = \cdots = b_n = 0$, which is the next subcase.

Subcase 1(iii): Recall that in Subcase 1, we fixed $\alpha_2, ..., \alpha_\ell$, and expanded a big matrix along the first row. We asserted that if none, or some (but not all), of the b_i vanished (i.e. subcase 1(i) and 1(ii), respectively), then we could use some orthogonality of characters arguments to derive a contradiction and thus be done. Well, if all the b_i vanished, then maybe we can search for other $\alpha_2, ..., \alpha_\ell$ such that either none of the b_i vanished, or some (but not all) of the b_i vanished, and then proceeded with Subcase 1(i) or Subcase 1(ii). That is, it seems plausible that there are some $\alpha_2, ..., \alpha_\ell$ out there such that none of the corresponding b_i vanish (or at least maybe we could find $\alpha_2, ..., \alpha_\ell$ such that not all of the corresponding b_i vanished which would be Subcase 1(ii)). We now prove this by way of contradiction. So suppose by way of contradiction that for any choice of $\alpha_2, ..., \alpha_\ell \in F^*(1+\mathfrak{p}_E)$, we get that all of the corresponding b_i vanish.

Then, in particular, $b_1 = 0$ for any choice of $\alpha_2, ..., \alpha_\ell$, so since b_1 is the determinant of the subminor

$$\begin{pmatrix} \chi_2(\alpha_2) & \chi_3(\alpha_2) & \chi_4(\alpha_2) & \cdots & \chi_\ell(\alpha_2) \\ \chi_2(\alpha_3) & \chi_3(\alpha_3) & \chi_4(\alpha_3) & \cdots & \chi_\ell(\alpha_3) \\ \chi_2(\alpha_4) & \chi_3(\alpha_4) & \chi_4(\alpha_4) & \cdots & \chi_\ell(\alpha_4) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \chi_2(\alpha_\ell) & \chi_3(\alpha_\ell) & \chi_4(\alpha_\ell) & \cdots & \chi_\ell(\alpha_\ell) \end{pmatrix}$$

then again, fixing $\alpha_3, \alpha_4, ..., \alpha_\ell$, and expanding this determinant by the top row, we get an equality

$$b_2'\chi_2(\alpha_2) + b_3'\chi_3(\alpha_2) + ... + b_n'\chi_\ell(\alpha_2) = 0 \; \forall \alpha_2 \in F^*(1 + \mathfrak{p}_E)$$

i.e.

$$b_2'\chi_2 + b_3'\chi_3 + ... + b_n'\chi' = 0$$

on the group $F^*(1 + \mathfrak{p}_E)$, where b_j' are the determinants of the obvious subminors. Therefore, if none of the b_j' vanish, again you use an orthogonality of characters argument to give that $\chi_{i'} = \chi_{j'}$ on $F^*(1 + \mathfrak{p}_E)$ for some i', j', and therefore for all i', j'. If some b_j' vanishes, then you proceed as before, but if all of the b_j' vanish, then in particular, some b_j' vanishes, so suppose $b_2' = 0$. Then, again, as in the beginning of subcase 1(iii) where we say there in the beginning that "$b_1 = 0$", we get another subminor

$$\begin{pmatrix} \chi_3(\alpha_3) & \chi_4(\alpha_3) & \chi_5(\alpha_3) & \cdots & \chi_\ell(\alpha_3) \\ \chi_3(\alpha_4) & \chi_4(\alpha_4) & \chi_5(\alpha_4) & \cdots & \chi_\ell(\alpha_4) \\ \chi_3(\alpha_5) & \chi_4(\alpha_5) & \chi_5(\alpha_5) & \cdots & \chi_\ell(\alpha_5) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \chi_3(\alpha_\ell) & \chi_4(\alpha_\ell) & \chi_5(\alpha_\ell) & \cdots & \chi_\ell(\alpha_\ell) \end{pmatrix}$$

whose determinant is zero $\forall \alpha_3, \alpha_4, ..., \alpha_\ell \in F^*(1+\mathfrak{p}_E)$. Continuing to proceed in the same way, we get to a situation where not all of the coefficients of an equation of the form $\sum c_i \chi_i = 0$ vanish, so that we can use an orthogonality of characters argument, or, eventually, we are left with continuing to consider more and more subminors, but eventually we are left with a subminor of the form

$$\begin{pmatrix} \chi_{i''}(\alpha_{n'}) & \chi_{j''}(\alpha_{n'}) \\ \chi_{i''}(\alpha_{n'+1}) & \chi_{j''}(\alpha_{n'+1}) \end{pmatrix}$$

whose determinant would be zero for some $i'', j'' \in \{1, 2, ..., n\}$ and some $n' \in \{1, 2, ..., n\}$. Then this would imply that

$$\frac{\chi_{i''}(\alpha_{n'})}{\chi_{j''}(\alpha_{n'})} = \frac{\chi_{i''}(\alpha_{n'+1})}{\chi_{j''}(\alpha_{n'+1})} \ \forall \alpha_{n'}, \alpha_{n'+1} \in F^*(1+\mathfrak{p}_E)$$

Then plugging in $\alpha_{n'+1} = 1$, we get that $\chi_{i''} = \chi_{j''}$ on $F^*(1+\mathfrak{p}_E)$, and thus $\chi_1 = \chi_2 = ... = \chi_\ell$ on $F^*(1+\mathfrak{p}_E)$, contradicting admissibility of $(E/F, \chi)$.

