ABSTRACT

Title of dissertation: Weakly Compressible Navier-Stokes Approximation of Gas Dynamics

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This dissertation addresses mathematical issues regarding weakly compressible approximations of gas dynamics that arise both in fluid dynamical and in kinetic settings. These approximations are derived in regimes in which (1) transport coefficients (viscosity and thermal conductivity) are small and (2) the gas is near an absolute equilibrium — a spatially uniform, stationary state. When we consider regimes in which both the transport scales and Re vanish, we derive the *weakly compressible Stokes approximation* — a *linear* system. When we consider regimes in which the transport scales vanish while Re maintains order unity, we derive the *weakly compressible Navier-Stokes approximation*—a *quadratic* system. Each of these weakly compressible approximations govern both the acoustic and the incompressible modes of the gas.

In the fluid dynamical setting, our derivations begin with the fully compressible Navier-Stokes system. We show that the structure of the weakly compressible Navier-Stokes system ensures that it has global weak solutions, thereby extending the Leray theory for the incompressible Navier-Stokes system. Indeed, we show that

this is the case in a general setting of hyperbolic-parabolic systems that possess an entropy under a structure condition (which is satisfied by the compressible Navier-Stokes system.) Moreover, we obtain a regularity result for the acoustic modes for the weakly compressible Navier-Stokes system.

In the kinetic setting, our derivations begin with the Boltzmann equation. Our work extends earlier derivations of the incompressible Navier-Stokes system by the inclusion of the acoustic modes. We study the validity of these approximations in the setting of the DiPerna-Lions global solutions. Assuming that DiPerna-Lions solutions satisfy the local conservation law of energy, we use a relative entropy method to justify the weakly compressible Stokes approximation which unifies the Acoustic-Stokes limits result of Golse-Levermore, and to justify the weakly compressible Navier-Stokes approximation modulo further assumptions about passing to the limit in certain relative entropy dissipation terms. This last result extends the result of Golse-Levermore–Saint-Raymond for the incompressible Navier-Stokes system.

Weakly Compressible Navier-Stokes Approximation of Gas Dynamics

by

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DEDICATION

To My Parents:

Mr. Sheng-Guo Jiang and Ms. Zhu-Yun Su

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CONTENTS

1. INTRODUCTION

This dissertation addresses mathematical issues regarding weakly compressible approximations of gas dynamics that arise both in fluid dynamical and in kinetic settings. These approximations are derived in regimes in which (1) transport coefficients (viscosity and thermal conductivity) are small and (2) the gas is near an absolute equilibrium — a spatially uniform, stationary state. The ratio of the size of the initial fluctuations of the gas about this absolute equilibrium to the transport scales is related to the initial Reynolds number Re. When we consider regimes in which both the transport scales and Re vanish, we derive the *weakly compressible Stokes approximation* — a *linear* system. When we consider regimes in which the transport scales vanish while Re maintains order unity, we derive the *weakly compressible Navier-Stokes approximation*—a *quadratic* system. Each of these weakly compressible approximations govern both the acoustic and the incompressible modes of the gas. When the acoustic modes are neglected, they reduce to the incompressible Stokes and Navier-Stokes systems respectively. These systems and their validity are the focus of this dissertation.

In the fluid dynamical setting, our derivations begin with the fully compressible Navier-Stokes system. The weakly compressible Navier-Stokes approximation is similar to one studied by Schochet [68] that derived from the barotropic Euler system. Our approximation differs from his “weakly barotropic” approximation by the inclusion of a heat equation and of dissipative terms. We show that the structure of the weakly compressible Navier-Stokes system ensures that it has global weak solutions, thereby extending the Leray theory [47] for the incompressible Navier-Stokes system. Indeed, we show that this is the case in a general setting of hyperbolic-parabolic systems that possess an entropy under a structure condition (which is satisfied by the compressible Navier-Stokes system.) Moreover, we obtain a regularity result for the acoustic modes for the weakly compressible Navier-Stokes system. This extends the results of Masmoudi [58] and Danchin [18] who studied the weakly barotropic system.

In the kinetic setting, our derivations begin with the Boltzmann equation. Our work extends earlier derivations of the incompressible Navier-Stokes system [7] by the inclusion of the acoustic modes. We study the validity of these approximations in the setting of the DiPerna-Lions global solutions [20]. (There is no analogous global theory for the compressible Navier-Stokes system.) Assuming that DiPerna-Lions solutions satisfy the local conservation law of energy, we use a relative entropy method to justify the weakly compressible Stokes approximation which unifies the Acoustic-Stokes limits result of Golse-Levermore [29], and to justify the weakly compressible Navier-Stokes approximation modulo further assumptions about passing to the limit in certain relative entropy dissipation terms. This last result extends the result of Golse-Levermore–Saint-Raymond [31] for the incompressible Navier-Stokes system.

1.1 *Fluid Setting*

To see how these weakly compressible approximations arise, we begin with the Navier-Stokes system for ideal gas dynamics:

$$\begin{aligned} \partial_t\rho + \nabla_x\cdot(\rho u) = 0\,, \\ \partial_t(\rho u) + \nabla_x\cdot(\rho u\otimes u) + \nabla_x p + \nabla_x\cdot\Sigma = 0\,, \\ \partial_t\left[\tfrac{1}{2}\rho|u|^2 + \rho e\right] + \nabla_x\cdot\left[(\tfrac{1}{2}\rho|u|^2 + \rho e + p)u\right] + \nabla_x\cdot[\Sigma\cdot u + q] = 0\,, \end{aligned} \tag{1.1.0.1}$$

with initial data $(\rho^{in}, u^{in}, \theta^{in})$, where ρ denotes the density, u the velocity, and e the specific internal energy, p the pressure, Σ the stress tensor, and q the heat flux. For polytropic gases, if we denote by θ temperature,

$$\begin{aligned} &\Sigma = -\mu\left[\nabla_x u + (\nabla_x u)^T - \tfrac{2}{D}(\nabla_x\cdot u)I\right]\,, \quad q = -\kappa\nabla_x\theta\,, \\ &e = \tfrac{1}{\gamma-1}\theta\,, \quad p = (\gamma-1)\rho e = \rho\theta\,. \end{aligned} \tag{1.1.0.2}$$

For monatomic gas, $\gamma = \frac{D+2}{D}$, where D denotes the spatial dimension. For this introduction we consider the case that the viscosity μ and heat conductivity κ are functions of temperature only. When μ and κ are zero, (1.1.0.1) reduces to the compressible Euler system.

The current mathematical understanding of solutions to both compressible Euler system and Navier-Stokes system is far from satisfactory. Solutions to the compressible Euler system are known to become singular in finite time even for very smooth initial data (see Sideris [69].) Recently, global existence theory of weak solutions, which is parallel to that of seminal work of Leray for incompressible Navier-Stokes [47], has been developed by P.-L. Lions for isentropic gases [55] and Feireisl for a special class of pressure laws depending on both density and temperature [23]. However, for ideal gases, which can be derived from kinetic equation for monatomic gas, the global existence for general initial data are not available.

We are interested in weakly compressible approximations which can be better understood mathematically. These are derived in regimes in which the gas is near an absolute equilibrium and the transport coefficients (viscosity and thermal conductivity) are small. For the periodic case, i.e., for the spatial domain is $\mathbb{T}^D$, the only stationary solutions to the compressible Navier-Stokes equations are absolute equilibria. This can be seen from the following argument.

From the compressible Navier-Stokes system, we have the internal energy equation:

$$\partial_t(\rho e) + \nabla_x \cdot (\rho e u) + p\nabla_x \cdot u = -\Sigma : \nabla_x u - \nabla_x \cdot q \,, \tag{1.1.0.3}$$

i.e.,

$$\rho \frac{De}{Dt} + p\nabla_x \cdot u = -\Sigma : \nabla_x u - \nabla_x \cdot q \,, \tag{1.1.0.4}$$

where the convective derivative

$$\frac{D}{Dt} \equiv (\partial_t + u \cdot \nabla_x) \,. \tag{1.1.0.5}$$

Thermodynamics tells us that e, ρ, and p are related to the entropy s and temperature θ by

$$\frac{De}{Dt} = \theta \frac{Ds}{Dt} + \frac{p}{\rho^2}\frac{D\rho}{Dt} \,. \tag{1.1.0.6}$$

It follows that

$$\begin{aligned} \rho \frac{De}{Dt} + p\nabla_x \cdot u &= \rho \frac{De}{Dt} - \frac{p}{\rho}\frac{D\rho}{Dt} \\ &= \rho\theta \left(\frac{1}{\theta}\frac{De}{Dt} - \frac{p}{\rho^2\theta}\frac{D\rho}{Dt} \right) \\ &= \rho\theta \frac{Ds}{Dt} \,. \end{aligned} \tag{1.1.0.7}$$

Combining (1.1.0.4) and (1.1.0.7), we have

$$\rho \frac{Ds}{Dt} = -\frac{1}{\theta} \Sigma : \nabla_x u + \frac{1}{\theta} \nabla_x \cdot (\kappa \nabla_x \theta) \,. \tag{1.1.0.8}$$

Integrating (1.1.0.8) over $\mathbb{T}^D$, and integrating by parts yields

$$\frac{d}{dt} \int_{\mathbb{T}^D} \rho s \, dx = \int_{\mathbb{T}^D} \frac{1}{\theta} \frac{1}{2\mu} \Sigma : \Sigma \, dx + \int_{\mathbb{T}^D} \frac{\kappa}{\theta^2} |\nabla_x \theta|^2 \, dx \,. \tag{1.1.0.9}$$

Stationary solutions therefore satisfy

$$\int_{\mathbb{T}^D} \frac{1}{\theta} \frac{1}{2\mu} \Sigma : \Sigma \, dx + \int_{\mathbb{T}^D} \frac{\kappa}{\theta^2} |\nabla_x \theta|^2 \, dx = 0 \,. \tag{1.1.0.10}$$

Because both terms are nonnegative, we see that $\nabla_x \theta = 0$, i.e., $\theta \equiv$constant and $\Sigma = 0$, then after some elementary (but nontrivial) calculations, we can derive $u \equiv$constant vector, when the domain is periodic. Finally, the momentum equation gives $\nabla_x(\rho\theta) = 0$, whereby $\rho \equiv$constant.

After a suitable Galilean transformation, these stationary homogeneous state can be fixed to be $(\rho_*, 0, \theta_*)$. To get the incompressible limit, we need to nondimensionize the compressible Navier-Stokes system. First, we determine the dimensional scales.

Dimensional Analysis

The volume of the periodic box determines a length scale L by setting

$$\int dx = L^D \,. \tag{1.1.0.11}$$

From the initial data we can determine the scales of the density and temperature

$$\begin{aligned} &\int \rho^{in} \, dx = \rho_* L^D \,, \quad \int \rho^{in} u^{in} \, dx = 0 \,, \\ &\int \tfrac{1}{2} \rho^{in} |u^{in}|^2 + \rho^{in} e^{in} \, dx = \rho_* \theta_* L^D \,. \end{aligned} \tag{1.1.0.12}$$

The scales of μ and κ can be defined as

$$\mu_* = \epsilon\mu(\theta_*)\,, \quad \kappa_* = \epsilon\kappa(\theta_*)\,, \tag{1.1.0.13}$$

where ϵ is a small number. The reference state introduces microscopic length and time scale into the problem that can be determined from the kinematic viscosity $\nu_* = \mu_*/\rho_*$ and a thermal velocity $v_* = \sqrt{c_{p*}\theta_*}$, where $c_{p*} = c_p(\rho_*, \theta_*)$ is the scale of specific heat at constant pressure c_p. Then, the mean-free path for the gas is on the order of ν_*/v_*, while the mean-free time scale as ν_*/v_*^2. Another dimensionless parameter can be derived directly from this reference state is the Prandtl number

$$\mathrm{Pr} = \frac{\mu_* c_{p*}}{\kappa_*}\,, \tag{1.1.0.14}$$

which relates the transport coefficients. For most gases, the Prandtl number is of order unity.

If U_* is the bulk velocity scale, we define two important dimensionless parameters, the Mach number Ma and Reynolds number Re:

$$\mathrm{Ma} = \frac{U_*}{v_*}\,, \quad \mathrm{Re} = \frac{U_* L}{\nu_*}\,. \tag{1.1.0.15}$$

Remark: our definition of the Mach number is different with the usual one, which is the ratio of bulk velocity to the sound speed. Because the thermal speed v_* is the same order with sound speed, so the Mach number Ma by our definition is same as the usual Mach number up to a factor of order one.

Now the compressible Navier-Stokes system can be reformulated in terms of dimensionless variables; these are introduced below adorned with hats. Dimensionless time, space, and the bulk velocity are defined by

$$t = \frac{1}{\tau}\frac{L}{v_*}\hat{t}\,, \quad x = L\hat{x}\,, \quad u = v_*\hat{u}\,; \tag{1.1.0.16}$$

In the above definition, $\tau = 1$ when we consider short time scale, while $\tau = \epsilon$, a small number when consider longer time scale.

The dimensionless density and temperature

$$\rho = \rho_*\hat{\rho}\,, \quad \theta = \theta_*\hat{\theta}\,; \tag{1.1.0.17}$$

the viscosity and heat conductivity

$$\mu = \epsilon\mu(\theta_*)\hat{\mu} = \mu_*\hat{\mu}\,, \quad \kappa = \epsilon\kappa(\theta_*)\hat{\kappa} = \kappa_*\hat{\kappa}\,, \tag{1.1.0.18}$$

where ϵ is some small number which measures the size of the dissipation terms.

Substituting all of these rescaled quantities into the original equations (1.1.0.1), and dropping all carets, yields

$$\begin{aligned} \tau\partial_t\rho + \nabla_x\cdot(\rho u) =& 0\,, \\ \tau\partial_t(\rho u) + \nabla_x\cdot(\rho u\otimes u) + \nabla_x(\rho\theta) =& -\tfrac{\mathrm{Ma}}{\mathrm{Re}}\nabla_x\cdot\Sigma\,, \\ \tau\partial_t\left[\tfrac{1}{2}\rho|u|^2 + \tfrac{D}{2}\rho\theta\right] + \nabla_x\cdot\left[(\tfrac{1}{2}\rho|u|^2 + \tfrac{D+2}{2}\rho\theta)u\right] =& -\tfrac{\mathrm{Ma}}{\mathrm{RePr}}\nabla_x\cdot(\kappa\nabla_x\theta) \\ & -\tfrac{\mathrm{Ma}}{\mathrm{Re}}\nabla_x\cdot[\Sigma\cdot u]\,, \end{aligned} \tag{1.1.0.19}$$

with initial data $(\rho^{in}, u^{in}, \theta^{in})$. From the famous von Karman relation:

$$\mathrm{Kn} = \frac{\mathrm{Ma}}{\mathrm{Re}}\,, \tag{1.1.0.20}$$

where Kn denotes the Knudsen number which is the ratio of the microscopic and macroscopic length scales. Knudsen number must be small in order to justify any use of a fluid description. We set $\mathrm{Kn} = \epsilon$. The selection of the time scale parameter τ depends on ϵ. So we set $\tau = \tau_\epsilon$.

Weakly compressible approximations are derived in regimes in which gas is near an absolute equilibrium which is a spatially uniform, stationary state. After a Galilean transformation and a suitable selection of units, we can assume that the absolute equilibrium is $(1, 0, 1)$. Suppose $(\rho_\epsilon, u_\epsilon, \theta_\epsilon)$ is a solution to the (scaled) compressible Navier-Stokes system (1.1.0.19) (with $\tau = \tau_\epsilon$, $\frac{\mathrm{Ma}}{\mathrm{Re}} = \epsilon$, and $\mathrm{Pr} = 1$,) and the size of the initial fluctuations is δ_ϵ, i.e.,

$$\rho_\epsilon^{in} = 1 + \delta_\epsilon \tilde{\rho}_\epsilon^{in}\,, \quad u_\epsilon^{in} = \delta_\epsilon \tilde{u}_\epsilon^{in}\,, \quad \theta_\epsilon^{in} = 1 + \delta_\epsilon \tilde{\theta}_\epsilon^{in}\,. \tag{1.1.0.21}$$

Suppose also that at later times, the size of fluctuations is also δ_ϵ, i.e.,

$$\rho_\epsilon = 1 + \delta_\epsilon \tilde{\rho}_\epsilon\,, \quad u_\epsilon = \delta_\epsilon \tilde{u}_\epsilon\,, \quad \theta_\epsilon = 1 + \delta_\epsilon \tilde{\theta}_\epsilon\,. \tag{1.1.0.22}$$

From the bulk velocity fluctuation we deduce that δ_ϵ has the same order with the Mach number Ma. Thus, from von Karman relation, we have

$$\mathrm{Re} = \frac{\delta_\epsilon}{\epsilon}\,. \tag{1.1.0.23}$$

In this dissertation, we consider the weakly compressible approximations in two regimes:

1, Both ϵ and the Reynolds number Re vanish, i.e., $\frac{\delta_\epsilon}{\epsilon} \to 0$;

2, ϵ vanishes and Reynolds number Re is of order 1, set $\delta_\epsilon = \epsilon$.

For $\tau_\epsilon = 1$, in both cases $\frac{\delta_\epsilon}{\epsilon} \to 0$ and $\delta_\epsilon = \epsilon$, suppose that the initial fluctuations $(\tilde{\rho}_\epsilon^{in}, \tilde{u}_\epsilon^{in}, \tilde{\theta}_\epsilon^{in}) \to (\tilde{\rho}^{in}, \tilde{u}^{in}, \tilde{\theta}^{in})$, then, $(\tilde{\rho}_\epsilon, \tilde{u}_\epsilon, \tilde{\theta}_\epsilon) \to (\tilde{\rho}, \tilde{u}, \tilde{\theta})$, where $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ satisfies the acoustic system:

$$\begin{aligned} \partial_t\tilde{\rho} + \nabla_x\cdot\tilde{u} = 0\,, &\quad \tilde{\rho}(0,x) = \tilde{\rho}^{in}(x)\,, \\ \partial_t\tilde{u} + \nabla_x(\tilde{\rho}+\tilde{\theta}) = 0\,, &\quad \tilde{u}(0,x) = \tilde{u}^{in}(x)\,, \\ \tfrac{D}{2}\partial_t\tilde{\theta} + \nabla_x\cdot\tilde{u} = 0\,, &\quad \tilde{\theta}(0,x) = \tilde{\theta}^{in}(x)\,, \end{aligned} \tag{1.1.0.24}$$

which can be written as

$$\partial_t\tilde{\mathbf{U}} + \mathcal{A}\tilde{\mathbf{U}} = 0\,, \quad \tilde{\mathbf{U}}(0,x) = \tilde{\mathbf{U}}^{in}(x)\,. \tag{1.1.0.25}$$

Here the acoustic operator $\mathcal{A}$ is

$$\mathcal{A}\tilde{\mathbf{U}} = \mathcal{A}\begin{pmatrix}\tilde{\rho}\\ \tilde{u}\\ \tilde{\theta}\end{pmatrix} = \begin{pmatrix}\nabla_x\cdot\tilde{u}\\ \nabla_x(\tilde{\rho}+\tilde{\theta})\\ \frac{2}{D}\nabla_x\cdot\tilde{u}\end{pmatrix}\,. \tag{1.1.0.26}$$

Now we consider the longer time scale $\tau_\epsilon = \epsilon$. In this time scale, the weakly compressible approximations in two regimes are different. In case 1, i.e., $\frac{\delta_\epsilon}{\epsilon} \to 0$, under the same assumption on the initial fluctuations, i.e., $(\tilde{\rho}_\epsilon^{in}, \tilde{u}_\epsilon^{in}, \tilde{\theta}_\epsilon^{in}) \to (\tilde{\rho}^{in}, \tilde{u}^{in}, \tilde{\theta}^{in})$, then weakly compressible approximation of the compressible Navier-Stokes system is

$$\begin{aligned} &\partial_t\tilde{\rho}_\epsilon + \frac{1}{\epsilon}\nabla_x\cdot\tilde{u}_\epsilon = 0\,, \\ &\partial_t\tilde{u}_\epsilon + \frac{1}{\epsilon}\nabla_x(\tilde{\rho}_\epsilon + \tilde{\theta}_\epsilon) = \nabla_x\cdot\left[\mu_*(\nabla_x\tilde{u}_\epsilon + \nabla_x\tilde{u}_\epsilon^T - \tfrac{2}{D}\nabla_x\cdot\tilde{u}_\epsilon I)\right]\,, \\ &\tfrac{D}{2}\partial_t\tilde{\theta}_\epsilon + \frac{1}{\epsilon}\nabla_x\cdot\tilde{u}_\epsilon = \nabla_x\cdot(\kappa_*\nabla_x\tilde{\theta}_\epsilon)\,. \end{aligned} \tag{1.1.0.27}$$

with initial data $(\tilde{\rho}^{in}, \tilde{u}^{in}, \tilde{\theta}^{in})$. (1.1.0.27) can be written as

$$\partial_t\tilde{\mathbf{U}} + \mathcal{A}\tilde{\mathbf{U}} = \mathcal{D}\tilde{\mathbf{U}}\,, \quad \tilde{\mathbf{U}}(0,x) = \tilde{\mathbf{U}}^{in}(x)\,, \tag{1.1.0.28}$$

where the diffusion operator $\mathcal{D}$ is

$$\mathcal{D}\tilde{\mathbf{U}} = \mathcal{D}\begin{pmatrix} \tilde{\rho} \\ \tilde{u} \\ \tilde{\theta} \end{pmatrix} = \begin{pmatrix} 0 \\ \nabla_x \cdot (\mu_* \sigma(\tilde{u})) \\ \frac{2}{D}\nabla_x \cdot (\kappa_* \nabla_x \tilde{\theta}) \end{pmatrix} . \tag{1.1.0.29}$$

Here $\sigma(u)$ is

$$\sigma(\tilde{u}) = \nabla_x \tilde{u} + (\nabla_x \tilde{u})^T - \tfrac{2}{D}(\nabla_x \cdot \tilde{u})I \, . \tag{1.1.0.30}$$

We call this linear system as **weakly compressible Stokes system** which is the linearization of the compressible Navier-Stokes system about $(1, 0, 1)$.

Incompressible Stokes Limit: As $\epsilon \to 0$, the behavior of the compressible Stokes system (1.1.0.27) is singular. The limit in the null space of the acoustic operator $\mathcal{A}$ is the incompressible Stokes system:

$$\begin{aligned} \nabla_x \cdot \tilde{u} &= 0 \, , \\ \partial_t \tilde{u} + \nabla_x \tilde{p} &= \mu \Delta_x \tilde{u} \, , \\ \tfrac{D+2}{2} \partial_t \tilde{\theta} &= \kappa \Delta_x \tilde{\theta} \, . \end{aligned} \tag{1.1.0.31}$$

with initial data $\Pi \tilde{U}^{in}$, where $\tilde{U}^{in} = (\tilde{\rho}^{in}, \tilde{u}^{in}, \tilde{\theta}^{in})$, and Π is the projection onto $\text{Null}(\mathcal{A})$, which we call "incompressible mode". When the initial data lies in the incompressible mode (for this case, we say the initial data is "well-prepared",) the convergence is strong. When the initial data is general, i.e., the projection onto the orthogonal complement of the incompressible mode $\text{Null}(\mathcal{A})^\perp$, which we call "acoustic mode", is nontrivial, the fast acoustic waves occur, then prevent strong convergence. We derive the so-called averaged equation which describes the propagation of the fast acoustic waves.

Remark: We can treat the acoustic and incompressible Stokes limits in a unified way. If we consider the weakly compressible Stokes system:

$$\tau_\epsilon \partial_t \tilde{\mathbf{U}}_\epsilon + \mathcal{A}\tilde{\mathbf{U}}_\epsilon = \epsilon \mathcal{D}\tilde{\mathbf{U}}_\epsilon \,. \tag{1.1.0.32}$$

then, when $\tau_\epsilon = 1$, solutions of (1.1.0.32) converge to solutions of the acoustic system with the same initial data; when $\tau_\epsilon = \epsilon$, solutions of (1.1.0.32) converge (weakly) to solutions of the incompressible Stokes system with initial data in incompressible mode.

For the case $\delta_\epsilon = \epsilon$, and $\tau_\epsilon = \epsilon$, the weakly compressible approximation of (1.1.0.19) is not the linearization about the absolute equilibrium, but a quadratic system. Using the method of multiple time scales and averaging, we derive the **weakly compressible Navier-Stokes system**, which is quadratic:

$$\partial_t \tilde{\mathbf{U}}_\epsilon + \frac{1}{\epsilon}\mathcal{A}\tilde{\mathbf{U}}_\epsilon + \overline{\mathcal{Q}}(\tilde{\mathbf{U}}_\epsilon, \tilde{\mathbf{U}}_\epsilon) = \overline{\mathcal{D}}\tilde{\mathbf{U}}_\epsilon \,, \quad \tilde{\mathbf{U}}_\epsilon(0,x) = \tilde{\mathbf{U}}^{in}(x) \,, \tag{1.1.0.33}$$

where $\overline{\mathcal{Q}}$ and $\overline{\mathcal{D}}$ are time averaging of the quadratic operator $\mathcal{Q}$ and $\mathcal{D}$:

$$\begin{aligned} \overline{\mathcal{Q}}(\hat{\mathbf{U}}, \hat{\mathbf{U}}) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{s\mathcal{A}} Q(e^{-s\mathcal{A}}\hat{\mathbf{U}} \,, e^{-s\mathcal{A}}\hat{\mathbf{U}})\, ds \,, \\ \overline{\mathcal{D}}(\hat{\mathbf{U}}) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{s\mathcal{A}} \mathcal{D}(e^{-s\mathcal{A}}\hat{\mathbf{U}})\, ds \,. \end{aligned} \tag{1.1.0.34}$$

Here the quadratic operator $\mathcal{Q}$ is

$$\mathcal{Q}(\tilde{\mathbf{U}}, \tilde{\mathbf{U}}) = \begin{pmatrix} \nabla_x \cdot (\tilde{\rho}\tilde{u}) \\ \tilde{u}\cdot\nabla_x \tilde{u} + (\tilde{\theta} - \tilde{\rho})\nabla_x \tilde{\rho} \\ \frac{D}{2}\tilde{u}\cdot\nabla_x\tilde{\theta} + \tilde{\theta}\nabla_x\cdot\tilde{u} \end{pmatrix} . \tag{1.1.0.35}$$

We show that (1.1.0.33) exists global weak solutions. Furthermore, the projection of (1.1.0.33) onto Null($\mathcal{A}$) is the incompressible Navier-Stokes system, i.e.,

$$\begin{aligned} \nabla_x \cdot \tilde{u} &= 0\,, \\ \partial_t \tilde{u} + \nabla_x \cdot (\tilde{u} \otimes \tilde{u}) + \nabla_x \tilde{p} &= \mu \Delta_x \tilde{u}\,, \\ \partial_t \tilde{\theta} + \nabla_x \cdot (\tilde{u} \tilde{\theta}) &= \kappa \Delta_x \tilde{\theta}\,, \end{aligned} \tag{1.1.0.36}$$

with the initial data $\Pi \tilde{\mathbf{U}}^{in}(x)$. The projection onto the acoustic mode Null$(\mathcal{A})^{\perp}$ is a nonlinear system with nonlocal terms with the initial data $\Pi^{\perp} \tilde{\mathbf{U}}^{in}(x)$. This averaged system in the acoustic mode is coupled with solutions to the incompressible Navier-Stokes system. Because its structure is complicated, we leave the details in Chapter 3.

1.2 Hyperbolic-Parabolic Systems with Entropy

The entropy structure of the compressible Navier-Stokes system plays the important role of providing a priori global estimates for the weakly compressible Navier-Stokes approximation. This relation extends to the analogous approximation in the more general setting. The compressible Navier-Stokes system can be considered as a special case of general hyperbolic-parabolic system with entropy

$$\begin{aligned} \partial_t \mathbf{U}_\epsilon + \sum_{\alpha=1}^{D} \partial_\alpha \left[\mathbf{F}^\alpha(\mathbf{U}_\epsilon) \right] &= \epsilon \sum_{\alpha,\beta=1}^{D} \partial_\alpha \left[\mathbf{B}^{\alpha\beta}(\mathbf{U}_\epsilon) \cdot \partial_\beta \mathbf{U}_\epsilon \right] \,, \\ \mathbf{U}_\epsilon^{in} &= \mathbf{U}_* + \epsilon \tilde{\mathbf{U}}_\epsilon^{in}\,, \end{aligned} \tag{1.2.0.37}$$

Indeed, the method to derive the weakly compressible Navier-Stokes approximation can be employed to derive the weakly nonlinear approximation of the general hyperbolic-parabolic system with entropy about absolute equilibrium of (1.2.0.37).

The solutions to the equation (1.2.0.37) depends on two different time scales. The oscillations on the short time scale persist which makes the asymptotic behavior over the long time scale singular if the initial data are not "well-prepared". The main techniques to deal with the singular-limit problems are the method of multiple scales and method of averaging. These techniques have been applied to numerous physical systems in fluid mechanics, gas dynamics, and MHD, etc. The method involves the introduction of slow and fast variables. The multiple scale expansions due to Krylov and Bogolibuov have been successfully employed in the context of ODEs (see [67],) as a variant the method used earlier by Poincaré and Lindsted to eliminate secular terms in the perturbation expansion of celestial mechanics. The method of multiple scales and method of averaging are equivalent for a single fast variable but the multiple time scales method applied to initial-value problems is often less efficient than the averaging. In singular perturbation theory however the use of multiple time scales is sometimes more attractive. For further details we refer to [67].

In this dissertation we employ the method of multiple time scales, i.e.,

$$\mathbf{U}_\epsilon(t) = \mathbf{U}_* + \epsilon \mathbf{U}^1(t,\tau)\big|_{\tau=\frac{t}{\varepsilon}} + \epsilon^2 \mathbf{U}^2(t,\tau)\big|_{\tau=\frac{t}{\varepsilon}} + \cdots . \tag{1.2.0.38}$$

Even though the expansion (1.2.0.38) may not always be valid (due to small-divisor or other problems,) it is still useful for deriving the correct limit equations. It is enough for our goal, because what we are concerned in this paper is not the convergence of $\mathbf{U}_\epsilon$ but the derivation of the limiting equation and its global existence. Substituting the expansion (1.2.0.38) into the equation (1.2.0.37) yields $\partial_\tau \mathbf{U}^1 +$

$\mathcal{A}\mathbf{U}^1 = 0$, where $\mathcal{A}$ is a first-order differential operator. Formally the solution is $\mathbf{U}^1(t,\tau) = e^{\tau\mathcal{A}}\hat{\mathbf{U}}(t)$. This solution for $\mathbf{U}^1$ separates the fast and slow variables, but can not completely solve $\mathbf{U}^1$, because $\hat{\mathbf{U}}(t)$ is an unknown function of the slow time scale t. So it is not sufficient to consider only the first-order term $\mathbf{U}^1$. To seek the equation obeyed by $\hat{\mathbf{U}}(t)$, we need to consider the next order equation which has of the form:

$$\partial_\tau \mathbf{U}^2 + \mathcal{A}\mathbf{U}^2 = \mathbf{f}(\mathbf{U}^1)\,. \tag{1.2.0.39}$$

$\mathbf{U}^2$ needs to be eliminated in order to obtain a closed set of equations for the linear solution $\mathbf{U}^1$. The standard method to do so is by the sublinearity condition. To fulfill this condition, the operator $\mathcal{A}$ needs to be skew-symmetric under some appropriate inner product. For the hyperbolic-parabolic system with entropy, we can define a natural inner product by the Hessian of the entropy at any constant state. For the compressible Navier-Stokes system, there exists a physical entropy. We rewrite the equations (1.2.0.37) such that the entropy plays a explicit role. Then employing the sublinear condition we derive the equation satisfied by $\hat{\mathbf{U}}(t)$, the so-called averaging equation:

Theorem 1: **(Formal Averaged Equation)** *The averaged system is:*

$$\partial_t \hat{\mathbf{U}} + \overline{\mathcal{Q}}(\hat{\mathbf{U}}, \hat{\mathbf{U}}) = \overline{\mathcal{D}}(\hat{\mathbf{U}})\,. \tag{1.2.0.40}$$

The detailed structures of $\overline{\mathcal{Q}}(\hat{\mathbf{U}}, \hat{\mathbf{U}})$ and $\overline{\mathcal{D}}(\hat{\mathbf{U}})$ will be described in Chapter 2 for general system, in Chapter 3 for Navier-Stokes system of compressible gas dynamics.

