ABSTRACT

Title of Document: THE EFFECTS OF EXPLICIT INSTRUCTION WITH DYNAMIC GEOMETRY SOFTWARE FOR SECONDARY STUDENTS WITH ADHD/LEARNING DISABILITIES

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The current study examined the effect of an instructional package on the ability of secondary students with mathematics difficulties to solve geometric similarity transformations. The instructional package includes a blend of research-based instructional practices including explicit instruction, the CRA sequence, and Dynamic Geometry Software. A multiple probe design across four participants was used to evaluate the intervention. The participants were four students with a history of mathematics difficulty in a suburban mid-Atlantic high school. Results of the study demonstrated that all four students improved their accuracy on geometric similarity transformations and maintained those skills four weeks after the completion of the intervention. Furthermore, providing multiple visual representations, including technology such as dynamic geometry software, as well as concrete manipulatives, allowed participants to make connections to geometric content and enhanced their metacognition, self-efficacy, and disposition toward geometry. This study supports the use of integrated instruction utilizing explicit instruction and visual representations for high school students with MD on grade-level geometry content.

THE EFFECTS OF EXPLICIT INSTRUCTION WITH DYNAMIC GEOMETRY SOFTWARE FOR SECONDARY STUDENTS WITH ADHD/LEARNING DISABILITIES

by

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Chapter I: Introduction

Mathematics is a critical life skill and has been referred to as a gatekeeper for technological advancement, post-secondary education, and economic growth (Stinson, 2004). Specifically, career opportunities in fields requiring higher mathematical and technological skills have grown and are expected to increase throughout the next decade (Dohm & Shniper, 2007). In addition, many colleges and universities require mathematics entrance and/or placement exams that include essential skills through trigonometry (American College Testing program [ACT], 2015; The College Board, 2011). Some trade schools, apprenticeships, and training programs, such as the International Brotherhood of Boilermakers and International Brotherhood of Electrical Workers/National Joint Apprenticeship Training Committee [IBEW/NJATC], require algebra and geometry skills necessary for reading blueprints, calculating measurements, and using formulas (Boilermakers National Apprenticeship Program [BNAP], 2011; NJATC/IBEW, 2011). Furthermore, taking more advanced mathematics courses increases potential income via earning postsecondary degrees (Rose & Betts, 2001). With the increase in technological occupations comes higher pay and therefore, more disposable income and overall economic growth. Additionally, some analysts estimate that increased income due to more advanced mathematical and technological skills translates into overall increased Gross Domestic Product (GDP; Hanushek, Jamison, Jamison, & Woessmann, 2008).

Though mathematics is essential, many high school graduates continue to lack the mathematics skills necessary for post-secondary education and future employment (American Diploma Project [ADP], 2004; Achieve, 2011). In part, this may be a result of differences in school expectations and graduation requirements. Specifically, in 1992 the U.S. Department of Labor issued a report based on data gathered from employers and universities, which identified skills necessary for all students exiting secondary school and reported the need for teamwork, technology, communication, and problem solving skills (U. S. Department of Labor, 1992). Additionally, Hui (2011) as well as the U. S. Bureau of Labor Statistics (1999; 2011) identified geometry skills (i.e., measurement, spatial rotations and transformations, similarity, congruence, and right triangle trigonometry) as critical to a myriad of careers within construction, agriculture, communications, art, criminal justice, and hospitality.

More recently, there has been a combination of efforts across policy makers and members of industry calling for increasing graduation requirements for college and career readiness, including those skills mentioned in the Department of Labor report. One specific consortium of businesses and governors dedicated to raising college and career readiness, Achieve, Inc. (2011), determined that although schools increased their graduation requirements, districts within and across states are inconsistent with their expectations, especially regarding mathematics requirements. Evidence for this position comes from finding that some districts require more rigorous mathematics courses (e.g., Algebra 1, Geometry, and Algebra 2) that include problem solving, reasoning, and analytic skills, while other districts require three years of mathematics without specifying individual courses, leaving room for less rigorous content.

Given the importance of achieving advanced mathematics skills, school systems have increased graduation requirements for students. Currently, 26 states require exit exams in mathematics in order to graduate, with 22 of those states including algebraic and geometric topics. Further, five states (Connecticut, Louisiana, Oklahoma, Tennessee, and Virginia) require an end of course geometry exam (Colasanti, 2007; Center on Education Policy [CEP], 2010). By 2018, 20 states and the District of Columbia will require four credits in mathematics, including Geometry and Algebra 2 (Dounay, 2007).

Although the number of students taking advanced mathematics and college entrance exams has increased, evidence suggests the majority of students continue to be unprepared. One reason for this may be that most geometry topics on state assessments (e.g., exit exams or No Child Left Behind Act [NCLB] assessments) do not include topics beyond middle school. For example, typical topics assessed include measurement and two-dimensional space instead of more complex topics including three-dimensional surface area and volume (Achieve, 2004). In addition, many colleges and universities have minimum entrance requirements that include Algebra 1, Algebra 2, and Geometry (Dounay, 2007). In 2013, though, only 44% of students taking the ACT college entrance exam met its mathematics benchmark for college readiness (i.e., 50% chance of earning a B or 75% chance of earning C in a credit bearing class; ACT, 2013). Furthermore, students' mathematics performance on the ACT has essentially been flat since 2009. Scores have changed minimally (e.g., 42 in 2009, 43 in 2010, 45 in 2011, 46 in 2012; ACT, 2013).

Therefore, additional research is needed that will help secondary students meet the increased academic demands in mathematics to prepare them for postsecondary requirements and future employment. A focus in geometry is critical, as there remains a dearth of research that focuses on this content area at the secondary level, particularly for students with special needs (Barrett & Clements, 2003; Battista, 2003; Cass, Cates, Smith, & Jackson, 2003; Kortering, deBettencourt, & Braziel, 2005; Maccini, Mulcahy, & Wilson, 2007; Maccini, McNaughton, & Ruhl, 1999).

Within the student population, some students, including those with learning disabilities (LD) have further challenges to their academic performance. Data from national studies (e.g., National Assessment of Educational Progress [NAEP], National Longitudinal Transition Study-2) report that students with disabilities continue to perform significantly below their peers in mathematics (Wagner, Newman, Cameto, & Levine, 2006; National Center for Educational Statistics [NCES], 2010). This is despite federal, state, and local legislation meant to improve access and performance of students with disabilities.

The Individuals with Disabilities Education Act (IDEA) provides access to the general education curriculum for students with disabilities, while NCLB requires *all* students to be assessed annually in grades 3-8 and once in high school, with reasonable adaptations and accommodations as needed. However, until recently basic computational skills were the primary focus of instruction for students with LD (Maccini & Gagnon, 2002).

With the increased focus on higher order reasoning skills, more rigorous graduation requirements, and demand for a more skilled workforce, geometry skills are essential for future competitiveness. Given that little attention has been given to developing and implementing effective mathematics interventions in geometry for secondary students with mathematics difficulties as well as those formally diagnosed with a disability, there is a significant gap in the research base. Therefore, in the remainder of this chapter, I discuss characteristics of students with mathematics learning difficulties and/or LD. I then review the development of proficiency in geometry and discuss a range of potential difficulties that students often experience. Third, I discuss the status of geometry proficiency in the United States and provide a broad overview of reforms in geometry, focusing on the last quarter century. I conclude with a description of existing research in geometry and students with mathematics difficulties, supporting the purpose of the study, guiding research questions, and definitions of terminology.

Characteristics of Students with Mathematics Difficulties or LD in Mathematics

Approximately 13% of U.S. students are identified as having one or more disabilities (Office of Special Education Programs [OSEP], 2011). Of those learners, 38% are identified as having a LD, which represents approximately 5% of the total student population (OSEP; Mazzocco, 2007), and is defined as, "a disorder in one or more of the basic psychological processes involved in understanding or in using language, spoken or written, that may manifest itself in the imperfect ability to listen, think, speak, read, write, spell or to do mathematical calculations" (IDEA, 2004, 118 Stat 2657-2658). Under IDEA, a LD is diagnosed by either a persistent lack of progress after use of researched-based interventions (e.g., Response to Intervention [RtI]) and/or an IQ-achievement discrepancy on standardized measures and evidence that demonstrate that the difficulties negatively impact education. Similarly, the Diagnostic and Statistical Manual of Mental Disorders, Fourth Edition (DSM-IV; American Psychiatric Association [APA], 2000) definition requires the use of formal standardized assessments as well as indications that the difficulties impact academic achievement or daily living.

However, implemented definitions for a *mathematics* learning disability (MLD) are based on local (state or district) criteria rather than specific national guidelines such that if districts use a discrepancy model or cut-scores they may not be the same across districts. Some discrepancies range from one to more than one standard deviations between IQ and achievement, with cut off mathematics performance below either the 10th, 15th, or 25th percentile (Mazzocco 2007; Vukovic & Siegel, 2010). These differences in diagnostic criteria make it unclear that students described in the literature are comparable.

Given the perceived biological basis of LD (i.e., IDEA and DSM definitions), it may be hard to distinguish students with a MLD from those who are low achievers (Mazzocco, 2007). Although difficulties in mathematics may be due to one or more problems with working memory, short-term memory, visual-spatial abilities, processing speed, mathematics background knowledge, and phonological processing, the research base has not sufficiently distinguished between those with persistent mathematics difficulties, which potentially may be a true MLD, and students with intermittent difficulties (Vukovic & Siegel, 2010). Therefore, my focus is on both students with mathematics disabilities and mathematics difficulties; in addition, I reference this population as MD.

The varied nature of MD can affect a student's ability to perform mathematically. In particular, students may have visual spatial processing difficulties, which impede their ability to spatially represent and interpret mathematical information such as geometric constructions, text illustrations, and diagrams of segments, lines and planes, two- and three-dimensional figures (Garnett, 1998; Geary, 2004; Steele, 2010). Language deficits may also interfere with the ability to make associations of words and symbols, which in turn impedes the ability to comprehend written and spoken language (Garnett, 2004).

Working memory and processing deficits, in particular, may impede students ability to problem solve, retain, and retrieve information (Passolunghi, Shiara, & Siegal, 2004; Swanson & Beebe-Frankenberger, 2004). Organization, sequencing and processing deficits can impede mathematical learning, such as with geometric proofs (Steele, 2010). Furthermore, aspects associated with productive disposition may adversely impact students' ability to perform mathematically, including their motivation, self-esteem, and self-monitoring (Gagnon & Maccini, 2001, Maccini & Gagnon, 2000; Montague, Bos, & Doucette, 1991). Given the wide range of difficulties that may affect mathematical performance for students with MD, proficiency in geometry is of great concern.

Development of Proficiency in Geometry

To be successful in mathematics, there are several areas that people must demonstrate proficiency. According to the National Research Council (NRC, 2001), five categories of proficiency rely on each other: (a) conceptual understanding, (b) procedural fluency, (c) strategic competence, (d) adaptive reasoning, and (e) productive disposition. (See Appendix J for further descriptions.) Conceptual understanding is the ability to comprehend mathematical concepts, operations, and relations and knowing what mathematical symbols, diagrams, and procedures mean. Procedural fluency refers to carrying out operations flexibly, accurately, efficiently and appropriately. Strategic competence is the ability to formulate problems and devise a plan to solve them, and includes the ability to understand concepts and fluently utilize procedures. Adaptive reasoning encompasses each of these, in addition to being able to logically explain and justify solutions. Lastly, productive disposition involves a person's ability to "see" that mathematics makes sense, is useful, is doable, and perseverance in thinking. These skills are similar to the National Council of Teachers of Mathematics' (NCTM; 2000) process standards that include problem solving, reasoning and proof, communication, connections, and representations. (See Appendix I for detailed descriptions.)

In geometry, the van Heile levels of geometric thinking posit that children's thinking moves from concrete to abstract concepts from early childhood through adulthood (Burger & Shaughnessy, 1986; Fuys, Geddes, & Tischler, 1988; Gutierrez & Jaime, 1998; Gutierrez, Jaime, & Fortuny, 1991; Mason, 1997; Mayberry, 1983; Senk, 1989, Usiskin, 1982; van Hiele, 1985). The five levels of geometric thinking include students moving up the levels from being able to: (Level 0) use visualizations or appearance rather than properties to reason about shapes; (Level 1) recognize properties but not relationships between classes of shapes; (Level 2) form abstract definitions, distinguish between necessary or sufficient conditions of concepts, and reason deductively to some extent; (Level 3) reason formally and deductively; to being able to (Level 4) apply rigorous deductive reasoning skills across various geometric systems without use of concrete models (Battista, 2007; Burger & Shaughnessy, 1986; Clements, 2003; Clements & Battista, 1992; van Hiele, 1985). Additionally, NRC proficiency standards and NCTM process standards (e.g., problem solving, reasoning and proof, communication, connections and representations) are evident throughout these levels of geometric thinking.

Van Hiele proposes that instruction move through the following five phases: (a) *inquiry* in which students explore materials and discover structure individually, (b) *guided orientation* which involves an instructor's providing additional structure for particular domains of exploration, (c) *explicitation* which involves the teacher's use of precise language to assist students in developing vocabulary and reasoning at the expected level of instruction, (d) *free orientation* which encompasses providing tasks that can be explored within a known system, and (e) *integration* which involves the student's analyzing and synthesizing all the information independently (Clements, 2003; van Hiele, 1985). The thinking levels and the five phases must work together for students to demonstrate proficiency. For example, when learning about the properties of shapes initially students learn what a square is by visually comparing and noting that two shapes are squares because they "look the same" (Level 0), then be able to classify based on properties such as four sides of the same length (Level 1), then relate properties, such as quadrilaterals that have opposite sides parallel (Level 2).

However, in order for students to move along the continuum, the levels of knowledge instruction must also move from exploring the concepts (i.e., looking at the shapes and making comparisons) with teachers using precise language and guiding students to understand and use the mathematical language, through further tasks to explore and expand student understanding. This exploration and guidance culminates in the student's ability to integrate the information and be able to analyze problems independently. Further, the rate of movement from one van Hiele thinking/reasoning level to the next varies with the particular concept (e.g., polygon properties and area, circles, right triangle trigonometry; Mayberry, 1983; Moyer, 2005; van Hiele, 1985).

It is also critical to understand possible disconnects between teacher's level of knowledge and student understanding when considering curricular design and teaching. Specifically, some researchers reported that if the teacher's geometric "level" were above the ability level of their students, it would be difficult for teachers to understand their student’s difficulty, which may impede the learner's growth and comprehension (Burger & Shaughnessy, 1986; Clements, 2003; Moyer, 2005; van Hiele, 1985). For instance, if the teacher’s level of understanding is at a Level 5 and the student is operating at Level 1-2, this resulting “disconnect” between teacher knowledge level and student understanding may be problematic (Burger & Shaughnessy, 1986; Clements, 2003; Moyer, 2005; van Hiele, 1985).

This disconnect may be due to the inability of students to meet the curricular expectations for deductive reasoning in a typical high school geometry course which exemplifies a van Hiele Level 2 entrance goal (i.e., being able to form abstract definitions and distinguish between necessary or sufficient conditions of concepts when beginning the geometry course in high school) and Level 3 exit goal (finishing the course with Level 3 understanding which is the ability to reason formally and deductively; Battista, 2009; Moyer, 2005). It is of great concern that for many students with or without disabilities, geometric thinking levels generally do not move beyond Level 2 (e.g., recognizing and comparing properties of figures such as how a square and a rhombus are the same or different) at the conclusion of secondary school (Burger & Shaughnessy, 1986; Moyer, 2005; Senk, 1989) and, for some students, into post-secondary education (Moyer, 2005).

Geometric Proficiency in the United States

Proficiency in foundational geometry concepts includes knowledge of similar triangles, properties of two- and three-dimensional shapes, formulas for area, perimeter, volume, and surface area, and finding unknown lengths, angles, and areas. These must be adequately addressed prior to high school (National Mathematics Advisory Panel

[NMAP], 2008; ADP, 2004; NCTM, 2000; Common Core State Standards Initiative [CCSS], 2010). However, many areas of geometry continue to be problematic for U.S. students as reported on international, national, and state assessments such as the nation's report card or NAEP (NCES, 2010). The following section includes a broad overview of student performance on international, national, and state assessments followed by a summary of student performance in geometry. Disaggregated data are shared when available, along with implications for research in geometry.

Internationally, the *Trends in International Mathematics and Science Study* [TIMSS] (Gonzales et al., 2008; Mullis, Martin, Foy, & Arora, 2012; Mullis, Martin, Ruddock, O'Sullivan, & Preuschoff, 2009) and the *Program for International Student Assessment* [PISA] (Fleischman, Hopstock, Pelczar, & Shelley, 2010; Lemke et al., 2004) compare student performance across countries in domains of numeracy, algebra, geometry/measurement, and data/probability. Overall, the data from the TIMSS indicate some growth over prior years in U.S. student performance on curriculum-based problems in discrete content domains (e.g., number, algebra, geometry,) as well as cognitive domains (e.g., knowing, applying, reasoning) as compared to students in other countries.

On the most recent TIMSS (2011), U.S. 8th graders mean score across all mathematical topics was better than the mean score for students in 36 of the 45 participating countries and had improved by 1 scale score point over the prior administration and 17 points since 1995; however the performance lagged behind the top performing country (Korea) by 104 points (international mean is set at 500; Mullis et al., 2012). Seven percent of U. S. students earned an advanced benchmark of proficiency (i.e., 625 or above) indicating the ability to apply their understanding and knowledge in a variety of complex situations and reason with geometric figures including recognizing rotations and reflections, visualizing a figure cut from a folded piece of paper, drawing the missing half of a symmetrical figure, reasoning with similar triangles and using the Pythagorean Theorem to find area of figures with only nine countries having a higher percentage of students scoring advanced (Mullis et al., 2012). Additionally, 24% of U.S. students performed with low proficiency (score of 400) indicating minimal skills with computation or graphing as compared to an international median of 29% (Mullis et al., 2012).

At the 8th grade level approximately 20% of the content is geometry, which includes the following skills: angle relationships; properties of 2-dimensional and 3-dimensional shapes (including symmetry); congruence of triangles and quadrilaterals; similarity of triangles; recognizing relationships between 3D shapes and 2D representations (e.g., nets); applying properties such as Pythagorean Theorem; drawing and using appropriate measurements such as perimeter, area volume, angles, including compound areas; locating Cartesian points and solving problems; recognizing and using geometric transformations of 2D shapes (Mullis et al., 2009). When considering geometry skills, U.S. students performed above the mean score of students in 29 countries, was the poorest score for all the mathematical topics assessed by approximately 30 points, and lagged behind the top performing country (Korea) by 127 points (Mullis et al., 2012). It is of note that geometry was the lowest content area for many countries. Disaggregated data were not available for students with disabilities.

Data from the most recent PISA (2012) show that, overall, U.S. students demonstrate poor performance related to problem solving and reasoning skills in real-world applications. The PISA addresses overarching areas of quantity, space/shape, change and relationships, and uncertainty rather than discrete mathematical domains such as algebra and geometry, with the scores adjusted so the mean is 500 and the standard deviation is 100 (Neidorf, Binkley, Gattis, & Nohara 2006). The overall mathematical literacy scores for U. S. students have not measurably increased since the prior administration in 2003 (Kelly, et al., 2013). U. S. students performed below the Organization for Economic Co-operation and Development (OECD) member countries' average of 494 (by 13 points) while 29 countries scored above the U.S. mean, 9 countries similar to the U.S. mean, and 26 countries scored below the U.S. mean (Kelly et al., 2013). Furthermore, 9% of U. S. students scored at or above Level 5 proficiency (minimum score of 607) indicating the ability to solve problems and use higher-order process skills, such as problem solving with visual-spatial reasoning in unfamiliar contexts, as compared to an OECD average of 13% while 26% of U.S. students performed below basic (level 2) indicating minimal accuracy with rote skills and procedural computation as compared to an OECD average of 23% (Kelly et al., 2013).

Difficulties with reasoning and problem solving also affect subtopics within geometry and measurement domains including congruence, similarity, properties of shapes, symmetry, transformations, spatial relationships, perimeter and area, surface area, volume, and angle relationships. In geometry topic areas, 35 countries scored above the U.S. mean, four were similar, and 25 were below (NCES, 2012). Similar to the TIMSS, geometry was the lowest of all topic areas for U.S. students, by approximately 20 points (NCES, 2012). Disaggregated data were not reported for students with disabilities.

In addition to international assessments, the NAEP in mathematics is given to students in grades 4 and 8 every other year and every four years in grade 12. The NAEP includes disaggregated data by English language learner (ELL), disability, gender, and race/ethnicity status (NCES, 2011). Reports show that although the scores of secondary students improved across time, the results are still below desirable proficiency levels. For example, in 2013 grade 8 students' mean score was 289 out of 500, just 7 points higher than the scores reported in 2003, while only 37% of all students scored at or above proficiency, indicating the ability to apply mathematical concepts and procedures consistently to complex problems (NCES, 2013; 2014). When comparing student mean scores, there was a 40-point difference between the performance of students with without disabilities (NCES, 2014). The results for geometry (e.g., transformations such as rotations, dilations, congruence, similarity; the Pythagorean Theorem; and properties of shapes) are similar to the overall scores with 33-point achievement gap and this gap has been steady since disaggregated data were initially reported (NCES, 2013; 2014).

Furthermore, on the most recent NAEP at the 12th grade level (2013), 28% of students without disabilities and 6% of students with disabilities scored at or above proficiency on more advanced mathematical topics including geometry and trigonometry (e.g., scale factors, dilations on the coordinate planes, proofs, height/length of figures; NCES, 2013). This gap is similar to that found in grade 8 (157 versus 123, gap of 34 points, max points 300; NCES, 2013). However, in contrast to TIMSS, PISA and grade 8 NAEP data grade 12 geometry skills were slightly higher than other mathematics subtopics for students with disabilities, but for students without disabilities scores were similar across each domain (NCES, 2013).

Given the expectation for students to be college or career ready, the TIMSS, PISA, and NAEP indicate this is not at the desired performance level, particularly for students with disabilities. Policies have been implemented to address the performance of all students in the U.S. Specifically, NCLB (2001) addressed the continued poor performance of all students, including those with disabilities. Prior to the legislation, students with disabilities were often excluded from state assessments as well as grade level participation in general education courses. The NCLB Act mandated that states assess the mathematical performance of all students, including students with disabilities annually in grades 3 through 8 and once in high school (NCLB, 2001). To date, five states have assessments specifically in geometry and five additional states include geometry skills in a general mathematics test given in grade 10 or 11 for NCLB purposes.

The available data corroborates national and international performance trends. For example, on Virginia's 2012-13 geometry assessment, 79% of students without disabilities passed, as compared to 43% of students with disabilities (Virginia Department of Education, n.d.). However, on Texas' most recent grade 11 mathematics assessment, 93% of students without disabilities passed, as compared to only 55% of students with disabilities (Texas Education Agency [TEA], n.d.). Students with disabilities and their nondisabled peers also scored lower on tasks related to geometry than on algebra and numeracy (TEA). The data from these two states highlight the disparities in performance for all students, but most especially the performance gap for students with disabilities.

Mathematics Reform and Geometry

Given the longstanding and continued poor performance of students in the United States based on intranational and international performance measures, particularly for students with disabilities, it is clear that in order to meet college and workforce expectations there is much room for growth. There have been a number of organizations and policy initiatives aimed at improving the mathematics performance for all students, such as NCTM, ADP, NMAP and CCSS. Each group or organization is will be discussed in the order listed, based on recent initiatives and the effects on mathematics instruction and curriculum, with emphasis on geometry when there is specific information available. However, it is prudent to include some of the earlier history to provide a background for the dramatic changes over the most recent 25 years.

Nathalie Sinclair (2008) provides an excellent overview of the geometry curriculum in the United States with suggestions and reform ideas that paved the way for the NCTM and CCSS. Prior to about 1850 geometry was exclusively taught in college, but European influences at that time led to expectations for knowledge of geometry for college entrance requirements. In 1892, the National Education Association’s Committee of 10 worked to identify pertinent elements of mathematics and made the first suggestions for geometry to be introduced at the elementary level with formal geometry at the high school level. Moving into the 1920s, additional committees and reports proposed a focus on transformations as integral to understanding geometry. With the Great Depression, geometry became less important, perhaps because people were focused on skills deemed important for work rather than those required for formal, deductive reasoning. Around 1935 the Bourbacki group, an influential group of mathematicians published geometry texts that were more focused on axioms (N. Sinclair, 2008).

With the advent of World War II, deficiencies in soldiers’ mathematics preparation fueled debate about the two core approaches to teaching geometry, practical or axiomatic. This disagreement was further fueled by scientific developments in other countries and the “space race” which lead to a major era of reform and the “New Math” that reflected much of the thinking of the Bourbacki group. However, there was a backlash that eventually led to a “Back to Basics” movement, particularly regarding arithmetic. This backlash was in part because so many children, parents and teachers discovered that students did not have the conceptual understanding of arithmetic and were unable to do basic calculations (Kline, 1973; N. Sinclair, 2008).

Regarding geometry, new curricular materials were developed with support of such organizations as the National Science Foundation. The 1960s reintroduced the idea of transformations; although, there were other competing views. In addition, instructional tools such as manipulatives and technology were first introduced during the 1960s and 1970s. However, with the introduction of and poor performance on mathematics measures such as TIMSS, and the backlash over “New Math”, the focus through the 1970s and early 1980s reverted to more basic arithmetic skills and applications to the real-world rather than an axiomatic and (for geometry) proofs-based curriculum (N. Sinclair, 2008). This paved the way for the most recent reform movements.

The NCTM was at the forefront of creating curricular standards for educators and districts. They published *Curriculum and Evaluation Standards for School Mathematics* (1989) and revised them as *Principles and Standards for School Mathematics* (2000). The 1989 Standards were groundbreaking in that no other organization had succinctly proposed curricular standards that could be used to unify the mathematics content in topical and grade level sequences. The Standards (1989) included recommendations for emphasizing the study of algebra in earlier grades and higher mathematical topics (e.g., geometry, trigonometry) with a balance of basics skills and applications to address the needs of students intending to pursue college or to pursue work upon completion of school. The NCTM *Principles and Standards* (2000) expanded the vision to a more integrated curriculum (e.g., geometry and algebraic thinking throughout grade bands) and to be more inclusive of all students including students with disabilities. The NCTM Standards became so influential that almost 100% of states aligned their mathematics curricular standards to them (Woodward, 2004).

The NCTM's Standards (2000) are guided by a constructivist instructional philosophy that involves a student-centered approach to learning with an emphasis on student exploration, discovery, and conceptual understanding. The NCTM Standards includes five process standards (problem solving, reasoning and proof, communication, connections, and representations) aligned across the five content standards (number and operations, algebra, data analysis/probability, and measurement/geometry). Within the geometry standard, the secondary curricular standards emphasize analyzing characteristics and properties of 2D and 3D shapes and developing arguments about such relationships, describing spatial relationships in Cartesian and other coordinate systems, applying transformations (e.g. rotations, dilations, congruence, similarity) and using transformations to analyze situations, and use of visualization, spatial reasoning, and geometric modeling to solve problems (NCTM, 2000). The NCTM (2000) views the study of geometry as intricately linked with other mathematical domains and critical to defining properties of the natural world.

The American Diploma Project (2004; Achieve, 2011), an initiative of Achieve, Inc., was organized by a consortium of businesses, governors, and education leaders to raise high school standards to better prepare all graduates for the demands of college and the workforce. The ADP emphasized the need for both consistency across states in terms of mathematics content, sequencing, and expected skill mastery, as well as increasing the content and rigor of graduation exams and earning a diploma (ADP, 2004). The ADP report includes benchmarks in geometry that are to be integrated across mathematics courses, including proving basic theorems to more advanced topics such as right triangle trigonometry and trigonometric functions (2004).

In light of continued poor performance of U.S. students and networks of organizations and leaders such and NCTM and ADP, President Bush signed an executive order in 2006 establishing the National Mathematics Advisory Panel. The NMAP included a variety of stakeholders, including parents, policy makers, and educators. The panel was charged with providing research-based recommendations for instruction and assessment within the K-12 mathematics curriculum (Bush, 2006). The final report included specific content recommendations and the need for a more coherent national curriculum and assessment system with a focus on critical competencies (NMAP, 2008). It also recommended specific geometry topics in the middle grades (e.g., similar triangles, surface area, volume, and properties of 2D and 3D figures), which students should master to be prepared for college or work (NMAP, 2008). The report was intended to be a starting point for both policy and future research on best practices.

Due in large part to the work of the ADP and the NMAP recommendations, the most recent initiative, the Common Core State Standards (CCSS, 2010), moved education systems toward establishing common curricular standards across states. To date, 43 states and the District of Columbia have adopted the CCSS. The CCSS emanated from the partnership of Achieve, the National Governors Association, and the Council of Chief State School Officers. The partnership developed K-12 standards for college and career readiness that are rigorous and based on research. The CCSS stresses conceptual understanding, reasoning and higher order thinking, and a more in-depth and integrated approach to content with limited repetition of material across grades.

The integration of content is important considering that geometry has typically been taught as an isolated course in grade 10 (or after Algebra 1). Furthermore, the CCSS mathematical content standards view transformations (e.g., rotations, dilations, congruence, similarity, symmetry) as a unifying concept across grade bands (CCSS, 2010). This is in contrast to many traditional geometry texts and curriculum trajectories that focus on a collection of seemingly unrelated topics at the elementary grades and then turn to an axiomatic and proofs-based secondary curriculum that many students do not understand (Kelly, 1971; Schuster, 1971; Wu, 1996).

The CCSS middle school topics feature transformations as a transition from informal, intuitive and conceptually based topics to the more formal reasoning often associated with a typical high school geometry course (Son, 2013). Although proofs and reasoning are included in the curriculum it is not in isolation and the conceptually based instruction of the prior elementary and middle school material should provide the foundation for further learning of the high school geometry that are divided into broad categories including congruence, similarity, right triangles and trigonometry, circles, expressing geometric properties with equations, and geometric measurement and dimension (CCSS, 2010). Each topic is further divided into more specific benchmarks with approximately 5-13 per topic such as representing and comparing transformations in a plane, using trigonometric ratios to solve right triangle problems, deriving the equation of a parabola given the focus and directrix, and identifying shapes of cross-sections of three-dimensional objects.

For more than 40 years, some researchers have advocated for the study of transformations as a different approach to geometry, for a variety of reasons. For example, integration of transformations in European mathematics teaching is the norm (Ada & Kurtulus, 2010; Usiskin, 1972) and the continued poor performance of U. S. students in geometry, as measured by international assessment, in relation to top performing countries that may use this transformational approach is one reason for supporting this view (Schmidt & Huoang, 2012). In addition, transformations are a foundation for advanced mathematics courses and the conceptual understanding (especially visual-spatial ability) required in order to succeed in such courses (Ada & Kurtulus, 2010; Friedman, 1995; Holzinger & Swineford, 1946; Martin, 1982; Thompson, 1985; Usiskin, 1972).

Furthermore, the shift in core conceptual understanding to transformations includes the idea that the spatial visualization skills of geometry are necessary for a variety of careers (CCSS, 2010; Eisenberg & McGinty, 1977; Hui, 2011; Thompson, 1985) and, more recently, there is a need for a more technical workforce with more advanced mathematics skills (Dohm & Shniper, 2007; NCTM, 2000). Given that the CCSS views transformations as a central connecting concept within the geometry curriculum, it is of concern that there is limited research in this topic area, particularly for students with MD.

Existing Research

In order for students with MD to be prepared for college and the workforce, they must have both access to the rigorous content, and the appropriate tools and instructional strategies to help “level the playing field.” Although Chapter 2 reviews the research in depth, certain strategies have been found to show promise for helping secondary students with MD in geometry. Overall, the following strategies were determined to be effective for students with MD, including the use of: (a) technology, such as computers and videos (Bottge, 1999; Bottge & Hasselbring, 1993; Bottge, Heinrichs, Chan, Mehta, & Watson, 2003; Bottge, Heinrichs, Chan, & Serlin, 2001; Bottge, Heinrichs, Mehta, & Hung, 2002; Bottge, Rueda, Kwon, Grant, & LaRoque, 2009; Bottge, Rueda, Serlin, Hung, & Kwon, 2007; Cihak & Bowlin, 2009; Satsangi & Bouck, 2015); (b) strategy instruction (Jitendra, et al., 2009); and (c) the concrete-representational-abstract sequence (Cass et al., 2003; Satsangi & Bouck, 2015). In particular, prior research reviews on interventions for students with MD have found that: (a) computer aided instruction (CAI) or technology such as games, tutorials for drill/practice, and simulations enhance student conceptual and procedural knowledge and/or problem solving by addressing student's individual needs (i.e., pacing, content) and motivation (e.g., Hughes & Maccini, 1997; Maccini & Hughes, 1997; Maccini et al., 2007); and (b) cognitive strategy instruction such as systematic teaching of mathematics or self-regulation strategies, steps, visuals, and memory aids to help students with MD to independently develop their own methods to successfully learn and utilize the strategies for immediate and long-term applications (e.g., Hughes, Maccini & Gagnon, 2003; Maccini & Hughes, 1997; Maccini et al., 2007).

These studies were limited by focusing solely on interventions that addressed skills appropriate for the elementary and lower middle school level (e.g., grade 6), such as measurement, area, and perimeter (Cass et al., 2003; Cihak & Bowlin, 2009; Satsangi & Bouck, 2015). Furthermore, only two studies (Cihak & Bowlin, 2009; Satsangi & Bouck, 2015) focused solely on high school age students. Therefore, future research is needed that addresses grade appropriate concepts at the high school level for students with MD.

Given the increasing technological skills necessary for college and careers (NMAP, 2008; CCSS, 2010) technology should also be incorporated into instruction and has been shown to be effective for students with MD, particularly for computation (e.g., Gleason, Carnine, & Boriero, 1990; Okolo, 1992; Shiah, Mastropieri, Scruggs, & Fulk, 1995), as well as general education students at all mathematics levels, including undergraduate (Baki, Kosa, Guven, 2011; Li & Ma, 2010). Future research should address interventions that focus on age-appropriate and grade appropriate content as advocated by NCTM, ADP, and CCSS, including spatial skills, reasoning and proof, congruence, similarity, transformations, coordinate geometry, as well as the relationship between algebra, geometry, and trigonometry.

Specifically, the critical topic of transformations (i.e., congruence, similarity, symmetry) is absent from the current literature on mathematics interventions for secondary students with MD. This topic is critical for three reasons: (a) transformations encompass several topics that require and expand students' reasoning and problem solving skills (Seago, Jacobs, & Driscoll, 2010; Wu, 2013), (b) students with disabilities perform markedly below peers in geometry and measurement skills (e.g., NAEP), (c) geometry skills have become important for college and career readiness (e.g., NMAP, CCSS). Given the CCSS place an increased emphasis on transformations throughout the K-12 geometry strand and 43 states currently have adopted the CCSS; it is of utmost importance for students to understand in order to be ready for college, careers, and life.

Fortunately, promising practices exist within the general education literature focused on the study of transformations and include the use of technology, such as dynamic geometry software, with programs that allow for construction and manipulation of 2D and/or 3D figures (Choi-Koh, 1999; Guven, 2012; Hollebrands, 2003; Hollebrands, 2007; Hungwe, Sorby, Drummer, & Molzon, 2007; Kirby & Boulter, 1999). Although described in detail in Chapter 2, dynamic geometry programs help students develop conceptual and/or procedural understanding, spatial relations, reasoning skills, and formula use. On the other hand, no studies have included students with MD. This is important given NCLB requirements that all students, including those with disabilities, be included in state assessments and meet minimum performance standards. Therefore, research must explore the promise of dynamic geometry programs with students with MD, particularly at the high school level.

Significance and Statement Purpose

Geometry is intricately linked with algebra and higher order reasoning and mathematics skills, and is critical for postsecondary education, employment, and life skills for *all* students, including students with disabilities. However, U.S. students perform significantly below proficiency expectations on intranational, international, and state assessments, particularly in geometry. Further, the performance of students with disabilities is significantly below their peers without disabilities. This poor performance may be due to certain characteristics of students with disabilities, including difficulties with visual spatial processing, language deficits, working memory and sequencing deficits, as well as difficulties with motivation, self-esteem, and self-monitoring. As such, it is critical that additional studies explore research-supported practices that show promise for helping students with MD in geometry, given their unique learner characteristics that enable students to be ready for post-secondary life.

Therefore, the study was designed to expand the existing literature on effective geometry interventions for students with MD, in light of reform efforts and legislation requiring research based methods. By expanding the existing research, the study examined the effects of an instructional package that includes technology, via dynamic geometry software for students with MD in geometric transformations, specifically similarity. This study addressed objectives aligned with the NCTM standards, NMAP and ADP benchmarks, as well as the CCSS for secondary geometry for all learners.

Research Questions

1. Do secondary students with MD taught an instructional intervention on geometric similarity transformation increase the accuracy of their performance in geometric similarity transformations and maintain performance four to six weeks after the intervention?
2. What conceptions of geometry do students hold before and after the intervention?
3. What connections or disconnections in the geometry content emerge during the intervention and how can these results be used to improve instruction?
4. To what extent do secondary students with MD find dynamic geometry technology beneficial to representing and solving geometric similarity

transformation tasks and in what ways does the intervention enhance metacognition, self-efficacy, and attitudes toward geometry?

Definitions

This section provides definitions of terms used in this study.

Computer enhanced instruction refers to the use of the computer within the lesson as an education tool, but not as the sole instructional source. Examples of this include spreadsheets, word processing programs, or software to explore specific examples within overall instruction.

Computer-assisted instruction refers to the use of computers as supplemental drill and practice on topics taught by the instructor.

Computer-based instruction refers to instructional learning systems that include videos-enhanced lessons, practice problems, and assessment as the primary mode of instruction, in contrast to instructor led content and pacing.

Conceptual knowledge refers to the idea that logical relationships are constructed internally and are constructed in the mind as part of a network of ideas.

Concrete phase refers to an instructional phase that utilizes physical manipulatives.

Concrete to representational to abstract (CRA) graduated instructional sequence refers to an instructional strategy that progresses through three phases of instruction (i.e., concrete, representational, abstract).

Dynamic geometry software refers to computer programs that allow users to construct and manipulate 2D and/or 3D figures.

Enhanced Anchored Instruction (EAI) is a video-based instructional program aimed at developing computation and problem solving skills through authentic contexts.

Explicit instruction refers to a highly structured, teacher-directed method for presenting new information that incorporates key variables such as curriculum-based assessment, advanced organizer, teacher modeling, guided practice, independent practice, and review for maintenance.

Integrated instruction refers to the use of multiple instructional strategies and materials within a lesson

Learning *disability* refers to a disorder in one or more of the basic psychological processes involved in understanding or in using language, which may manifest itself in the imperfect ability to listen, think, speak, read, write, spell, or do mathematical calculations (IDEA, 2004).

Manipulatives include both physical objects as well as virtual (i.e., computerized) representations that support mathematical thinking and represent a mathematical concept (i.e., counters, beads, algebra tiles, geoboards).

Mathematics difficulties (MD) refers to the various difficulties that people may have with understanding and using mathematics and includes those formally diagnosed with a disability as well as those who have not (e.g., at-risk for mathematics failure due to a history of below age or grade level performance, low-achieving, and/or placement in remedial courses)

Procedural fluency refers to the flexible, accurate, and efficient use of mathematical procedures and algorithms (NRC, 2001).

Problem solving refers to the process of applying previously learned concepts and skills to novel situations (NCTM, 2000).

Representational phase refers to an instructional phase that utilizes visual representations, such as drawings and virtual manipulatives, to represent abstract mathematical concepts and is synonymous with the term semi-concrete.

Secondary students are students in grades 7 through 12.

Strategy instruction provides students with a plan for solving a problem (e.g., providing students with a memory device or a cue card).

Student-centered instruction refers to instruction in which the students are primarily responsible for their learning.

Teacher-directed instruction refers to instruction in which the teacher is primarily communicating the mathematics to students.

Chapter 2: Review of the Literature

Despite the increased demand for proficiency in mathematics, including geometry, international assessments suggest secondary students in the United States continue to experience difficulty and score below students in other countries. Data from TIMMS and PISA indicate students in the U.S. are performing below the level of many other industrialized (or rapidly developing) countries in mathematics. Additionally, geometry performance was the lowest of all assessed areas (i.e., algebra, number sense, statistics; Lemke et al., 2004; Mullis, Martin, & Foy, 2008). Education experts and policy makers express concern that the U.S. will lose its position as an international leader because of the lack of mathematical proficiency (NMAP, 2008).

In addition, U.S. students are not making sufficient gains on national assessments. Data from the NAEP indicate that although the scores of secondary students have improved over time, they remain below desirable levels. For example, while 38% of eighth graders without disabilities scored at the proficient level, defined as solid academic competency on challenging content, only 7% of students with disabilities scored proficient (NCES, 2014). At the twelfth grade, the performance of students deteriorates, within both the general student population and students with disabilities, with 28% and 6% of students scoring at or above proficiency, respectively (NCES, 2014). Additionally, while all students have made some progress over time, the performance gap between students with and without disabilities has persisted at both grade levels (NCES, 2010; 2014). Clearly, students are not performing at an acceptable level.

In response to these concerns, the NCTM addressed the need for more rigorous standards with the publication of *Principles and Standards for School Mathematics*

(2000). The NCTM's Standards focus on a student-centered, constructivist approach to learning with an emphasis on authentic, real-world problem solving and conceptual understanding for all students. The *Content* (i.e., number sense, algebra, geometry, measurement, data analysis, probability) and *Process Standards* (i.e., problem solving, reasoning and proof, communication, connections, representations) describe the topics as well as the ways of learning the content.

In addition to the NCTM standards, the ADP and NMAP reports highlight the need for a more coherent and rigorous curriculum for all students. The authors recommend benchmarks and a focus on critical competencies (NMAP, 2008), as well as sequencing of content topics and mastery of skills (ADP, 2004). In geometry, it is recommended that foundational skills be integrated at the elementary and secondary levels in order to build and solidify conceptual understanding and problem solving skills (ADP, 2004) to prepare all learners for postsecondary education and the work force (NMAP, 2008). The recommendations from the NCTM, ADP, and NMAP are evident in the CCSS, which includes a more coherent curriculum across K-12 mathematics with a heavy focus on the interdependence of both content and process standards (CCSS, 2010). Furthermore, to date, 43 states and the District of Columbia have adopted the CCSS with rigorous expectations for all students, including students with disabilities. As outlined in the CCSS, the study of transformations is a central concept across all grade bands.

In addition to the increased academic standards, the Individuals with Disabilities Education Act (IDEA) amendments in 1997 and 2004 increased accountability requirements for students with disabilities. Students with disabilities are to have access to a grade-appropriate curriculum and standards that are on par with their non-disabled peers. Furthermore, states must report progress toward meeting these accountability goals annually. NCLB (2001) also mandated access to a general education curriculum and accountability for proficiency for all students, including students with disabilities. States must adopt rigorous and challenging curriculum and accountability measures. All students must demonstrate significant improvement on yearly measures of reading and mathematics (grades 3-8 and high school). As mandated by NCLB (2001) and IDEA (2004), students with disabilities are to be included with their general education peers as much as possible and to be assessed on grade level material (Mandlawitz, 2007).

To date, 27 states require graduation exams that include geometry topics or an end of course exam in geometry (Colasanti, 2007; CEP, 2010, 2012) and by 2018, 20 states will require a geometry course for a diploma (Dounay, 2007). However, many areas of geometry continue to be problematic for typically developing students, and especially for students with MD. Areas of difficulty within geometry include perimeter, area, surface area, and volume (Barrett, Clements, Klanderman, Pennisi, & Polaki, 2006; Battista 2003; Battista & Clements, 1996), reasoning with properties of figures (Fujita, 2012; Monaghan, 2000; Usiskin, Griffin, Witonsky, & Willmore, 2008), 2D and 3D visualization (Battista & Clements, 1996), transformations such as translations, reflections, rotations, dilations (Lean & Clements, 1981; Gorgorio, 1998), and proofs (Burger & Shaughnessy, 1986; McCrone & Martin, 2004; Senk, 1989).

However, given the various content in geometry it is too extensive a topic to cover in depth with one review or study. Although the topic of transformations is a central topic in the CCSS, it is necessary to narrow this topic for the intervention. I have chosen to focus on the subtopic of similarity for two reasons: (1) similarity is a large portion of the curriculum in the district where I teach and is used to make connections between other topics, and (2) similarity is a difficult topic for students to understand, as researchers and my experience with students have shown. In the next section, I explain common difficulties students experience, and in particular students with MD.

Similarity is a difficult area for students for a variety of reasons including vocabulary and the integration of prerequisite skills such as proportional reasoning. Students must have an understanding of basic vocabulary. There are several issues with terminology in mathematics and geometry in particular. Some terms are vague, such as abstract concepts of point, line and plane (Stone, 1971), or specific to mathematics or geometry and may have not meaning or a different meaning otherwise, including similar, plane, or angle (Vollrath, 1977), which should be explored so that students develop their understanding, rather than learning my rote (Stone, 1971; Vollrath, 1977). Furthermore, an intuitive and conceptually based view of similarity can be built on foundations from early experiences people have of scaling and manipulation of objects such as model toys (Lehrer, Strom & Confrey, 2002), photograph or font enlargements/reductions, (Cox, Lo, & Mingus, 2007), and illustrations in books, movies or television (Van den Brink & Streefland, 1979). This can assist students in developing both conceptual as well as procedural knowledge, which are equally important for the topic of similarity.

There are multiple ways to define geometric similarity, which to mathematicians or teachers, may be understood to be the same idea, but how the concept is defined for students to be able to internalize the concept may be confusing. For example, Miyakawa and Winslow's (2009) interpretation of Euclid's definition: *Two polygons are called similar if they have their angles severally equal and the sides about the equal angles* *proportional;* points out that there are two parts to this definition, angles and sides, and that for triangles each part implies the other, but for other polygons this is not the case. However, Vollrath (1977) defines similarity as a “relation between figures (point sets). A figure F1 is similar to figure F2 if there exists a similarity transformation (composition of a dilation and an isometry) such that: s(F1)=F2” (p. 211). This definition may be simplified as Lehrer et al. (2002) states that “the mathematics of similarity defines objects as similar if they differ only in position and scale” (p. 360). While in contrast Kelly (1971) simply states that objects are similar if “they are identical in shape but not necessarily in size” (p. 478). Battista and Clements (1995) note that visual and general definitions such as the prior same shape/not same size are a beginning and as students develop their conceptions, a more precise mathematical definition can be obtained.

I have experienced this issue with my high school students. When we begin with a general definition such as Kelly's same shape-different size, some students are unable to move beyond a naïve definition and leave out the importance of proportionality; however, with the advent of the CCSS this may begin to change. Vollrath's definition is most closely aligned with the CCSS view of similarity via transformations, which fits with how my school district intends for the concept to be taught. Currently, in my school district the curriculum places congruence (i.e. same shape same size) and isometric transformations in order to lead into dilations as a type of non-rigid transformation and the idea of scale factor. This trajectory moves students from an elementary and concrete or visual understanding of similarity to a more Euclidean view that does not necessarily rely on a coordinate grid, as is used with most isometric transformations, and focuses on not only the congruency of angles but also the proportionality of corresponding side lengths. Most textbooks and curriculum materials I use include the essential features of congruent angles and proportional side lengths, and when implementing the curriculum we are expected to link the ideas of the current unit to the prior units, which for similarity includes congruence, isometries and dilations.

Proportionality and proportional reasoning are an integral part of defining similarity. In order for objects to be deemed similar, not only must they be the same shape, such as a triangle or rectangle, but also the ratio of the corresponding side lengths must be the same. Proportional reasoning, including the application to similarity, is a difficult concept for students (Chazan, 1987; Hart, 1984; Lesh, Post & Behr, 1989). Some areas of difficulty include: using additive rather than multiplicative strategies (Hart, 1984); conceptualizing and applying multiplication of fractions less than one (Taber, 1999) which is important to scaling or reduction dilations; incorrectly relying on visual perception (Lamon, 1993); and the misuse of proportionality in non-proportional problems, especially word problems and applications to multi-dimensional and irregular figures (De Bock, Van Dooren, Janssens & Verschaffel, 2002).