SubCase 2: Suppose there was some a_i that vanished. More precisely, we make a similar argument as in the first paragraph in Subcase 1(iii). That is, suppose we couldn't find any $z_2, z_3, ..., z_\ell$ such that no a_i vanished (because if we could, then we would be in Subcase 1). Suppose without loss of generality that $a_1 = 0$ for all choices of $z_2, ..., z_\ell$. Then this says that the determinant of

$$\begin{pmatrix} \chi_2(z_2) & \chi_3(z_2) & \chi_4(z_2) & \cdots & \chi_\ell(z_2) \\ \chi_2(z_3) & \chi_3(z_3) & \chi_4(z_3) & \cdots & \chi_\ell(z_3) \\ \chi_2(z_4) & \chi_3(z_4) & \chi_4(z_4) & \cdots & \chi_\ell(z_4) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \chi_2(z_\ell) & \chi_3(z_\ell) & \chi_4(z_\ell) & \cdots & \chi_\ell(z_\ell) \end{pmatrix}$$

is zero $\forall z_2, z_3, ..., z_\ell \in B_1$. Then again expanding this determinant by the top row, we get an equality

$$a_2'\chi_2(z_2) + a_3'\chi_3(z_2) + ... + a_\ell'\chi_\ell(z_2) = 0 \ \forall z_2 \in B_1$$

where a_j' are the determinants of the obvious subminors. Therefore, if none of the a_j' vanish (i.e. if we can find some $z_3, ..., z_\ell$ such that none of the a_j' vanish), we can proceed as in SubCase 1. If some a_j' vanishes for all choices of $z_3, ..., z_\ell$, then suppose without loss of generality that $a_2' = 0$ for all choice of $z_3, z_4, ..., z_\ell$. Then, again, we get another subminor

$$\begin{pmatrix} \chi_3(z_3) & \chi_4(z_3) & \chi_5(z_3) & \cdots & \chi_\ell(z_3) \\ \chi_3(z_4) & \chi_4(z_4) & \chi_5(z_4) & \cdots & \chi_\ell(z_4) \\ \chi_3(z_5) & \chi_4(z_5) & \chi_5(z_5) & \cdots & \chi_\ell(z_5) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \chi_3(z_\ell) & \chi_4(z_\ell) & \chi_5(z_\ell) & \cdots & \chi_\ell(z_\ell) \end{pmatrix}$$

whose determinant is zero for all $z_3, z_4, ..., z_\ell \in B_1$. Continuing in this way, we will eventually have that none of the coefficients we have will vanish, so that we are reduced back to SubCase 1, or we eventually end up with a subminor

$$\begin{pmatrix} \chi_{i''}(z_{n'}) & \chi_{j''}(z_{n'}) \\ \chi_{i''}(z_{n'+1}) & \chi_{j''}(z_{n'+1}) \end{pmatrix}$$

whose determinant would be zero for some $i'', j'' \in \{1, 2, ..., n\}$ and some $n'' \in \{1, 2, ..., n\}$. Then this would imply that

$$\frac{\chi_{i''}(z_{n'})}{\chi_{j''}(z_{n'})} = \frac{\chi_{i''}(z_{n'+1})}{\chi_{j''}(z_{n'+1})} \ \forall z_{n'}, z_{n'+1} \in B_1$$

But then if we fix $z'_{n'+1} \in B_1$, then

$$\frac{\chi_{i''}(z_{n'})}{\chi_{j''}(z_{n'})} = \frac{\chi_{i''}(z'_{n'+1})}{\chi_{j''}(z'_{n'+1})} = c \ \forall z_{n'} \in B_1$$

where we define $c := \frac{\chi_{i''}(z'_{n'+1})}{\chi_{j''}(z'_{n'+1})}$. But then since $y^{n+1} \in B_1 \forall y \in B_1$, this implies that

$$c^{n+1} = \left(\frac{\chi_{i''}(z'_{n'+1})}{\chi_{j''}(z'_{n'+1})}\right)^{n+1} = \frac{\chi_{i''}(z_{n'}^{n+1})}{\chi_{j''}(z_{n'}^{n+1})} = c \ \forall z_{n'} \in B_1$$

Therefore, $c^n = 1$. But then if you let $a_1, ..., a_\ell \in B_1$, then you get

$$\frac{\chi_{i''}(a_1 a_2 ... a_\ell)}{\chi_{j''}(a_1 a_2 ... a_\ell)} = \frac{\chi_{i''}(a_1)}{\chi_{j''}(a_1)} \frac{\chi_{i''}(a_2)}{\chi_{j''}(a_2)} \frac{\chi_{i''}(a_3)}{\chi_{j''}(a_3)} \cdots \frac{\chi_{i''}(a_\ell)}{\chi_{j''}(a_\ell)} = c^n = 1$$

But this implies that $\frac{\chi_{i''}}{\chi_{j''}}|_{F^*(1+\mathfrak{p}_E)}$ since $F^*(1+\mathfrak{p}_E) = \{x_1 x_2 ... x_n : x_i \in B_1 \ \forall i\}$. Therefore, again we get that $\chi_1 = \chi_2 = ... = \chi_\ell$ on $F^*(1+\mathfrak{p}_E)$, contradicting admissibility of the pair $(E/F, \chi)$.

This concludes the proof of Proposition (8.6.2) □

Proposition 8.6.3. *Suppose E/F is ramified non-Galois, and E_1/F is unramified. Suppose $(E/F, \chi)$ and $(E_1/F, \chi_1)$ are admissible pairs. Then $\exists w \in E^* : n(w) = 0$ such that $F(\tilde{\chi})(w) \neq \theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w)$.*

Proof. It is shown in [22] that $\theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w) = 0 \ \forall w \in E^* : n(w) = 0$. Thus, if we can find a single element of $\{w \in E^* : n(w) = 0\}$ such that $F(\tilde{\chi})(w) \neq 0$, then we'd be done.

But this is clear, because the numerator $\chi(w)\Delta_\chi(w)$, of $F(\tilde{\chi})$, takes values in $\mathbb{C}^*$. □

Proposition 8.6.4. *Suppose E/F is unramified, and E_1/F is ramified (Galois or non-Galois, it doesn't matter). Suppose $(E/F, \chi)$ and $(E_1/F, \chi_1)$ are admissible pairs. Then $\exists w \in E^* : n(w) = 0$ such that $F(\tilde{\chi})(w) \neq \theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w)$.*

Proof. It is shown in [23] that $\theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w) = 0 \ \forall w \in E^* : n(w) = 0$. Thus, if we can find a single element of $\{w \in E^* : n(w) = 0\}$ such that $F(\tilde{\chi})(w) \neq 0$, then we'd be done. So suppose by way of contradiction that $F(\tilde{\chi})(w) = 0 \ \forall w \in E^* : n(w) = 0$.

In fact, there is always an element $w \in E^* : n(w) = 0$ such that $F(\tilde{\chi})(w) \neq 0$. The following proof was communicated to me by Loren Spice and can be found in his thesis ([21, page 11-12, 16]):

We first start with some results on finite fields. We introduce some new notation. Let F be a local field and $\mathbb{F}$ its residue field. Let $\mathbb{E}$ be the degree ℓ extension of of $\mathbb{F}$. If ψ is a character of $\mathbb{E}^*$, then $\tilde{\psi}$ denotes $\sum_{\upsilon \in Aut(\mathbb{E}/\mathbb{F})} \psi^{\upsilon}$. In this section, we call a subgroup $\mathbb{K}$ of $\mathbb{E}^*$ *acceptable* if it is not contained in $\mathbb{F}^*$ and if $[\mathbb{K} : \mathbb{K} \bigcap \mathbb{F}^*]$ is coprime to $|\mathbb{F}^*|$.