From above averaged equation and the formal multi-time scales expansion

(1.2.0.38), the fluctuation around the constant state $\mathbf{U}_*$ behaves like

$$\frac{\mathbf{U}_\epsilon(t,\mathbf{x})-\mathbf{U}_*}{\epsilon}\sim e^{-\frac{t}{\epsilon}\mathcal{A}}\hat{\mathbf{U}}(t,\mathbf{x}) \tag{1.2.0.41}$$

asymptotically, where $\hat{\mathbf{U}}(t,\mathbf{x})$ obeys the above averaged equation. Here " $\sim$ " means asymptotically as $\epsilon\to 0$. So the right-hand side of (1.2.0.41) describes the large-time behavior of the fluctuation. If we define $\mathbf{V}=e^{t\mathcal{A}}\hat{\mathbf{U}}$ where $\hat{\mathbf{U}}$ is a solution to the averaged equation, then $\mathbf{V}$ satisfies

$$\partial_t\mathbf{V}+\mathcal{A}\mathbf{V}+\overline{\mathcal{Q}}(\mathbf{V},\mathbf{V})=\overline{\mathcal{D}}(\mathbf{V})\,. \tag{1.2.0.42}$$

In the case of the Navier-Stokes equations for compressible gases, solutions $\mathbf{V}(t,\mathbf{x})$ to (1.2.0.42) has the same large-time behavior with solutions to the Navier-Stokes equations around a constant state. This problems has been extensively studied by many people, such as [40], [41], [43], [44], [56], [57]. Most of these works focus on the estimates of the Green's function of the linearized Navier-Stokes system $\partial_t\mathbf{V}+\mathcal{A}\mathbf{V}=\mathcal{D}\mathbf{V}$, or the linear version of (1.2.0.42), i.e., $\partial_t\mathbf{V}+\mathcal{A}\mathbf{V}=\overline{\mathcal{D}}\mathbf{V}$. The new system (1.2.0.42) includes quadratic terms. It should capture more precise information about the large-time behavior of the compressible Navier-Stokes equations.

This averaged system (1.2.0.40) shares many properties with the usual Navier-Stokes. Its quadratic term $\overline{\mathcal{Q}}$ is non-local, and has divergence form. More importantly, it does not contribute to the energy estimate. Generally, the diffusion term $\overline{\mathcal{D}}$ is only semi-dissipative, not strictly dissipative. We give a sufficient structure condition which guarantees the strict dissipativity of the diffusion term so that we can prove the following global existence theorem:

Theorem 2: *If* $-\mathcal{A}\mathcal{D}\mathcal{A}|_{\mathrm{Null}(\mathcal{D})} > 0$*, then for any* L^2 *initial data* $\mathbf{U}^{in}$*, there exists a global weak solution to the averaged system* (1.2.0.40)*.*

Applying the above general theorem to the Navier-Stokes system for the viscous ideal gas, we can derive the explicit form of the averaging equation whose projection onto the null space of the operator $\mathcal{A}$ is the incompressible Navier-Stokes equation as expected. The projection of the averaged equation onto the fast mode, the orthogonal complement of the null space of $\mathcal{A}$ is a nonlinear equation with two non-local terms which have divergence form. One of the nonlocal terms depends on the solution of the incompressible Navier-Stokes equation. It is easy to verify that the compressible Navier-Stokes system satisfies the structure condition, then its averaged equation exists global weak solution.

Corollary 1: **(Formal Averaged Equation)** *The solution to the averaged equation* $\mathbf{U}(t,x) = \Pi\mathbf{U} + \Pi^{\perp}\mathbf{U}$ *, where* $\Pi\mathbf{U}$ *satisfies the incompressible Navier-Stokes system with initial data* $\Pi\mathbf{U}^{in}$*.* $\Pi^{\perp}\mathbf{U}$ *satisfies the equation:*

$$\begin{aligned} \partial_t(\Pi^{\perp}\mathbf{U}) + \nabla_x \cdot \mathbf{Q}^{\flat}_{2r}(\Pi\mathbf{U}, \Pi^{\perp}\mathbf{U}) + \nabla_x \cdot \mathbf{Q}^{\flat}_{3r}(\Pi^{\perp}\mathbf{U}, \Pi^{\perp}\mathbf{U}) = \tilde{\mu}\Delta_x \Pi^{\perp}\mathbf{U}\,, \\ \Pi^{\perp}\mathbf{U}(0,x) = \Pi^{\perp}\mathbf{U}^{in}(x)\,. \end{aligned} \tag{1.2.0.43}$$

Corollary 2: **(Global Existence)** *For any* L^2 *initial data* $\mathbf{U}^{in}$*, there exists at least one global weak solution to the averaged equation.*

1.3 Kinetic Setting

Fluid dynamical systems, can be formally derived from the Boltzmann equation through a scaling in which the density F is close to the absolute Maxwellian

M. So it is natural to introduce the relative density $G = G(t,x,v)$, defined as $F = MG$. More precisely, we consider families of solutions parametrized by the Knudsen number ϵ that have the form

$$G_\epsilon^{in} = 1 + \delta_\epsilon g_\epsilon^{in}\,, \quad G_\epsilon = 1 + \delta_\epsilon g_\epsilon\,, \tag{1.3.0.44}$$

where G_ϵ satisfies the scaled Boltzmann equation with initial data G_ϵ^{in},

$$\tau_\epsilon \partial_t G + v \cdot \nabla_x G = \frac{1}{\epsilon}\mathcal{Q}(G, G,) \qquad G(0,x,v) = G^{in}(x,v)\,, \tag{1.3.0.45}$$

whose fluctuations g_ϵ satisfy

$$\frac{\delta_\epsilon}{\epsilon} \to 0, \quad \text{or} \quad 1, \quad \text{or} \quad \infty \quad \text{as } \epsilon \to 0\,. \tag{1.3.0.46}$$

the time scale $\tau_\epsilon = \epsilon$ for slow time (Stokes) scale, $\tau_\epsilon = 1$ for fast time (acoustic) scale.

The fluctuations g_ϵ formally satisfy the local conservation laws

$$\begin{aligned} \partial_t \langle g_\epsilon \rangle + \frac{1}{\tau_\epsilon}\nabla_x \cdot \langle v g_\epsilon \rangle = 0\,, \\ \partial_t \langle v g_\epsilon \rangle + \frac{1}{\tau_\epsilon}\nabla_x \cdot \langle v \otimes v g_\epsilon \rangle = 0\,, \\ \partial_t \langle \tfrac{1}{2}|v|^2 g_\epsilon \rangle + \frac{1}{\tau_\epsilon}\nabla_x \cdot \langle v \tfrac{1}{2}|v|^2 g_\epsilon \rangle = 0\,. \end{aligned} \tag{1.3.0.47}$$

If we define the fluid variables associated with the fluctuation of the number density g_ϵ:

$$\tilde{\rho}_\epsilon = \langle g_\epsilon \rangle\,, \quad \tilde{u}_\epsilon = \langle v g_\epsilon \rangle\,, \quad \tilde{\theta}_\epsilon = \tfrac{2}{D}\left\langle \left(\tfrac{1}{2}|v|^2 - \tfrac{D}{2}\right) g_\epsilon \right\rangle\,. \tag{1.3.0.48}$$

If we define $\tilde{U}_\epsilon = (\tilde{\rho}_\epsilon, \tilde{u}_\epsilon, \tilde{\theta}_\epsilon)$, then after tedious calculations, the local conservation laws become:

$$\begin{aligned} \partial_t \tilde{U}_\epsilon + \frac{1}{\tau_\epsilon}\mathcal{A}\tilde{U}_\epsilon + \frac{\delta_\epsilon}{\tau_\epsilon}\mathcal{Q}(\tilde{U}_\epsilon, \tilde{U}_\epsilon) = \frac{\epsilon}{\tau_\epsilon}\mathcal{D}\tilde{U}_\epsilon + \tilde{R}_\epsilon\,, \\ \tilde{U}_\epsilon(0,x) = \tilde{U}^{in}\,, \end{aligned} \tag{1.3.0.49}$$

where

$$\tilde{U}^{in} = \left(\langle g_\epsilon^{in}\rangle, \langle v g_\epsilon^{in}\rangle, \tfrac{2}{D}\left\langle \left(\tfrac{1}{2}|v|^2 - \tfrac{D}{2}\right) g_\epsilon^{in}\right\rangle\right), \qquad (1.3.0.50)$$

the remainder term $\tilde{R}_\epsilon$ will vanish as $\epsilon \to 0$.

In the **Stokes scaling**, i.e., $\frac{\delta_\epsilon}{\epsilon} \to 0$, the leading behavior of (1.3.0.49) is governed by

$$\begin{aligned} \partial_t \tilde{U}_\epsilon + \frac{1}{\tau_\epsilon}\mathcal{A}\tilde{U}_\epsilon &= \frac{\epsilon}{\tau_\epsilon}\mathcal{D}\tilde{U}_\epsilon\,, \\ \tilde{U}_\epsilon(0,x) &= \tilde{U}^{in}\,. \end{aligned} \qquad (1.3.0.51)$$

In the short time scale $\tau_\epsilon = 1$, its limit as $\epsilon \to 0$ is obviously the acoustic system. In the longer time scale $\tau_\epsilon = \epsilon$, the asymptotic behavior will be singular. We can show that the projection onto the slow mode $\mathrm{Null}(\mathcal{A})$, is the incompressible Stokes system with the Boussinesq balance law. When the initial data are not "well-prepared", the projection onto the fast mode $\mathrm{Null}(\mathcal{A})^\perp$ will oscillate very fast. Applying the method of multiple time scales and averaging, we can derive the averaged equation that describes the propagation of the fast waves which prevent the strong convergence from weakly compressible Stokes to incompressible Stokes system. Thus, the weakly compressible Stokes system (1.3.0.51) governs the behavior of both acoustic and Stokes system at different time scales.

Inspired by this observation, we construct a family of local Maxwellians

$$M_\epsilon(t) = \mathcal{M}_{(1+\delta_\epsilon\rho_\epsilon, \delta_\epsilon u_\epsilon, 1+\delta_\epsilon\theta_\epsilon)}\,, \qquad (1.3.0.52)$$

where $(\rho_\epsilon, u_\epsilon, \theta_\epsilon)$ is the solutions to the weakly compressible Stokes system (1.3.0.51). Under some technical assumption on the collision kernel and solutions of the Boltzmann equation, we prove that this family of local Maxwellians govern the fluid behavior of the Boltzmann equation in the sense of the relative entropy. We prove that

Theorem 3: **(Weakly Compressible Stokes Approximation)** *Assume that the collision kernel satisfies the hard sphere potential with a small deflection cut-off condition, and DiPerna-Lions solutions satisfy the local energy conservation law. If initially, the relative entropy* $\frac{1}{\delta_\epsilon^2}H(F_\epsilon^{in}|M_\epsilon^{in}) \to 0$*, then, for the later time* $t > 0$*,*

$$\frac{1}{\delta_\epsilon^2}H(F_\epsilon(t)|M_\epsilon(t)) \to 0\,. \tag{1.3.0.53}$$

We shall show that the relative entropy can control the distance between the fluctuation of the number density g_ϵ and the infinitesimal Maxwellian

$$g_\epsilon^S = \rho_\epsilon + u_\epsilon \cdot v + \theta_\epsilon(\tfrac{1}{2}|v|^2 - \tfrac{D}{2})\,. \tag{1.3.0.54}$$

Combining with the convergence results from the weakly compressible Stokes to acoustic system (in short time scale) and incompressible Stokes system (in long time scale,) we unify the Golse-Levermore acoustic-Stokes limit theorem [29]. The weakly Stokes approximation results will be discussed in details in chapter 6.

In the **Navier-Stokes scaling**, i.e., $\tau_\epsilon = \epsilon$ and $\delta_\epsilon \sim \epsilon$, the problem will be much harder. Formally, the leading behavior of (1.3.0.49) would be

$$\begin{aligned} \partial_t\tilde{U}_\epsilon + \frac{1}{\epsilon}\mathcal{A}\tilde{U}_\epsilon + \mathcal{Q}(\tilde{U}_\epsilon, \tilde{U}_\epsilon) = \mathcal{D}\tilde{U}_\epsilon\,, \\ \tilde{U}_\epsilon(0,x) = \tilde{U}^{in}\,. \end{aligned} \tag{1.3.0.55}$$

Unfortunately, this system is not well-posed. It does not satisfied conservation of energy even in the formal level. We can not apply this system directly to describe the fluid dynamics of the Boltzmann equation. Applying the method of multiple time scales and averaging again, we derive the averaged equation which has the same long time behavior with (1.3.0.55):

$$\begin{aligned} \partial_t \tilde{U}_\epsilon + \frac{1}{\epsilon}\mathcal{A}\tilde{U}_\epsilon + \overline{\mathcal{Q}}(\tilde{U}_\epsilon, \tilde{U}_\epsilon) &= \overline{\mathcal{D}}\tilde{U}_\epsilon\,, \\ \tilde{U}_\epsilon(0,x) &= \tilde{U}^{in}(x)\,. \end{aligned} \tag{1.3.0.56}$$

Results in chapter 3 provide the global existence of the above averaged system. As for the weakly compressible Stokes approximation, we construct a family of local Maxwellians $M_\epsilon(t) = \mathcal{M}_{(1+\delta_\epsilon\rho_\epsilon, \delta_\epsilon u_\epsilon, 1+\delta_\epsilon\theta_\epsilon)}$, where $(\rho_\epsilon, u_\epsilon, \theta_\epsilon)$ is the solutions to the averaged system (1.3.0.55). We expect it would be a good approximation to solutions to the Boltzmann equation in Navier-Stokes scaling. Unfortunately, because of some technical difficulties, so far we could not prove results parallel to Theorem 3. Even in the formal level, the long-time stability of the relative entropy $\frac{1}{\epsilon^2}H(F_\epsilon(t)|M_\epsilon(t))$ is not trivial; see Golse, Levermore and Saint-Raymond's result in [31]. Their result is about the well-prepared initial data. We generalize it to the non-well-prepared initial data. This formal theorem on the weakly compressible Navier-Stokes approximation will be proved in chapter 7.

1.4 Organization of the Dissertation

We now lay out the organization of this dissertation. Chapter 2 is about the general hyperbolic-parabolic system with entropy. We use the method of multiple time scales and averaging to derive weakly nonlinear approximation around a constant state. We derive the averaged system and analyze its structure. We give an easy-to-check condition which guarantees the global existence of the averaged system.

In Chapter 3, we apply our general theory to the compressible Navier-Stokes system. We present a more detailed formulation of the averaged system whose projection to the null space of the acoustic operator is the incompressible Navier-Stokes equations. Use the Littlewood-Paley theory, we prove a higher regularity of the projection of the averaged system on the fast mode $\mathrm{Null}(\mathcal{A})^{\perp}$, where $\mathcal{A}$ is the acoustic operator.

Chapter 4 is a review of the Boltzmann equation, which includes the formal structure of the Boltzmann equation and the DiPerna-Lions theory.

Chapter 5 is an introduction to the fluid dynamics of the Boltzmann equation. We use the multiple timescales and averaging method, formally derive weakly compressible fluid limits of the Boltzmann equation for the general initial data, i.e., those which do not satisfy the incompressibility and Boussinesq relations.

Chapter 6 includes two parts. The first part gives a detailed analysis of the asymptotic behavior of the weakly compressible Stokes system to acoustic and incompressible Stokes system. In the second part, we use the relative entropy method to prove the long time stability of the relative entropy constructed from solutions to the weakly compressible Stokes system.

Chapter 7 is a formal result about the weakly compressible Navier-Stokes approximation of the Boltzmann equation. Using solutions to the averaged system of the weakly compressible Navier-Stokes system, we construct a family of local Maxwellians and prove that under some assumptions about passing to the limit in certain relative entropy dissipation terms and remainders, it governs the long time behavior of solutions to the Boltzmann equation in the sense of the relative entropy.

Finally, we state some possible future work.

2. WEAKLY NONLINEAR APPROXIMATION OF HYPERBOLIC-PARABOLIC SYSTEMS WITH ENTROPY

In this chapter, we study the weakly nonlinear approximation of hyperbolic-parabolic systems with entropy. The compressible Navier-Stokes system, which is endowed with a natural physical entropy, is a special case of such systems. We show that the entropy structure leads to a quadratic global a priori estimates and under a mild structural assumption the existence of global weak solutions to the weakly nonlinear approximation. As introduced in Chapter 1, we are interested in the asymptotics as ϵ goes to zero of the following system:

$$
\begin{aligned}
&\partial_t \mathbf{U}_\epsilon + \sum_{\alpha=1}^{D} \partial_\alpha \left[\mathbf{F}^\alpha(\mathbf{U}_\epsilon)\right] = \epsilon \sum_{\alpha,\beta=1}^{D} \partial_\alpha \left[\mathbf{B}^{\alpha\beta}(\mathbf{U}_\epsilon) \cdot \partial_\beta \mathbf{U}_\epsilon\right] \,, \\
&\mathbf{U}_\epsilon(0,x) = \mathbf{U}_* + \delta_\epsilon \mathbf{U}_\epsilon^{in}(x)\,.
\end{aligned}
\tag{2.0.0.1}
$$

Suppose the hyperbolic-parabolic system (2.0.0.1) has an entropy-entropy flux pair (Φ, Ψ). Assume that the fluctuations of $\mathbf{U}_\epsilon(t,x)$ to the absolute equilibrium $\mathbf{U}_*$ have the same order δ_ϵ as the initial fluctuations, i.e., $\mathbf{U}_\epsilon(t,x) = \mathbf{U}_* + \delta_\epsilon \tilde{\mathbf{U}}_\epsilon(t,x)$. Suppose also that $\tilde{\mathbf{U}}_\epsilon \to \tilde{\mathbf{U}}$, and formally all the small terms vanish as $\epsilon \to 0$. Then, the limit $\tilde{\mathbf{U}}$ satisfies the system

$$
\partial_t \tilde{\mathbf{U}} + \mathcal{A}\tilde{\mathbf{U}} = 0\,, \tag{2.0.0.2}
$$

where the first-order operator $\mathcal{A}$ is defined as $\mathcal{A}\tilde{\mathbf{U}} = \mathbf{F}^{\alpha}_{\mathbf{U}}(\mathbf{U}_*) \cdot \partial_\alpha \tilde{\mathbf{U}}$, (we will use an equivalent but different form later to emphasize the role played by the entropy). In the compressible Navier-Stokes system, $\mathcal{A}$ is the acoustic operator which has a nontrivial null space. $\mathcal{A}$ is skew-symmetric with respect to the inner product defined by the Hessian of the entropy $\mathbf{G} = \Phi_{\mathbf{uu}}(\mathbf{U}_*)$. Then $e^{t\mathcal{A}}$ is a semigroup which preserve norm defined by $\mathbf{G}$. Using this semigroup, the solution to (2.0.0.2) is $\tilde{\mathbf{U}} = e^{-t\mathcal{A}}\hat{\mathbf{U}}$, which represents acoustic waves in compressible Navier-Stokes system.

The small dissipation term is not negligible when we consider the longer time scale. In that case, the fluctuations $\tilde{\mathbf{U}}_\epsilon$ satisfy

$$\epsilon\partial_t\tilde{\mathbf{U}}_\epsilon + \mathcal{A}\tilde{\mathbf{U}}_\epsilon + \delta_\epsilon\mathcal{Q}(\tilde{\mathbf{U}}_\epsilon, \tilde{\mathbf{U}}_\epsilon) + O(\delta_\epsilon^2) = \epsilon\mathcal{D}\tilde{\mathbf{U}}_\epsilon + O(\epsilon\delta_\epsilon)\,. \tag{2.0.0.3}$$

In the case $\delta_\epsilon \ll \epsilon$, the quadratic term is small. When $\delta_\epsilon = \epsilon$, the correctors in the quadratic and diffusion terms are the same order. Using the semigroup $e^{t\mathcal{A}}$, define $\tilde{\mathbf{U}}_\epsilon = e^{-\frac{t}{\epsilon}\mathcal{A}}\hat{\mathbf{U}}_\epsilon$. Suppose $\hat{\mathbf{U}}_\epsilon \to \hat{\mathbf{U}}$, then $\hat{\mathbf{U}}$ satisfies the so-called averaged system

$$\partial_t\hat{\mathbf{U}} + \overline{\mathcal{Q}}(\hat{\mathbf{U}}, \hat{\mathbf{U}}) = \overline{\mathcal{D}}\hat{\mathbf{U}}\,. \tag{2.0.0.4}$$

In this chapter we use the averaging method systematically derive the averaged system, and use a priori estimates given by entropy bound to prove global existence of weak solutions to the averaged system (2.0.0.4) under a structural condition.

In this chapter, section 2.1, we first introduce some basic structure of the hyperbolic system of conservation laws, then in section 2.2, we use the method of multiple time scales and averaging to derive the weakly nonlinear approximation of the general system which is described by averaged system . In section 2.3, we analyze the formal properties of the averaged system. We prove the formal energy identity and give a condition on the structure of the original system which guarantees the strict parabolicity of the averaged diffusion term. In the last section 2.4 of this chapter, we prove global existence of the weak solutions to the averaged system.

2.1 Hyperbolic-Parabolic Systems with Entropy

First-order nonlinear hyperbolic systems of conservation laws are the equations of the form

$$\partial_t \mathbf{U} + \sum_{\alpha=1}^{D} \partial_\alpha \left[\mathbf{F}^\alpha(\mathbf{U})\right] = 0\,. \tag{2.1.0.5}$$

Here $\mathbf{F}^\alpha(\mathbf{U}) = (\mathbf{F}_1^\alpha(\mathbf{U})\,,\cdots,\mathbf{F}_N^\alpha(\mathbf{U}))^T$, $\alpha = 1\,,\cdots,D$, $\mathbf{U} = (\mathbf{U}_1(x,)\cdots,\mathbf{U}_N(x))^T$ denote $n-$dimensional vectors, with $\mathbf{F}^\alpha$ smooth functions of $\mathbf{U}$, and $\mathbf{U}$ functions of the time t and the space coordinate $x = (x_1\,,\cdots,x_D)$. As is well-known, the existence of an entropy function for (2.1.0.5) is characterized by the property that (2.1.0.5) can be symmetrized by introducing a new dependent variable. We owe these results to Godunov [26] and Friedrichs-Lax [24].

We consider the initial-value problem for the second-order nonlinear systems associated with (2.1.0.5), i.e.,

$$\begin{gathered}\partial_t \mathbf{U} + \sum_{\alpha=1}^{D} \partial_\alpha \left[\mathbf{F}^\alpha(\mathbf{U})\right] = \sum_{\alpha,\beta=1}^{D} \partial_\alpha \left[\mathbf{B}^{\alpha\beta}(\mathbf{U})\cdot\partial_\beta \mathbf{U}\right]\,,\\ \mathbf{U}(0,x) = \mathbf{U}^{in}(x)\,,\end{gathered} \tag{2.1.0.6}$$

where $\mathbf{B}^{\alpha\beta}(\mathbf{U})$ denote $N\times N$ matrices depending smoothly on $\mathbf{U}$ for each $\alpha,\beta = 1\,,\cdots,D$. Both $\mathbf{F}^\alpha(\mathbf{U})$ and $\mathbf{B}^{\alpha\beta}(\mathbf{U})$ are defined on an open set $\Omega\subset\mathbb{R}^N$. A typical example of such systems arise as the conservation law of viscous compressible fluid. The notion of the entropy function has a natural extension to the second-order system (2.1.0.6) and the fact that the symmetrizability of the system can be characterized by the existence of an entropy function remains valid for (2.1.0.6). For the convenience of the readers, we give a brief review of these observation in this section. We assume that the following properties hold for system (2.1.0.6),

- The flux vectors $\mathbf{F}^\alpha$, $1 \le \alpha \le D$, the dissipation matrices $\mathbf{B}^{\alpha\beta}$, $1 \le \alpha, \beta \le D$, are smooth functions of the variables $\mathbf{U} \in \mathcal{O}$, where $\mathcal{O}$ is a convex open set of $\mathbb{R}^N$.

Definition 1: Let $\Phi(\mathbf{U})$ be a real-valued smooth function defined on a convex open set $\mathcal{O} \subset \Omega$. Then Φ is called an entropy function for the system (2.1.0.6) if the following properties hold:

(E_1*)* The function $\Phi(\mathbf{U})$ is a strictly convex on $\mathcal{O}$ in the sense that the Hessian matrix $\Phi_{\mathbf{uu}}$ is positive definite on $\mathcal{O}$;

(E_2*)* There exits real-valued smooth functions $\Psi^\alpha = \Psi^\alpha(\mathbf{U})$ such that

$$\Phi_{\mathbf{u}}(\mathbf{U}) \cdot \mathbf{A}^\alpha(\mathbf{U}) = \Psi^\alpha_{\mathbf{u}}(\mathbf{U})\,, \tag{2.1.0.7}$$

where $\mathbf{A}^\alpha(\mathbf{U}) = (\mathbf{A}^\alpha_{ab}(\mathbf{U}))$ is $N \times N$ matrix, and $\mathbf{A}^\alpha_{ab}(\mathbf{U}) = \frac{\partial F^\alpha_a(\mathbf{U})}{\partial \mathbf{U}_b}$;

(E_3*)* We have the property

$$\mathbf{D}^{\alpha\beta}(\mathbf{U})^T = \mathbf{D}^{\beta\alpha}(\mathbf{U})\,, \tag{2.1.0.8}$$

where

$$\mathbf{D}^{\alpha\beta}(\mathbf{U}) = \mathbf{B}^{\alpha\beta}(\mathbf{U}) \cdot \Phi_{\mathbf{uu}}(\mathbf{U})^{-1}\,; \tag{2.1.0.9}$$

(E_4) The matrix $\tilde{\mathbf{B}}(\mathbf{U}, w) = \sum_{\alpha,\beta=1}^{D} \mathbf{D}^{\alpha\beta}(\mathbf{U}) w_\alpha w_\beta$ is symmetric positive semi-definite for $\mathbf{U} \in \mathcal{O}_{\mathbf{U}}$ and $w \in \mathbb{S}^{D-1}$ in the sense that

$$\sum_{\alpha,\beta=1}^{D} \sum_{a,b=1}^{N} \xi^a \xi^b \mathbf{D}_{ab}^{\alpha\beta}(\mathbf{U}) w_\alpha w_\beta \geq 0\,, \quad \forall \xi \in \mathbb{R}^N\,, \quad \forall w \in \mathbb{S}^{D-1}\,. \tag{2.1.0.10}$$

Let $\Phi^* : \mathcal{O}^* \to \mathbb{R}$ be the Legendre dual function of the strictly convex entropy function Φ. Its domain is given by

$$\mathcal{O}^* \equiv \{\mathbf{V} \in \mathbb{R}^{N*} | \mathbf{V} = \Phi_{\mathbf{u}}(\mathbf{U}) \quad \text{for some} \quad \mathbf{U} \in \mathcal{O}\}\,, \tag{2.1.0.11}$$

and for every Φ it satisfies

$$\Phi^*(\mathbf{V}) + \Phi(\mathbf{U}) = \mathbf{U} \cdot \mathbf{V} \tag{2.1.0.12}$$

where $\mathbf{U} \in \mathcal{O}$ and $\mathbf{V} \in \mathcal{O}^*$ are related by

$$\mathbf{U} = \Phi^*_{\mathbf{v}}(\mathbf{V})\,, \qquad \mathbf{V} = \Phi_{\mathbf{u}}(\mathbf{U})\,. \tag{2.1.0.13}$$

We call Φ^* as the entropy potential of Φ. Similarly, We can introduce the entropy flux potential:

$$\Psi^{*\alpha}(\mathbf{V}) + \Psi^\alpha(\mathbf{U}) = \mathbf{F}^\alpha(\mathbf{U}) \cdot \mathbf{V}\,. \tag{2.1.0.14}$$

for $\alpha = 1\,, \cdots, D$.