Furthermore, numerical proportional reasoning should be distinguished from geometric proportional reasoning (see Appendix A). For example, grouping numerically cannot be applied in the same manner geometrically, particularly when considering multidimensional objects and irregular or embedded images (Cox, 2013). Typical numerical reasoning with grouping and quantities do not translate the same to a geometric context (Cox, 2013). For example, when considering objects such as a paper clips, fingernail file (i.e. emery board), staples, toothpicks et cetera that perhaps many view as flat, two-dimensional objects really are not. So when scaling them there are three-dimensions-length, height *and* width. Furthermore, if the shapes are irregular, rather than simple 2D polygons, or if the figure is on a coordinate grid where not only a scale factor is necessary but also a center of dilation, then simple numerical reasoning with proportions do not take these additional factors into consideration.

Consider another difference between the use of proportions in general and geometrically. Suppose that a manufacturer samples 100 items and finds that 8 are defective, then extrapolates that to conclude that out of a production run of 1 million items 80000 items would be defective then uses this to make adjustments to production processes. This is different from how a proportion would be used in a geometric context (such as what an artist might use when scaling up a model for a larger sculpture) checking perhaps mathematically and visually that both figures are proportional. What an issue that would be if the final product were something major like Mount Rushmore and it was not scaled properly!

Lastly, in most units taught in school with geometric proportions it is important to develop an understanding of within and between ratios, which is slightly different that how numerical proportions may be understood. Take the defects example above. The proportion could be set up as defective sample amount/sample items total = defective total/total items OR defective sample/defective total = sample item total/total items. This would give the equations $\frac{8}{100} = \frac{x}{1000000}$ OR $\frac{8}{x} = \frac{100}{1000000}$, which still gives the same solution of 80,000 defective items. However, when dealing with geometric objects while on the surface it may seem that you are essentially doing the same thing-using two different set ups for the same comparisons (e.g. within comparison using the ratio of the two sides of one triangle set up equal to the ratio of the two sides of another triangle to check that they are proportional OR using a between comparison using the ratio of left sides of both triangles set equal to the ratio of the right sides of both triangles) understanding the different set-ups can assist in evaluating acceptability of the scaled figures, particularly with 3D solids.

Given all the sub skills necessary to build geometric understanding, and of similarity in particular, certain characteristics of students with MD may impede their performance, including visual-spatial processing deficits (Garnett, 1998; Geary, 2004; Steele, 2010), language deficits (Garnett, 1998), working memory and processing deficits (Passolunghi et al., 2004; Swanson & Beebe-Frankenberger, 2004), as well as organization, sequencing, and processing deficits (Steele, 2010). Additionally, students may have difficulty with motivation, self-esteem, and self-monitoring (Gagnon & Maccini, 2001; Maccini & Gagnon, 2000; Montague et al., 1991). To help students meet the increased demands of college and career preparation, it is critical to provide effective, research-based instructional strategies (Scheuermann, Deshler, & Schumaker, 2009).

Organization of the Review of the Literature

In the remainder of this chapter, I present a comprehensive review of current research involving geometry interventions for secondary students with MD. This review serves two purposes: (a) to describe the current status of and the need for effective geometry interventions for secondary students with MD, and (b) to examine empirically based instructional variables to inform the current study.

Studies meeting the following search criteria were included in this review: (a) published in a peer reviewed journal between 1989 and 2015 (since the publication of the NCTM *Principles and Standards*), (b) examined the effects of an instructional intervention in geometry on the performance of secondary students with MD, (c) used qualitative or quantitative methods (experimental, quasi-experimental, single-subject design, or case study), and (d) included students with MD from grades 7-12. Studies were excluded that involved students from multiple grade levels that included those below grade 7, as several states begin secondary school and/or secondary teacher certification with grade 7. A database search using ERIC, Education Research Complete (EBSCO), PsychINFO, PsychARTICLES, Social Sciences Citation Index, as well as Google Scholar, identified possible studies for inclusion in this review. Combinations of the following descriptors were used*: mathematics*, *geometry*, *transformations*, *symmetry, similarity, congruence, rotation, reflection, dilation, learning disability*, *instruction, intervention, teaching*, *learning*, *secondary*, *computer*, and *technology*. An ancestral search of the reference lists of the articles obtained in the automated search was conducted to locate additional studies.

Lastly, a hand search of journals in the field of special education and mathematics education was done to identify the most relevant articles for the topic of this review. Specific journals were chosen due to their frequency of citation in the literature for students with disabilities and/or mathematics instruction. The journals included were *The Journal of Special Education, Journal of Learning Disabilities, Learning Disabilities Research and Practice, Learning Disability Quarterly, Journal of Mathematics Education*, *Exceptional Children,* and *Remedial and Special Education.* Eleven articles met all the criteria for inclusion (Bottge, 1999; Bottge & Hasselbring, 1993; Bottge et al., 2001, 2002, 2003, 2007, 2009; Cass et al., 2003; Cihak & Bowlin, 2009; Jitendra et al., 2009; Satsangi & Bouck, 2015).

Overview of Studies

A total of eleven studies met the criteria for inclusion in this literature review (see Table 1). Out of the total sample of participants, 74 (11%) were identified as having a LD, 105 (16%) were identified as having any disability, including a LD; and 187 (23%) were at-risk for mathematics failure, including those with disabilities. Three of the 11 studies included only students with LD (Cass et al., 2003; Cihak & Bowlin, 2009, Satsangi & Bouck, 2015). Six studies included students with and without disabilities (Bottge, 1999; Bottge & Hasselbring, 1993; Bottge et al., 2001, 2002, 2003, 2007, 2009; Jitendra et al., 2009). Four of those studies (Bottge et al., 2001, 2003, 2009; Jitendra et al., 2009) included students at risk for mathematics failure (i.e., remedial).

Eight studies utilized a group design (Bottge, 1999; Bottge & Hasselbring, 1993; Bottge et al., 2001, 2002, 2003, 2007, 2009; Jitendra et al., 2009) and three studies utilized a single-subject design (Cass et al., 2003; Cihak & Bowlin, 2009; Satsangi & Bouck, 2015). The following review of the literature is divided into three sections: (a) nature of the sample, (b) instructional content and focus, and (c) instructional activities.

Nature of Sample

This section includes a description of the participants including identification criteria for MD status, demographic information, gender, age, grade level, and setting. These variables were chosen based on previous analyses of mathematics interventions for students with disabilities (Gersten et al., 2009; Templeton, Neel, & Blood, 2008).

Table 1
Geometry Interventions for Secondary Students with MD

Author (year)	Sample, Setting, Duration	Research Design	Intervention	Dependent Measure	Instructional Content[a]/ Instructional Focus	Results	M/G
Bottge (1999)	N= 66, LD = 3, SWD = 7 (total), M = 37, F = 29, age = NS ; grade 8, 2 pre-algebra, 1 remedial class, 2 teachers; 10 consecutive instructional days plus 2 assessment days	Quasi-experimental	a) Contextualized instruction b) Word problem instruction	Researcher designed tests of fraction computation, linear measurement word problems, contextualized problem, transfer contextualized problem.	Fraction and decimal computation and problem solving, perimeter, 2D to 3D models[a] Conceptual Focus + Problem Solving	a>b on contextualized (ES large .56) and transfer problems (ES medium .37) a = b computation and word problems.	--/G
Bottge & Hasselbring (1993)	N= 36, SWD= 17, age = NS; Grade 9, 2 remedial math classes; 5 instructional days	Quasi-experimental	a) Contextualized instruction b) Word problem instruction	Researcher designed tests of fraction computation, linear measurement word problems; contextualized problems; transfer contextualized problems.	Fraction and decimal computation and problem solving, 2D to 3D models Conceptual Focus + Problem Solving	Both groups improved but a) more than b) on contextualized and transfer measures.	--/G

Author (year)	Sample, Setting, Duration	Research Design	Intervention	Dependent Measure	Instructional Content[a]/ Instructional Focus	Results	M/G
Bottge, Heinrichs, Chan & Serlin (2001)	N= 75, LD = 16, ED = 2, OHI = 1, M = 45, F = 30, Age = 13-15; Grade 8, 1 remedial math class, 3 pre-algebra classes; 12 90-minute classes, maintenance 10-days later	Quasi-experimental	a) Contextualized instruction (EAI) b) Traditional problem solving instruction (TPI)	WRAT-III arithmetic; Researcher designed problem -solving test on distance, rate, time, reading graphs/tables, estimating speeds including whole numbers and decimals. Maintenance tests designed to align with method of instruction EAI or TPI-similar to problem solving test.	Fraction and decimal computation and problem solving, perimeter, 2D to 3D models[a] Conceptual Focus + Problem Solving	Problem solving test: EAI = TPI (gains), RM=PA on post (RM made greater gains than PA in EAI.) WRAT-III: RM decreased, TPI no change, RM <PA (both EAI & TPI) EAI/TPI-based measure: EAI=TPI, RM=PA on both application and traditional types.	--/--

Author (year)	Sample, Setting, Duration	Research Design	Intervention	Dependent Measure	Instructional Content[a]/ Instructional Focus	Results	M/G
Bottge, Heinrichs, Chan, Mehta, Watson (2003)	N= 37, LD[b] = 4, ED = 3, Other SWD =1, M = 23, F = 14, Age = NS; Grade 8 , 1 inclusive pre-algebra, 1 remedial (divided 2 groups); 12 sessions(AA), 22 sessions (LA),30 sessions (LA)	Quasi-experimental, multiple baseline repeated measures across groups; individuals in 2 of 3 groups randomly assigned	a) Video-based instruction b) Applied instruction	Researcher designed word problems and computation probes	Fraction and decimal computation and problem solving, perimeter, 2D to 3D models[a] Conceptual , Procedural Focus + Problem Solving	(a) > (b) for LA and AA. Inconsistent performance on individual computation types, particularly across LA participants .	-/-
Bottge, Heinrichs, Mehta, Hung (2002)	N= 42, LD[b] = 7, ED = 2, M = 20, F = 22 (LD: M = 3, F = 4), Age = NS; Grade 7, 2 inclusive classes, 3 teachers, 4 groupings; 12 sessions	Quasi-experimental, non-equivalent control group design	a) Video-based instruction b) Traditional problem instruction	Researcher designed fraction computation, text-based word problem, and contextualized problem assessments	Fraction and decimal computation, unit conversion, percents, problem solving, 2D to 3D models[a] Conceptual Focus + Problem Solving	(a) = (b) computation and word problems. (a) > (b) contextualized problems and transfer. SWD from pre- to post- tests in (a) decreased in computation; increased in word problem, contextualized problems; in (b) there was	--/G

Author (year)	Sample, Setting, Duration	Research Design	Intervention	Dependent Measure	Instructional Content[a]/ Instructional Focus	Results	M/G
						not a significant change, for students with disabilities on any measure. SWD: (b) > (a) on computation (a decreased), (a) = (b) word problems, (a) > (b) contextualized and transfer. SWOD increased in (a) for contextualized problems and (a) > (b), and (a) > (b) on transfer. On the contextualized problems, scores were similar (and low) for SWD and SWOD. In all other areas SWD did not make gains to near level SWOD and on computation and contextual problems the difference	

Author (year)	Sample, Setting, Duration	Research Design	Intervention	Dependent Measure	Instructional Content[a]/ Instructional Focus	Results	M/G
						widened in (a). Effect sizes: 0.81 (large) contextualized assessment and 0.62 transfer assessment (medium).	
Bottge, Rueda, Kwon, Grant & LaRoque (2009)	N= 109, LD = 8, HA = 26, AA = 57, LA = 36, M = 56, F = 53, Age = NS; Grade 7, 2 teachers, 6 classes; 14 days	Random assignment (to test form) via alternate matched pairs	Video based instruction	Researcher designed paper-based tests and computerized test with hyperlinks on targeted topics.	Fraction & decimal computation, percents, estimation & problem solving with measurement & 2D to 3D models[a] Conceptual Focus + Problem Solving	PPT = CBT All groups improved after EAI but there were not differences in gains from HA, AA and LA based on test form. HA= AA on pre and post and HA/AA > LA on pre and posttests.	--/--
Bottge, Rueda, Serlin, Hung, & Kwon (2007)	N= 128, LD[b] = 12, ADD = 1 M = 60 (13 SWD), F = 68, Age (LD)= 12-14, mean 12.5; Grade 7,	Quasi-experimental, non-equivalent dependent variables design with multiple	Video based instruction	Researcher developed problem solving assessment	Nonlinear[a] /linear functions[a], line of best fit[a], variables, slope, proportions[a] , graphing,	All students improved. Students with LD had larger gains on algebraic tasks than students without LD and LD = non-LD on	M/-

Author (year)	Sample, Setting, Duration	Research Design	Intervention	Dependent Measure	Instructional Content[a]/ Instructional Focus	Results	M/G
	6 classes: 1 inclusive, 1 pre-algebra, 4 typical, 2 teachers; 7 months	measures in repeated waves			measurement error, fraction, percent & decimal computation; 2D to 3D models[a] Conceptual Focus + Problem solving	post-test. No difference between students with LD and without LD on maintenance.	
Cass, Cates, Smith, & Jackson (2003)	N= 3, LD = 3, M = 2, F = 1, Age= 13,15,16; Grades 7, 9 & 10, 1 teacher; 20 minute sessions daily	Single subject, Multiple baseline across participants	Manipulatives	Researcher designed geometry problem solving	Area[a-gr7] and perimeter word problems Conceptual + Procedural focus	All students met criteria gains, generalized and maintained performance.	M/G
Cihak & Bowlin (2009)	N= 3,LD = 3, M = 1, F = 2, Age= 15,16, 18; Grades= High School, 1 teacher, special ed class; 20 sessions	Single subject, multiple baseline across participants	Computer video models	Teacher designed curriculum based geometry assessments	Geometry: perimeter of triangles and polygons Procedural Focus	All participants improved to above 90% and maintained above 80%.	M/-

Author (year)	Sample, Setting, Duration	Research Design	Intervention	Dependent Measure	Instructional Content[a]/ Instructional Focus	Results	M/G
Jitendra, Star, Starosta, Leh, Sood, Caskie, Hughes, & Mack (2009)	N= 148, LD = 15, LA = 28 (LD=6), AA= 75 (LD = 9), HA= 45, M = 69, F = 79, Age = 11-14, mean 12.75; Grade 7, 6 teachers, 8 inclusive classes; Ten days, 40 minutes daily	Quasi-experimental, non-equivalent control group design Pretest-intervention-posttest-retention	a) Schema based instruction b) Direct, explicit instruction	Researcher designed problem solving assessment; PSSA	Ratios, rates, solving proportions[a], scale drawings[a], fractions, decimals, and percents; word problems Conceptual + Procedural Focus	(a) > (b) on problem solving assessment. (a) = (b) on PSSA. For LA group there was not a significant difference in post test or maintenance, but there was for AA and HA. Effect size: favoring (a) =0.52 post (medium), 0.69 maintenance (medium); on problem solving test; 0.65 on PSSA (medium).	M/--

Author (year)	Sample, Setting, Duration	Research Design	Intervention	Dependent Measure	Instructional Content[a]/ Instructional Focus	Results	M/G
Satsangi & Bouck, 2015	N = 3; LD = 3; M = 3 Age = 14, 16, 18; Grade = 9, 11, 11; 1 teacher; 5-10 sessions for 40 minutes	Single subject, Multiple baseline across participants	Virtual manipulatives	Researcher designed assessment of area and perimeter	Geometry: area and perimeter of regular & irregular polygons Conceptual + Procedural Focus	All participants increased over baseline for area and perimeter; All 3 M 70% + perimeter; 2 participants M 80%+ area; 2 participants G 100% area; 1 participant G 100% perimeter	M/G

NOTE: M = Maintenance, G = generalization; [a]content was at or above grade level for all participants based on the Common Core State Standards; [b]some participants were diagnosed with more than one disability; N= total number of participants; ED = number of participants identified with emotional disability; ADHD = number of participants identified with attention deficit hyperactivity disorder; ID = number of participants identified with Intellectual Disability; LA = low achieving; AA = average achieving; HA = high achieving; SWD = students with disabilities; SWOD = students without disabilities; Sample: M = male, F = female; NS = not specified; WRAT-III = Wide Range Achievement Test, 3rd Edition; MOT-R = Mathematics Operations Test-Revised; MCAT = Mathematics Concepts and Applications Test; TCAP = Tennessee Comprehensive Achievement Test; PAT = Progressive Achievement Test in Mathematics; TOSCA = Test of Scholastic Abilities; PSSA = Pennsylvania System of School Assessment; PPT = paper-pencil test; CBT = computer-based test.

MD Definition and Identification

For the purposes of this study, students with MD include students formally identified with a LD as well as students with mathematics difficulties who may not have been formally diagnosed. There are no uniform criteria for diagnosing a LD or definition for mathematics difficulties. However, as the studies reviewed included participants that are formally diagnosed with a LD as well as those at-risk of mathematics failure both are addressed in the section and the categories are as the authors describe.

Of the 11 studies, 64% reported criteria for a LD identification for students in the sample. Identification criteria varied across the studies: (a) two studies (Bottge & Hasselbring, 1993; Cass et al., 2003) identified students with LD based on a discrepancy between intellectual ability and academic achievement, (b) one study (Bottge et al., 2002) provided the state criteria of achievement at or below 50% of expectations (formula: IQ *Years in school*.5 = grade score) and fail to achieve equivalent with age and ability, (c) one study (Bottge et al., 2007) provided the state criteria of 1.75 standard deviations below expected achievement based on standardized achievement and ability tests, (d) two studies (Bottge, 1999; Cihak & Bowlin, 2009) stated that students met state and/or district criteria, without providing more specific information, and (e) one study (Satsangi & Bouck, 2015) included the specific school district criteria for each participant such that both a discrepancy between intellectual ability and academic achievement as well as a Response to Intervention (RtI) model were used. Four studies (Bottge et al., 2001, 2003, 2009; Jitendra et al., 2009) did not report specific criteria for identifying students with a LD.

Furthermore, several studies included students who may be considered at-risk for mathematics failure along with students who were identified as having a LD. Of the 11 studies, six (Bottge, 1999; Bottge & Hasselbring, 1993; Bottge et al., 2001, 2003, 2009; Jitendra et al., 2009) referenced students who may struggle in mathematics without specific definitions or criteria. Bottge (1999) included students in remedial mathematics courses without a LD diagnoses. Bottge and Hasselbring (1993) included students called "at-risk" although all participants were enrolled in a remedial mathematics course. Bottge et al. 2003 included low-achieving students. Bottge et al. 2001 included students in remedial math courses, including students with disabilities, but did not define criteria for remedial placement. Bottge et al. 2009 included high, average, and low achieving students but did not provide defining information for the categories. Jitendra et al. (2009) included students of various abilities that were previously grouped into classes based on the prior year's mathematics course performance (high, average, low), but did not specify placement criteria.

Including specific criteria for MD, including at-risk or formally diagnosed with a LD is critical because of the variety of definitions and identification criteria used across districts (Colker, 2011). Regarding a LD, using a discrepancy between IQ and achievement is problematic due to: (a) the variables not being independent of one another (Fuchs, Fuchs, Compton, & Bryant, 2005), (b) the variation across discrepancy values (e.g., 1, 1.5, 1.75 or 2 standard deviations, 3 years below grade level; Colker, 2011), (c) cut-off scores may inaccurately exclude students with a LD while including low performing students (Colker, 2011; Mazzocco, 2007), and (d) a discrepancy does not necessarily rule out other factors influencing achievement which may exclude students who have a LD (Mazzocco, 2007). Further, the discrepancy model appears to favor identifying students from higher SES backgrounds, Caucasians, and males as having a LD; whereas, there is an overrepresentation of students from minority backgrounds, ELL students and those from lower SES backgrounds identified as having an intellectual disability (Fletcher, Lyon, Fuchs, & Barnes, 2007; Speece, Case, & Molloy, 2003).

To address the increasing number of students diagnosed with a LD, special education policy and the IDEA have moved toward a Response to Intervention model. This approach typically involves three tiers of instruction: (a) general education classroom instruction, (b) specialized small group instruction for students not progressing with the classroom instruction only, and (c) intensive individualized instruction for students who have not made sufficient progress with the prior tiers (Fuchs et al., 2005). However, school psychologists and the APA consider that measures of intellectual function are still essential in the accurate diagnosis of LD and in providing multiple measures of student performance (Colker, 2011; Schrank, Miller, Caterino, & Desrocher, 2006). Therefore, to improve the generalization of the results and support the interpretation of effects of an intervention, future studies should include the specific district criteria used to identify students as having a LD, as well as the IQ achievement discrepancy data and/or RtI methods, whichever is used by the state for a LD identification (Gersten et al., 2005; Horner et al., 2005).

There seems to be even less consensus on what at-risk means. Often researchers provide their own definitions for their study rather than a standard definition, because there is not a standard definition for at-risk (Baker, Gersten, & Lee, 2002). This lack of consensus can be seen within the articles reviewed here where authors state that the participants are students with disabilities and at-risk of failure, or include students who are in remedial courses or low achieving, but may not provide standardized assessment information, as typically used when formally diagnosing a student with a disability. Additionally, there has been much research on risk factors for poor school performance including measures of poverty, race/ethnicity, immigrant, or ELL status (Pungello, Kupersmidt, Burchinal, & Patterson, 1996). It is important to have a standard definition of terms for comparison of results across studies, especially if the expectation is for teachers to use the methods in their practice. Unfortunately, at this time, there is not a consensus, so for the purposes of this study and review MD includes those formally diagnosed with a LD as well as those that have difficulties with mathematics.

Demographics

Of the 11 studies, 64% provided race/ethnicity information for students in the sample. Authors of six studies (Bottge et al., 2002, 2003, 2009; Cass et al., 2003; Jitendra et al., 2009; Satsangi & Bouck, 2015) reported the race and/or ethnicity of the total sample of students, but did not provide the information separately for students with a MD. Conversely, Bottge et al. (2007) reported the race/ethnicity for those with a LD only (Caucasian, n=11; African American, n=1). A total 642 students participated in the studies, of which race/ethnicity information was provided for 351 (55%) participants. Of those, the samples included 279 (79%) Caucasian students, 35 (10%) African-American students, 32 (9%) Latino students, 4 (1%) Asian/Pacific Islander students, 1 (<1%) Native American students and 3 (1%) other/unspecified race/ethnicity students. Additionally, two studies (Cass et al. 2003; Satsangi & Bouck, 2015) provided race/ethnicity information on the entire school population where the study was conducted. Jitendra et al.'s study was the only one that included information on ELLs (n=5; 2009).

Of the 11 studies, 91% reported some location information where the study took place, while 18% included measures of socioeconomic status of the participants. Eight studies (Bottge, 1999; Bottge & Hasselbring, 1993; Bottge, et al., 2001, 2002, 2003, 2007, 2009; Cass et al., 2003) took place in the rural, Midwest United States, one study (Satsangi, & Bouck, 2015) took place in an urban Midwestern city, while one study (Jitendra et al., 2009) took place in an unspecified urban area. Additionally, socioeconomic status was reported in two studies (Cass et al., 2003; Jitendra et al., 2009) and Cass et al. measured SES by the free and reduced price meal (FARMS) status of participants and included a general statement on SES regarding the type of parental employment (blue collar/low skills) for the area.

Including demographic information is critical for generalization of results across participants in various geographic regions, across a diverse racial/ethnic population, and for addressing potential sample bias (Colker, 2011; Fletcher et al., 2007, Gersten et al., 2005; Horner et al., 2005). For example, there should be a match between the number of students in each demographic category (i.e., SES, race/ethnicity, and IEP status) within the sample and the student body as a whole, as well as the general population (Colker, 2011; Fletcher et al., 2007, Gersten et al., 2005; Horner et al., 2005). Furthermore, ELLs are a growing segment of the nation's population, but vary greatly across geographic areas. Overrepresentation of this group in special education is also a concern (Colker, 2011; Hollenbeck, 2007; Johnson, Mellard, & Byrd, 2006). Therefore, future studies should provide information regarding student background information at the school and district levels as well as for the study sample in order to address generalization and overrepresentation concerns.

Gender

Of the 11 studies, 91% reported gender information for total students in the sample, while 45% report information specifically for students with LD. For the 10 studies (Bottge, 1999; Bottge et al., 2001, 2002, 2003, 2007, 2009; Cass et al., 2003; Cihak & Bowlin, 2009; Jitendra et al., 2009; Satsangi & Bouck, 2015) that reported total gender data (n=614), 316 (51%) participants were male and 298 (49%) were female. In the total sample, 13% (N=75) of participants were identified with a LD (some comorbid with other disabilities) and four studies (Bottge et al., 2002, 2007; Cass et al., 2003; Cihak & Bowlin, 2009; Satsangi & Bouck, 2015) reported gender information for students with a LD (n=29), including 22 (76%) males and 7 (24%) females, accounting for just 39% of the participants with a LD.

Including gender information is as important as other participant information for generalizability of results and replication (Gersten et al., 2005; Horner et al., 2005). Although research indicates that approximately 5-8% of people have a LD (Geary, 2004), males are more likely to be identified with a LD than females (Colker, 2011; Fletcher et al., 2007; Speece et al., 2003). Given that more than half the sample participants with a LD were male, there may be a lack of ability for the researchers to have access to willing participants or there may be an imbalance in gender due to the identification processes that perhaps shows bias toward males (e.g., Colker, 2011; Fletcher et al., 2007; Speece et al., 2003). To the extent possible, future research should include participants that match the gender identification rates within the school district and nationally to improve generalization of the intervention and provide an explanation of any differences.

Age, Grade, and Setting

Of the 11 studies, 55% reported age information for students in the sample, 91% reported the school type of school setting and 100% reported the class level setting in which the intervention took place. Students' age was reported in six studies (Bottge et al., 2001, 2007; Cass et al., 2003, Cihak & Bowlin, 2009; Jitendra et al., 2009; Satsangi & Bouck, 2015) and ranged from 11 to 18 years, with only 7 participants older than 14 (see Table 1). Six studies were conducted in a middle school setting (Bottge, 1999; Bottge et al., 2001, 2002, 2003, 2007, 2009), three studies were conducted in a high school setting (Bottge & Hasselbring, 1993; Cihak & Bowlin, 2009; Satsangi & Bouck, 2015), and one study was conducted in a combined middle/high school setting (Cass et al., 2003). Jitendra et al. (2009) did not specify the setting but the grade reported was consistent with middle school. All studies specified the total number of students per grade, with the exception of Cihak and Bowlin (2009). A total of 428 (66%) students were reported in grade 7, 178 (27%) in grade 8, 38 (6%) in grade 9, 1 (<1%) in grade 10 and 2 (<1%) in grade 11. Settings in which the intervention took place included inclusive general education classes for 8 studies (Bottge, 1999; Bottge & Hasselbring, 1993; Bottge et al., 2001, 2002, 2003, 2007, 2009; Jitendra et al., 2009), while three studies (Cass et al., 2003; Cihak & Bowlin, 2009; Satsangi & Bouck, 2015) were conducted in special education or individual settings.

For generalization and interpretation of results, it is important to include information about the age, grade, and setting (Gersten et al., 2005; Horner et al., 2005).

Age is as important as the grade, especially as students in special education often repeat grades, which provides information for comparability of groups, especially in group research (Gersten et al., 2005). It is also important to know the location of the instruction. IDEA requirements address the need to educate students in general education as much as possible (IDEA, 2004), as such, it is promising that the majority of interventions were conducted in the general education setting. In fact, nationally 97% of students with LD are educated in their home school and 61% of students with LD spend 80% or more of the day in general education (NCES, 2011). However, the geometry interventions at the high school setting (Cass et al., 2003; Cihak & Bowlin, 2009; Satsangi & Bouck, 2015) were not in a general education, therefore future studies should address this need by focusing on high school geometry content within general education.

Summary: Nature of Sample

The current review of the research shows promise with regard to the nature of the setting in which the interventions were conducted; however future research needs to address criterion used for MD identification, diversity within the sample of participants (i.e., SES, race/ethnicity, gender) as well as inclusion of information about grade level of participants. First, an area of particular strength is that 80% of the studies were conducted in general education classrooms. This increase is also an improvement from information reported in prior literature reviews of mathematics interventions for students with MD as the majority of interventions reviewed were mostly conducted in segregated settings or remedial classes only (Hughes & Maccini, 1997; Maccini & Hughes, 1997; Lessen, Dudzinski, Karsh, & van Acker, 1989; Maccini et al., 2007; Mastropieri, Scruggs, & Shiah, 1991; Pereira & Winton, 1991). Five studies have been published since Maccini et al.'s review, with three of those studies conducted in general education mathematics classrooms. Future research should continue in general education settings.

Second, the criterion used to identify students with MD should be specified. Definitions and methods (such as cut scores) vary across districts and it is critical to include this information for accuracy of comparison (across studies of similar nature or comparison of groups used within a study). Additionally, the reauthorization of IDEA (2004) has moved the field of special education away from reliance on a discrepancy formula and a "wait-to-fail" approach, to use of Response to Intervention to address concerns with over identification, and to assist students with academic difficulties sooner (Gersten, et al., 2005; Horner, et al., 2005). Therefore, future research should include the specific criteria used to identify students with MD, such as district requirements, standardized assessments used and scores, academic performance (e.g., teacher reports, class grades) and/or RtI methods.

Third, the limited details regarding demographic information specific to students with MD should be addressed in future research. Data on the NAEP for grades 8 and 12 indicated that while all groups (i.e., gender, racial/ethnic, disability, ELL, SES) have increased performance, the gap between Caucasians or Asians and all other races remains as does the gap between students with and without disabilities, ELL and native English speakers, and students from low SES and average/above average SES backgrounds (NCES, 2010; 2013). These data exemplify the need to address a diverse group of students. In this review, general participant SES, race, ELL, and gender information was included; however, the student school level data should also be included for students with MD in order to make adequate comparisons and conclusions about the effectiveness of interventions. Lastly, the majority of the studies included middle school students; therefore, future research should address grade level content with high school students.

Instructional Content and Focus

In this section, interventions are identified by the nature of their instructional content and focus. Instructional content refers to the nature of the geometry skills and related mathematics topics that are addressed in the studies. Instructional focus refers to the type of mathematical knowledge (i.e., conceptual, procedural, problem solving) necessary for proficiency in geometry (CCSS, 2010; Hudson & Miller, 2006; NCTM, 2000; NMAP, 2008; NRC, 2001; van Hiele, 1959).

Instructional Content

The majority of the interventions focused on the following geometry or related skills: (a) basic measurement skills including computation of fractions, decimals and measurement concepts, and unit conversions (Bottge, 1999; Bottge & Hasselbring, 1993; Bottge et al., 2001, 2002, 2003, 2007, 2009; Jitendra, et al., 2009); (b) 2D to 3D modeling (Bottge, 1999; Bottge & Hasselbring, 1993; Bottge et al., 2001, 2002, 2003, 2007, 2009); and (c) area and perimeter of triangles or polygons (Bottge, 1999; Bottge et al., 2001, 2003; Cass et al., 2003; Cihak & Bowlin, 2009; Satsangi & Bouck, 2015). Although all studies included geometry content only three focused solely on geometry skills (Cass et al., 2003; Cihak & Bowlin, 2009; Satsangi & Bouck, 2015), but at a basic level.

Instruction on grade level material is of great importance (e.g., NCLB, 2001; IDEA, 2004; CCSS, 2010). Access to grade level curriculum is critical for students with MD because many states require several years of upper level mathematics courses for graduation, college, and career readiness (ADP, 2004; Colasanti, 2007; CEP, 2010; Dounay, 2007). However, students with MD have had a history of being educated in remedial or segregated settings and exposed to a mathematics curriculum focused on drill and practice of basic skills (Hughes & Maccini, 1997; Maccini & Hughes, 1997; Lessen et al., 1989; Maccini et al., 2007; Mastropieri et al., 1991; Pereira & Winton, 1991). To determine if the mathematical content of the studies was on grade level, participants' grade level was compared to the grade in which the content was placed according to the CCSS.

Seven studies included at least some grade appropriate content in geometry (Bottge, 1999; Bottge et al., 2001, 2002, 2003, 2007, 2009; Jitendra et al., 2009). For instance, all of the Bottge et al. studies addressed 2D to 3D modeling which align with the CCSS content standard for the targeted grade; however, one study (Bottge & Hasselbring, 1993) included grade 9 students and therefore the content focus for these older learners was below grade level. It is of concern, though, that five of the studies (Bottge, 1999; Bottge et al., 2001, 2002, 2003, 2009) addressed primarily computational skills rather than geometry, which limited the grade level appropriateness. However, one study (Bottge et al., 2007) involved students in grade 7 and targeted grade-appropriate content standards involving algebraic topics. Similarly, Jitendra et al. (2009) addressed transformational geometry (i.e. scale factors), which is a critical geometry topic woven throughout spatial reasoning, coordinate geometry, and higher level connections necessary in later courses such as trigonometry (CCSS, 2010, NCTM, 2000); however the main topics in the study were mostly computation and therefore below grade level.

It is disappointing that the three studies (Cass et al., 2003; Cihak & Bowlin, 2009; Satsangi & Bouck, 2015) with high school students addressed middle school geometry content rather than grade appropriate content, such as transformations (e.g. dilations, symmetry, congruence, similarity, 2D to 3D) of polygons, right triangle trigonometry, circles (e.g. arcs, sectors), conic sections, proofs, and applications of geometric concepts to design within constraints (CCSS, 2010). There is a distinct lack of research on grade level geometry interventions for students with MD, particularly for high school students when a full year geometry course is typically taken in grade 10 or 11 (Domina & Saldana, 2012; NMAP, 2008). It is critical that future research address this gap.

Instructional Focus

Interventions can also be identified by the nature of the instructional focus. The instructional focus refers to three types of interconnected knowledge or skills that are necessary for geometric proficiency: procedural fluency, conceptual knowledge, and problem solving. Procedural fluency involves executing procedures flexibly, accurately, efficiently, and appropriately (NRC, 2001) and includes rules, facts, or sequential steps used to compute proficiently (CCSS, 2010; Hudson & Miller, 2006; NCTM, 2000). Conceptual knowledge involves the comprehension of mathematical concepts, operations, and relations (NRC, 2001) and being able to apply newly learned concepts across novel problems (i.e., generalize) in a flexible manner (Hudson & Miller, 2006). The generalization extends across disciplines both to abstract concepts and to real-world application (CCSS, 2010; NCTM, 2000). Problem solving includes the ability to apply previously learned concepts and skills to novel situations (NCTM, 2000), including word problems as well as abstract notations and authentic real-world applications (CCSS, 2010; Hudson & Miller, 2006; NRC, 2001). Problem solving often includes making connections between new and familiar tasks, looking for multiple entry points to a situation, as well as considering multiple solutions, while monitoring and revising strategies (CCSS, 2010; Hudson & Miller, 2006; NCTM, 2000).

Although each type of mathematical knowledge is defined separately, they need to develop simultaneously to achieve mathematical proficiency (CCSS, 2010; Hudson & Miller, 2006; NCTM, 2000; van Hiele, 1959/1985). Procedural fluency enhances conceptual understanding and vice versa (CCSS, 2010; Hudson & Miller, 2006; NRC, 2001), as a student advances in his or her mathematic education, the lack of procedural fluency can impede development of conceptual understanding (NCTM, 2000; NRC, 2001). Both procedural fluency and conceptual understanding are intricately linked with problem solving, particularly with having connections between concepts and being able to access multiple strategies to solve open-ended real-world problems (CCSS, 2010; NRC, 2001). In the following sections, studies are grouped by the type of knowledge addressed, and in most cases, more than one area is addressed prominently in the intervention.

Procedural fluency. Only Cihak and Bowlin (2009) focused solely on developing procedural fluency. Three students with LD were taught step-by-step procedures for calculating perimeter of polygons via video clips. For instance, rather than demonstrating the procedures in person via lecture and class discussion, the instructor recorded the written procedures along with audio explanations of the steps with several example problems. There was no in-class instruction or explanation provided. Students were sent home with the videos to review as often as needed while completing homework problems independently. The homework was checked in class the next day. If the students made errors, they were told to review the video and redo the practice problems. If students scored 100% on the homework, they proceeded to the next lesson via video.

Conceptual knowledge and procedural fluency. Authors of three studies (Cass et al., 2003; Jitendra et al., 2009; Satsangi & Bouck, 2015) focused on developing conceptual knowledge as well as procedural fluency. Cass et al. (2003) incorporated the concrete-representational-abstract (CRA) sequence to represent and solve area and perimeter of quadrilaterals. The CRA sequence represents concepts in a variety of ways (e.g., concrete, pictorial representations, abstract notation), thus making connections with representations to develop conceptual understanding (Maccini & Hughes, 2000; Maccini & Ruhl, 2000; NRC, 2001). Cass et al. included a generalization task that measured participants' ability to solve a task that they were not directly exposed to during the intervention, which addressed conceptual understanding (CCSS, 2010; Gersten et al., 2009). The task required students to apply the area and perimeter skills to a model house and then to scale up to actual size measures to order floor and window coverings.

Satsangi and Bouck (2015) also incorporated elements of the CRA sequence to represent and solve area and perimeter of regular and irregular polygons, integrating the use of virtual manipulatives and pictorial representations. The researchers included a generalization task in which students applied area and perimeter to word problems rather than the static images utilized during the intervention.

Jitendra et al. (2009) utilized schema-based instruction to solve ratio, proportion, and scale drawing word problems. For instance, diagrams were used to align the parts of the word problem with the appropriate place in the ratio as well as a checklist of steps to set up the problem, solve the problem, and check the solution. The diagrams fit to certain schemas so that students would be able to make choices about the most appropriate ratios and proportions for the problems. The schema-based instruction enhanced students' conceptual understanding by connecting representations with ratios or proportions and addressing the underlying structure of the problem (Hudson & Miller, 2006).

Problem solving and conceptual knowledge. Authors of four studies (Bottge, 1999; Bottge et al., 2001, 2007, 2009) incorporated both conceptual knowledge and problem solving via use of video anchors for solving authentic problems with multiple solution paths. Bottge (1999) included generalization tasks that measured participants' ability to solve problems that they were not directly exposed to during the intervention (CCSS, 2010; Gersten et al., 2009), which assessed conceptual understanding and problem solving ability. For example, Bottge (1999) included two applied tasks that involved students' integrating the skills they used during the intervention. Students had to explain how they could afford to build a kite frame within given parameters (i.e., money and materials) and develop a schematic drawing to construct a skateboard ramp. The tasks required students to independently assimilate the requisite mathematical skills and solve tasks going beyond what was taught during the intervention, therefore assessing both conceptual knowledge and problem solving.

Problem solving, conceptual knowledge and procedural fluency. Authors of three studies (Bottge & Hasselbring, 1993; Bottge et al., 2002, 2003) included problem solving and conceptual knowledge as previously noted in their similar studies, but in order to assess procedural fluency included instruction and assessment of computational procedures. For instance, Bottge and Hasselbring (1993) developed a fraction computation test and word problem test to measure student performance prior to and subsequent to use of a fraction videodisc program. This was prior to the contextualized portion of the intervention in order to improve computation skills that may have otherwise hampered performance on the contextualized and transfer tasks that required computation of fractions for linear measurement. Therefore, future research should include explicit or direct instruction on prerequisite skills that are essential for the contextualized, authentic mathematical tasks.

Summary of Instructional Content and Focus

The studies in this review included geometry topics of measurement and 2D to 3D modeling. Only one study addressed transformational geometry in addition to algebra content (Jitendra et al., 2009). Most topics were appropriate for the grade level; however, none of the high school level interventions addressed grade appropriate geometry topics. Future research should include high school students in grade level geometry skills, particularly transformations, as this is a core skill (CCSS, 2010; NCTM, 2000).

Regarding instructional focus, one study (Cihak & Bowlin, 2009) focused on procedural fluency, three studies (Cass et al., 2003; Jitendra et al., 2009; Satsangi & Bouck, 2015) addressed conceptual and procedural knowledge, four studies (Bottge, 1999; Bottge et al., 2001, 2007, 2009) focused on conceptual knowledge and problem solving while another three studies (Bottge & Hasselbring, 1993; Bottge et al., 2002, 2003) included all three types of mathematical knowledge necessary for proficiency in geometry (CCSS, 2010; Hudson & Miller, 2006; NCTM, 2000; NRC, 2001). Given there is a long history of instruction on computation via drill and practice for students with MD (Maccini et al., 2007), the current research is promising in that 91% of the studies included other types of knowledge, 64% included the critical skill of problem solving, and 27% addressed all three types of knowledge. Future research should continue this trend.

Instructional Activities

This section is organized based on the nature of the instructional methods, type of delivery approach, and materials used in the intervention, as adapted from a recent analysis of mathematics interventions for students with MD (Gersten et al., 2009). For each instructional activity, a definition and review are provided, followed by a discussion of the potential benefits of the instructional activities to assist students with MD.

Instructional Practices

Instructional practices refer to the methods that promote geometric proficiency. The types of practices are categorized by delivery (teacher-directed or student-centered) and instructional methods (sequence and range of examples, strategy instruction, and Enhanced Anchored Instruction [EAI]) as shown in Table 2.

Delivery. Teacher-directed instruction involves the teacher primarily communicating the mathematical content directly to the students (NMAP, 2008). Each of the teacher directed studies included elements of explicit instruction (EI) such as: (a) an advanced organizer, to provide structure for the lesson, (b) teacher demonstration (c) guided practice such as prompts to support student learning while gradually fading support, (d) independent practice (e.g., progress monitoring and corrective feedback), and (e) maintenance reviews (Hudson & Miller, 2006). Student-centered instruction occurs when students are primarily responsible for learning, with little or no direct guidance from the teacher (NMAP, 2008; Kirschner, Sweller & Clark, 2006).

Table 2

Geometry Interventions by Instructional Methods, Delivery and Materials

Author (year)	Instructional Methods	Delivery	Materials
Bottge (1999)	Enhanced Anchored Instruction	Student-centered	Video disc, worksheets (both groups), wood (and other building supplies)
Bottge & Hasselbring (1993)	Enhanced Anchored Instruction	Student-centered	Video disc Worksheets
Bottge, Heinrichs, Chan & Serlin (2001)	Enhanced Anchored Instruction	Student-centered	Video disc, model cars, model ramp worksheets (both groups)
Bottge, Heinrichs, Chan, Mehta & Watson (2003)	Enhanced Anchored Instruction	Student-centered	Video disc, wood (and other building supplies), worksheets
Bottge, Heinrichs, Mehta & Hung (2002)	Enhanced Anchored Instruction	Student-centered	Video disc, wood (and other building supplies), worksheets
Bottge, Rueda, Kwon, Grant & LaRoque (2009)	Enhanced Anchored Instruction	Student-centered	Video disc, wood (and other building supplies), worksheets
Bottge, Rueda, Serlin, Hung & Kwon (2007)	Enhanced Anchored Instruction	Student-centered	Videodisc, model cars, model ramp
Cass, Cates, Smith & Jackson (2003)	CRA	Teacher-directed (Explicit instruction)	Manipulatives (geoboard), doll house model
Cihak & Bowlin (2009)	Video Modeling Sequencing of examples	Teacher-directed (Explicit instruction)	Video
Jitendra, Star, Starosta, Leh, Sood, Caskie, Hughes & Mack (2009)	Schema-based Instruction	Teacher-directed (Explicit instruction)	Strategy checklist
Satsangi & Bouck (2015)	CRA	Teacher-directed (Explicit instruction)	Virtual manipulatives

Teacher-directed instruction. Authors of four studies utilized teacher-directed instruction, including elements of the explicit teaching method (Cass et al., 2003; Cihak & Bowlin, 2009; Jitendra et al., 2009; Satsangi & Bouck, 2015). Three studies utilized a single-subject multiple baseline across participants design (Cass et al. 2003; Cihak & Bowlin, 2009; Satsangi & Bouck, 2015), while Jitendra et al. (2009) utilized a quasi-experimental nonequivalent control group design.

Cass et al. (2003) investigated the use of manipulatives (i.e., geoboards) on solving area and perimeter problems utilizing a single-subject design with one middle school and two high school students with MD. Each lesson included four EI components: (a) teacher modeling how to represent given shapes on the geoboard and calculate the area or perimeter, (b) prompting/guided practice to represent shapes on the geoboard and calculate the area or perimeter, (c) independent practice representing shapes on the geoboard and calculating area and perimeter with corrective feedback, and (d) a post-test.

Following two to three weeks of intervention, students reached a criterion level of 80% accuracy or greater on three consecutive probes. Twice weekly maintenance measures were administered for three weeks after the termination of the intervention, followed by a paper-pencil longer-term maintenance measure given two weeks later. At this time, an applied generalization task was also administered in which students measured rooms in a model doll house and converted the scale measures to order floor and window coverings. Scores were greater than 90% on both the maintenance and generalization measures. Using the manipulatives with high school students appears promising based on this single-subject study, and future research should explore this further. However, a note of caution is warranted given that elements of EI were prominently utilized, thus it is difficult to determine if EI rather than the use of the manipulative was responsible for the increase in student performance.

Similar to Cass et al. (2003), Satsangi and Bouck (2015) investigated the use of virtual manipulatives on solving area and perimeter problems utilizing a single-subject design across three high school students with MD. Elements of EI used in the study included: (a) teacher modeling how to represent given shapes using the virtual manipulative program and calculate area and perimeter, (b) prompting/guided practice of how to represent shapes in the virtual program and calculate the area or perimeter, (c) independent practice representing shapes in the virtual program and calculating area and perimeter with corrective feedback, and (d) a post-test.

Following one to two weeks of intervention, students reached stability. Stability was defined as either 80% of the data points falling within 20% of the mean or, if more than 8 sessions, the intervention ended once three consecutive probes were identical for area and perimeter. Two weeks after the end of the intervention three maintenance probes were administered. Finally, a generalization measure was administered for three sessions in which participants applied the skills to word problems While all students improved performance over baseline, there was some variation, with one student earning 100% on all measures for area and perimeter, one participant earned above 70% for maintenance and generalization for one topic but decelerated on maintenance and generalization for the other topic, while the third participant was flat or variable for maintenance and generalization. Using the virtual manipulatives with high school students appears promising based on this single subject study, and future research should explore this further. However, a note of caution is warranted given that elements of EI were prominently utilized, thus it is difficult to determine if EI rather than the use of the manipulative was responsible for the increase in student performance.

Cihak and Bowlin (2009) investigated investigate the effects of procedures involving video models with narrative for solving perimeter of polygons utilizing a single-subject design across three high school students with MD. The lessons included three elements of EI: (a) teacher modeling of procedures for how to calculate perimeter of various polygons, (b) independent practice (i.e., homework practice problems solving perimeter of polygons), and (c) post-test.

The recorded models involved the teacher's solving a variety of perimeter problem exemplars on the targeted topic of the day using step-by-step procedures. The target problems involved perimeter of rectangles/squares, triangles/trapezoids, and various polygons. There was no instruction at school, rather, students were provided recordings on a tablet computer and told to use the models as needed to solve 10 practice problems at home. The assignment was graded the next day and a quiz was given if all problems were correct or the teacher instructed the student to redo the assignment after reviewing the video models. Once a student scored 100% on three consecutive quizzes, instruction moved on to the next target shape. All participants maintained 80% accuracy or greater six weeks after the conclusion of the intervention. The use of technology provided some element of independence to the students, and should be considered in future research for students with MD.

Jitendra et al. (2009) investigated the use of schema-based strategy instruction to teach grade 7 general education students to recognize structure and solve ratio and proportion word problems involving unit rates, scale drawings, percents and fractions utilizing a quasi-experimental nonequivalent control group design. The study compared the strategy-based instruction (SBI) to traditional key word problem solving. Each lesson included the following elements of EI: (a) an advance organizer with review and teacher introduction of key concepts, (b) teacher modeling the problem type and use of the schematic diagram to solve the ratio or proportion (i.e. unit rates or percents), (c) guided practice and feedback on solution methods and correction (i.e. of computation or incorrect schema use), and (d) independent practice and feedback on the homework.

Although scripted lessons were provided, instructors were encouraged to add their own explanations and discussion. Additionally, a checklist was provided to students for self-monitoring of the strategy steps: *Find the problem type*, *Organize the information using diagrams*, *Plan to solve*, *Solve*. The intervention group showed greater improvement than the control group from pre- to post-test. Students maintained their performance on measure four months after the conclusion of the study. Although the unit of measure was the individual student, disaggregated data for the students with MD were not included; therefore, future research should compare the effects of SBI for students with MD and without MD. Additionally, future research should include a generalization measure on skills beyond scale factor recognition and make connections to more complex transformational skills within authentic problem situations in order to address higher-level skills such as in the Common Core mathematics standards.