Lemma 8.6.5. *Acceptable subgroups exist.*

Proof. [21, Theorem 5.4] □

Note that if $\mathbb{K}$ is acceptable, then so is $\mathbb{F}^*\mathbb{K}$ and any $\mathbb{H} \leq \mathbb{K}$ which is not contained in $\mathbb{F}^*$.

Restricting our attention to acceptable subgroups allow us to work only with characters which are trivial on $\mathbb{F}^*$. We make this more precise. Since $\mathbb{E}^*$ is cyclic, if $\mathbb{K}$ is acceptable, then there is a subgroup $\mathbb{H}$ of $\mathbb{E}^*$ so that $\mathbb{K} = \mathbb{H} \times (\mathbb{K} \bigcap \mathbb{F}^*)$. For any character ψ of $\mathbb{K}$, we denote by $T_{\mathbb{K}}(\psi)$ the character of $\mathbb{K}$ which agrees with ψ on $\mathbb{H}$ but is trivial on $\mathbb{K} \bigcap \mathbb{F}^*$. Put $S_{\mathbb{K}}(\psi) = \psi^{-1} T_{\mathbb{K}}(\psi)$. Note that $S_{\mathbb{K}}(\psi)$ is completely determined by $\psi|_{\mathbb{K} \bigcap \mathbb{F}^*}$. Since $S_{\mathbb{K}}(\psi)$ is trivial on $\mathbb{H}$, it satisfies $S_{\mathbb{K}}(\psi) = S_{\mathbb{K}}(\psi)^{\upsilon}$ for all $\upsilon \in Aut(\mathbb{E}/\mathbb{F})$. Thus we have that $T_{\mathbb{K}}(\psi))^{\sim} = S_{\mathbb{K}}(\psi)\tilde{\psi}$. This fact will allow us to replace ψ by $T_{\mathbb{K}}(\psi)$ in our calculations.

The next lemma plays a technical role.

Lemma 8.6.6. *If $\mathbb{K} \leq \mathbb{E}^*$ is acceptable, then, for any character of $\mathbb{E}^*$, there is some $\alpha \in \mathbb{K} \setminus \mathbb{F}^*$ so that $\tilde{\psi}(\alpha) \neq 0$.*

Proof. Put $\psi' = T_{\mathbb{K}}(\psi)$ and $\chi = S_{\mathbb{K}}(\psi)$. Since $\tilde{\psi}' = \chi\tilde{\psi}$, it suffices to prove that there is some $\alpha \in \mathbb{K} \setminus \mathbb{F}^*$ so that $\tilde{\psi}'(\alpha) \neq 0$.

If ψ' is trivial on $\mathbb{K}$, then this is obvious. Thus, without loss of generality, ψ' is nontrivial on $\mathbb{K}$, so that

$$l|\mathbb{K} \bigcap \mathbb{F}^*| + \sum_{\alpha \in \mathbb{K} \setminus \mathbb{F}^*} \tilde{\psi}'(\alpha)$$

$$= \sum_{\alpha \in \mathbb{K} \bigcap \mathbb{F}^*} \tilde{\psi}'(\alpha) + \sum_{\alpha \in \mathbb{K} \setminus \mathbb{F}^*} \tilde{\psi}'(\alpha) = \sum_{\alpha \in \mathbb{K}} \tilde{\psi}'(\alpha) = \sum_{\alpha \in \mathbb{K}} \sum_{i=0}^{l-1} \psi'(\alpha^{q^i})$$
$$= \sum_{i=0}^{l-1} \sum_{\alpha \in \mathbb{K}} \psi'(\alpha^{q^i}) = \sum_{i=0}^{l-1} \sum_{\alpha \in \mathbb{K}} \psi'(\alpha) = 0.$$

Since the first summand (in the first sum) is nonzero, so is the second summand. This can happen only if there is *some* $\alpha \in \mathbb{K} \setminus \mathbb{F}^*$ so that $\tilde{\psi}'(\alpha) \neq 0$. $\square$

Since acceptable subgroups exist, in particular, for any character ψ of $\mathbb{E}^*$ there is some $\alpha \in \mathbb{E}^* \setminus \mathbb{F}^*$ so that $\tilde{\psi}(\alpha) \neq 0$.

Now, back to our setting where E/F is a degree ℓ extension of local fields and $\mathbb{E}$, $\mathbb{F}$ are their residue fields. Let χ be a character of E^*, and let ψ be the restriction of χ to $\mathbb{E}^*$, where we view $\mathbb{E}^*$ as embedded in E^* as a subset. (We may therefore regard $\mathbb{E}^*$ as a subset of E^*). Then $\mathbb{E}^* \setminus \mathbb{F}^* \subset E^* \setminus F^*(1+\mathfrak{p}_E) = \{w \in E^* : n(w) = 0\}$. Therefore, by the remark following the above lemma, there is some $\alpha \in E^* : n(w) = 0$ satisfying $\tilde{\psi}(\alpha) \neq 0$, and therefore $\chi(\alpha)+\chi^{\upsilon}(\alpha)+...+\chi^{\upsilon^{\ell-1}}(\alpha) \neq 0$ since $\tilde{\psi}(\alpha) = \chi(\alpha) + \chi^{\upsilon}(\alpha) + ... + \chi^{\upsilon^{\ell-1}}(\alpha)$. Therefore, $F(\tilde{\chi})(\alpha) \neq 0$, which is what we sought out to prove.

Therefore, our unramified character formula can't come from a ramified character formula. □

Proposition 8.6.7. *Suppose E/F is ramified Galois, and E_1/F is ramified Galois such that $E \ncong E_1$. Suppose $(E/F, \chi)$ and $(E_1/F, \chi_1)$ are admissible pairs. Then $\exists w \in E^* : n(w) = 0$ such that $F(\tilde{\chi})(w) \neq \theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w)$.*

Proof. Then, it is shown in [23] that $\theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w) = 0 \ \forall w \in E^* : n(w) = 0$. Thus, if we can find a single element of $\{w \in E^* : n(w) = 0\}$ such that $F(\tilde{\chi})(w) \neq 0$, then we'd be done.