The entropy-entropy flux potentials obey the following relations:

$$\begin{aligned} \Phi_{\mathbf{uu}}(\mathbf{U}) \cdot \Phi^*_{\mathbf{vv}}(\mathbf{V}) &= \mathrm{I} \\ \mathbf{F}^\alpha(\mathbf{U}) &= \Psi^{*\alpha}_{\mathbf{v}}(\Phi_{\mathbf{u}}(\mathbf{U}))\,. \end{aligned} \tag{2.1.0.15}$$

Applying above relations, we can rewrite the system (2.1.0.6) into the following form, emphasizing the entropy structure in an explicit way:

$$\partial_t \mathbf{U} + \sum_{\alpha=1}^{D} \partial_\alpha \left[\Psi^{*\alpha}_{\mathbf{v}}(\Phi_{\mathbf{u}}(\mathbf{U}))\right] = \sum_{\alpha,\beta=1}^{D} \partial_\alpha \left[\mathbf{D}^{\alpha\beta}(\mathbf{U}) \cdot \partial_\beta \Phi_{\mathbf{u}}(\mathbf{U})\right] \tag{2.1.0.16}$$

. Formally, taking inner product $\Phi_{\mathbf{u}}$ with (2.1.0.16), every classical solution satisfies

$$\partial_t\Phi + \nabla_x \cdot \Psi = -(\partial_\alpha \Phi_{\mathbf{u}})^T \cdot \mathbf{D}^{\alpha\beta} \cdot \partial_\beta \Phi_{\mathbf{u}} + \partial_\alpha[\mathbf{D}^{\alpha\beta}_{ij} \Phi_{\mathbf{u}_i} \partial_\beta \Phi_{\mathbf{u}_j}] \,, \tag{2.1.0.17}$$

Because of the semi-definite positivity of $\mathbf{D}^{\alpha\beta}$, (see condition E_4 in definition 1,) taking integral spatially on above equation, the global entropy $\int \Phi \, dx$ is dissipated under suitable boundary condition, for example, periodic condition. More precisely, the global entropy inequality is satisfied:

$$\int_{\mathbb{T}^D} \Phi(\mathbf{U})(t) \, dx + \int_0^t \int_{\mathbb{T}^D} \mathbf{D}^{\alpha\beta}_{ij} \Phi_{\mathbf{u}_i} \partial_\alpha \Phi_{\mathbf{u}_i} \partial_\beta \Phi_{\mathbf{u}_j} \, dxds \leq \int_{\mathbb{T}^D} \Phi(\mathbf{U}^{in}) \, dx \tag{2.1.0.18}$$

The weakly nonlinear approximation is derived in regime in which the solution $\mathbf{U}$ is near an absolute equilibrium $\mathbf{U}_*$. Since the entropy inequality is the only *a priori* estimate we have, it is natural to use entropy to measure the size of fluctuations to the absolute equilibrium. We define the so-called **relative entropy** of Φ with respect to the absolute equilibrium $\mathbf{U}_*$ as

$$\tilde{\Phi}(\mathbf{U}) = \Phi(\mathbf{U}) - \Phi(\mathbf{U}_*) - \Phi_{\mathbf{U}}(\mathbf{U}_*)(\mathbf{U} - \mathbf{U}_*) \,. \tag{2.1.0.19}$$

We have the global entropy inequality for the relative entropy

$$\int_{\mathbb{T}^D} \tilde{\Phi}(\mathbf{U}_\epsilon)(t) \, dx + \int_0^t \int_{\mathbb{T}^D} \tilde{\mathbf{D}}(\mathbf{U}_\epsilon) \, dxds \leq \int_{\mathbb{T}^D} \tilde{\Phi}(\mathbf{U}^{in}_\epsilon) \, dx \,. \tag{2.1.0.20}$$

Suppose initially, the global relative entropy $\int_{\mathbb{T}^D} \tilde{\Phi}(\mathbf{U}^{in}_\epsilon) \, dx = O(\delta_\epsilon^2)$, then using Young's inequality, we can show that $\mathbf{U}^{in}_\epsilon = \mathbf{U}_* + \delta_\epsilon \tilde{\mathbf{U}}^{in}_\epsilon$, where the fluctuations $\tilde{\mathbf{U}}^{in}_\epsilon$ are relatively compact in w-$L^1_{\text{loc}}(dt, L^1(dx))$ [49]. The relative entropy inequality (2.1.0.20) then implies $\int_{\mathbb{T}^D} \tilde{\Phi}(\mathbf{U}_\epsilon(t)) \, dx = O(\delta_\epsilon^2)$, for $t > 0$. Thus, using Young's inequality again, $\mathbf{U}_\epsilon(t) = \mathbf{U}_* + \delta_\epsilon \tilde{\mathbf{U}}_\epsilon(t)$, where the fluctuations $\tilde{\mathbf{U}}_\epsilon$ is relatively compact in w-$L^1_{\text{loc}}(dt, L^1(dx))$.

2.2 The Weakly Nonlinear Approximations

In this section we consider the weakly nonlinear approximation around an absolute equilibrium $\mathbf{U}_*$ of the general hyperbolic-parabolic equation (2.1.0.6) with small dissipation term:

$$\partial_t \mathbf{U} + \sum_{\alpha=1}^{D} \partial_\alpha \left[\mathbf{F}^\alpha(\mathbf{U})\right] = \epsilon \sum_{\alpha,\beta=1}^{D} \partial_\alpha \left[\mathbf{B}^{\alpha\beta}(\mathbf{U}) \cdot \partial_\beta \mathbf{U}\right] \,, \qquad (2.2.0.21)$$
$$\mathbf{U}(0,x) = \mathbf{U}_* + \epsilon \mathbf{U}^{in}(x)\,.$$

As $\epsilon \to 0$, the dissipation term vanishes. So the limiting system is the hyperbolic system of conservation law (2.1.0.5). In the longer time $[0\,,\frac{1}{\epsilon}]$, the small dissipation term of order ϵ is not negligible. So we consider the longer time scale. Using the entropy and entropy flux introduced last section, we rewrite (2.2.0.21) in the longer time scale:

$$\partial_t \mathbf{U}_\epsilon + \frac{1}{\epsilon} \sum_{\alpha=1}^{D} \partial_\alpha \left[\Psi^{*\alpha}_{\mathbf{v}}(\Phi_{\mathbf{u}}(\mathbf{U}))\right] = \sum_{\alpha,\beta=1}^{D} \partial_\alpha \left[\mathbf{D}^{\alpha\beta}(\mathbf{U}) \cdot \partial_\beta \Phi_{\mathbf{u}}(\mathbf{U})\right] \,, \qquad (2.2.0.22)$$
$$\mathbf{U}(0,x) = \mathbf{U}_* + \epsilon \mathbf{U}^{in}(x)\,.$$

The goal in this section is to derive systematically (but formally) a simplified averaged system for the limiting dynamics of (2.2.0.22) as $\epsilon \to 0$, valid on a time interval $0 < t < \frac{T}{\epsilon}$ with T fixed. We will utilize the method of multiple scales. We assume that the solution $\mathbf{U}_\epsilon(t)$ for (2.2.0.22) depends on the fast scale $\tau = \frac{t}{\varepsilon}$, and on the slow scale t, i.e., $\mathbf{U}_\epsilon(t) = \mathbf{U}_\epsilon(t, \frac{t}{\varepsilon})$. Thus, for $\varepsilon \ll 1$ the solution $\mathbf{U}_\epsilon$ has the formal expression

$$\mathbf{U}_\epsilon(t) = \mathbf{U}_* + \epsilon \mathbf{U}^1(t,\tau)\big|_{\tau=\frac{t}{\varepsilon}} + \epsilon^2 \mathbf{U}^2(t,\tau)\big|_{\tau=\frac{t}{\varepsilon}} + \cdots \,, \qquad (2.2.0.23)$$

where $\mathbf{U}_*$ is an absolute equilibrium.

We plug our ansatz (2.2.0.23) into the equation (2.2.0.22), using the rule $\partial_t \mathbf{U}(t, \frac{t}{\varepsilon}) = \partial_t \mathbf{U} + \frac{1}{\varepsilon}\partial_\tau \mathbf{U}$ and match powers of ϵ. The lowest order is $O(1)$:

$$O(1): \qquad \partial_\tau \mathbf{U}^1 + \mathcal{A}\mathbf{U}^1 = 0\,. \tag{2.2.0.24}$$

where the first order differential operator $\mathcal{A}$ is defined as

$$\mathcal{A}\mathbf{U} := \overline{\Psi}^{*\alpha}_{\mathbf{vv}} \cdot \mathbf{G} \cdot \partial_\alpha \mathbf{U} \tag{2.2.0.25}$$

where $\mathbf{G} = \overline{\Phi}_{\mathbf{uu}}$. In the above expression, the notation $\overline{\Phi}$ denotes the function (or matrix) Φ evaluated at the constant state $\mathbf{U}_*$. For simplicity, from now on we drop the bar and consequently $\Phi, \Psi, \Phi^*, \Psi^*$ will denote their evaluations at the constant state.

Now we denote by $\mathcal{H}$ the set

$$\mathcal{H} = \left\{ \mathbf{U} \in \mathcal{D}(\mathbb{T}^D)^N : \int_{\mathbb{T}^D} \mathbf{U}\, d\mathbf{x} = 0 \right\}\,, \tag{2.2.0.26}$$

where $\mathcal{D}(\mathbb{T}^D)$ denotes the test function space on $\mathbb{T}^D$, i.e., $\mathcal{C}^\infty(\mathbb{T}^D)$. The entropy Hessian $\mathbf{G}$ defines a natural inner product on $\mathcal{H}$:

$$\langle \mathbf{U}, \mathbf{V} \rangle = \int_{\mathbb{T}^D} \mathbf{U}^T \cdot \mathbf{G} \cdot \overline{\mathbf{V}}\, d\mathbf{x}\,, \tag{2.2.0.27}$$

where $\overline{\mathbf{V}}$ denotes the complex conjugate of the vector $\mathbf{V}$. The most important property of $\mathcal{A}$ is skew-symmetry under this inner product:

Lemma 1: *Operator $\mathcal{A}$ is skew-symmetric under the inner product* (2.2.0.27), *i.e.,*

$$\langle \mathcal{A}\mathbf{U}, \mathbf{V} \rangle = -\langle \mathbf{U}, \mathcal{A}\mathbf{V} \rangle\,. \tag{2.2.0.28}$$

The proof of Lemma 1 is trivial. Using the inner product we can define a norm on $\mathcal{H}$ as $\|\mathbf{U}\|^2 := \langle \mathbf{U}, \mathbf{U} \rangle$.

The solution of (2.2.0.24) is formally given by

$$\mathbf{U}^1(t,\tau) = e^{-\tau\mathcal{A}}\hat{\mathbf{U}}^1(t)\,, \tag{2.2.0.29}$$

where $\hat{\mathbf{U}}^1(t) = \mathbf{U}^1(t,0)$.

Note that in the leading-order solution of the asymptotic equation in (2.2.0.29), the fast scales and the slow scales are factored in a special form that we exploit below.

With the ansatz from (2.2.0.23) substituted into (2.2.0.22), the terms of order $O(\epsilon)$ vanish provided that

$$O(\epsilon): \qquad \partial_\tau \mathbf{U}^2 + \mathcal{A}\mathbf{U}^2 = \mathbf{f}(\mathbf{U}^1)\,, \tag{2.2.0.30}$$

where

$$\begin{aligned}\mathbf{f}(\mathbf{U}^1) = -\partial_t \mathbf{U}^1 + \sum_{\alpha,\beta=1}^{D} \partial_\alpha \left[\mathbf{D}^{\alpha\beta}\cdot\mathbf{G}\cdot\partial_\beta\mathbf{U}^1\right] \\ -\tfrac{1}{2}\sum_{\alpha=1}^{D}\tfrac{\partial}{\partial x_\alpha}\{\Psi^{*\alpha}_{\mathbf{vv}}\cdot\left[\Phi_{\mathbf{uuu}}:(\mathbf{U}^1\otimes\mathbf{U}^1)\right] + \Psi^{\star\alpha}_{\mathbf{vvv}}:\left[\mathbf{G}\cdot\mathbf{U}^1\otimes\mathbf{G}\cdot\mathbf{U}^1\right]\}\,,\end{aligned} \tag{2.2.0.31}$$

Using the entropy structure, we have the following relation:

Lemma 2:

$$\Psi^{\star\alpha}_{\mathbf{vv}}\cdot[\Phi_{\mathbf{uuu}}:(\mathbf{U}\otimes\mathbf{U})] = -\Psi^{\alpha}_{\mathbf{vv}}\cdot\Phi_{\mathbf{uu}}\cdot\Phi^{*}_{\mathbf{vvv}}:[\Phi_{\mathbf{uu}}\cdot\mathbf{U}\otimes\Phi_{\mathbf{uu}}\cdot\mathbf{U}]\;. \tag{2.2.0.32}$$

Proof of Lemma 2: Let $\mathbf{V} = \mathbf{G}\cdot\mathbf{U}$. From the relation $\Phi_{\mathbf{uu}}(\mathbf{U})\cdot\Phi^*_{\mathbf{vv}}(\mathbf{V}) = \mathrm{I}$, we have $\mathbf{U} = \Phi^*_{\mathbf{vv}}\cdot\mathbf{V}$. Then

$$\Psi^{*\alpha}_{ij}\Phi_{jkn}\mathbf{U}_n\mathbf{U}_k = \Psi^{*\alpha}_{ij}\Phi_{jkn}\Phi^*_{kq}\Phi^*_{np}\mathbf{V}_p\mathbf{V}_q\,. \tag{2.2.0.33}$$

Differentiate the relation $\Phi_{\mathbf{uu}}(\mathbf{U}) \cdot \Phi^*_{\mathbf{vv}}(\mathbf{V}) = \mathrm{I}$. This yields

$$\partial_{\mathbf{U}_n}\Phi_{jk}(\mathbf{U})\Phi^*_{kq}(\Phi_{\mathbf{u}}(\mathbf{U})) = -\Phi_{jk}(\mathbf{U})\partial_{\mathbf{V}_m}\Phi^*_{kq}(\mathbf{V})\partial_{\mathbf{U}_n}\mathbf{V}_m\,, \tag{2.2.0.34}$$

i.e.,

$$\Phi_{jkn}\Phi^*_{kq} = -\Phi_{jk}\Phi^*_{kqm}\Phi_{mn}\,. \tag{2.2.0.35}$$

Combining this with (2.2.0.33), and recalling the relation $\Phi_{mn}\Phi^*_{np} = \delta_{mp}$, we derive

$$\begin{aligned}\Psi^{*\alpha}_{ij}\Phi_{jkn}\mathbf{U}_n\mathbf{U}_k &= -\Psi^{*\alpha}_{ij}\Phi_{jk}\Phi^*_{kqm}\Phi_{mn}\Phi^*_{np}\mathbf{V}_p\mathbf{V}_q\\ &= -\Psi^{*\alpha}_{ij}\Phi_{jk}\Phi^*_{kqm}\delta_{mp}\mathbf{V}_p\mathbf{V}_q\\ &= -\Psi^{*\alpha}_{ij}\Phi_{jk}\Phi^*_{kpq}\mathbf{V}_p\mathbf{V}_q\,.\end{aligned} \tag{2.2.0.36}$$

Thus we proved (2.2.0.32). □

Using the relation (2.2.0.32), we can rewrite (2.2.0.30)

$$\partial_\tau\mathbf{U}^2 + \mathcal{A}\mathbf{U}^2 = -\partial_t\mathbf{U}^1 - \mathcal{Q}(\mathbf{U}^1\,,\mathbf{U}^1) + \mathcal{D}(\mathbf{U}^1)\,, \tag{2.2.0.37}$$

where

$$\begin{aligned}\mathcal{Q}(\mathbf{U}\,,\mathbf{U}) &:= \mathcal{Q}_1(\mathbf{U}\,,\mathbf{U}) - \mathcal{Q}_2(\mathbf{U}\,,\mathbf{U})\\ &= \tfrac{1}{2}\Psi^*_{\mathbf{vvv}} : \nabla_x(\mathbf{G}\cdot\mathbf{U}\otimes\mathbf{G}\cdot\mathbf{U}) - \tfrac{1}{2}\mathcal{A}\Phi^*_{\mathbf{vvv}} : (\mathbf{G}\cdot\mathbf{U}\otimes\mathbf{G}\cdot\mathbf{U})\,,\end{aligned} \tag{2.2.0.38}$$

$$\mathcal{D}(\mathbf{U}) := \mathbf{D}^{\alpha\beta}\cdot\mathbf{G}\cdot\partial_{\alpha\beta}\mathbf{U}\,. \tag{2.2.0.39}$$

As is typical of singular perturbation problems, the $O(\epsilon)$ equation (2.2.0.30) involve the second-order perturbation $\mathbf{U}^2$, which therefore need to be eliminated in order to obtain a closed set of equations for the limit solution $\mathbf{U}^1$. The standard method to do so is by the sublinear growth condition, i.e., the condition that the second-order perturbation terms be $o(\tau)$, so that the ordering of the expansion remain correct up through the value $\frac{1}{\epsilon}$ of the fast time τ actually occurring in the expansion (2.2.0.23). More precisely, we require the sublinear growth condition for the fast variable,

$$|\mathbf{U}^2(t,\tau)| = o(\tau) \quad \text{uniformlly for} \quad 0 < t < \tfrac{T}{\epsilon}\,. \tag{2.2.0.40}$$

In order to calculate the sublinear growth condition, we find the solution of (2.2.0.37) explicitly by the Duhamel formula:

$$\begin{aligned} e^{\tau\mathcal{A}}\mathbf{U}^2 = \mathbf{U}^2(t,\tau)|_{\tau=0} - \tau\partial_t\tilde{\mathbf{U}}^1(t) - \int_0^\tau e^{s\mathcal{A}}\mathcal{Q}(e^{-s\mathcal{A}}\tilde{\mathbf{U}}^1\,, e^{-s\mathcal{A}}\tilde{\mathbf{U}}^1)\,ds \\ + \int_0^\tau e^{s\mathcal{A}}\mathcal{D}(e^{-s\mathcal{A}}\tilde{\mathbf{U}}^1)\,ds\,. \end{aligned} \tag{2.2.0.41}$$

Since $\mathcal{A}$ is a skew-symmetric operator on $\mathcal{H}$, the operator $e^{\tau\mathcal{A}}$ preserves the norm in $\mathcal{H}$, so $e^{\tau\mathcal{A}}\mathbf{U}^2$ satisfies the sublinear growth condition in (2.2.0.40) if and only if $\mathbf{U}^2$ does. In addition, $\mathbf{U}^2(t,\tau)|_{\tau=0}$ is independent of τ, so $\lim_{\tau\to 0}\frac{1}{\tau}\mathbf{U}^2(t,\tau)|_{\tau=0} = 0$. Thus, from the explicit formula (2.2.0.41), we observe that (2.2.0.40) is satisfied provided that $\mathbf{U}(x,t)$ satisfies the following averaged system :

$$\partial_t\mathbf{U} + \overline{\mathcal{Q}}(\mathbf{U},\mathbf{U}) = \overline{\mathcal{D}}(\mathbf{U})\,, \tag{2.2.0.42}$$

where $\overline{\mathcal{Q}}(\mathbf{U},\mathbf{U}) = \overline{\mathcal{Q}}_1(\mathbf{U},\mathbf{U}) - \overline{\mathcal{Q}}_2(\mathbf{U},\mathbf{U})$ and

$$\begin{aligned} \overline{\mathcal{Q}}_1(\mathbf{U},\mathbf{U}) &= \tfrac{1}{2}\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{s\mathcal{A}}\Psi^*_{\mathbf{vvv}}:\nabla_x(\mathbf{G}\cdot e^{-s\mathcal{A}}\mathbf{U}\otimes\mathbf{G}\cdot e^{-s\mathcal{A}}\mathbf{U})\,ds\,, \\ \overline{\mathcal{Q}}_2(\mathbf{U},\mathbf{U}) &= \tfrac{1}{2}\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{s\mathcal{A}}\mathcal{A}(\Phi^*_{\mathbf{vvv}}:\mathbf{G}\cdot e^{-s\mathcal{A}}\mathbf{U}\otimes\mathbf{G}\cdot e^{-s\mathcal{A}}\mathbf{U})\,ds\,, \\ \overline{\mathcal{D}}(\mathbf{U}) &= \lim_{T\to\infty}\frac{1}{T}\int_0^T e^{s\mathcal{A}}\mathbf{D}^{\alpha\beta}\cdot\mathbf{G}\cdot\partial_{\alpha\beta}(e^{-s\mathcal{A}}\mathbf{U})\,ds\,. \end{aligned} \tag{2.2.0.43}$$

Remark: Actually $\mathcal{Q}(\mathbf{U},\mathbf{U})$ has a much simpler form. It is exactly the quadratic term in the Taylor expansion of $\mathbf{F}^\alpha(\mathbf{U})$, i.e.,

$$\mathbf{F}^\alpha_{\mathbf{uu}} = [\Psi^{*\alpha}_{\mathbf{vvv}} - (\Psi^\alpha_{\mathbf{vv}}\cdot\Phi_{\mathbf{uu}})\cdot\Phi^*_{\mathbf{vvv}}] : \Phi_{\mathbf{uu}}\otimes\Phi_{\mathbf{uu}}\,. \tag{2.2.0.44}$$

Proof of (2.2.0.44): Using the relation $\Phi_{\mathbf{uu}}(\mathbf{U}) \cdot \Phi^*_{\mathbf{vv}}(\mathbf{V}) = \mathrm{I}$, we have

$$\Psi^{*\alpha}_{\mathbf{vv}} = (\Psi^{*\alpha}_{\mathbf{vv}} \cdot \Phi_{\mathbf{uu}}) \cdot \Phi^*_{\mathbf{vv}} \,. \tag{2.2.0.45}$$

Differentiating this respect to $\mathbf{V}$, we derive

$$\Psi^{*\alpha}_{\mathbf{vvv}} - (\Psi^{*\alpha}_{\mathbf{vv}} \cdot \Phi_{\mathbf{uu}}) \cdot \Phi^*_{\mathbf{vvv}} = (\Psi^{*\alpha}_{\mathbf{vv}} \cdot \Phi_{\mathbf{uu}})_{\mathbf{v}} \cdot \Phi^*_{\mathbf{vv}} \,. \tag{2.2.0.46}$$

So the right-hand side of (2.2.0.44) equals to $(\Psi^{*\alpha}_{\mathbf{vv}} \cdot \Phi_{\mathbf{uu}})_{\mathbf{v}} \cdot \Phi_{\mathbf{uu}}$. Here we again use the relation $\Phi_{\mathbf{uu}}(\mathbf{U}) \cdot \Phi^*_{\mathbf{vv}}(\mathbf{V}) = \mathrm{I}$. We know that $\mathbf{F}^\alpha(\mathbf{U}) = \Psi^{*\alpha}_{\mathbf{v}}(\Phi_{\mathbf{u}})$. Differentiating with respect to $\mathbf{U}$ again, and noting $\mathbf{V}_{\mathbf{U}} = \Phi^*_{\mathbf{vv}}$, finally we have $\mathbf{F}^\alpha_{\mathbf{uu}} = (\Psi^{*\alpha}_{\mathbf{vv}} \cdot \Phi_{\mathbf{uu}})_{\mathbf{v}} \cdot \Phi^*_{\mathbf{vv}}$. Thus we proved (2.2.0.44). □

Although $\mathbf{F}^\alpha_{\mathbf{uu}}$ is easier to calculate than the expression in (2.2.0.38), from the explicit form of $\mathbf{F}^\alpha_{\mathbf{uu}}$, we could not see the role of the entropy-entropy flux and their Legendre transformations which play the key role in our later analysis; see the next section.

2.3 Properties of the Averaged System

In this section, we will derive the detailed form of the averaged system. The hyperbolicity at the constant state $\mathbf{U}_*$ implies that the $N \times N$ matrix $\hat{A}(\mathbf{k}) := \mathbf{k} \Psi^*_{\mathbf{vv}} \mathbf{G}$ has N real eigenvalues: $\lambda_1(\mathbf{k}) \le \lambda_2(\mathbf{k}) \le \cdots \le \lambda_N(\mathbf{k})$, for any wave number $\mathbf{k} \in \mathbb{Z}^D$. The corresponding right eigenvectors are $\eta_1(\mathbf{k},)\cdots, \eta_N(\mathbf{k})$ which can be normalized as

$$\langle \eta^a(\mathbf{k},) \eta^b(\mathbf{k}) \rangle = \delta_{ab}, \qquad a, b = 1, 2, \cdots, N \,. \tag{2.3.0.47}$$

Define $\phi^b(\mathbf{x},\mathbf{k}) = \eta^b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}$. We claim that $\phi^b(\mathbf{k})$ is an eigenvector of the first order differential operator $\mathcal{A}$ which is defined in (2.2.0.25), i.e., $\mathcal{A}\phi^b(\mathbf{k}) = i\lambda^b(\mathbf{k})\phi^b(\mathbf{k})$.

Proof of the claim:

$$\begin{aligned}\mathcal{A}\phi^b(\mathbf{k}) &= \Psi^{*,\alpha}_{\mathbf{vv}}\cdot\mathbf{G}\cdot\partial_\alpha\phi^b \\ &= i\hat{A}(\mathbf{k})\cdot\eta^b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= i\lambda^b(\mathbf{k})\phi^b(\mathbf{k})\,.\end{aligned} \tag{2.3.0.48}$$

and from linear algebra we know the $\{\phi^b(\mathbf{k})\}$, for $b = 1, 2, \cdots, N$ and $\mathbf{k} \in \mathbb{Z}^N$ generates an orthonormal base of the Hilbert space $\mathcal{H}$ which is defined in (2.2.0.26).

Proposition 1: *The averaged diffusion operator $\overline{\mathcal{D}}$ has the structure:*

$$\overline{\mathcal{D}}(\mathbf{U}) = \sum_{\alpha,\beta=1}^{D} \partial_{\alpha\beta}\overline{D}^{\alpha\beta}(\mathbf{U})\,, \tag{2.3.0.49}$$

where $\overline{D}^{\alpha\beta}(\mathbf{U})$ is given by its Fourier coefficient

$$\widehat{\overline{D}^{\alpha\beta}(\mathbf{U})}(\mathbf{k}) = \sum_{\substack{a,b=1 \\ \lambda^a(\mathbf{k})=\lambda^b(\mathbf{k})}}^{N} C^{\alpha\beta}_{ab}(\mathbf{k})\mathbf{U}^a(\mathbf{k})\eta^b(\mathbf{k})\,, \tag{2.3.0.50}$$

and

$$\begin{aligned}C^{\alpha\beta}_{ab}(\mathbf{k}) &= \langle \mathbf{D}^{\alpha\beta}\cdot\mathbf{G}\cdot\eta^a(\mathbf{k})\,, \eta^b(\mathbf{k})\rangle\,, \\ \mathbf{U}^a(\mathbf{k}) &= \langle \mathbf{U}\,, \phi^a(\mathbf{k})\rangle\,.\end{aligned} \tag{2.3.0.51}$$

Proof of the Proposition: Note that

$$e^{s\mathcal{A}}\mathbf{U} = e^{s\mathcal{A}}(\mathbf{U}^a(\mathbf{k})\phi^a(\mathbf{k})) = \mathbf{U}^b(\mathbf{k})e^{-is\lambda^a(\mathbf{k})}\phi^a(\mathbf{k})\,. \tag{2.3.0.52}$$

Then

$$\mathbf{D}^{\alpha\beta}\cdot\mathbf{G}\cdot\partial_{\alpha\beta}(e^{s\mathcal{A}}\mathbf{U}) = -\sum_{a=1}^{N}\sum_{\mathbf{k}\in\mathbb{Z}^{\mathbb{D}}} \mathbf{k}^\alpha\mathbf{k}^\beta\mathbf{U}^a(\mathbf{k})e^{-is\lambda^a(\mathbf{k})}\mathbf{D}^{\alpha\beta}\cdot\mathbf{G}\cdot\phi^a(\mathbf{k})\,. \tag{2.3.0.53}$$

Note that

$$\begin{aligned}\mathbf{D}^{\alpha\beta}\cdot\mathbf{G}\cdot\phi^a(\mathbf{k}) &= \sum_{b=1}^{N}\sum_{\mathbf{l}\in\mathbb{Z}^{\mathbb{D}}}\langle\mathbf{D}^{\alpha\beta}\cdot\mathbf{G}\cdot\phi^a(\mathbf{k})\,,\phi^b(\mathbf{l})\rangle\phi^b(\mathbf{l}) \\ &= \sum_{b=1}^{N}\langle\mathbf{D}^{\alpha\beta}\cdot\mathbf{G}\cdot\phi^a(\mathbf{k})\,,\phi^b(\mathbf{k})\rangle\phi^b(\mathbf{k})\,,\end{aligned} \tag{2.3.0.54}$$

then

$$e^{s\mathcal{A}}(\mathbf{D}^{\alpha\beta}\cdot\mathbf{G}\cdot\phi^a(\mathbf{k})) = \sum_{b=1}^{N} C_{ab}^{\alpha\beta}(\mathbf{k})e^{is\lambda^b(\mathbf{k})}\phi^b(\mathbf{k})\,. \tag{2.3.0.55}$$

Thus

$$\overline{\mathcal{D}}(\mathbf{U}) = -\mathbf{k}^\alpha\mathbf{k}^\beta\lim_{T\to\infty}\frac{1}{T}\int_0^\infty C_{ab}^{\alpha\beta}(\mathbf{k})\mathbf{U}^a(\mathbf{k})e^{-i(\lambda^a(\mathbf{k})-\lambda^b(\mathbf{k}))}\phi^b(\mathbf{k})\,ds\,. \tag{2.3.0.56}$$

Recalling that the Riemann-Lebesgue lemma guarantees that the only nontrivial contribution in the above time averaging is the case $\lambda^a(\mathbf{k}) = \lambda^b(\mathbf{k})$, this yields

$$\overline{\mathcal{D}}(\mathbf{U}) = -\mathbf{k}^\alpha\mathbf{k}^\beta\sum_{\lambda^a(\mathbf{k})=\lambda^b(\mathbf{k})} C_{ab}^{\alpha\beta}(\mathbf{k})\mathbf{U}^a(\mathbf{k})\phi^b(\mathbf{k})\,. \tag{2.3.0.57}$$

Noting that $-\mathbf{k}^\alpha\mathbf{k}^\beta\phi^b(\mathbf{k}) = \partial_{\alpha\beta}\phi^b(\mathbf{k})$, we finish the proof of the proposition (1). □

Remark: The special case of the numbers $C_{ab}^{\alpha\beta}(\mathbf{k})$ has been used in the construction of diffusion waves when considering the large-time decay of compressible Navier-Stokes equations by T.-P. Liu and others [56, 57]. They considered 1-D and strictly hyperbolic case. It is reasonably to expect that $C_{ab}^{\alpha\beta}(\mathbf{k})$ will play an important role when consider the diffusion waves of multidimensional, fully compressible Navier-Stokes system which is a non-strict hyperbolic.

Now it is easy to derive that

$$\langle\overline{\mathcal{D}}(\mathbf{U})\,,\mathbf{U}\rangle = -\mathbf{k}^\alpha\mathbf{k}^\beta\sum_{\lambda^a(\mathbf{k})=\lambda^b(\mathbf{k})} C_{ab}^{\alpha\beta}(\mathbf{k})\mathbf{U}^a(\mathbf{k})\overline{\mathbf{U}}^b(\mathbf{k})\,. \tag{2.3.0.58}$$

Define a family of vector-valued functions $\mathbf{W}^\alpha = \partial_\alpha \mathbf{U}$, where $\alpha = 1, 2, \cdots, D$. Then it is easily to derive

$$\begin{aligned}\mathbf{W}^\alpha_a(\mathbf{k}) = \langle \mathbf{W}^\alpha \,, \phi^a(\mathbf{k})\rangle &= -\int_\Omega \mathbf{U}^T \!\cdot\! \mathbf{G} \!\cdot\! \partial_\alpha \phi^a(\mathbf{k})\, d\mathbf{x} \\ &= -i\mathbf{k}^\alpha \mathbf{U}^a(\mathbf{k})\,.\end{aligned} \tag{2.3.0.59}$$

Thus

$$\langle \overline{\mathcal{D}}(\mathbf{U})\,, \mathbf{U}\rangle = -\sum_{\mathbf{k}\in\mathbb{Z}^D} \sum_{\lambda^a(\mathbf{k})=\lambda^b(\mathbf{k})} C^{\alpha\beta}_{ab}(\mathbf{k}) \mathbf{W}^\alpha_a(\mathbf{k}) \overline{\mathbf{W}}^\beta_b(\mathbf{k})\,. \tag{2.3.0.60}$$

Note that $\hat{A}(\mathbf{k})$ is homogeneous in $\mathbf{k}$, then $\lambda^b(\mathbf{k}/|\mathbf{k}|) = \lambda^b(\mathbf{k})/|\mathbf{k}|$, and $\eta^b(\mathbf{k}) = \eta^b(\mathbf{k}/|\mathbf{k}|)$, then $C^{\alpha\beta}_{ab}(\mathbf{k}) = C^{\alpha\beta}_{ab}(\mathbf{k}/|\mathbf{k}|)$. When the spatial dimension $D = 1$, and the inviscid part of the hyperbolic-parabolic system is strictly hyperbolic,

$$-\langle \overline{\mathcal{D}}(\mathbf{U})\,, \mathbf{U}\rangle = \sum_{\mathbf{k}\in\mathbb{Z}} C^b(\mathbf{k}) |\widehat{\partial_x \mathbf{U}}(\mathbf{k})|^2\,, \tag{2.3.0.61}$$

where $C^b(\mathbf{k}) = \langle \mathbf{D}\cdot\mathbf{G}\cdot\eta^b(\mathbf{k})\,, \eta^b(\mathbf{k})\rangle$. If $C^b(\mathbf{k}) > 0$, because $C^b(\mathbf{k}) = C^b(\mathbf{k}/|\mathbf{k}|)$, then from the compactness of the unit sphere, there exists a $\delta > 0$, such that $C^b(\mathbf{k}) > \delta$. Thus $-\langle \overline{\mathcal{D}}(\mathbf{U})\,, \mathbf{U}\rangle \geq \delta \|\partial_x \mathbf{U}\|_2^2$, i.e., the averaged diffusion operator $\overline{\mathcal{D}}$ is strictly parabolic. The condition $C^b(\mathbf{k}) = \langle \mathbf{D}\cdot\mathbf{G}\cdot\eta^b(\mathbf{k})\,, \eta^b(\mathbf{k})\rangle > 0$ is the so-called Kawashima condition [43, 44] which has a wide application in the large-time behavior of the 1-D diffusion waves [56, 57].