Student-centered instruction. In contrast to studies in which the teacher was the primary means of modeling or showing procedures, seven studies (Bottge, 1999; Bottge & Hasselbring, 1993; Bottge et al., 2001, 2002, 2003, 2007, 2009) utilized student-centered learning in their intervention in which students were primarily responsible for their learning with teacher facilitation. All of Bottge and colleagues' studies involved the use of EAI in which students worked in pairs or groups to solve real-world, authentic, open-ended problems based on a videodisc anchor, with the teacher asking probing questions to guide discussion and problem solving. The videos allowed students to search for relevant information as many times as they wanted in order to solve problems in which the solution methods were not explicit.

The studies took place in a variety of settings and with varied groups. One study (Bottge & Hasselbring, 1993) took place in remedial classes, one study (Bottge et al., 2002) took place in general education classes, while the remaining five studies (Bottge, 1999; Bottge et al., 2001, 2003, 2007, 2009) compared student performance in general education and remedial settings. On contextualized measures the EAI group(s) outperformed students provided traditional word problem instruction involving identifying key words. However, based on results of some of EAI studies (e.g., Bottge, 1999; Bottge et al., 2001) which showed that student problem solving performance was hampered by difficulties with fraction computation, two studies (Bottge & Hasselbring, 1993; Bottge et al., 2002) specifically included lessons on fraction computation prior to the beginning of the EAI activities, which brought the pre-intervention fraction computation performance of student groups to a similar level.

This additional instruction (via either computerized individual instruction or teacher supplemental instruction) may have positively impacted overall student performance. Therefore, Bottge and colleagues' research in EAI appears to support the NMAP (2008) findings and CCSS (2010) view that high quality mathematics education should include elements of both student-centered and teacher-directed instruction. For instance, two EAI studies (Bottge & Hasselbring, 1993; Bottge et al, 2002) addressed the issue raised in prior EAI studies in which students improved on contextualized problems but did not improve computation by including supplemental instruction in computation.

Teacher directed instruction, such as EI, has proven to be effective for teaching students with MD (Hudson & Miller, 2006; Maccini & Hughes, 1997). Explicit instruction assists students with MD in learning because it compensates for difficulties in memory via review of prerequisite skills, multiple practice opportunities, and thorough review. However, the benefit of student-centered instruction such as EAI, is not universal because while students in Bottge and colleagues' studies improved their contextualized problem performance, assessments of computation showed no improvement, even with the pre-teaching (e.g., Bottge et al., 2002) for some students with MD.

Given this conflicting information, future research is needed to determine if explicit instruction in procedural or computational skills followed by or in conjunction with student-centered authentic problem solving tasks will improve achievement for students with MD rather than student-centered instruction only, particularly in high school level mathematics content such as transformational geometry. This is a critical area of research, as more students with MD are in general education classes, likely using a student centered approach (Woodward & Montague, 2002) and state graduation requirements are increasing to include up to four years of mathematics, including geometry, in order to prepare students for college and careers (Dounay, 2007).

Instructional methods. In addition to the method of delivery, the specific instructional methods varied within the current review. The instructional methods included sequencing and/or providing a range of examples, strategy instruction, and

Enhanced Anchored Instruction. Each type of method is defined followed by a description of the methods employed in each study.

Sequence and range of examples. Interventions in this category needed to include: (a) a specified sequence or pattern of examples, such as the CRA sequence; or (b) variation in the range of examples, such as teaching proper fractions prior to improper fractions rather than teaching concepts simultaneously (Gersten et al., 2009). Four studies in the current review incorporated sequencing and range of examples to improve student performance with ratio and proportions (Jitendra et al., 2009), perimeter (Cihak & Bowlin, 2009), and perimeter and area (Cass et al., 2003; Satsangi & Bouck, 2015).

The CRA sequence is a three-part sequence of instruction in which students learn via manipulation of physical objects, followed by pictorial depictions of the objects and concepts they represent, concluding with solving problems using abstract symbols (Witzel, 2005). Use of the CRA sequence has been shown to help students with MD improve their ability to represent and solve problems such as integer word problems (Maccini & Ruhl, 2000; Maccini & Hughes, 2000) as well as solving multi-step one-variable equations (Witzel, 2005; Witzel, Mercer, & Miller, 2003). Two studies in the present review (Cass et al., 2003; Satsangi & Bouck, 2015) utilized the CRA sequence or portions thereof, with systematic examples as part of an intervention to improve student's performance with computation of perimeter and area.

Cass et al. (2003) focused on quadrilaterals (e.g., squares, rectangles, composite figures of those two shapes). The sequencing began with perimeter then proceeded to area problems, including teacher modeling for each type of calculation prior to student practice. The CRA steps included the following: (a) concrete level instruction using geoboards to represent and solve for the perimeter of rectangles, squares and composite figures until students reached mastery of 80% or greater; (b) representation level instruction using drawings of the rectangles, squares, and composite figures using those shapes until students reached mastery of 80% or greater; and (c) abstract level instruction using numbers and symbols to solve perimeter of rectangles, squares, and composite figures until students reached mastery of 80% or greater.

The intervention continued the same steps and performance criteria with area calculation. Generalization involved students determining the area and perimeter of rooms in a dollhouse and scaling up the measurements as if ordering carpet and materials for a real home. All students met and maintained criterion on immediate, maintenance, and generalization measures.

Similarly, Satsangi and Bouck (2015) focused on area and perimeter of regular and irregular polygons. The sequencing began with teacher modeling of perimeter then proceeded to area problems, followed by instruction on the use of the virtual manipulatives prior to student practice. The CRA sequence did not include a concrete or abstract steps but focused on representation level instruction using drawings of the irregular shapes with unit squares as the base shape (similar to those shapes found in the game Tetris) to re-create the image using the virtual manipulative program and solving for perimeter and area on five problems until students reached stability.

Generalization involved students determining the area and perimeter applied to word problems in which students would create the shapes required based on the written information/descriptions. Although all three participants increased their performance over baseline for intervention, maintenance, and generalization, the performance was variable or decelerating for two of the participants on area, perimeter or both.

Providing a range of examples means progressing from less complex or easy to more complex or difficult problems (Scheurmann, Deshler, & Shumaker, 2009), which is critical for students with MD who may have difficulties with working memory (e.g., Jitendra, Kameenui, & Carnine, 1994; Woodward, 1991). Providing a range of examples has been shown to help students with MD to improve their procedural and conceptual understanding, as well as their ability to generalize from simple to more complex problems, including algebra concepts involving solving multi-step equations (Scheurmann, et al., 2009). Two studies in the current review (Cihak & Bowlin, 2009; Jitendra et al., 2009) included sequencing of simple to more complex problem types and provided a range of topics or figures to help students with MD to solve perimeter problems (Cihak & Bowlin, 2009) and word problems involving ratio and proportions (Jitendra, et al., 2009).

Cihak and Bowlin (2009) embedded sequencing and a range of successively more complex perimeter of polygon problems into the instructional design via computer video modeling. Students had to demonstrate proficiency on one type of problem prior to advancing to the next set of problems involving squares and rectangles, triangles and trapezoids, and other polygons with missing information (i.e., pentagons and hexagons). As a result of the intervention, all students performed at or above 90% during the intervention phase for each shape. Further, students maintained performance of 80% or greater on maintenance probes six weeks after the conclusion of the intervention.

Jitendra et al. (2009) utilized sequencing of more complex ratio and proportion problems using schema-base instruction, which involved using diagrams and “think-alouds" modeling the various schemas used such as multiplicative comparisons. Ten lessons progressed from basic terminology instruction on ratios to equivalency, simplifying, unit rates, proportions, scale factors and percents. It was determined that the intervention group outperformed the traditional instruction (i.e., key word) group on immediate post-test and maintenance measures four months post-instruction.

In summary, the CRA sequence and systematic variation in the range of examples are promising practices for students with MD. In the CRA sequence, the concrete materials and representations scaffold students' learning of abstract concepts, which are often challenging for students with MD (Bley & Thornton, 2001; Geary, 2004). Further, providing systematic variation in the range of examples into instruction helps to build a foundation to more complex problems, an area of critical need for students with MD (Gagnon & Maccini, 2001). Future research and instruction should aim to increase the conceptual as well as procedural competence of students with MD by providing ample opportunity for students to make connections between simple and more complex figures or problems and incorporating the CRA sequence. Furthermore, future research should also include grade appropriate high school content such as transformational geometry, an area absent from the literature base and of central importance in the CCSS (2010).

Strategy instruction. The use of a strategy involves a general approach to solving a variety of problems and has been found to be effective for students with MD in both general education and alternative settings (Maccini, Strickland, Gagnon, & Malmgren, 2008). Strategy instruction (SI) includes the use of memory aids (e.g., mnemonics, cue cards, checklists) and graphic organizers (e.g., graphs and charts) to provide students with a strategic plan to solve problems (Gersten et al., 2009). One type of strategy instruction, schema-based instruction, assists students in recognizing the underlying context and structure within a content domain to help students represent and solve problems (Hegarty & Kozhevnikov, 1999). Characteristics of schema-based instruction include authentic real-life problems, representational diagrams of concepts, as well as self-monitoring components such as a checklist or think-aloud in order to engage students and improve their conceptual understanding (Hegarty & Kozhevnikov, 1999; Kramarski & Mizrachi, 2006).

Jitendra et al. (2009) was the only study that primarily examined the use of schema-based instruction (SBI) with students with MD. Students were taught to recognize general forms of ratio and proportion problems that fit a particular visual comparison and solution method (i.e., equivalent fractions or cross multiplication) using a first letter mnemonic strategy, FOPS, to help students remember the steps to follow when solving ratio and proportion word problems. The strategy included the following steps: (a) Find the problem type, (b) Organize the problem, (c) Plan to solve the problem, and (d) Solve the problem. As compared to the keyword method, students in the SBI group improved their performance from pre-test to post-test and maintained their improved performance four months after the conclusion of the intervention.

Although only one study focusing on SI included geometry content (i.e., scale factor), research has shown it to be useful with secondary students' solving of proportions (e.g., Hutchinson, 1993) and integer word problems (e.g., Maccini & Hughes, 2000; Maccini & Ruhl, 2000). Strategy instruction is a promising instructional practice that supports metacognitive processes such as self-regulation, planning, monitoring, and evaluating a task, which are often difficult for students with MD (Bley & Thornton, 2001; Montague, 2008). Future research should include SI to support student's metacognition.

Enhance anchored instruction. Enhanced anchored instruction is a video-based instructional program for developing computational and problem solving skills via authentic problem situations. After viewing a video portraying a real-life scenario, students solve the problem presented or similar problems, such as determining the costs to build a skateboard ramp, a kite, or a pet cage. Students can review the videos as often as needed to locate the essential information in order to solve the problem in the scenario. Seven studies (Bottge, 1999; Bottge & Hasselbring, 1993; Bottge et al., 2001, 2002, 2003, 2007, 2009) utilized EAI and compared the performance of students with MD and students without MD.

A variety of computation, algebraic, and geometric skills were addressed in the studies utilizing authentic problems to engage students in discussion and problem solving in small groups. For example, in one study (Bottge et al., 2001) the video *Kim's Komet* was used to facilitate student discussion and application of concepts involving linear functions, slope, acceleration, line of best fit, and measurement in order to predict from a graph the location a car should be released to successfully complete a model car race. Through this activity students discovered their own formulas (e.g., Distance = Rate*Time), created their own representations (e.g., sketches, graphs), and used various student-created solution methods.

Furthermore, conceptual and procedural knowledge of reading schematic plans, fraction computation, money computation, estimation, and unit conversions were targeted in five studies. In *Bart's Pet Project* students decided if they could afford to build a pet cage (Bottge, 1999; Bottge & Hasselbring, 1993; Bottge et al., 2003). In *A Fraction of the Cost* students decided on the most cost-effective way to build a skateboard ramp (Bottge et al., 2009). Additionally *Kim's Komet*, as well as *A Fraction of the Cost*, were used in one study (Bottge et al., 2007) to target computation as well as algebraic applications using the student-centered problem solving approach as in earlier EAI studies. To further enhance student engagement and generalization, six studies (Bottge, 1999; Bottge et al., 2001, 2002, 2003, 2007, 2009) included extensions in which students independently generalized the skills with a culminating project (e.g., building a model car, compost bin, or skateboard ramp).

The results were mixed in terms of the overall effectiveness of EAI. For example, EAI instruction was found to be more effective than traditional instruction involving word problems with key words on contextualized problems; however, neither group improved on computation measures. Furthermore, while in some cases students with MD made greater gains than students without disabilities (Bottge et al., 2001, 2007), the performance gap remained across the groups as seen in the mean pre- and post-test scores of the MD versus non-MD groups (Bottge, 1999; Bottge et al., 2002, 2003, 2009).

In summary, EAI has shown promise for improving performance of students with and without MD on contextualized problems. EAI provides real-world applications and group cooperation as advocated for in the CCSS (2010) and NCTM (2000). However, a particular area of need remains with planning for review or embedded instruction on foundational skills as seen in the poor post-intervention computation scores in each EAI study, except Bottge and Hasselbring (1993) in which pre-intervention instruction on computation was provided.

Although Bottge and Hasselbring's (1993) approach improved computation performance, it was noted that this did not necessarily improve the computation skills within the contextualized problems. This is problematic in that the problem solving performance continued to be hindered by poor computational skills. This may indicate that in isolation computational skills improved, but perhaps more explicit methods need to be employed during EAI to harness the improved computation skills. Future research should incorporate technology such as the video-anchors and open-ended authentic problems along with explicit instruction or remediation of foundational skills in order for students with MD to achieve maximum benefits from the instruction.

Instructional Materials

Instructional materials are physical devices, equipment, technology, or other non-technical items (e.g., charts, colored overlays) used in the classroom to support growth in knowledge (Dash & Dash, 2007; Reys, Suydam, & Lindquist, 1992). The use of physical material has led to increased achievement (Hudson & Miller, 2006). Materials included in this review are manipulatives, structured worksheets, and technology.

Manipulatives. Manipulatives are usually concrete materials such as chips, tiles, or other objects (commercial or created by teachers and/or students), that represent mathematical relationships (Maccini & Gagnon, 2000). For example, in geometry, geoboards are boards of various shapes with a lattice of pegs used to create segments or polygons using rubber bands stretched between the pegs to examine geometric relationships (Gattegno, 1971). However, manipulatives may also include computerized or virtual manipulatives that are either static (i.e., representations or pictures) or dynamic (i.e., able to be moved or changed) and often modeled after concrete manipulatives

(Moyer, Bolyard & Spikell, 2002). For example, Java applets or the National Library of Virtual Manipulatives (NVLM) provide a variety of resources for exploring concepts at all levels and topics, including geometry (Moyer, Bolyard & Spikell, 2002).

The use of manipulatives helps students to: (a) make connections between concrete and abstract concepts, (b) increase reasoning skills, and (c) increase task engagement (Boggan, Harper & Whitmire, 2010). In this review, four studies (Bottge et al., 2001, 2007; Cass et al., 2003; Satsangi & Bouck, 2015) incorporated manipulatives in their interventions. Both Bottge et al. (2001, 2007) studies used model cars and a ramp as part of their intervention using authentic problems targeting distance, rate, and time calculations. Cass et al. (2003) included the use of geoboards to explore perimeter and area concepts, while Satsangi and Bouck (2015) utilized virtual manipulatives to explore perimeter and area. Students in all four studies improved on the target skills and concepts, which supports previous research and the benefits of manipulatives in secondary mathematics (Maccini & Ruhl, 2000; Maccini & Hughes, 2000; Witzel, 2005; Witzel et al., 2003). Future research should incorporate manipulatives to teach higher-level mathematics (e.g., transformational geometry) to assist students with MD in developing conceptual understanding of abstract concepts essential for advanced mathematics (Banchoff, 2008).

Structured worksheets. Structured worksheets or cue cards, help students develop a strategic plan to solve problems by prompting students to think about important parts of the problem (Hudson & Miller, 2006; Maccini et al., 2008) and ask themselves questions about known, unknown and relevant information. Structured worksheets may include checklists, including for procedures (Maccini & Hughes, 2000; Maccini & Ruhl, 2000). Structured worksheets address the need for students to develop self-regulation and metacognitive skills, which are critical for students with MD (Gurganus, 2007). One study in this review (Jitendra et al., 2009) included a structured worksheet and checklist for the problem solving strategy mnemonic, *FOPS*, for solving ratio and proportions. The teacher modeled and then students utilized the checklist with gradual fading of its explicit use. The worksheet included the strategy steps (i.e., figuring out the problem) and space for students to write in responses.

The immediate and maintenance measures indicated the intervention was significantly more effective than traditional key word problem solving instruction. Given the difficulties that many students with MD experience with problem solving and metacognition, this study shows promise for instruction within the general education setting. Future research should explore the use of structured strategy worksheets with more advanced secondary content, particularly as the material becomes more complex with multiple steps.

Technology. NCTM (2000) and CCSS (2010) support the use of technology as tools for increasing students' conceptual understanding including calculators, computer algebra systems, interactive geometry software, applets, spreadsheets, and interactive presentation devices. Nine studies included technology as a central component of the intervention (Bottge, 1999; Bottge & Hasselbring, 1993; Bottge et al., 2001, 2002, 2003, 2007, 2009; Cihak & Bowlin, 2009; Satsangi & Bouck, 2015). Cihak and Bowlin (2009) utilized video recordings of the teacher demonstrating procedural solutions to calculate perimeter of polygons, including composite figures. Students were provided the recordings to take home and view as often as needed while independently solving similar problems. Students maintained performance above 80% up to six weeks post-intervention. Satsangi and Bouck (2015) addressed perimeter as well as area of regular and irregular polygons utilizing virtual manipulatives. All students improved and maintained performance approximately four weeks after intervention as well as generalized to word problems.

Bottge and colleagues (1993, 1999, 2001, 2002, 2003, 2007, 2009) utilized EAI, which addresses the CCSS (2010) and NCTM (2000) goals for the use of technology integration. EAI utilized videodiscs to introduce an authentic problem-solving scenario, which lead into small group discussion and application of a variety of mathematics skills, often culminating in the building of 3D objects from the work in the group problems. Students were able to review the video recordings as often as needed in order to find the relevant information and work within the groups to solve contextualized problems, with teacher facilitation. The EAI technology enables students to learn mathematics conceptually by using multiple representations and authentic scenarios. Although EAI improved the problem solving ability of students, future research needs to address the computational performance of students, which was not consistently improved across all the EAI studies.

The NCTM (2000) and CCSS (2010) advocate for the appropriate use of technology to improve the computational and/or conceptual understanding of students, which have long been recognized as areas of difficulty for students with MD (Geary, 2004). The independent use of video recordings shows promise for the computational performance, while the independent use of virtual manipulatives shows promise for conceptual as well as procedural knowledge of secondary students with MD.

Additionally, the integration of technology into authentic contextualized problem solving situations shows promise in a variety of settings and with students of all ability levels. Future research should explore the potential benefits of technology for secondary students with grade appropriate geometry content (CCSS, 2010).

Summary of Instructional Activities

Authors of the current studies incorporated a variety of instructional practices, methods, and materials to improve the geometry performance of secondary students with MD (see Table 2). Four studies (Cass et al., 2003; Cihak & Bowlin, 2009; Jitendra et al., 2009; Satsangi & Bouck, 2015) utilized teacher-directed instruction as the primary method of instructional delivery, while seven studies (Bottge, 1999; Bottge & Hasselbring, 1993; Bottge et al., 2001, 2002, 2003, 2007, 2009) utilized a student-centered approach to instruction. Teacher-directed instruction, particularly within resource and remedial settings, has dominated instructional delivery for students with special needs for decades (Hudson & Miller, 2006). However, the majority of students with MD are currently included in general education settings (NCES, 2011) with the rigorous content promoted by CCSS (2010) and NCTM (2000). Given the varied needs of students with MD, a blended (or integrated) approach has been advocated for by some researchers and policy makers (e.g., Hudson, Miller, & Butler, 2006; NMAP, 2008). For example, utilizing EI to address foundational concepts such as fraction computation (either prior to the main instruction or supplementary, as needed, individualized instruction), students are then able to actively participate in the student-centered instruction in order to develop conceptual understanding of more complex topics such as scale factor or dilations.

Instructional practices include sequence and/or range of examples (e.g., CRA sequence), strategy instruction, and EAI. All studies in the current review included more than one approach. This is important as each practice addresses areas of need for students with MD. For example, teacher-directed methods such as EI provide scaffolding and multiple opportunities for practice in order to increase mastery (Hudson & Miller, 2006) to support students with memory deficits. Strategy instruction, such as schema-based instruction, aids students in recognizing the underlying context and structure within a particular content domain to help students represent and solve problems (Hegarty & Kozhevnikov, 1999; Kramarski & Mizrachi, 2006). The CRA sequence promotes conceptual understanding as students move from concrete to abstract concepts, which can be conceptually difficult for students with MD due to organization, sequencing, and processing difficulties (Steele, 2010). However, students with MD are a heterogeneous group, with difficulties due to a variety of intraindividual factors, and require supports based on those needs. Interventions that incorporate multiple practices can address a variety of difficulties that students with MD may exhibit to support achievement.

Materials in the current studies included the use of manipulatives (e.g., geoboards, model cards, ramps, virtual manipulatives), structured worksheets, and technology. Six studies (Bottge, 1999; Bottge et al., 2001, 2002, 2003, 2007, 2009) included multiple types of materials to support students' procedural and conceptual learning. Therefore, future research should include a variety of materials that provide structure and support to assist students with MD who may experience problems with memory, sequencing, metacognition, visuo-spatial processing, organization, and understanding abstract concepts.

Summary of Literature

The current review of the literature on geometry interventions for secondary students with MD identified instructional approaches, practices, and materials that lead to improved performance on a variety of geometric concepts. In the following section, limitations of the current literature and suggestions for future research are summarized.

Limitations

Overall, the authors of the studies reviewed here include many beneficial instructional practices that led to improved student performance. However, there are several limitations to the current research:

1. The existing studies focused primarily on basic geometry content rather than critical content and age-appropriate tasks as recommended by the NCTM, NMAP, ADP, and CCSS.
2. Few studies include high school participants; therefore, the effectiveness of the instructional practices in this review are uncertain for older students charged with learning advanced geometry content.
3. There is no consistent criterion for identifying students with MD, which limits generalization of current findings.
4. Two studies (Bottge et al., 2007; Jitendra et al., 2009) did not disaggregate data for students with MD. Additionally, while six EAI studies compared students in remedial classes to typical or advanced students, further disaggregation of the data by type of disability is needed.
5. Few studies provided race/ethnicity information for students with MD and those that did were overwhelming lacking in minority representation. However, Cass et

al. (2003) and Satsangi & Bouck (2015) provided the race/ethnicity of the participants and the school, which is important information to include for equity and generalization.

Future Research

Traditionally, there has been more research on reading disabilities than mathematical difficulties, with a 14 to 1 ratio of reading studies as compared to mathematics studies (Gersten, Clarke, & Mazzocco, 2007). Additionally, geometry difficulties have received little to no attention from researchers in the field of special education with the majority of research focusing on computation or pre-algebra skills as evidenced by multiple research reviews and meta-analyses for over 20 years (e.g., Lessen et al., 1989; Pereira & Winton, 1991; Mastropieri et al., 1991; Maccini & Hughes, 1997; Maccini et al., 2007). Given the limitations in the existing research base, future research should examine:

1) Student performance on high school geometry content, such as similarity transformations.
2) Inclusion of high school students with MD in general education classes, including geometry and subsequent geometry dependent courses such as trigonometry.
3) The use of an instructional package including instructional practices and materials supported by the current research such as student centered instruction, EI (supplementary, as needed, or as an integral part of delivery), technology (e.g., computer software, videos), CRA sequence (including manipulatives), and structured worksheets/cue cards.

Summary. The current literature review synthesizes research findings involving geometry interventions for secondary students with MD. These interventions address instructional practices, methods, and materials that provide students with disabilities access to the general education curriculum. This is a critical area of research as more secondary students with disabilities are included in general education classes (NCES, 2011). Given this inclusion of students with disabilities in general education classes there also must be integration of the special education literature with the research in geometry from a general education perspective.

Special education research and interventions have a long history with behaviorist theories of learning that focus on teacher-direct instruction and repeated practice (Woodward & Montague, 2002). However, the NCTM (2000) and CCSS (2010) promote a more student-centered approach that enhances conceptual understanding rather than an overreliance on procedural or rote skills. Additionally, the CCSS (2010) also views geometry from a transformation-centered approach, which is particularly absent from the special education literature. The next section provides an overview of the general education intervention literature in transformational geometry in a similar manner to the review of special education literature and concludes with an integration of the findings and rationale for the current study.

Organization of General Education Literature Review

In this section, I present a review of the current research involving transformational geometry interventions for secondary students. This review serves two purposes: (a) to determine the current status of effective transformational geometry research with secondary students, and (b) to examine empirically based instructional variables to inform the current study for students with MD.

Studies meeting the following search criteria were included in this review: (a) published in a peer reviewed journal between 1989 and 2015 (1989 was chosen to reflect the NCTM Principles and Standards), (b) examined the performance of secondary students in transformational geometry (either an intervention impacting achievement or examined the nature of student learning), (c) used qualitative or quantitative methods (experimental, quasi-experimental, single-subject design, or case study), and (d) included secondary students from grades 7-12. Studies were excluded that involved students from multiple grade levels that included those below grade 7, as several states begin secondary school and/or secondary teacher certification with grade 7. An automated database search was conducted using Educational Resources Information Center (ERIC), Psychological Articles by the American Psychological Association (PsychARTICLES), Education Research Complete, as well as Google Scholar, to identify possible studies for inclusion in this review. The terms used for the search were combinations of *reasoning, geometry, spatial reasoning, problem solving, transformations, symmetry, similarity, congruence, rotation, reflection, dilation, instruction, teaching, learning, secondary, representation, computer* and *technology*. An ancestral search was conducted within the reference lists of the articles obtained in the automated search results to locate additional studies.

Lastly, a hand search of journals was conducted to identify the most relevant articles for the topic of this review. Specific journals were chosen due to their frequency of citation in literature for mathematics instruction and general education research. The journals included were *Journal for Research in Mathematics Education, Educational Studies in Mathematics, Journal of Mathematical Behavior, Journal of Educational Research,* and *Journal of Educational Psychology*. Nine articles were identified as meeting all the criteria for inclusion (Boulter & Kirby, 1994; Choi-Koh, 1999; Gorgorio, 1998; Guven, 2012; Hollebrands, 2003; Hollebrands, 2007; Hungwe et al., 2007; Kirby & Boulter, 1999; Rowell & Mansfield, 2001).

Overview of General Education Studies

Nine studies met the criteria for inclusion in this literature review (see Table 3). Four studies utilized qualitative methodology, including case studies, clinical interviews, and task analysis (Boulter & Kirby, 1994; Choi-Koh, 1999; Hollebrands, 2003; Hollebrands, 2007), four studies utilized quantitative comparisons (Guven, 2012; Hungwe, et al., 2007; Kirby & Boulter, 1999; Rowell & Mansfield, 2001), while one study used a mixed-methods design (Gorgorio, 1998). Five studies included middle school populations (Boulter & Kirby, 1994; Choi-Koh, 1999; Kirby & Boulter, 1999; Guven, 2012; Rowell & Mansfield, 2001), two studies high school students (Hollebrands, 2003, 2007) and two studies included participants from both types of secondary locations (Gorgorio, 1998; Hungwe, et al., 2007). The following review of the literature is divided into three major sections: (a) nature of the sample, (b) instructional content and focus, and (c) instructional activities.

Nature of Sample

The section below includes a description of the participants including demographic information, gender, age, grade level, and setting. These variables were chosen to be consistent with the earlier sections from the special education literature.

Table 3

Transformation Geometry Instruction for Secondary Students

Authors (year)	Sample, Setting and Duration	Research Design	Independent Variables	Dependent Measures	Instructional Content/ Instructional Focus	Results
Boulter & Kirby (1994)	N= 10; Grade 7-8; Duration not specified	Qualitative-task analysis	(NA-Study investigated student strategy use after regular instruction, not specified)	Researcher design test	Transformations[a], translation, rotation, reflection Conceptual Focus	Holistic or analytic strategy use varied with the task but overall analytic strategies correlated with higher test scores.
Choi-Koh (1999)	N=1; Grade 7; 21 hours	Qualitative-clinical interview	Dynamic Geometry Software	Researcher designed pre and post test	Transformations[a], similarity, symmetry, congruence of triangles and polygons Conceptual Focus	Improved level of geometric thinking (van Hiele) from1/2 to 4 (recognition/ analysis to deductive rigor).
Gorgorio (1998)	N= 24 ; Ages 12-16; Various schools (not specified)	Mixed Methods	(NA-study investigated study strategy use in paper-pencil activities, instructional methods and prior experiences not specified)	Interviews, task analysis of researcher designed items	Transformations[a], rotations of 2D representations of 3D figures Conceptual Focus	Visual and non-visual strategies used did not vary by age or gender; structuring of task influenced strategy use rather than preferred learning style.

Authors (year)	Sample, Setting and Duration	Research Design	Independent Variables	Dependent Measures	Instructional Content/ Instructional Focus	Results
Guven, 2012	N = 68; Grade 8; Age 14-15; 2 classes; 8 instructional hours	Quasi-experimental Pre-test/post test 2 classes randomly assigned per method	Dynamic Geometry software versus traditional instruction (paper-pencil worksheets)	assessments of transformations 1-Researcher designed multiple choice assessment 2-Researcher designed (based on other researchers) Open-ended assessment of van Heile levels of thinking	Transformations[b], translations, rotations, symmetry, composites Conceptual Focus	MC test=Students in experimental group improved significantly more than those in control group from pre to post test; Moderate ES vanHeile= experimental group improved significantly more than control group; Highly effective ES
Hollebrands (2003)	N= 6; Grade 10; 4 days weekly, 50 minutes per session 7 weeks	Qualitative - case study	Dynamic Geometry Software	Interview and tasks items (researcher designed)	Transformations[a], translations, reflections, rotations, and dilations Conceptual Focus	Students moved from rigid to dynamic views of figures
Hollebrands (2007)	N= 6; Grade 10; 4 days weekly, 3 50 minute sessions, 1 90 minute session 7 weeks	Qualitative- case study	Dynamic Geometry Software	Interview and tasks items	Transformations[a], translations, reflections, rotations, shears, similarity Conceptual Focus	Strategy use changed from reactive/random to proactive/ reflective and increased awareness of geometric properties.

Authors (year)	Sample, Setting and Duration	Research Design	Independent Variables	Dependent Measures	Instructional Content/ Instructional Focus	Results
Hungwe, Sorby, Drummer & Molzon (2007)	(Study 1) N= 37; Grade 8; 2-3 days per week , 54 minutes per class, 8 modules (Study 2) N= 40; Grade 9-10; duration NS	Quantitative one-group pretest/post test	a)Workbook, software, manipulatives b)Technology integrated in geometry course along with manipulatives and workbooks	Pre-post test (Study 1): PSVT-R (Study 2): PSVT-R; MCT; DAT	Transformations[a], rotation, reflection, symmetry; solid cross-sections; isometries Conceptual Focus	All students greatly improved from pre-test to post test however a gender gap in skills remained, favoring males for rotations.
Kirby & Boulter (1999)	N= 70; Grade 7/8, 2 classes; 40 minutes daily 9 sessions 2 weeks	Randomized stratified	a) Traditional direct instruction with computerized support b) Concrete manipulatives with computerized support	Pre-/post test Researcher designed based on state curriculum in geometry topics; pre-tests/post tests of spatial ability (researcher designed)	Transformations[a], rotations, flips, slides Conceptual Focus	a=b Both groups made comparable gains from pre to post test.

Authors (year)	Sample, Setting and Duration	Research Design	Independent Variables	Dependent Measures	Instructional Content/ Instructional Focus	Results
Rowell & Mansfield (2001)	N= 245; Grade 8, 8 intact classes 1 teacher; 7 lessons	Quasi-experimental Pre-test/post test 2 classes randomly assigned per method	a) deduction-student activity b) induction-student activity (c) deduction-teacher demonstration (d) induction-teacher demonstration	Researcher designed assessments of lesson objectives in two formats: paper-pencil or paper-pencil with manipulatives available to students; half of each class randomly assigned to type of test	Transformations[a], reflection, translation, rotation including on the coordinate plane Conceptual Focus	Students who were formal thinkers at pretest as opposed to those at the concrete operational level made greater gains (i.e. the gap widened);(c) and (d) made greater gains from pre to post test

NOTE: M = maintenance; G = generalization; [a]content was grade appropriate based on the Common Core State Standards; [b] all transformations from grade 6-8 used so only some topics were on grade level based on CCSS; NS= not specified; N= number of participants; PSVT-R=Purdue Visual Spatial Visualization Test-Revised; MCT= Mental Cutting Test; DAT=Differential Aptitude Test

Demographics

Of the nine studies, only Hollebrands' (2007) included race/ethnicity information for students in the sample. The author reported the majority of the students were Caucasian but did not provide specifics on the other races/ethnicities for the sample of 16 participants. However, it is of note that some studies did not take place in the United States, which may influence the availability of this information.

Four studies (Choi-Koh, 1999; Kirby & Boulter, 1999; Guven, 2012; Rowell & Mansfield, 2001) reported general information on where the study took place. Locations included both urban and rural areas. Choi-Koh's (1999) study took place in an urban city in the United States, Kirby and Boulter's (1999) study took place in a rural school in Canada, Guven's (2012) study took place in an urban school in Turkey, while Rowell and Mansfield's (2001) study took place in a large urban school in Southern Australia.

This lack of information is in contrast to the special education literature in which more than 50% of the studies included such information. A variety of demographic information such as race/ethnicity, location, and socioeconomic status would be helpful, particularly for teachers who may want to implement within their own classes and may have concerns about generalization to their student population.

Gender

Six studies (Choi-Koh, 1999; Gorgorio, 1998; Guven, 2012; Hungwe et al., 2007; Kirby & Boulter, 1999; Rowel & Mansfield, 2001) reported gender information for the students in the sample. Of the total 881 participants, gender information was available for 839 of which 447 (53%) of the participants were male and 392 (47%) were female. This is similar to the gender breakdown for the participants in the special education literature.

Age, Grade, and Setting

All studies reported one or more dimensions of age, grade, and setting. Two studies included age information (Gorgorio, 1998; Guven, 2012). However, for Gorgorio's (1998) study it was not clear if the age represented a grade level or if that was the chronological age of the students. Seven studies (Boulter & Kirby, 1994; Choi-Koh, 1999; Guven, 2012; Hollebrands, 2003, 2007; Kirby & Boulter, 1999; Rowell & Mansfield, 2001) included grade level information. Of those, two studies (Hollebrands, 2003, 2007) were with grade 10 students; two studies (Guven, 2012; Rowell & Mansfield, 2001) were with grade 8 students; two studies (Boulter & Kirby, 1994; Kirby & Boulter, 1999) included both grades 7 and 8; while Choi-Koh' (1999) study included grade 7 students.

Eight studies specifically included setting information. Setting was the entire class for six studies (Guven, 2012; Hollebrands, 2003, 2007; Hungwe et al., 2007; Kirby & Boulter, 1999; Rowell & Mansfield, 2001), while two studies were conducted with students individually (Boulter & Kirby, 1994; Gorgorio, 1998). Choi-Koh (1999) did not specify the setting, but based on other descriptions it appears to be individualized instruction. This is similar to the special education literature results in that most studies were with middle school students and included whole class instruction. However, in contrast to the special education literature only two studies provided age information.

Summary: Nature of Sample

In contrast to the expansive information provided within the special education research, the general education literature reviewed here lacks essential information. First, few studies included race/ethnicity information. Second, although the provided gender information indicates that the participants were roughly equal across genders, this information was provided in only six of the nine studies. Third, the age was provided in only two studies, which is a critical piece of data considering some students, particularly those with MD, may repeat grades. Fourth, few studies were with high school age students. This lack of critical characteristics of the participants limits generalization.

Instructional Content and Focus

As with the special education literature review, the general education literature can be described by the nature of the content and the type of knowledge that is the focus of the investigation or intervention of the study. For the general education studies, the geometry content reviewed was limited to topics that fit under transformations because transformational geometry is a unifying concept in the CCSS (2010) and the visuo-spatial skills necessary to succeed on transformational tasks such as mental rotations, dilations, reflections, are poorly developed in many students (Lean & Clements, 1981).

All of the studies were concerned with the conceptual knowledge. Researchers utilized a variety of strategies, materials, and methods to improve students' conceptual knowledge or attempt to understand how students' concept development. The results indicate that the interventions improved the conceptual understanding of students as evidenced by their improved scores on transformational tasks. The qualitative studies also provide information on the methods students employ when solving tasks and point to the type of tasks that teachers could use that promote conceptual understanding.

Instructional Activities

In contrast to the organization of the activities for special education literature, the general education studies rarely provided sufficient information to align with similar categories and used a variety of methodology to assess student understanding or to investigate the effects on an intervention. The activities are divided into the following groups: (a) problem tasks, (b) instructional method variation, (c) manipulatives, (d) dynamic geometry programs, and (e) geometry software, workbook, and manipulatives.

Two studies (Boulter & Kirby, 1994; Gorgorio, 1998) investigated the solution methods and problem solving of students as they worked through transformational geometry tasks. Boulter and Kirby (1994) provided students with various transformational tasks on paper and asked them to solve the problems (e.g., "flip the figure over the line and draw the new figure"). The authors analyzed the strategies and characterized students as holistic or analytic thinkers, then compared strategy use to performance. Results indicated that analytic strategy users performed better than holistic strategy users. Gorgorio (1998) also utilized problems presented on paper (e.g., "Construct, with the wooden cubes, the object presented in the figure, as it would remain after rotating it 180 degrees on its base.") to assess the understanding of students spatial skills, however students were also provided real (3D) objects along with the 2D (paper copies) of the objects so that students could use them while solving the tasks. Results indicated that students use visual or non-visual strategies to solve the problems and those skills can be taught so that students may improve their spatial skills.

Rowell and Mansfield (2001) however, investigated the methods employed by the teacher to instruct students on transformational tasks. Each class was taught using either deduction instruction followed by student activities, induction instruction followed by student activity, deduction instruction with teacher demonstration, or induction instruction followed by teacher demonstration. Students made greater gains on the post-assessment when the teacher provided demonstration rather than student investigation.

Kirby and Boulter (1999) compared traditional textbook, paper-pencil task instruction to manipulatives (e.g., tiles, geoboards, mirrors) instruction. Both groups also utilized an unspecified computer program for review. The results show that student performance improved similarly for both groups regardless of the type of instruction.

Four studies (Choi-Koh, 1999; Guven, 2012; Hollebrands, 2003, 2007) utilized dynamic geometry programs (DGS), specifically *Geometer's Sketchpad*, to investigate the nature of students' understandings of transformational geometry. Choi-Koh's (1999) case study results indicated that dynamic geometry software improved the student's geometry understanding from a concrete recognition level (i.e., van Hiele level 1/2) to a deductive level of thinking necessary for formal proofs (i.e., van Hiele level 3/4). Guven's (2012) quasi-experimental design study compared traditional (e.g., paper pencil) instruction to DGS and found that DGS not only improved students conceptual understanding of transformations (based on the van Heile levels of thinking) but also their procedural performance.

Hollebrands (2003) investigated the changing nature of student understanding while using DGS. Student thinking changed from rigid (i.e., concrete) to more flexible views of the figures. Similarly, Hollebrands (2007) investigated how students used the various aspects of the DGS along with the effect on student understanding of transformations. Students began with random and unorganized use of the DGS and became more reflective in their use of the DGS over time and their performance on transformational tasks increased. Students in all four studies improved their geometric similarity performance based on either quantitative measures or qualitative analysis. This improvement supports previous research on the benefits of dynamic geometry software for learners at the elementary level (Ng & Sinclair, 2015b; N. Sinclair & Moss, 2012), secondary level (Ng & Sinclair, 2015a; M. Sinclair, 2003, 2006) and pre-service teachers (M. Sinclair, Mamolo & Whiteley, 2011).

Lastly, Hungwe and colleagues (2007) utilized a computerized program (designed by the authors) along with a structured workbook and manipulatives (i.e., cubes). Student performance, as measured by a paper-pencil assessment of transformational geometry tasks such as rotations, isometries, and slicing improved. Teacher ratings indicated that the program and materials were easy to use and were helpful for students.

Summary of the General Education Literature

The current review of the literature on transformational geometry interventions for secondary students identified instructional approaches, practices, and materials that lead to improved performance and conceptual understanding of students. Promising instructional activities and materials include technology (e.g., DGS) and manipulatives along with elements of teacher-directed instruction such as demonstration/modeling. However, there were several limitations in the current literature that are discussed in the following sections along with suggestions for future research.

Limitations

Overall, the authors of the studies in this review included many beneficial instructional practices and procedures that led to improvement in student performance. However, there are several limitations to the current research.

1) Limited information was provided on the nature of the sample. Providing demographic information such as race/ethnicity, age, setting, gender, and socioeconomic status is essential for generalization, particularly if the aim is for practitioners to implement with their own classes.
2) Few studies included participants in high school, therefore, the effectiveness of the instructional practices in geometric transformations are uncertain for this population of students.
3) Many studies lacked sufficient description of the methods and procedures, making replication of the research difficult or impossible.
4) Most of the studies did not indicate the nature of the student population. Particularly, it is unclear if any of the students were struggling mathematics students or students with a diagnosed disability. It is especially important to include students with MD, particularly as the majority of students with MD are educated within general education.

Summary of All Geometry Literature

Although there are promising practices from both the general and special education literature, it is necessary to combine this in order to provide appropriate instruction that benefits *all* students, as advocated for by NCTM, NMAP, ADP, and CCSS. Promising practices from both bodies of literature include: technology (e.g., computer software, videos), direct-instruction or teacher modeling (supplementary, as needed, or as an integral part of delivery), CRA sequence (including manipulatives), and structured worksheets/cue cards. Additionally, several studies included a combination of the aforementioned practices, pointing to the benefits of an instructional package.

Limitations

Overall, the authors of studies in this review employed many beneficial practices that lead to improved student performance and/or researcher insight into students' thinking. However, there are limitations in the geometry research for students with MD.

1) There are no studies on transformational geometry specifying the inclusion of students with MD.
2) Few studies in transformational geometry include high school students.
3) Procedures or descriptions of the intervention or activities were insufficient for replication.

Future Research

Historically, there has been much more research on instruction and learning of students with disabilities in reading than in mathematics. Furthermore, special education mathematics research has focused on lower level skills, particularly computation, and the technology interventions for students with disabilities have predominantly been used for drill and practice (e.g., Hughes & Maccini, 1997; Maccini et al., 2007; Maccini & Hughes, 1997). Although the general education research focuses on conceptual understanding, an area of difficulty for students with MD (e.g., Geary, 2004), the inclusion of students with MD is not evident in the studies for the current review. Additionally, the NCTM and CCSS advocate for the inclusion of technology as a tool for increasing student conceptual understanding.

Gersten and Edyburn (2007) provided quality indicators for special education technology research. Their seven categories of essential information include conceptualization, disclosure, sample selection, description of participants, implementation, outcome measures, and data analysis. This includes: evidence of the importance of the research that builds on previous studies; sample selection that matches the target population; detailed participant descriptions; clear descriptions of the technology, procedures and fidelity of implementation; multiple measures of assessment that are reliable and valid for the appropriate analysis procedures and methods for adjustment. While the special education literature included many of these quality indicators, the general education literature did not. Future research should address this.

Rationale for Current Study

Competency in geometry is necessary for all students to succeed in school and transformations are an essential life skill and necessary for many careers. Furthermore, students who attend college must demonstrate knowledge of geometry, and transformations, including similarity, are an essential topic. Therefore, to prepare students with MD for post-secondary education and careers, instructional interventions for accessing transformation geometry are critical. Currently, no studies within the special education literature address transformational geometry for high school students with MD. To develop competency in transformational geometry, students must develop conceptual knowledge and procedural competency. Procedural skills necessary in transformation geometry focus on the computation necessary for dilations and comparing figures to decide if a figure is similar or congruent to another, in particular proportional reasoning and associated computations. Conceptual knowledge in transformational geometry includes the ability to see relationships between figures and actions on figures to transform them across various manipulations.

Conceptual knowledge and procedural fluency are equally important (CCSS, 2010) and without their interconnected growth, student's knowledge is hindered. Based on the van Hiele theory of geometry thinking and instruction, student understanding moves along a continuum from concrete to abstract ability (van Hiele, 1985). However, the instruction must be in line with optimal growth in student understanding (van Hiele, 1985), Vygotsky's zone *of proximal development* (Gurganus, 2007). Procedural fluency and conceptual understanding of transformations are essential for students to demonstrate competency on state assessments and college placement exams. Currently, no studies address transformational geometry instruction for students with MD.

The current study examined the effects of a contextualized instructional package on the performance of high school students with MD in transformational geometry. The van Hiele theory of geometry learning and instruction provided the theoretical foundation for the study (van Hiele, 1985; Moyer, 2005). The intervention included instructional components found to be effective in this review, including explicit instruction, CRA sequence, and technology (i.e., DGS). A combination of teacher-directed and student-directed instruction was implemented, emphasizing procedural and conceptual understanding of transformational geometry processes. Problem solving was addressed as students applied knowledge of transformations to more complex figures.

Chapter 3: Methodology

This chapter outlines the methodology used in the current study, which was developed in response to reform efforts and legislation that promote use of research-based instructional methods with students who struggle learning mathematics. This study focuses on higher-level geometry content (i.e., similarity transformations) integrating research-based instructional strategies with dynamic visual representations. Critical instructional supports for students with MD infused in the instructional package include: (a) components of explicit instruction, (b) graduated instructional sequence (i.e., CRA instruction), (c) cue cards, and (d) dynamic geometry software. This study addresses the geometry content aligned with the NCTM, ADP, CCSS, and State of Maryland framework standards and benchmarks for geometry for *all* learners (see Appendices E-H); as well as the NCTM process standards, CCSS practices, and the NRC proficiency strands (see Appendices I-K). The specific geometry content addressed includes: (a) procedural fluency addressing similarity transformations of polygons, and (b) conceptual understanding of similarity transformations of polygons (see Appendix L for unit objectives and Appendix M for lesson plans) and aimed to answer the following research questions:

1. Do secondary students with MD taught an instructional intervention on geometric similarity transformation increase the accuracy of their performance in geometric similarity transformations and maintain performance four to six weeks after the intervention?
2. What conceptions of geometry do students hold before and after the intervention?

3. What connections or disconnections in the geometry content emerge during the intervention and how can these results be used to improve instruction?
4. To what extent do secondary students with MD find dynamic geometry technology beneficial to representing and solving geometric similarity transformation tasks and in what ways does the intervention enhance metacognition, self-efficacy, and attitudes toward geometry?

In order to answer the above questions, the study utilized a single-subject design, which has a lengthy history in special education research (Cakiroglu, 2012; Tawney & Gast, 1984). Single-subject research is useful when establishing evidence-based practices for: (a) students with disabilities who represent a small percentage of the student population, (b) measuring individual participant performance, and (c) addressing ethical concerns (Kazdin, 2011). Single-subject research is experimental and aims to determine if there is a functional relation between an independent variable (i.e., an intervention) and dependent variable(s) (Guralnick, 1978; Horner et al., 2005; Riley-Tillman & Burns, 2009).

See Figure 1 for the conceptual framework for the study, which examined the effects of an instructional package including cue cards, explicit instruction, manipulatives, and dynamic geometry software on the procedural fluency with and conceptual understanding of similarity transformations of polygons. The conceptual framework indicates how the characteristics of students with MD led to the choice of the specific components of the intervention to improve students' geometry performance, while the more general logic model (see Figure 2) includes the broader policies, context, and factors that influence mathematical performance.