By way of contradiction, suppose $F(\tilde{\chi})(w) = 0 \ \forall w \in E^* : n(w) = 0$. Then, the same proof from Proposition (8.6.2) works here.

□

Proposition 8.6.8. *Suppose E/F is ramified non-Galois, and E_1/F is ramified Galois. Suppose $(E/F, \chi)$ and $(E_1/F, \chi_1)$ are admissible pairs. Then $\exists w \in E^*$:*

$n(w) = 0$ *such that* $F(\tilde{\chi})(w) \neq \theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w)$.

Proof. This is tautological, since if F is a local field, then it can't simultaneously have a degree ℓ ramified non-Galois extension and a degree ℓ ramified Galois extension. □

Proposition 8.6.9. *Suppose E is ramified Galois, and E_1 is ramified non-Galois. Suppose $(E/F, \chi)$ and $(E_1/F, \chi_1)$ are admissible pairs. Then $\exists w \in E^* : n(w) = 0$ such that $F(\tilde{\chi})(w) \neq \theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w)$.*

Proof. This is tautological, since if F is a local field, then it can't simultaneously have a degree ℓ ramified non-Galois extension and a degree ℓ ramified Galois extension. □

Proposition 8.6.10. *Suppose E/F is ramified non-Galois, and E_1/F is ramified non-Galois such that $E \ncong E_1$. Suppose $(E/F, \chi)$ and $(E_1/F, \chi_1)$ are admissible pairs. Then $\exists w \in E^* : n(w) = 0$ such that $F(\tilde{\chi})(w) \neq \theta_{\pi_{\chi_1 \Delta_{\chi_1}}}(w)$.*

Proof. The same argument as in Proposition (8.6.3) works here. □

Now we are finished all cases. Therefore, we have finished the proof of Theorem (8.6.1). Summing up, we have altogether shown that if $(E/F, \chi)$ is a regular pair such that χ has positive level, then there is a unique positive depth supercuspidal representation, $\pi_{\chi \Delta_\chi}$, whose character, on the range $\{z \in T(F)^{reg} : 0 \leq n(z) \leq r/2\}$, agrees with $F(\tilde{\chi})$. There is one minor point here to resolve. Is there possibly a depth zero supercuspidal representation whose character, on the range $\{z \in T(F)^{reg} : 0 \leq n(z) \leq r/2\}$, also equals $F(\tilde{\chi})$? We will prove in the next chapter that if $(E_1/F, \chi_1)$

is a regular pair corresponding to a depth zero supercuspidal representation π via Theorem (7.3.1), then its character formula, on the range $\{z \in T(F)^{reg} : 0 \leq n(z) \leq r/2\}$, is

$$F(\tilde{\chi}_1)(w) = (-1)^{\ell+1} \frac{deg(\pi)}{deg(\sigma)} \tau_o((-1)^{\sum_{k=1}^{\frac{\ell-1}{2}} k}) \left(\frac{\sum_{i=0}^{\ell-1} \chi_1(\upsilon^i(w))}{\tau(\Delta^0(w, \Delta^+))\rho_\tau(w)} \right)$$

$$\forall w \in E_1^* \setminus F^*(1 + \mathfrak{p}_{E_1})$$

where υ is a generator of $Aut(E_1/F)$. Then, the same arguments as in Theorems (8.5.1) and (8.6.1) show that the character of π cannot equal $F(\tilde{\chi})$, on the range $\{z \in T(F)^{reg} : 0 \leq n(z) \leq r/2\}$, unless $\pi \cong \pi(\tilde{\chi})$.

Therefore, combining Theorems (8.5.1), (8.6.1), and (8.3.3) and (8.4.1), we obtain the following result.

Theorem 8.6.11. *The assignment*

$$\left\{ \text{irreducible } \phi : W_F \to GL(\ell, \mathbb{C}) \right\} \mapsto \tilde{\chi} \in \widehat{T(F)}_{\tau \circ \rho} \mapsto \pi(\tilde{\chi})$$

from Section (8.1) is the Local Langlands correspondence for positive depth supercuspidal representations of $GL(\ell, F)$, where $\pi(\tilde{\chi})$ is the unique supercuspidal representation whose character, on the range $\{z \in T(F)^{reg} : 0 \leq n(z) \leq r/2\}$, is $F(\tilde{\chi})$.

Chapter 9

Our constructions in the depth zero case for $GL(\ell, F)$, ℓ an odd prime

9.1 On the proof that our conjectural character formulas agree with depth zero supercuspidal characters

In the following two sections, we prove Theorems (8.1.1) and (8.1.2) for the case of depth zero supercuspidal representations of $GL(\ell, F)$, where ℓ is an odd prime. We prove that our conjectured character formula $F(\tilde{\chi})$ agrees with a supercuspidal character, again, on the range $\{z \in T(F)^{reg} : 0 \leq n(z) \leq r/2\}$. Note that for depth zero representations, $r = 0$. We show that this supercuspidal character is the character of the supercuspidal representation that occurs in the local Langlands correspondence $\pi_{\chi\Delta_\chi}$. We then prove that there are no other supercuspidal characters that agree with $F(\tilde{\chi})$, on the range $\{z \in T(F)^{reg} : 0 \leq n(z) \leq r/2\} = \{z \in T(F)^{reg} : n(z) = 0\}$.

Let us make a few preliminaries. Since regular pairs in the setting of depth zero supercuspidal representations of $GL(\ell, F)$ are of the form $(E/F, \chi)$ where in particular E/F is unramified, we have that $\delta_{E/F} = 1$. Moreover, since $\delta_{E/F} = 1$, we have that $\Delta_\chi = 1$.