In the general case, similarly we need only to consider the diffusion wave numbers $C^{\alpha\beta}_{ab}(\mathbf{k})$ on the unit sphere $|\mathbf{k}| = 1$. From the compactness of the unit sphere, $-\langle \overline{\mathcal{D}}(\mathbf{U})\,, \mathbf{U}\rangle > 0$ will imply $-\langle \overline{\mathcal{D}}(\mathbf{U})\,, \mathbf{U}\rangle > \delta \langle \nabla_x \mathbf{U}\,, \nabla_x \mathbf{U}\rangle$ for some $\delta > 0$. We will give a sufficient condition that guarantees this strict parabolicity.

Proposition 2: *If $-\mathcal{A}\mathcal{D}\mathcal{A}|_{\mathrm{Null}(\mathcal{D})} > 0$, then there exits a $\delta > 0$, so that*

$$-\langle \overline{\mathcal{D}}(\mathbf{U})\,, \mathbf{U}\rangle > \delta \langle \nabla_x \mathbf{U}\,, \nabla_x \mathbf{U}\rangle\,. \tag{2.3.0.62}$$

Proof of the Proposition: From the argument above, we need only show that

$$-\langle\overline{\mathcal{D}}(\mathbf{U})\,,\mathbf{U}\rangle > 0 \tag{2.3.0.63}$$

under the condition $-\mathcal{A}\mathcal{D}\mathcal{A}|_{\mathrm{Null}(\mathcal{D})} > 0$. We prove this by contradiction. Suppose there exits a nonzero $\mathbf{U} \in \mathbb{R}^N$ so that $\overline{\mathcal{D}}\mathbf{U} = 0$, i.e.,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \langle e^{t\mathcal{A}}\mathcal{D}e^{-t\mathcal{A}}\mathbf{U}\,,\mathbf{U}\rangle\,dt = 0\,. \tag{2.3.0.64}$$

Define a function $F(t) = -\langle e^{t\mathcal{A}}\mathcal{D}e^{-t\mathcal{A}}\mathbf{U}\,,\mathbf{U}\rangle$, then $F(t)$ has the following properties:

1. $F(t) \geq 0$;
2. $\lim_{T\to\infty}\frac{1}{T}\int_0^T F(t)\,dt = 0$;
3. $F(t)$ is an almost-periodic function in t.

Then from the standard theory of almost-periodic functions (see page 20, [2],) $F(t) \equiv 0$, for all $t \geq 0$. In particular, $F^{''}(t) = 0$, for all $t \geq 0$. It can be easily shown that

$$\begin{aligned} F^{''}(t) &= \langle\mathcal{A}^2\mathcal{D}e^{-t\mathcal{A}}\mathbf{U}\,,e^{-t\mathcal{A}}\mathbf{U}\rangle + \langle\mathcal{D}\mathcal{A}^2e^{-t\mathcal{A}}\mathbf{U}\,,e^{-t\mathcal{A}}\mathbf{U}\rangle \\ &\quad + \langle -2\mathcal{A}\mathcal{D}\mathcal{A}e^{-t\mathcal{A}}\mathbf{U}\,,e^{-t\mathcal{A}}\mathbf{U}\rangle\,. \end{aligned} \tag{2.3.0.65}$$

Note that $F(t) = 0$ implies $e^{-t\mathcal{A}}\mathbf{U} \in \mathrm{Null}(\mathcal{D})$ and the relation $\langle\mathcal{D}\mathbf{U}\,,\mathbf{V}\rangle = \langle\mathcal{D}\mathbf{V}\,,\mathbf{U}\rangle$. Then the first two terms in above identity vanish. This yields that

$$F^{''}(t) = \langle -2\mathcal{A}\mathcal{D}\mathcal{A}e^{-t\mathcal{A}}\mathbf{U}\,,e^{-t\mathcal{A}}\mathbf{U}\rangle\,. \tag{2.3.0.66}$$

The condition $-\mathcal{A}\mathcal{D}\mathcal{A}|_{\mathrm{Null}(\mathcal{D})} > 0$ implies that $F^{''}(t) > 0$. Contradiction. Thus we finish the proof of lemma. □

Remark: the condition $-\mathcal{A}\mathcal{D}\mathcal{A}|_{\text{Null}(\mathcal{D})} > 0$ is slightly stronger than Kawashima condition in multi-D case. Kawashima condition states that the diffusion operator $\mathcal{D}$ is dissipative if and only if the eigenvector of $\mathcal{A}$ is not in Null($\mathcal{D}$). It heavily depends on the spectrum of the operator $\mathcal{A}$. Our condition is only sufficient, but is much easier to check.

Now we calculate the convection term $\overline{\mathcal{Q}}(\mathbf{U}\,,\mathbf{U})$ in the averaged system .

$$
\begin{aligned}
&\overline{\mathcal{Q}}_1(\mathbf{U}\,,\mathbf{U})\\
&=\sum_{a,\mathbf{k}}\langle\overline{\mathcal{Q}}_1(\mathbf{U}\,,\mathbf{U})\,,\phi^a(\mathbf{k})\rangle\phi^a(\mathbf{k})\\
&=-\tfrac{1}{2}\lim_{T\to\infty}\frac{1}{T}\int_0^T\langle\Psi^{*,\alpha}_{\mathbf{vvv}}:\mathbf{G}\cdot e^{s\mathcal{A}}\mathbf{U}\otimes\mathbf{G}\cdot e^{s\mathcal{A}}\mathbf{U}\,,\partial_\alpha e^{s\mathcal{A}}\phi^a(\mathbf{k})\rangle\,ds\phi^a(\mathbf{k})\\
&=\tfrac{1}{2}i\mathbf{k}^\alpha\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_\Omega I^{abc,\alpha}_{\mathbf{kmn}}\mathbf{U}^b(\mathbf{m})\mathbf{U}^c(\mathbf{n})e^{is[\lambda_b(\mathbf{m})+\lambda_c(\mathbf{n})-\lambda_a(\mathbf{k})]}e^{-i(\mathbf{m}+\mathbf{n}-\mathbf{k})\cdot\mathbf{x}}\,ds\,d\mathbf{x}\phi^a(\mathbf{k})\\
&=\tfrac{1}{2}i\mathbf{k}^\alpha\sum_{\substack{\mathbf{m}+\mathbf{n}=\mathbf{k}\\ \lambda^b(\mathbf{m})+\lambda^c(\mathbf{n})=\lambda^a(\mathbf{k})}}I^{abc,\alpha}_{\mathbf{kmn}}\mathbf{U}^b(\mathbf{m})\mathbf{U}^c(\mathbf{n})\phi^a(\mathbf{k})\,,
\end{aligned}
\tag{2.3.0.67}
$$

where

$$
I^{abc,\alpha}_{\mathbf{kmn}}=\Psi^{*,\alpha}_{\mathbf{vvv}}:\mathbf{G}\cdot\eta^a(\mathbf{k})\otimes\mathbf{G}\cdot\eta^b(\mathbf{m})\otimes\mathbf{G}\cdot\eta^c(\mathbf{n})\,. \tag{2.3.0.68}
$$

Note that $\lambda_a(-\mathbf{k})=-\lambda_a(\mathbf{k})$ and $\eta^a(-\mathbf{k})=\eta^a(\mathbf{k})$. Then $I^{abc,\alpha}_{\mathbf{kmn}}=I^{abc,\alpha}_{-\mathbf{kmn}}$ and

$$
\overline{\mathcal{Q}}_1(\mathbf{U}\,,\mathbf{U})=-\tfrac{1}{2}i\mathbf{k}^\alpha\sum_{\substack{\mathbf{k}+\mathbf{m}+\mathbf{n}=0\\ \lambda^a(\mathbf{k})+\lambda^b(\mathbf{m})+\lambda^c(\mathbf{n})=0}}I^{abc,\alpha}_{\mathbf{kmn}}\mathbf{U}^b(\mathbf{m})\mathbf{U}^c(\mathbf{n})\overline{\phi}^a(\mathbf{k})\,. \tag{2.3.0.69}
$$

We can easily derive

$$
\langle\overline{\mathcal{Q}}_1(\mathbf{U}\,,\mathbf{U})\,,\mathbf{U}\rangle=-\tfrac{1}{2}i\mathbf{k}^\alpha\sum_{\substack{\mathbf{k}+\mathbf{m}+\mathbf{n}=0\\ \lambda^a(\mathbf{k})+\lambda^b(\mathbf{m})+\lambda^c(\mathbf{n})=0}}I^{abc,\alpha}_{\mathbf{kmn}}\mathbf{U}^a(\mathbf{k})\mathbf{U}^b(\mathbf{m})\mathbf{U}^c(\mathbf{n})\,. \tag{2.3.0.70}
$$

Note that the 3-tensor $\Psi^\alpha_{\mathbf{vvv}}$ is symmetric, then the interaction numbers $I^{abc,\alpha}_{\mathbf{kmn}}$ is invariant under the permutation of $\mathbf{k}\,,\mathbf{m}$, and $\mathbf{n}$. Then

$$
\begin{aligned}
&\langle \overline{\mathcal{Q}}_1(\mathbf{U}\,,\mathbf{U})\,,\mathbf{U}\rangle \\
&= -\tfrac{1}{12} i(\mathbf{k}+\mathbf{m}+\mathbf{n})^{\alpha} \sum_{\substack{\mathbf{k}+\mathbf{m}+\mathbf{n}=0 \\ \lambda^a(\mathbf{k})+\lambda^b(\mathbf{m})+\lambda^c(\mathbf{n})=0}} I^{abc,\alpha}_{\mathbf{kmn}} \mathbf{U}^a(\mathbf{k})\mathbf{U}^b(\mathbf{m})\mathbf{U}^c(\mathbf{n}) \\
&= 0\,.
\end{aligned}
\tag{2.3.0.71}
$$

Similarly, we can derive

$$
\overline{\mathcal{Q}}_2(\mathbf{U}\,,\mathbf{U}) = -\tfrac{1}{2} i\lambda^a(\mathbf{k}) \sum_{\substack{\mathbf{k}+\mathbf{m}+\mathbf{n}=0 \\ \lambda^a(\mathbf{k})+\lambda^b(\mathbf{m})+\lambda^c(\mathbf{n})=0}} J^{abc}_{\mathbf{kmn}} \mathbf{U}^b(\mathbf{m})\mathbf{U}^c(\mathbf{n})\overline{\phi}^a(\mathbf{k})\,, \tag{2.3.0.72}
$$

where

$$
J^{abc}_{\mathbf{kmn}} = \Phi^*_{\mathbf{vvv}} : \mathbf{G}\cdot\eta^a(\mathbf{k}) \otimes \mathbf{G}\cdot\eta^b(\mathbf{m}) \otimes \mathbf{G}\cdot\eta^c(\mathbf{n})\,, \tag{2.3.0.73}
$$

and

$$
\begin{aligned}
&\langle \overline{\mathcal{Q}}_2(\mathbf{U}\,,\mathbf{U})\,,\mathbf{U}\rangle \\
&= -\tfrac{1}{2} i\lambda^a(\mathbf{k}) \sum_{\substack{\mathbf{k}+\mathbf{m}+\mathbf{n}=0 \\ \lambda^a(\mathbf{k})+\lambda^b(\mathbf{m})+\lambda^c(\mathbf{n})=0}} J^{abc}_{\mathbf{kmn}} \mathbf{U}^a(\mathbf{k})\mathbf{U}^b(\mathbf{m})\mathbf{U}^c(\mathbf{n}) \\
&= -\tfrac{1}{12} i(\lambda^a(\mathbf{k})+\lambda^b(\mathbf{m})+\lambda^c(\mathbf{n})) \sum_{\substack{\mathbf{k}+\mathbf{m}+\mathbf{n}=0 \\ \lambda^a(\mathbf{k})+\lambda^b(\mathbf{m})+\lambda^c(\mathbf{n})=0}} J^{abc}_{\mathbf{kmn}} \mathbf{U}^a(\mathbf{k})\mathbf{U}^b(\mathbf{m})\mathbf{U}^c(\mathbf{n}) \\
&= 0\,.
\end{aligned}
\tag{2.3.0.74}
$$

Our calculations show that the quadratic term $\overline{\mathcal{Q}}(\mathbf{U},\mathbf{U})$ has some good properties. Actually we have proved the following lemma which states that the nonlinear term $\overline{\mathcal{Q}}(\mathbf{U},\mathbf{U})$ is orthogonal to $\mathbf{U}$, so it does not contribute to the energy estimate. Furthermore, it has a divergence form. This property is an analogue of that for convection term in the incompressible Navier-Stokes equations. It will play a key role in the proof of global weak solutions.

Proposition 3: *The convection term in the averaged system has the following properties:*

1. $\overline{\mathcal{Q}}_1$ *has divergence form*

$$\overline{\mathcal{Q}}_1(\mathbf{U},\mathbf{U}) = \nabla_x \cdot \mathcal{B}_1(\mathbf{U},\mathbf{U})\,; \tag{2.3.0.75}$$

2. $\overline{\mathcal{Q}}_2$ *has of the form*

$$\overline{\mathcal{Q}}_2(\mathbf{U},\mathbf{U}) = \mathcal{A}\mathcal{B}_2(\mathbf{U},\mathbf{U})\,; \tag{2.3.0.76}$$

3. $\overline{\mathcal{Q}}$ *does not contribute in energy estimate*

$$\langle \overline{\mathcal{Q}}(\mathbf{U},\mathbf{U},)\mathbf{U}\rangle \equiv 0\,, \tag{2.3.0.77}$$

where $\mathcal{B}_1(\mathbf{U},\mathbf{U})$ and $\mathcal{B}_2(\mathbf{U},\mathbf{U})$ are given by their Fourier coefficients.

$$\mathcal{B}_1\widehat{(\mathbf{U},\mathbf{U})}(\mathbf{k}) = \tfrac{1}{2}\sum_{a=1}^{N}\sum_{b,c=1}^{N}\sum_{\substack{\mathbf{m}+\mathbf{n}=\mathbf{k}\\ \lambda^b(\mathbf{m})+\lambda^c(\mathbf{n})=\lambda^a(\mathbf{k})}} I^{abc}_{\mathbf{kmn}}\mathbf{U}^b(\mathbf{m})\mathbf{U}^c(\mathbf{n})\eta^a(\mathbf{k})\,, \tag{2.3.0.78}$$

and

$$\mathcal{B}_2\widehat{(\mathbf{U},\mathbf{U})}(\mathbf{k}) = \tfrac{1}{2}\sum_{a=1}^{N}\sum_{b,c=1}^{N}\sum_{\substack{\mathbf{m}+\mathbf{n}=\mathbf{k}\\ \lambda^b(\mathbf{m})+\lambda^c(\mathbf{n})=\lambda^a(\mathbf{k})}} J^{abc}_{\mathbf{kmn}}\mathbf{U}^b(\mathbf{m})\mathbf{U}^c(\mathbf{n})\eta^a(\mathbf{k})\,. \tag{2.3.0.79}$$

In the original hyperbolic-parabolic system, the first-order operator $\mathcal{A}$ and second-order operator $\mathcal{D}$ do not commute. The time averaging brings some nice properties that the skew symmetric operator $\mathcal{A}$ commutes with the averaged diffusion and convection operators.

Proposition 4: *The averaged diffusion and convection terms satisfy:*

1. $\mathcal{A}\overline{\mathcal{D}} = \overline{\mathcal{D}}\mathcal{A}\,;$

2. $\overline{\mathcal{Q}}(e^{t\mathcal{A}}\mathbf{U}, e^{t\mathcal{A}}\mathbf{U}) = e^{t\mathcal{A}}\overline{\mathcal{Q}}(\mathbf{U}\,,\mathbf{U})$ *, for any* $\mathbf{U}$.

Proof: As shown in Proposition (1), for any $\mathbf{U}$,

$$\overline{\mathcal{D}}(\mathbf{U}) = -\mathbf{k}^\alpha\mathbf{k}^\beta \sum_{\lambda^a(\mathbf{k})=\lambda^b(\mathbf{k})} C^{\alpha\beta}_{ab}(\mathbf{k})\mathbf{U}^a(\mathbf{k})\eta^b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\,. \tag{2.3.0.80}$$

Then

$$\begin{aligned}\mathcal{A}\overline{\mathcal{D}}(\mathbf{U}) &= -i\mathbf{k}^\alpha\mathbf{k}^\beta\mathbf{k}^\gamma\Psi^{*,\gamma}_{\mathbf{vv}}\cdot\mathbf{G}\cdot \sum_{\lambda^a(\mathbf{k})=\lambda^b(\mathbf{k})} C^{\alpha\beta}_{ab}(\mathbf{k})\mathbf{U}^a(\mathbf{k})\eta^b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= -i\mathbf{k}^\alpha\mathbf{k}^\beta \sum_{\lambda^a(\mathbf{k})=\lambda^b(\mathbf{k})} C^{\alpha\beta}_{ab}(\mathbf{k})\mathbf{U}^a(\mathbf{k})\lambda^b(\mathbf{k})\eta^b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\,.\end{aligned} \tag{2.3.0.81}$$

Here we used the fact that $\mathbf{k}^\gamma\Psi^{*,\gamma}_{\mathbf{vv}}\cdot\mathbf{G}\cdot\eta^b(\mathbf{k}) = \lambda^b(\mathbf{k})\eta^b(\mathbf{k})$.

$$\begin{aligned}\overline{\mathcal{D}}\mathcal{A}(\mathbf{U}) &= -\mathbf{k}^\alpha\mathbf{k}^\beta \sum_{\lambda^a(\mathbf{k})=\lambda^b(\mathbf{k})} C^{\alpha\beta}_{ab}(\mathbf{k})\langle\mathcal{A}\mathbf{U}\,,\phi^a(\mathbf{k})\rangle\eta^b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= \mathbf{k}^\alpha\mathbf{k}^\beta \sum_{\lambda^a(\mathbf{k})=\lambda^b(\mathbf{k})} C^{\alpha\beta}_{ab}(\mathbf{k})\langle\mathbf{U}\,,\mathcal{A}\phi^a(\mathbf{k})\rangle\eta^b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= -i\mathbf{k}^\alpha\mathbf{k}^\beta \sum_{\lambda^a(\mathbf{k})=\lambda^b(\mathbf{k})} C^{\alpha\beta}_{ab}(\mathbf{k})\mathbf{U}^a(\mathbf{k})\lambda^a(\mathbf{k})\eta^b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\,.\end{aligned} \tag{2.3.0.82}$$

Recalling the summation is taken on the resonant set $\lambda^a(\mathbf{k}) = \lambda^b(\mathbf{k})$ (because of averaging!,) we prove that $\mathcal{A}\overline{\mathcal{D}} = \overline{\mathcal{D}}\mathcal{A}$. To prove part (2), we use the formulas derived in Proposition (3).

$$\begin{aligned}
\overline{\mathcal{Q}}_1(e^{t\mathcal{A}}\mathbf{U}, e^{t\mathcal{A}}\mathbf{U}) &= \tfrac{1}{2}i\mathbf{k}^\alpha \sum_{\substack{\mathbf{m}+\mathbf{n}=\mathbf{k}\\ \lambda^b(\mathbf{m})+\lambda^c(\mathbf{n})=\lambda^a(\mathbf{k})}} I^{abc,\alpha}_{\mathbf{kmn}} \langle e^{t\mathcal{A}}\mathbf{U}, \phi^b(\mathbf{m})\rangle \langle e^{t\mathcal{A}}\mathbf{U}, \phi^c(\mathbf{n})\rangle \phi^a(\mathbf{k}) \\
&= \tfrac{1}{2}i\mathbf{k}^\alpha \sum_{\substack{\mathbf{m}+\mathbf{n}=\mathbf{k}\\ \lambda^b(\mathbf{m})+\lambda^c(\mathbf{n})=\lambda^a(\mathbf{k})}} I^{abc,\alpha}_{\mathbf{kmn}} e^{it(\lambda^b(\mathbf{m})+\lambda^c(\mathbf{n}))}\mathbf{U}^b(\mathbf{m})\mathbf{U}^c(\mathbf{n})\phi^a(\mathbf{k}) \\
&= \tfrac{1}{2}i\mathbf{k}^\alpha \sum_{\substack{\mathbf{m}+\mathbf{n}=\mathbf{k}\\ \lambda^b(\mathbf{m})+\lambda^c(\mathbf{n})=\lambda^a(\mathbf{k})}} I^{abc,\alpha}_{\mathbf{kmn}} \mathbf{U}^b(\mathbf{m})\mathbf{U}^c(\mathbf{n}) e^{it\lambda^a(\mathbf{k})}\phi^a(\mathbf{k}) \\
&= \tfrac{1}{2}i\mathbf{k}^\alpha \sum_{\substack{\mathbf{m}+\mathbf{n}=\mathbf{k}\\ \lambda^b(\mathbf{m})+\lambda^c(\mathbf{n})=\lambda^a(\mathbf{k})}} I^{abc,\alpha}_{\mathbf{kmn}} \mathbf{U}^b(\mathbf{m})\mathbf{U}^c(\mathbf{n}) e^{t\mathcal{A}}\phi^a(\mathbf{k}) \\
&= e^{t\mathcal{A}}\overline{\mathcal{Q}}_1(\mathbf{U}, \mathbf{U})\,.
\end{aligned} \tag{2.3.0.83}$$

Similarly,

$$\begin{aligned}
\overline{\mathcal{Q}}_2(e^{t\mathcal{A}}\mathbf{U}, e^{t\mathcal{A}}\mathbf{U}) &= \tfrac{1}{2}i\lambda^a(\mathbf{k}) \sum_{\substack{\mathbf{m}+\mathbf{n}=\mathbf{k}\\ \lambda^b(\mathbf{m})+\lambda^c(\mathbf{n})=\lambda^a(\mathbf{k})}} J^{abc}_{\mathbf{kmn}} \mathbf{U}^b(\mathbf{m})\mathbf{U}^c(\mathbf{n}) e^{t\mathcal{A}}\phi^a(\mathbf{k}) \\
&= e^{t\mathcal{A}}\overline{\mathcal{Q}}_2(\mathbf{U}, \mathbf{U})\,.
\end{aligned} \tag{2.3.0.84}$$

This completes the proof of the proposition. □

As a consequence, the following corollary can be easily derived.

Corollary 3: *If* $\mathbf{U}$ *is a solution to the averaged system , then* $\mathbf{V} = e^{t\mathcal{A}}\mathbf{U}$ *obeys the following equation:*

$$\partial_t \mathbf{V} + \mathcal{A}\mathbf{V} + \overline{\mathcal{Q}}(\mathbf{V}, \mathbf{V}) = \overline{\mathcal{D}}\mathbf{V}\,. \tag{2.3.0.85}$$

Proof of Corollary: $\partial_t \mathbf{V} = -\mathcal{A}e^{t\mathcal{A}}\mathbf{U} + e^{t\mathcal{A}}\partial_t \mathbf{U}$. Then,

$$\begin{aligned}
\partial_t \mathbf{V} + \mathcal{A}\mathbf{V} &= e^{t\mathcal{A}}(-\overline{\mathcal{Q}}(\mathbf{U}, \mathbf{U}) + \overline{\mathcal{D}}\mathbf{U}) \\
&= -\mathcal{Q}(e^{t\mathcal{A}}\mathbf{U}, e^{t\mathcal{A}}\mathbf{U}) + \overline{\mathcal{D}}(e^{t\mathcal{A}}\mathbf{U}) \\
&= -\mathcal{Q}(\mathbf{V}, \mathbf{V}) + \overline{\mathcal{D}}\mathbf{V}\,.
\end{aligned} \tag{2.3.0.86}$$

$\square$ **Remark:** In many applications, the first order operator $\mathcal{A}$ has 0 as eigenvalue. Then the null space of $\mathcal{A}$ is nontrivial. Usually, Null($\mathcal{A}$) and its orthogonal complement Null$(\mathcal{A})^\perp$ have different physical meanings. Null($\mathcal{A}$) is usually called the "slow mode" and Null$(\mathcal{A})^\perp$ is called "fast mode" because the operator $e^{\tau\mathcal{A}}$ generated by the fast time scale $\tau = \frac{t}{\epsilon}$ does not affect Null($\mathcal{A}$). The averaged system usually behaves significantly differently on slow and fast modes. In applications, for example, the Navier-Stokes system for compressible gas dynamics, or other nonlinear system in plasma physics, continuum mechanics, we project the averaged system onto these two modes to get so-called "slow equation" and "fast equation". In many cases, the slow equation could be completely decoupled from the fast equation. We will see this from the following sections on the Navier-Stokes system for compressible gas dynamics.

2.4 *Global Weak Solutions to the Averaged System*

The main goal of this section is to prove the existence of global weak solutions to the averaged system

$$\begin{aligned} \partial_t \mathbf{U} + \overline{\mathcal{Q}}_1(\mathbf{U},\mathbf{U}) - \overline{\mathcal{Q}}_2(\mathbf{U},\mathbf{U}) &= \overline{\mathcal{D}}\mathbf{U}\,, \\ \mathbf{U}(0,x) &= \mathbf{U}^{in}(x)\,, \end{aligned} \tag{2.4.0.87}$$

where the averaged diffusion operator $\overline{\mathcal{D}}$ is given by (2.3.0.49), while the averaged quadratic terms $\overline{\mathcal{Q}}_1$ and $\overline{\mathcal{Q}}_2$ are expressed by (2.3.0.75), (2.3.0.78) and (2.3.0.76), (2.3.0.79) respectively. The weak solutions we seek are in the same spirit as the Leray theory of global weak solutions to the initial-value problem for Navier-Stokes system [47]. Before we state and prove our main theorem of this chapter, we set up some rigorous mathematical notions, such as the function spaces in which weak solution are defined, the definition of weak solutions, and the statement of the main theorem.

2.4.1 Mathematical Setting for Averaged System

In this section, we present some function spaces in which weak solutions are defined. First we define the Hilbert space $\mathbb{H}$

$$\mathbb{H} = \left\{ \mathbf{U} \in \mathbf{L}^2(d\mathbf{x}\,, \mathbb{C}^N) : \int_{\mathbb{T}^D} \mathbf{U}\, d\mathbf{x} = 0 \right\}, \tag{2.4.1.1}$$

and a subspace of $\mathbb{H}$, called $\mathbb{V}$,

$$\mathbb{V} = \left\{ \mathbf{U} \in \mathbb{H} : \int_{\mathbb{T}^D} |\nabla_x \mathbf{U}|^2\, d\mathbf{x} < \infty \right\}. \tag{2.4.1.2}$$

In order to define the weak solutions of the averaged system , we need to introduce the principle spaces involved. Given any Banach space $\mathbb{X}$ with norm $\|\cdot\|_{\mathbb{X}}$ and $1 \le p \le \infty$, the space of (equivalence classes of) measurable functions $\mathbf{V} = \mathbf{V}(t)$ from $[0, \infty)$ into $\mathbb{X}$ such that $\|\mathbf{V}\|_{\mathbb{X}} \in L^p([0, T])$ for every $T > 0$ will be denoted $L^p_{\text{loc}}([0, \infty,)\mathbb{X})$, and finally $C([0, \infty); \text{w-}\mathbb{X})$ will denote the space of continuous functions from $[0, \infty)$ into w-$\mathbb{X}$, which denotes $\mathbb{X}$ equipped with its weak topology. This means that $\mathbf{V} \in C([0, \infty); \text{w-}\mathbb{X})$ if for every $\psi \in \mathbb{X}^*$ the function $t \mapsto \langle \mathbf{V}(t,)\psi \rangle$ is in $C([0, \infty))$ endowed with the usual topology of uniform convergence over compact intervals. We remark that $L^p_{\text{loc}}([0, \infty,)\mathbb{X})$ and $C([0, \infty); \text{w-}\mathbb{X})$ are Fréchet spaces rather than Banach spaces. As such, their topologies are completely determined by the class of convergence sequences.

Definition 2: **A weak solution** to the averaged system (2.4.0.87) is a vector valued function $\mathbf{U}(t,x)$ that belongs to $C([0,\infty);\text{w-}\mathbb{H}) \cap L^2([0,\infty);\mathbb{V})$, and satisfies the averaged system in the following weak sense:

$$\begin{aligned} &\langle \mathbf{U}(t_2,)\chi\rangle - \langle \mathbf{U}(t_1,)\chi\rangle - \int_{t_1}^{t_2} \langle \mathcal{B}_1(\mathbf{U},\mathbf{U},)\nabla_x\chi\rangle \, dt \\ &- \int_{t_1}^{t_2} \langle \mathcal{B}_2(\mathbf{U},\mathbf{U},)\mathcal{A}\chi\rangle \, dt + \int_{t_1}^{t_2} \langle \partial_\beta \overline{D}^{\alpha\beta}(\mathbf{U},)\partial_\alpha\chi\rangle \, dt = 0 \,, \end{aligned} \tag{2.4.1.3}$$

for every $[t_1,t_2] \subset [0,\infty)$, and every $\chi \in \mathbb{V} \cap C^1(\mathbb{T}^D)$.