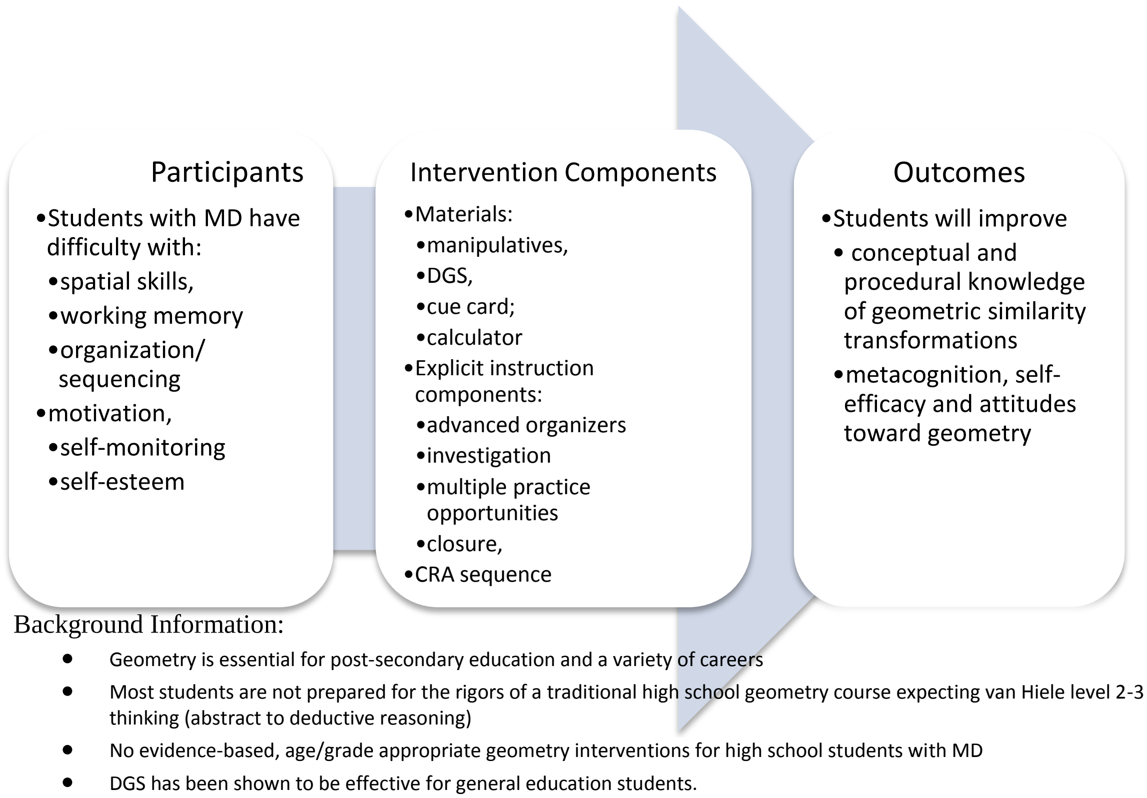


Figure 1. Conceptual model for the instructional package

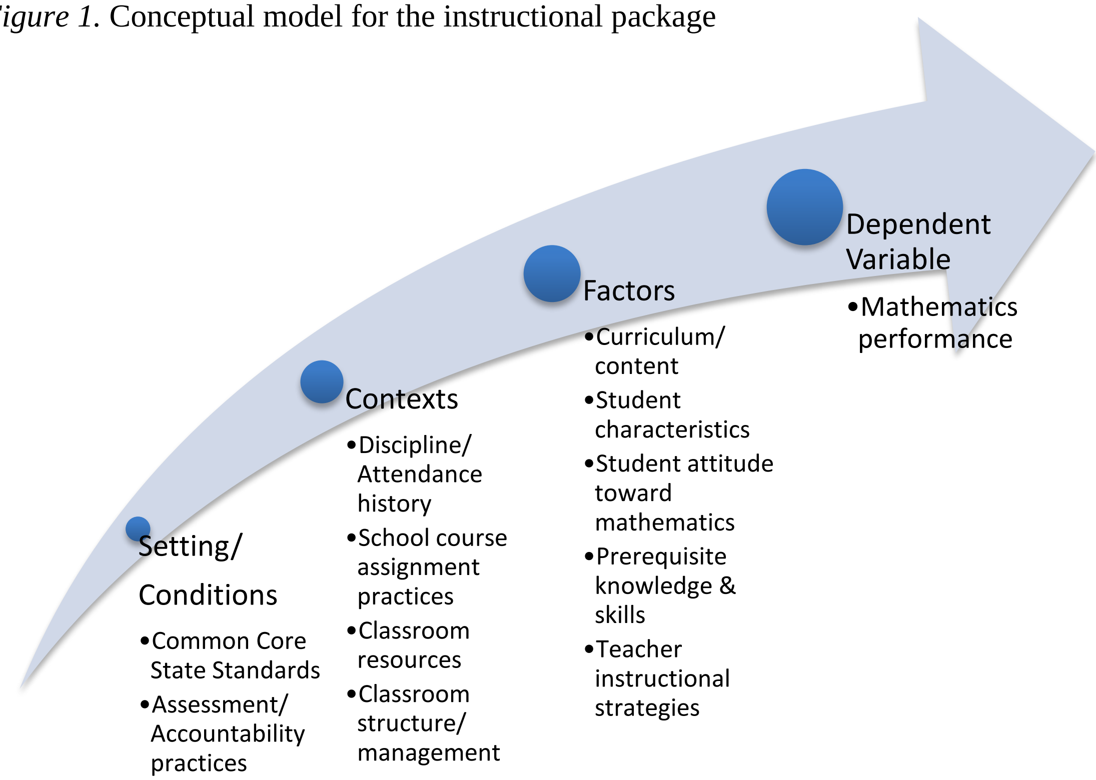


Figure 2. Logic model for mathematics performance (based on Horner & Odom, 2014)

This chapter provides a description of the (a) experimental design procedures; (b) curricular development; (c) independent variable development, procedures and materials; (d) dependent measures development and procedures; and (e) participants and setting.

Single-Subject Design

The research utilized a single subject design. Single subject research is an experimental design that documents a functional relation between the intervention and behavior change (Guralnick, 1978; Horner et al., 2005; Kratochwill et al., 2010; Riley-Tillman & Burns, 2009; Tawney & Gast, 1984) by comparing patterns of performance across stages of the study (i.e., baseline and intervention). There are many types of single-subject designs (e.g., withdrawal/reversal., time lagged, comparison); however, all designs include the following: (a) continuous assessment; (b) replication of intervention effects over multiple behaviors, settings or participants; and (c) evaluation of data via visual analysis (Hammond & Gast, 2010; Tawney & Gast, 1984).

Single subject research has a long history in special education research, particularly in behavior analysis (Hammond & Gast, 2010; O'Neill, McDonnell, Billingsley & Jensen, 2011) and includes three major features. First, the individual is the unit of analysis (Tawney & Gast, 1984, Horner et al., 2005). Second, characteristics under investigation, such as participants, independent and dependent variables, must be operationally defined (Cakiroglu, 2012; Horner et al. 2005; Tawney & Gast, 1984). Third, data are collected, compared, and analyzed over time (Cakiroglu, 2012; Horner et al. 2005; Tawney & Gast, 1984). It is essential to define, measure, and document all aspects of the intervention to determine effectiveness.

I chose to use a single-subject design for two core reasons. First, single-subject research has an extensive history of establishing evidence-based practices for students with disabilities, a critical federal mandate (Cakiroglu, 2012; Horner et al., 2005). Second, I am interested in the effect of the intervention on individual students and single-subject research allows for detailed analysis of responders as well as nonresponders to the intervention (Horner et al. 2005; Kratochwill et al., 2010). Furthermore, with the expectation that all students will be proficient in mathematics skills (e.g. NCLB and state graduation requirements), I feel it is necessary to address instruction for the students who have a history of difficulty in mathematics and who may need methods and materials that are not readily available for whole class or small group instruction at my current school. The following section describes the (a) experimental design procedures, (b) independent variable, (c) dependent variable and measurement procedures, and (d) data analysis.

Experimental Design and Study Procedures

The study incorporated a multiple probe design across four participants to establish if the intervention was effective for the target population. The minimum recommended number of replications has traditionally been three (Kazdin, 2011). However, if the replications do not clearly show that the participants improve on the target behavior then the data are problematic. Therefore, more replications could provide evidence of an effective intervention if, for example, three participants' data were trending similarly, such as having a stable baseline, increasing trend and low variability, but one additional participant's data patterns do not show the same (i.e., stable baseline but slower acceleration in trend or high variability in probe scores; Kazdin, 2011).

Furthermore, by including four participants (i.e., replications) this should reduce several threats to internal validity such as history (i.e., events outside of the study that may affect participants, particularly if a study lasts an extended period of time), maturation (i.e., naturally occurring physical, intellectual or emotional changes as participants increase in age), and attrition (i.e., participants dropping out of the study; Gay, Mills, & Airasian, 2006; Richards, Taylor, & Ramasamy, 2014).

Additionally, I chose the multiple probe design, rather than a multiple baseline design because prolonged baseline measures may be unnecessary or impractical (Kazdin, 2011; Tawney & Gast, 1984). The impracticality of prolonged measures is my primary reason for choosing a multiple probe design. For example, the time available during the school day to administer the baseline probes must be during the student's resource classes so that students are not missing critical instruction in their regular academic courses. In addition, utilizing the resource class for extended periods would take time away from their study skills instruction and potential time for receiving their extra time testing accommodations, again making prolonged baselines impractical. Furthermore, participants may improve their performance based on prolonged exposure to the probes and repeated practice with the assessed material (Kazdin, 2011; Tawney & Gast, 1984), making the use of parallel versions of the probes particularly important, as was limiting the number of probes given to the students.

The study design consisted of three phases: (a) baseline phase - students were involved in their regular coursework with no instruction on the unit objectives and are assessed on the domain objectives periodically, (b) intervention phase – students were provided individualized instruction on the unit objectives; and (c) post-intervention phase with post-intervention and maintenance data collection (Kazdin, 2011; Tawney & Gast, 1984). This design involved the systematic and sequential introduction of the independent variable (i.e., the unit objective instruction) to one participant at a time, staggered across participants until all four participants completed the baseline data domain probes, instructional intervention, post-instruction domain probes, and maintenance probes.

Baseline measures were collected intermittently prior to the intervention phase and included at least four domain probes. Before beginning the intervention, the baseline domain probes were expected to show that the first student's performance is stable by the following means: (a) scores are consistently below 60% (i.e. non-mastery as defined by the local school district), (b) trend of the data are flat or not increasing (i.e., the participant did not gain knowledge of the targeted objectives), and (c) there was little to no variability in the probe scores (i.e. domain probe scores were expected to be 10% or less different than the mean probe scores for the baseline phase). Internal validity was demonstrated when the participant's performance changed significantly upon the introduction of the intervention (e.g., domain probe performance increased from 10% at baseline to 40% at intervention), with improved performance replicated over multiple participants (Kazdin, 2011; Tawney & Gast, 1984). See Figure 5 for the graph of the design as implemented.

Baseline phase. In order to show the effect of the intervention, baseline data must show stability prior to the introduction of the intervention. According to O'Neill et al. (2011), stability includes level (i.e., mean of the data points), trend (i.e., direction and slope of the line of best fit through the data points), and variability (i.e., how much the data points vary from the mean) but also must include a sufficient number of baseline data points (i.e., probes) in order to be certain that the level, trend, and variability are stable. Specifically, the *What Works Clearinghouse* design standards require three to four data points per phase (baseline or intervention) to meet standards with reservations, but at least five data points per phase to meet standards (Kratochwill, et al., 2010; 2013). However, Vannest, Davis, and Parker (2011) stated that five points might be sufficient if there is 10% or less variability, but nine or more baseline probes would be needed if there were variability.

A minimum of four probes were administered during the baseline phase for each participant, and checked for level, trend, and variability. There were at least three consecutive probes during baseline just prior to beginning the intervention to ensure the level was below 60%, trend was flat or not increasing, and the data had low variability. All participants completed the initial domain probe at the same point, while the first participant continued baseline domain probes. The intervention was introduced once the performance of the first participant was stable (i.e., trend not increasing or showing improvement in the target skill, the data exhibits little variability, and mean performance level below 60%). Participant 2 took continuous baseline domain probes (at least three) with the final baseline probe coinciding with Participant 1 beginning the post-intervention probes (Richards et al., 2014). This pattern continued for participants 3 and 4, making sure a baseline probe coincided with the prior participant beginning post-intervention probes, until all participants started the intervention (Richards et al., 2014). Specific information on domain probe development and procedures are in a subsequent section.

Intervention phase. This section explains the procedures and criteria for implementing the intervention and collecting data. During the intervention phase, participants were removed from the resource class and taken to the conference room or unused classroom (e.g., students having a resource class during one period utilized the conference room while students having a resource class during the other period utilized the same available unused classroom) to receive instruction on target objectives as listed in the scope and sequence approximately four times per week. I provided all instruction during the intervention phase.

Each intervention lesson concluded with a brief assessment of student learning of the objective (e.g. exit ticket), which was reviewed with the student for clarification of any errors or misunderstanding prior to moving on to the next lesson. Domain probes were not administered at the end of each lesson, or intermittently during the intervention, due to the sequential academic nature of the intervention. This was for two reasons: (a) the domain probes included questions representative of the entire unit and it would be obvious that once a subtopic was introduced the score on domain probes should increase, and (b) the danger of testing fatigue of the participants. At the conclusion of the instructional unit, there was a final review (session 10) prior to the administration of a post-intervention domain probe, similar to how a usual class unit would proceed-daily lessons, review, and unit test. Therefore, while it was desired that a student would improve over baseline, a specific numerical criterion was not set. While some researchers utilizing CBAs (e.g. Hudson & Miller, 2006), recommend a criterion of 80%, I decided that 80% was too high due to 60% being the school district standard for passing, and the rigorous academic nature of the intervention, including the student's level of anxiety and extent of the participants' difficulties learning mathematics.

Finally, the introduction of the intervention was staggered across participants. Participant 1 received the intervention after stability was demonstrated during baseline probes. Participants 2 through 4 were administered a baseline probe when Participant 1 began the post-intervention probes. Participant 1 continued the intervention until completion of the 10 intervention sessions and five post-intervention probes. Once Participant 1 showed a change in progress based on the post-intervention domain probes, participant 2 began the consecutive baseline probes leading up to beginning the intervention, once the series of baseline probes showed stability (Richards et al., 2014). Participant 2 began consecutive baseline probes with one aligning with participant 1 taking post-intervention probe, as well as participants 3 and 4 taking a baseline probe. When Participant 2 began the post-intervention probes, Participants 3 and 4 took another baseline probe. This sequence continued until all participants complete the intervention.

Post-intervention phase. After the conclusion of the intervention for each participant a minimum of three data points are required to meet the *What Works Clearinghouse* standards with reservations, but five data points to meet standards (Kratochwill, et al., 2010, 2013). Participants were administered five domain probes directly after the conclusion of the intervention. This phase was visually analyzed for a change in level (over baseline), overlap with baseline, trend, variability, and consistency between and within phases.

Curriculum Development

I developed the independent variable materials and dependent measures using feedback and input from mathematics educators and special educators who have expertise in working with high school students with MD in my school. The unit objectives, lesson planning, instructional materials, procedures and measures were revised multiple times based on critical comments and suggestions. The following sections include descriptions of the instructional unit, materials, and the development of my dependent measures.

Independent Variable

In this study, the independent variable combined the content, practice, and process standards promoted by members of the mathematics education community (CCSS, 2010; NCTM, 2000) and critical instructional practices identified by special education research within an instructional unit on geometric similarity transformations. Specifically, the independent variable combined the use of explicit instruction and multiple visual representations (e.g., manipulatives and dynamic geometry software) to develop conceptual knowledge of and procedural fluency with similarity transformations. The following sections describe the elements of the instructional package including the unit content, instructional procedures and materials, as well as fidelity of treatment methods.

Instructional Unit

The investigator-developed instructional unit included lesson plans that address age- and grade-level appropriate geometry content consistent with the CCSS (2010), NCTM (2000), the State Curriculum (2011), and the ADP Benchmarks (2004). The goal of the instructional unit was to promote the conceptual understanding of similarity transformations, which support and enhance the procedural fluency with the algebraic concepts embedded in the geometric relationships. Furthermore, the NCTM Process Standards and NRC Strands of Mathematical Proficiency (2000), which describe ways students acquire and apply content knowledge, are embedded to promote values held by the mathematics education community. The standards and proficiencies are also found within the CCSS for mathematical practice (CCSS, 2010) and include such skills as reasoning, making connections, communication, and representations. Additionally, the lesson plans incorporate instructional practices and materials that have demonstrated positive effects for the acquisition of various mathematics processes for students with MD within the special education research base. These instructional practices include components of explicit instruction (e.g., investigations, multiple practice opportunities) while the instructional materials used simultaneously include visual representations (i.e., concrete and virtual manipulatives, sketches, cue cards). The activities and specific problems were created based on a variety of references including school district documents and textbooks, consultations with colleagues, the reviewed literature, and online sources such as NCTM Illuminations (http://illuminations.nctm.org), Annenberg Learner (http://www.learner.org) and William McCallum's Illustrative Mathematics (https://www.illustrativemathematics.org/).

The unit began with an exploration of similarity of both concrete and representations of solids and polygons as students developed an initial definition of similarity. Through the subsequent lessons, students explored and refined their definition to discover the importance of ratios and proportions for similar figures. Over the next several lessons, the focus shifted to triangle similarity theorems (e.g., Angle-Angle). Once established, students explored the theorems' usefulness in contextualized problems.

The unit culminated with applications to measurement in 2- and 3-dimensions, which wrapped back to the explorations from the initial lessons with solids. Furthermore, the unit objectives addressed in this study were not presented within the participants regular courses during this study to avoid the multiple treatment interference validity threat (Gast, 2010). At the end of this section, Table 4 provides a summary of the lesson objectives with the core instructional components (e.g. concrete, DGS, representations, abstract), which are described in more detail in the following sections, while the lesson plans in Appendix M provide more detailed information on the integration of the components.

Instructional procedures

Critical instructional practices identified by special education researchers (e.g., Archer & Hughes, 2011; Hudson & Miller, 2006) and the content and process standards promoted by the mathematics education community (CCSS, 2010; NCTM, 2000) were used in the instructional intervention. Furthermore, as suggested by the NMAP (2008) as well as the CCSS and NCTM, lessons include real-world situations in addition to abstract applications. Lesson plans included each of the following components to ensure systematic implementation:

a) Advance organizers, which include review of pre-requisite skills, lesson objectives, link to the current lesson, and the rationale for the current lesson topic;
b) Investigation, which includes the investigator demonstrating or facilitating the completion of a new task using critical instructional practices and materials from special education, mathematics education research, and the CCSS mathematical practice standards;

c) Multiple practice opportunities with scaffolding, including opportunities for guided and independent task completion;

d) Closure, which includes a review of the lesson and objective assessment via questioning and discussion with the participant or an exit ticket.

Throughout the instructional sessions, I acted as facilitator to guide participants to explore concepts and skills through discussion of the tasks and investigations. For example, when participants completed a problem, such as solving for the unknown side of a triangle similar to a given triangle, they had to verify that their solution was correct and explain how they arrived at their conclusion. Thinking processes were emphasized by asking students to justify their responses using questions such as, “Explain how you got your solution,” or “How do you know that your solution is correct?” These prompts were provided regardless of whether the students’ responses were correct. Teacher modeling and think-alouds were provided if students were unable to provide reasonable responses through the questioning.

Integrated instruction. Procedural fluency as well as conceptual understanding were targeted via the combined instructional format, as many math skills are interrelated with knowledge in one area leading to improvement in others (Rittle-Johnson & Alibali, 1999). Furthermore, the main instructional components consisted of elements of explicit instruction, which include (a) advance organizer, (b) investigations, (c) multiple practice opportunities, and (d) explicit task sequencing (Archer & Hughes; 2011; Hudson & Miller, 2006). Each component is important for setting the direction of the lesson and enhancing student performance.

First, an advance organizer was provided to review prior relevant skills, the current lesson objective, and purpose for learning the targeted skill. For instance, dilations are typically taught at the end of unit on transformations, after students have been introduced to isometries (i.e., rotations, reflections, translations). A lesson may have begun with a review of these concepts, followed by a discussion leading into the lesson objective involving how a figure can change in another way by showing a picture that has been through each of the transformations and how the dilation picture shown is different. Finally, the discussion of examples included other situations when using this new type of transformation is important (i.e., expanding or shrinking document copies, blueprints).

Second, the investigations included either (a) teacher modeling and think-alouds for procedures along with maximizing student engagement using questions and prompts, or (b) student discovery of concepts using variety of explorations utilizing the dynamic geometry software depending on the nature of the lesson objective. For instance, students engaged in a series of explorations using Geometer's Sketchpad to discover that dilations change the size of a figure, but not the general shape (e.g., rectangle, regular pentagon) or the angle measures and there is a relationship between the length of sides of the pre-image and the image (i.e., scale factor). Third, multiple practice opportunities were embedded in each lesson and included one or more of the following: hands-on activities using manipulatives, dynamic geometry software, and/or real-world problems completed with or without assistance. For instance, while students may have explored dilations using the DGS, the practice problems included varied examples using different shapes (e.g. triangles, rectangles, pentagons,), in order to lead the student to generalize to the definition of a dilation transformation and scale factor. Throughout the lessons, student understanding was monitored via prompts, questioning, or re-teaching, as needed.

Finally, explicit task sequencing was included across the unit topics and within the lessons. Explicit task sequencing refers to dividing concepts into manageable and concise subtopics in a predetermined order (Scheuermann et al., 2009). For instance, rigid transformations (in a coordinate plane) are typically taught prior to dilations, and congruence is taught prior to similarity because isometries (i.e. rigid transformations) result in the “same shape, same size” notion of congruence, which is often easier for students to understand, while dilations result in the “same shape, different size” notion of similarity which is a more complex concept and students have more difficulty understanding. Therefore, for this study the first lesson began with the advance organizer relating the previously learned concept of congruence/isometries (addressed in middle school), followed by the instructional and practice phases linking this to the notion of similarity, at first used in contexts outside of mathematics (e.g. similar means something different in an English class such as comparing characters, or in science/art with shades of color). The sequencing of subsequent lessons on similarity of polygons led to a focus on triangle similarity, including each of the specific triangle similarity theorems (i.e. Angle-Angle, Side-Side-Side, Side-Angle-Side). Finally, study of similarity in the 2D figures extended to 3D solids and measurement (i.e., perimeter, area, surface area, volume).

Furthermore, as part of the integrated approach, the NCTM process standards and CCSS standards for mathematical practice were embedded throughout the instructional lessons. This was done to provide opportunities for students to do one or more of the following: (a) persevere in problem solving; (b) demonstrate reasoning by formulating conjectures and justifying reasoning; (c) communicate; (d) make connections to previously learned mathematics and/or to other mathematical, scientific or real-world situations; (e) model and represent mathematically; and (f) use tools strategically. Many of the processes and practices were addressed simultaneously. For instance, to demonstrate and justify reasoning as well as communication, students used the dynamic geometry software to complete a task aimed at discovering similarity relationships, and were required to explain in writing and/or orally the choices they made and what led them to their conclusion. Furthermore, strategic tool use was encouraged by allowing students to choose any material (e.g., cue cards, calculator, manipulatives) to complete the practice problems, exit tickets or domain probes; however, they were asked to explain why their choice was the most appropriate.

Multiple visual representations. In the study, multiple visual representations included use of concrete manipulatives, sketches of figures and manipulatives, and dynamic geometry software (i.e., *Geometer's Sketchpad*). Multiple visual representations of mathematics concepts are recommended by CCSS (2010) and NCTM (2000) as well as special education research (e.g., Jitendra et al., 2009). Furthermore, research in special education has identified the concrete-representational-abstract graduated instructional sequence as an effective strategy for algebra (Maccini & Hughes, 2000; Maccini & Ruhl, 2000) and geometry (Cass et al., 2003; Satsangi & Bouck, 2015).

For instance, multiple visuals in the lessons included (a) concrete objects or materials (e.g., AngLegs that create a triangle), (b) a drawing of a figure and the manipulation of a paper version created by the students (i.e., concrete or semi-concrete) or (c) the manipulation of the dynamic visuals using the computer software to dilate (i.e., expand or shrink) the figures in order for the student to discover a rule or generalization.

It is important to note that the CRA sequence was not used in a continuum with beginning lessons using only concrete materials, intermediary lessons using representations, and concluding lessons with an abstract focus, but within each lesson some or all of the CRA sequence was used. For example, lesson 1 exploring a preliminary definition of similarity primarily utilized concrete materials, lessons 4-7 exploring the Triangle Similarity Theorems utilized the AngLegs as concrete materials, DGS and sketches or pre-printed images on paper as representations, while lesson 9 exploring multi-dimensional measurement utilized the DGS and images on paper as representations and the formulas at the abstract level. This modification to simultaneously introduce the similarity concepts through concrete or virtual manipulatives, static representations (e.g., images pre-printed on paper or sketched by participants), and symbolic notation (e.g. formulas for multidimensional measurement) were in response to recommendations in a report from the Institute of the Education of the Sciences on instructional practices for improving students learning (Pashler et al. 2007).

Instructional Materials

The intervention included an investigator-developed instructional unit that incorporates materials and instructional supports to help students with MD access the general education mathematics curriculum including calculators, cue cards, manipulatives and computerized dynamic geometry software.

Calculators. Participants were provided access to the use of graphing calculators throughout the unit for three reasons. First, participants' IEP accommodations included the use of a calculator for instruction and assessment. Second, lesson objectives do not address computational skills but rather conceptual understanding of geometric similarity transformations. As many students with MD have difficulty with computational skills (Geary, 2004), it is important to allow participants to strategically choose to perform computations using this tool, if desired, in order to focus on the conceptual understanding of the mathematical topic (Bouck, Joshi, & Johnson, 2013; Steele, 2010). Lastly, as part of their regular mathematics instruction students in the school were allowed access to calculators at anytime, primarily due to the focus on conceptual understanding rather than rote computation.

Cue card. A cue card with the major topics, such as the triangle similarity theorems, was provided to students for use during the unit lessons and post-intervention probes. Cue cards are often used for procedural strategies (Hudson & Miller, 2006), but can be beneficial for recall of information, which is an area of difficulty for students with MD (Geary, 2004). Cue cards have been found to be beneficial for students with disabilities to analyze and solve problems (Butler, Miller, Crehan, Babbitt, & Pierce, 2003; Maccini & Gagnon, 2000), including geometry (Mulcahy & Krezmien, 2009). Additionally, as part of their regular mathematics instruction students in the school were provided with a formula sheet that they were allowed to use during instruction and assessments. Examples of the information on the formula sheet include perimeter, area, volume of polygons and polyhedrons, the Pythagorean Theorem, and Trigonometric ratios. Triangle congruence and similarity information was not provided on the sheet, however some teachers that I have worked with in the past have allowed the non-honors level students to use notes with this information and it helped some students. Therefore, the cue card included the major ‘look-fors’ (e.g., the two core criteria to check that figures are similar), similarity theorems with associated images (e.g., Angle-Angle similarity theorem, Triangle proportionality Theorem) and formulas (e.g., scale factor related to area and volume). The cue card was modified so that only the concepts already taught were available on the card for the current lesson, as noted in the lesson plans. The complete cue card is Appendix N.

Dynamic geometry software (DGS). There are varieties of technological choices for instruction, as evidenced by the literature review. I could have opted to design my own software, make demonstration videos, or utilize commercially created programs or free online activities such as Java applets. Although the literature review focused on transformations and technology with participants in grades 7-12 only, there have been many studies focused on a range of geometry topics with elementary through post-secondary participants that have shown the benefits of not only DGS but Logo or researcher created programs (e.g., Edwards, 1991; N. Sinclair & Moss, 2012; M. Sinclair, et al., 2011; Hungwe et al, 2007). I chose to use DGS for my intervention.

DGS represents geometry explorations performed with interactive computer software (Jiang, White, & Rosenwasser, 2011). It is available in free forms via the internet, such as the National Library of Virtual Manipulatives (NLVM) and GeoGebra, as well as commercially purchased formats such as *Geometer’s Sketchpad* (Jackiw, 2009). DGS aids students in constructing mathematical concepts via explorations and investigations that are active, such as dragging, measuring, observing, conjecturing, conjecture testing, reasoning, and proving, rather than static representations (Jiang et al. 2011).

I chose the program *Geometer's Sketchpad* to aid in the development of student conceptual understanding of similarity transformations for several reasons. First, the program has activities that are available free on the internet for individual exploration of specific concepts, such as dilations and discovery of the similarity theorems. Second, the school computers include the *Geometer's Sketchpad* software, making it readily available for use during the study, as well as being available for student use after the conclusion of the study, if desired. Third, *Geometer's Sketchpad* provides users the ability to create their own figures for manipulation. Fourth, I have found that the program is user friendly and does not require as much instruction on how to use as more training intensive programs (e.g., Cabri, TI-Inspire). This is critical for students who may have reading difficulties that would make using the help menu or instructions arduous (Garnett, 1998) and/or memory or processing deficits that impede their ability to retain and retrieve information (Swanson & Beebe-Frankenberger, 2004). Lastly, utilizing several different programs could be confusing to the students, due to such as difficulties with processing or working memory (Swanson & Beebe-Frankenberger), organization and sequencing (Steele, 2010) or self-monitoring (Gagnon & Maccini, 2001).

In this intervention the DGS software was used in concert with the concrete manipulatives through the use of integrated instruction in the lessons. Two sample activities are included to clarify the general instructions included in the lesson plans provided in the appendix. For example, in Lesson 5 students explore Side-Side-Side, Angle-Angle-Side, Angle-Side-Angle and Side-Side-Angle are viable theorems after discovering (in lessons 1-3) that for figures to be similar all pairs of corresponding angles must be congruent while all pairs of corresponding sides must be proportional, while in lesson 4 they had discovered that the AA ‘short-cut’ allows one to conclude that two triangles are similar if two angles are congruent. In addition to viewing figures separately as shown in these two examples, the figures can be manipulated so that they overlap. This feature is useful in the initial discovery that corresponding angle are congruent (e.g. lessons 2, 3 and 4) and in later lessons when figures may be embedded, such as with the Triangle Midsegment or Proportionality Theorems. As shown in Figure 3 students can manipulate the sides in the figure on the right by moving point E so that eventually they get a triangle (as shown) that is not similar to the triangle on the left. (The initial figures show a completed triangle on the left but an open figure on the right). Then students would have the program measure BC and EF, calculate the ratio and find that it is NOT 1.63 (nor is $\angle A \cong \angle D$ or $\angle B \cong \angle E$) so the triangles are not similar and therefore SSA is not a viable short-cut (Triangle Similarity Theorem). They repeat a similar set-up for each theorem (AA, SSS and SSA that work but no other combinations do or the other theorems provide sufficient information) either utilizing preformed figures or creating their own, depending on the lesson objectives.

In lesson 9 students are conjecturing about the multidimensional measurement relationships. Specifically students explore the relationship between scale factor, a linear measurement, and area, perimeter and volume. While some DGS programs can be set up to visualize in 3-dimensions, this program was not for this lesson. Students created and explored various figures (triangles, quadrilaterals or irregular polygons) to discover that if the scale factor (linear/perimeter) is a/b then the relationship is a^2/b^2 for area.

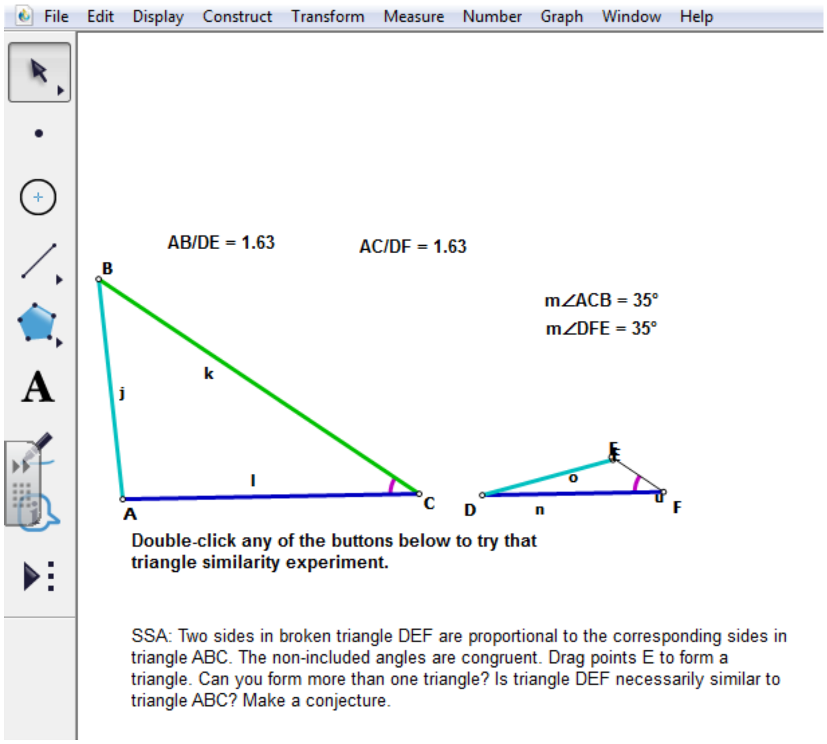


Figure 3: DGS SSA Similarity Theorem example

An example of a figure that students may have created is shown in Figure 4. Students then were asked to conjecture what the relationship would be for a 3D figure. If students were not able to surmise that the formula for volume would be a^3/b^3 then further scaffolding and exploration with another program or concrete objects would be available options

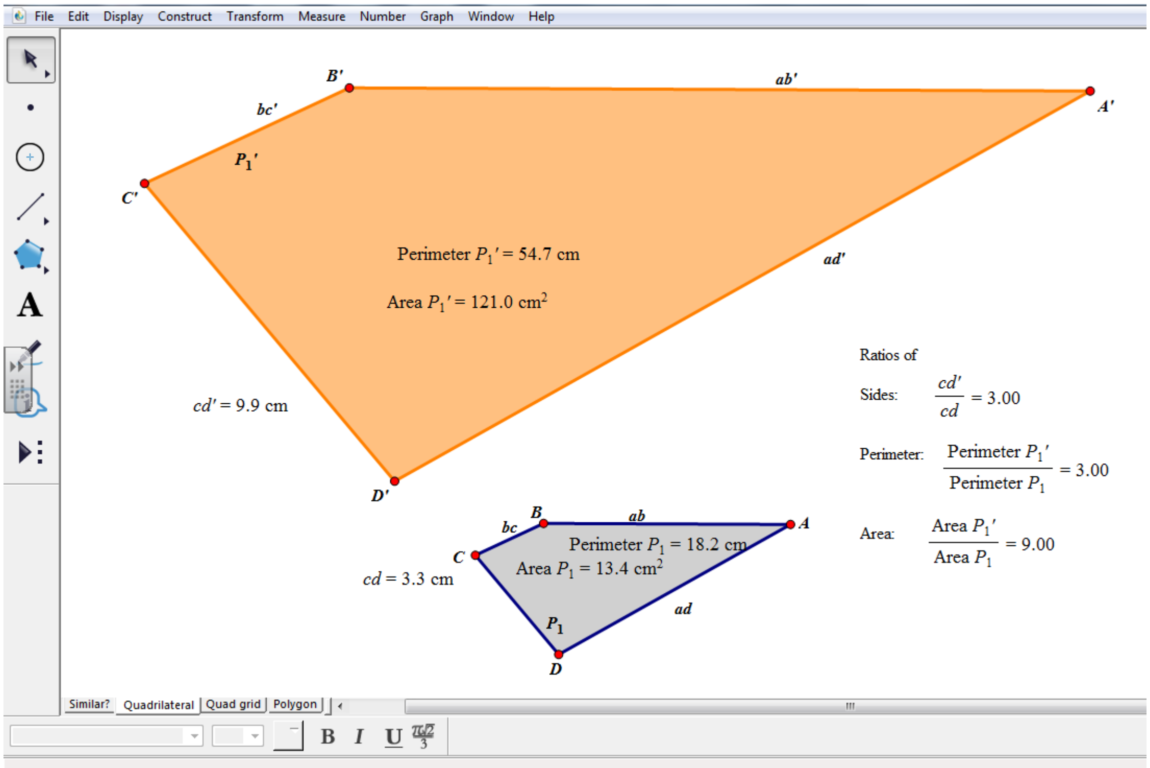


Figure 4: DGS multidimensional measurement example

Manipulative materials. Concrete manipulatives and measurement tools used in this intervention include everyday items and commercially purchased materials. Commercially purchased geometric 2D shapes and 3D solids, protractors, rulers, and AngLegs (i.e., plastic “popsicle” sticks of varying length for making angles, triangles, and other two-dimensional polygons) were used to explore properties of shapes and discover relationships (e.g., which similarity theorems work as shortcuts and which potential shortcuts do not create similar figures). Various everyday items were also used as examples of specific shapes (e.g., tennis ball as a sphere, cereal box as a prism, oatmeal container as a cylinder). Additionally, instructor and students created representations using paper and the aforementioned tools.

Table 4
Lesson objectives and core components

Lesson	Objective	Core components
1	Defining similarity	Concrete Representations
2	Defining similarity of triangles	Concrete DGS Representations
3	Defining similarity of polygons	DGS Representations (modified cue card)
4	Triangle similarity theorem-AA	Concrete DGS Representations
5	Triangle similarity theorem- SSS, not AAS/ASA/SSA	Concrete DGS Representations
6	Triangle similarity theorem-SAS	Concrete DGS Representations
7	Triangle similarity theorems- midsegment and proportionality	Concrete DGS Representations
8	Indirect measurement	DGS Representations
9	Linear, 2D, 3D measurement	DGS Representations Abstract
10	Review	Representations Abstract

Fidelity of treatment

Fidelity of treatment refers to the extent the intervention was consistently implemented during the study (O’Neill et al., 2011). According to O’Neill and colleagues, treatment fidelity should be conducted on at least 25% of intervention sessions, with a score of at least 90% for each observation. Fidelity of treatment was calculated by dividing the number of components present by the total number of components and multiplying by 100 (O’Neill et al., 2011). Two independent observers (i.e., trained graduate students) conducted fidelity of treatment observations. A checklist of the essential components of the intervention (see Appendix Q) was used to train the independent observers and was also used for the treatment fidelity checks. Training consisted of an explanation and review of the lesson plans, fidelity checklist, and accompanying video-recorded instructional sessions (videos were used for the fidelity of treatment checks to ensure inclusion of essential lesson elements).

The training criteria included the expectation that the rater must be able to identify the components of the intervention and agree with my ratings with at least 90% agreement on at least three consecutive viewings of the video-recorded lessons. Observers completed fidelity data on 30% of the sessions (approximately 3 of the 10 intervention sessions per participant for a total of 12 sessions) using the video recordings and the treatment fidelity checklist. The video recordings also reduced the possibility of participant reactivity, as participants were not intermittently observed by an unfamiliar person (Richards et al., 2014). Fidelity of treatment was calculated by dividing the number of components present by the total number of components and multiplying by 100 (O'Neill). Based on the independent observations I implemented the intervention as intended 100% of the time.

Inter-observer agreement

In addition to ensuring that the treatment was carried out as intended, it was important to have verification of this by independent person(s) not involved in the study. Therefore, comparisons were made between the fidelity of treatment checklists from the two observers of the videos. Inter-observer agreement was calculated by (a) summing the total number of agreements; (b) summing the total number of agreements and disagreements; (c) dividing (a) by (b); then multiplying by 100 (O'Neill, 2011). Inter-observer agreement was 100% (although this may have not been necessary, as the fidelity of treatment was 100% across observers).

Dependent Variable

Change in the dependent variable must be measured in a reliable and observable manner to show a functional relation between the independent and dependent variables (Richards et al., 2014). Dependent measures included investigator-developed domain probes specifically designed to align with the target objective topics for assessment of the dependent variable- student understanding of geometric similarity transformations (specifically growth of knowledge). Domain probes were used to (a) establish baseline performance prior to the intervention, (b) determine performance immediately after the intervention, and (c) determine maintenance of performance four to six weeks after the conclusion of the intervention. Furthermore, pre-test and post-test assessments based on a standardized mathematics test (e.g. NAEP) were also utilized, with specific questions that focused on similarity skills and applications. The purpose of this assessment was to compare the results of the researcher-created curriculum-based probes with a standardized assessment.

Development of Measures

Domain probes focused specifically on the content of the unit were developed as well as a standardized measure based on the NAEP. The domain probes were similar in nature to how a unit test would be designed with approximately 10 open-ended questions with one per major unit idea, such as scale factor and missing portions of a diagram, figuring out if triangles are similar or not and justifying reasoning, contextualized problems involving indirect measurement or applications to 2D and 3D. The 10 NAEP items included multiple choice and open-ended questions. Experts in mathematics education and special education reviewed each probe to determine content validity, with revisions made based on their feedback. Both measures were piloted with students not participating in the study to determine reliability.

Domain probes. I sampled from all of the objectives in the instructional unit and developed eleven parallel versions of the domain probe. Eleven different probes were designed to limit a participant from receiving the same form of a probe, as the probes were randomly selected. To address conceptual and procedural knowledge, questions include contextualized, non-contextualized, and open-ended questions. For instance, in a non-contextualized open-ended problem a student was asked to determine if two triangles (embedded/overlapping) are similar or not and explain why (using the definition of similar figures, similarity theorems) and if not then to provide additional information/ what needs to change for the figures to be similar. A contextualized problem required the student to use similar triangles to measure the distance across a river or other location such as a surveyor or engineer would do to build a bridge (see Appendix O for a sample domain probe).

I established content validity of the parallel measures by expert review (Huck, 2008). Specifically, two experts in the field of mathematics and special education reviewed each version of the domain probes and determined that they were comparable in addressing the unit content and difficulty level. Additionally, the probes were piloted with students not participating in the study, alternate forms reliability calculated and found to be high ($r = 0.903$).

Standardized Mathematics Assessment (NAEP) probes. I also administered a probe based on the NAEP. This assessment consisted of 10 questions aligned with the similarity transformations topics targeted in the intervention. However, unlike the entirely open-ended domain probes, six of the NAEP-based questions were multiple choice with justifications requested, with the remaining four questions open-ended. In order to choose the questions that aligned with the unit I reviewed the approximately 13 public released items from the grade 8 and grade 12 assessments and compared them with the unit objectives. However, the NAEP also includes items in which measurement is applied within proportional applications (i.e., using a ruler and scaling), which was not a focus of the intervention as those problems were not part of the high school geometry curriculum. Furthermore, the majority of the NAEP public release items were in topics other than geometry, with only two grade 12 items available as samples. As with the domain probes, content validity of the measures was established by expert review (Huck, 2008). Specifically, the same two experts in mathematics and special education who reviewed the domain probes also reviewed the proposed NAEP items, provided feedback on item selection, and gave suggestions for revisions of the assessment. For example, I eliminated several items from grade 8, as the participants may have been instructed on that material and this could potentially bias the results. Lastly, although the NAEP uses a variety of analysis methods to ensure reliability and validity of assessment items (NCES, n.d.) it seemed prudent to establish reliability of the measure because it was only a subset of the NAEP. Prior to the intervention the NAEP instrument was piloted with students not participating in the study to establish test-retest reliability, which was found to be high ($r = 0.94$). See Appendix P for the NAEP-based probe.

Measurement Procedures

I employed the following procedures when administering domain and NAEP probes in order to provide consistency in their administration and to address threats to internal validity. Many of the procedures were the same during baseline, intervention, and post-intervention; however, some differences between the measures required use of different procedures and explanation of those follow in separate sections. Lastly, interrater reliability and fidelity of assessment are provided.

Threats to validity addressed within the design included multiple treatment interference, practice/testing effects, and treatment diffusion. First, to avoid issues with multiple treatment interference, during the baseline phase participants received instruction during the same class period, in the same location and I did not provide any instruction related to the target topic, similarity transformations (Gast, 2010). Participants received the intervention during their resource class, or directly after school (on a few occasions when requested by the student and with parental agreement due to unforeseen scheduling conflicts) so that they did not miss essential course material due to being removed from a regular content class and use of the resource class was the least disruptive to their schedules. Furthermore, it was normal for students to utilize the resource class to complete assessments for other courses and receive their extended time accommodation. Second, I randomly selected ten parallel versions of the domain probes and administered them to address practice or testing effects (O'Neill, et al., 2011). Finally, to address treatment diffusion, domain probes were administered in the same room as the instructional intervention (i.e., conference room or classroom), during the resource class period, to minimize distractions that could affect student performance (O'Neill, et al., 2011).

During probe sessions, I provided participants with pencil, paper, calculator, and measurement tools (i.e., ruler, protractor). If asked, I read students the directions and probe questions verbatim, as my goal was not to assess students' reading ability. If a participant asked for clarification or interpretation of the instructions or probe questions, I responded, "Do the best that you can." At the conclusion of the probes, I thanked students for their participation and attempts of the problems but gave no feedback regarding performance. If a student asked about his or her performance, I said that the results could be shared at the end of the study. I administered and collected the probes for scoring and inter-rater reliability checks. I also video recorded sessions for fidelity checks.

Domain probes. I administered parallel versions of the domain probes randomly during baseline, post-intervention, and maintenance. I administered a minimum of four domain probes to participants during baseline and immediately after the intervention (Kratochwill, et al., 2013) to assess students' mastery, and a maintenance probe four to six weeks after the intervention. While only three post-intervention probes are required if the data are stable, (Kratochwill, et al., 2010, 2013; Vannest, et al., 2011) as can be seen from Figure 5, post-intervention probes were not stable, so I administered additional probes to students.

Standardized Mathematics Assessment (NAEP) probes. I administered the probe once to participants during baseline as well as once at the end of the intervention, as explained in the dependent measures procedures above.

Interrater reliability. Criteria established by the *What Works Clearinghouse* requires more than one assessor to verify performance on at least 20% of the sessions per condition (i.e., baseline and intervention) and document accuracy of scoring using an acceptable statistical measure of consistency (Kratochwill, et al., 2010; 2013). Historically minimum acceptable inter-assessor agreement has been 80% based on percentage agreement; however, some researchers recommend 90% as desirable, with checks on 20-33% of the data per phase (O'Neill et al., 2011; Richards et al., 2014).

For the study, interrater reliability was obtained on at least 33% of the domain and standardized assessments in each phase (baseline and post-intervention) to monitor the consistency of measurement of the dependent variable. Specifically, interrater reliability was calculated using eight of the baseline and eight of the post-intervention domain probes (of 46 total) plus four of the eight standardized pre-post tests, one per participant. I taught a graduate student to score data. During training, I provided mock probes with an answer key for the assessor to practice scoring. I considered him trained after he was able to score three mock probes with at least 90% agreement with me. This person then independently scored each probe. We scored each permanent product probe using an answer key with specific point assignment for each problem. The percentage of inter-rater reliability on the students' permanent products was calculated by (a) totaling the number of items marked as correct for one scorer, (b) totaling the number of items marked as correct for another scorer, (c) dividing the smaller of (a) or (b) by the larger of (a) or (b), then multiplying the result in (c) by 100 (O'Neill et al., 2011). This calculation for permanent products provides similar results to the method typically used in research for interval agreement (i.e., when observations are made), in which the number of agreements between two observers is divided by the total number of possibilities (agreements plus disagreements) then multiplied by 100 (O'Neill et al., 2011). Since an assessment is a permanent product, I used the former method.

Initial reliability on domain probes was 96.8% (range = 85.6%-100%), with 100% agreement on baseline scores and 93.7% for post-intervention (range = 85.6%-100%). Following a discussion of disagreements, reliability was 100%. Disagreements happened because of errors in scoring due to different decisions about when to assign full or partial credit and two instances of scorer error in assigning full credit instead of partial credit for a participant answering half of a two-part question. After discussion, the trained assessor scored two more probes from any participant where there was initial disagreement and achieved 100% reliability with my scores. Reliability on the NAEP measure was 100%, with no disagreements.

Fidelity of assessment. Similar to the fidelity of treatment, the collection of assessment data also should be uniform and should be conducted on at least 25% of sessions, with an interobserver agreement of at least 80%, with 90% agreement highly desirable (O'Neill et al 2011; Richards et al, 2014). Seven each of the baseline probes (one NAEP =25%, and six domain = 27%) and post-intervention (five domain = 25%, one NAEP = 25% and one maintenance = 25%) probes were selected for viewing by two independent observers. Both independent observers indicated that all elements were included as planned; there were no disagreements, so inter-observer agreement was 100% See Appendix R for the checklist.

Incentive. Furthermore, since the multiple probe design required participants to complete numerous probes, I gave each participant an incentive for each probe that he or she completed, regardless of the level of performance. The students earned one ticket for each probe, which was then turned in for nominal rewards, such as a coupon for a free ice cream in the school cafeteria, coupon for a small item from a local business (e.g., personal pizza), or items from the school store.