Let us recall from the previous chapter that the proposed character formula

$$F(\tilde{\chi})(z) = \epsilon(\tilde{\chi}, \Delta^+, \tau) \frac{\sum_{s \in W} \epsilon(s)\tilde{\chi}({}^s w)}{\tau(\Delta^0(z, \Delta^+))(\tau \circ \rho)(w)}, \quad w \in T(F)^{reg}$$

reduces to

$$F(\tilde{\chi})(w) = \epsilon(\tilde{\chi}, \Delta^+, \tau) \frac{\sum_{s \in Aut(E/F)} \chi({}^s w)}{\tau(\Delta^0(w, \Delta^+))\rho_\tau(w)}, \quad w \in T(F)^{reg}$$

where W denotes the relative Weyl group $W(G(F), T(F))$. For depth zero representations, we define $\epsilon(\tilde{\chi}, \Delta^+, \tau) := (-1)^{\ell+1} \frac{deg(\pi)}{deg(\sigma)} \tau_o((-1)^{\sum_{k=1}^{\frac{\ell-1}{2}} k}) \epsilon(\Delta^+)$, where $\epsilon(\Delta^+)$ is as in Section (8.2) and $deg(\pi), deg(\sigma)$ are as in Theorem (6.1.1). Therefore, the formula simplifies to

$$F(\tilde{\chi})(w) = (-1)^{\ell+1} \frac{deg(\pi)}{deg(\sigma)} \tau_o((-1)^{\sum_{k=1}^{\frac{\ell-1}{2}} k}) \left(\frac{\sum_{i=0}^{\ell-1} \chi(\upsilon^i(w))}{\tau(\Delta^0(w, \Delta^+))\rho_\tau(w)} \right) \quad w \in T(F)^{reg}$$

where υ is a generator of $Aut(E/F)$.

Recall that the following theorem gives the formula for the depth zero supercuspidal representations of $GL(\ell, F)$:

Theorem 9.1.1. [1] *[9, Theorem 5.4.1]*

Suppose $\gamma \in F^* K_0^{reg}$.

$$\frac{\theta_\pi(\gamma)}{deg(\pi)} = \begin{cases} \chi_\pi(z) \frac{\chi_\sigma(\gamma)}{deg(\sigma)} & \text{if } \gamma = zw \text{ is unramified elliptic and } \gamma \\ & \text{is not in } F^* K_1, z \in Z, w \in K_0 \\ \chi_\pi(z) L.C.E. & \text{if } \gamma = z(1 + {}^g X) \text{ with } X \in \mathfrak{b}_1, g \in G, \text{ and } z \in Z \\ 0 & \text{otherwise} \end{cases}$$

[1] Notice that this theorem is slightly different than the one from [9]. It is because there are a few typos in [9].

Here, χ_π is the central character of the representation π.

We will compare our conjectured character formula to the supercuspidal characters of Theorem (9.1.1), on elements on the range $\{z \in T(F)^{reg} : n(z) = 0\}$. We will show that this means that we are only interested in character values on the elements $\gamma = zw$ where $z \in Z, w \in K_0$ where γ is unramified elliptic and γ is not in $F^* K_1$.

We will first simplify the supercuspidal characters of Theorem (9.1.1). They are written in terms of regular pairs, so we recall a few notions regarding regular pairs corresponding to depth zero supercuspidal representations. We will then compare the depth zero supercuspidal characters with our proposed character formula.

We now compute the supercuspidal characters of Theorem (9.1.1). They are written in terms of regular pairs, so we recall a few notions regarding the regular pairs for depth zero supercuspidal representations. We will then compare the formula of Theorem (9.1.1) with our proposed character formula. Let $(E/F, \chi)$ be a regular pair corresponding to a depth zero supercuspidal representation via Theorem (7.3.1). This means that E/F is unramified and χ has level zero, so $\chi|_{U_E}$ gives rise to a character θ of the multiplicative group of the residue field $\mathbb{F}_{q^\ell}$ of E. Note that when E/F is unramified, $(E/F, \chi)$ is regular if and only if $(E/F, \chi)$ is admissible. Let $\mathbb{G} := GL(\ell, \overline{\mathbb{F}_q})$. Let $\mathbb{T}$ be the maximal torus of $\mathbb{G}$ defined over $\mathbb{F}_q$ such that $\mathbb{T}^\Phi = \mathbb{F}_{q^\ell}^*$ is the elliptic torus in $GL(\ell, \mathbb{F}_q)$. Then, by Deligne-Lusztig theory, the pair $(\mathbb{T}, \theta)$ yields a generalized character $R_{\mathbb{T},\theta}$ of $\mathbb{G}(\mathbb{F}_q) = GL(\ell, \mathbb{F}_q)$. We need the values of this character on the semisimple elements elements.

Proposition 9.1.2. *If $s \in \mathbb{G}^{\Phi}$ is semisimple, then*

$$R_{\mathbb{T},\theta}(s) = \frac{\epsilon_{\mathbb{T}}\epsilon_{C^0(s)}}{|\mathbb{T}^{\Phi}||C^0(s)^{\Phi}|_p} \sum_{g \in \mathbb{G}^{\Phi}, g^{-1}sg \in \mathbb{T}^{\Phi}} \theta(g^{-1}sg)$$

Proof. [8, Proposition 7.5.3]. □

We will define the various terms in this formula throughout the section. Let us calculate the values of $R_{\mathbb{T},\theta}$ on the elliptic torus. Let $\mathbb{T}_s$ denote the split torus in $\mathbb{G}$. Then $\mathbb{T}$ is obtained from $\mathbb{T}_s$ by twisting by the canonical generator of the Weyl group, the permutation $w = (123...n)$ (cf [10, Definition 3.24]). That is,

$$t \in \mathbb{T}^{\Phi} \Longleftrightarrow g^{-1}tg \in \mathbb{T}_s^{w\Phi}$$

where $g^{-1}\Phi(g) \in N_{\mathbb{G}}(\mathbb{T}_s)$ maps to w in $W(\mathbb{G}, \mathbb{T}_s)$. Then, $\mathbb{T}_s^{w\Phi}$ is the group of fixed points of the morphism

$$\begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix} \mapsto \begin{pmatrix} a_n^q & 0 & 0 & \cdots & 0 \\ 0 & a_1^q & 0 & \cdots & 0 \\ 0 & 0 & a_2^q & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1}^q \end{pmatrix}$$

This says that

$$a_1 = a_n^q, a_2 = a_1^q, a_3 = a_2^q, ..., a_n = a_{n-1}^q$$

which says that $a = a^{q^n}$ so that $a \in \mathbb{F}_{q^n}^*$. Therefore, the group of fixed points is

$$\left(\begin{array}{ccccc} a & 0 & 0 & \cdots & 0 \\ 0 & a^q & 0 & \cdots & 0 \\ 0 & 0 & a^{q^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a^{q^{n-1}} \end{array}\right) : a \in \mathbb{F}_{q^n}^*$$

Proposition 9.1.3.

$$R_{\mathbb{T},\theta}(s) = (-1)^{\ell+1} \sum_{i=0}^{\ell-1} \theta(\upsilon^i(s))$$

for all regular semsimple s in $\mathbb{T}^\Phi$, where υ is a generator of $Gal(\mathbb{F}_{q^\ell}/\mathbb{F}_q)$.