The main theorem of this section is

Theorem 4: **(Leray-Type Global Weak Solutions)** *If the diffusion operator $\overline{\mathcal{D}}$ in the averaged system* (2.4.0.87) *is strictly dissipative, i.e., there exists a $\delta > 0$, such that*

$$-\langle \overline{\mathcal{D}}\mathbf{U} \,, \mathbf{U}\rangle \geq \delta \langle \mathbf{U} \,, \mathbf{U}\rangle \,. \tag{2.4.1.4}$$

Then, for any given initial data $\mathbf{U}^{in} \in \mathbb{H}$, there exists at least one $\mathbf{U} \in C([0,\infty);\text{w-}\mathbb{H}) \cap L^2([0,\infty);\mathbb{V})$ that is a weak solution to the averaged system (2.4.0.87)*. Moreover, for every $t > 0$, $\mathbf{U}$ satisfies the dissipation inequality:*

$$\tfrac{1}{2}\|\mathbf{U}(t)\|^2 + \int_0^t \langle \partial_\alpha \overline{D}^{\alpha\beta}(\mathbf{U},)\partial_\beta \mathbf{U}\rangle \, dt \leq \tfrac{1}{2}\|\mathbf{U}^{in}\|^2 \,. \tag{2.4.1.5}$$

From Proposition 2, we immediately have the following corollary

Corollary 4: *Given an absolute equilibrium $\mathbf{U}_*$, if the hyperbolic-parabolic system* (2.2.0.22) *satisfies the structure condition*

$$-\mathcal{A}\mathcal{D}\mathcal{A}|_{\mathrm{Null}(\mathcal{D})} > 0 \,, \tag{2.4.1.6}$$

Then, for any given initial data $\mathbf{U}^{in} \in \mathbb{H}$*, there exists at least one* $\mathbf{U} \in C([0,\infty); \text{w-}\mathbb{H}) \cap L^2([0,\infty); \mathbb{V})$ *that is a weak solution to the averaged system* (2.4.0.87)*. Moreover, for every* $t > 0$*,* $\mathbf{U}$ *satisfies the dissipation inequality* (2.4.1.5)*.*

2.4.2 Proof of the Existence of Global Weak Solutions

The strategy of the proof follows that introduced by Leray in the context of the Navier-Stokes equations, as well as to that of many other existence proofs for the weak solutions of other equations. Roughly, the idea is to construct a sequence of solutions to equations that approximate the averaged system , then show that the sequence is relatively compact in a topology that is strong enough to allow us pass from the approximate equation to the limit for any converging subsequence. This involves striking a balance between the facts that compactness is easier to establish for weaker topologies, while convergence is easier to prove in stronger topology. Uniqueness can never be asserted by such a compactness argument, but often requires the knowledge of additional regularity of the solution. In Chapter 3, when we apply our theory for the general hyperbolic-parabolic system with entropy to Navier-Stokes system of compressible gas dynamics, the averaged system depends by a solution to the Navier-Stokes equation which does not has good regularity to guarantee uniqueness. So its solutions are not unique. Then, for general hyperbolic-parabolic system, Leray-type solutions are the best we can expect.

Proof of the Theorem: We use the classical Galerkin approximation method. The proof proceeds in four distinct steps.

Step 1. Construct a family of approximation solutions $\mathbf{U}^{(n)}$ by any method that yields a consistent weak formulations and an energy relation. We use here the Fourier-Galerkin method. Let $\mathrm{P}_n : \mathbb{H} \to \mathbb{H}$ denote the L^2- orthogonal projection onto the span of slow Fourier modes of wave number $\mathbf{k}$ with $|\mathbf{k}| \leq n$:

$$\mathbf{P}_n \mathbf{U} = \sum_{\substack{\mathbf{k} \\ |\mathbf{k}| \leq n}} \sum_{a=1}^{N} \mathbf{U}^a_{\mathbf{k}} \phi^a_{\mathbf{k}} \,, \tag{2.4.2.1}$$

where $\mathbf{U}^a_{\mathbf{k}} = \mathbf{U}^a(\mathbf{k})$ and $\phi^a_{\mathbf{k}} = \phi^a(\mathbf{k})$.

The Galerkin system of order n is the system

$$\begin{aligned} \partial_t \mathbf{U}^{(n)} + \mathrm{P}_n \nabla_x \cdot \mathcal{B}_1(\mathbf{U}^{(n)}, \mathbf{U}^{(n)}) + \mathrm{P}_n \mathcal{A} \mathcal{B}_2(\mathbf{U}^{(n)}, \mathbf{U}^{(n)}) &= \overline{\mathcal{D}} \mathbf{U}^{(n)} \,, \\ \mathbf{U}^{(n)}(0, x) = \mathrm{P}_n \mathbf{U}^{in}(x) \,. \end{aligned} \tag{2.4.2.2}$$

Let $\mathbf{U}^{(n)} = \sum_{\mathbf{k}} \sum_{a=1}^{N} \mathbf{U}^{(n,)a}_{\mathbf{k}}(t) \phi^a_{\mathbf{k}}$ with $|\mathbf{k}| \leq n$. The above Galerkin system, after taking inner product with $\phi^a_{\mathbf{k}}$, for $1 \leq a \leq N$, can be reduced to

$$\begin{aligned} &\frac{d}{dt} \mathbf{U}^{(n,)a}(t) + \mathbf{k}^\alpha \mathbf{k}^\beta \sum_{\substack{a,b=1 \\ \lambda^a(\mathbf{k}) = \lambda^b(\mathbf{k})}}^{N} C^{\alpha\beta}_{ab} \mathbf{U}^{(n,)a}(t) \\ &+ \tfrac{1}{2} i \mathbf{k}^\alpha \sum_{\substack{\mathbf{m}+\mathbf{n}=\mathbf{k} \\ \lambda^b(\mathbf{m}) + \lambda^c(\mathbf{n}) = \lambda^a(\mathbf{k})}} I^{abc,\alpha}_{\mathbf{kmn}} \mathbf{U}^{(n,)b}(\mathbf{m}) \mathbf{U}^{(n,)c}(\mathbf{n}) \\ &+ \tfrac{1}{2} i \lambda^\alpha(\mathbf{k}) \sum_{\substack{\mathbf{m}+\mathbf{n}=\mathbf{k} \\ \lambda^b(\mathbf{m}) + \lambda^c(\mathbf{n}) = \lambda^a(\mathbf{k})}} J^{abc}_{\mathbf{kmn}} \mathbf{U}^{(n,)b}(\mathbf{m}) \mathbf{U}^{(n,)c}(\mathbf{n}) = 0 \,, \end{aligned} \tag{2.4.2.3}$$

with the initial data $\mathbf{U}^{in,a}_{\mathbf{k}} = \langle \mathbf{U}^{in}, \phi^a_{\mathbf{k}} \rangle$, where $\mathbf{k} \in \mathbb{T}^D$, and $|\mathbf{k}| \leq n$. Thus the Galerkin system of order n (2.4.2.3) is a constant coefficient ODE system. The nonlinearities are quadratic polynomials, hence locally Lipschitz. Then guaranteed by the Picard-Linderof existence theorem, there exists a $T^* > 0$, such that the Galerkin system has a solution on $[0, T^*)$.

It can be shown that $\mathbf{U}^{(n)}$ satisfy the weak form of the regularized system

(2.4.2.2):

$$\begin{aligned}&\langle \mathbf{U}^{(n)}(t_2,)\chi\rangle - \langle \mathbf{U}^{(n)}(t_1,)\chi\rangle - \int_{t_1}^{t_2} \langle \mathbf{B}_1(\mathbf{U}^{(n)},\mathbf{U}^{(n)},)\nabla_x\chi\rangle\, dt \\ &- \int_{t_1}^{t_2} \langle \mathbf{B}_2(\mathbf{U}^{(n)},\mathbf{U}^{(n)},)\mathcal{A}\chi\rangle\, dt + \int_{t_1}^{t_2} \langle \partial_\alpha \overline{D}^{\alpha\beta}(\mathbf{U}^{(n)},)\partial_\beta\chi\rangle\, dt = 0\,,\end{aligned} \tag{2.4.2.4}$$

for all $\chi \in \mathbb{H} \cap C^1(\mathbb{T}^D)$. Taking inner product $\mathbf{U}^{(n)}$ with the Galerkin system, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\mathbf{U}^{(n)}\|_{\mathbb{H}}^2 + \langle \partial_\alpha \overline{D}^{\alpha\beta}(\mathbf{U}^{(n)},)\partial_\beta \mathbf{U}^{(n)}\rangle + \langle \mathrm{P}_n \overline{\mathcal{Q}}_1(\mathbf{U}^{(n)},\mathbf{U}^{(n)})\rangle \\ &\qquad\qquad + \langle \mathrm{P}_n \overline{\mathcal{Q}}_2(\mathbf{U}^{(n)},\mathbf{U}^{(n)})\rangle = 0\,.\end{aligned} \tag{2.4.2.5}$$

To finish the step 1, we need to derive the energy identity for the approximate equation, which is an consequence of the key orthogonality property of the averaged quadratic terms $\overline{\mathcal{Q}}_1$ and $\overline{\mathcal{Q}}_2$, i.e., the part 3 of the Proposition (3). We state it again here:

$$\langle \overline{\mathcal{Q}}_1(\mathbf{U},\mathbf{U})\,,\mathbf{U}\rangle = 0\,, \quad \text{and} \quad \langle \overline{\mathcal{Q}}_2(\mathbf{U},\mathbf{U})\,,\mathbf{U}\rangle = 0\,. \tag{2.4.2.6}$$

Applying the above identities to (2.4.2.5), we immediately derive the energy identity for the solutions to the Galerkin system (2.4.2.2):

$$\tfrac{1}{2}\|\mathbf{U}^{(n)}(t_2)\|_{\mathbb{H}}^2 + \int_{t_1}^{t_2} \langle \partial_\alpha \overline{D}^{\alpha\beta}(\mathbf{U}^{(n)},)\partial_\beta \mathbf{U}^{(n)}\rangle\, dt = \tfrac{1}{2}\|\mathbf{U}^{(n)}(t_1)\|_{\mathbb{H}}^2\,, \tag{2.4.2.7}$$

for every $[t_1, t_2] \subset [0,\infty)$.

The above energy identity immediately yields a global L^2-bound on the solutions to the approximated equations.

Step 2. Show that the sequence $\mathbf{U}^{(n)}$ is a relatively compact set (has a compact closure) in

$$\mathrm{C}([0\,,\infty)\,,\text{w-}\mathbb{H}) \wedge \text{w-}L^2_{\mathrm{loc}}([0\,,\infty)\,,\text{w-}\mathbb{V}) \tag{2.4.2.8}$$

Proof of Step 2. The energy identity for $\mathbf{U}^{(n)}$, along with $\|\mathbf{U}^{(n)}\|_{\mathbb{H}}^2 \leq \|\mathbf{U}^{(n)}\|_{\mathbb{H}}^2$, implies that

1. The sequence $\mathbf{U}^{(n)}(t)$ is uniformly bounded in $\mathbb{H}$, for any $t \in [0, \infty)$, i.e.,

$$\sup_{0\leq t\leq T} \sum_{|\mathbf{k}|\leq n} |\mathbf{U}^{(n)}(t)|^2 \leq C\,, \quad \text{uniformly in} \quad n\,, \tag{2.4.2.9}$$

2. $\int_0^T \langle \partial_\alpha \overline{D}^{\alpha\beta}(\mathbf{U}^{(n)},)\partial_\beta \mathbf{U}^{(n)}\rangle \, dt \leq C$, uniformly in n.

for any $0 < T < \infty$. The uniform bound 1 above implies that

- $\mathbf{U}^{(n)}$ is relatively compact in w-$\mathbb{H}$ for every $t > 0$.

Recall that the key structure assumption $-\mathcal{A}\mathcal{D}\mathcal{A}|_{\mathrm{Null}(\mathcal{D})} > 0$, Proposition (2) implies that there exists a $\delta > 0$, such that

$$\delta \int_0^T \|\nabla_x \mathbf{U}^{(n)}(t)\|_{\mathbb{H}}^2 \, dt \leq \int_0^T \langle \partial_\alpha \overline{D}^{\alpha\beta}(\mathbf{U}^{(n)},)\partial_\beta \mathbf{U}^{(n)}\rangle \, dt \leq C\,, \quad \text{uniformly in} \quad n\,. \tag{2.4.2.10}$$

This implies that

$$\mathbf{U}^{(n)} \quad \text{is bounded in} \quad L^2(0, T; \mathbb{V}) \quad \text{uniformly in} \quad n\,, \tag{2.4.2.11}$$

for any $0 < T < \infty$. This uniform bound implies immediately that

- $\mathbf{U}^{(n)}$ is relatively compact in w-$L^2_{\mathrm{loc}}([0, \infty,)\text{w-}\mathbb{V})$.

In order to complete Step 2, it must be shown that $\mathbf{U}^{(n)}$ is a relatively compact set in $C([0, \infty); \text{w-}\mathbb{H})$. This compactness requires more than just boundedness because of the strong topology over t. Because $\mathbb{H}$ is $L^2(\mathbb{T}^D)$, we appeal to the Arzelà-Ascoli theorem which asserts that $\mathbf{U}^{(n)}$ is a relatively compact set in $C([0, \infty); \text{w-}\mathbb{H})$ if and only if

A1 $\mathbf{U}^{(n)}$ is a relatively compact set in w-$\mathbb{H}$ for every $t \geq 0$;

A2 $\mathbf{U}^{(n)}$ is equicontinuous in $\mathrm{C}([0\,,\infty)\,,\text{w-}\mathbb{H})$.

The condition **A1** has also been proved from the energy identity. In order to establish **A2**, we must show that for every $\chi \in \mathbb{H}$ we have

A2' $\langle \mathbf{U}^{(n)}(t)\,,\chi\rangle$ is equi-continuous in $\mathrm{C}([0\,,\infty))$.

This is done by first establishing **A2′** for χ in C^∞ and then using a density argument to extend to the general case of $\chi \in \mathbb{H}$.

Proof of A2': From the weak form of the regularized system (2.4.2.4):

$$\begin{aligned}&\langle \mathbf{U}^{(n)}(t_2,)\chi\rangle - \langle \mathbf{U}^{(n)}(t_1,)\chi\rangle \\ &= \int_{t_1}^{t_2} \langle \mathbf{B}_1(\mathbf{U}^{(n)}, \mathbf{U}^{(n)},)\nabla_x\chi\rangle\, dt + \int_{t_1}^{t_2} \langle \mathbf{B}_2(\mathbf{U}^{(n)}, \mathbf{U}^{(n)},)\mathcal{A}\chi\rangle\, dt \\ &- \int_{t_1}^{t_2} \langle \partial_\alpha \overline{D}^{\alpha\beta}(\mathbf{U}^{(n)},)\partial_\beta\chi\rangle\, dt\,,\end{aligned} \tag{2.4.2.12}$$

for all $\chi \in \mathbb{H} \cap C^1(\mathbb{T}^D)$.

We need to estimate the three terms on the right-hand side. Notice that

$$\begin{aligned}\mathbf{B}_1(\mathbf{U}^{(n)}, \mathbf{U}^{(n)}) &= \tfrac{1}{2}\sum_{a=1}^{N}\sum_{b,c=1}^{N} \sum_{\substack{\mathbf{m}+\mathbf{n}=\mathbf{k} \\ \lambda^b(\mathbf{m})+\lambda^c(\mathbf{n})=\lambda^a(\mathbf{k})}} I^{abc}_{\mathbf{kmn}} \mathbf{U}^{(n,)b}_{\mathbf{m}} \mathbf{U}^{(n,)c}_{\mathbf{n}} \phi^a(\mathbf{k})\,, \\ \mathbf{B}_2(\mathbf{U}^{(n)}, \mathbf{U}^{(n)}) &= \tfrac{1}{2}\sum_{a=1}^{N}\sum_{b,c=1}^{N} \sum_{\substack{\mathbf{m}+\mathbf{n}=\mathbf{k} \\ \lambda^b(\mathbf{m})+\lambda^c(\mathbf{n})=\lambda^a(\mathbf{k})}} J^{abc}_{\mathbf{kmn}} \mathbf{U}^{(n,)b}_{\mathbf{m}} \mathbf{U}^{(n,)c}_{\mathbf{n}} \phi^a(\mathbf{k})\,,\end{aligned} \tag{2.4.2.13}$$

where the wave number $\mathbf{k}$ is taken summation over the set $|\mathbf{k}| \leq n$. From the definitions of the interaction wave number $I^{abc}_{\mathbf{kmn}}$ and $J^{abc}_{\mathbf{kmn}}$, it is easy to see both are uniformly bounded, i.e.,

$$|I^{abc}_{\mathbf{kmn}}| \leq C\,, \quad |J^{abc}_{\mathbf{kmn}}| \leq C\,. \tag{2.4.2.14}$$

Then

$$|\langle \mathbf{B}_1(\mathbf{U}^{(n)},\mathbf{U}^{(n)},)\nabla_x\chi\rangle| \leq C[\sum_{|\mathbf{m}|\leq n}(\mathbf{U}_{\mathbf{m}}^{(n,)b})^2]^{\frac{1}{2}}[\sum_{|\mathbf{n}|\leq n}(\mathbf{U}_{\mathbf{m}}^{(n,)c})^2]^{\frac{1}{2}} \\ \leq C\,. \tag{2.4.2.15}$$

similarly

$$|\langle \mathbf{B}_2(\mathbf{U}^{(n)},\mathbf{U}^{(n)},)\mathcal{A}\chi\rangle| \leq C[\sum_{|\mathbf{m}|\leq n}(\mathbf{U}_{\mathbf{m}}^{(n,)b})^2]^{\frac{1}{2}}[\sum_{|\mathbf{n}|\leq n}(\mathbf{U}_{\mathbf{m}}^{(n,)c})^2]^{\frac{1}{2}} \\ \leq C\,. \tag{2.4.2.16}$$

Thus we have the estimates

$$\left|\int_{t_1}^{t_2}\langle \mathbf{B}_1(\mathbf{U}^{(n)},\mathbf{U}^{(n)},)\nabla_x\chi\rangle\,dt\right| \leq C|t_2-t_1|\,, \\ \left|\int_{t_1}^{t_2}\langle \mathbf{B}_2(\mathbf{U}^{(n)},\mathbf{U}^{(n)},)\mathcal{A}\chi\rangle\,dt\right| \leq C|t_2-t_1|\,. \tag{2.4.2.17}$$

Under the structure condition $-\mathcal{A}\mathcal{D}\mathcal{A}_{\mathrm{Null}(\mathcal{D})} > 0$, the third term on the right-hand side

$$\left|\int_{t_1}^{t_2}\langle \partial_\alpha\overline{D}^{\alpha\beta}(\mathbf{U}^{(n)},)\partial_\beta\chi\rangle\,dt\right| \leq C\int_{t_1}^{t_2}\|\partial_\alpha\overline{D}^{\alpha\beta}(\mathbf{U}^{(n)})\|_{\mathbb{H}}\|\partial_\beta\chi\|_2\,dt \\ \leq C|t_2-t_1|^{\frac{1}{2}}\int_{t_1}^{t_2}\|\nabla_x\mathbf{U}^{(n)}\|_{\mathbb{H}}^2\,dt \\ \leq C|t_2-t_1|^{\frac{1}{2}}\int_0^T\|\mathbf{U}^{(n)}\|_{\mathbb{V}}^2\,dt\,. \tag{2.4.2.18}$$

(2.4.2.17) and (2.4.2.18) yield the equi-continuity of $\langle \mathbf{U}^{(n)}(t)\,,\chi\rangle$:

$$|\langle \mathbf{U}^{(n)}(t_2)\,,\chi\rangle - \langle \mathbf{U}^{(n)}(t_1)\,,\chi\rangle| \leq C|t_2-t_1| + C|t_2-t_1|^{\frac{1}{2}}\,. \tag{2.4.2.19}$$

Hence the equi-continuity for every $\chi \in C^\infty(\mathbb{T}^D)$. To extend the class of test functions from C^∞ to $\mathbb{H}$ we use a standard density argument. Let $\eta > 0$ be arbitrary small number. Choose a $\chi_\eta \in \mathrm{C}^\infty$ with

$$\|\chi-\chi_\eta\|_2 < \frac{\eta}{3}\frac{1}{\|\mathbf{U}^{in}\|_{\mathbb{H}}}\,. \tag{2.4.2.20}$$

By the triangle inequality

$$\begin{aligned}
&|\langle \mathbf{U}^{(n)}(t_2,)\chi\rangle - \langle \mathbf{U}^{(n)}(t_1,)\chi\rangle| \\
&= |\langle \mathbf{U}^{(n)}(t_2) - \mathbf{U}^{(n)}(t_1)\,,\chi_\eta\rangle + \langle \mathbf{U}^{(n)}(t_2) - \mathbf{U}^{(n)}(t_1)\,,\chi - \chi_\eta\rangle| \\
&\le \|\chi - \chi_\eta\|_2 \|\mathbf{U}^{(n)}(t_2)\|_{\mathbb{H}} + \|\chi - \chi_\eta\|_2 \|\mathbf{U}^{(n)}(t_1)\|_{\mathbb{H}} \\
&+ |\langle \mathbf{U}^{(n)}(t_2) - \mathbf{U}^{(n)}(t_1)\,,\chi_\eta\rangle|
\end{aligned} \tag{2.4.2.21}$$

Recalling the uniform bound in **i**, we have

$$|\langle \mathbf{U}^{(n)}(t_2,)\chi\rangle - \langle \mathbf{U}^{(n)}(t_1,)\chi\rangle| \le \frac{\eta}{3} + \frac{\eta}{3} + |\langle \mathbf{U}^{(n)}(t_2) - \mathbf{U}^{(n)}(t_1)\,,\chi_\eta\rangle| \tag{2.4.2.22}$$

Applying (2.4.2.19) to the C^∞ function χ_η, we may choose $|t_2 - t_1|$ small enough so that the last term in (2.4.2.22) is less than $\frac{\eta}{3}$. This establish the $\mathbf{A2}'$ and completes the proof of step 2.

Step 3. Show that the sequence $\mathbf{U}^{(n)}$ is a relatively compact set $L^2_{\text{loc}}([0\,,\infty)\,,\mathbb{H})$ considered with its usual strong topology.

Proof of step 3. The crucial point is to use the results of step 2 along with the following imbedding lemma.

Lemma 3: *The injection*

$$C([0\,,\infty)\,,\text{w-}\mathbb{H}) \wedge \text{w-}L^2_{\text{loc}}([0\,,\infty)\,,\text{w-}\mathbb{V}) \hookrightarrow L^2_{\text{loc}}([0\,,\infty)\,,\mathbb{H}) \tag{2.4.2.23}$$

is continuous.

Proof of the Lemma: see [21] page 293, Lemma.

Step 2 states that $\mathbf{U}^{(n)}$ is a relatively compact set in both $C([0\,,\infty)\,,\text{w-}\mathbb{H})$ and $\text{w-}L^2_{\text{loc}}([0\,,\infty)\,,\text{w-}\mathbb{V})$, and because the continuous image of a compact set is compact, it follows that $\mathbf{U}^{(n)}$ is a relatively compact set in $L^2_{\rm loc}([0\,,\infty)\,,\mathbb{H})$. Hence, any subsequence of $\mathbf{U}^{(n)}$ that converges in both $\mathrm{C}([0\,,\infty)\,,\text{w-}\mathbb{H})$ and w-$L^2_{\rm loc}([0\,,\infty)\,,\text{w-}\mathbb{V})$ will be strongly convergent in $L^2_{\rm loc}([0\,,\infty)\,,\mathbb{H})$. This completes the proof of Step 3. □

Step 4. Passage to the limit. That is, the weak solution $\mathbf{U}$ in the Main Theorem is defined as the limit of a convergent subsequence of $\mathbf{U}^{(n)}$. The fact that this subsequence converges in the various function spaces is used to verify the weak form of (2.4.0.87) and the energy relation (2.4.1.5).

Proof of Step 4. Step 2 ensures that there is a subsequence of $\mathbf{U}^{(n)}$, which we also refer to as $\mathbf{U}^{(n)}$, that simultaneously converges to a limit $\mathbf{U}$ in $\mathrm{C}([0\,,\infty)\,,\text{w-}\mathbb{H})$ and w-$L^2_{\rm loc}([0\,,\infty)\,,\text{w-}\mathbb{V})$. Thus, $\mathbf{U} \in \mathrm{C}([0\,,\infty)\,,\text{w-}\mathbb{H}) \cap L^2_{\rm loc}([0\,,\infty)\,,\mathbb{V})$. Step 3 ensures the strong convergence of $\mathbf{U}^{(n)}$ to $\mathbf{U}$ in $L^2_{\rm loc}([0\,,\infty)\,,\mathbb{H})$. All that remains is to show that the limit $\mathbf{U}$ satisfies the weak form of the averaged system (2.4.1.3) as well as the energy inequality (2.4.1.5) . Toward this end we check convergence of each term in the respective regularized versions, (2.4.2.4) and (2.4.2.7) .

For any $\chi \in \mathbb{H} \cap C^1(\mathbb{T}^D)$, the convergence of $\mathbf{U}^{(n)}$ in $\mathrm{C}([0\,,\infty)\,,\text{w-}\mathbb{H})$ to $\mathbf{U}$ yields

$$\langle \mathbf{U}^{(n)}(t,)\chi\rangle \to \langle \mathbf{U}(t,)\chi\rangle\,, \tag{2.4.2.24}$$

for every $t \geq 0$. The convergence of $\mathbf{U}^{(n)}$ in $L^2_{\rm loc}([0\,,\infty)\,,\mathbb{V})$ implies

$$\tilde{\mu}\int_{t_1}^{t_2} \langle \partial_\alpha \overline{D}^{\alpha\beta}(\mathbf{U}^{(n)},)\partial_\beta\chi\rangle\, dt \to \tilde{\mu}\int_{t_1}^{t_2} \langle \partial_\alpha \overline{D}^{\alpha\beta}(\mathbf{U},)\partial_\beta\chi\rangle\, dt\,. \tag{2.4.2.25}$$

The averaged convection terms $\overline{\mathcal{Q}}_1(\mathbf{U},\mathbf{U})$ and $\overline{\mathcal{Q}}_1(\mathbf{U},\mathbf{U})$ are more involved. They are quadratic in $\mathbf{U}$. The strong convergence $\mathbf{U}^{(n)}$ in $L^2_{\rm loc}([0\,,\infty)\,,\mathbb{H})$ leads to

$$\overline{\mathcal{Q}}_1(\mathbf{U}^{(n)},\mathbf{U}^{(n)},)\nabla_x\chi\rangle\, dt \to \overline{\mathcal{Q}}_1(\mathbf{U},\mathbf{U},)\nabla_x\chi\rangle\, dt\,. \tag{2.4.2.26}$$

It is the same for $\overline{\mathcal{Q}}_2$.

Thus the limit $\mathbf{U}$ satisfies the weak form of the averaged system (2.4.1.3). Recalling that $\mathbf{U}^{(n)} \in \mathbb{H}$, it is easy to see its limit is also in $\mathbb{H}$,.

Now, to recover the energy relation (2.4.1.5) start from its regularized version

$$\tfrac{1}{2}\|\mathbf{U}^{(n)}(t)\|_{\mathbb{H}}^2 + \int_0^T \langle \partial_\alpha \overline{D}^{\alpha\beta}(\mathbf{U}^{(n)})\,,\partial_\beta \mathbf{U}^{(n)}\rangle\, ds = \tfrac{1}{2}\|\mathrm{P}_n \mathbf{U}^{in}\|_{\mathbb{H}}^2\,. \tag{2.4.2.27}$$

First examining the right side of above identity, the strong convergence of the initial data in $\mathbb{H}$ implies

$$\|\mathrm{P}_n \mathbf{U}^{in}\|_{\mathbb{H}}^2 \to \|\mathbf{U}^{in}\|_{\mathbb{H}}^2\,, \tag{2.4.2.28}$$

while the strong convergence of $\mathbf{U}^{(n)}$ in $\mathrm{C}([0\,,\infty)\,,\text{w-}\mathbb{H})$, together with the fact that the norm of the weak limit is an eventual lower bound to the norms of the sequence, yields

$$\tfrac{1}{2}\|\mathbf{U}(t)\|_{\mathbb{H}}^2 \le \tfrac{1}{2}\liminf_{n\to\infty} \|\mathbf{U}^{(n)}(t)\|_{\mathbb{H}}^2\,. \tag{2.4.2.29}$$

Similarly, the convergence of $\mathbf{U}^{(n)}$ in $\text{w-}L^2_{\text{loc}}([0\,,\infty)\,,\text{w-}\mathbb{V})$ implies

$$\int_0^T \langle \partial_\alpha \overline{D}^{\alpha\beta}(\mathbf{U})\,,\partial_\beta \mathbf{U}\rangle\, ds \le \liminf_{n\to\infty} \int_0^T \langle \partial_\alpha \overline{D}^{\alpha\beta}(\mathbf{U}^{(n)})\,,\partial_\beta \mathbf{U}^{(n)}\rangle\, ds\,. \tag{2.4.2.30}$$

Hence, the energy relation is satisfied and the main theorem is proved. □

3. WEAKLY COMPRESSIBLE NAVIER-STOKES SYSTEM

In this chapter, we study the weakly nonlinear approximation of the compressible Navier-Stokes system about an absolute equilibrium. Because the compressible Navier-Stokes system is a special case of the hyperbolic-parabolic system with entropy, we can apply the general theory developed in Chapter 2 to this specific system.

For the compressible Navier-Stokes system, the first-order operator $\mathcal{A}$ is the acoustic operator. We precisely describe the null space and its orthogonal complement space of $\mathcal{A}$ by explicitly formulating the eigenspace of $\mathcal{A}$. The incompressible mode $\mathrm{Null}(\mathcal{A})$ includes the incompressibility and Boussinesq relations, while the acoustic mode $\mathrm{Null}(\mathcal{A})^{\perp}$ describes the propagation of fast acoustic waves.

In this chapter, we explicitly derive the averaged system of the compressible Navier-Stokes system, and describe its behavior in the incompressible and acoustic modes respectively. We show that the projection of on $\mathrm{Null}(\mathcal{A})$ is the corresponding incompressibility model, while the projection on $\mathrm{Null}(\mathcal{A})^{\perp}$ describes how fast oscillations propagate. The averaged system of compressible Navier-Stokes system satisfies the structure condition given in Chapter 2, so the existence of global weak solutions to the averaged system is a consequence of the general existence theorem proven in Chapter 2. Furthermore, we use a Littlewood-Paley decomposition, we show that in the time interval of the existence of the regular solution of the incompressible

Navier-Stokes equations, the averaged system in the acoustic mode $\text{Null}(\mathcal{A})^\perp$ has higher regularity.

In section 1 of this chapter, we introduce the general Navier-Stokes system for compressible gas dynamics. In section 2, we analyze in details the acoustic operator $\mathcal{A}$. In section 3 and 4, we derive the projection of the averaged system onto the slow mode $\text{Null}(\mathcal{A})$ and fast mode $\text{Null}(\mathcal{A})^\perp$ respectively. In section 5, we prove the global existence of Leray-type solutions to the orthogonal averaged system. All results of this section are application of the general theory of Chapter 2, so could be considered as a concrete example of the last chapter. In the final section, we use Littlewood-Paley decomposition to prove higher regularity.