Participants and Setting

This section provides an overview of the participant eligibility for inclusion in the study, participant characteristics, setting, and the instructor description. Additionally, information is discussed relative to obtaining Internal Review Board approvals, informed consent from parents/legal guardian, and informed assent from participants.

Eligibility

Participants met the following criteria to be eligible for participation in the study: (a) enrolled in grades 9-12, (b) had not successfully completed a credit in geometry with a minimum final average of a C (70%), (c) demonstrated difficulties in mathematics via educational reports (e.g., report cards showing mathematics courses failed, repeated or with a final grade below a C; cognitive [WISC/WAIS] or academic measures [Woodcock Johnson] scores below the 25^{th} percentile; educator recommendation), (d) demonstrated a need for the intervention as evidenced by scoring below 60% on an investigator-developed domain probe targeting objectives related to similarity transformations of polygons and (e) turned in signed parent permission and student assent forms.

Four students who meet the criteria were randomly selected to receive the intervention. Although random selection is a concern with many research designs in order to reduce threats to external validity (O'Neill, et al., 2011) single-subject research usually addresses this concern via replication. I still believe that it was important to randomly select the participants and the order to work with the students primarily because I am familiar to the students (and their course teachers) and did not want it to appear there were reasons to work with students in any particular order. Table 5 displays information obtained from school records regarding characteristics of participants, including gender, race/ethnicity, age, grade level, IEP/ELL, FARMS status, cognitive and academic achievement scores. Furthermore, the participants were representative of a suburban school district, located in the mid-Atlantic United States. The district served a diverse population with approximately 50% non-Caucasian (including 10% Hispanic/Latino), 17% received free or reduced-price meals (a measure of poverty), 5% ELL, and 7.5% received special education services. The school where the study took place served approximately 1500 students with 70% of the population non-Caucasian (including 15% Hispanic/Latino), 30% received free or reduced-price meals, 5% ELL, and 9% received special education services.

Participant characteristics

All four participants were receiving special education services during the study. Specific individual participant information regarding demographics, special educational status, and educational history follows. Table 5 provides a summary of participant demographic information.

Jason. This student was a 15-year-old African-American and Hispanic tenth grader. He was diagnosed with a LD in reading, writing and mathematics in the 6th grade utilizing the discrepancy method as well as a review of his classroom performance and lack of response to prior interventions (RtI). When diagnosing a LD the state allows for the use of either a discrepancy method (e.g., 1.5 standard deviations difference between

Table 5
Demographic Information

	*Participants**			
Characteristics	Jason	Kenneth	Khafila	Sara
Gender	Male	Male	Female	Female
Race/Ethnicity	AA/H	A	AA	C/AA
Age (years)	15	14	18	16
Grade	10	9	11	9
Disability	SLD	ADHD	SLD/ADHD	SLD/SL
ELL status	--	Released	--	--
FARMS	No	No	No	Yes
Aptitude (WISC-IV)	88	96	73	89
Achievement (WJIII-Math)	77	105	72	67

*All student names were changed to maintain confidentiality and are listed alphabetically in the table, not by order of participation in the intervention.
C=Caucasian; AA=African American; A = Asian; H = Hispanic; SLD = specific learning disability; SL= speech or language impairment; ADHD = Attention Deficit Hyperactivity Disorder; FARMS = Free and reduced price meals student; ELL = English language learner released are still followed for two years

intellectual functioning and academic achievement) or an RtI method. As defined by the State code of regulations (COMAR), IEP teams may determine a diagnoses of a SLD (a) if a student does not make adequate progress to meet (i) age or (ii) State-approved grade-level standards in one or more of the areas identified in the regulations (e.g. reading, writing, math) when using a process based on the student's response to scientific research-based intervention or (b) the student exhibits a pattern of strengths and weaknesses in performance, achievement, or both, relative to (i) age, (ii) State-approved grade-level standards, or (iii) intellectual development (COMAR *13A.05.01.06).*

Based on the WISC-IV, an individually administered test of cognitive functioning with a mean score of 100 and a standard deviation of 15, his overall IQ was found to be in the low average range of achievement (SS = 88) with some variability as seen by subtest scores as follows: average Verbal Comprehension (SS = 99) and Perceptual Reasoning (SS= 96), low average Working Memory (SS = 88) and borderline Processing

Speed (SS = 75). His scores on the Woodcock-Johnson III Tests of Achievement, and individually administered test of academic achievement, with a mean score of 100 and a standard deviation of 15, are divided into reading, written language and mathematics rather than an overall score and are as follows: low average reading (SS= 84) and written language (SS = 83) achievement and low math achievement (SS = 77).

Jason received additional interventions in mathematics and reading during middle school, but continued to perform below expectancy. At the time of initial eligibility, Jason was doing more poorly in mathematics than other subjects, but was struggling in all academic areas. Teacher reports also described him as wanting to do well but exhibiting signs of anxiety such as shutting down and not working. Later special education annual reviews and re-evaluations note similar areas of concern. The year prior to this study, he was enrolled in a co-taught double period algebra course in order to provide additional support and skills practice; he did not pass the course although he did pass the end-of-course exam in mathematics required by the state. The end-of-course exams are statewide criterion-referenced assessment administered to students completing instruction in Algebra 1/Data Analysis, Biology, English 10 and American Government. At the time of the study, Jason was enrolled in Algebra 1 for the second time. Jason received all his academic instruction in general education and attended a resource class daily for assistance with study skills the year prior to and during the study.

Kenneth. This student was a 14-year-old ninth grader. Kenneth attended schools outside of the United States, prior to enrollment in high school this year, due to his parents' work requirements. Based on prior school records he was previously diagnosed with Attention Deficit Hyperactivity Disorder, at age 10. However, the most recent evaluation available was from grade 8, which included a battery of assessment, including WISC-IV, WJ-III, Gray Oral Reading Test (GORT-5), and ADHD rating scales. Results of the WISC-IV indicate average overall intellectual function (SS=96), with superior verbal comprehension (SS=126), average Perceptual Reasoning (SS=92), low average Working Memory (SS=80) and Processing Speed (SS=80). On the GORT Kenneth's fluency was better than his comprehension (score information was not in the records just a summary). Results of the WJ-III did not include a full scale score but included individual results for reading, math and written language as follows: average Reading (SS=100), Math (SS=105) and written language (SS=108). The examiner noted some specific subsections of the assessments that required visual tracking were areas of weakness, this matches with a history of visual difficulties and that Kenneth had vision therapy around age 9.

Records reviewed during his transfer and eligibility noted difficulties with organization and planning that specifically impacted mathematics performance. In his prior schools, he had been provided additional tutoring and support for mathematics, although his performance in his mathematics courses in grades 6-8 were Ds, and the lowest performance of all his courses. At the time of the study, Kenneth received all his academic instruction in general education and attended a resource class daily for assistance with study skills, was enrolled in Algebra 1, and would take the required state assessment at the end of the year.

Khafila. This student was an 18-year-old African-American eleventh grader. She was diagnosed with a LD in the 8^{th} grade, specifically a Non-Verbal Learning Disability, due to the discrepancy method. She was also diagnosed with ADHD. Khafila was administered several assessments of cognition, memory and achievement. Based on the WISC-IV her overall IQ was found to be in the borderline range of intellectual functioning as compared to same age peers (SS=73). She was also given another test of cognitive ability, the Comprehensive Test of Nonverbal Intelligence (CTONI-2), which measures analogic reasoning, categorical classification, and sequential reasoning. The assessment has a mean of 100 and a standard deviation of 15. Her score was in the poor range (SS = 70). On the Wide Range Assessment of Memory and Learning (WRAML-2), an individually administered memory assessment with a mean of 100 and a standard deviation of 15, her memory skills were in the average range (SS = 92). On the Beery-Buktenica Developmental Test of Visual Integration (VMI-6), a test of visual-motor integration with a mean of 100 and a standard deviation of 15, her skills were in the very low range of achievement (SS = 45). Her academic achievement measures (WJ-III) indicated average reading skills (SS = 99), low average written language skills (SS=87), while her mathematics skills were in the low range of achievement (SS = 72), however an overall score was not reported. At the time of her initial evaluation she was failing mathematics and specific comments included that she had difficulty with arithmetic (e.g., uses fingers for counting), math concepts (e.g., fractions, measurement, reading clock, percent), multi-step equations (e.g., completed first step correctly, but does not consistently and accurately complete a second step).

At the time of the study, Khafila received all her academic instruction in general education, attended a resource class daily for assistance with study skills, and was enrolled in Algebra 1. Khafila had failed this course twice, and given her history of failure she was enrolled in a co-taught double period algebra course in order to provide additional support and skills practice. She had also taken the state end-of-course exam, a requirement for a diploma, three times and failed each time (by a 15-point margin). Khafila had passed the Biology and Government exams by a wide margin, which provided additional points to combine with the scores in English 10 and Algebra so she met a minimum combined score the state allowed as an option to passing the individual assessments.

Sara. This student was a 16-year-old African-American and Caucasian ninth grader. She was diagnosed with a LD in reading, written language and mathematics, during the 6th grade using a discrepancy method as well as review of classroom performance and lack of response to interventions (RtI). Records indicate a history of academic concerns beginning in kindergarten (she was retained) including memory, reading comprehension and mathematics; however, it was noted that her basic math facts, such as multiplication, were a relative strength.

On the WISC-IV, Sara scored in the low average range of intellectual functioning (SS = 89). However, Sara performed in the average range on each of the subtests that measure verbal abstract reasoning (Verbal Comprehension SS=93), non-verbal problem solving (Perceptual Reasoning SS=90), speed of information processing (Processing Speed SS=91) and auditory working memory (Working Memory SS=94). Based on the WRAML2, Sara's memory was low average to average as indicated by the following standard scores: Verbal Memory SS=94, Visual Memory SS=91, Attention Concentration SS= 82, General Memory SS=85. Academic achievement scores, as measured by the WJ-III, were in the low to very low range of achievement with the following scores: low achievement in reading (SS =76) and written language (SS=76) and very low achievement in mathematics (SS = 67). At the time of the study, Sara received all her academic instruction in general education, attended a resource class daily for assistance with study skills, was enrolled in Algebra 1 and would take the required state assessment at the end of the year.

Setting

I taught students during their resource study skills class period; on occasion, some lessons were given after school per student request. This included the administration of the baseline probes, post-intervention probes, and the intervention instruction. All aspects of the intervention took place in the main office conference room or an available classroom depending on the time of the student's resource class and available space. The location was consistent for the individual student. The room contained sufficient seating and work space (e.g., a large table or several student desks) and I brought other necessary materials such as computer with the software, manipulatives, and writing materials. The length of the intervention was designed to reflect the length of a typical secondary mathematics topic, 10 lessons, including review, for 45-minutes per lesson. Furthermore, during the baseline condition participants did not receive instruction related to the content of the study, similarity transformations, during their regular mathematics class.

Instructor

I implemented the intervention and assumed the role of teacher and researcher for the duration of the study. This was done for three reasons: (a) importance of teacher research, (b) personal experiences, and (c) addressing validity threats. First, teacher research involves "systematic and intentional inquiry" (Cochran-Smith & Lytle, 1999, p. 22) and is important in bridging the research to practice gap (Klinger & Boardman, 2011). This type of action research takes place in context (i.e., by other teachers), which may enhance learning, teaching, and policy as teachers, as consumers of research, may find research conducted by fellow teachers to be more persuasive (Gay, Mills, & Airasian, 2006).

Second, the ideas for this research study resulted from my previous 17 years experience as a secondary teacher, first as a mathematics educator, then as a special educator in both public and private schools. During my years as a teacher, my classes included many students with mathematics difficulties, including those with formally diagnosed disabilities and those who experienced low mathematics achievement. Specifically, I observed that students struggle to demonstrate competence in geometry, especially with visuo-spatial dependent skills, when taught using abstract, static, manipulation without an emphasis on conceptual understanding. Finally, a reactive arrangement validity threat is a concern if an unfamiliar investigator implemented the intervention (Gast, 2010). I have either worked with some of the participants as course teacher or assisted periodically in their resource class. My familiarity with the students reduces the reactive arrangement validity threat that could occur if a person unfamiliar to the students were to implement the intervention.

Internal Review Board

Prior to the beginning the investigation, I submitted plans for approval to the Internal Review Board at the University of Maryland, College Park. I also submitted plans and received approval from the public school system where the study took place.

Informed consent

Participants and their parents/legal guardians received a letter (see Appendix B) that stated the purpose of the study, the geometry content addressed during the study, and the risks and benefits with participation. Furthermore, I requested access to each participant's educational records (i.e., IEP/504, scores for IQ and achievement tests, grades in current and prior mathematics courses). Participants and their parents/legal guardians were informed that participants could withdraw from the study at any time without penalty, although none did. Parents/legal guardians signed a permission form (see Appendix C) and participants signed an assent form (see Appendix D).

Data Analysis

I collected and graphed data for each participant. Patterns in the data baseline and post-intervention data were analyzed to: (a) determine the next steps in the design; (b) make adjustments to the intervention, if needed; and (c) determine the reliability of the findings (Kazdin, 2011; Tawney & Gast, 1984; Vannest et al., 2013). The traditional approach to analyzing single-subject research is a visual analysis of graphic displays of the data (Gast, 2010); however, there are also several statistical analyses for determining effect size (Kazdin, 2011; Parker, Vannest & Davis, 2011; Richards et al., 2014). In recent years with the increased focus on evidence-based practices the use of effect sizes as a measure of the magnitude of an intervention's effectiveness have increased (O'Neill et al., 2011). I utilized both the visual approach and appropriate effect size statistics for this intervention. In the sections below, I explain and clarify aspects of the visual method followed by the effect size measures that I utilized.

Visual analysis

Visual analysis is typically used with continuous numerical data in order to make formative and summative analyses of study results (Richards et al., 2014), which suits the individualized data collection and graphical display for this intervention. Specifically, I focused on within-phase patterns and between phase patterns (Richards et al., 2014; Tawney & Gast, 1984) to determine that a functional relation existed between the independent and dependent variables (Guralnick, 1978; Horner et al., 2005; Kratochwill, et al., 2010; Riley-Tillman & Burns, 2009).

Within-phase patterns. For within-phase patterns (both baseline and post-intervention), I analyzed data points in regard to (a) level or the mean of the data points within a phase (Richards et al., 2014), (b) trend, which refers to the direction and slope of the data points (Gast, 2010), and (c) variability of the data points or the degree they deviate from the mean or general trend (Kazdin, 2011). For this study, I anticipated that the baseline within-phase patterns consisted of a mean level below 60% performance on domain probes, low variability with the last three data probes being within 10% of the mean, and a somewhat flat trend line (Tawney & Gast, 1984); or all data probes within one standard deviation of the mean (Kazdin, 2011). For the post-intervention phase, I anticipated that there would be a higher mean performance than at baseline, a stable, or increasing trend, and low variability.

Between-phase patterns. The between phase pattern is also important to determine the functional relationship between the independent and dependent variable by examining the data to see if there is an immediate change in level and trend (Richards et al., 2014) as well as the extent of any overlap of the data (Kennedy, 2005). Following the introduction of the intervention, I conducted a visual analysis of the graph to determine if there was: (a) an increase in level of performance on the domain probes as compared to baseline; (b) an increasing trend (i.e., slope of the line of best fit) of the probe scores post- intervention as compared to baseline trend being low and flat; and (c) low variability of data points within each phase. It is important to compare level, trend, variability, and overlap to adequately evaluate whether or not a functional relationship between the independent and dependent variable exists (Kennedy, 2005).

Effect Size

With the increased emphasis on evidence-based interventions, measures of the extent of the impact of the intervention are becoming increasingly important. This may enhance the interpretation of the visual analysis or address issues with the interpretation of the results viewed graphically (Parker et al., 2011). For example, the baseline may not be stable but perhaps there is a trend, or the rate of change may be slow or it is unclear as to how effective (or not) the intervention was (Kazdin, 2011). There are several statistical analyses that could be used, including parametric and nonparametric methods. Parametric tests used in single case design include regression and multi-level models, however these are typically used for larger data sets and most single case designs do not meet the additional assumptions such as score scale and data distribution (Vannest, et al., 2013). To support the visual analysis of the data, there are several nonoverlap methods that have been specifically designed for single subject research that are promising effect size measures, such as the improvement rate difference (IRD), split middle, percent non-overlapping data (PND), non-overlap all pairs (NAP), percentage of data exceeding a median trend (PEM-T), and Tau-U (Parker et al., 2011; Vannest et al, 2013).

Given that there were so few participants and there was a clear change post-intervention, PND was utilized first as it has been determined to be the most meaningful method of measuring treatment effectiveness for single subject research (Tawney & Gast, 1984). The PND was calculated by (a) identifying the highest data point in the baseline phase, (b) identifying the number of data points in the post-intervention phase that exceed the highest data point in baseline, (c) dividing the number of post-intervention phase points exceeding this data point by the total number of intervention data points, and (d) multiplying the result of (c) by 100 (Scruggs & Mastropieri, 2001). The closer PND is to 100%, the more effective the intervention (Scruggs & Mastropieri, 1998).

Multiple visual and statistical methods, including IRD, PEM-T and Tau-U, were utilized to make comparisons and more effectively interpret the data (Kratochwill et al, 2013). IRD was calculated using the following method: 1) identify the smallest number of data points among each phase that need to be removed to eliminate overlap between phases, 2) subtract the number of data points removed from each phase from the total number of data points in the phase to obtain the number of data points remaining in each phase, 3) write an improvement rate fraction for each phase 4) calculate the difference between the values from step 3, 5) multiply difference from Step 4 by 100 (Rakap, 2015). For PEM-T, after graphing the data: 1) calculate and draw the split middle line of trend estimation through the baseline data and extend it through post-intervention 2) count the number of data points in post-intervention above the trend line from step 2, 3) divide the count from step 3 by the total number of data points in the post-intervention phase, and 4) multiply by 100 (Rakap, 2015). The closer the IRD, PEM-T and Tau-U are to 100% the more effective the intervention (Rakap, 2015).

Social Validity

Social validity measures indicate the importance, effectiveness, appropriateness, and/or satisfaction of the participant's experiences in relation to the intervention (Kazdin, 2011; Richards et al., 2014). At the end of each student's participation in the study, participants were asked to share their thoughts on the intervention by completing a questionnaire to assess their perceptions regarding their learning of the content, the usefulness of the instructional tools (e.g., manipulatives, computer programs), and their likes and dislikes of the intervention. An instrument was developed based on social validity measures within the field of mathematics special education research (Cihak & Bowlin, 2009; Maccini, 1998; Mulcahy, 2007; Strickland, 2011).

I administered the questionnaire to all study participants during their resource period, upon the completion of the student participation in the study (i.e., after the post-intervention assessments). The measure included nine questions on a five-point Likert scale regarding the effectiveness of various features of the intervention. All questions are phrased so that a "1" indicates a negative opinion of the effectiveness, while a "5" indicates a positive opinion. Participants were told to indicate a score of "1" if they strongly disagree with a statement, "2" if they disagree, "3" if the feel neutral, "4" if they agree or "5" if they strongly agree. Additionally, participants were asked to respond to seven open-ended questions, soliciting suggestions for improving the intervention. The students were permitted to have the questionnaire read to them verbatim, in whole or in part, and allowed to dictate responses. I analyzed data by determining the mean score for each item on the Likert scale and reporting themes from the open-ended response questions (see Appendix S for the instrument).

Mathematical Disposition

I administered to all study participants a Likert scale questionnaire similar to the design of the social validity measure to complement the quantitative data and address the research questions regarding conceptions of geometry pre- and post-intervention as well as metacognition, self-efficacy, and attitudes toward geometry. This was administered once prior to any pre-intervention baseline probes and again after the post-domain assessments. Participants were asked to share their thoughts about mathematics and geometry, including their perceived ability, experiences, and approach to learning. I based this 16-question questionnaire on other mathematics attitude measures within the field of mathematics education research and mathematics special education research (Bottge, 1999; Bottge, et al., 2001, 2002; Fennema & Sherman, 1976; Jitendra et al., 2009; Utley, 2007). I analyzed the data by determining the mean score change for each item, as well as comparing with the oral and written work and feedback. Furthermore, individual or groups of questions specifically related to geometry were sub-analyzed in relation to additional research questions, such as conceptions of geometry pre and post-intervention. See Appendix T for the instrument.

Chapter 4: Results

This chapter presents the results of the data analysis conducted to assess the independent variables (explicit instruction with DGS, CRA) and dependent measures (domain and standardized probes) of this study. Additionally social validity and mathematics disposition questionnaires and measures are included that enhance that quantitative data. The results of the analysis in this study are organized into sections based on the research questions that guided the study. First, Question 1 is most directly addressed by the single subject design and divided into two sections based on the domain and NAEP probes. Second, for Question 2 the work samples from the domain and NAEP measures are supplemented with information from the mathematics attitude measure relevant to geometry. Next, Question 3 is addressed with work samples and excerpts from discussion. Finally, Question 4 was addressed by the social validity and mathematics attitudes measures. The chapter concludes with a summary of the results. The following research questions guided the investigation:

1. Do secondary students with MD taught an instructional intervention on geometric similarity transformation increase the accuracy of their performance in geometric similarity transformations and maintain performance four to six weeks after the intervention?
2. What conceptions of geometry do students hold before and after the intervention?
3. What connections or disconnections in the geometry content emerge during the intervention and how can these results be used to improve instruction?
4. To what extent do secondary students with MD find dynamic geometry technology beneficial to representing and solving geometric similarity

transformation tasks and in what ways does the intervention enhance metacognition, self-efficacy, and attitudes toward geometry?

Research Question 1:

Accuracy and Maintenance on Geometric Similarity Tasks

I measured student's accuracy on geometric tasks by monitoring their performance on domain probes and the NAEP measures pre- and post-intervention. Maintenance was measured utilizing a final domain probe 4 weeks after the conclusion of the intervention. This section presents data from all participants, first regarding the increased accuracy then the maintenance.

Increased Accuracy

First, results of the domain probes aligned with the target lesson objects are presented. Next, information on the pre-test post-test NAEP measures are shared along with some pertinent comparisons between the measures.

Domain probes. Regarding the domain probes, as shown in Figure 5 and Table 6, all students increased their accuracy from an average of 2.3% during baseline to an average of 71.8% after the intervention. Specifically, the baseline scores ranged from 0% to 11%., while the post-intervention scores ranged from 46-96%, with an average increase of 69.6 percentage points over baseline.

I also analyzed graphs of the data using visual analysis to identify patterns within and between phases. All participants demonstrated a stable baseline pattern prior to beginning the intervention and with-in phase analysis showed a pattern of responses for all participants with a flat or near flat trend, level well below 60% (non-mastery) and low

Geometric Similarity Transformation Performance

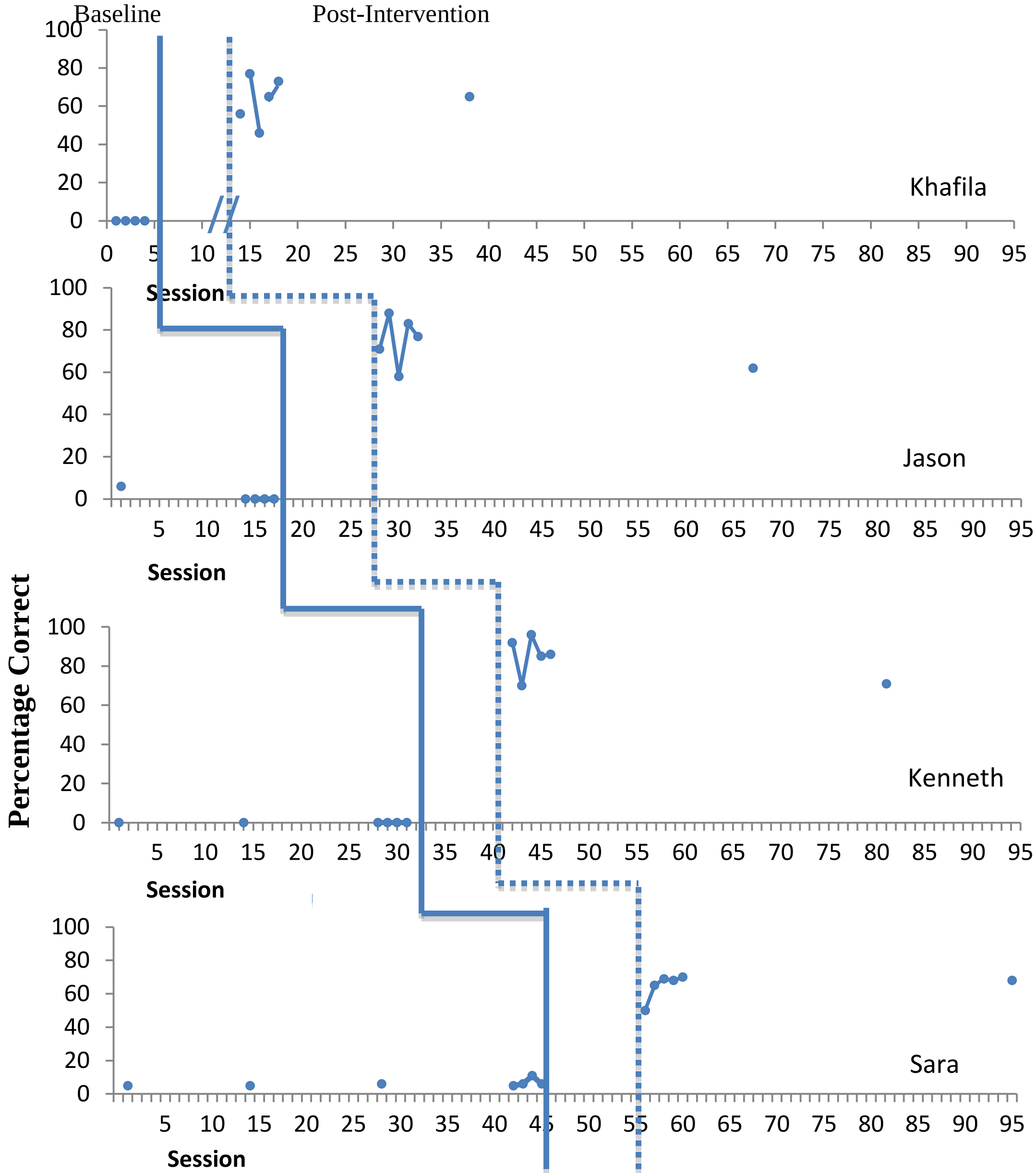


Figure 5. Effects of the IV on the DV across 4 participants

Note: Khafila had 1 booster session between post-intervention probes 1- 2 and another between 3-4 due to extended scheduling interruptions.

Table 6

Average Percentage Accuracy & Increases in Percentages on Domain Probes

Participant	**Baseline**	**Post-Intervention**	**Increase**
Khafila	0%	63.4% (r = 46%-77%)	63.4%
Jason	1.2% (r = 0%-6%)	75.4% (r = 58%-88%)	74.2%
Kenneth	0%	85.8% (r = 70%-96%)	85.8%
Sara	6.3% (r = 5%-11%)	64.4% (r = 50%-70%)	58.1%
All participants	2.3 % (r = 0%-11%)	71.8% (r = 46%-96%)	69.6% (r = 58%-86%)

variability (difference in scores less than 10%). An analysis of between-phase patterns indicated an immediate increase in level for all participants upon post-intervention, specifically an initial average of 67% (range = 50%-92%) which is well above the average baseline performance of 2.3% (range 0%-11%), There was no overlap of the post-intervention and baseline domain performance. However, there was some variability in most participants' performance, with an overall flat or slightly increasing trend.

In order to corroborate the visual analysis several measures of effect size were also used. The PND for all participants as well as the aggregated PND was 100%, meaning none of the baseline data points were above (or overlapping) the post-intervention data points (and conversely none of the intervention points were below, or overlapping, the baseline data points). The improvement rate difference (IRD) was 1 (100%), indicating improvement over baseline. Additionally, since Sara showed a slight improvement in baseline performance close to commencing intervention checking the

Percentage of data Exceeding a Median Trend (PEM-T) seemed a wise choice, although this rate was also 100%. Furthermore, the aggregated Tau-U effect size across the four participants was 1 (95% Confidence interval 0.6299<>1.3701) meaning that 100% of the data from baseline to post-intervention assessment did not overlap. Each of these visual and nonparametric measures (PND, IRD, PEM-T) and Tau-U effect size indicate that the participants improved their performance.

NAEP probes. I administered the NAEP based assessment to participants before they began the intervention (at the point all participants took the first domain pre-assessment) and approximately one week following the conclusion of the intervention (after completing the domain probes). Three of the participants improved their score from pre-test to post-test; however not all were meaningful differences. Khafila improved from 10% to 30%, Jason improved from 10% to 70%, and Kenneth improved from 30% to 40%, while Sara's score remained at 30%.

Item analysis shows some differences that are not readily apparent from the scores alone. For example, while Kenneth's score changed little, the items he answered correctly were not all the same from pre to post-test; this also was evident in other participant's scores as shown in Table 7. Given that several questions were multiple choice on the pre-assessment, a participant may have gotten the correct answer, but not had a valid reason. For example, on the pre-assessment Kenneth indicated for number 8 that he multiplied by 3 (correct reason) but for item 10 he guessed (correctly), while on the post-assessment for number 8 he multiplied only one of the dimensions by 3 (not both), resulting in an incorrect response but for item 10 he stated he did not know what to do and guessed incorrectly. It was noticeable that for the pre-assessment the participants

Table 7
Correct/Incorrect on NAEP pre-post tests.

Question	Khafila		Jason		Kenneth		Sara	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post
1								
2				x			x	x
3		x				x		
4		x		x		x	x	x
5				x		x		
6		x		x	x	x		
7								
8	x			x	x			
9				x			x	
10			x	x	x			x

Note: An x indicates the participant answered the question correctly

often stated they did not know how to solve many of the problems or attempted to use measurements in some manner even though those particular problems were not drawn to scale. However, on the post-assessments, they did use multiplicative strategies, but may have set up the proportions incorrectly, thus not earning credit (see Figures 5 and 6).

Maintenance of performance. Students also completed a final domain probe 4-6 weeks after the conclusion of the intervention to measure their maintenance of performance. Participants had an average retention rate of 66.5% (range = 62%-71%). This improvement is clinically significant meaning the results are important to the individual as it is a measure of the degree to which the intervention makes a meaningful difference in participants' lives (Bothe & Richardson, 2011). For the present study this gain is important and meaningful if it meant the student progressed from failing to passing based on school system criteria (i.e., 60% or higher). All participants maintained a passing score (60% or higher). The importance is also corroborated by the students' comments (either written feedback with the social validity measures or verbally), especially in regards to wanting to do well in their current or subsequent mathematics courses required for graduation.

Research Question 2:

Geometric Similarity Conceptions Pre- and Post-intervention

Conceptions, or understanding, of a concept is not always evident by the quantitative scores, but often the information that provides the best insight to the understanding a student has are their work samples and dialogue. In this section, selected work samples from the domain probes and NAEP assessment that provide insight into their understanding (or lack thereof) prior to the intervention and after the intervention are shared. Furthermore, relevant responses to the mathematics attitude measure (MAM) are included that support the evidence for participant conceptions of geometric similarity.

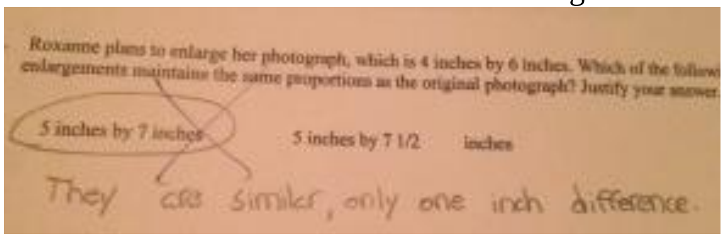
Probes

Prior to the intervention on the NAEP pre-assessment as well as the domain probes, students often indicated they did not know what to do to solve the problem (i.e., literally writing 'IDK' or leaving the problem blank and stating they did not know how to solve it). However, when attempts were made it was based on measurement, numerical reasoning, or additive strategies. In contrast to the lack of response on the pre-intervention probes, participants attempted all of the problems on the post-intervention assessments. The participants used multiplicative relationships and proportions for the bulk of the problems in which it was required. Errors were primarily due to setting up the proportions incorrectly or calculation errors. Sample responses for the pre-intervention probes are compared with their responses on the post-intervention probes.

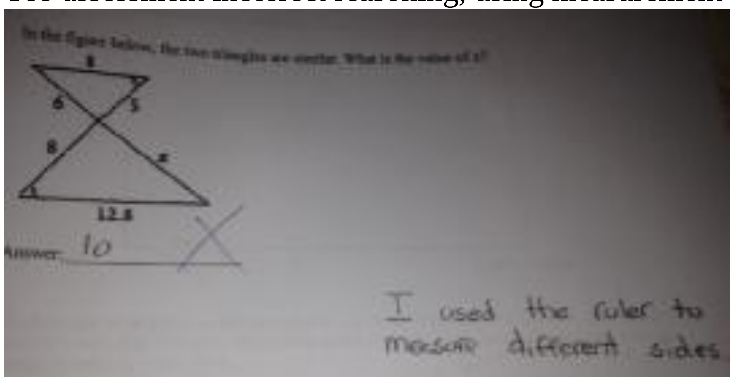
Kenneth. The first participant followed the instructions and provided explanations more often and with more detail on the pre-assessment than the other participants provided. However, his strategies consisted of measurement or additive strategies inappropriate to the situation, rather than multiplicative, proportional reasoning. Figure 6 shows several examples of Kenneth's responses on the NAEP pre-assessment that are indicative of limited or no understanding of geometric similarity. There were some inconsistencies where he may have gotten a question correct on the pre-assessment but incorrect on the post-assessment as shown in the last two examples in Figure 6. Also, on the post-assessment for the problem at the top of Figure 6 Kenneth correctly calculated the within ratios and concluded that the 4 x 6 and the 5 x 7 ½ were both .66 and therefore the 5 x 7 ½ is in the same proportion.

Jason. The next participant attempted a question on a domain probe that asked him to decide if a TV is wide screen (based on an aspect ratio of 16:9) or standard (with an aspect ratio of 4:3) given an example 60 " TV with dimension of 133 cm by 75 cm (a sample TV image was provided with the problem. His response: *wide screen because the ratio is high*, shows some number sense, but not the necessary proportional reasoning and application of problem solving. In contrast, on the post-intervention probe he recognized that the aspect ratios were to be compared to the ratio of the measurements for the given situation. He converted the standard aspect ratio to 1.3 and the wide screen ratio to 1.7 then calculated the ratio for the given TV (133 by 75) to be approximately 1.7. He stated

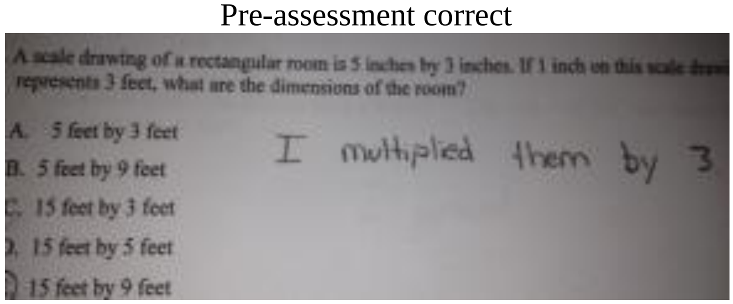
Pre-assessment additive reasoning



Pre-assessment incorrect reasoning, using measurement



Pre-assessment correct



Post assessment error in contrast to correct pre-assessment

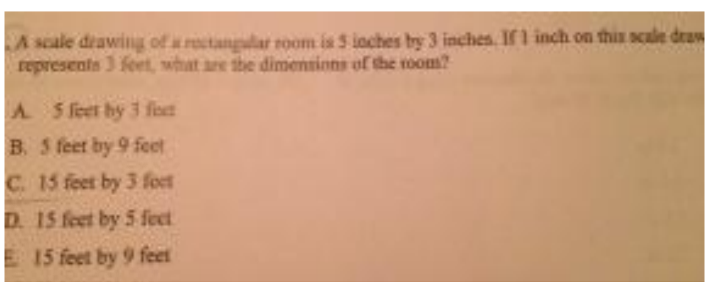


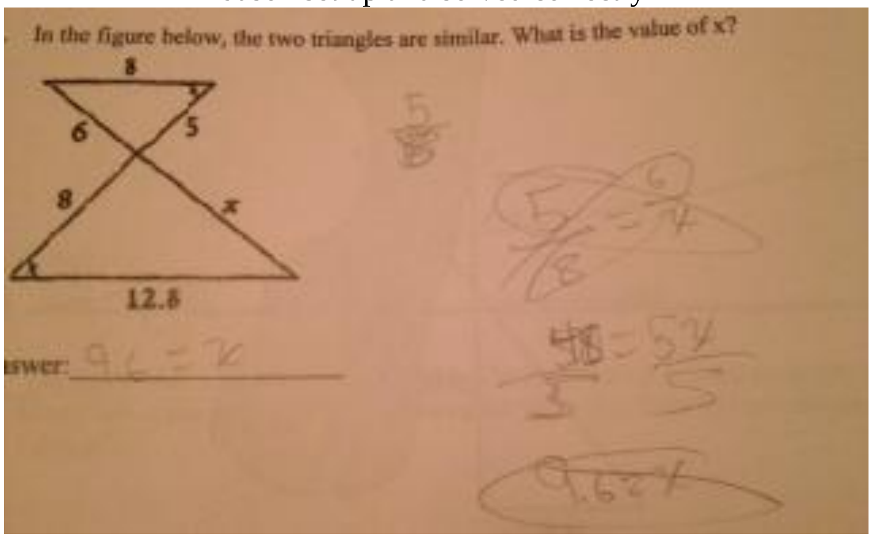
Figure 6. Kenneth NAEP pre and post assessment error examples

the TV was wide screen because the ratios were the same. By using the within ratios he was demonstrating more geometric proportional reasoning than the general number sense utilized pre-intervention.

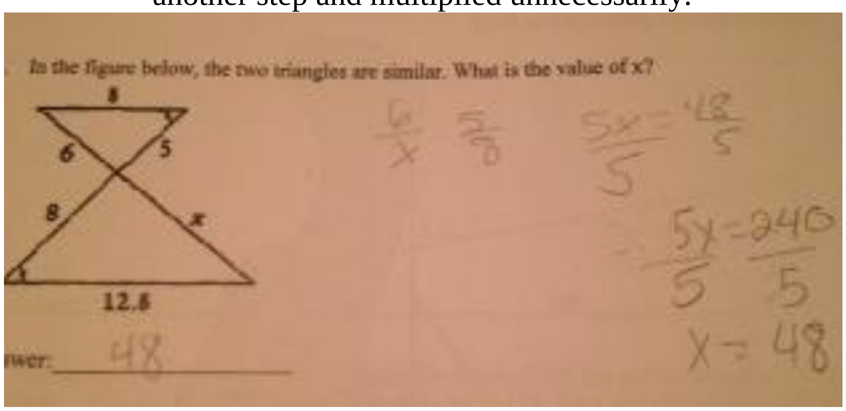
Sara. The third participant attempted a NAEP problem in which a room image was given with a scale factor (1 inch = 4 feet). The question asked how many boxes of tile would be needed if one box covers 25 square feet. Sara divided 25 by 4 (which gives 6.25) then she round up to 6 ½ boxes. This shows some number sense but not appropriate problem solving skills (using the ratio of 1 inch = 4 feet), measuring with the available tools (i.e. ruler) and then using the proportions in a geometric manner. In fact, it appears that she picked the larger number (25) and the smaller number (4) and assumed that the most logical step was to divide. However, she did not provide much elaboration or explanation other than 'having a bit more, just in case". This does, on the other hand show some real-life understanding that perhaps when buying tile (or something similar) it might be a good idea to have more in case one makes mistakes. However, on the post assessment it appears that she did not utilize all of the information in the problem. She measured the sides of the image and got 4 by 2 ½ (inches). She then stated that area is length x width and showed 4 x 2 ½ = 40 boxes of tile. She did not appear to use the information about one box covering 25 square feet. Just comparing these problems would indicate that she has not improved either her proportional reasoning skills or her ability to apply such skills geometrically.

Furthermore, Kenneth was the only participant to attempt the second problem in Figure 6 on the pretest and he utilized measurement or non-proportional reasoning. In contrast, on the post-assessment Kenneth set up the proportion and solved it correctly. As

Jason set up and solved correctly



Sara set up correctly but then did not follow through by dividing, but added another step and multiplied unnecessarily.



Khafila post assessment error in aligning corresponding parts

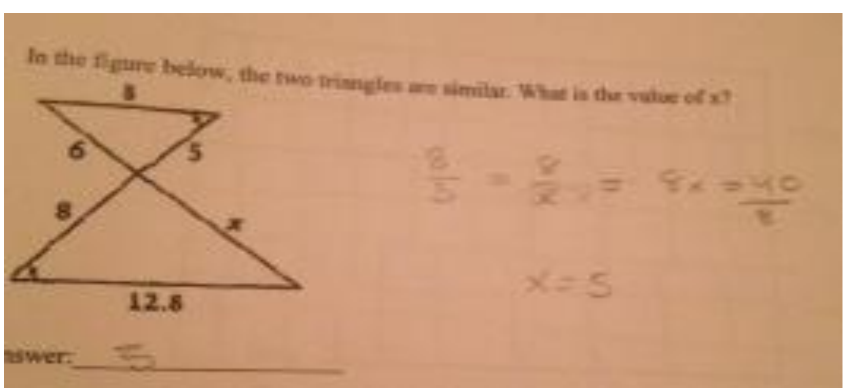


Figure 7. NAEP post-assessment: Jason, Sara and Khafila solution comparison

shown in Figure 7, all participants attempted to use proportions for this problem on the post-assessment. Jason also set it up correctly and solved correctly. However, Khafila did not compare the correct parts while Sara set up the correct proportion but did not follow through with correct calculations.

On the baseline domain probes no attempts were made on the problems related to the triangle similarity theorems or applications to multi-dimensional measurement, however, all participants attempted in whole or part on the post-tests. Sample responses showing full or partial understanding are included next.

First, on each probe there were two questions related to using the triangle similarity theorems such as Angle-Angle, Side-Angle-Side, Side-Side-Side, midsegment or Triangle Proportionality. The set up of each of those is as shown in Figure 10, where students needed to decide if the triangles are similar, explain (i.e., give the theorem), and if similar provide the similarity statement and if not similar state what was missing/needed to be changed.

One example problem showed two separate triangles with two angles marked showing two pairs of corresponding and congruent angles. In this case that meant the triangles were similar by AA (i.e., Angle-Angle Similarity Theorem). Khafila and Kenneth earned full points for answering all three parts of the question; however, Jason did not include the similarity statement so earned partial credit. On each probe asking this type of problem there were inconsistencies as sometimes they would earn full credit, but other times might earn partial credit due to providing some of the required parts.

Second, regarding applications to measurement students did not consistently make comparisons within the context of the problem. For example, each participant had a problem comparing a small ball (e.g., wiffle ball or golf ball) to the Spaceship Earth at Epcot Center. Students were given the diameter of the small ball (different for each probe), the diameter of Spaceship Earth (i.e., height of 180 feet) then given either the surface area of volume of the small ball. Students were asked to calculate the scale factor of the small ball to Spaceship Earth, then to calculate the surface area of volume of Spaceship Earth.

While Jason did write that the relationship for volume needed to be cubed, he did not pick out the information from the narrative to have the scale factor so then was not able to answer the second part (calculate the volume of Spaceship Earth). Khafila had the scale factor reversed (Spaceship Earth to golf ball instead of golf ball to Spaceship Earth) so when she set up the proportion and solved for the volume of Spaceship Earth it was incorrect. Sara had the correct scale factor but set up the proportion incorrectly by switching the numerator and denominator so the surface area result was incorrect, although she did recognize that the proportion was to be squared for area. Kenneth set up both parts of the problem correctly and earned full credit. On other probes with similar applications, students may have made either more errors (e.g., calculations, incorrectly set up proportions, not picking out relevant information from the written scenario) or fewer errors; hence, participants were inconsistent.

These examples from the NAEP and domain probes show that in comparison to the distinct lack of geometric understanding prior to the intervention, there were changes or growth. In particular, students used additive or basic numeric strategies prior to the intervention. After the intervention, students utilized proportions and multiplicative comparisons as well as definitions and theorems rather than relying on naïve visuals (e.g., saying it 'looks' like a figure is or is not similar).

MAM

Results from the mathematics attitude measure are provided in Table 8. There were seven questions (#4, 9, 12-16) most relevant to geometry conceptions. Regarding statements about the usefulness of geometry in future, seeing geometry in daily activities, geometry having multiple solution paths, hands-on problem solving or images being helpful to understanding, mean scores across participants were under a 3 (on a scale of 1-5) prior to the intervention, while after the intervention the mean was above a 3. The most improvement was regarding the perception that geometry problems have more than one solution method.

Overall, the responses showed a slightly more positive perception of the usefulness of geometry or aspects that enhance geometric thinking post-intervention. For example, images are important in much of geometry, and for this similarity unit in particular. Almost every domain or NAEP assessment included images or it would have been useful for solving the problems (in addition to the lesson activities). Prior to the intervention Khafila and Jason stated they did not make pictures, drawings or sketches to solve problems, while Kenneth and Sara did; however after the intervention Khafila and Jason stated they did a bit more, while Kenneth decreased slightly and Sara stayed the same. The work samples corroborated the statements by Khafila and Jason, as through the intervention they did make more sketches or outline with markers with decreased prompting. However, Sara rarely made sketches, even with prompting, which was in contrast to her statement that she often did. On the related question regarding hands-on

Table 8

Participants' Responses on Mathematics Attitude Measure

Questions	**Khafila**		**Jason**		**Kenneth**		**Sara**		**Mean gain/loss**
	Pre	*Post*	*Pre*	*Post*	*Pre*	*Post*	*Pre*	*Post*	
My mathematics teachers have encouraged me to do well.	2	4	3	5	3	3	5	5	1
I am confident when I use mathematics.	2	3	2	4	3	3	5	4	0.5
I am better at mathematics than other subjects.	1	1	1	2	1	2	3	5	1
I often make pictures, drawings, or sketches to help me figure out problems.	1	3	1	2	4	3	5	5	0.5
If I cannot do a problem, I keep trying different ideas and try to think of how it might be similar to other problems.	3	4	4	3	3	3	5	5	0
If I make mistakes, I work until I have corrected them.	4	3	4	2	3	3	5	4	-1
I would rather figure out the answer to a math problem on my own than have the teacher (or peer) tell me the answer.	3	4	3	1	3	3	3	4	0
Projects and hands-on on activities are better for me to understand a concept.	4	4	3	5	4	4	2	3	0.75

Questions	Khafila		Jason		Kenneth		Sara		Mean gain/loss
	Pre	*Post*	*Pre*	*Post*	*Pre*	*Post*	*Pre*	*Post*	
Most of the problems that my teachers use that are supposed to be real-world problems or applications *do* seem like something useful rather than fake, made up problems just for class.	2	4	2	1	3	3	3	3	0.25
I do not really get anxious or worried about coming to math class or taking tests.	4	2	2	5	4	3	4	4	0
I see geometry in everyday things.	1	2	1	3	3	4	2	2	1
Understanding and using geometry will be useful in my future.	4	2	1	5	3	3	3	3	0.5
Geometry problems often have more than one method/way to find a solution.	3	3	2	5	4	3	1	4	1.25
Geometry is easier for me to understand than Algebra.	3	2	1	2	2	4	1	1	0.5
Math, and especially geometry, helps develop a person's mind and teaches him/her to be a better thinker.	4	2	1	5	3	3	5	2	-0.25

**Note:* *1- Strongly Disagree, 2 – Disagree, 3- Neutral/In-between, 4 – Agree, 5 – Strongly Agree* activities while pre-intervention Sara was the only participant to indicate this was *not* a preferred learning style, all participants indicated a positive view toward the benefits of hands-on learning.

The work samples and the geometry related items from the MAM indicate that overall students had slightly negative views toward geometry and that their conceptions or understanding of the nature of geometry was limited prior to the intervention while post-intervention students had a slightly more positive view toward geometry and that their conceptions or understanding of the nature of geometry increased. Prior to the intervention for most of the pre-assessment problems on both the domain and NAEP probes students indicated they did not know how to solve them and did not attempt them and the problems that were attempted showed measurement, numerical or additive reasoning rather than proportional or geometric reasoning. In contrast, post-intervention participants attempted all problems and included aspects of geometric and proportional thinking.