Proof. We first need a lemma:

Lemma 9.1.4. *Suppose S is a maximal torus in a connected reductive group G and $s \in S$ is regular. Suppose $g \in G$ satisfies $g^{-1}sg \in S$. Then $g \in N_G(S)$.*

Proof. Let $C^0(s)$ be the connected centralizer of s in G. Let $h \in C^0(s)$. Then defining $s' := g^{-1}sg$, we get that $hs' = s'h$, since $C^0(s) = S$. Therefore, $g \in N_G(C^0(s)) = N_G(S)$.

□

Let $s \in \mathbb{T}^\Phi$. Since we are assuming that s is regular semisimple, we have by the above Lemma that

$$\sum_{g \in \mathbb{G}^\Phi \;:\; g^{-1}sg \in \mathbb{T}^\Phi} \theta(g^{-1}sg) = \sum_{g \in N_{\mathbb{G}^\Phi}(\mathbb{T}^\Phi)} \theta(g^{-1}sg) =$$
$$|\mathbb{T}^\Phi| \sum_{w \in N_{\mathbb{G}^\Phi}(\mathbb{T}^\Phi)/\mathbb{T}^\Phi} \theta({}^w s)$$

Therefore, the values of $R_{\mathbb{T},\theta}$ on regular semisimple elements simplify to

$$R_{\mathbb{T},\theta}(s) = \frac{\epsilon_{\mathbb{T}}\epsilon_{C^0(s)}}{|\mathbb{T}^{\Phi}||C^0(s)^{\Phi}|_p}|\mathbb{T}^{\Phi}| \sum_{w \in N_{\mathbb{G}^{\Phi}}(\mathbb{T}^{\Phi})/\mathbb{T}^{\Phi}} \theta({}^w s) = \frac{\epsilon_{\mathbb{T}}\epsilon_{C^0(s)}}{|C^0(s)^{\Phi}|_p} \sum_{i=0}^{\ell-1} \theta(\upsilon^i(s))$$

where υ is a generator of $Gal(\mathbb{F}_{q^\ell}/\mathbb{F}_q)$, since the relative Weyl group is $W(\mathbb{G}(\mathbb{F}_q), \mathbb{T}(\mathbb{F}_q)) = Aut(\mathbb{F}_{q^\ell}/\mathbb{F}_q)$ (see Theorem (A.0.3)). It remains to calculate the constants in front.

Now, since $s \in \mathbb{T}^{\Phi}$ is regular semisimple, then $C^0(s) = \mathbb{T}$. Therefore, $|C^0(s)^{\Phi}| = q^\ell - 1$, so $|C^0(s)^{\Phi}|_p = 1$, where $|\ |_p$ denotes the p-part of $|\ |$. Now, ϵ_H is defined to be $(-1)^{\mathbb{F}_q-rank\ of\ H}$, for any algebraic group H [10, page 66]. Therefore, $\epsilon_{\mathbb{T}} = -1$ and $\epsilon_{C^0(s)} = (-1)^\ell$. Therefore, the values of $R_{\mathbb{T},\theta}$ on regular semisimple elements of the elliptic torus are

$$R_{\mathbb{T},\theta}(s) = \epsilon_{\mathbb{T}}\epsilon_{C^0(s)} \sum_{i=0}^{\ell-1} \theta(\upsilon^i(s)) = (-1)^{\ell+1} \sum_{i=0}^{\ell-1} \theta(\upsilon^i(s))$$

□

We wish to show that our proposed character formula agrees with a depth zero supercuspidal character on the ranges where they are defined, i.e. $(F^*K_0 \setminus F^*K_1) \bigcap E^*$. In the next section we will prove that this supercuspidal character is unique. We need the following Lemma:

Lemma 9.1.5. $(F^*K_0 \setminus F^*K_1) \bigcap E^* = E^* \setminus F^*(1 + \mathfrak{p}_E)$

Proof. Note that $(F^*K_0 \setminus F^*K_1) \bigcap E^* = (F^*K_0 \bigcap E^*) \setminus (F^*K_1 \bigcap E^*)$. Then we claim that the set $F^*K_0 \bigcap E^* \setminus F^*K_1 \bigcap E^*$ is precisely the set $E^* \setminus F^*(1 + \mathfrak{p}_E)$.

To see this, one can check that $F^*K_0 \bigcap E^* = F^*(K_0 \bigcap E^*)$. Now, one knows that $K_0 \bigcap E^* = \mathfrak{o}_E^*$ (cf [9, Lemma 3.2.1 (2)]). Therefore, $F^*K_0 \bigcap E^* = F^*\mathfrak{o}_E^* = E^*$. Likewise, you $F^*K_1 \bigcap E^* = F^*(K_1 \bigcap E^*) = F^*(1+\mathfrak{p}_E)$. Therefore, we have that $(F^*K_0 \setminus F^*K_1) \bigcap E^* = E^* \setminus F^*(1+\mathfrak{p}_E)$. □

We now can prove

Theorem 9.1.6. *$F(\tilde{\chi})$ agrees with the character of the depth zero supercuspidal representation $\pi_{\chi\Delta_\chi}$ on the range $E^* \setminus F^*(1+\mathfrak{p}_E) = \{z \in E^* : n(z) = 0\}$.*

Proof. Since E/F is unramified, $\Delta_\chi \equiv 1$. Therefore, we need to show that

$$\tau_o((-1)^{\sum_{k=1}^{\frac{\ell-1}{2}} k}) \left(\frac{\sum_{i=0}^{\ell-1} \chi(\upsilon^i(w))}{\tau(\Delta^0(w,\Delta^+))\rho_\tau(w)} \right) = \sum_{i=0}^{\ell-1} \chi(\upsilon^i(w)) \quad \forall w \in E^* \setminus F^*(1+\mathfrak{p}_E)$$