3.1 Navier-Stokes System for Compressible Gas Dynamics

The Navier-Stokes equations for compressible gas dynamics can be written in the conservation laws:

$$
\begin{aligned}
&\partial_t \rho + \nabla_x \cdot (\rho u) = 0\,, \\
&\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x p = -\nabla_x \cdot \Sigma\,, \qquad (3.1.0.1)\\
&\partial_t \left[\rho(\tfrac{|u|^2}{2} + e)\right] + \nabla_x \cdot \left[(\rho(\tfrac{|u|^2}{2} + e) + p)u\right] = -\nabla_x \cdot (\Sigma \cdot u + q)\,,
\end{aligned}
$$

where $\rho = \rho(t,x)$ denotes the density, $u = u(t,x)$, the velocity field and $e = e(t,x)$, the internal energy per unit mass (see for example the introduction of P.-L.Lions' book [54].) We are concerned with the evolution of (3.1.0.1) for positive time t. We restrict ourselves to the case of a Newtonian gas so that the viscous stress tensor reduces to

$$\begin{aligned}\Sigma(u) &= -\mu[\nabla_x u + (\nabla_x u)^T] - \lambda \nabla_x \cdot u I \\ &= -\mu[\nabla_x u + (\nabla_x u)^T - \tfrac{2}{D}\nabla_x \cdot u I] - (\lambda + \tfrac{2}{D}\mu)\nabla_x \cdot u I \,, \\ &= -\mu\sigma(u) - (\lambda + \tfrac{2}{D}\mu)\nabla_x \cdot u I \,,\end{aligned} \tag{3.1.0.2}$$

where $\sigma(u) = \nabla_x u + (\nabla_x u)^T - \frac{2}{D}\nabla_x \cdot uI$, and λ and μ are the so-called Lamé coefficients. In general λ and μ are functions of ρ and θ. For viscous gases, the strict dissipativity of the viscous term implies that $\Sigma(u) \cdot \nabla_x u \geq 0$, which equivalently means

$$\mu\sigma(u) : \sigma(u) + (\lambda + \tfrac{2}{D}\mu)(\nabla_x \cdot u)^2 \geq 0 \,. \tag{3.1.0.3}$$

Then $\mu > 0$ and $\lambda + \frac{2}{D}\mu \geq 0$. The heat flux vector q is given by $q = -\kappa \nabla_x \theta$ where κ in general is a positive scalar function of ρ and θ. For a general gas, the internal energy per unit mass e is a function of ρ and θ, while the pressure p is a function of e (thus θ) and ρ, i.e.,

$$p = p(\rho, \theta) \,, \quad e = e(\rho, \theta) \,. \tag{3.1.0.4}$$

As we mentioned in the introduction, our primary interests are the hydrodynamics from the Boltzmann equations, the kinetic theory of gases (monatomic gases) indicates that the Stokes relation should hold. namely

$$\lambda + \tfrac{2}{D}\mu = 0 \,, \tag{3.1.0.5}$$

where D is the spatial dimension. We also assume $e = \frac{1}{\gamma-1}\theta$, the pressure $p = (\gamma - 1)\rho e = \rho\theta$, for a monatomic gas, $c_{\mathbf{v}} = \frac{D}{2}$, $\gamma = \frac{D+2}{D}$. We want to remark that all the calculations in this chapter could be generalized to general gases without any serious modifications.

Now we define the conservative variables for the Navier-Stokes equations for (polytropic) ideal gas:

$$\begin{aligned}\mathbf{U} &= (\mathrm{U}_0, \mathrm{U}, \mathrm{U}_4) \\ &= (\rho, \rho u, \tfrac{1}{2}\rho|u|^2 + \tfrac{1}{\gamma-1}\rho\theta)\end{aligned} \tag{3.1.0.6}$$

or

$$\begin{aligned}&\rho = \mathrm{U}_0\,, \quad u = \tfrac{\mathrm{U}}{\mathrm{U}_0}\,, \quad \theta = (\gamma-1)(\tfrac{\mathrm{U}_4}{\mathrm{U}_0} - \tfrac{|\mathrm{U}|^2}{2\mathrm{U}_0^2})\,, \\ &P = (\gamma-1)(\mathrm{U}_4 - \tfrac{|\mathrm{U}|^2}{2\mathrm{U}_0})\,.\end{aligned} \tag{3.1.0.7}$$

For the viscous ideal gas, there is a natural physical entropy:

$$S := \rho \ln\left(\frac{\theta}{\rho^{\gamma-1}}\right) . \tag{3.1.0.8}$$

Expressed in terms of conservative variables, the entropy becomes

$$\Phi(\mathbf{U}) := -\mathrm{U}_0 \ln\left(\frac{(\gamma-1)(\mathrm{U}_4 - \frac{|\mathrm{U}|^2}{2\mathrm{U}_0})}{\mathrm{U}_0^\gamma}\right) . \tag{3.1.0.9}$$

As introduced in Chapter 2 (2.1.0.13), we define the entropy variable $\mathbf{V}$ as $\mathbf{V} = \nabla_{\mathbf{u}}\Phi(\mathbf{U})$:

$$\mathbf{V} := \frac{1}{\rho e}\begin{pmatrix} -\mathrm{U}_4 + \rho e(\gamma + 1 - s) \\ \mathrm{U}_1 \\ \mathrm{U}_2 \\ \mathrm{U}_3 \\ -\mathrm{U}_0 \end{pmatrix} . \tag{3.1.0.10}$$

where ρe and s in terms of $\mathbf{U}$ are

$$\rho e = \mathrm{U}_4 - \tfrac{|\mathrm{U}|^2}{2\mathrm{U}_0}\,, \quad s = \ln\left(\tfrac{(\gamma-1)\rho e}{\mathrm{U}_0^\gamma}\right) . \tag{3.1.0.11}$$

The inverse mapping $\mathbf{V} \mapsto \mathbf{U}$ is given by:

$$\mathbf{U} = \rho e \begin{pmatrix} -\mathrm{V}_4 \\ \mathrm{V}_1 \\ \mathrm{V}_2 \\ \mathrm{V}_3 \\ 1 - \frac{|\mathrm{V}|^2}{2\mathrm{V}_4} \end{pmatrix} . \tag{3.1.0.12}$$

where ρe and s in terms of $\mathbf{V}$ are

$$\rho e = \left(\frac{\gamma-1}{(-\mathrm{V}_4)^\gamma}\right)^{\frac{1}{\gamma-1}} \exp\left(\frac{-s}{\gamma-1}\right), \quad s = \gamma - \mathrm{V}_0 + \frac{|\mathrm{V}|^2}{2\mathrm{V}_4} . \tag{3.1.0.13}$$

By the definition (2.1.0.14), after tedious but straightforward calculations, the entropy potential $\Phi^*(\mathbf{V})$ is

$$\Phi^*(\mathbf{V}) = (\gamma-1)^{\frac{\gamma}{\gamma-1}} (-\mathrm{V}_4)^{\frac{1}{1-\gamma}} \exp\left(\frac{-\gamma + \mathrm{V}_0 - \frac{|\mathrm{V}|^2}{2\mathrm{V}_4}}{\gamma - 1}\right) . \tag{3.1.0.14}$$

The entropy-flux $\Psi(\mathbf{U}) = \Phi(\mathbf{U})\frac{\mathrm{U}}{\mathrm{U}_0}$, so the entropy-flux potential $\Psi^*(\mathbf{V})$ is defined as

$$\Psi^*(\mathbf{V}) = (\gamma-1)\mathrm{V} \left(\frac{\gamma-1}{(-\mathrm{V}_4)^\gamma}\right)^{\frac{1}{\gamma-1}} \exp\left(\frac{-\gamma + \mathrm{V}_0 - \frac{|\mathrm{V}|^2}{2\mathrm{V}_4}}{\gamma - 1}\right) . \tag{3.1.0.15}$$

After complicated algebraic calculations, we find the explicit expressions of the quadratic and the dissipation terms in the weakly nonlinear asymptotics of the compressible Navier-Stokes system. (see definitions in (2.2.0.38) and (2.2.0.39)) Denoting $\mathbf{W} = \mathbf{G} \cdot \mathbf{U}$, and $\mathbf{W} = (\mathrm{W}_0, \mathrm{W}, \mathrm{W}_4)^T$, we obtain

$$\mathcal{Q}_1(\mathbf{U}, \mathbf{U}) = \begin{pmatrix} \frac{1}{(\gamma-1)^2} \nabla_x \cdot [\mathrm{W}(\mathrm{W}_0 + \frac{\gamma}{\gamma-1}\mathrm{W}_4)] , \\ \frac{1}{(\gamma-1)^2} \nabla_x \cdot [\mathrm{W} \otimes \mathrm{W}] + \frac{1}{2(\gamma-1)^2} \nabla_x |\mathrm{W}|^2 \\ + \frac{\gamma}{2(\gamma-1)^3} \nabla_x \mathrm{W}_4^2 + \frac{1}{2(\gamma-1)^2} \nabla_x (\mathrm{W}_0 + \frac{\gamma}{\gamma-1}\mathrm{W}_4)^2 , \\ \frac{\gamma}{(\gamma-1)^3} \nabla_x \cdot [\mathrm{W}(\mathrm{W}_0 + \frac{2\gamma-1}{\gamma-1}\mathrm{W}_4)] \end{pmatrix} , \tag{3.1.0.16}$$

$$\mathcal{Q}_2(\mathbf{U},\mathbf{U}) = \begin{pmatrix} \frac{1}{(\gamma-1)^2}\nabla_x \cdot [\mathrm{W}(\mathrm{W}_0 + \frac{\gamma}{\gamma-1}\mathrm{W}_4)] \\ \frac{\gamma}{2(\gamma-1)^2}\nabla_x|\mathrm{W}|^2 + \frac{\gamma}{2(\gamma-1)^3}\nabla_x \mathrm{W}_4^2 + \frac{1}{2(\gamma-1)^2}\nabla_x(\mathrm{W}_0 + \frac{\gamma}{\gamma-1}\mathrm{W}_4)^2 \\ \frac{\gamma}{(\gamma-1)^3}\nabla_x \cdot [\mathrm{W}(\mathrm{W}_0 + \frac{\gamma}{\gamma-1}\mathrm{W}_4)] \end{pmatrix}. \tag{3.1.0.17}$$

Note that

$$\mathbf{W} = (\gamma \mathrm{U}_0 - (\gamma-1)\mathrm{U}_4\,, (\gamma-1)\mathrm{U}_1\,, (\gamma-1)\mathrm{U}_2\,, (\gamma-1)\mathrm{U}_3\,, -(\gamma-1)\mathrm{U}_0 + (\gamma-1)^2\mathrm{U}_4)^T\,, \tag{3.1.0.18}$$

Using $\mathcal{Q} = \mathcal{Q}_1 - \mathcal{Q}_2$, finally we have

$$\mathcal{Q}(\mathbf{U},\mathbf{U}) = \begin{pmatrix} 0 \\ \nabla_x \cdot (\mathrm{U} \otimes \mathrm{U}) - \frac{\gamma-1}{2}\nabla_x(|\mathrm{U}|^2) \\ \frac{\gamma}{\gamma-1}\nabla_x \cdot [\mathrm{U}((\gamma-1)\mathrm{U}_4 - \mathrm{U}_0)] \end{pmatrix}, \tag{3.1.0.19}$$

$$\mathcal{D}(\mathbf{U}) = \begin{pmatrix} 0 \\ \mu\Delta_x \mathrm{U} + (\mu+\lambda)\nabla_x(\nabla_x \cdot \mathrm{U}) \\ \kappa\Delta_x((\gamma-1)\mathrm{U}_4 - \mathrm{U}_0) \end{pmatrix}. \tag{3.1.0.20}$$

Remark: The expression (3.1.0.19) of $\mathcal{Q}$ are different with (1.1.0.35). This is because we expand conservative $(\mathrm{U}_0, \mathrm{U}, \mathrm{U}_4)$ variables here while expand (ρ, u, θ) in Chapter 1. Indeed, after taking averaging, they are coincide [42].

Then, the averaged system of the Navier-Stokes system of compressible gas dynamics is:

$$\partial_t \mathbf{U} + \lim_{T\to\infty}\frac{1}{T}\int_0^T e^{s\mathcal{A}}\mathcal{Q}(e^{-s\mathcal{A}}\mathbf{U}, e^{-s\mathcal{A}}\mathbf{U})\,ds = \lim_{T\to\infty}\frac{1}{T}\int_0^T e^{s\mathcal{A}}\mathcal{D}(e^{-s\mathcal{A}}\mathbf{U})\,ds\,. \tag{3.1.0.21}$$

in other words,

$$\partial_t \mathbf{U} + \overline{\mathcal{Q}}(\mathbf{U}, \mathbf{U}) = \overline{\mathcal{D}}\mathbf{U}\,, \tag{3.1.0.22}$$

where $\mathcal{Q}(\mathbf{U}\,, \mathbf{U})$ and $\mathcal{D}(\mathbf{U})$ are given by (3.1.0.19) and (3.1.0.20) respectively.

3.2 Skew-symmetric Operator $\mathcal{A}$

We denote by $\mathcal{H}$ the set

$$\mathcal{H} = \left\{ \mathbf{U} \in \mathcal{D}(\mathbb{T}^D)^5 : \int_{\mathbb{T}^D} \mathbf{U}\, dx = 0 \right\}, \tag{3.2.0.23}$$

where $\mathcal{D}(\mathbb{T}^D)$ denotes the test function space on $\mathbb{T}^D$, i.e., $\mathcal{C}^\infty(\mathbb{T}^D)$.

The entropy Hessian $\mathbf{G}$ at a constant state $\mathbf{U}^* = (1, 0, \frac{1}{\gamma-1})$ of the Navier-Stokes system is a positive definite symmetric matrix

$$\mathbf{G} = \begin{bmatrix} \gamma & 0 & 0 & 0 & -(\gamma-1) \\ 0 & \gamma-1 & 0 & 0 & 0 \\ 0 & 0 & \gamma-1 & 0 & 0 \\ 0 & 0 & 0 & \gamma-1 & 0 \\ -(\gamma-1) & 0 & 0 & 0 & (\gamma-1)^2 \end{bmatrix}, \tag{3.2.0.24}$$

For monatomic gases, $\gamma = \frac{D+2}{D}$.

Let $\mathbf{U}(x)$,$\mathbf{V}(x)$ be two measurable vector-valued functions

$$\mathbf{U} = (\mathrm{U}_0, \mathrm{U}, \mathrm{U}_4)^T\,, \quad \mathbf{V} = (\mathrm{V}_0, \mathrm{V}, \mathrm{V}_4)^T\,, \tag{3.2.0.25}$$

where $\mathrm{U} = (\mathrm{U}_1, \mathrm{U}_2, \mathrm{U}_3)^T$. $\mathbf{G}$ defines a natural inner product between $\mathbf{U}$ and $\mathbf{V}$:

$$\langle \mathbf{U}, \mathbf{V} \rangle = \int_{\mathbb{T}^D} (\mathbf{G} \cdot \mathbf{U}, \overline{\mathbf{V}})\, dx\,, \tag{3.2.0.26}$$

i.e.,

$$\langle \mathbf{U}, \mathbf{V} \rangle = \int_{\mathbb{T}^D} \left[\gamma \mathrm{U}_0 \overline{\mathrm{V}}_0 + (\gamma - 1)^2 \mathrm{U}_4 \overline{\mathrm{V}}_4 - (\gamma - 1) \mathrm{U}_0 \overline{\mathrm{V}}_4 - (\gamma - 1) \mathrm{U}_4 \overline{\mathrm{V}}_0 + (\gamma - 1) \mathrm{U} \cdot \overline{\mathrm{V}} \right] d\mathbf{x} \,. \tag{3.2.0.27}$$

Where $(\cdot\,,\cdot)$ is the canonical inner product on Eulidean space, $\overline{\mathrm{V}}$ denotes the complex conjugate of the vector V. By this definition,

$$\langle \mathbf{U}, \mathbf{U} \rangle = (\gamma - 1) \int_{\mathbb{T}^D} |\mathrm{U}_0|^2 \, dx + \int_{\mathbb{T}^D} |\mathrm{U}_0 - (\gamma - 1)\mathrm{U}_4|^2 \, dx + (\gamma - 1) \int_{\mathbb{T}^D} |\mathrm{U}|^2 \, dx \,, \tag{3.2.0.28}$$

then $\langle \mathbf{U}, \mathbf{U} \rangle \geq 0$, " $=$ " if and only $\mathbf{U} \equiv 0$.

Now, we can define a first-order linear differential operator on the space $\mathcal{H}$

$$\mathcal{A} : \mathcal{H} \longrightarrow \mathcal{H} \tag{3.2.0.29}$$

by

$$\mathcal{A}(\mathbf{U}) = \sum_{\alpha=1}^{3} \tfrac{\partial}{\partial x_\alpha} \left[(\Psi^{\star\alpha}_{\mathbf{vv}}(\Phi_{\mathbf{u}}(\mathbf{U}^*)) \cdot \Phi_{\mathbf{uu}}(\mathbf{U}^*) \cdot \mathbf{U} \right] , \tag{3.2.0.30}$$

where

$$\Phi_{\mathbf{uu}}(\mathbf{U}^*) = \mathbf{G} \,, \tag{3.2.0.31}$$

and the Hessian of the three entropy flux at $\mathbf{V}^* = \Phi_{\mathbf{u}}(\mathbf{U}^*)$ are

$$\overline{\Psi}^{*1}_{\mathbf{vv}} = \begin{bmatrix} 0 & \frac{1}{\gamma-1} & 0 & 0 & 0 \\ \frac{1}{\gamma-1} & 0 & 0 & 0 & \frac{\gamma}{(\gamma-1)^2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\gamma}{(\gamma-1)^2} & 0 & 0 & 0 \end{bmatrix}, \quad \overline{\Psi}^{*2}_{\mathbf{vv}} = \begin{bmatrix} 0 & 0 & \frac{1}{\gamma-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\gamma-1} & 0 & 0 & 0 & \frac{\gamma}{(\gamma-1)^2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\gamma}{(\gamma-1)^2} & 0 & 0 \end{bmatrix}, \tag{3.2.0.32}$$

$$\overline{\Psi}^{*3}_{\mathbf{vv}} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{\gamma-1} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\gamma-1} & 0 & 0 & 0 & \frac{\gamma}{(\gamma-1)^2} \\ 0 & 0 & 0 & \frac{\gamma}{(\gamma-1)^2} & 0 \end{bmatrix}. \qquad (3.2.0.33)$$

Where " $-$ " denotes evaluation at the constant state $\mathrm{U}^* = (1, 0, \frac{1}{\gamma-1})$. After a straightforward calculation, we obtain

$$\mathcal{A}\mathbf{U} = \begin{pmatrix} \nabla_x \cdot \mathrm{U} \\ (\gamma-1)\nabla_x \mathrm{U}_4 \\ \frac{\gamma}{(\gamma-1)}\nabla_x \cdot \mathrm{U} \end{pmatrix}. \qquad (3.2.0.34)$$

We know that the operator $\mathcal{A}$ is skew-symmetric, so that its eigenvalues are purely imaginary. The symbol of the operator $\mathcal{A}$ is:

$$\hat{\mathcal{A}}(\mathbf{k}) = \begin{bmatrix} 0 & i\mathbf{k}_1 & i\mathbf{k}_2 & i\mathbf{k}_3 & 0 \\ 0 & 0 & 0 & 0 & i(\gamma-1)\mathbf{k}_1 \\ 0 & 0 & 0 & 0 & i(\gamma-1)\mathbf{k}_2 \\ 0 & 0 & 0 & 0 & i(\gamma-1)\mathbf{k}_3 \\ 0 & i\frac{\gamma}{\gamma-1}\mathbf{k}_1 & i\frac{\gamma}{\gamma-1}\mathbf{k}_2 & i\frac{\gamma}{\gamma-1}\mathbf{k}_3 & 0 \end{bmatrix}. \qquad (3.2.0.35)$$

The 5 eigenvalues of $\mathcal{A}$ are $i\sqrt{\gamma}|\mathbf{k}|, -i\sqrt{\gamma}|\mathbf{k}|, 0, 0, 0$, so that its kernel in $\mathcal{H}$ is non-trivial. Null($\mathcal{A}$) and its orthogonal complement in $\mathcal{H}$ which is denoted by Null$(\mathcal{A})^{\perp}$ will play significantly different roles in weakly nonlinear asymptotics of the Navier-Stokes system. So we need understand their structures which are described in the following lemma.

Lemma 4: *The null space of the linear operator $\mathcal{A}$ and its orthogonal complement in $\mathcal{H}$ can be characterized as:*

1. $\mathrm{Null}(\mathcal{A}) = \Big\{\mathbf{U} \in \mathcal{H} : \mathbf{U} = (\mathrm{U}_0, \mathrm{U}, 0)^T, \nabla_x \cdot \mathrm{U} = 0\Big\}$;

2. $\mathrm{Null}(\mathcal{A})^\perp = \Big\{\mathbf{U} \in \mathcal{H} : \mathbf{U} = (\mathrm{U}_0, \mathrm{U}, \mathrm{U}_4)^T, \mathrm{U}_4 = \frac{\gamma}{\gamma-1}\mathrm{U}_0, \mathrm{U} = \nabla_x \varphi\Big\}$.

Proof of Lemma: The proof of (1) easily follows the definition of the operator $\mathcal{A}$. $\mathcal{A}\mathbf{U} = 0$ implies that U is divergence free and U_4 is a constant. Since every vector-valued function in $\mathcal{H}$ has mean value 0, this constant should be 0.

The proof of part (2) follows from (1) and the definition of the inner product associated with H. When $\mathcal{A}\mathbf{U} = 0$,

$$\langle \mathcal{A}\mathbf{U}, \mathbf{V} \rangle = \int_{\mathbb{T}^D} \left[\gamma \mathrm{U}_0(\overline{\mathrm{V}}_0 - \tfrac{\gamma-1}{\gamma}\overline{\mathrm{V}}_4) + \gamma - 1\mathrm{U} \cdot \overline{\mathrm{V}}\right] dx\,, \tag{3.2.0.36}$$

Thus the relations $\mathrm{V}_4 = \frac{\gamma}{\gamma-1}\mathrm{V}_0$ and $\mathrm{V} = \nabla_x \varphi$ follow. □

Definition 3: We define two subspaces of $\mathcal{H}$:

$$\mathcal{H}_1 = \mathrm{Null}(\mathcal{A})\,, \quad \mathcal{H}_2 = \mathrm{Null}^\perp(\mathcal{A})\,. \tag{3.2.0.37}$$

Since the operator $\mathcal{A}$ is skew-symmetric, we have the following decomposition of $\mathcal{H}$:

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2\,. \tag{3.2.0.38}$$

For any $\mathbf{U} \in \mathcal{H}$, $\mathbf{U}$ has the unique decomposition:

$$\mathbf{U} = \Pi\mathbf{U} + \Pi^\perp\mathbf{U}\,, \tag{3.2.0.39}$$

where Π and $\Pi^\perp$ are projections onto $\mathrm{Null}(\mathcal{A})$ and $\mathrm{Null}(\mathcal{A})^\perp$ respectively with

$$\Pi : \mathcal{H} \longrightarrow \mathcal{H}_1\,, \tag{3.2.0.40}$$

defined by

$$\Pi(\mathrm{U}_0, \mathrm{U}, \mathrm{U}_4)^T = (\mathrm{U}_0 - \tfrac{\gamma-1}{\gamma}\mathrm{U}_4, P\mathrm{U}, 0)^T\,, \tag{3.2.0.41}$$

and

$$\Pi^{\perp} : \mathcal{H} \longrightarrow \mathcal{H}_2\,, \tag{3.2.0.42}$$

defined by

$$\Pi^{\perp}(\mathrm{U}_0, \mathrm{U}, \mathrm{U}_4)^T = (\tfrac{\gamma-1}{\gamma}\mathrm{U}_4, Q\mathrm{U}, \mathrm{U}_4)^T\,, \tag{3.2.0.43}$$

where P is the usual Leray projection onto the space of divergence-free vector fields and Q is the projection onto the space of gradients defined by

$$P = I - Q, \qquad Q = \nabla\Delta^{-1}\nabla\cdot\,, \tag{3.2.0.44}$$

where Δ^{-1} denotes the inverse Laplace operator on $\mathbb{T}^D$:

$$f = \Delta^{-1}g, \quad \text{if} \quad \Delta f = g, \quad \text{and} \quad \int_{\mathbb{T}^D} f\,dx = 0\,. \tag{3.2.0.45}$$

To understand the averaged system (3.1.0.21), the first difficulty we encounter is that its structure is unclear. We need more explicit expressions of the equations. The averaged system is involved the exponential operators $e^{s\mathcal{A}}$ and $e^{-s\mathcal{A}}$, so a natural idea is to find the spectrum of $\mathcal{A}$, and to represent the equations in terms of eigenfunctions. The exponential operators $e^{s\mathcal{A}}$ does not have any effect on the eigenspace associated with the eigenvalue 0, so the concrete form of the eigenvectors corresponding to 0 is not important. We only care about the spectral space associated with the nontrivial eigenvalues. The next lemma provides the construction of the orthonormal basis of $\mathrm{Null}(\mathcal{A})^{\perp}$, and the orthogonality is with respect to the inner product $\langle\cdot,\cdot\rangle$.

Let $\{\lambda_{\mathbf{k}}^2\}_{\mathbf{k}\in\mathbb{Z}^3}$ ($\lambda_{\mathbf{k}} > 0$) be the nondecreasing sequences and $\{\varphi_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^3}$ the orthonormal basis of $L^2(\mathbb{T}^D)$ functions with zero mean value of eigenvectors of the Laplace operator $-\Delta$ on $\mathbb{T}^D$:

$$-\Delta\varphi_{\mathbf{k}} = \lambda_{\mathbf{k}}^2\varphi_{\mathbf{k}}, \quad \text{in} \quad \mathbb{T}^3, \quad \int_{\mathbb{T}^D} \varphi_{\mathbf{k}}\, dx = 0\,. \tag{3.2.0.46}$$

The Fourier transform of the equation is, $|\tilde{\mathbf{k}}|^2\hat{\varphi}(\mathbf{k}) = \lambda_{\mathbf{k}}^2\hat{\varphi}(\mathbf{k})$. We deduce that $\lambda_{\mathbf{k}} = |\mathbf{k}|$, for each $\mathbf{k}\in\mathbb{Z}^3$, where $\tilde{\mathbf{k}} = (\frac{\mathbf{k}_1}{a_1}, \frac{\mathbf{k}_2}{a_2}, \frac{\mathbf{k}_3}{a_3})$. For simplicity, in the rest of the dissertation, we still use $\mathbf{k}$ to denote $(\frac{\mathbf{k}_1}{a_1}, \frac{\mathbf{k}_2}{a_2}, \frac{\mathbf{k}_3}{a_3})$. We can take the eigenfunctions $\varphi_{\mathbf{k}} = e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$, with normalization $\int_{\mathbb{T}^D} e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} = 1$ and $\int_{\mathbb{T}^D} \nabla_x\varphi_{\mathbf{k}}\cdot\nabla_x\varphi_{\mathbf{l}} = 0$, for any $\mathbf{k},\mathbf{l}\in\mathbb{Z}^3$ with $\mathbf{k}\neq\mathbf{l}$.

Now, we can construct the eigenfunctions of the operator $\mathcal{A}$ from that of $-\Delta$ which is described by the following Lemma:

Lemma 5: *For each $\mathbf{k}\in\mathbb{Z}^3$, the operator $\mathcal{A}$ has eigenvalues and corresponding eigenvectors $(\mathrm{i}\omega_{\mathbf{k}}, \Phi_{\mathbf{k}})$ and $(-\mathrm{i}\omega_{\mathbf{k}}, \overline{\Phi}_{\mathbf{k}})$, i.e.,*

$$\mathcal{A}\Phi_{\mathbf{k}} = \mathrm{i}\omega_{\mathbf{k}}\Phi_{\mathbf{k}}\,, \quad \mathcal{A}\overline{\Phi}_{\mathbf{k}} = -\mathrm{i}\omega_{\mathbf{k}}\overline{\Phi}_{\mathbf{k}}\,, \tag{3.2.0.47}$$

where $\omega_{\mathbf{k}} = \sqrt{\gamma}\lambda_{\mathbf{k}} = \sqrt{\gamma}|\mathbf{k}|$, and

$$\begin{aligned}
\Phi_{\mathbf{k}} &= \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{\gamma(\gamma-1)}}\varphi_{\mathbf{k}} \\ \frac{1}{i\lambda_{\mathbf{k}}\sqrt{\gamma-1}}\nabla_x\varphi_{\mathbf{k}} \\ \frac{\gamma}{\sqrt{\gamma(\gamma-1)^3}}\varphi_{\mathbf{k}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{\gamma(\gamma-1)}} \\ \frac{1}{\sqrt{\gamma-1}}\frac{\mathbf{k}}{|\mathbf{k}|} \\ \frac{\gamma}{\sqrt{\gamma(\gamma-1)^3}} \end{pmatrix} e^{\mathbf{i}\mathbf{k}\cdot\mathbf{x}}\,, \\
\overline{\Phi}_{\mathbf{k}} &= \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{\gamma(\gamma-1)}}\varphi_{\mathbf{k}} \\ \frac{1}{-i\lambda_{\mathbf{k}}\sqrt{\gamma-1}}\nabla_x\varphi_{\mathbf{k}} \\ \frac{\gamma}{\sqrt{\gamma(\gamma-1)^3}}\varphi_{\mathbf{k}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{\gamma(\gamma-1)}} \\ \frac{1}{\sqrt{\gamma-1}}\frac{\mathbf{k}}{|\mathbf{k}|} \\ \frac{\gamma}{\sqrt{\gamma(\gamma-1)^3}} \end{pmatrix} e^{-\mathbf{i}\mathbf{k}\cdot\mathbf{x}}\,.
\end{aligned} \tag{3.2.0.48}$$

Furthermore, $\{\Phi_{\mathbf{k}}, \overline{\Phi}_{\mathbf{k}}\}$ *is an orthonormal basis of* $\mathrm{Null}(\mathcal{A})^{\perp}$ *under the inner product* $\langle \cdot, \cdot \rangle$ *in the following sense:*

- $\langle \Phi_{\mathbf{k}}, \Phi_{\mathbf{l}} \rangle = \delta_{\mathbf{kl}}\,;$
- $\langle \Phi_{\mathbf{k}}, \overline{\Phi}_{\mathbf{l}} \rangle = 0\,;$
- $\langle \overline{\Phi}_{\mathbf{k}}, \overline{\Phi}_{\mathbf{l}} \rangle = \delta_{\mathbf{kl}}.$

Proof of Lemma: Straightforward calculations. □

Now, we have the following orthogonal decomposition: for every $\mathbf{U} \in \mathcal{H}$,

$$\begin{aligned} \mathbf{U} &= \Pi\mathbf{U} + \Pi^{\perp}\mathbf{U} \\ &= \begin{pmatrix} \mathrm{U}_0 - \frac{\gamma-1}{\gamma}\mathrm{U}_4 \\ P\mathrm{U} \\ 0 \end{pmatrix} + \sum_{\mathbf{k}\in\mathbb{Z}^3} (\mathbf{U}_{\mathbf{k}}\Phi_{\mathbf{k}} + \overline{\mathbf{U}}_{\mathbf{k}}\overline{\Phi}_{\mathbf{k}})\,, \end{aligned} \tag{3.2.0.49}$$

where $\mathbf{U}_{\mathbf{k}}$ is the coefficient of $\mathbf{U}$ with respect to the basis $\Phi_{\mathbf{k}}$ under the inner product $\langle \cdot, \cdot \rangle$. i.e.,

$$\mathbf{U}_{\mathbf{k}} = \langle \mathbf{U}, \Phi_{\mathbf{k}} \rangle = \tfrac{\gamma-1}{\sqrt{2}}\sqrt{\tfrac{\gamma-1}{\gamma}}\hat{\mathrm{U}}_4(\mathbf{k}) + \tfrac{1\sqrt{\gamma-1}}{\sqrt{2}}\tfrac{\mathbf{k}}{|\mathbf{k}|} \cdot \hat{\mathrm{U}}(\mathbf{k})\,. \tag{3.2.0.50}$$

3.3 Averaged System in the Incompressible Mode

In this section, we start to calculate the explicit form of the averaged system (3.1.0.21) of the Navier-Stokes system for compressible gas dynamics. The asymptotic behavior on the slow mode $\mathrm{Null}(\mathcal{A})$ and on the fast mode $\mathrm{Null}(\mathcal{A})^{\perp}$ are significantly different. The basic idea is to project the averaged system (3.1.0.21) onto $\mathrm{Null}(\mathcal{A})$ and $\mathrm{Null}(\mathcal{A})^{\perp}$ respectively. Recalling the projection operators Π and

$\Pi^{\perp}$ defined in (3.2.0.41) and (3.2.0.43), we decompose the diffusion and quadratic terms

$$
\begin{aligned}
&\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{s\mathcal{A}}\mathcal{D}(e^{-s\mathcal{A}}\mathbf{U})\,ds\\
=&\lim_{T\to\infty}\frac{1}{T}\int_0^T \Pi[e^{s\mathcal{A}}\mathcal{D}(e^{-s\mathcal{A}}\mathbf{U})]\,ds+\lim_{T\to\infty}\frac{1}{T}\int_0^T \Pi^{\perp}[e^{s\mathcal{A}}\mathcal{D}(e^{-s\mathcal{A}}\mathbf{U})]\,ds\\
=&\Pi\overline{\mathcal{D}}\mathbf{U}+\Pi^{\perp}\overline{\mathcal{D}}\mathbf{U}\,,
\end{aligned}
\tag{3.3.0.51}
$$

and

$$
\begin{aligned}
&\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{s\mathcal{A}}\mathcal{Q}(e^{-s\mathcal{A}}\mathbf{U},e^{-s\mathcal{A}}\mathbf{U})\,ds\\
=&\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{s\mathcal{A}}\Pi\mathcal{Q}(e^{-s\mathcal{A}}\mathbf{U},e^{-s\mathcal{A}}\mathbf{U})\,ds+\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{s\mathcal{A}}\Pi^{\perp}\mathcal{Q}(e^{-s\mathcal{A}}\mathbf{U},e^{-s\mathcal{A}}\mathbf{U})\,ds\\
=&\Pi\overline{\mathcal{Q}}(\mathbf{U},\mathbf{U})+\Pi^{\perp}\overline{\mathcal{Q}}(\mathbf{U},\mathbf{U})\,.
\end{aligned}
\tag{3.3.0.52}
$$

Then the averaged system (3.1.0.21) is decomposed into

$$
\partial_t\Pi\mathbf{U}+\partial_t\Pi^{\perp}\mathbf{U}+\Pi\overline{\mathcal{Q}}(\mathbf{U},\mathbf{U})+\Pi^{\perp}\overline{\mathcal{Q}}(\mathbf{U},\mathbf{U})=\Pi\overline{\mathcal{D}}\mathbf{U}+\Pi^{\perp}\overline{\mathcal{D}}\mathbf{U}\,. \tag{3.3.0.53}
$$

Its orthogonal projection onto the slow and fast modes are

$$
\partial_t\Pi\mathbf{U}+\Pi\overline{\mathcal{Q}}(\mathbf{U},\mathbf{U})=\Pi\overline{\mathcal{D}}\mathbf{U},\qquad \text{equation on Null}(\mathcal{A})\,, \tag{3.3.0.54}
$$

and

$$
\partial_t\Pi^{\perp}\mathbf{U}+\Pi^{\perp}\overline{\mathcal{Q}}(\mathbf{U},\mathbf{U})=\Pi^{\perp}\overline{\mathcal{D}}\mathbf{U},\qquad \text{equation on Null}(\mathcal{A})^{\perp}\,, \tag{3.3.0.55}
$$

respectively. The projection on the slow mode Null($\mathcal{A}$) will be derived in this section. Our formal calculation illustrates that the averaged system on the slow mode is the **Navier-Stokes system for incompressible flow**.