Research Question 3: Connections and Disconnections

While the quantitative results give one indication of student understanding of the mathematical topics, work samples and conversations provide a richer context for conceptual understanding. Furthermore, given these students have history of difficulty with mathematics often with underlying anxiety; formal assessments alone may not adequately measure their understanding (Gagnon & Maccini, 2001; Maccini & Gagnon, 2000; Montague, et al., 1991). This section discusses the aspects of the intervention (e.g. representations) that potentially helped or hindered the conceptual understanding of the participants and student internal (or external) development of CCSS, NCTM and NRC proficiency standards such as for metacognition and includes exemplars.

In this intervention representations include concrete objects, virtual sketches or drawings (e.g., computerized), as well as student created sketches or drawings on paper. This includes physical images as well as the processes internally and thoughts expressed while engaged in doing mathematics (NCTM, 2000). The processes, as well as the products, are equally important to developing conceptual understanding via connections as well as disconnections. It is important not only for an instructor to address correct mathematical methods but also to explore when students are not making connections or making incorrect connections (i.e., disconnections).

CCSS (2010), NCTM (2000) and research with students who have difficulty with mathematics recommend multiple visual representations of mathematics concepts, specifically the concrete-representational-abstract graduated instructional sequence for Algebra (Maccini & Hughes, 2000; Maccini & Ruhl, 2000) and geometry (Cass et al., 2003). For example, in the instructional lessons utilized concrete manipulatives (i.e., AngLegs) and *Geometer's Sketchpad* for developing conceptual understanding of the definition of similar figures, and Triangle Similarity Theorems and the relationship between scale factor and area of similar figures. Prior to the intervention, none of the participants had experience with AngLegs or *Geometer's Sketchpad* so it was important to provide practice within the lessons, particularly for the software.

When using the AngLegs to explore concepts then transitioning to the representations on paper proved difficult for some participants. For example, when the images of the triangles were overlapping (and thus three-dimensional), the participants were easily able to see the angles that were congruent and shared between the triangles; however when the same situation was portrayed on paper the participant was not able to make the connection between the representations. Figure 8 shows what the AngLeg set up looks like with overlapping triangles, while Figure 9 displays two examples of a similar on paper for a problem where Khafila made the connection, while for a Sara there was a disconnect.

In this situation when the triangles are similar, the non-overlapping sides of the triangles are parallel and the corresponding angles are congruent. As shown in Figure 8, with AngLegs the overlapping angles are on the left, while in the paper images (as shown in Figure 9) this is angle A to the left in Khafila's problem and angle X at the top in Sara's image. If the triangles are similar then the side lengths are proportional. For the AngLeg example the proportion (and hence the scale factor) is 2. In Khafila's problem there is additional information given (i.e. the third sides are parallel) so she was able to figure out that the triangles were similar by AA, then made a correct similarity statement by ordering the corresponding sides and angles for the two triangles. Khafila was able to visualize the two triangles; otherwise, it is unlikely that she would have made a correct similarity statement.

In Sara's problem, the figures are not similar because the sides are not proportional. However, Sara did not make the connection that there were two triangles, ΔXZV and ΔXYW, although when using the AngLegs she readily was able to make comparisons between the corresponding angles (which in the problem posed would be $\angle ZXV \cong \angle YXW$, $\angle Z \cong \angle Y$, $\angle V \cong \angle W$) and the corresponding sides. It is of note that during instruction participants were shown how to use colored markers to break the

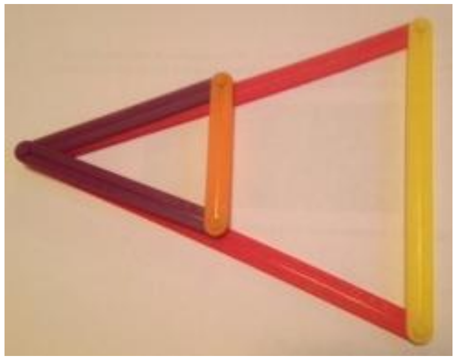
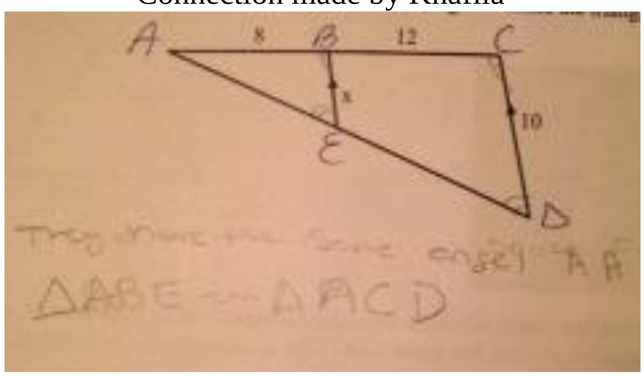


Figure 8. AngLeg representation of overlapping triangles

Domain Probe problem posed: Are the two triangles similar? Explain how you know and include relevant similarity theorem if appropriate. If they are similar, write the similarity statement. If they are not similar, what additional information is needed or what could be changed to make the triangles similar?

Connection made by Khafila



Disconnect by Sara

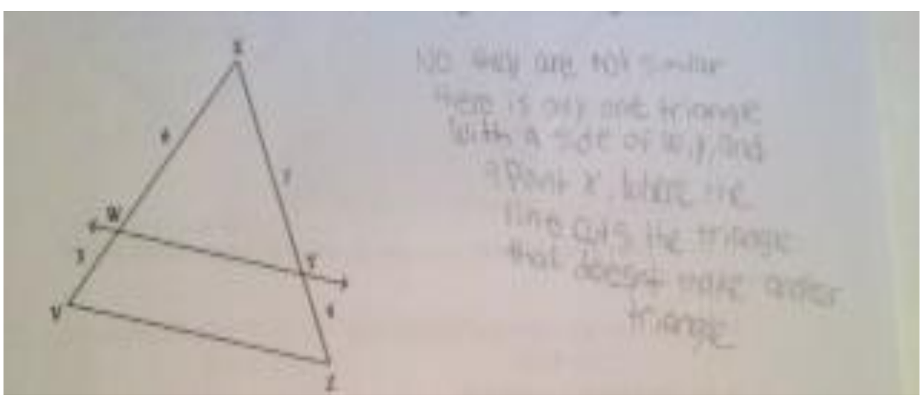


Figure 9. Connection and Disconnection concrete to pictorial representations

overlapping figures into the two distinct triangles in order to make the necessary comparisons and encouraged to do this independently. Sara was unable to visualize the separate triangles and did not attempt to outline even the outer triangle (which possibly could have helped her realize there was another “inner” triangle.

Although this is one example, when presented with a flat image, not all participants were consistent in visualizing two triangles, either on the probes or on the practice problems during the lessons. *Geometer’s Sketchpad* allowed for the manipulation of overlapping triangles similar to the AngLegs, but users are able to drag and separate the images or take two separate images and drag to overlap. This enables the user to discover that the corresponding angles are congruent without having to measure each angle, which is time consuming. Furthermore, the program allows for the calculation of perimeter and area of figures, even if the figures were not usual shapes with easy to use formulas (i.e., squares, rectangles, trapezoids) that one could calculate the perimeter or area by hand. If those images were only on paper (even with a Coordinate grid) it would be time consuming, and for some students, impossible calculate. See Figure 10 for an example showing the polygons before and after this manipulation along with the other measurements. Although all participants were able to manipulate the figures using the software to determine whether the figures were similar (i.e., corresponding angles congruent as well as measuring the side lengths and calculating the scale factor), this did not consistently translate into success on the practice problems or probes.

For example, Kenneth and Jason were both relatively successful and quick to figure out how to utilize the DGS to compare figures, get the program to provide side

Are polygons ABCDE and KLMNO similar? How do you know?

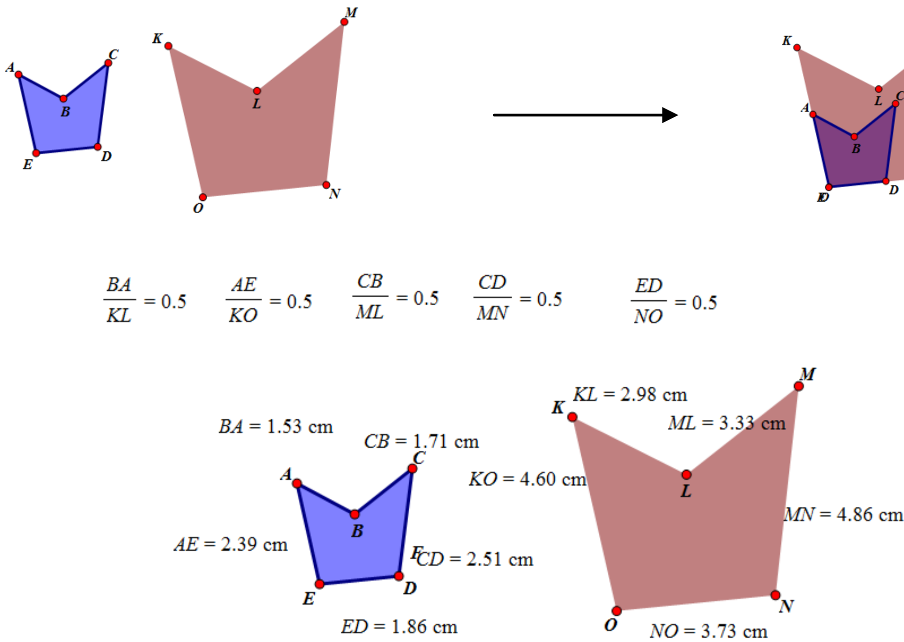
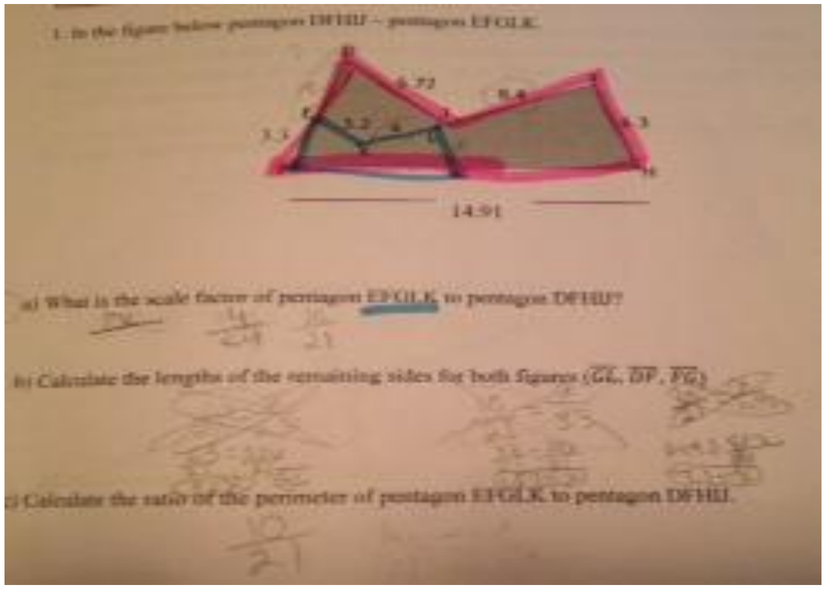


Figure 10. Geometer's Sketchpad example

measurements and ratios in order to determine if figures were similar. However, on a comparable problem on paper, even after guided practice, they were not consistently successful. In Figure 11 is a problem that Jason was able to be partially successful on, however Kenneth was not. The image was of overlapping, non-regular pentagons. While both were able to calculate the scale factor, each made errors. Kenneth used the calculator to change the fraction that had a decimal denominator to an equivalent fraction but the conversion was not correct. This made subsequent calculations incorrect. While, at first Jason did not compare the two figures properly (he flipped the scale factor and did not compare EFGLK to DFHIJ but had it reversed). He erased his original error and attempted to correct it after he used the markers to outline the figures. In part b, he set up the proportions and solved for the missing sides correctly twice but did so incorrectly

Jason connection



Kenneth disconnect

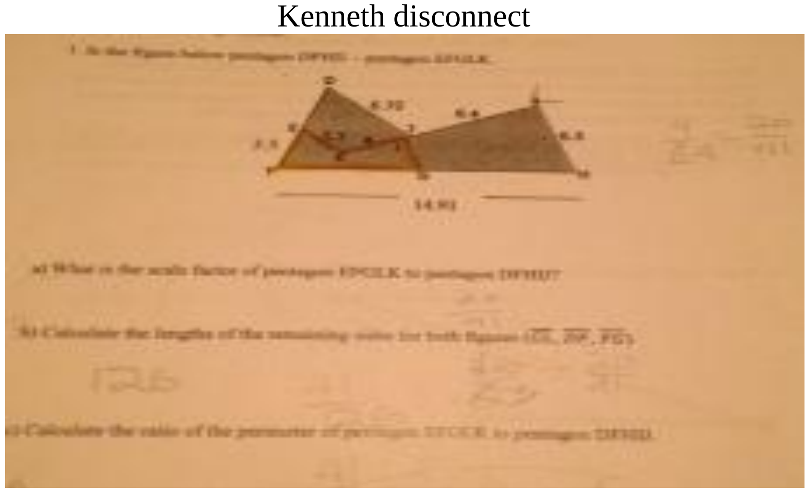


Figure 11. Jason and Kenneth Geometer's Sketchpad connections and disconnections

when solving for the third side because he reversed the digits representing the numerator and denominator, which resulted in an incorrect answer. Kenneth however did not set up the proportions necessary to solve the problem. He did note on the diagram that angle L corresponds with angle I but rather than setting up the proportion with the unknown (often designated with the variable *x*), he used L and then got stuck and did not know how to proceed with the cross multiplying. He stated that the image was “confusing” because angle L in the inner pentagon was close to angle J.

These inconsistencies in the written work did not necessarily agree with the statements made by the participants during the lessons or in their written feedback upon completion of the intervention. For example, Sara indicated she liked the computer program because she could ‘try it” out many ways but when asked what she like *least* about the intervention she stated using the AngLegs was “kind of easy to understand” but that a suggestion for improving the intervention was to use objects more. However, during the lesson activities she easily matched up the corresponding angles and sides she was not as consistent in doing so independently (e.g., on the probes) and she rarely would make notations on the figures (such as color-coding), whereas other participants would readily do so without prompting.

Research Question 4: Social Validity and Enhancing Metacognition, Self-efficacy, and Attitudes toward Geometry

This section provides results from the social validity measure as evidence of the students’ perceptions of the usefulness of the intervention as well as their disposition toward geometry. Relevant student oral or written commentary from the lessons and/or assessments supplemented the social validity and mathematics attitudes measures.

Social validity

The mean score from the social validity measure was 3.7 (range 1-5; mode = 4; see Table 9). Students agreed or strongly agreed that the intervention was worth their time and would be willing to use it for other geometry topics. Three students felt

Table 9

Participants' Responses on Social Validity Measure

Questions	**Khafila**	**Jason**	**Kenneth**	**Sara**	**Mean**
I learned to successfully solve geometric similarity problems.	3	4	3	4	3.5
The use of manipulatives helped me to solve geometric similarity problems.	4	3	4	5	4
The use of the dynamic geometry software helped me to solve geometric similarity problems.	4	3	3	3	3.25
The use of the cue card helped me to solve geometric similarity problems.	3	3	3	1	3.25
The word problem scenarios helped me understand the relationship between the concepts and the mathematical representations.	2	4	5	5	4
This intervention was worth my time.	4	4	4	4	4
I would recommend this intervention to other students.	4	3	3	4	3.5
As a result of the intervention, I feel better about my geometry skills.	3	5	3	3	3.5
I would be willing to use this intervention for additional math topics.	4	5	4	4	4.25

**Note: 1- Strongly Disagree, 2 – Disagree, 3- Neutral/In-between, 4 – Agree, 5 – Strongly Agree*

manipulatives were helpful in solving problems and the contextualized problems were helpful in understanding the topic. Students' feelings were mixed regarding the other aspects of the instruction. For example, Sara did not feel the cue card was helpful and the others were neutral. However, Sara did not seem to know how to utilize the cue card as she attempted to rely on her memory unnecessarily when solving problems and required prompting to refer to the cue card during instruction. Other participants occasionally required prompting to utilize the cue card.

Most participants responded positively to the open-ended questions from the social validity measure. However, Jason indicated that geometry is “boring” although the intervention “helped him learn” the topics and he liked being able to “get the hang of” geometry before he would take the course next school year. The remaining students were more specific with their comments on the usefulness of the topics taught and the instructional materials. Kenneth liked “computer and the shapes” because it helped him understand the material and he felt that geometry is “easier than other math” such as algebra. Khafila liked the AngLegs because being able to snap them together helped her “with measuring and comparing” similar figures, while Sara liked the computer program because you could “try moving the shapes and understand the proportions.”

The three students who provided the most detail indicated the computer program helped them visualize the shapes and it was quicker to understand concepts, such as the relationship between scale factor, perimeter, and area because the computer could calculate those measurements faster than they could have done if the information was on a worksheet and they had to figure it out (for example, using composite area). All students disliked the requirement that they complete so many probes (due to the single subject design) and felt that focusing on either the hands-on manipulatives (i.e., AngLegs) or the computer program alone would have helped them understand the topic.

Student attitudes

Metacognition, self-efficacy, and attitudes toward a subject are interrelated concepts. Metacognition is a complex process involving an individual’s knowledge and control of his or her own cognitive processes while learning (Flavell, 1979). Self-efficacy, often associated with self-esteem, is confidence in one's abilities in one or more domains, summed up as a 'can do' attitude (Schwarzer, Bassler, Kwiatek, Schroder, & Zhang, 1997). Whether a person is aware of their thought processes when working through complex and challenging problems and is able to feel positive in their abilities greatly impacts their perceptions and ultimately, in this study, whether or not the person likes geometry in particular or mathematics in general.

Overall, the students had generally positive attitudes toward mathematics instruction and their abilities prior to the intervention and there was a slight increase in their perception post-intervention, however there was some variability (see Table 8). Regarding confidence in their ability, Khafila and Jason had slightly negative attitudes before the intervention but improved slightly afterwards, while Kenneth remained ambivalent, and Sara was more slightly more positive before the intervention. One reason for Sara's negative belief might be that she learned that she had failed her regular mathematics course during this intervention, a fact that could have influenced her final responses; consequently, an unfortunate validity threat.

However, when comparing mathematics performance to experiences in other subjects it is interesting that in spite of having failed her math course for the year, Sara continued to indicate she felt she performed better in math than in other subjects and felt more strongly that this was the case after the intervention, while all other participants felt more strongly that they performed better in subjects other than mathematics. In addition, given some research indicates that students with a history of difficulty in mathematics may have issues with anxiety (Ashcraft, 2002); this was the case only with

Jason prior to the intervention (based on his survey response as well as teacher observation/input and educational history); however, after the intervention he indicated that he felt much less anxious. Conversely, Khafila indicated the opposite. However, based on discussion with her this did not necessarily seem to be the case; as she was doing well in her current math course, although it was her third time in the course, but she did need to take geometry the next year (grade 12) in order to graduate on time. So, it is possible that this influenced her thoughts at the end of the study.

Regarding specific questions related to geometry, there was overall improvement in student thoughts about the usefulness of geometry and their abilities. For example there was an average of one level increase in participants seeing geometry as useful in everyday life and that there is more than one way to solve problems. Most students were neutral in their seeing geometry as important to their future, however in conversation they indicated that they had to take and pass Geometry to graduate but that for a career they did not think that math (and geometry in particular) would be a major factor.

Summary

In summary, students improved their performance on geometric similarity utilizing explicit instruction with CRA and DGS and retained that knowledge approximately 4 weeks after the intervention. Prior studies have shown that CRA improves students' performance in algebra (e.g., Maccini & Ruhl, 2000; Maccini & Hughes, 2000; Witzel, 2005; Witzel et al., 2003) and in geometric measurement (i.e., perimeter and area; Cass et al., 2003; Satsangi & Bouck, 2015). Furthermore, the use of DGS has been shown to be effective for general education students in understanding geometric transformations, (Choi-Koh, 1999; Guven, 2012; Hollebrands, 2003, 2007). In this study, the quantitative results and positive student comments support the prior research on the benefits of CRA for students with MD and extend the research on DGS for use with students with MD. Additionally, student comments and responses to the social validity and mathematics attitude measures indicate that the use of both CRA and DGS may have improved their disposition toward mathematics and geometry in particular.

Chapter 5: Discussion

The purpose of this study was to investigate the effectiveness of integrated instruction including explicit instruction, representations and DGS on the ability of students with MD to solve geometric similarity transformations. Overall, the intervention successfully helped participants develop an improved understanding of geometric similarity transformations including using proportions for solving contextualized and non-contextualized linear, 2D, and 3D problems, and applying triangle similarity theorems. Furthermore, participants maintained these skills 4-6 weeks after the intervention ended. This chapter begins with a summary of the research findings and a discussion of the importance of the findings in relation to prior literature. Next, I interpret my findings. Finally, I conclude with study limitations and implications for research and practice.

Summary of the Results

Since the groundbreaking publication of the National Council of Teachers of Mathematics (NCTM) *Curriculum and Evaluation Standards for School Mathematics* in 1989, there have been 11 published studies examining geometric skills for secondary students with MD. Additionally, none of those studies solely focused on the critical topic of similarity transformations. The studies focused on lower level geometry skills such as measurement, area and perimeter (Cass et al, 2003; Cihak & Bowlin, 2009; Satsangi & Bouck, 2015) or geometry skills were a limited portion of the intervention with Algebra the primary focus (Bottge, 1999, Bottge & Hasselbring, 1993; Bottge et al., 2001, 2002, 2003, 2007, 2009; Jitendra et al, 2009). These studies did not examine student performance on more advanced geometry topics as advocated for by NMAP (2008), Achieve (2004), and CCSS (2010). Furthermore, only four studies included high school grade students (Bottge & Hasselbring, 1993; Cass et al, 2003; Cihak & Bowlin, 2009; Satsangi & Bouck, 2015). Additionally, with the increased focus on integration of technology (e.g., CCSS, NCTM) within instruction and the advent of specific geometry software along with the dearth of geometric similarity transformations research for students with MD it was prudent to include general education research utilizing technology for this specific topic as only one study included high school students with MD that utilized technology, but not with geometric similarity transformations (Satsangi & Bouck, 2015).

This expansion of the literature base to include general education studies resulted in an additional nine studies for the literature review that focused on geometric transformations. Six of those studies incorporated technology (Choi-Koh, 2001; Guven, 2012; Hollebrands 2003, 2007; Hungwe, et al, 2007; Kirby & Boulter, 1999) while only three studies included high school age participants (Gorgorio, 1998; Hollebrands 2003, 2007). The current study was designed to address this research gap for high school students with MD utilizing a package of research-based instructional practices within upper secondary geometry content (i.e., geometric similarity transformations) including technology (i.e., *Geometer's Sketchpad*) as a primary component of instruction.

An extensive review of the current literature led to the development of an instructional package that incorporated the following research-based practices for general education students as well as for students with MD. These strategies included: (a) explicit instruction (Cass et al., 2003; Cihak & Bowlin, 2009; Jitendra et al., 2009; Satsangi & Bouck, 2015); (b) CRA graduated instructional sequence (Cass et al., 2003; Satsangi &

Bouck, 2015) and technology (Bottge, 1999; Bottge & Hasselbring, 1993; Bottge et al, 2001, 2002, 2003, 2007, 2009; Choi-Koh, 1999; Cihak & Bowlin, 2009; Guven, 2012; Hollebrands, 2003, 2007; Hungwe et al, 2007; Kirby & Boulter, 1999; Satsangi & Bouck, 2015). As recommended by NMAP (2008), NCTM (2000) NRC (2001) and CCSS (2010) students had the opportunity to engage in activities that required them to problem solve, justify solution methods, reason, make connections between topics (i.e., algebra and geometry; mathematics and real world), utilize models and representations, and use tools appropriately and strategically. The current study incorporated research-supported practices to address the advanced geometry topic of similarity transformations.

Emphasis on Geometric Similarity Transformations

This study is the first to focus on the topic of similarity transformations with high school level students with MD. While students with MD have been involved in single-subject or group design studies that included geometry skills, either the focus was on lower level geometry skills (e.g., area and perimeter) or the primary focus was on computation with minimal attention to transformations related topics (e.g., proportions, scale factor), therefore not appropriate for high school students. While the CCSS middle school curriculum addresses the topic of transformations, the focus is on rigid or isometric transformations and dilations, not the integration with similarity as found in the major reform organizations and state curriculum. The general education literature did include studies that primarily focused on transformations; however, only two studies (Choi-Koh, 1999; Hollebrands, 2007) addressed the topic of similarity. In contrast to the special education literature, the participants were on or above the expected grade level for the topic, rather than below grade level. Unfortunately, general education studies do not typically include information regarding the disability status of students and neither of those studies included information about participant MD status.

Research bridging the gap between special education and general education is critical in developing effective strategies for students with MD on grade-level mathematics topics, such as similarity transformations in high school. Nationally, more than 60% of students with diagnosed disabilities spend 80% or more of their time in general education (NCES, 2011). Furthermore, there are additional students at-risk for mathematics failure, low achieving or underperforming but not formally diagnosed with any disability (Mazzocco, 2007), and therefore in general education so the need for instructional strategies that address this is of the utmost importance.

Use of Cue Card

Cue cards or strategy checklists have been found to be beneficial for students with MD in various mathematics topics such as problem solving and pre-algebra (Butler et al., 2003; Jitendra et al., 2009; Maccini & Gagnon, 2000) as well as geometric measurement (Mulcahy & Krezmien, 2009). In this study, the use of the cue card was effective for three of the four participants, as they referred to it primarily for the similarity theorems and formulas needed for applications to 2D and 3D scenarios, with minimal prompting during instruction and utilized it regularly on the post-intervention probes. However, although no formal collection of frequency data on the use of the cards occurred, on the social validity measure three participants were neutral on their opinion about the usefulness of the cue card. The one participant who did not find it useful required multiple prompts during instruction to refer to it rather than relying on her recall-which teacher input as well as formal assessment information indicated was an area of weakness.

Use of Multiple Visual Representations

The instructional package implemented in this study included the use of visual representations of similarity of polygons and polyhedrons. The visuals included concrete manipulatives (e.g., solids such as rectangular prisms, AngLegs), virtual manipulatives (i.e., images created using the DGS) and sketches or 2D representations (e.g., various pictures and sketches of shapes such as triangles). The use of manipulatives, concrete or virtual (i.e., computer generated), has proven an effective strategy for both special education (Gersten et al., 2009) and general education students (NRC, 2001) and encourages students to model and use tools strategically as prescribed by the CCSS (2010). Recent studies identified the use of representations as effective for teaching geometry to students with and without MD. Specifically, Cass et al. (2003) utilized concrete manipulative and the CSA sequence, while Satsangi and Bouck (2015) utilized virtual manipulatives with the CSA sequence to teach area and perimeter to students with MD. Virtual manipulatives, specifically DGS, has been found to be effective for teaching upper secondary content such as similarity transformations (Choi-Koh, 1999; Hollebrands, 2007). The current study joined the use of concrete manipulatives with virtual manipulatives in similarity transformations for students with MD.

Use of Technology

Both NCTM (2000) and CCSS (2010) support the use of technology for increasing student's conceptual understanding. Prior research has shown the use of technology, particularly computerized programs or applications, to be effective for general education at all levels (e.g., Li & Ma, 2010) and students with MD especially for computation (e.g., Mastropieri et al, 1995; Cihak & Bowlin, 2009; Satsangi & Bouck, 2015). Additionally, technology such as games, tutorials and simulations that allow for adjustments based on individual needs can enhance both conceptual and procedural knowledge as well as motivation for students with MD (e.g., Hughes & Maccini, 1997; Maccini et al., 2007). In geometry, DGS such as *Geometer's Sketchpad*, has been shown to be effective for both procedural and conceptual understanding for general education students, particularly for transformational geometry topics (e.g., Choi-Koh, 1999; Guven, 2012; Hollebrands, 2003; 2007). This study is unique because it investigated the use of dynamic geometry software for students with MD in high school on grade-level content, which had not previously been investigated.

Use of Integrated Instruction

The current study investigated the effects of integrated instruction with visual representations, including technology via DGS, on participants' accuracy when solving geometric similarity transformations. The core instructional components of explicit instruction (i.e., advance organizer, investigations, practice, task sequencing; Hudson & Miller, 2006; Scheuermann et al., 2009), with CCSS (2010) standards for mathematical practice and NCTM (2000) process standards, address both procedural fluency and conceptual understanding. This integrating of instructional strategies from general and special education research is critical in helping student with MD gain meaningful experiences within grade level content (Hitchcock, Meyer, Rose, & Jackson, 2002).

Interpretation of Findings

This study utilized a multiple probe design across four participants to answer Research Question 1. For Research Question 2, item analysis of domain probes and a standardized pre-post test supplemented by geometric specific questions from the MAM was used to compare pre-intervention conceptions of geometry with post-intervention conceptions of geometry. Research Question 3 regarding connections and disconnections was answered using responses to the social validity and mathematical attitudes measures combined with work samples. Lastly, surveys of the participants were used to gather social validity data and mathematical disposition to answer question four.

Research Question 1: accuracy and maintenance on geometric similarity tasks.

The effectiveness of the intervention was measured by the domain probes pre-to-post intervention as well as the NAEP pre and post-assessments, which is also a generalization measure to some extent, while the maintenance measure was a domain probe administered 4 weeks after the conclusion of the intervention. The results are discussed in that order: domain, NAEP, maintenance. Error analysis, work samples, and other factors that may enhance or impede performance are discussed.

Domain probes. The effectiveness of this intervention on the geometric similarity accuracy is evident by the change in level demonstrated by each participant from baseline phase probes to post-intervention assessment (see Figure 5). All participants scored well below the inclusionary criterion on baseline, pre-intervention domain probes. Participant improvement from baseline to post-intervention was 58-86 percentage points (average 69.6). Specifically the baseline scores ranged from 0% to 11% (average 2.3%), while the post-assessment scores ranged from 46% to 96% (average 71.8%). Individually post-intervention averages were as follows: Khafila 63%, Jason 75%, Kenneth 86%, and Sara 64%. These scores are above the school system minimum required passing score of 60%, and therefore show minimal to above average mastery of the topic.

The baseline condition for each participant was stable, relatively flat, with little to no variability (i.e., less than 10%) prior to entering intervention. The large change in level demonstrated from pre-intervention (i.e., baseline) to post-intervention suggests that the instructional package, including explicit instruction and multiple visual representations with DGS, positively affected students' performance as they solved geometric similarity problems. Furthermore, this effect was evident across four different participants at different points in time. This replication of the findings across participants demonstrated experimental control and generalization while the between-phase patterns showing an immediate increase in scores suggests a functional relation between the independent and the dependent variables (Kennedy, 2005; Kratochwill et al., 2013).

However, there was some variability in performance on the post-intervention probes that could have been addressed differently (and are addressed in the limitations and future research section). Khafila was provided booster sessions due to unanticipated scheduling interruptions, which may have affected her results. Since she was the first participant I made a decision, when school was twice missed due to multiple snow days, to provide booster sessions. I did not include additional post-intervention probes for several reasons, that she would have taken additional probes that she had already taken during baseline (as there were eleven parallel probes and it was planned that she would take 10), but primarily to address testing fatigue. No other participants had a similar situation, but the results were similar for Jason and Kenneth, with a dip in scores then improvement (i.e., a zigzag pattern in the graph).

The data on student's ability to solve problems related to geometric similarity transformation aligned with previous research in which students with MD demonstrated gains after explicit instruction (Cass et al., 2003; Cihak & Bowlin, 2009; Jitendra et al., 2009; Satsangi & Bouck, 2015) with the CRA sequence (Cass et al., 2003, Satsangi & Bouck, 2015) and technology (Bottge, 1999; Bottge & Hasselbring, 1993; Bottge, et al., 2001, 2002, 2003, 2007, 2009; Cihak & Bowlin, 2009; Satsangi & Bouck, 2015). Furthermore, while the use of DGS had not been explored with upper secondary students with MD, prior research has shown the benefits of DGS, which is both technology and visual representations, for high school students (Choi-Koh, 1999; Guven, 2012; Hollebrands, 2003, 2007). Furthermore, the results were similar to the results in the three single subject design studies with students with MD (Cass et al., 2003; Cihak & Bowlin, 2009; Satsangi & Bouck, 2015) in which there was a distinct change in level from baseline phase to intervention, post-intervention, maintenance and/or generalization.

An error analysis revealed that participants missed points primarily because they did not answer all parts of a question (e.g., did not explain/justify, did not calculate all missing parts of a figure) or set up proportions incorrectly (e.g., flipped one of the ratios so that the parts of a figure were not aligned). All participants had one or more of these errors on each probe. First, Kenneth often either did not include justification or explanations or his reasoning was incomplete-leaving the evaluator to infer what he meant. During the lessons, he did not want to write and would ask if it was necessary to put such justifications in writing. However, accommodating such requests on the probes would have violated posttest protocols, and such incomplete responses meant a loss of credit. This inattention to detail (e.g., not reading all the information clearly and skipping steps), rushing to finish without being thorough, and lack of self-monitoring is characteristic of ADHD (Daley & Birchwood, 2010). Furthermore, Kenneth, and to some extent the other participants would ask “I did it on the calculator, why do I have to write it down?) or state “I got the answer, why do I need to check (redo)?” These responses are common with students with MD who may struggle with evaluating reasonableness and/or accuracy of their solutions (Miller & Mercer, 1997).

Second, all students were inconsistent in setting up proportions to answer the questions, particularly when embedded in a contextualized or real-world context with sentences (e.g., the Spaceship Earth problem) rather than a simpler, more procedural context (e.g., Figure 7 or Figure 9). Difficulties with proportional reasoning for all students has been well documented in the literature (e.g., Chazan, 1987; Lesch et al, 1989), as are applications with word problems (De Bock et al., 2002). Furthermore, students with MD have difficulty with word problems (e.g., Maccini & Ruhl, 2000; Maccini & Hughes, 2000) due to the variety of reasons including language deficits (Garnett, 1998), organization and sequencing (Steele, 2010) or working memory and processing (Passolunghi et al., 2004).

NAEP. Similar errors to those on the domain probes were also apparent on the NAEP, such as not providing sufficient detail/explanation or not setting up proportions properly. For example, there was a question that asked students to decide which two pairs of figures must be similar (5 choices) then explain their reasoning. Two of the participants provided a reason; Jason chose the correct answer (two equilateral triangles) and provided an appropriate reason (AA triangle similarity theorem), Khafila picked the wrong answer (two right triangles) and provided a naïve definition of similarity (same shape) while Sara chose the correct answer without an appropriate reason but Kenneth did not choose the correct answer or provide an appropriate reason.

Incorrect proportions were more problematic on the problems with overlapping or embedded triangles (see Figure 9) than on problems with separate shapes to compare (see Figure 7). There were two problems with overlapping triangles and three of the participants did not get either question correct, while Jason got one correct. It is notable that none of the participants utilized the highlighters to trace and match up the corresponding parts in order to assist in their solution methods, even though those tools were on the desk available to use if they chose. This was in contrast to their regular use on the domain probes for Jason and Khafila. Furthermore, during instruction Kenneth and Sara needed prompting to either outline the figures or break the figures into images of the two triangles in order to match up the corresponding parts. This difficulty with visuo-spatial skills is well documented in the special education literature (e.g., Garnett, 1998; Geary, 2004) as well as general education research (e.g., Lean & Clements, 1981).

There were other factors that potentially influenced student performance on this measure, particularly for the post-test. Kenneth and Sara participated more closely to the end of the year than the other two participants, which was problematic. This was because three mandatory state assessment windows happened during the spring, which disrupted our ability to work together. In addition, for those students involved in the state testing, this was an element of potential anxiety (and mentioned by at least one participant as an area of concern). For example, on some occasions when we met Kenneth indicated he had many other things he needed to work on for his other courses (e.g., tests or projects) and sometimes he stated that rather than rescheduling for another day he wanted to push through so that we could finish. This was particularly evident when he was taking the post-intervention domain probes because he asked each time we met how many more (tests) he needed to take and could he just do them all and “get it over with.” Since the NAEP-based assessment was the last of the probes, he likely was influenced by this testing fatigue and his stress over getting other things accomplished. Likewise, Sara took the NAEP last and this coincided with another heavy testing time.

In contrast to the poor performance on the post intervention NAEP for Khafila, Kenneth and Sara, Jason made remarkable improvement from the pre-intervention probe. He scored 10% on the initial probe but increased to 70% on the post-assessment. I am not sure what contributed to his improvement while the others did not. For the most part the questions were similar to the domain probes, requiring setting up proportions to solve for missing information with and without context. However, when considering his approach to mathematics in general and his work ethic he seemed to benefit the most from his positive performance which spurred him to be energetic and willing to persevere. During the lessons when he was correct or confident in his ability to solve the problems he would smile and say things like “ I got this!” and set to work on the next problem with relish. As time went on he was less discouraged when he did not understand or know how to proceed and would re-read the problems and at least make an attempt, rather than shutting down-as indicated in his educational reports. Perhaps this carried over to the NAEP measure in addition to the domain probes. While the other participants were not necessarily discouraged, their personalities were, perhaps, more reserved than Jason's and I simply did not see evidence of their positive attitude in this manner.

Furthermore two of the NAEP items required applications of measurement that were not directly addressed within the intervention. For example, there were two questions that required students to measure a figure with a ruler and utilize the scale factor then apply it to a real-world application (buying tile to cover a floor). On the post-test Jason and Kenneth stated they did not know how to complete the problem (same as stated on the pre-test). Khafila attempted the problem but did not appear to know how to complete it properly as she divided the figure into squares (unevenly) and counted them rather than using a scale factor. Sara was the closest to solving the problem as she multiplied the lengths of the sides (to get 10) then multiplied by the scale factor (4) to get 40. However, this did not take into account the scale factor should have been multiplied by both the side lengths nor did she consider the information given that each box of tile covered 25 feet. This was in contrast to the pre-test in which she had calculated 6.5 boxes were needed but did not attend to the directions to round to the nearest whole number of boxes. This inability to transfer skills to novel situations as well as to read and incorporate relevant aspects to solve word-problems is well documented (e.g. Jitendra & Xin, 1997).

Lastly, while the NAEP assessment was sampled from publicly released questions from the topic of similarity, most of the questions were multiple choice rather than open-ended, in contrast to the domain probe questions. It is possible that some students did not attempt the problems in the same manner as some students did not attempt to set up proportions to solve problems that were presented similarly to those in the domain probes for which they DID set up proportions to solve. In a few instances there was no attempt at any work and it appeared the student guessed; even on questions that were multiple choice but another portion requested an explanation none was given. This difference in performance on multiple choice versus open-ended assessments, particularly for students with academic difficulties, including MD, is consistent with prior literature (e.g., Abedi, Leon, & Kao, 2008).

Maintenance. All of the participants demonstrated maintenance on geometric similarity tasks four-to-six weeks after conclusion of the intervention. The average score on the maintenance probe was 67% with a range of 62%-71%. This is especially important given that students with MD may have difficulty with memory and retention of learning (Passolunghi & Siegal, 2004; Swanson & Beebe-Frankenberger, 2004) and based on the history of these participants the long-term retention of learning, particularly in mathematics, has been an area of weakness. These finding are consistent with research in which students with MD maintained an increase in performance 2-6 weeks post-intervention that resulted from explicit/systematic instruction (Cass et al., 2003; Cihak & Bowlin, 2009; Satsangi & Bouck, 2015), the CRA sequence (Cass et al., 2003), and technology (Bottge et al, 2007; Cihak & Bowlin, 2009; Satsangi & Bouck, 2015).

Research Question 2: Geometric similarity conceptions pre- and post-intervention

The primary measure related to the question was the NAEP-based pre-test and the baseline domain probes, but some information from the mathematics attitude measure also is relevant to their definitions of what constitutes geometry and their feelings toward mathematics, and geometry in particular. All participants had limited knowledge about geometric similarity pre-intervention and few agreed or strongly agreed with statements indicating positive conceptions of geometry. However, post-intervention the results were mixed with a lot of improvement on domain probes (range 58%-86%), and positive changes in the MAM, but less improvement in performance on the NAEP (range 0%-60%; average 23%). Taken together these measures indicate that participants improved their conceptions of geometry from pre-to post intervention. More detailed information and discussion regarding why there was or was not growth are discussed in relation to student characteristics that enhance or impede their understanding.

Probes. Much of the unit on similarity relied heavily on proportional reasoning and specifically geometric proportional reasoning, which has been well established as an area of difficulty for many students (e.g., Cox, 2013; Hart, 1984). Students needed to be able to manipulate the images to compare corresponding parts, set up appropriate proportions and solve for missing information, compare complex figures and apply theorems to determine if figures were similar, and apply proportional reasoning to multidimensional scenarios. Regarding the NAEP probes Jason improved most, 10% pre-intervention to 70% post-intervention, while the other participants' scores remained low. On the pre-test, most students indicated they did not know how to complete problems, and even if there were multiple-choice options it appears they guessed, as there was no work to support the answer chosen. In contrast, on the post-assessment, all students attempted to set up proportions for the majority of the problems, but most of them were incorrectly set up such that the corresponding parts were not in the corresponding places in the proportions. Kenneth provided explanations that are more detailed or showed more work than any other participants, even for questions that were incorrect. For example, Kenneth utilized measurement or additive strategies for making comparisons rather than multiplicative relationships such as ratios or proportions (see Figure 6). In addition, two questions required measuring (a drawing of an irregularly shaped room) and then utilizing a scale factor to determine the lengths or to determine the amount of tile to purchase. Although those skills (with the exception of measuring with a ruler) were addressed in the intervention there are several reasons students may not have performed as well, which were already addressed under research question one (such as this was the last probe-other than the maintenance, the multiple choice format in contrast to the open-ended domain probes).

In contrast to the lack of improvement on the NAEP students improved rapidly and distinctly on the domain probes. As opposed to additive strategies or no response on the pre-intervention probes, on the post-intervention probes students attempted all problems in whole or part. While students may have been inconsistent in setting up proportions, this was a marked improvement from the pre-intervention probe responses. Students moved from more naïve use of visuals to more geometric proportional reasoning, using not only between ratios (see Figure 7 Jason's example) but within ratios (see Figure 9 Khafila's example). Students also moved from not knowing what similarity meant in the context of geometry, as evidenced by not being able to answer any pre-intervention problems asking for scale factor and solving for missing information in a figure or statements in the initial lessons such as 'similar means sort of alike' to not only stating that similar figures have "corresponding angles that are congruent and corresponding side lengths that are proportional: but making comparisons between figures to check that all those conditions are met in order to conclude that figures are similar.

MAM. In general, students did not view making sketches or drawings as important prior to the intervention, but did post-intervention (average gain across participants = 0.5). This is important because many topics in geometry involve images and manipulations of them, either on paper or mentally, and this is a topic that many students struggle with (Lean & Clements, 1981). Similarly, given the need to manipulate images hands-on materials or activities might be important, it is encouraging that most students felt that hands-on activities helped them understand concepts and two students felt this more so after the intervention (average gain 0.75). On the questions related to seeing geometry in everyday things, the usefulness of geometry, geometry having more than one method to solve problems there was a mean gain from pre-intervention to post-intervention of .5-1.25. These responses indicate a better understanding of the purpose of geometry post-intervention. In contrast the final question about geometry helping develop better thinking, there was a slight decrease from pre-to post-intervention. There is some research relating student's attitudes and beliefs, specifically regarding geometry, to achievement however, there are many additional factors, such as teaching methods that may also influence achievement (Burstein, 1980; Mogari, 1994). Furthermore, given that the mathematics instruction for most students has been focused on algebraic skills for at least the last year (and for three years in the case of Khafila) and the algebra course is not integrated with geometry, it is not surprising that they had limited conceptions of geometry prior to the intervention.

Overall these items along with the work samples from the domain and NAEP probes show the growth in participant understanding of the geometric concepts, in particular the importance of proportional reasoning rather than additive. Proportional reasoning is an area of difficulty for typically developing students (e.g., Hart, 1984) as well as students with MD (e.g., Jitendra et al, 2009). Furthermore, the specific mathematical terminology that has a different meaning than used outside of mathematics is also an area of difficulty for students (e.g., Vollrath, 1971) and language is an area that students with MD may have difficulty that could further impede their understanding (e.g. Garnett, 1998). The inconsistent performance of students on the NAEP and domain probes could be explained in part to difficulties with the abstract vocabulary and the proportional reasoning, as well as the visuo-spatial skills required with many of the problems.

Research Question 3: Connections and Disconnections.

A core component of the intervention was the use of the CRA sequence, which included concrete manipulatives (e.g., AngLegs) but also virtual manipulatives via the DGS program, *Geometer's Sketchpad.* It was interesting that all students showed inconsistencies in their ability to transfer their success when manipulating and comparing with both the concrete and virtual manipulatives.

This connection or disconnection was evident using the AngLegs more when the triangles were overlapping (see Figures 7 and 8). When presented with the AngLegs students often made the connections to their prior knowledge and the concept of similarity, using the manipulative tools as a bridge to new information. Being able to build on existing knowledge is a critical component of explicit instruction (Hudson & Miller, 2006). After developing naïve 'working definitions' of similarity the AngLegs were used a lot when exploring the Triangle Similarity Theorems to determine which were 'short-cuts' and would mean that rather than requiring the students to compare all the angles and calculating all the side proportions, that this would save time. Furthermore, the midsegment and Triangle Proportionality Theorems always include the added element of the overlapping triangles. However, any of the other theorems might apply to a given overlapping image depending on what information was provided on the diagram.

In this way, the concrete manipulatives also served as an anchor to the more abstract but still representational-images on paper. When provided with the AngLegs, they easily were able to compare the corresponding angles and sides, then use those to make conjectures and to solve for missing parts (i.e. although the AngLegs have specific side lengths marked I would ask the students "well what if this side were…and the scale factor was…what would …side be on this other triangle"). However, this skill did not consistently transfer to the paper representations. Furthermore, students were provided with highlighters they could use whenever they chose students were inconsistent in doing so. When comparing student responses, when they chose to use the highlighters they were more successful in matching up the corresponding parts of figures in order to then set up a proportion.

Similar connections and disconnections were seen with the virtual manipulatives in the DGS. The DGS program allowed for the creation and manipulation of figures with more than three sides. This allowed for not only exploration to discover the definition of similar figures in the earlier lessons but for calculating the perimeter and area of various non-regular shapes for students to discover the relationship between the scale factor with perimeter and area. Utilizing the program students quickly made connections and generalizations, however as with the AngLegs, this did not consistently translate to being able to apply this to the paper-based tasks (see Figure 11).

These findings are consistent with research that students with MD have difficulty with visuo-spatial processing (e.g., being able to separate the overlapping figures into two different triangles; Steele, 2010), abstract reasoning (Miller & Mercer, 1997) as well as the strategic use of tools (e.g., highlighters; Bottge et al., 2007). Although students were not always successful in making the connections, all students showed some growth in their ability to make connections (as seen most readily in their growth on the domain probes from pre-to post-test), which is consistent with research on the CRA sequence, including concrete and virtual manipulatives, that has been shown to be beneficial for students with MD (Cass et al., 2003; Satsangi & Bouck, 2015).

Research Question 4: Social Validity and Enhancing Metacognition, Self-efficacy, and Attitudes toward Geometry

This section analyzes results from the social validity measure as well as information on their disposition toward the subject. The social validity aspects are presented first, followed by the mathematics disposition. Lastly, any similarity or differences with respect to prior literature are compared.

Social Validity. At the conclusion of the intervention participants completed a questionnaire. The measure consisted of two parts: (a) a 5-point Likert scale, and (b) open-ended questions. All participants reported that the intervention was worth their time and would be willing to use it for additional math topics. Participants were generally positive about successfully solving similarity problems as well as using manipulatives, both concrete and virtual, which is consistent with previous research (Maccini & Ruhl, 2000; Satsangi & Bouck, 2015). However, some of the comments were vague (e.g., “geometry is boring”) and had I asked them more specific questions about how the intervention instruction compared to instruction in their courses I may have received more detailed information. Results were mixed regarding the use of a cue card, as Sara strongly disagreed. However, she continually needed to be shown what information on the cue card would apply to a given problem, while the other students used it independently. This may be a reason why she did not do as well on the post-assessments as the others, in particular Jason and Kenneth. The use of a cue card has been shown to be beneficial for students with MD (Maccini et al., 2008), and at least for those participants in the current study that may have been true.