Recall that $\tau(\Delta^0(w,\Delta^+))\rho_\tau(w) = \tau_o(\Delta^0(w,\Delta^+))\rho_{\tau_o}(w)|\Delta^0(w,\Delta^+)\rho(w)|$. Let $w \in E^*$. Recall that

$$\Delta^0(w,\Delta^+)\rho(w) =$$
$$(-1)^{\sum_{k=1}^{\frac{\ell-1}{2}} k} \frac{N_{EL/L}((w-\upsilon(w))(w-\upsilon^2(w))(w-\upsilon^3(w))...(w-\upsilon^{\frac{\ell-1}{2}}(w)))}{N_{E/F}(w)^{\frac{\ell-1}{2}}}$$

Therefore, $\tau_o(\Delta^0(w,\Delta^+))\rho_{\tau_o}(w) = \tau_o((-1)^{\sum_{k=1}^{\frac{\ell-1}{2}} k})$ for any element $w \in E^*$. Now, let $w \in E^* : n(w) = 0$. We claim that $|\Delta^0(w,\Delta^+)\rho(w)| = 1$. For let $w = p^n u$, where $u \in \mathfrak{o}_E^*$ and $n \in \mathbb{Z}$. Then

$$|\Delta^0(w,\Delta^+)\rho(w)| =$$
$$\left| \frac{N_{EL/L}((p^n u-\upsilon(p^n u))(p^n u-\upsilon^2(p^n u))(p^n u-\upsilon^3(p^n u))...(p^n u-\upsilon^{\frac{\ell-1}{2}}(p^n u)))}{N_{E/F}(p^n u)^{\frac{\ell-1}{2}}} \right| =$$

$$\left|\frac{N_{EL/L}(p^n)^{\frac{\ell-1}{2}} N_{EL/L}((u-\upsilon(u))(u-\upsilon^2(u))(u-\upsilon^3(u))...(u-\upsilon^{\frac{\ell-1}{2}}(u)))}{N_{EL/L}(p^n)^{\frac{\ell-1}{2}} N_{E/F}(u)^{\frac{\ell-1}{2}}}\right| =$$
$$\left|\frac{N_{EL/L}((u-\upsilon(u))(u-\upsilon^2(u))(u-\upsilon^3(u))...(u-\upsilon^{\frac{\ell-1}{2}}(u)))}{N_{E/F}(u)^{\frac{\ell-1}{2}}}\right|$$

Now, since $n(w) = 0$, this means that the leading coefficient of u is in $k_E \setminus k_F$, where k_E is the residue field of E and k_F is the residue field of F. Therefore, $u - \upsilon^i(u) \in \mathfrak{o}_E^* \; \forall i = 1, 2, ... \frac{\ell-1}{2}$. Therefore,

$$\frac{N_{EL/L}((u-\upsilon(u))(u-\upsilon^2(u))(u-\upsilon^3(u))...(u-\upsilon^{\frac{\ell-1}{2}}(u)))}{N_{E/F}(u)^{\frac{\ell-1}{2}}} \in \mathfrak{o}_F^*$$

and therefore it's absolute value is 1. □

Note that in constructing the character formula $F(\tilde{\chi})$, we have chosen Δ^+ to be the standard set of positive roots. The same argument as in the case of positive depth supercuspidal representations shows that our overall character formula $F(\tilde{\chi})$ remains the same regardless of the choice of positive roots.

9.2 On whether there are two character formulas coming from the same Cartan

In this section, we show that if the distribution characters of two depth zero supercuspidal representations, both coming from the unramified Cartan, agree on the $n(w) = 0$ range, then the supercuspidal representations are isomorphic. Note that $(F^*K_0 \setminus F^*K_1) \bigcap E^* = E^* \setminus F^*(1 + \mathfrak{p}_E) = \{w \in E^* : n(w) = 0\}$.

Theorem 9.2.1. *Suppose $(E/F, \chi_1), (E/F, \chi_2)$ are admissible pairs such that $F(\tilde{\chi}_1)(w) = F(\tilde{\chi}_2)(w)$ on the set $E^* \setminus F^*(1 + \mathfrak{p}_E) = \{w \in E^* : n(w) = 0\}$. Then $\chi_1 = \chi_2^\upsilon$ for some $\upsilon \in Aut(E/F)$.*

Proof. Assume $F(\tilde{\chi}_1)(w) = F(\tilde{\chi}_2)(w)$ on the set $E^* \setminus F^*(1+\mathfrak{p}_E)$. Then this implies (by [21, page 16]) that $\chi_1 = \chi_2^{\upsilon}$ for some $\upsilon \in Aut(E/F)$. □

Summing up, we have altogether shown that if $(E/F, \chi)$ is a regular pair such that χ has level zero, then there is a unique depth zero supercuspidal representation, $\pi_{\chi\Delta_\chi}$, whose character, on the range $\{z \in T(F)^{reg} : 0 \leq n(z) \leq r/2\} = \{z \in T(F)^{reg} : n(z) = 0\}$, agrees with $F(\tilde{\chi})$. There is one minor point here to resolve. Is there possibly a positive depth supercuspidal representation whose character, on the range $\{z \in T(F)^{reg} : n(z) = 0\}$, also equals $F(\tilde{\chi})$? Suppose $(E_1/F, \chi_1)$ is a regular pair corresponding to a positive depth supercuspidal representation π via Theorem (7.3.1). Then, the same arguments as in Theorems (8.5.1) and (8.6.1) show that the character of π cannot equal $F(\tilde{\chi})$, on the range $\{z \in T(F)^{reg} : n(z) = 0\}$, unless $\pi \cong \pi(\tilde{\chi})$.

Therefore, combining Theorems (9.2.1) and (9.1.6), we obtain the following result.

Theorem 9.2.2. *The assignment*

$$\left\{ \textit{irreducible } \phi : W_F \to GL(\ell, \mathbb{C}) \right\} \;\mapsto\; \tilde{\chi} \in \widehat{T(F)}_{\tau\circ\rho} \mapsto \pi(\tilde{\chi})$$

from Section (8.1) is the Local Langlands correspondence for depth zero supercuspidal representations of $GL(\ell, F)$, where $\pi(\tilde{\chi})$ is the unique supercuspidal representation whose character, on the range $\{z \in T(F)^{reg} : 0 \leq n(z) \leq r/2\} = \{z \in T(F)^{reg} : n(z) = 0\}$, is $F(\tilde{\chi})$.