We start with computing the diffusion term. Because $\Pi\overline{\mathcal{D}}\in$ Null($\mathcal{A}$), it follows that $\Pi\overline{\mathcal{D}}=\langle\Pi\overline{\mathcal{D}},\eta\rangle\eta$, where η is the eigenvector associated with eigenvalue 0.

Null($\mathcal{A}$) is spanned by such $\eta's$. Although we can explicitly find its expression, as we did for Null$(\mathcal{A})^{\perp}$, we do not need the explicit form in our calculation. The reason is that Null($\mathcal{A}$) is generated from the eigenfunctions corresponding to zero eigenvalues, so the exponential operator $e^{s\mathcal{A}}$ does not affect Null($\mathcal{A}$). The inner product of $\Pi\overline{\mathcal{D}}$ with $\vec{e}$ is:

$$\begin{aligned}\langle \Pi\overline{\mathcal{D}}, \eta\rangle &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \left\langle \Pi[e^{s\mathcal{A}}\mathcal{D}(e^{-s\mathcal{A}}\mathbf{U})], \eta\right\rangle\, ds \\ &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \left\langle e^{s\mathcal{A}}[\Pi\mathcal{D}(e^{-s\mathcal{A}}\mathbf{U})], \eta\right\rangle\, ds \\ &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \left\langle \Pi\mathcal{D}(e^{-s\mathcal{A}}\mathbf{U},)e^{-s\mathcal{A}}\eta\right\rangle\, ds \\ &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \left\langle \Pi\mathcal{D}(e^{-s\mathcal{A}}\mathbf{U},)\eta\right\rangle\, ds\,.\end{aligned} \tag{3.3.0.56}$$

In the second equality, we used the identity $\langle e^{s\mathcal{A}}\mathbf{U}, \mathbf{V}\rangle = \langle \mathbf{U}, e^{-s\mathcal{A}}\mathbf{V}\rangle$, which is a simple consequence of the skew-symmetry of the operator $\mathcal{A}$, (see Lemma 1.) In the first equality above, we applied the commutativity between Π and $e^{s\mathcal{A}}$, i.e., $\Pi\circ e^{s\mathcal{A}} = e^{s\mathcal{A}}\circ\Pi$. The following simple proof of this commutativity is based on the orthogonal decomposition of $\mathcal{H}$.

For any $\mathbf{U}\in\mathcal{H}$,

$$\begin{aligned}\Pi(e^{s\mathcal{A}}\mathbf{U}) &= \Pi(\Pi\mathbf{U} + \sum_{\mathbf{k}\in\mathbb{Z}^3}\{\mathbf{U}_{\mathbf{k}}e^{\mathrm{i}s\omega_{\mathbf{k}}}\Phi_{\mathbf{k}} + \overline{\mathbf{U}}_{\mathbf{k}}e^{-\mathrm{i}s\omega_{\mathbf{k}}}\overline{\Phi}_{\mathbf{k}}\}) \\ &= \Pi\mathbf{U}\,.\end{aligned} \tag{3.3.0.57}$$

Notice that $\Phi_{\mathbf{k}}\in$ Null$^{\perp}(\mathcal{A})$, so $\Pi\Phi_{\mathbf{k}} = 0$. It is obvious that $e^{s\mathcal{A}}\Pi\mathbf{U} = \Pi\mathbf{U}$. Then $\Pi(e^{s\mathcal{A}}\mathbf{U}) = e^{s\mathcal{A}}(\Pi\mathbf{U})$, for any $\mathbf{U}\in\mathcal{H}$. Recalling the definition of the diffusion operation $\mathcal{D}$, we can derive

$$\begin{aligned}
&\left\langle \Pi\mathcal{D}(e^{-s\mathcal{A}}\mathbf{U},)\eta\right\rangle \\
=&\left\langle \begin{pmatrix} -\frac{\kappa(\gamma-1)}{\gamma}\Delta_x[(\gamma-1)(e^{-s\mathcal{A}}\mathbf{U})_4-(e^{-s\mathcal{A}}\mathbf{U})_0] \\ \mu P\Delta_x[(e^{-s\mathcal{A}}\mathbf{U})_I] \\ 0 \end{pmatrix}, \eta \right\rangle .
\end{aligned} \tag{3.3.0.58}$$

Here we used the notation $e^{s\mathcal{A}}\mathbf{U} = ((e^{s\mathcal{A}}\mathbf{U})_0, (e^{s\mathcal{A}}\mathbf{U})_I, (e^{s\mathcal{A}}\mathbf{U})_4)^T$. A straightforward manipulation yields

$$\begin{aligned}
&\Delta_x[(\gamma-1)(e^{-s\mathcal{A}}\mathbf{U})_4-(e^{-s\mathcal{A}}\mathbf{U})_0] \\
=&\Delta_x(\tfrac{\gamma-1}{\gamma}\mathrm{U}_4-\mathrm{U}_0)+\tfrac{1}{\sqrt{2}}\sqrt{\tfrac{\gamma-1}{\gamma}}\sum_{\mathbf{k}}[\mathbf{U}_{\mathbf{k}}e^{-\mathrm{i}s\omega_{\mathbf{k}}}+\overline{\mathbf{U}}_{\mathbf{k}}e^{\mathrm{i}s\omega_{\mathbf{k}}}]\Delta_x\varphi_{\mathbf{k}} .
\end{aligned} \tag{3.3.0.59}$$

Notice that under the periodic boundary condition the Leray projection P commutes with Δ, it follows that

$$P\Delta_x[(e^{-s\mathcal{A}}\mathbf{U})_I] = \Delta_x P\mathrm{U} . \tag{3.3.0.60}$$

Here we used the identity

$$(e^{-s\mathcal{A}}\mathbf{U})_I = P\mathbf{U} + \sum_{\mathbf{k}} \tfrac{1}{\mathrm{i}\lambda_{\mathbf{k}}}(\mathbf{U}_{\mathbf{k}}e^{-\mathrm{i}s\omega_{\mathbf{k}}}-\overline{\mathbf{U}}_{\mathbf{k}}e^{\mathrm{i}s\omega_{\mathbf{k}}})\nabla_x\varphi_{\mathbf{k}} . \tag{3.3.0.61}$$

Thus

$$\begin{aligned}
&\left\langle \Pi\mathcal{D}(e^{-s\mathcal{A}}\mathbf{U},)\eta\right\rangle \\
=&\left\langle \begin{pmatrix} -\frac{\kappa(\gamma-1)}{\gamma}\Delta_x(\frac{\gamma-1}{\gamma}\mathrm{U}_4-\mathrm{U}_0) \\ \mu\Delta_x P\mathrm{U} \\ 0 \end{pmatrix}, \eta \right\rangle \\
+&\left\langle \begin{pmatrix} \frac{1}{\sqrt{2}}\sqrt{\frac{\gamma-1}{\gamma}}\sum_{\mathbf{k}}[\mathbf{U}_{\mathbf{k}}e^{-\mathrm{i}s\omega_{\mathbf{k}}}+\overline{\mathbf{U}}_{\mathbf{k}}e^{\mathrm{i}s\omega_{\mathbf{k}}}]\Delta_x\varphi_{\mathbf{k}} \\ 0 \\ 0 \end{pmatrix}, \eta \right\rangle .
\end{aligned} \tag{3.3.0.62}$$

The first term is the resonant term which is independent of s, so is not affected by time averaging. The second is non-resonant, which is filtered by time averaging. The key underlying mathematical fact is the following Riemann-Lebesgue lemma which guarantees that:

Lemma 6:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{\mathrm{i}s\omega_{\mathbf{k}}}\phi(s)\,ds = 0\,, \tag{3.3.0.63}$$

for any integrable function $\phi(t)$.

Thus, we have

$$\begin{aligned}\langle\Pi\overline{\mathcal{D}},\eta\rangle &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \left\langle\Pi\mathcal{D}(e^{-s\mathcal{A}}\mathbf{U},)\eta\right\rangle\,ds\\ &= \left\langle\begin{pmatrix}-\frac{\kappa(\gamma-1)}{\gamma}\Delta_x(\frac{\gamma-1}{\gamma}\mathrm{U}_4-\mathrm{U}_0)\\ \mu\Delta_x P\mathrm{U}\\ 0\end{pmatrix},\eta\right\rangle.\end{aligned} \tag{3.3.0.64}$$

$\begin{pmatrix}-\frac{\kappa(\gamma-1)}{\gamma}\Delta_x(\frac{\gamma-1}{\gamma}\mathrm{U}_4-\mathrm{U}_0)\\ \mu\Delta_x P\mathrm{U}\\ 0\end{pmatrix}$ is in $\mathrm{Null}(\mathcal{A})$, together with $\eta\in\mathrm{Null}(\mathcal{A})$, this yields the projection of the diffusion term of the averaged system on the null space of $\mathcal{A}$ is:

$$\Pi\overline{\mathcal{D}}(\mathbf{U}) = \begin{pmatrix}-\frac{\kappa(\gamma-1)}{\gamma}\Delta_x(\frac{\gamma-1}{\gamma}\mathrm{U}_4-\mathrm{U}_0)\\ \mu\Delta_x P\mathrm{U}\\ 0\end{pmatrix}. \tag{3.3.0.65}$$

The next step is to formally derive the projection of the quadratic term in the averaged system . Since $\Pi\overline{\mathcal{Q}}\in\mathrm{Null}(\mathcal{A})$, $\Pi\overline{\mathcal{Q}} = \langle\Pi\overline{\mathcal{Q}},\eta\rangle\eta$. As we did for diffusion term, We need only compute

$$\begin{aligned}\langle \Pi\overline{\mathcal{Q}}, \eta\rangle &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \langle e^{s\mathcal{A}}\Pi\mathcal{Q}(e^{-s\mathcal{A}}\mathbf{U}, e^{-s\mathcal{A}}\mathbf{U},)\eta\rangle ds \\ &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \langle \Pi\mathcal{Q}(e^{-s\mathcal{A}}\mathbf{U}, e^{-s\mathcal{A}}\mathbf{U},)\eta\rangle ds\,,\end{aligned} \tag{3.3.0.66}$$

where

$$\begin{aligned}&\left\langle \Pi\mathcal{Q}(e^{-s\mathcal{A}}\mathbf{U}, e^{-s\mathcal{A}}\mathbf{U},)\eta\right\rangle \\ &=\left\langle \begin{pmatrix} -\nabla_x\cdot\{(e^{-s\mathcal{A}}\mathbf{U})_I[(\gamma-1)(e^{-s\mathcal{A}}\mathbf{U})_4-(e^{-s\mathcal{A}}U)_0]\} \\ P\nabla_x\cdot[(e^{-s\mathcal{A}}\mathbf{U})_I\otimes(e^{-s\mathcal{A}}\mathbf{U})_I] \\ 0\end{pmatrix}, \eta\right\rangle .\end{aligned} \tag{3.3.0.67}$$

Now we compute the above quantity term by term.

$$\begin{aligned}P\nabla_x\cdot[(e^{-s\mathcal{A}}\mathbf{U})_I\otimes(e^{-s\mathcal{A}}\mathbf{U})_I] &= \nabla_x\cdot(P\mathrm{U}\otimes P\mathrm{U}) \\ &+\tfrac{1}{2(\gamma-1)}\sum_{\mathbf{k}}\tfrac{1}{\mathrm{i}\lambda_{\mathbf{k}}}(\mathbf{U}_{\mathbf{k}}e^{-\mathrm{i}s\omega_{\mathbf{k}}}-\overline{\mathbf{U}}_{\mathbf{k}}e^{\mathrm{i}s\omega_{\mathbf{k}}})P\nabla_x\cdot(P\mathrm{U}\otimes\nabla_x\varphi_{\mathbf{k}}+\nabla_x\varphi_{\mathbf{k}}\otimes P\mathrm{U}) \\ &+\tfrac{1}{2(\gamma-1)}\sum_{\mathbf{k},\mathbf{l}}\tfrac{1}{-\lambda_{\mathbf{k}}\lambda_{\mathbf{l}}}[\mathbf{U}_{\mathbf{k}}\mathbf{U}_{\mathbf{l}}(e^{-\mathrm{i}s(\omega_{\mathbf{k}}+\omega_{\mathbf{l}})}+\overline{\mathbf{U}}_{\mathbf{k}}\overline{\mathbf{U}}_{\mathbf{l}}e^{\mathrm{i}s(\omega_{\mathbf{k}}+\omega_{\mathbf{l}})})]P\nabla_x\cdot(\nabla_x\varphi_{\mathbf{k}}\otimes\nabla_x\varphi_{\mathbf{l}}) \\ &+\tfrac{1}{2(\gamma-1)}\sum_{\mathbf{k},\mathbf{l}}\tfrac{1}{\lambda_{\mathbf{k}}\lambda_{\mathbf{l}}}[\mathbf{U}_{\mathbf{k}}\overline{\mathbf{U}}_{\mathbf{l}}e^{-\mathrm{i}s(\omega_{\mathbf{k}}-\omega_{\mathbf{l}})}]P\nabla_x\cdot(\nabla_x\varphi_{\mathbf{k}}\otimes\nabla_x\varphi_{\mathbf{l}}) \\ &+\tfrac{1}{2(\gamma-1)}\sum_{\mathbf{k},\mathbf{l}}\tfrac{1}{\lambda_{\mathbf{k}}\lambda_{\mathbf{l}}}[\overline{\mathbf{U}}_{\mathbf{k}}\mathbf{U}_{\mathbf{l}}e^{\mathrm{i}s(\omega_{\mathbf{k}}-\omega_{\mathbf{l}})}]P\nabla_x\cdot(\nabla_x\varphi_{\mathbf{k}}\otimes\nabla_x\varphi_{\mathbf{l}})\,,\end{aligned} \tag{3.3.0.68}$$

and

$$\begin{aligned}\nabla_x\cdot\{(e^{-s\mathcal{A}}\mathbf{U})_I[(\gamma-1)(e^{-s\mathcal{A}}\mathbf{U})_4-(e^{-s\mathcal{A}}U)_0]\} &= \nabla_x\cdot[(P\mathrm{U})(\tfrac{\gamma-1}{\gamma}\mathrm{U}_4-\mathrm{U}_0)] \\ &+\sum_{\mathbf{k}}\sqrt{\tfrac{\gamma-1}{2\gamma}}(\mathbf{U}_{\mathbf{k}}e^{-\mathrm{i}s\omega_{\mathbf{k}}}+\overline{\mathbf{U}}_{\mathbf{k}}e^{\mathrm{i}s\omega_{\mathbf{k}}})\nabla_x\cdot[(P\mathrm{U})\varphi_{\mathbf{k}}] \\ &+\tfrac{1}{\sqrt{2(\gamma-1)}}\sum_{\mathbf{k}}\tfrac{1}{\mathrm{i}\lambda_{\mathbf{k}}}(\mathbf{U}_{\mathbf{k}}e^{-\mathrm{i}s\omega_{\mathbf{k}}}-\overline{\mathbf{U}}_{\mathbf{k}}e^{\mathrm{i}s\omega_{\mathbf{k}}})\nabla_x\cdot[\nabla\varphi_{\mathbf{k}}(\tfrac{\gamma-1}{\gamma}\mathrm{U}_4-\mathrm{U}_0)] \\ &+\tfrac{1}{2(\gamma-1)}\sqrt{\tfrac{\gamma-1}{\gamma}}\sum_{\mathbf{k},\mathbf{l}}\tfrac{1}{\mathrm{i}\lambda_{\mathbf{k}}}(\mathbf{U}_{\mathbf{k}}e^{-\mathrm{i}s\omega_{\mathbf{k}}}-\overline{\mathbf{U}}_{\mathbf{k}}e^{\mathrm{i}s\omega_{\mathbf{k}}})(\mathbf{U}_{\mathbf{l}}e^{-\mathrm{i}s\omega_{\mathbf{l}}}+\overline{\mathbf{U}}_{\mathbf{l}}e^{\mathrm{i}s\omega_{\mathbf{l}}})\nabla_x\cdot[(\nabla_x\varphi_{\mathbf{k}})\varphi_{\mathbf{l}}]\,.\end{aligned} \tag{3.3.0.69}$$

We are concerning with terms of the form

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{\mathrm{i}s(\alpha\omega_{\mathbf{k}}+\beta\omega_{\mathbf{l}})}\phi(x,t)dt\,. \tag{3.3.0.70}$$

In the above integral, α, β take values in 1 or -1. Whenever $\alpha\omega_{\mathbf{k}} + \beta\omega_{\mathbf{l}} \neq 0$, the corresponding term above is oscillatory and the averaging procedure above will give a zero contribution as described in Lemma (6). Consequently, the only nonzero contributions that survive the averaging process are the two-wave resonances with $\alpha\omega_{\mathbf{k}} + \beta\omega_{\mathbf{l}} = 0$. Recall the definition of $\omega_{\mathbf{k}}$, i.e., $\omega_{\mathbf{k}} = \sqrt{\gamma}\lambda_{\mathbf{k}} > 0$. Then the only two-wave resonance is $\omega_{\mathbf{k}} = \omega_{\mathbf{l}}$. Thus, we have

$$\begin{aligned}
\langle \Pi\overline{\mathcal{Q}}, \eta\rangle &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \left\langle \Pi\mathcal{Q}(e^{-s\mathcal{A}}\mathbf{U}, e^{-s\mathcal{A}}\mathbf{U},)\eta\right\rangle ds \\
&= \left\langle \begin{pmatrix} -\nabla_x\cdot[(P\mathrm{U})(\frac{\gamma-1}{\gamma}\mathrm{U}_4 - \mathrm{U}_0)] \\ \nabla_x\cdot(P\mathrm{U}\otimes P\mathrm{U}) \\ 0 \end{pmatrix}, \eta\right\rangle \\
&+ \left\langle \begin{pmatrix} \frac{1}{2}\sqrt{\frac{\gamma-1}{\gamma}}\sum_{\omega_{\mathbf{k}}=\omega_{\mathbf{l}}}\frac{1}{\mathrm{i}\lambda_{\mathbf{k}}}(\mathbf{U}_{\mathbf{k}}\overline{\mathbf{U}}_{\mathbf{l}} - \overline{\mathbf{U}}_{\mathbf{k}}\mathbf{U}_{\mathbf{l}})\nabla_x\cdot[(\nabla_x\varphi_{\mathbf{k}})\varphi_{\mathbf{l}}] \\ \sum_{\omega_{\mathbf{k}}=\omega_{\mathbf{l}}}\frac{1}{\lambda_{\mathbf{k}}\lambda_{\mathbf{l}}}(\mathbf{U}_{\mathbf{k}}\overline{\mathbf{U}}_{\mathbf{l}} + \overline{\mathbf{U}}_{\mathbf{k}}\mathbf{U}_{\mathbf{l}})P\nabla_x\cdot(\nabla_x\varphi_{\mathbf{k}}\otimes\nabla_x\varphi_{\mathbf{l}}) \\ 0 \end{pmatrix}, \eta\right\rangle .
\end{aligned} \tag{3.3.0.71}$$

We claim that the last term above actually vanishes, based on the following argument. In the first component, if we exchange the index $\mathbf{k}$ and $\mathbf{l}$, $\mathbf{U}_{\mathbf{k}}\overline{\mathbf{U}}_{\mathbf{l}} - \overline{\mathbf{U}}_{\mathbf{k}}\mathbf{U}_{\mathbf{l}}$ will change sign, but $\nabla_x\cdot[(\nabla_x\varphi_{\mathbf{k}})\varphi_{\mathbf{l}}] = \nabla_x\cdot[(\nabla_x\varphi_{\mathbf{l}})\varphi_{\mathbf{k}}]$, whenever $\omega_{\mathbf{k}} = \omega_{\mathbf{l}} = L$. The reason is that

$$\begin{aligned}
\nabla_x\cdot[(\nabla_x\varphi_{\mathbf{k}})\varphi_{\mathbf{l}}] &= \Delta_x\varphi_{\mathbf{k}}\varphi_{\mathbf{l}} + \nabla_x\varphi_{\mathbf{k}}\cdot\nabla_x\varphi_{\mathbf{l}} \\
&= -L^2\varphi_{\mathbf{k}}\varphi_{\mathbf{l}} + \nabla_x\varphi_{\mathbf{k}}\cdot\nabla_x\varphi_{\mathbf{l}}\,,
\end{aligned} \tag{3.3.0.72}$$

since $\varphi_{\mathbf{k}}$ is an eigenfunction of Δ, with eigenvalue $\lambda_{\mathbf{k}}^2 = L^2$. So the first component above vanishes.

In the resonant set $\{\omega_{\mathbf{k}} = \omega_{\mathbf{l}} = L\}$,

$$\nabla_x \cdot (\nabla_x \varphi_{\mathbf{k}} \otimes \nabla_x \varphi_{\mathbf{l}}) = -\tfrac{1}{2} L^2 \nabla_x (\varphi_{\mathbf{k}} \varphi_{\mathbf{l}}) + \tfrac{1}{2} \nabla_x (\nabla_x \varphi_{\mathbf{k}} \cdot \nabla_x \varphi_{\mathbf{l}}) \,. \tag{3.3.0.73}$$

It is a gradient term, and vanishes under the action of the Laray projection P. Thus,

$$\Pi \overline{\mathcal{Q}}(\mathbf{U}, \mathbf{U}) = \begin{pmatrix} -\nabla_x \cdot [(P\mathrm{U})(\frac{\gamma-1}{\gamma}\mathrm{U}_4 - \mathrm{U}_0)] \\ \nabla_x \cdot (P\mathrm{U} \otimes P\mathrm{U}) + \nabla_x p \\ 0 \end{pmatrix} \tag{3.3.0.74}$$

for some smooth function p. Finally, we derive the projection of the averaged system onto Null($\mathcal{A}$),

$$\partial_t \Pi \mathbf{U} + \Pi \overline{\mathcal{Q}}(\mathbf{U}, \mathbf{U}) = \Pi \overline{\mathcal{D}} \mathbf{U} \,, \tag{3.3.0.75}$$

i.e.,

$$\partial_t \begin{pmatrix} \mathrm{U}_0 - \frac{\gamma-1}{\gamma}\mathrm{U}_4 \\ PU \\ 0 \end{pmatrix} + \begin{pmatrix} \nabla_x \cdot [(P\mathrm{U})(\mathrm{U}_0 - \frac{\gamma-1}{\gamma}\mathrm{U}_4)] \\ \nabla_x \cdot (P\mathrm{U} \otimes P\mathrm{U}) + \nabla_x p \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\kappa(\gamma-1)}{\gamma} \Delta_x (\mathrm{U}_0 - \frac{\gamma-1}{\gamma}\mathrm{U}_4) \\ \mu \Delta_x P\mathrm{U} \\ 0 \end{pmatrix} . \tag{3.3.0.76}$$

This implies that $(P\mathrm{U}, \mathrm{U}_0 - \frac{\gamma-1}{\gamma}\mathrm{U}_4)$ satisfies the Navier-Stokes system for incompressible flow:

$$\begin{aligned} \nabla_x \cdot u = 0 \,, \\ \partial_t u + u \cdot \nabla_x u + \nabla_x p = \mu \Delta_x u \,, \\ \partial_t \theta + u \cdot \nabla_x \theta = \tfrac{\kappa(\gamma-1)}{\gamma} \Delta_x \theta \,, \\ (u, \theta)(t=0) = (P\mathrm{U}, \mathrm{U}_0^{in} - \tfrac{\gamma-1}{\gamma}\mathrm{U}_4^{in}) \,. \end{aligned} \tag{3.3.0.77}$$

This confirms formally a well-known fact: the incompressible Navier-Stokes equation is a weakly nonlinear asymptotics of the compressible Navier-Stokes equation.

Remark: Note that $\Pi\mathbf{U} = (\mathrm{U}_0 - \frac{\gamma-1}{\gamma}\mathrm{U}_4, P\mathrm{U}, 0)^T$. The above equation depends *only* on $\Pi\mathbf{U}$. This means that the projection of the averaged system on Null($\mathcal{A}$) is completely decoupled from fast mode. So the equation (3.3.0.75) can written as:

$$\partial_t \Pi\mathbf{U} + \Pi\overline{\mathcal{Q}}(\Pi\mathbf{U}, \Pi\mathbf{U}) = \Pi\overline{\mathcal{D}}\Pi\mathbf{U}\,. \tag{3.3.0.78}$$

3.4 Averaged System in the Acoustic Mode

This section will be devoted to characterizing the projection of averaged system onto the orthogonal complement of the null space of the linear operator $\mathcal{A}$. It behaves quite differently with the slow motion. It describes the nonlinear interactions between slow motion and fast waves. We will precisely analyze the structure of the resonant set and its influence on the averaged system .