MAM. Prior to and after the intervention participants complete a questionnaire. The measure consisted of 16 questions using a 5-point Likert scale. Before the intervention, the mean score was 2.8, indicating a slightly negative attitude toward mathematics, while after the intervention the average was 3.2 indicating a neutral to slightly positive attitude toward mathematics with the same results for the seven questions most directly related to geometry. Results were mixed and including comments and work samples enhances the information provided by students, as some research has shown that students’ self-reporting of ability is not as accurate as independent observers or other measures (Tousignant & DesMarchais, 2002). For example, Sara stated she often makes pictures or drawings, but based on her work during the lessons and on the assessments this was rarely the case. While only Jason indicated anxiety as an issue prior to the intervention, and this was consistent with his history, Kenneth stated orally that he has some issues with anxiety but did not note this on the instrument. After the intervention Jason noted that math did not make him anxious or worried, which may be due to his time spent with the researcher, but also may be due to his experiences with his course teachers, as he also noted a change from pre-to post-intervention regarding his math teachers encouraging him to do well. This is important considering prior research indicating that math anxiety can impede the performance and disposition toward mathematics of students with MD (Ashcraft, 2002). Additionally, students indicated if they make mistakes, they work to correct them and if they cannot do a problem, they keep trying. However, based on student work during the lessons and their inconsistent solutions on the probes this may not be the case. This lack of self-monitoring and accuracy of evaluating solutions is characteristic of students with MD (Gagnon & Maccini, 2001; Mazzocco, 2007).

Summary. These results suggest integrated instruction with explicit instruction, multiple representations and dynamic geometry software can improve the performance of students with MD as they solve geometric similarity transformations. All participants increased their accuracy scores significantly from baseline, with two students essentially earning a D, one a C and one a B based on school system scores. Additionally, all students continued to score above passing on a probe administered four to six weeks after the conclusion of the intervention. However, only one student, Jason, made significant improvement on a standardized measure of related geometry skills. This was slightly surprising, given that the others did not do as well, and there were no readily apparent differences. However, perhaps the confidence he expressed during the time we worked together was elevated that day. Based on my observations from our sessions together it seems that when he was more confident he was better able to grasp the concepts and accurately complete the work. Unfortunately, I did not have the opportunity to review the results with him (or the other participants) and was unable to explore this aspect.

Limitations and Implications for Future Research

Although the results of this study are promising, there are limitations and suggestions for future research. First, I conducted this research outside of the general education classroom setting. While this is not a limitation for a single-subject design, it does not provide an indication of the applicability to other settings (i.e. general education). Working with students individually allowed me to allot more time to a student than would be possible within the general education classroom, or possibly in a small group setting. Furthermore, according to some research (Chazan, & Yerushalmy, 1995; Kutluca, 2013) a benefit of the DGS program is the ability to have students to work together, discuss strategies and make discoveries. Future studies with pairs or small groups or larger groups of intact classes would allow researchers to explore the benefits of students working together as well as with larger groups being able to generalize findings to a typical classroom, particularly an inclusive setting. In addition, future research could compare the performance of students with and without MD to gauge the effectiveness of the intervention and its potential effect on achievement gaps between students with MD and non-MD students. Replication with additional geometry topics (congruence, circles, symmetry, pure transformations), upper level mathematics topics from trigonometry or advanced algebra) and with other students with mathematics difficulties would be useful for generalization (Kennedy, 2005). Further, single-subject research can assist in establishing evidence-based practices, and scaling up to a group design is recommended (Gersten, et al., 2008).

Second, the study participants were all students who demonstrated a history of difficulty with mathematics and who earned low scores on a baseline domain probe. This selection resulted in a homogeneous group of students. While this is not a limitation of the single-subject design, there are some other factors that were thus not explored and future research may address. All of the participants had been identified as a student with a disability, including ADHD, speech or language impairment and a specific learning disability. All of these student were included as students with MD and were in need of specialized instruction, by virtue of being identified as a student with a disability and having an IEP. These high incidence disabilities are the primary population I work with co-teaching in general education, and thus are the focus of the intervention. Furthermore, given the difficulty students have with the topic of similarity that is why so much time was devoted at the beginning of the unit to developing conceptual understanding and defining similarity. Although three of the participants were formally diagnosed with a disability in mathematics and the other was diagnosed with ADHD, caution is warranted when generalizing these findings to other students with disabilities. Future research should address each of the following: (a) a range of disabilities (e.g. autism, EBD of which none of the students in the present study were diagnosed) within a single study, (b) a study focused solely on students with one disability (e.g. LD or ADHD), or (c) include students with and without MD to compare the effectiveness of the package for all students within the topic of similarity.

Third, I developed the domain probes that were aligned with the instructional unit. Although the probes were reviewed by two experts in the field of mathematics and special education for content validity and piloted with a group of students to establish parallel forms reliability prior to implementation the number of reviewers and students used was relatively small (less than 60). Additionally the probes aligned with the content of the intervention, similar to how a unit assessment would be written, and did not include tasks that would involve transfer of knowledge or application to unfamiliar situations, as is often included in special education research (e.g., Cass et al, 2003).

While a standardized measure based on the NAEP was included for comparison of growth on the topic of similarity and to some extend a generalization measure that was lacking on the domain probes, this assessment was based on the public released NAEP items from grades 8 and 12. Although reliability and validity data on the entire NAEP and the researcher for this study piloted the instrument for test-retest reliability, this 10-question assessment was only a sample of what the NAEP might actually address on the topic of similarity. Furthermore, there is no standardized mathematics assessment that solely focuses on the unit topic. In addition, I administered the NAEP assessment as a pre and post-test measure with no comparison group. Future studies should use measures that have well-established validity and reliability, particularly with larger sample sizes, include questions not directly taught in the intervention, and utilize a control group. Additionally, as the NAEP includes more transfer of skills, particularly to real-world problems, and the two problems on the NAEP were the ones that these students either did not complete or did not complete correctly future research should incorporate skills that not only include the use of scale factor in isolation but also with skills (such as measurement) that should have been mastered in prior grade levels.

Fourth, there were scheduling interruptions that the design did not uniformly take into consideration. The post domain probes should have been administered on consecutive days, which is what occurred for three participants. However, multiple snow days occurred during the time that Khafila participated in the intervention. If this had been a regular class instructional unit with a test planned and there were multiple days off, such as for snow, then a teacher likely would postpone the test and spend more time reviewing with students. This is why I added a booster/review between the first two post-instruction probes and another booster between the third and fourth probes. This is consistent with recommendations from some researchers (e.g. Montague, 2004). However, by doing this Khafila's results may not be comparable to the other three participants. Fortunately none of the participants were absent during the post-intervention probe sessions. This could have had a similar impact on the performance as the snow days. Therefore, future research should anticipate such interruptions by planning for booster reviews such as with a criteria that if there is an interruption of two or more days during the post-intervention probe period then a booster session is utilized; or, conversely do not include boosters at all.

Fifth, my relationships with the participants may have influenced the results from the social validity and mathematics attitude measures. I worked in the school where the study took place. Furthermore, I had academic relationships with the students either as a co-teacher, study skills assistance or after school tutoring. While there is a benefit to action research, for generalization, future studies should collect social validity data in settings in which the researcher is not familiar to the participants.

Lastly, while work samples and student commentary improved the quantitative information additional qualitative data would further enhance the data. Qualitative data could explain the *how* students did or did not understand the concepts. The collection of additional information via structured interviews or utilizing transcripts of the lessons to analyze the work more in depth would help determine how the DGS or other instructional materials assist students in understanding geometric similarity. Future research should include both quantitative and qualitative data.

Implications for Practice

The current study contributes to the literature in several was: 1) it addresses the need for an intervention to teach geometric similarity transformation at the high school level; 2) includes procedural fluency and conceptual understanding of geometric content; 3) it incorporates research-based strategies for accessibility; 4) it addresses social validity First, this study addressed geometric similarity transformations with high school students with MD. While several general education studies addressed transformations (Boulter & Kirby, 1994; Choi-Koh, 1999; Gorgorio, 1998; Guven, 2012; Hollebrands, 2003, 2007; Hungwe et al, 2007; Kirby & Boulter, 1999; Rowel & Mansfield, 2001) none indicated that students with MD were included in the participants. The content of the instructional unit aligned with Common Core Standards as suggested by the NMAP and ADP. Most of the previous research in geometry for students with MD focused on lower level concepts such as perimeter and area (Cass et al., 2003; Cihak & Bowlin, 2009; Satsangi & Bouck, 2015). The current study extends to concepts that are more complex. This is critical, as most students with MD are educated in general education and accountable for the curriculum, including state high school exit exams.

Second, this study addresses both procedural fluency related to proportions and conceptual understanding of geometric similarity. Procedural fluency is critical to developing conceptual understanding. Both procedural fluency and conceptual knowledge are necessary and often-interdependent skills, needed to link prior knowledge with newly developing skills and problem solving (NRC, 2001). Procedural fluency was addressed primarily within the proportions and solving for the missing information. The visual representations assisted students in developing their initial concepts of similarity, then revising the initial concepts to more mature definitions and applying the necessary procedures to new applications. The retention of procedures are enhanced with the expansion of conceptual understanding and vice versa (Hudson & Miller, 2006)

Third, this study blended research-based strategies from both general education and special education literature to provide access to the CCSS for students with MD. This study included CRA sequence (Cass et al, 2003, Satsangi & Bouck, 2015), explicit instruction (Cass et al, 2003; Cihak & Bowlin, 2009; Jitendra et al, 2009; Satsangi & Bouck, 2015) and a cue card (Jitendra et al, 2009) which have been shown to be beneficial for students with MD, while DGS has been shown to be beneficial for students in general education (Choi-Koh, 1999; Guven, 2012; Hollebrands 2003, 2007). The AngLegs as concrete manipulatives and the DGS as virtual manipulatives address the NCTM and CCSS standards for multiple representations. Furthermore, tasks included opportunities for direct instruction, as needed particularly for procedures (Maccini et al, 2008) and student-directed investigations (NCTM, 2000). NMAP (2008) recommends a blending of instruction for students with MD.

Fourth, this study addresses the social validity of the intervention. Only two studies reviewed reported social validity (Cihak & Bowlin, 2009; Satsangi & Bouck, 2015). One additional study noted the use of a measure of social validity, but did not report the results (Cass et al, 2003). It is important to include a measure of the importance, effectiveness, appropriateness, and/or satisfaction of the participant's experiences in relation to the intervention (Kazdin, 2011; Richards et al., 2014).

Conclusions

The majority of students with disabilities are educated in general education and federal legislation not only requires access to the curriculum, but all students are accountable for meeting proficiency standards (IDEA, 2004; NCLB, 2001). However, students with disabilities continue to perform poorly when compared to peers on both national and international assessments (Wagner et al, 2006). The current study investigated the effects of integrated instruction including explicit instruction with DGS on student's understanding of geometric similarity transformations. Prior to this study, no research targeted geometric similarity transformations for high school students with MD. The results of this study provide promising evidence that students with MD can increase their performance on and understanding of geometric similarity utilizing an instructional package.

Further research is crucial to identify instructional practices that can make the general education geometry curriculum not only accessible, but also achievable for students with MD. All students must demonstrate competency in rigorous mathematics courses, even beyond geometry, not only to graduate from high school but to pursue employment either immediately post-high school or post-secondary education. Therefore, research-supported instructional practices for students with mathematics difficulties may contribute to improvement in high school courses but improvement on national and international assessments, increased graduation rates, increased post-secondary training and employment.

Appendix A: Behaviors of a Geometric Proportional Thinker

(Cox, 2013)

Proportional Thinker	*Geometric Proportional Thinker*
Knowing the mathematical characteristics of proportional situations	Knowing the properties of similar figures
Being able to differentiate mathematical characteristic of proportional thinking from nonproportion contexts	Being able to recognize or surmise the presence and absence of distortion
Understanding realistic and mathematical examples of proportional situations	Understanding the principles of scale in both realistic and mathematical contexts
Realizing that multiple methods can be used to solve proportional tasks and that these methods are related to each other	Realizing that both within and between ratios can be used to differentiate figures and that these ratios also help judge the reasonableness of constructed figures
Knowing how to solve quantitative and qualitative proportional reasoning tasks	Knowing how to scale images quantitatively and qualitatively and realizing the continuous nature of the scaling functions
Being unaffected by the context of the numbers in the task	Being unaffected by the complexity or simplicity of the figure, the relationship of the labeled measurements, and the integral or non-integral nature of the numbers in the task

Appendix B: Letter to Parents/Guardians and Students

[Date]

Dear Parent/Guardian and Students:

We are conducting a study on the effectiveness of a geometry instructional package for high school students who may be struggling in math. The instructional package will target the transformational geometry topic of similarity of polygons (e.g. triangles) skills that are important in high school mathematics courses, but also certain aspects of art, science, and technology beyond secondary school. The content addressed in the study is aligned with the county and state mathematics curiculum as well as the Common Core State standards for mathematical content and practice. The instructional package will be taught by Ms. Toronto, who is a certified teacher (special educataion and mathematics) at Long Reach High School and a doctoral student from the University of Maryland, College Park.

We are looking for students to participate in this study. The study will last about four weeks. Students will be taught 3-4 days per week for approximately 45 minutes during their regular scheduled tutorial period during normal school hours. Ms. Toronto will access confidential student education records to obtain pertinent data related to the study including IEP or 504 information, cognitive skills (i.e., IQ) and academic acheivement (i.e., report cards). All data regarding your child will be kept confidential and only accessed by Ms. Toronto. Data will be destroyed five years after the study ends.

Risks associated with this study include possible frustration with difficult tasks and the possibility of your child's image being viewed in research presentations, publications, and/or teacher trainings, if permission for video recording is granted. Participantion will not affect your child's grades in their current courses. You may request that your child be withdrawn from participating in the study at any time without penalty. Benefits may include improvements in understanding and performance on grade level mathematics objectives.

By signing the attached permission form, you are agreeing to allow your child to participate in this study, if your child meets all of the eleigibility requirements.

If you have questions about this study, please contact Allyson Toronto at:
atoronto@umd.edu (email)
301-437-2747 (phone)

Allyson P. Toronto
Student Investigator
Special Education/Mathematics Teacher
Long Reach High School

Dr. Susan De La Paz
Faculty Advisor
Associate Professor
University of Maryland,
College Park

David Burton
Principal
Long Reach High School

Frank Weisberg
Special Education ITL
(Instructional Team Leader)
Long Reach High School

Appendix C: Parent/Guardian Consent Form

<table>
<tr><td>Project Title</td><td>Effects of Explicit Instruction with Dynamic Geometry Software for Secondary Students with ADHD/Learning Disabilities</td></tr>
<tr><td>Purpose of the Study</td><td>This is a research project being conducted by Allyson Toronto, a special educaation and mathematics teacher at Long Reach High school, as part of her doctoral studies at the University of Maryland, College Park, under the supervision of Dr. Susan De La Paz. We are inviting your child to participate in this research because he or she has a history of difficulty in mathematics. The purpose of this research project is to advance current knowledge on effective geometry interventions for secondary students having difficulty with mathematics.</td></tr>
<tr><td>Procedures</td><td>The procedures involve the following:
I will:
collect information from your child’s confidential school file, including cognitive (e.g. IQ) and academic acheivement scores, and grades from past and current mathematics courses to determine if your child is eligible for the intervention.

Your child will be asked:
to complete a minimum of five pretests before instruction is provided.
to participate in mathematics instruction for 3-4 days per week, for 45 minutes per session for a period of approximately 3 weeks. Sessions will be scheduled during your child’s regular school day (i.e. tutorial class) or directly after school and content will be directly related to the mathematics curriculum on similarity transformations.
to complete periodic assessments related to the study over approximately 2-3 weeks, during the tutorial class. After completing all instructional sessions, your child will complete a minimum of five post-tests to determine if there are any changes in his or her understanding of the topics addressed in the study and one additional test to determine if he or she is able to apply what was learned to new similarity trransformations questions.
to complete a short assessment to determine if he or she remembers the content that was taught; four to six weeks after the end of the intervention,
his or her opinion regarding the instruction. For example, your child will be asked if the intervention helped him or her learn the targeted mathematics topics and what he or she liked most and least about the intervention.</td></tr>
</table>

	Furthermore, participants will receive a nominal remuneration, such as tickets or tokens, for each assessment completed, which can be used at the school store or local businesses. The compensation is for the time committment of the participant. During this study, I will be video recording the instructional and assessment sessions. I would like your permission to use portions of these videos in four ways: 1. To determine your child's thinking about the mathematical topics; 2. To determine if the intervention is being implemented as planned; 3. To determin if the assessment is being implemented as planned; 4.In research presentations, publications, and/or teacher trainings. If you choose not to have your child video recorded, he or she may still participate in the study.
Potential Risks and Discomforts	There may be some risks from participating in this research study. Risks associated with this study include possible frustration with difficult tasks and the possibility of your child's likeness being viewed in research presentations, publications, and/or teacher trainings.
Potential Benefits	This research is not specifically designed to help your child personally, but the results may help me learn more about instruction for students who have difficulty with mathematics. I hope that, in the future, other people might benefit from this study through improved understanding of instructional practices in mathematics, specifically similarity transformations. Your child may benefit by participaing because the study is designed to improve understanding of mathematics, specifically similarity transformations
Confidentiality	Any potential loss of confidentiality will be minimized by storing data in a locked file cabinet at Long Reach High School or digitally on a password protected computer. If I write a report or article about this research project, your child's identity will be protected to the maximum extent possible and your child's name will not be used. Your child's information may be shared with representatives of the University of Maryland, College Park or governmental authorities if your child or someone else is in danger or if we are required to do so by law.

Right to Withdraw and Questions	Your child's participation in this research is completely voluntary. You may choose for your child not to take part at all. If you decide for your child to participate in this research, you and/or he/she may request to stop participating at any time. If you decide not to have your child participate in this study or if you and/or he/she requests to stop participating at any time, your child will not be penalized or lose any benefits to which he/she otherwise qualifies. Your child's participation or non-participation in this study will not affect his or her grades. If your child withdraws from the study prior to the conclusion, he/she will receive remuneration based on the number of pre-tests or post-tests completed. If you decide for your child to stop taking part in the study or your child decides to stop taking part in the study, or if you have questions about the research study itself, please contact: **Allyson P. Toronto** **Long Reach High School** **6101 Old Dobbin Lane** **Columbia MD, 20145** **410-313-7117** **atoronto@umd.edu** If you have any questions about the study's implementation at Long Reach High School please contact: **David Burton** **Principal** **Long Reach High School** **6101 Old Dobbin Lane** **Columbia MD, 21045** **410-313-7117**
Participant Rights	If you have questions about your rights as a research participant or wish to report a research-related injury, please contact: **University of Maryland College Park** **Institutional Review Board Office** **1204 Marie Mount Hall** **College Park, Maryland, 20742** **E-mail: irb@umd.edu** **Telephone: 301-405-0678** This research has been reviewed according to the University of Maryland, College Park IRB procedures for research involving human subjects.

<table>
<tr><td>Statement of Consent</td><td colspan="2">Your signature indicates that you are at least 18 years of age; you have read this consent form or have had it read to you; your questions have been answered to your satisfaction and you voluntarily agree to allow your child or legal ward to participate in this research study. You will receive a copy of this signed consent form.

If you agree to allow your child to participate, sign your name below.</td></tr>
<tr><td rowspan="5">Signature and Date</td><td>NAME OF CHILD PARTICIPANT (Please Print)</td><td></td></tr>
<tr><td>I agree to:</td><td>(please initial each blank, as appropriate)

________have my child video recorded to determine his or her thinking processes about the geometry topics.

________have my child video recorded to determine if the intervention is being implemented as planned.
OR
_____ have my child audio recorded in place of video recording

________have my child’s likeness used in research presentations, publications, and/or teacher trainings.</td></tr>
<tr><td>NAME OF PARTICIPANT'S PARENT/LEGAL GUARDIAN (Please Print)</td><td></td></tr>
<tr><td>SIGNATURE OF PARTICPANT'S PARENT/LEGAL GUARDIAN</td><td></td></tr>
<tr><td>DATE</td><td></td></tr>
</table>

Appendix D: Student Assent Form

Effects of Explicit Instruction with Dynamic Geometry Software for Secondary Students with ADHD/Learning Disabilities

We are requesting your participation in an educational project conducted by Ms. Allyson Toronto, a teacher at Long Reach High School and doctoral student at the University of Maryland, College Park. You are under 18 years of age, and your parent or legal guardian has agreed that you can participate in this study.

The purpose of this study is to learn more about good geometry instruction for high school students with learning difficulties in mathematics. You will participate in instructional sessions last 1 period of approximately 45 minutes, 3 to 4 times per week, for about 4 weeks and participate in short assessments periodically over the course of approximately 6 weeks. Instruction will take place at school, during regular school hours and the instructional sessions will be video recorded. Video recordings may be used for three reasons: (1) to determine how you think about the mathematics questions; (2) to determine how I am teaching the topics; and (3) to use your likeness in research presentations, publications, and/or teacher trainings. If you do not want to be video recorded, you may still participate in the study. You will complete assessments before, during, and after the study. You will also be asked your opinion about the study, such as what you like best and what you would change. Ms. Toronto will also collect information from your confidential school records such as IQ scores, academic acheivement scores, and current or prior math grades. Any information collected by Ms. Toronto will be confidential, which means it will not be shared with anyone.

Participation in this study will not affect your math or other course grades. You may feel frustrated with some of the math work. You may benefit from this study because the project is designed to improve your math skills. You are free to ask questions anytime and you may stop participating at any time. If you stop participating, your grades in your classes will not be affected.

______________________________   ____________________

Print Name   *Date*

____

Signature

Appendix E: Common Core State Standards Addressed in the Intervention

Similarity, Right Triangles, and Trigonometry

Understand similarity in terms of similarity transformations:

1. (G-STR1) Verify experimentally the properties of dilations given by a center and a scale factor:
    a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.
    b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.
2. (G-SRT2) Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.
3. (G-SRT3) Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.

Prove theorems involving similarity
4. (G-STR4) Prove theorems about triangles
5. (G-SRT5) Use similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Note: Formal proves were not addressed but foundational concepts and reasoning that would lay the groundwork for formal proofs.

Appendix F: National Council of Teachers of Mathematics Standards Addressed in the Intervention

In grades 9-12 all students should:
Geometry Standard:

1) Analyze characteristics and properties of two-and three-dimensional geometric shapes and develop mathematical arguments about geoemtric relationships
   a) explore relationships (including similarity) among classes of two- and three -dimensional geometric objects, make and test conjectures about them, and solve problems involving them
   b) establish the validity of geometric conjectures using deduction, prove theorems and critique arguements made by others
2) Specify locations and describe spatial relationships using coordinate geometry and other representational systems
   a) use Cartesian coordinates and other coordinate systems, such as navigational, polar, or spherical systems to analyze geometric situations
   b) investiage conjectures and solve problems involving two- and three -dimensional objects represented with Cartesian coordinates
3) Apply transformations and use symmetry to analyze mathematical situations
   a) understand and represent translations, reflections, rotations, and dilations of objects in the plane by using sketches and coordinates,
      b) use various representations to help understand the effects of simple transformatons and their compositions
4) Use visualization, spatial reasoning and geometric modeling to solve problems
   a) draw and construct represenatioans of two- and three-dimentional geometric objects using a variety of tools
   b) use geometric models to gain insights into, and answer questions in other areas of mathematics;
   c) use geometric ideas to solve problems in, and gain insights into, other disciplines and other areas of interest such as art and architecture.

Measurement Standard

1) Apply appropriate techniques, tools, and formulas to determine measurements
   a) understand and use formulas for the area, surface area, and volume of geometric figures, including cones, spheres, and cylinders

Note: Formal proofs were not addressed but the initial reasoning and justification that lay the gourndwork for formal proofs. Transformations (e.g. rotations, translations, dilations) and cartesion coordinates were not addressed as distinct concepts but were integrated within the unit.

Appendix G: Maryland Common Core State Curriclum Framework Addressed in the Intervention

Unit 2: Similarity, Proof, and Trigonometry
Cluster: Understand similarity in terms of similarity transformations
1) Verify experimentally the properties of dilations given by a center and a scale factor.
    a) A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged
    Essential Skills and Knowledge: Ability to connect experiences with dilations and orientation to experiences with lines
    b) The dilation of a line segment is longer or shorter in the ratio given by the scale factor.
    Essential Skills and Knowledge: Ability to develop a hypothesis based on observation.
2) Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of sides.
    Essential Skills and Knowledge:
    a) Ability to make connections between the definition of similarity and the attributes of two given figures
    b) Ability to set up and use appropriate ratios and proportions
3) Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.
    Essential Skills and Knowledge: Ability to recognize why particular combinations of corresponding parts establish similarity and why others do not
Cluster: Prove theorems involving similarity
4) Prove theorems about triangles. *Theorems include a line parallel to one side of a triangle divides the other two sides proportionally, and conversely.*
    Essential Skills and Knowledge: Ability to construct proof using one of a variety of methods
5) Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures
    Essential Skills and Knowledge: Ability to use information given in verbal or pictorial form about geometric figures to set up a proportion that accurately models the situation
Cluster: Apply geometric concepts in modeling situations
    Note: These are overarching standards that have application in multiple units.
6) Use geometric shapes, their measures, and their properties to describe objects (e.g. modeling a tree trunk or a human torso as a cylinder).
    Essential Skills and Knowledge: See the skills and knowledge that are stated in the Standard
7) Apply geometric methods to solve design problems (e.g. designing an object or structure to satisfy physical constraints or minimize cost; work with typographic grid system based on ratios
Unit 3: Extending to Three Dimentions
Cluster: Explain volume formulas and use them to solve problems

Cluster Note: Informal arguments for area and volume formulas can make use of the way in which area and volume scale under similarity tranfomations: when one figure in the plan resulsts from another by applying a similarity transformation with scale factor k, its area is k^2 times the area of the first. Simlarly, volumes of solid figures scale by k^3 under a similarity transformation with scale factor k.

8) Use volume formulas for cylinders, pyramids, cones and spheres to solve problems

Note: Formal proves were not addressed but foundational concepts and reasoning that would lay the groundwork for formal proofs.

Appendix H: American Diploma Project Skills Addressed in the Intervention

1. Scaling, dilation, and dimension
    a. Analyze and represent the effects of multiplying the linear dimensions of an object in the plane or in space by a constant scale factor, r.
    Example: Multiplying the lengths of the sides of a polygon by r results in a polygon having the same shape as the original. Mathematically, having the same shape means that the image will have the same number of angles and sides as the pre-image, that all angles will preserve their measure, and that corresponding sides will be proportional.
    b. Use ratios and proportional reasoning to apply a scale factor to a geometric object, a drawing, a three-dimensional space, or a model, and analyze the effect.
    c. Interpret and represent origin-centered dilations of objects in the coordinate plane. A dilation centered at the origin with scale factor r maps the point (x, y) to the point (rx, ry).
    Example: In triangle A'B'C' with A'(9,3), B'(12,6), and C'(15,0) is the dilation of triangle ABC with A(3,1), B(4,2), and C(5,0). The scale factor for this dilation is 3.

2. Similarity
    a. Interpret the definition and characteristics of similarity for triangles in the plane. Informally, two geometric objects in the plane are similar if they have the same shape. More formally, having the same shape means that one figure can be mapped onto the other by means of rigid transformations and/or an origin-centered dilation.
        i. Know that two triangles are similar if their corresponding angles have the same measure.
        ii. Know that the ratio formed by dividing the lengths of corresponding sides of similar triangles is a constant, often called the constant of proportionality, and determine this constant for given similar triangles.
    b. Apply similarity in practical situations.
        i. Calculate the measures of corresponding parts of similar figures.
        ii. Use the concepts of similarity to create and interpret scale drawings.
    c. Identify and apply conditions that are sufficient to guarantee similarity of triangles.
        i. Identify two triangles as similar if the ratios of the lengths of corresponding sides are equal (SSS criterion), if the ratios of the lengths of two pairs of corresponding sides and the measures of the corresponding angles between them are equal (SAS criterion), or if two pairs of corresponding angles are congruent (AA criterion).
        ii. Apply the SSS, SAS, and AA criteria to verify whether or not two triangles are similar.

iii. Apply the SSS, SAS, and AA criteria to construct a triangle similar to a given triangle using straightedge and compass or geometric software.
iv. Identify the constant of proportionality and determine the measures of corresponding sides and angles for similar triangles.
v. Recognize, use and explain why a line drawn inside a triangle parallel to one side forms a smaller triangle similar to the original one.

d. Extend the concepts of similarity to other polygons in the plane. A closed plane figure is called a polygon if all of its edges are line segments, every vertex is the endpoint of two sides, and no two sides cross each other.
   i. Identify two polygons as similar if they have the same number of sides and angles, if corresponding angles have the same measure, and if corresponding sides are proportional.
   ii. Determine whether or not two polygons are similar.
   iii. Use examples to show that analogues of the SSS, SAS, and AA criteria for similarity of triangles do not work for polygons with more than three sides.

Appendix I: National Council of Teachers of Mathematics Process Standards

NCTM Standard	Definition
Communication	Using the language of mathematics precisely and coherently to express mathematical ideas or evaluate the strategies of others.
Connections	Recognizing connections among mathematical ideas (i.e., geometry and algebra) and applying mathematics in contexts outside of mathematics (e.g., science, social studies, consumer activities).
Problem solving	Engaging in a task for which the solution is not known in advance and for which there may be multiple solution paths.
Reasoning	Making and investigating mathematical conjectures, developing and evaluating arguments and selecting various solution pathways.
Representations	Representations include student created diagrams, graphs, models, or symbols that are applied to a mathematical problem task that aids students in their ability to solve a problem.

Appendix J: National Research Council Strands of Mathematical Proficiency

Proficiency Strand	Definition
Adaptive reasoning	The ability to think logically, reflect, explain and justify.
Conceptual understanding	The integrated and functional grasp of mathematical ideas, which enables students to make connections between prior knowledge and new skills.
Procedural fluency	Carrying out procedures flexibly, accurately, efficiently and appropriately.
Productive disposition	The inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.
Strategic competence	The ability to formulate, represent and solve mathematical problems.

Appendix K: Common Core Mathematical Practice Standards

Practice	Definition
Make sense of problems and persevere in solving them	Students use multiple methods to understand and solve problems; monitoring, evaluating, and changing their approach, as needed.
Reason abstractly and quantitatively	Students make sense of quantities and their relationships in problem situations by decontextualizing and contextualizing representations and referents.
Construct viable arguments and critique the reasoning of others	Students justify conclusions, communicate to others, and respond to arguments of others, asking questions to clarify.
Model with mathematics	Students use context to interpret problems and use diagrams, table, graphs flowcharts or formulas while solving problems.
Use appropriate tools strategically	Students use the most appropriate tool (calculator, software, manipulatives, paper-pencil) for a given situation, revising as needed.
Attend to precision	Students use definitions, symbols, labels, and units of measure accurately and appropriately for a problem context and communicate this with others.
Look for and make use of structure	Students look closely for patterns to solve problems.
Look for and express regularity in repeated reasoning.	Students notice if calculations are repeated and look for general methods and shortcuts to solve problems.

Appendix L: Unit Objectives

Lesson 1: Students will use concrete objects and 2D representations to explore and define similarity of figures.

Lesson 2: Students will use visual representations and dynamic geometry software to explore and define similarity of triangles.

Lesson 3: Students will use visuals and dynamic geometry software to explore and define similarity of polygons.

Lesson 4: Students will use visuals and dynamic geometry software to explore and define theorems (AA not need AAA) to prove triangles are similar.

Lesson 5: Students will use visuals and dynamic geometry software to explore, define, and apply triangle similarity theorems (SSS, not SSA).

Lesson 6: Students will use visuals and dynamic geometry software to explore, define, and apply triangle similarity theorems (SAS, not AAS, not ASA,)

Lesson 7: Students will use visuals and dynamic geometry software to explore, define and apply triangle similarity theorems (midsegment and triangle proportionality).

Lesson 8: Students will us visuals and dynamic geometry software to apply triangle similarity theorems to indirect measurements.

Lesson 9: Students will use visuals and dynamic geometry software to determine multidimensional measurements of similar figures (1D-length/perimeter, 2D-area, 3D-surface area/volume).

Lesson 10: Review of all material in the unit

Appendix M: Unit Lesson Plans

Lesson 1

CCSSM Content: G-SRT2, G-SRT5

CCSSM Practice: 1-Make sense of problems and persevere in solving them, 2-Reason abstractly and quantitatively, 3-construct viable arguments, 5-Use appropriate tools, 6-Attend to precision, 7-Look for and make use of structure

Materials: handouts with various shapes, various 3D objects (e.g. prisms/boxes/cubes, cylinders/cans), writing materials (e.g. markers, pencils)

Objective: Students will use concrete objects and 2D representations to explore and define similarity of figures.

I. Advance organizer:

Review Pre-requisite Skills: Provide a handout with various shapes and ask the student: *Which of the figures shown are congruent? Explain why you chose the particular figures-what does it mean for figures to be congruent.*

Objective and Link

T: *Great work on the review. We just reviewed important math skills to help us with our lesson today. Recently you have been studying rigid transformations such as congruence. Today we are going to explore similarity. What does it mean for two (or more) things to be similar?* [depending on student response use the English/Language arts definition as a springboard and probe for examples.]

S: *To be alike.*

Rationale:

T: *Excellent. You really understand that congruent figures are exactly the same and only the position of the figure might be different. This is useful when making exact copies of figures, such as on a production of a variety of products you might buy such as games, books, phones, even clothes. Similarity is also useful, especially when companies are developing models of products, such as [any of those just mentioned] and making comparisons. This is why it is a core concept in a geometry class and also assessed on college entrance exams such as the SAT/ACT.*

II.Investigation:

Have students explore various representations of figures and concrete objects to decide what makes them similar. Provide a handout with dilations (reductions/enlargements) of figures as well as some that are distortions. Have students explain why the figures are/are not similar. Provide probing questions as they explore so that they come up with a definition of similar figures akin to “same shape, different size”, eventually want them to discover that similar figures have the same angle measures and that the sides are proportional (extends to lessons 2 and 3).

a. **Model Thinking and Action:** Think aloud while explaining how to make comparisons of the figures (i.e. side lengths, angle measures, corresponding parts).

Handouts (Representations)=

Activity 1-fox distortions (not proportional)

Activity 2-funhouse mirrors (similar by literature definition but not mathematically-not proportional); only use this if students have difficulty with the fox activity

Concrete objects=

Activity 3-provide a selection of objects such as rectangular prisms (boxes), cubes (boxes), cylinders (cans) and have students sort the objects into groups that they think are similar then explain their groupings

Representations=

Activity 4-provide a selection of cutouts/cards with images of regular and irregular objects (polygons and non-polygons) and have students sort into groups they think are similar, then explain their groupings; have students compare their reasoning to the concrete objects activity

b. **Maximize Student Engagement and Monitor Student Understanding*:*** Involve students in the process as teacher continues to think aloud while guiding student through comparing the figures and eliciting their solution strategies and potential definitions for similarity of figures.

III. Guided and Independent Practice

a. Provide guidance as students perform 3-6 more problems (on paper) using the definitions they have developed in the prior activities (to see if their definition holds)

b. Scaffolded inquiry or varied *level of support* as students assume more responsibility for the learning (e.g. *High*: Verbalize the procedures and have students restate and/or apply, *Medium*: Have students verbalize each procedure and apply, *Low*: Have students verbalize all of the steps (chunk together) and apply).

IV. Closure: summarizes the lesson (daily even if the topic is over multiple days) and provides a snapshot of student comprehension.

a. **Big Ideas:** Teacher review with student: *What does it mean for figures to be similar?* [preliminary definition=same shape but different size; may have gotten to idea that angles congruent, sides proportional] *Define:* corresponding angles, corresponding sides, scale factor (depending on what student has gleaned).

Exit Ticket: Draw two figures that are congruent and two figures that are similar. Explain how you know.

b. **Link to Future Instruction:**

T: *Next time we will work on similarity of specific figures, such as triangles.*

Lesson 2

CCSSM Content: G-SRT2, G-SRT5

CCSSM Practice: 1-Make sense of problems and persevere in solving them, 2-Reason abstractly and quantitatively, 3-construct viable arguments, 5-Use appropriate tools, 6-Attend to precision, 7-Make use of structure

Materials: triangle cutouts, rulers, calculator, computer (with dgs program), writing materials (e.g. markers, pencils), cue card

Objective: Students will use visual representations and dynamic geometry software to explore and define similarity of triangles.

I. Advance Organizer:

Review Pre-requisite Skills: Provide a handout with various shapes and ask the student: *Which of the figures shown are congruent, similar, or neither? Explain why you chose the particular figures.*

Objective and Link

T: *Great work on the warm-up. Last time we explored concrete objects, such as boxes, and representations, such as drawings of different shapes. Today we are going to explore similarity of triangles. Remind me of what you came up with last time we met of what does it mean for two (or more) things to be similar?* [depending on student response use the English/Language arts definition as a springboard and probe for examples.]

S: *To be sort of the same size and shape.*

Rationale:

T*:* *Excellent. Just like with congruence, similarity is also useful, especially when companies are developing models of products, such as [any of those just mentioned] and making comparisons. This is why it is a core concept in a geometry class and also assessed on college entrance exams such as the SAT/ACT.*

II. Investigation: Have students explore triangles to determine specific features that make them similar. Have students explain why the figures are/are not similar. Provide probing questions as they explore so that they build on the ideas of the prior lesson that similar figures are the "same shape, different size", eventually want them to discover that similar figures have the same angle measures and that the sides are proportional.

a. **Model Thinking and Action**: Think aloud while explaining how to make comparisons of the figures.

Handouts=

Activity 1-triangle cutout activity (depending on time may not do group 2-3 of the triangles)

DGS=

Activity 2-use Geometer's Sketchpad to compare triangles and decide if the relationships discovered in the cutout activity hold for other triangles *b.* **Maximize Student Engagement and Monitor Student Understanding***:* Involve students in the process as teacher continues to think aloud while guiding student through comparing the figures and eliciting their solution strategies and potential more formal definitions for similarity of figures.

III. Guided and Independent Practice

c. Provide guidance as students perform 3-6 more problems (on paper) using the ratios for triangles. (Note: cue card modified from later lessons to have only the two criteria for similarity: three congruent angles and three pairs of proportional sides.This can be used once students discover this through initial explorations.)

d. Scaffolded inquiry or varied *level of support* as students assume more responsibility for the learning (e.g. *High*: Verbalize the procedures and have students restate and/or apply, *Medium*: Have students verbalize each procedure and apply, *Low*: Have students verbalize all of the steps (chunk together) and apply).

IV. Closure: summarizes the lesson (daily even if the topic is over multiple days) and provides a snapshot of student comprehension.

a. **Big Ideas:** Teacher review with student**:** *What does it mean for figures to be similar?* [same shape but different size is a basic definition but more specifically, angles congruent, sides proportional] *Define:* corresponding angles, corresponding sides, scale factor.

b. **Exit ticket:** Have students answer two questions about which sets of triangles are similar, explaining why or why not.

c. **Link to Future Instruction: T:** *Great work today! Next time we will work on similarity of figures other than triangles and more practice with proportions.*

Lesson 3

CCSSM Content: G-SRT1, G-SRT2, G-SRT5

CCSSM Practice: 1-Make sense of problems and persevere in solving them, 2-Reason abstractly and quantitatively, 3-construct viable arguments, 5-Use appropriate tools, 6-Attend to precision, 7-Look for and make use of structure, 8-look for and express regularity in repeated reasoning

Materials: protractor, ruler, calculator, computer (with dgs program), writing materials (e.g. markers, pencils), cue card

Objective: Students will use visuals and dynamic geometry software to explore and define similarity of polygons.

I. **Launch:**

Review Pre-requisite Skills: Provide a handout with two problems to review from prior lesson on triangles. One in which triangles are similar and the other set are not. Have students explain why the figures are/are not similar. [gets at understanding of the definition of similarity].

Objective and Link

T: *Great work on the review. This will prepare you for further explorations of similar figures today. Remind me again; What does it mean for two triangles to be similar?*

S: *To have the same shape, with congruent angles but the sides are proportional.*

Rationale:

T*: Excellent. You really understand that similar triangles have the same angles, but unlike congruent figures that have the same side lengths the sides of similar figures are related by ratios. As seen in our previous discussion , this is important for reducing or enlarging and also comes in handy in art or design.*

II. **Investigation:** Have students explore various representations of figures other than triangles decide if the definition from the prior lesson holds [angles are congruent but corresponding sides are proportional] to decide what makes figures similar. Provide a handout with dilations (reductions/enlargements) of figures as well as some that are distortions. Have students explain why the figures are/are not similar. Provide probing questions as they explore so that they discover that similar figures have the same angle measures and that the sides are proportional, not only for triangles but other figures as well.

a. **Model Thinking and Action**: Think aloud while explaining how to make comparisons of the figures.

Handouts=

Activity 1-provide a handout for conjecturing about which figures are similar or not (can let them use word to drag and manipulate the figures or tools to measure)

DGS=

Activity 2-use dgs examples for them to explore similar figures and conjecture on if the definition from the prior lesson on triangles applies to other polygons 3-(link to the meanings for scale factor and calculations for the guided then independent practice using the definitions)

b. **Maximize Student Engagement and Monitor Student Understanding*:*** Involve students in the process as teacher continues to think aloud while guiding student through comparing the figures (on paper #1 and using dgs #2) and eliciting their solution strategies and potential definitions for similarity of figures.

III. Guided and Independent Practice

c. Provide guidance as students perform 3-6 problems (on paper) using ratios for polygons. (Note: cue card modified from later lessons to have only the two criteria for similarity: three congruent angles and three pairs of proportional sides.)

d. Scaffolded inquiry or varied *level of support* as students assume more responsibility for the learning (e.g. *High*: Verbalize the procedures and have students restate and/or apply, *Medium*: Have students verbalize each procedure and apply, *Low*: Have students verbalize all of the steps (chunk together) and apply).

IV. Closure: summarizes the lesson (daily even if the topic is over multiple days) and provides a snapshot of student comprehension.

a. **Big Ideas:** Teacher review with student**:** *What does it mean for figures to be similar?* [same shape but different size, angles congruent, sides proportional] *Define:* corresponding angles, corresponding sides, scale factor.
*b.*Complete **EXIT ticket** on similar polygons (2 problems).
c. **Link to Future Instruction: T:** *Next time we will explore proportional relationships more in depth and with applications to specific problems.*

Lesson 4

CCSSM Content: G-SRT2, G-SRT3, G-SRT5

CCSSM Practice: 1-Make sense of problems and persevere in solving them, 2-Reason abstractly and quantitatively, 3-construct viable arguments, 5-Use appropriate tools, 6-Attend to precision, 7-Look for and make use of structure, 8-look for and express regularity in repeated reasoning

Materials: protractor, ruler, AngLegs, calculator, computer (with dgs program), writing materials (e.g. markers, pencils), cue card

Objective: Students will use visuals and dynamic geometry software to explore and define theorems to prove triangles are similar (AA not need AAA).

I. **Launch:**

Review Pre-requisite Skills: Provide a handout with two problems to review from prior lessons on polygon similarity. One in which polygons are similar and the other set are not. Have students explain why the figures are/are not similar. [to make sure students understand the definition of similarity]

Objective and Link

T: *Great work on the review. This will prepare you for further explorations of similar figures today. Remind me again; What does it mean for two figures to be similar?*

S: *To have the same shape, with congruent angles but the sides are proportional.*

Rationale:

T*: Excellent. You really understand that similar figures have the same angles, but unlike congruent figures that have the same side lengths the sides of similar figures are proportional. As a reminder, this is important for reducing or enlarging, comparing scale figures, and is also useful in art. Today we will be exploring triangular figures more in depth to discover if there are any additional special relationships.*

II. **Investigation:** Have students explore various drawings or computer generated images of triangles to discover any shortcuts to showing (proving) that triangles are similar; rather than having to show that all the angles are congruent and all the sides are proportional for every comparison. Is there a minimum amount of information that is needed to be certain the triangles are similar? Provide students with one or more of the following tools: paper/pencil, ruler, protractor, patty paper, spaghetti (if anglegs not available), anglegs, and dgs. Have students create sets of triangles that are similar using the tools (first the hands-on materials then further with the dgs due to human errors). Provide probing questions as they explore so that they discover that it is not necessary to measure/use all three angles to have similar figures (only two angles are needed as it follows that the third will be congruent, and as a result the sides will be proportional).

a. **Model Thinking and Action**: Think aloud while explaining how to make comparisons of the figures.

Concrete=

Activity 1-Provide paper and the AngLegs and show/have students to create one triangle, with pre-chosen angle measures. Then use the tools to create another triangle that has the same angle measures and see if it is also similar. Make sure to discuss the scale factor (side proportions), as well as any irregularities due to human error in using the tools. Have students repeat this with additional sets of triangles (guide to make some that are acute, obtuse etc.).

DGS=

Activity 2- Once students are comfortable with their conjectures based on hand-on materials transition to the dgs and have them explore and confirm their assumptions.

DGS/Representation=

Activity 3-(along with maximizing engagement) once students have discovered that AA is sufficient then transition to using this with figures that have measurements given and stating if the triangles are/are not similar based on AA (and if not what else is needed). Also, transition to images on paper (to move into the guided/independent practice)

b. **Maximize Student Engagement and Monitor Student Understanding*:*** Involve students in the process as teacher continues to think aloud while guiding student through creating additional sets of triangles with the hands-on tools, then transitioning to the dgs examples, eliciting their thoughts/ideas for any shortcuts to showing that triangles are similar if you only know two sets of congruent angles.

III. Guided and Independent Practice

c. Provide guidance as students perform 3-6 more problems (on paper) identifying similar triangles using the AA theorem. (Note: cue card available through the theorems completed is available-covered or printed without the information not yet introduced-for lesson 4 used modified card with only the two criteria for similarity: three congruent angles and three pairs of proportional sides.)

d. Scaffolded inquiry or varied *level of support* as students assume more responsibility for the learning (e.g. *High*: Verbalize the procedures and have students restate and/or apply, *Medium*: Have students verbalize each procedure and apply, *Low*: Have students verbalize all of the steps (chunk together) and apply).

IV. Closure: summarizes the lesson (daily even if the topic is over multiple days) and provides a snapshot of student comprehension.

a. **Big Ideas:** Teacher review with student**:** *What does it mean for figures to be similar?* [same shape but different size, angles congruent, sides proportional] *How many pairs of angles of a set of triangles do you need to know in order to determine if the triangles are similar?* [two because the third angle will be easily figured out/known due to the triangle sum being 180 degrees, and the sides will be proportional-this could in some cased be calculated but not needed]

b. Complete **EXIT ticket** (2 problems).
c. **Link to Future Instruction: T:** *Next time we will see if there are any other triangle similarity shortcuts.*

Lesson 5

CCSSM Content: G-SRT-1, G-SRT2, G-SRT-4, G-SRT5

CCSSM Practice: 1-Make sense of problems and persevere in solving them, 2-Reason abstractly and quantitatively, 3-construct viable arguments, 5-Use appropriate tools, 6-Attend to precision, 7-Look for and make use of structure, 8-look for and express regularity in repeated reasoning

Materials: protractor, ruler, calculator, AngLegs, computer (with dgs program), writing materials (e.g. markers, pencils), cue card

Objective: Students will use visuals and dynamic geometry software to explore, define, and apply triangle similarity theorems (SSS, not $AAS/ASA/SSA$).

I. **Launch:**

Review Pre-requisite Skills: Provide a handout with two problems to review from prior lessons on triangle similarity using AA theorem . One in which triangles are similar and the other set are not. Have students explain why the figures are/are not similar. [to make sure students understand the definition of similarity and the shortcut AA].