Appendix A

Miscellaneous and Weil Index Notions

Here we list some various helpful lemmas that we needed throughout the thesis.

The following result and proof was communicated to me by Loren Spice.

Theorem A.0.3. *Let F be a local non-archimedean field of characteristic zero or a finite field. Suppose T is the elliptic torus in $G(F) = GL(n, F)$ whose F-points are $T(F) = E^*$ where E/F is a degree n extension. Then the relative Weyl group $W(G(F), T(F)) = Norm_{G(F)}(T(F))/T(F)$ is isomorphic to $Aut(E/F)$, the group of field automorphisms of E that fix F pointwise, where $Norm$ denotes normalizer.*

Proof. First note that $Norm_{G(F)}(E^*)/T(F) = Norm_{G(F)}(E)/T(F)$, where we are viewing E and E^* as embedded in the group of all n by n matrices.. There is a canonical map

$$Norm_{G(F)}(E) \xrightarrow{\Xi} Aut(E/F)$$

$$g \mapsto \lambda_g$$

where $\lambda_g(w) := gwg^{-1}$. We need the following Lemma, which is a Corollary of the Noether-Skolem theorem (see Milne's notes on Class Field Theory).

Lemma A.0.4. *Let A be a central simple algebra over F, and let B_1, B_2 be simple F-subalgebras of A. Any isomorphism $f : B_1 \to B_2$ is induced by an inner au-*

tomorphism of A. That is, there exists an $a \in A$ such that $f(b) = aba^{-1}$ for all $b \in B_1$.

Setting $A = M_n(F)$ and $B_1 = B_2 = E$ in the lemma, we conclude that Ξ is surjective. It is clear that the kernel of Ξ is $Cent_{G(F)}(E) = Cent_{G(F)}(T(F))$. It remains to show that $Cent_{G(F)}(T(F)) = T(F)$. But this is clear because $Cent_{G(F)}(T(F)) = Cent_G(T(F))^{Gal(\overline{F}/F)} = Cent_G(T(F)) \cap G(F) = T \cap G(F) = T(F)$. Note that since $T(F)$ contains strongly regular semisimple elements x, we have $Cent_G(x) = T$, which implies that $Cent_G(T(F)) = T$. $\square$

Lemma A.0.5. *Let F be a local field whose residual characteristic is not 2. Suppose λ is a character of F^* whose order is a power of 2. Then $\lambda|_{U_E^1} \equiv 1$.*

Proof. Let $w \in U_E^1$. Then w is a square since the leading term in the power series expansion is 1. Moreover, one of the square roots of w is in U_E^1. We may then proceed inductively to conclude that w is a 2^n-th power for any $n \in \mathbb{N}$. $\square$

Lemma A.0.6. *Suppose the residual characteristic of the local field F is not 2. Let $x = p^n u$ be an element of F^*, with $n \in \mathbb{Z}, u \in U_F$. For x to be a square, it is necessary and sufficient that n is even and the image $\overline{u}$ of u in $\mathbb{F}_p^* = U_E/U_E^1$ is a square.*

Proof. See [19, Section 3.3] $\square$

We now collect some basic properties of the Weil index, which we need for various computations (see [17] for more details):

Throughout, F denotes either a local field or a finite field of characteristic $\neq 2$, and $(x,y)\in\mu_2$ the Hilbert symbol of F. Let η be a nontrivial additive character of F. For any $a\in F$, we write $a\eta$ for the character $a\eta : x\mapsto \eta(ax)$.

Definition A.0.7. Define

$$\gamma_F(\eta) := Weil\ index\ of : x\mapsto \eta(x^2)$$

$$\gamma_F(a,\eta) := \gamma_F(a\eta)/\gamma_F(\eta) \quad a\in F^*$$

We have:

Lemma A.0.8. (1)$\gamma_F(ac^2,\eta)=\gamma_F(a,\eta)$ *and* $\gamma_F(ab,\eta)\gamma_F(a,\eta)^{-1}\gamma_F(b,\eta)^{-1}=(a,b)_F$.

(2)$\gamma_F(-1,\eta)=\gamma_F(\eta)^{-2}$

(3)$\{\gamma_F(a,\eta)\}^2=(-1,a)_F=(a,a)_F$

Let Q be a nondegenerate quadratic form of degree n over F.

Definition A.0.9. The Hasse invariant $h_F(Q)$ is defined as follows:

$$h_F(Q)=\gamma(\eta\circ Q)\{\gamma_F(\eta)\}^{-n}\{\gamma_F(detQ,\eta)\}^{-1}$$

Here $\gamma(\eta\circ Q)$ is the Weil index of $x\mapsto\eta(Q(x))$.

We then have

Lemma A.0.10. (1) *If* $n=2$, *and* $Q=a_1x_1^2+a_2x_2^2, a_1,a_2\in F^*$, *then* $h_F(Q)=(a_1,a_2)_F$.

Lemma A.0.11. *Let F be a finite field of characteristic $\neq 2$. Then*

$$\gamma_F(a,\eta)=\left(\frac{a}{F}\right)$$

where $\left(\frac{a}{F}\right)$ *equals* 1 *or* -1*, according to whether* a *is a square or not.*

For the next Lemma, let F be a nonarchimedean local field with characteristic $\neq 2$, R the ring of integers of F, π a generator of the maximal ideal of R, $\overline{F}$ the residue field of F. Let η be a nontrivial additive character of F and let $ord\ \eta$ denote the largest integer m such that $\eta = 1$ on $\pi^{-m}R$. Let $\alpha(\eta)$ denote the parity of $ord\ \eta$, i.e. $\alpha(\eta) = 0\ or 1$ according to whether $ord\ \eta$ is even or odd. We then have

Lemma A.0.12. *Suppose the characteristic of $\overline{F}$ is not 2. Let*

$$\overline{\eta} : x + \pi R \mapsto \eta(\pi^{-m-1}x).$$

Then $\overline{\eta}$ is a nontrivial character of $\overline{F}$ and

$$\gamma_F(\eta) = \{\gamma_{\overline{F}}(\overline{\eta})\}^{\alpha(\eta)}.$$

Moreover,

$$\gamma_F(a, \eta) = \{\left(\frac{\overline{u}}{\overline{F}}\right)\gamma_{\overline{F}}(\overline{\eta})\}^{\alpha(a)}$$

where $a = \pi^{ord\ a}u$, u being a unit of R.

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