Objective and Link

T: *Great work on the review. This will prepare you for further explorations of similar figures today. Remind me again; What does it mean for two triangles to be similar? Why does the AA shortcut work?*

S: *To have the same shape, with congruent angles but the sides are proportional. AA works because if you know two pairs of angles then the third angle is easily known-sum of the interior angles of a triangle are 180*, this makes the lengths of the sides work out to be proportional.*

Rationale:

T*: Excellent. Short-cuts such as AA can really come in handy when planning and designing, besides saving a lot of time making rather than making calculations for all the angles and sides!*

II. **Investigation:** Have students explore various representations/drawings or computer generated images of triangles to discover any shortcuts to showing (proving) that triangles are similar; rather than having to show that all the angles are congruent and all the sides are proportional for every comparison. Is there a minimum amount of information that is needed to be certain the triangles are similar? Provide students with one or more of the following tools: paper/pencil, ruler, protractor, patty paper, spaghetti (if AngLegs not available), anglegs, and dgs. Have students create sets of triangles that are similar using the tools (first the hands-on materials then further with the dgs due to human errors). Provide probing questions as they explore so that they discover that it is not necessary to measure/use all three angles to have similar figures (Only proportional sides-3 pairs, as this will make the angles across be congruent; but two side pairs and the outside/not included angle will NOT necessarily make similar figures)

a. **Model Thinking and Action**: Think aloud while explaining how to make comparisons of the figures.
Concrete=
Activity 1-Provide paper and/or the tools and show/have students to create one triangle, with pre-chosen side lengths. Then use the tools to create another triangle that is similar with side lengths that are proportional (i.e. choose a scale factor such as 2 or 1/2 that would not be too difficult to manipulate). Students should measure/compare the angles to see if they are congruent. Make sure to discuss any irregularities due to human error in using the tools. Have students repeat this with additional sets of triangles (guide to make some that are acute, obtuse etc.). Student may use AngLegs and realize that overlapping the angles will be sufficient and not need to measure the angles directly.
DGS=
Activity 2-Once students are sufficiently comfortable with their hands-on conjectures transition to dgs comparisons for corroboration of their assumptions.
Concrete/DGS
Activity 3-Next ask students to consider if they think there might be additions shortcuts in addition to AA and SSS that might work (depending on what they suggest indicate that we will explore those later-SAS- or that we will check it out if SSA; also that AAS/ASA are not needed due to AA covering it) and then Repeat investigations 1 & 2 to discover that unlike SSS (or AA) there is not workable shortcut for SSA/ASS (provide two side lengths for two sets of triangles to compare and have students notice that one set is similar but the other set is not)
Representations
Activity 4-(along with maximizing engagement) once students have discovered that SSS is sufficient but SSA is not, then transition to using this with figures (on paper) that have measurements given and stating if the triangles are/are not similar based on SSS/SSA (and if not what else is needed).

b. **Maximize Student Engagement and Monitor Student Understanding*:*** Involve students in the process as teacher continues to think aloud while guiding student through comparing the figures and eliciting thoughts/ideas for any shortcuts to showing that triangles are similar if sides are known, but not angles or two sides and a non-included angle.

III. Guided and Independent Practice

c. Provide guidance as students perform 3-6 problems (on paper) identifying similar triangles using the SSS theorem, and excluding SSA as triangles that cannot be proved similar. (Note: cue card available through the theorems completed is available-covered or printed without the information not yet introduced.)

d.

e. Scaffolded inquiry or varied *level of support* as students assume more responsibility for the learning (e.g. *High*: Verbalize the procedures and have students restate and/or apply, *Medium*: Have students verbalize each procedure and apply, *Low*: Have students verbalize all of the steps (chunk together) and apply).

IV. Closure: summarizes the lesson (daily even if the topic is over multiple days) and provides a snapshot of student comprehension.

a. **Big Ideas:** Teacher review with student**:** *Which short-cuts have we explored so far?* [AA and SSS] *Why do those short-cuts work?* [if two angles are known then the third must be the same because of the triangle sum of 180 degrees; if the sides are all proportional then the angles must be congruent] *Why does SSA not work?* Because the included angle between the stated sides will possibly be different so the third side may not be proportional.

b. **Exit ticket:** Have students answer two questions using the SSS/SSA/AA shortcuts.

c. **Link to Future Instruction: T:** *Next time we will explore and find out if there are additional short-cuts for triangle (SAS-if mentioned earlier) that work for showing triangles are similar.*

Lesson 6

CCSSM Content: G-SRT-1, G-SRT2, G-SRT-4, G-SRT5

CCSSM Practice: 1-Make sense of problems and persevere in solving them, 2-Reason abstractly and quantitatively, 3-construct viable arguments, 5-Use appropriate tools, 6-Attend to precision, 7-Look for and make use of structure, 8-look for and express regularity in repeated reasoning

Materials: protractor, ruler, calculator, AngLegs, computer (with dgs program), writing materials (e.g. markers, pencils), cue card

Objective: Students will use visuals and dynamic geometry software to explore, define, and apply triangle similarity theorems (*SAS*)

I. Launch:

Review Pre-requisite Skills: Provide a handout with two problems to review from prior lessons on triangles (SSS, SSA, AA). One in which some triangles are similar and the other set are not. Have students explain why the figures are/are not similar. .

Objective and Link

T: *Great work on the review. This will prepare you for further explorations of similar triangles today. Remind me again; What does it mean for two triangles to be similar? What shortcuts have we explored so far?*

S: To have the same shape, with congruent angles but the sides are proportional. AA, SSS.

Rationale:

T*: Excellent. You really understand that similar triangles have the same angles, but unlike congruent figures that have the same side lengths the sides of similar figures are related by ratios. The short-cuts that we have explored so far are important for art as well as architecture. Today we will be doing more exploration to find out if there are additional combinations of sides and angles for shortcuts to proving triangles are similar.*

II. Investigation: Have students explore various representations/drawings or computer generated images of triangles to discover any shortcuts to showing (proving) that triangles are similar; rather than having to show that all the angles are congruent and all the sides are proportional for every comparison. Is there a minimum amount of information that is needed to be certain the triangles are similar? Provide students with one or more of the following tools: paper/pencil, ruler, protractor, patty paper, spaghetti, anglegs, and dgs. Have students create sets of triangles that are similar using the tools (first the hands-on materials then further with the dgs due to human errors). Provide probing questions as they explore so that they discover that it is not necessary to measure/use all three angles to have similar figures (Only 2 pairs of proportional sides-as this will make the angles across be congruent; and the angle between the given sides)

a. **Model Thinking and Action**: Think aloud while explaining how to make comparisons of the figures.
Concrete=
Activity 1-Provide paper and/or the AngLegs and show/have students to create one triangle, with 2 pre-chosen side lengths and a pre-chosen angle measure in between. Then use the tools to create another triangle that is similar with side lengths that are proportional (i.e. choose a scale factor such as 2 or 1/2 that would not be too difficult to manipulate) and the same included angle measure. Students should measure the two other angles (or overlap figures to compare) and compare the lengths of the third side to see if they are also proportional (as for the given two side pairs). Make sure to discuss any irregularities due to human error in using the tools. Have students repeat this with additional sets of triangles (guide to make some that are acute, obtuse etc.).
DGS=
Activity 2-Once students are sufficiently comfortable with their hands-on conjectures transition to dgs comparisons for corroboration of their assumptions.
Activity 3-Next ask students to consider if they think there might be additional shortcuts that might work or have we discussed/explored all combinations. (We have covered all the shortcuts)
Representations=
Activity 4-(along with maximizing engagement) once students have discovered that SAS is then transition to using this with figures (on paper) that have measurements given and stating if the triangles are/are not similar based on SAS etc. (and if not what else is needed).

b. **Maximize Student Engagement and Monitor Student Understanding*:*** Involve students in the process as teacher continues to think aloud while guiding student through comparing the figures and eliciting thoughts/ideas for any shortcuts to showing that triangles are similar using SAS.

III. Guided and Independent Practice

c. *c.* Provide guidance as students perform 3-6 problems (on paper) identifying similar triangles using the SAS theorem. (Note: cue card available through the theorems completed is available-covered or printed without the information not yet introduced.)
d. Scaffolded inquiry or varied *level of support* as students assume more responsibility for the learning (e.g. *High*: Verbalize the procedures and have students restate and/or apply, *Medium*: Have students verbalize each procedure and apply, *Low*: Have students verbalize all of the steps (chunk together) and apply).

IV. Closure: summarizes the lesson (daily even if the topic is over multiple days) and provides a snapshot of student comprehension.

a. **Big Ideas:** Teacher review with student**:** *Which short-cuts have we explored so far?* [AA/SSS/SAS,] *Why do those short-cuts work?* [i.e. if two angles are known then the third must be the same because of the triangle sum of 180 degrees; if the sides are all proportional then the angles must be congruent] *Why does SSA not work?* [If the angle is not included then the angles between the two given sides can 'swing' and still connect to the third side to make a triangle-one that is similar and one that is not similar-it is ambiguous.

b. **Exit ticket:** Have students answer two questions using the SAS shortcut.

c. **Link to Future Instruction: T:** *Next time we will explore additional triangle relationships that are useful for comparisons, particularly for art, architecture etc.*

Lesson 7

CCSSM Content: G-SRT2, G-SRT-4, G-SRT5

CCSSM Practice: 1-Make sense of problems and persevere in solving them, 2-Reason abstractly and quantitatively, 3-construct viable arguments, 5-Use appropriate tools, 6-Attend to precision, 7-Look for and make use of structure, 8-look for and express regularity in repeated reasoning

Materials: computer (with dgs program), AngLegs, calculator, protractor/ruler, writing materials (e.g. markers, pencils), cue card

Objective: Students will use visuals and dynamic geometry software to explore, discover and apply triangle similarity theorems (midsegment and proportionality).

I. **Launch:**

Review Pre-requisite Skills: Provide a handout with two problems to review from prior lessons on triangles. One in which triangles are similar and the other set are not. Have students explain why the figures are/are not similar. [Which shortcuts are applicable and why.].

Objective and Link

T: *Great work on the review. This will prepare you for further explorations of similar figures today. Remind me again; What does it mean for two triangles to be similar? And which shortcuts have we discovered?*

S: *To have the same shape, with congruent angles but the sides are proportional.* SSS, AA, SAS

Rationale:

T*: Excellent. You really understand that similar triangles have the same angles, but unlike congruent figures that have the same side lengths the sides of similar figures are related by ratios. This is important for reducing or enlarging and also comes in handy in art or design.*

II. **Investigation:** Following the warm-up which showed the overlapping triangles use this to springboard into the midsegment and triangle proportionality theorems. Primarily use dgs to explore the relationships as any side and angle measurements can be used but first have a visual with the AngLegs and have a conjecture about relationships.

a. **Model Thinking and Action**: Think aloud while explaining how to create the figures in dgs (primarily) and make comparisons of the triangle parts.

Concrete=

Activity 1- Have the student make 1-2 sets of triangles that are similar (using AngLegs). Then overlap them and compare to see if they can discover the midsegment relationship. (In any triangle, a segment joining the midpoints of any two sides will be parallel to the third side and half its length.)

DGS=

Activity 2-Transition to the dgs to have student compare what happens when the angles and lengths are changed to discover the midsegment/confirm it if

discovered in step 1. Relate to some of the earlier problems done when discovering/using the shortcuts and link the two ideas.
Activity 3-Link this to the proportionality theorem. Guide students to notice that the third sides are parallel (get at the idea of correspond/alt inter etc.) Then shift to the proportions of the sides using the midsegment (with midsegment it is half but with triangle proportionality it is just proportional).
Representations=
Activity 4-Once students are sufficiently comfortable with the relationships transition to using that information to solve for missing information in triangles (i.e. side lengths).

b. **Maximize Student Engagement and Monitor Student Understanding*:*** Involve students in the process as teacher continues to think aloud while guiding student through comparing the figures (on paper #1 and using dgs #2) and eliciting their solution strategies.

III. Guided and Independent Practice

d. Provide guidance as students perform 3-6 more problems (on paper) using midsegment and proportionality of triangles. (Note: cue card available through the theorems completed is available-cover or printed without the information not yet introduced.)

e.

f. Scaffolded inquiry or varied *level of support* as students assume more responsibility for the learning (e.g. *High*: Verbalize the procedures and have students restate and/or apply, *Medium*: Have students verbalize each procedure and apply, *Low*: Have students verbalize all of the steps (chunk together) and apply).

IV. Closure: summarizes the lesson (daily even if the topic is over multiple days) and provides a snapshot of student comprehension.

a. **Big Ideas:** Teacher review with student**:** *What were the two main ideas we learned today?* [midsegment and proportionality of triangle parts]
b. **Exit ticket:** Have students answer two questions using the midsegment and proportionality theorems.
c. **Link to Future Instruction: T:** *Next time we will explore proportional relationships with applications to specific problems. This is what all the work with the shortcuts have been leading up to-when is this useful in the real-world? Is it used often in construction and surveying such as when roads, bridges and buildings are being made.*

Lesson 8

CCSSM Content: G-SRT2, G-SRT-4, G-SRT5

CCSSM Practice: 1-Make sense of problems and persevere in solving them, 2-Reason abstractly and quantitatively, 3-construct viable arguments, 5-Use appropriate tools, 6-Attend to precision, 7-Look for and make use of structure, 8-look for and express regularity in repeated reasoning

Materials: computer (with dgs program), calculator, protractor/ruler, writing materials (e.g. markers, pencils), cue card

Objective: Students will use visuals and dynamic geometry software to apply triangle similarity theorems to indirect measurements.

I. **Launch:**

Review Pre-requisite Skills: Provide a handout with two problems to review from prior lessons on triangles (one of either midsegment or proportionality and one of the shortcuts).

Objective and Link

T: *Great work on the review. This will prepare you for further explorations of similar figures today with applications. Remind me again; what information do you need to check to know if triangles are similar? Which shortcuts or theorems have we explored?*

S: *Similar triangles have congruent angles but the sides are proportional. The shortcuts are AA, SAS and SSS and also midsegment and proportionality within triangles.*

Rationale:

T*: Excellent. You really understand that similar triangles have the same angles, but unlike congruent figures that have the same side lengths the sides of similar figures are related by ratios. This is important for reducing or enlarging and also comes in handy in art or design, which we are going to explore today.*

II. **Investigation:** Show the student some concrete objects that are models of real-world objects such as miniatures of objects. Ask questions such as how artists, architects engineers etc. make plans for building things that are not easily measured (like the shown objects.) Or did you ever wonder how tall something is but have no easy way to measure? Link to the many lessons already explored.

a. **Model Thinking and Action**: Think aloud while explaining how to make comparisons of the figures.

DGS=

Activity 1 Ask student how you could model such a scenario using gsp. (May need to show them one set up and then make some calculations).

Representations=

Activity 2 Then transition to doing problems on paper: do a variety with embedded, separate and flipped. Such as measuring across a lake/river, shadow height versus flag pole etc.

b. **Maximize Student Engagement and Monitor Student Understanding*:*** Involve students in the process as teacher continues to think aloud while guiding student through comparing the figures and eliciting their solution strategies and potential definitions for similarity of figures.

III. Guided and Independent Practice

c. Provide guidance as students perform 3-6 more application problems (on paper with cue card thorough the theorems available-cover or printed without the last/formula information from lesson 9).

d. Scaffolded inquiry or varied *level of support* as students assume more responsibility for the learning (e.g. *High*: Verbalize the procedures and have students restate and/or apply, *Medium*: Have students verbalize each procedure and apply, *Low*: Have students verbalize all of the steps (chunk together) and apply).

IV. Closure: summarizes the lesson (daily even if the topic is over multiple days) and provides a snapshot of student comprehension.

a. **Big Ideas:** How are similar triangles useful in real-life? **S:** bridges etc.

b. **Exit ticket:** Have students answer two application problems.

c. **Link to Future Instruction: T:** *Next time we will explore apply proportions in application problems 2D and 3D figures in addition to linear measurements focused on today.*

Lesson 9

CCSSM Content: G-SRT2, G-SRT5

CCSSM Practice: 1-Make sense of problems and persevere in solving them, 2-Reason abstractly and quantitatively, 3-construct viable arguments, 5-Use appropriate tools, 6-Attend to precision, 7-Look for and make use of structure, 8-look for and express regularity in repeated reasoning

Materials: computer (with dgs program), calculator, protractor/ruler, writing materials (e.g. markers, pencils), cue card

Objective: Students will use visuals and dynamic geometry software to explore linear, 2D and 3D measurements of polygons and polyhedrons.

I. Launch:

Review Pre-requisite Skills: Provide a handout with two problems to review from prior lessons on similar figures. One problem reviews indirect measurement and one problem reviews scale factor and proportions for polygons [to review the immediately prior lesson but then goes back to earlier lessons to be able to recall relevant information for the current lesson]

.

Objective and Link

T: *Great work on the review. This will prepare you for further applications of similarity. Remember that scale factor is useful when comparing any similar figures, not just triangles. We haven't practiced with this in a while. Remind me again what are the two big 'look fors' with similar figures?*

S: *Congruent angles but proportional sides.*

Rationale:

T: *Excellent. You really seem to be getting this idea. For today's lessons we are going back to looking at other figures, not just triangles. It has been a while but when we first started working on this topic we looked at various 2D and 3D figures and this is especially important in advertizing, manufacturing, building as well as art.*

II. Investigation: Have students go back to the 3D figures from lesson 1 to see if they would group the items differently based on newly discovered criteria of similar figures. Then look at perimeter, area, surface area and volume using concrete materials and dgs. Provide probing questions as they explore so that they discover that similar figures that are 1D (linear) are proportional and this includes perimeter. But area (and surface area) are the scale factor squared, while volume is the scale factor cubed.

a. **Model Thinking and Action**: Think aloud while explaining how to make comparisons of the figures.
DGS=
Activity 1-use dgs examples for them to explore similar figures (1D/2D) and conjecture on the relationships of the scale factors etc. Have some preset

figures and they also may create their own. Explore perimeter using the established figures. The perimeter is in the same ratio as the scale factor.
Activity 2- Conjecture on what other measurements of the figures could be made besides the angles and side lengths. This moves into area (2D) then lead into discussion on surface area. (Do not need to calculate but do discuss how to do so-that it is the sum of all the areas). Then extend to volume.
Representations=
Activity 3-Surface area and volume examples using representations not dgs. (abstract=using formulas, 1D =a/b, 2D = a^2/b^2, 3D=a^3/b^3)

b. **Maximize Student Engagement and Monitor Student Understanding*:*** Involve students in the process as teacher continues to think aloud while guiding student through comparing the figures and eliciting their solution strategies and potential definitions for similarity of figures.

III. Guided and Independent Practice

c. Provide guidance as students perform 3-6 more problems (on paper) using ratios for polygons. (Note: Once students have figured out the formulas the cue card is available for all practice problems.)

d. Scaffolded inquiry or varied *level of support* as students assume more responsibility for the learning (e.g. *High*: Verbalize the procedures and have students restate and/or apply, *Medium*: Have students verbalize each procedure and apply, *Low*: Have students verbalize all of the steps (chunk together) and apply).

IV. Closure: summarizes the lesson (daily even if the topic is over multiple days) and provides a snapshot of student comprehension.

a. **Big Ideas:** Teacher review with student**:** *What does it mean for figures to be similar? How does this apply to linear, 2D and 3D?* [same shape but different size, angles congruent, sides proportional; perimeter/linear is the same as scale factor but area/surface area is squared while volume is cubed]

b. ***Exit ticket:*** Have students answer two application problems.

b. **Link to Future Instruction: T:** *Next time we will review all of the similarity material for this unit!*

Lesson 10

CCSSM Content: G-SRT2, G-SRT5

CCSSM Practice: 1-Make sense of problems and persevere in solving them, 2-Reason abstractly and quantitatively, 3-construct viable arguments, 5-Use appropriate tools, 6-Attend to precision, 7-Look for and make use of structure, 8-look for and express regularity in repeated reasoning

Materials: calculator, protractor/ruler, AngLegs, DGS, writing materials (e.g. markers, pencils), cue card

Objective: Students will review similarity of polygons, using representations as appropriate.

I. **Launch:**

   Review Pre-requisite Skills: Verbally review the meaning of similarity with the student.
   T: What does it mean for two figures to be similar? How do you know?
   S: Two figures are similar if the matching (corresponding) angles are congruent (the same measure) but the side (lengths) are proportional..
   Objective, Link and Rationale:
   T: *Excellent recall. Today we are tying together all the information from this unit on similarity. Afterward you will complete the post-assessments for the entire topic.*

II. **Investigation:** Assist the student as he/she completes several problems to review the information on similarity. Provide the cue card/cheat sheet and provide probing questions as they complete the problems. Have available the dgs program if they want to use that to model the problems, as well as other materials they could use to represent the scenarios.

   a. **Model Thinking and Action**: Students will have a review packet (of approximately 10 questions) that include the main points of the unit. Assist students as needed, using think-alouds while explaining how to make comparisons of the figures (if provided) and/or to create appropriate representations in order to answer the questions posed.
   b. **Maximize Student Engagement and Monitor Student Understanding*:*** Involve students in the process as teacher continues to think aloud while guiding students (if the student is unable to independently complete the problems) and eliciting their solution strategies.

III. **Guided and Independent Practice**

   c. Provide guidance as students complete the review problems.
   d. Scaffolded inquiry or varied *level of support* as students assume more responsibility for the learning (e.g. *High*: Verbalize the procedures and have students restate and/or apply, *Medium*: Have students verbalize each

procedure and apply, *Low*: Have students verbalize all of the steps (chunk together) and apply).

IV. Closure: summarizes the lesson (daily even if the topic is over multiple days) and provides a snapshot of student comprehension.

a. **Big Ideas:** Teacher review with student**:** *What does it mean for figures to be similar?* [S: angles congruent, sides proportional]

b. **Link to Future Instruction: T:** *You have done very well with the notion of similarity. Next, you will complete the post-assessments. I wish you good luck and I hope that this will prepare you for Geometry next year.*

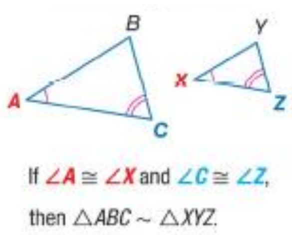
Appendix N: Cue Card

Are figures similar? Check for these:

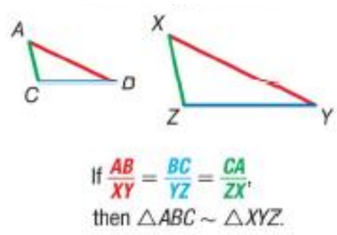
- ☐ 1) The pairs of corresponding ANGLES are CONGRUENT
- ☐ 2) The pairs of corresponding SIDES are PROPORTIONAL

Triangle shortcuts (Similarity Theorems)

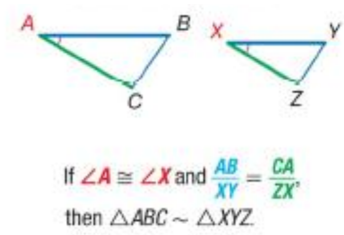
AA



SSS



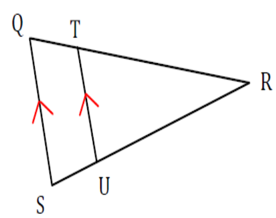
SAS



Triangle Proportionality and converse

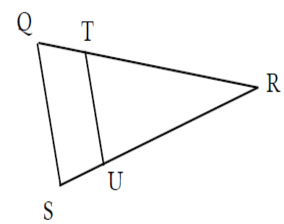
If $\overline{TU} \parallel \overline{QS}$, then

$\frac{RT}{TQ} = \frac{RU}{US}$



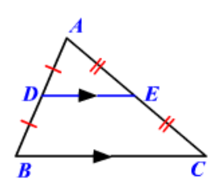
If $\frac{RT}{TQ} = \frac{RU}{US}$, then

$\overline{TU} \parallel \overline{QS}$



Midsegment

If $AD = DB$ and $AE = EC$, then $\overline{DE} \parallel \overline{BC}$ and $DE = \frac{1}{2}BC$.



Measurements of similar figures:

Given the scale factor is $\frac{a}{b}$

Linear: $\frac{a}{b}$ Area: $\frac{a^2}{b^2}$ Volume: $\frac{a^3}{b^3}$

Appendix O: Sample Domain Probe A

Name____Date__________

Directions: Read the specific instructions for each question or set of questions. Answer to the best of your ability. Keep the following in mind:

a) Show all calculations and explain your reasoning.

b) Round decimal answers to the nearest thousandth (three decimal places).

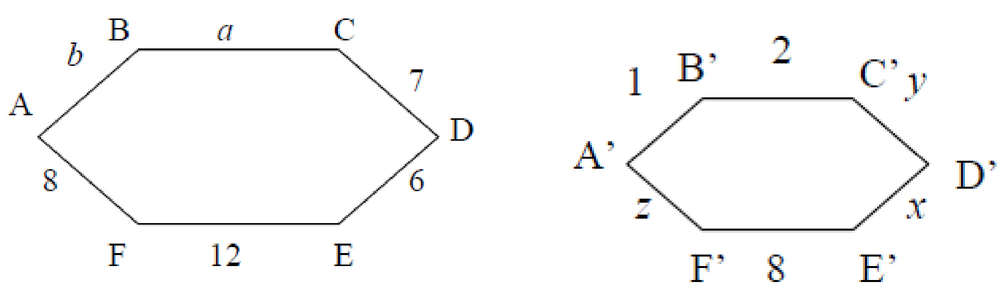
c) Reduce fractions to lowest terms.

d) Make your own sketches to help you if pictures are not provided.

e) Figures are NOT drawn to scale.

f) Use fractions or decimals (rounded appropriately) that are most suited and reasonable for the contextual, real-life problems.

1. Hexagon ABCDEF ~ Hexagon A'B'C'D'E'F'.



a) What is the scale factor of Hexagon ABCDEF to Hexagon A'B'C'D'E'F'?

b) Calculate the values for each of the side lengths for both hexagons (a,b,x,y,z).

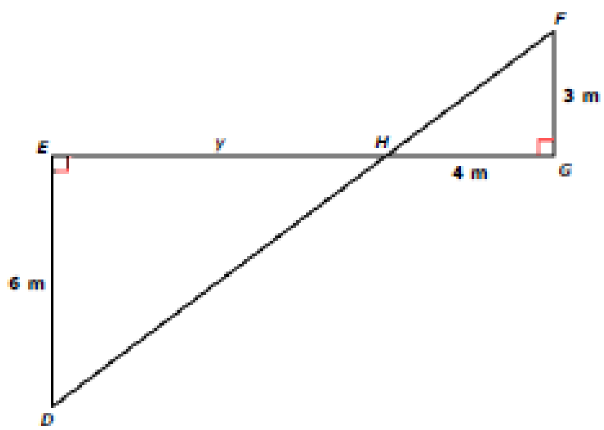
c) Calculate the ratio of the perimeter of Hexagon ABCDEF to Hexagon A'B'C'D'E'F'.

2. Typical school portrait packages include the following size (in inches) photographs:

8 x 10 5 x 7 3 x 5 2 x 3

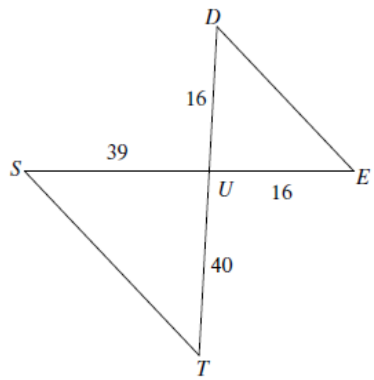
Which size photos are the most similar? Explain and show all calculations to support your reasoning.

3. In the diagram below Δ EDH ~ Δ GFH. What is the value of y?

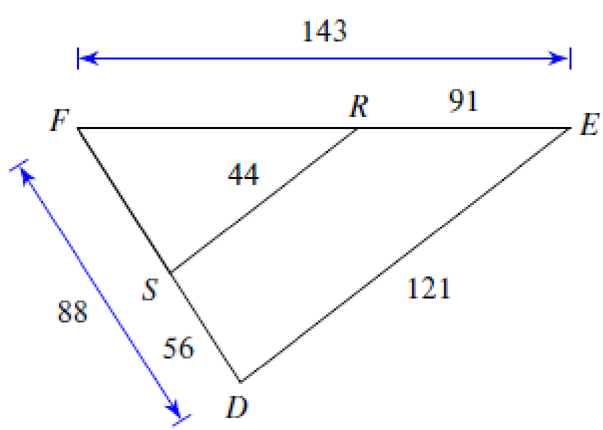


4. Are the two triangles similar? Explain how you know and include relevant similarity theorems if appropriate. If they are similar, write the similarity statement. If they are not

similar, what additional information is needed or what could be changed to make the triangles similar?

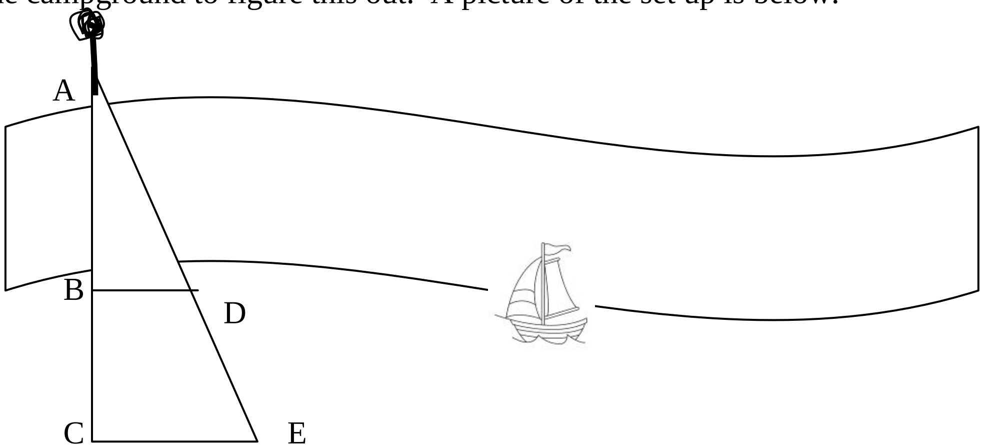


5. Are the two triangles similar? Explain how you know and include relevant similarity theorems if appropriate. If they are similar, write the similarity statement. If they are not similar, what additional information is needed or what could be changed to make the triangles similar?



6. You and your family are staying at a riverside campground. You see a flag at another campground across the river and wonder how wide the river is in case you wanted to

swim over with some friends. The ranger tells you that you use some markers already at the campground to figure this out. A picture of the set up is below.



BC is 45 meters. CE is 90 meters. BD is 60 meters. BD ⊥ AB and CE ⊥ AC.
How wide is the river?

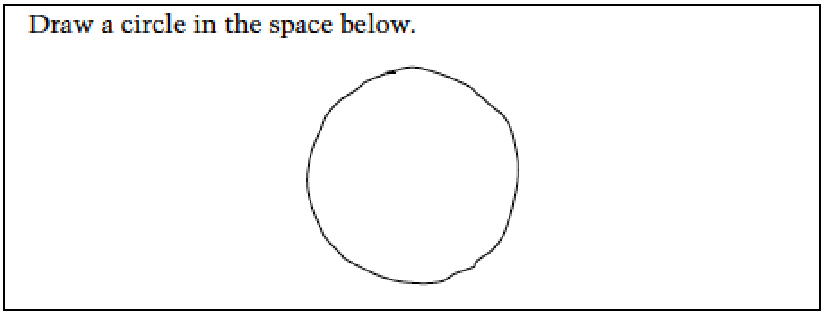
7. Mrs. Smith planted a garden that was 6 feet by 10 feet last year. She had to put a fence around the garden to keep out the deer. She has some left over fencing material, about 50 feet, which she wants to use for this year's garden. She wants to make her garden larger, but similar to the previous garden. What is the most likely dimension her new garden would be?

Appendix P: NAEP Probe

Name____Date________

Directions

Read each question carefully and answer it as well as you can. Do not spend too much time on any one question. For some of the questions you may need to write or draw the answer. You can see how this is done in the example below.



If you need to use a ruler to measure round to the nearest half inch or nearest half centimeter.

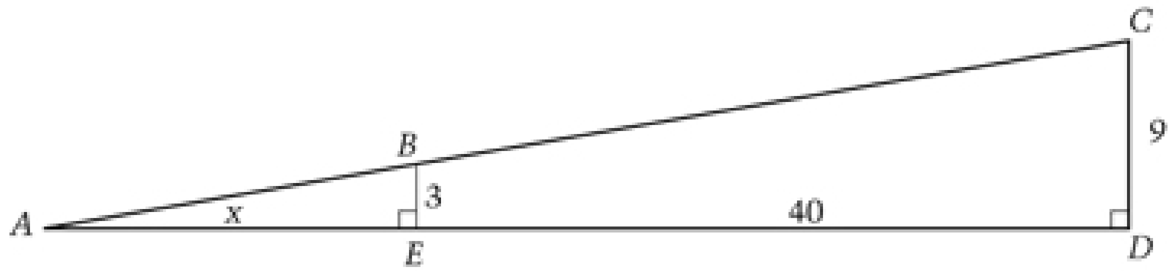
You may be permitted to use a calculator for at least one part. You may use either your own calculator or the calculator provided. If you are permitted to use a calculator, you will have to decide when to use it in each section where its use is permitted. For some questions using the calculator is helpful, but for other questions the calculator may not be helpful. If you are using the calculator provided, make sure you know how to use it. If the calculator does not work or if you do not know how to use it, raise your hand and ask for help.

REMEMBER:

Read each question CAREFULLY. CIRCLE only ONE choice for each question or write your answer in the space provided. If you change your answer, ERASE your first answer COMPLETELY. CHECK OVER your work if you finish a section early.

Do not go past the STOP sign at the end of each section until you are told to do so.

STOP

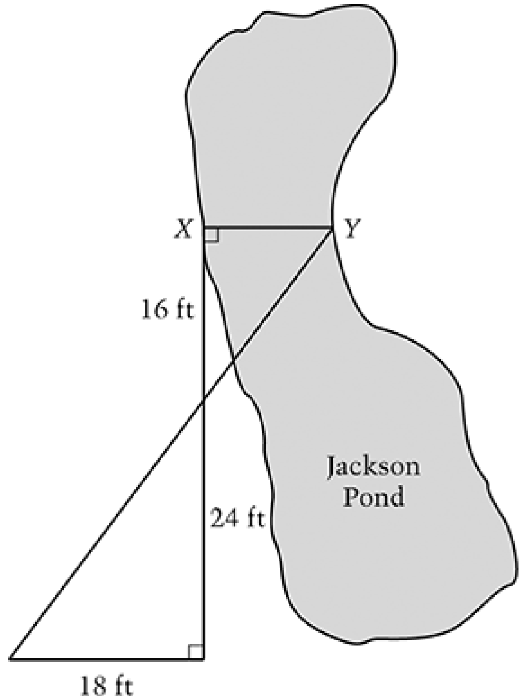


1. The figure above shows two right angles. The length of *AE* is *x* and the length of *DE* is 40. Show all of the steps that lead to finding the value of *x*. Your last step should give the value of *x*.

2. Which of the following pairs of geometric figures must be similar to each other? Explain your reasoning.

   A. Two equilateral triangles
   B. Two isosceles triangles
   C. Two right triangles
   D. Two rectangles
   E. Two parallelograms

3. The figure below shows Jackson Pond.



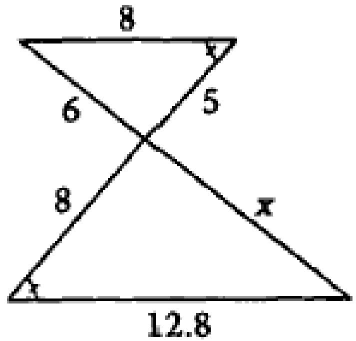
What is the distance across Jackson Pond from point X to point Y?

A. 8 feet
B. 10 feet
C. 12 feet
D. 14 feet
E. 22 feet

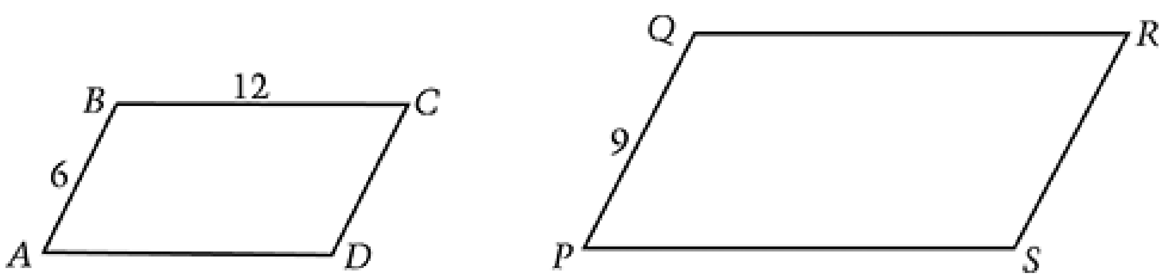
4. Roxanne plans to enlarge her photograph, which is 4 inches by 6 inches. Which of the following enlargements maintains the same proportions as the original photograph? Justify your answer.

5 inches by 7 inches   5 inches by 7 ½ inches

5. In the figure below, the two triangles are similar. What is the value of x?



Answer:____________________



6. Parallelograms ABCD and PQRS above are similar. What is the length of side QR?

A. 4.5
B. 9
C. 12
D. 15
E. 18

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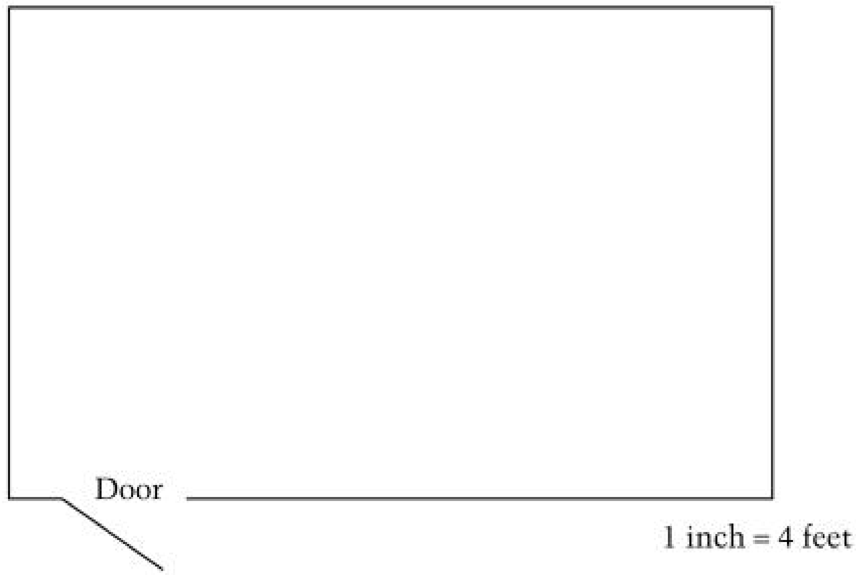


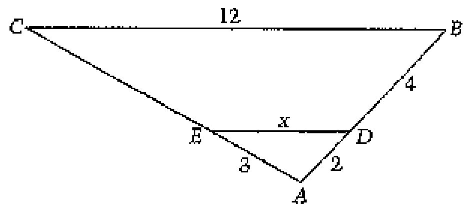
7. The floor of a room shown in the figure above is to be covered with tiles. One box of floor tiles will cover 25 square feet. Use your ruler to determine how many whole boxes of these tiles must be bought to cover the entire floor.

________ boxes of tiles

Explain your reasoning in the space below.

8. A scale drawing of a rectangular room is 5 inches by 3 inches. If 1 inch on this scale drawing represents 3 feet, what are the dimensions of the room?

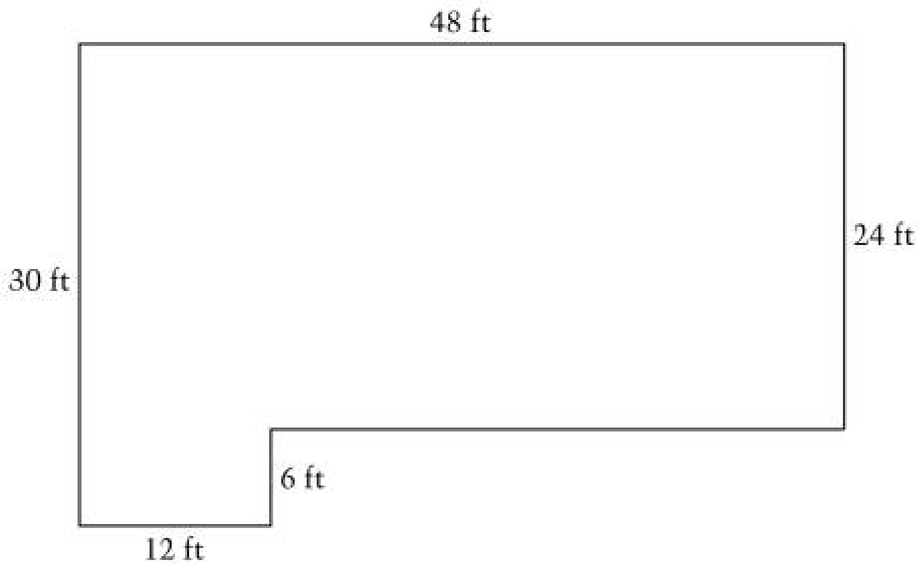
A. 5 feet by 3 feet
B. 5 feet by 9 feet
C. 15 feet by 3 feet
D. 15 feet by 5 feet
E. 15 feet by 9 feet



9. If triangles *ADE* and *ABC* shown in the figure above are similar, what is the value of *x*?

A. 4
B. 5
C. 6
D. 8
E. 10

The following question refers to the following diagram.



10. If you were to redraw the diagram using a scale of 3/4 inch = 10 feet, what would be the length of the side that is 48 feet?

A. 3.0 in
B. 3.6 in
C. 5.6 in
D. 7.5 in
E. 12.0 in

STOP

Appendix P: Treatment Fidelity Checklist

<table>
<tr><td colspan="2">Observer:</td><td>Participant</td><td>Lesson</td></tr>
<tr><td colspan="4">Directions:
Indicate the observed behaviors by placing a check mark in the spaces below.</td></tr>
<tr><td>Item</td><td>Description</td><td>Observed?</td><td>Notes</td></tr>
<tr><td colspan="4">Advance Organizer:</td></tr>
<tr><td>1</td><td>Review of prerequisite skills</td><td></td><td></td></tr>
<tr><td>2</td><td>Lesson objective stated at the beginning of instruction</td><td></td><td></td></tr>
<tr><td>3</td><td>Rationale for learning the topic</td><td></td><td></td></tr>
<tr><td colspan="4">Investigation:</td></tr>
<tr><td>4</td><td>Maximizing student engagement via questions and prompts</td><td></td><td></td></tr>
<tr><td>5</td><td>Modeling the thinking and action for procedures needed to solve the problem</td><td></td><td></td></tr>
<tr><td>6</td><td>Prompting questions to facilitate student exploration</td><td></td><td></td></tr>
<tr><td colspan="4">Multiple Practice Opportunities:</td></tr>
<tr><td>7</td><td>Opportunities for students to practice tasks demonstrated or explored. Teacher acts as facilitator. May include guided practice and/or individual work.</td><td></td><td></td></tr>
<tr><td colspan="4">Visual Representations:</td></tr>
<tr><td>8</td><td>Multiple opportunities to utilize a variety of visual representations (i.e. concrete, sketch, DGS)</td><td></td><td></td></tr>
</table>

NCTM Process Standards and CCSS Practice Standards:			
9	Reasoning, which includes making mathematical conjectures, justifying solutions and methods, and/or using multiple solution methods		
10	Communication, using the language of mathematics (i.e., transformations, dilation, theorem) is used to express mathematical ideas and/or articulating reasoning to self or teacher		
11	Making connections to previously learned mathematics and/or to other mathematical (e.g., algebra) or contexts outside of mathematics or real-world (e.g., language arts, science social studies)		
12	Creating or using models and representations such as diagrams, graphs, symbolic notations, or concrete materials to solve a problem		
13	Use tools strategically by choosing the most appropriate tool (i.e., paper-pencil, calculator, DGS, manipulatives) to solve problems		
Closure:			
14	Review the main ideas at the end of the lesson		
15	Assessment, which includes student completing an independent task or responding orally to teacher's questions.		

Appendix R: Assessment Fidelity Checklist

<table>
<tr><td colspan="2">Observer:</td><td>Participant</td><td>Session/Probe</td></tr>
<tr><td colspan="4">Directions:
Indicate the observed behaviors by circling the best response in the spaces below.</td></tr>
<tr><td>Item</td><td>Description</td><td>Observed?</td><td>Notes</td></tr>
<tr><td>1</td><td>Teacher read instructions and questions verbatim to student</td><td>Yes
No
NA</td><td></td></tr>
<tr><td>2</td><td>Instructor did not provide any assistance to students other than saying, “do the best that you can.”</td><td>Yes
No</td><td></td></tr>
<tr><td>3</td><td>Instructor provided calculator, ruler or other supplies requested OR provided reasonable explanation as to why items could not be supplied</td><td>Yes
No</td><td></td></tr>
</table>

Appendix S: Social Validity Measure

Section 1: Directions: Indicate the degree to which you agree or disagree with each statement below by **circling** the best choice.

	Strongly Disagree	**Disagree**	**Neutral/ In-Between**	**Agree**	**Strongly Agree**
I learned to successfully solve geometric similarity problems.	**1**	**2**	**3**	**4**	**5**
The use of manipulatives helped me to solve geometric similarity problems.	**1**	**2**	**3**	**4**	**5**
The use of the dynamic geometry software (*Geometer's Sketchpad*, GeoGebra) on the computer helped me to solve geometric similarity problems.	**1**	**2**	**3**	**4**	**5**
The use of the cue card helped me to solve geometric similarity problems.	**1**	**2**	**3**	**4**	**5**
The word problem scenarios helped me understand the relationship between the concepts and the mathematical representations.	**1**	**2**	**3**	**4**	**5**
This intervention was worth my time.	**1**	**2**	**3**	**4**	**5**
I would recommend this intervention to other students.	**1**	**2**	**3**	**4**	**5**
As a result of the intervention, I feel better about my geometry skills.	**1**	**2**	**3**	**4**	**5**
I would be willing to use this intervention (i.e., manipulatives and computer) for additional geometry topics.	**1**	**2**	**3**	**4**	**5**

Section 2: Write your response to each question below:

1) Tell me ways in which you can see or use geometric similarity outside of school.

2) Do you enjoy/like geometry? Why or why not?

3) Are you interested in learning more about geometric transformations? Why or why not?

4) How did the intervention help you understand geometric similarity transformations?

5) What aspects of the intervention did you <u>like best?</u>

6) What aspects of the intervention did you <u>like least</u>?

7) What suggestions do you have for improving the intervention?

Appendix T: Mathematics Attitude Measure

Directions: Each of the statements expresses a feeling that a particular person has toward mathematics, or geometry in particular. You are to express, on a five-point scale, the extent of your agreement with the statement. Please use the scale below to help you circle the best choice regarding your feelings.

	Strongly Disagree	Disagree	Neutral/ In-Between	Agree	Strongly Agree
My mathematics teachers have encouraged me to do well.	**1**	**2**	**3**	**4**	**5**
I am confident when I use mathematics.	**1**	**2**	**3**	**4**	**5**
I am better at mathematics than other subjects.	**1**	**2**	**3**	**4**	**5**
I often make pictures, drawings, or sketches to help me figure out problems.	**1**	**2**	**3**	**4**	**5**
If I cannot do a problem, I keep trying different ideas and try to think of how it might be similar to other problems.	**1**	**2**	**3**	**4**	**5**
If I make mistakes, I work until I have corrected them.	**1**	**2**	**3**	**4**	**5**
I would rather figure out the answer to a math problem on my own than have the teacher (or peer) tell me the answer.	**1**	**2**	**3**	**4**	**5**
Using a calculator *is not* essential to help me understand mathematical concepts better.	**1**	**2**	**3**	**4**	**5**
Projects and hands-on on activities are better for me to understand a concept.	**1**	**2**	**3**	**4**	**5**
Most of the problems that my teachers use that are supposed to be real-world problems or applications *do* seem like something useful rather than fake, made up problems just for class.	**1**	**2**	**3**	**4**	**5**
I do not really get anxious or worried about coming to math class or taking tests.	**1**	**2**	**3**	**4**	**5**
I see geometry in everyday things.	**1**	**2**	**3**	**4**	**5**
Understanding and using geometry will be useful in my future.	**1**	**2**	**3**	**4**	**5**
Geometry problems often have more than one method/way to find a solution.	**1**	**2**	**3**	**4**	**5**
Geometry is easier for me to understand than Algebra.	**1**	**2**	**3**	**4**	**5**
Math, and especially geometry, helps develop a person's mind and teaches him/her to be a better thinker.	**1**	**2**	**3**	**4**	**5**

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