We start with the easier part, the fast oscillation of the diffusion term, denoted by $\tilde{\mathcal{D}}^{\perp}$, as indicated in the last section. We recall that

$$\Pi^{\perp}\overline{\mathcal{D}}\mathbf{U} = \lim_{T\to\infty}\frac{1}{T}\int_0^T \Pi^{\perp}[e^{s\mathcal{A}}\mathcal{D}(e^{-s\mathcal{A}}\mathbf{U})]\,ds\,. \tag{3.4.0.79}$$

Since $\mathrm{Null}^{\perp}(\mathcal{A}) = \mathrm{span}\{\Phi_{\mathbf{k}}, \overline{\Phi}_{\mathbf{k}}\}$, it follows that

$$\begin{aligned}\Pi^{\perp}\overline{\mathcal{D}}\mathbf{U} &= \sum_{\mathbf{k}}\langle \Pi^{\perp}\overline{\mathcal{D}}\mathbf{U}, \Phi_{\mathbf{k}}\rangle\Phi_{\mathbf{k}} + \sum_{\mathbf{k}}\langle \Pi^{\perp}\overline{\mathcal{D}}\mathbf{U}, \overline{\Phi}_{\mathbf{k}}\rangle\overline{\Phi}_{\mathbf{k}} \\ &= 2Re\sum_{\mathbf{k}}\overline{\mathcal{D}}^{\perp}_{\mathbf{k}}(\mathbf{U})\Phi_{\mathbf{k}}\,,\end{aligned} \tag{3.4.0.80}$$

For each $\mathbf{k} \in \mathbb{Z}^3$, the time averaging $\overline{\mathcal{D}}^{\perp}_{\mathbf{k}}(\mathbf{U})$

$$\begin{aligned}
\overline{\mathcal{D}}^{\perp}_{\mathbf{k}}(\mathbf{U}) &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \langle \Pi^{\perp}[e^{s\mathcal{A}} \mathcal{D}(e^{-s\mathcal{A}}\mathbf{U})], \Phi_{\mathbf{k}} \rangle \, ds \\
&= \lim_{T\to\infty} \frac{1}{T} \int_0^T \langle \mathcal{D}(e^{-s\mathcal{A}}\mathbf{U},) e^{-s\mathcal{A}} \Phi_{\mathbf{k}} \rangle \\
&= \lim_{T\to\infty} \frac{1}{T} \int_0^T \left\langle \begin{pmatrix} 0 \\ \mu \nabla_x (e^{-s\mathcal{A}}\mathbf{U})_I + (\mu+\lambda)\nabla_x[\nabla_x \cdot (e^{-s\mathcal{A}}\mathbf{U})_I] \\ \kappa \Delta_x[(\gamma-1)(e^{-s\mathcal{A}}\mathbf{U})_4 - (e^{-s\mathcal{A}}\mathbf{U})_0] \end{pmatrix}, e^{-\mathrm{i}s\omega_{\mathbf{k}}} \Phi_{\mathbf{k}} \right\rangle .
\end{aligned} \tag{3.4.0.81}$$

To apply Lemma (6) as in last section, we decompose the first term in $\langle \cdot\,, \cdot \rangle$ into two parts, of which one is independent of s, the other depends on s. Using again the orthogonal decomposition of $\mathbf{U}$ into $\Pi\mathbf{U}$ and $\sum_{\mathbf{k}\in\mathbb{Z}^3}(\mathbf{U}_{\mathbf{k}}\Phi_{\mathbf{k}} + \overline{\mathbf{U}}_{\mathbf{k}}\overline{\Phi}_{\mathbf{k}})$, we have

$$\begin{aligned}
\overline{\mathcal{D}}^{\perp}_{\mathbf{k}}(\mathbf{U}) = \lim_{T\to\infty} \frac{1}{T} \int_0^T & \left\langle \begin{pmatrix} 0 \\ \mu \Delta_x P\mathrm{U} \\ \kappa \Delta_x (\frac{\gamma-1}{\gamma}\mathrm{U}_4 - \mathrm{U}_0) \end{pmatrix}, \frac{e^{-\mathrm{i}s\omega_{\mathbf{k}}}}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{\gamma(\gamma-1)}}\varphi_{\mathbf{k}} \\ \frac{1}{i\lambda_{\mathbf{k}}\sqrt{\gamma-1}} \nabla_x \varphi_{\mathbf{k}} \\ \frac{\gamma}{\sqrt{\gamma(\gamma-1)^3}}\varphi_{\mathbf{k}} \end{pmatrix} \right\rangle ds \\
& + \lim_{T\to\infty} \frac{1}{T} \int_0^T \left\langle \Box(s,) \frac{e^{-\mathrm{i}s\omega_{\mathbf{k}}}}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{\gamma(\gamma-1)}}\varphi_{\mathbf{k}} \\ \frac{1}{i\lambda_{\mathbf{k}}\sqrt{\gamma-1}} \nabla_x \varphi_{\mathbf{k}} \\ \frac{\gamma}{\sqrt{\gamma(\gamma-1)^3}}\varphi_{\mathbf{k}} \end{pmatrix} \right\rangle ds \\
& = \overline{\mathcal{D}}^{\perp}_{\mathbf{k},1}(\mathbf{U}) + \overline{\mathcal{D}}^{\perp}_{\mathbf{k},2}(\mathbf{U}) \, ,
\end{aligned} \tag{3.4.0.82}$$

where

$$\Box(s) = \begin{pmatrix} 0 \\ (2\mu+\lambda)\frac{1}{\sqrt{2(\gamma-1)}} \sum_{\mathbf{l}} \frac{1}{\mathrm{i}\lambda_{\mathbf{l}}} (\mathbf{U}_{\mathbf{l}} e^{-\mathrm{i}s\omega_{\mathbf{l}}} - \overline{\mathbf{U}}_{\mathbf{l}} e^{\mathrm{i}s\omega_{\mathbf{l}}}) \nabla_x \Delta_x \varphi_{\mathbf{l}} \\ \kappa \sqrt{\frac{\gamma-1}{2\gamma}} \sum_{\mathbf{l}} (\mathbf{U}_{\mathbf{l}} e^{-\mathrm{i}s\omega_{\mathbf{l}}} - \overline{\mathbf{U}}_{\mathbf{l}} e^{\mathrm{i}s\omega_{\mathbf{l}}}) \end{pmatrix} . \tag{3.4.0.83}$$

Applying Lemma (6) to $\overline{\mathcal{D}}^{\perp}_{\mathbf{k},1}(\mathbf{U})$, immediately we have

$$\overline{\mathcal{D}}^{\perp}_{\mathbf{k},1}(\mathbf{U}) = 0\,. \tag{3.4.0.84}$$

To evaluate $\overline{\mathcal{D}}^{\perp}_{\mathbf{k},2}$, we may apply Lemma (6) as before. After some tedious but straightforward calculations, we have

$$\begin{aligned}\overline{\mathcal{D}}^{\perp}_{\mathbf{k},2}(\mathbf{U}) = {} & \tfrac{\kappa(\gamma-1)^2}{2\gamma} \sum_{\substack{\mathbf{l} \\ \omega_{\mathbf{l}}=\omega_{\mathbf{k}}}} \mathbf{U}_{\mathbf{l}} \int_{\mathbb{T}^D} (\Delta_x \varphi_{\mathbf{l}}) \varphi_{\mathbf{k}}\, dx \\ & + \tfrac{2\mu+\lambda}{2} \sum_{\substack{\mathbf{l} \\ \omega_{\mathbf{l}}=\omega_{\mathbf{k}}}} \mathbf{U}_{\mathbf{l}} \tfrac{1}{\lambda_{\mathbf{k}}\lambda_{\mathbf{l}}} \int_{\mathbb{T}^D} \nabla_x \Delta_x \varphi_{\mathbf{l}} \cdot \nabla_x \varphi_{\mathbf{k}}\, dx\,.\end{aligned} \tag{3.4.0.85}$$

In view of the orthogonality of the eigenfunctions $\varphi_{\mathbf{k}}$ under the L^2 inner product on $\mathbb{T}^D$, we derive

$$\begin{aligned}\overline{\mathcal{D}}^{\perp}_{\mathbf{k},2}(\mathbf{U}) &= -\tfrac{\kappa(\gamma-1)^2}{2\gamma} \sum_{\substack{\mathbf{l} \\ \omega_{\mathbf{l}}=\omega_{\mathbf{k}}}} \mathbf{U}_{\mathbf{l}} \lambda_{\mathbf{l}}^2 \delta_{\mathbf{kl}} - \tfrac{2\mu+\lambda}{2} \sum_{\substack{\mathbf{l} \\ \omega_{\mathbf{l}}=\omega_{\mathbf{k}}}} \mathbf{U}_{\mathbf{l}} \tfrac{1}{\lambda_{\mathbf{k}}\lambda_{\mathbf{l}}} \lambda_{\mathbf{l}}^2 \lambda_{\mathbf{k}}^2 \delta_{\mathbf{kl}} \\ &= -\tfrac{1}{2}\left[\tfrac{\kappa(\gamma-1)^2}{\gamma} + 2\mu + \lambda\right] \mathbf{U}_{\mathbf{k}} \lambda_{\mathbf{k}}^2\,.\end{aligned} \tag{3.4.0.86}$$

The identity

$$\begin{aligned}\langle \Delta_x \Pi^{\perp}\mathbf{U}, \Phi_{\mathbf{k}}\rangle &= \langle \Pi^{\perp}\mathbf{U}, \Delta_x \Phi_{\mathbf{k}}\rangle \\ &= -\lambda_{\mathbf{k}}^2 \langle \Pi^{\perp}\mathbf{U}, \Phi_{\mathbf{k}}\rangle \\ &= -\lambda_{\mathbf{k}}^2 \mathbf{U}_{\mathbf{k}} \Phi_{\mathbf{k}}\,.\end{aligned} \tag{3.4.0.87}$$

yields that

$$\Pi^{\perp}\overline{\mathcal{D}}(\mathbf{U}) = \tfrac{1}{2}\left[\tfrac{\kappa(\gamma-1)^2}{\gamma} + 2\mu + \lambda\right] (\sum_{\mathbf{k}\in\mathbb{Z}^3} \langle \Delta_x \Pi^{\perp}\mathbf{U}, \Phi_{\mathbf{k}}\rangle \Phi_{\mathbf{k}} + \langle \Delta_x \Pi^{\perp}\mathbf{U}, \overline{\Phi}_{\mathbf{k}}\rangle \overline{\Phi}_{\mathbf{k}})\,. \tag{3.4.0.88}$$

Finally, we derive that

$$\Pi^{\perp}\overline{\mathcal{D}}(\mathbf{U}) = \tfrac{1}{2}\left[\tfrac{\kappa(\gamma-1)^2}{\gamma} + 2\mu + \lambda\right] \Delta_x \Pi^{\perp}\mathbf{U} \tag{3.4.0.89}$$

Combining this with the result in last section, we have a complete structure for the diffusion term in the averaged system :

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{s\mathcal{A}}\mathcal{D}(e^{-s\mathcal{A}}\mathbf{U})ds$$
$$= \begin{pmatrix} -\frac{\kappa(\gamma-1)}{\gamma}\Delta_x(\frac{\gamma-1}{\gamma}\mathrm{U}_4-\mathrm{U}_0) \\ \mu\Delta_x P\mathrm{U} \\ 0 \end{pmatrix} + \frac{1}{2}\left[\frac{\kappa(\gamma-1)^2}{\gamma}+2\mu+\lambda\right]\Delta_x \begin{pmatrix} \frac{\gamma-1}{\gamma}\mathrm{U}_4 \\ Q\mathrm{U} \\ \mathrm{U}_4 \end{pmatrix}. \tag{3.4.0.90}$$

Remark: The second order operator $\mathcal{D}$, coming from the diffusion term of the compressible Navier-Stokes system, is only partially parabolic. That is one of the difficulty for the equations of compressible model because the equation of continuity is just a transport equation, does not have dissipation. From our derivation, after taking time averaging, the second order term in the averaged system is strictly parabolic. Actually, this averaged diffusion term already appeared in Hoff-Zumbrun's work [40, 41]. They used the name artificial viscosity term, applied to the isentropic gas, no equation of energy involved. So our averaged system , when we ignore the nonlinearity, is a natural generalization of the Hoff-Zumbrun's so-called "linear effective artificial viscosity system" [40, 41].

Finally, we are ready to compute the convection term in the averaged system . In the last section, we derived its projection on the slow mode, the kernel of $\mathcal{A}$, and conclude that it is actually the convection term of Navier-Stokes system for an incompressible flow. We will see in this section that its behavior in the fast mode is quite complicated. It depends on a nontrivial resonant set. We continue to use the notation in the last section.

$$\Pi^{\perp}\overline{\mathcal{Q}}(\mathbf{U},\mathbf{U}) = 2Re\sum_{\mathbf{k}\in\mathbb{Z}^3}\overline{\mathcal{Q}}^{\perp}_{\mathbf{k}}(\mathbf{U},\mathbf{U})\Phi_{\mathbf{k}}\,, \tag{3.4.0.91}$$

where

$$\begin{aligned}\overline{\mathcal{Q}}^{\perp}_{\mathbf{k}}(\mathbf{U},\mathbf{U}) &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \langle e^{s\mathcal{A}}\mathcal{Q}(e^{-s\mathcal{A}}\mathbf{U},e^{-s\mathcal{A}}\mathbf{U},)\Phi_{\mathbf{k}}\rangle\,ds\\ &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \langle \mathcal{Q}(e^{-s\mathcal{A}}\mathbf{U},e^{-s\mathcal{A}}\mathbf{U},)e^{-\mathrm{i}s\omega_{\mathbf{k}}}\Phi_{\mathbf{k}}\rangle\,ds\,.\end{aligned} \tag{3.4.0.92}$$

After some complicated algebraic calculation, the fast oscillation term $\overline{\mathcal{Q}}^{\perp}_{\mathbf{k}}(\mathbf{U},\mathbf{U})$ consists the terms of the form

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{\mathrm{i}s(\alpha\omega_l+\beta\omega_m+\gamma\omega_{\mathbf{k}})}\phi(x,t)dt\,, \tag{3.4.0.93}$$

where α,β,γ take values 1 or -1. Apply the Lemma (6) as in last section, the only nontrivial contributions that survive the averaging process are the two-waves and three-waves resonances defined as

$$\begin{aligned}\mathrm{R} &= \{(\mathbf{l},\mathbf{m},\mathbf{k})\in\mathbb{Z}^3\times\mathbb{Z}^3\times\mathbb{Z}^3\ :\alpha\omega_{\mathbf{l}}+\beta\omega_{\mathbf{m}}+\gamma\omega_{\mathbf{k}}=0\}\,,\\ &= \mathrm{R}_{2r}\cup\mathrm{R}_{3r}\,,\end{aligned} \tag{3.4.0.94}$$

where R_{2r} is the 2-resonant set

$$\mathrm{R}_{2r} := \{(\mathbf{l},\mathbf{k})\in\mathbb{Z}^3\times\mathbb{Z}^3\ :\omega_{\mathbf{l}}=\omega_{\mathbf{k}}\}\,, \tag{3.4.0.95}$$

while R_{3r} is the 3-resonant set

$$\begin{aligned}\mathrm{R}_{3r}\ :&= \{(\mathbf{l},\mathbf{m},\mathbf{k})\in\mathbb{Z}^3\times\mathbb{Z}^3\times\mathbb{Z}^3\ :\alpha\omega_{\mathbf{l}}+\beta\omega_{\mathbf{m}}=\omega_{\mathbf{k}}\}\,,\\ &= \mathrm{R}_{3r,1}\cup\mathrm{R}_{3r,2}\cup\mathrm{R}_{3r,3}\,,\end{aligned} \tag{3.4.0.96}$$

where

$$
\begin{aligned}
\mathrm{R}_{3r,1} &:= \{(\mathbf{l},\mathbf{m},\mathbf{k}) \in \mathbb{Z}^3 \times \mathbb{Z}^3 \times \mathbb{Z}^3 \ : \omega_{\mathbf{l}} + \omega_{\mathbf{m}} = \omega_{\mathbf{k}}\}\,, \\
\mathrm{R}_{3r,2} &:= \{(\mathbf{l},\mathbf{m},\mathbf{k}) \in \mathbb{Z}^3 \times \mathbb{Z}^3 \times \mathbb{Z}^3 \ : \omega_{\mathbf{l}} - \omega_{\mathbf{m}} = \omega_{\mathbf{k}}\}\,, \\
\mathrm{R}_{3r,3} &:= \{(\mathbf{l},\mathbf{m},\mathbf{k}) \in \mathbb{Z}^3 \times \mathbb{Z}^3 \times \mathbb{Z}^3 \ : -\omega_{\mathbf{l}} + \omega_{\mathbf{m}} = \omega_{\mathbf{k}}\}\,.
\end{aligned}
\tag{3.4.0.97}
$$

After the time averaging, the resonant term $\overline{\mathcal{Q}}_{\mathbf{k}}^{\perp}(\mathbf{U},\mathbf{U})$ is the summation of the 2-wave and 3-wave resonant terms

$$
\overline{\mathcal{Q}}_{\mathbf{k}}^{\perp}(\mathbf{U},\mathbf{U}) = \mathcal{Q}_{2r,\mathbf{k}}^{\perp}(\mathbf{U},\mathbf{U}) + \mathcal{Q}_{3r,\mathbf{k}}^{\perp}(\mathbf{U},\mathbf{U})\,,
\tag{3.4.0.98}
$$

so

$$
\begin{aligned}
\Pi^{\perp}\overline{\mathcal{Q}}(\mathbf{U},\mathbf{U}) &= \mathcal{Q}_{2r}^{\perp}(\mathbf{U},\mathbf{U}) + \mathcal{Q}_{3r}^{\perp}(\mathbf{U},\mathbf{U}) \\
&= 2Re\sum_{\mathbf{k}\in\mathbb{Z}^3} \mathcal{Q}_{2r,\mathbf{k}}^{\perp}(\mathbf{U},\mathbf{U})\Phi_{\mathbf{k}} + 2Re\sum_{\mathbf{k}\in\mathbb{Z}^3} \mathcal{Q}_{3r,\mathbf{k}}^{\perp}(\mathbf{U},\mathbf{U})\Phi_{\mathbf{k}}\,,
\end{aligned}
\tag{3.4.0.99}
$$

where the 2-wave resonant term $\mathcal{Q}_{2r,\mathbf{k}}^{\perp}(\mathbf{U},\mathbf{U})$

$$
\begin{aligned}
\mathcal{Q}_{2r,\mathbf{k}}^{\perp}(\mathbf{U},\mathbf{U}) = c_1 &{\sum_{\mathbf{l}}}^{\prime,\mathbf{k}} \mathbf{U}_{\mathbf{l}} \int_{\mathbb{T}^D} \nabla_x \cdot [(P\mathrm{U})\varphi_{\mathbf{l}}]\varphi_{\mathbf{k}} d\mathbf{x} \\
+ c_2 &{\sum_{\mathbf{l}}}^{\prime,\mathbf{k}} \tfrac{1}{\mathrm{i}\lambda_{\mathbf{l}}} \mathbf{U}_{\mathbf{l}} \int_{\mathbb{T}^D} \nabla_x \cdot [\nabla_x\varphi_{\mathbf{l}}(\mathrm{U}_4 - \tfrac{\gamma}{\gamma-1}\mathrm{U}_0)]\varphi_{\mathbf{k}}\, dx \\
+ c_3 &{\sum_{\mathbf{l}}}^{\prime,\mathbf{k}} \tfrac{1}{\lambda_{\mathbf{l}}\lambda_{\mathbf{k}}} \mathbf{U}_{\mathbf{l}} \int_{\mathbb{T}^D} \nabla_x(P\mathrm{U}\cdot\nabla_x\varphi_{\mathbf{l}}) \cdot \nabla_x\varphi_{\mathbf{k}}\, dx \\
+ c_4 &{\sum_{\mathbf{l}}}^{\prime,\mathbf{k}} \tfrac{1}{\lambda_{\mathbf{l}}\lambda_{\mathbf{k}}} \mathbf{U}_{\mathbf{l}} \int_{\mathbb{T}^D} [\nabla_x \cdot (P\mathrm{U}\otimes\nabla_x\varphi_{\mathbf{l}} + \nabla_x\varphi_{\mathbf{l}}\otimes P\mathrm{U})] \cdot \nabla_x\varphi_{\mathbf{k}}\, dx\,,
\end{aligned}
\tag{3.4.0.100}
$$

and the 2-wave resonant term $\mathcal{Q}^{\perp}_{3r,\mathbf{k}}(\mathbf{U},\mathbf{U})$

$$\begin{aligned}\mathcal{Q}^{\perp}_{3r,\mathbf{k}}(\mathbf{U},\mathbf{U}) = c_5 \sum_{\mathbf{l},\mathbf{m}}^{\prime\prime,\mathbf{k}} \tfrac{1}{\mathrm{i}\lambda_{\mathbf{l}}} \mathbf{U}^{\alpha}_{\mathbf{l}} \mathbf{U}^{\beta}_{\mathbf{m}} \int_{\mathbb{T}^D} \nabla_x \cdot [(\nabla_x \varphi^{\alpha}_{\mathbf{l}}) \varphi^{\beta}_{\mathbf{m}}] \overline{\varphi}_{\mathbf{k}} \, dx \\ + c_6 \sum_{\mathbf{l},\mathbf{m}}^{\prime\prime,\mathbf{k}} \tfrac{1}{\mathrm{i}\lambda_{\mathbf{l}}\lambda_{\mathbf{m}}\lambda_{\mathbf{k}}} \mathbf{U}^{\alpha}_{\mathbf{l}} \mathbf{U}^{\beta}_{\mathbf{m}} \int_{\mathbb{T}^D} \nabla_x \cdot (\nabla_x \varphi^{\alpha}_{\mathbf{l}} \otimes \nabla_x \varphi^{\beta}_{\mathbf{m}}) \cdot \nabla_x \overline{\varphi}_{\mathbf{k}} \, dx \\ + c_7 \sum_{\mathbf{l},\mathbf{m}}^{\prime\prime,\mathbf{k}} \tfrac{1}{\mathrm{i}\lambda_{\mathbf{l}}\lambda_{\mathbf{m}}\lambda_{\mathbf{k}}} \mathbf{U}^{\alpha}_{\mathbf{l}} \mathbf{U}^{\beta}_{\mathbf{m}} \int_{\mathbb{T}^D} \nabla_x (\nabla_x \varphi^{\alpha}_{\mathbf{l}} \cdot \nabla_x \varphi^{\beta}_{\mathbf{m}}) \cdot \nabla_x \overline{\varphi}_{\mathbf{k}} \, dx \,,\end{aligned} \tag{3.4.0.101}$$

where α, β stand for $+$ or $-$, $\mathbf{U}^{+}_{\mathbf{l}}$ denotes $\mathbf{U}_{\mathbf{l}}$, while $\mathbf{U}^{-}_{\mathbf{l}}$ denotes $\overline{\mathbf{U}}_{\mathbf{l}}$, and

$$\begin{aligned} c_1 = \tfrac{\gamma-1}{2}\,, \quad c_2 = \tfrac{\sqrt{\gamma-1}}{2}\sqrt{\tfrac{\gamma-1}{\gamma}}\,, \quad c_3 = -\tfrac{\gamma-1}{2}\,, \quad c_4 = \tfrac{1}{2}\,, \\ c_5 = \tfrac{\sqrt{\gamma-1}}{2\sqrt{2}}\,, \quad c_6 = \tfrac{1}{2\sqrt{2(\gamma-1)}}\,, \quad c_7 = -\tfrac{\gamma-1}{4\sqrt{2(\gamma-1)}}\,. \end{aligned} \tag{3.4.0.102}$$

and $\sum_{\mathbf{l}}^{\prime,\mathbf{k}}$ and $\sum_{\mathbf{l},\mathbf{m}}^{\prime\prime,\mathbf{k}}$ denotes the summation over the 2-resonant set and 3-resonant respectively, i.e.,

$$\sum_{\mathbf{l}}^{\prime,\mathbf{k}} := \sum_{\substack{\mathbf{l}\in\mathbb{Z}^3 \\ (\mathbf{l},\mathbf{k})\in\mathrm{R}_{2r}}}\,, \qquad \sum_{\mathbf{l},\mathbf{m}}^{\prime\prime,\mathbf{k}} := \sum_{\substack{(\mathbf{l},\mathbf{m})\in\mathbb{Z}^3\times\mathbb{Z}^3 \\ (\mathbf{l},\mathbf{m},\mathbf{k})\in\mathrm{R}_{3r}}}\,. \tag{3.4.0.103}$$

Using the facts that $\lambda_{\mathbf{k}} = |\mathbf{k}|$ and $\varphi_{\mathbf{k}} = e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$, we can clarify the above expressions.

$$\begin{aligned}&\mathcal{Q}^{\perp}_{2r,\mathbf{k}}(\mathbf{U},\mathbf{U}) \\ &= \mathrm{i}\mathbf{k} \cdot \sum_{\mathbf{l}}^{\prime,\mathbf{k}} \left(c_2 \frac{\mathbf{l}}{|\mathbf{l}|} \int_{\mathbb{T}^D} (\mathrm{U}_4 - \tfrac{\gamma}{\gamma-1}\mathrm{U}_0) e^{\mathrm{i}(\mathbf{l}-\mathbf{k})\cdot\mathbf{x}} \, dx + \frac{\mathbf{l}}{|\mathbf{l}||\mathbf{k}|} \int_{\mathbb{T}^D} (\mathrm{PU}\cdot\mathbf{k}) e^{\mathrm{i}(\mathbf{l}-\mathbf{k})\cdot\mathbf{x}} \, dx \right) \mathbf{U}_{\mathbf{l}}\,,\end{aligned} \tag{3.4.0.104}$$

and

$$\mathcal{Q}^{\perp}_{3r,\mathbf{k}}(\mathbf{U},\mathbf{U}) = \mathrm{i}\mathbf{k} \cdot \sum_{\mathbf{l},\mathbf{m}}^{\prime\prime,\mathbf{k}} \left(c_5 \frac{\mathbf{l}}{|\mathbf{l}|} + c_6 \frac{\mathbf{l}(\mathbf{m}\cdot\mathbf{k})}{|\mathbf{l}||\mathbf{m}||\mathbf{k}|} + c_7 \frac{\mathbf{k}(\mathbf{l}\cdot\mathbf{m})}{|\mathbf{l}||\mathbf{m}||\mathbf{k}|} \right) \mathbf{U}^{\alpha}_{\mathbf{l}} \mathbf{U}^{\beta}_{\mathbf{m}}\,, \tag{3.4.0.105}$$

Remark: The summations $\sum_{\mathbf{l}}{}^{\prime,\mathbf{k}}$ and $\sum_{\mathbf{l},\mathbf{m}}{}^{\prime\prime,\mathbf{k}}$ above are still understood in the sense of (3.4.0.103). Considering $\lambda_{\mathbf{k}} = |\mathbf{k}|$ and the orthogonality of $\varphi_{\mathbf{k}} = e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$, we have more precise description of the 2-wave and 3-wave resonant sets:

$$
\begin{aligned}
&\mathrm{R}_{2r} := \{(\mathbf{l},\mathbf{k}) \in \mathbb{Z}^3 \times \mathbb{Z}^3 \ : |\mathbf{l}| = |\mathbf{k}|\}\,, \\
&\mathrm{R}_{3r} := \{(\mathbf{l},\mathbf{m},\mathbf{k}) \in \mathbb{Z}^3 \times \mathbb{Z}^3 \times \mathbb{Z}^3 \ : \alpha\mathbf{l} + \beta\mathbf{m} = \mathbf{k}\,, \alpha|\mathbf{l}| + \beta|\mathbf{m}| = |\mathbf{k}|\}\,.
\end{aligned}
\tag{3.4.0.106}
$$

In the rest of the chapter, the resonant sets will be understood in the above sense.

From the expression (3.4.0.104), $\overline{\mathcal{Q}}^{\perp}_{2r,\mathbf{k}}(\mathbf{U},\mathbf{U})$ can be represented as

$$
\mathcal{Q}^{\perp}_{2r,\mathbf{k}}(\mathbf{U},\mathbf{U}) = \mathrm{i}\mathbf{k} \cdot \sum_{\substack{\mathbf{l}\in\mathbb{Z}^3 \\ (\mathbf{l},\mathbf{k})\in \mathrm{R}_2}} \mathcal{Q}^{\perp}_{2r,\mathbf{k},\mathbf{l}}(\Pi\mathbf{U})\mathbf{U}_{\mathbf{l}}\,. \tag{3.4.0.107}
$$

the coefficients $\mathcal{Q}^{\perp}_{2r,\mathbf{k},\mathbf{l}}(\Pi\mathbf{U})$ are the integral terms in (3.4.0.104). These terms are real and quadratic, depending on $P\mathrm{U}$ and $\mathrm{U}_0 - \frac{\gamma-1}{\gamma}\mathrm{U}_4$, which as we formally derived in the last section, satisfy the incompressible Navier-Stokes equation. Noting that $\mathbf{U}_{\mathbf{l}}$ depends only on $\mathrm{Null}(\mathcal{A})^{\perp}$, $\mathcal{Q}^{\perp}_{2r,\mathbf{k},\mathbf{l}}(\Pi\mathbf{U})$ is the product of one term from $\mathrm{Null}(\mathcal{A})$, the other from $\mathrm{Null}(\mathcal{A})^{\perp}$. Physically $\mathcal{Q}^{\perp}_{2r,\mathbf{k}}(\mathbf{U},\mathbf{U})$ can be understood as nonlinear interactions between two waves, one from the incompressible mode, the other from the fast mode. So it is convenient to represent $\mathcal{Q}^{\perp}_{2r}(\mathbf{U},\mathbf{U})$ as

$$
\begin{aligned}
\mathcal{Q}^{\perp}_{2r}(\mathbf{U},\mathbf{U}) &= 2Re \sum_{\mathbf{k}\in\mathbb{Z}^3} \mathcal{Q}^{\perp}_{2r,\mathbf{k}}(\mathbf{U},\mathbf{U})\Phi_{\mathbf{k}}\,, \\
&= \nabla_x \cdot (\sum_{\mathbf{k}\in\mathbb{Z}^3} \hat{\mathcal{Q}}^{\flat}_{2r}(\mathbf{k}) e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}})\,, \\
&= \nabla_x \cdot \mathcal{Q}^{\flat}_{2r}(\Pi\mathbf{U}, \Pi^{\perp}\mathbf{U})\,.
\end{aligned}
\tag{3.4.0.108}
$$

where the Fourier coefficients $\hat{\mathcal{Q}}^{\flat}_{2r}(\mathbf{k})$ is given by

$$
\hat{\mathcal{Q}}^{\flat}_{2r}(\mathbf{k}) = \sum_{\substack{\mathbf{l} \\ (\mathbf{l},\mathbf{k})\in \mathrm{R}_2}} \mathcal{Q}^{\perp}_{2r,\mathbf{k},\mathbf{l}}(\Pi\mathbf{U})\mathbf{U}_{\mathbf{l}} \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{\gamma(\gamma-1)}} \\ \frac{1}{\sqrt{\gamma-1}}\frac{\mathbf{k}}{|\mathbf{k}|} \\ \frac{\gamma}{\sqrt{\gamma(\gamma-1)^3}} \end{pmatrix} - \overline{\mathcal{Q}}^{\perp}_{2r,-\mathbf{k},\mathbf{l}}(\Pi\mathbf{U})\overline{\mathbf{U}}_{\mathbf{l}} \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{\gamma(\gamma-1)}} \\ \frac{1}{\sqrt{\gamma-1}}\frac{-\mathbf{k}}{|\mathbf{k}|} \\ \frac{\gamma}{\sqrt{\gamma(\gamma-1)^3}} \end{pmatrix}\,.
\tag{3.4.0.109}
$$

here $\overline{\mathcal{Q}}^{\perp}_{2r,-\mathbf{k},\mathbf{l}}(\Pi\mathbf{U})$ means the complex conjugate of $\mathcal{Q}^{\perp}_{2r,-\mathbf{k},\mathbf{l}}(\Pi\mathbf{U})$.

The other terms $\mathcal{Q}^{\perp}_{3r,\mathbf{k}}(\mathbf{U},\mathbf{U})$ have of the form

$$\mathcal{Q}^{\perp}_{3r,\mathbf{k}}(\mathbf{U},\mathbf{U}) = \mathrm{i}\mathbf{k}\cdot \sum_{\substack{\mathbf{l},\mathbf{m}\\ (\mathbf{l},\mathbf{m},\mathbf{k})\in \mathrm{R}_3}} \mathcal{Q}^{\perp}_{3r,\mathbf{k},\mathbf{l},\mathbf{m}} \mathbf{U}^{\alpha}_{\mathbf{l}} \mathbf{U}^{\beta}_{m}\,, \tag{3.4.0.110}$$

where as before α and β stand for $+$ or $-$, $\mathbf{U}^{+}_{\mathbf{l}}$ denotes $\mathbf{U}_{\mathbf{l}}$, while $\mathbf{U}^{-}_{\mathbf{l}}$ denotes $\overline{\mathbf{U}}_{\mathbf{l}}$. These terms are quadratic in $\mathrm{Null}^{\perp}(\mathcal{A}.)$ The coefficients $\mathcal{Q}^{\perp}_{3r,\mathbf{k},\mathbf{l},\mathbf{m}}$ are terms in the bracket of (3.4.0.105) which do not depend on the incompressible mode. They describe the nonlinear three wave interactions in fast mode. The 3-wave terms $\mathcal{Q}^{\perp}_{3r}(\mathbf{U},\mathbf{U})$ can be represented as

$$\begin{aligned}\mathcal{Q}^{\perp}_{3r}(\mathbf{U},\mathbf{U}) &= 2Re\sum_{\mathbf{k}\in\mathbb{Z}^3} \mathcal{Q}^{\perp}_{3r,\mathbf{k}}(\mathbf{U},\mathbf{U})\Phi_{\mathbf{k}}\,,\\ &= \nabla_x\cdot\big(\sum_{\mathbf{k}\in\mathbb{Z}^3} \hat{\mathcal{Q}}^{\flat}_{3r}(\mathbf{k}) e^{\mathbf{i}\mathbf{k}\cdot\mathbf{x}}\big)\,,\\ &= \nabla_x\cdot \mathcal{Q}^{\flat}_{3r}(\Pi^{\perp}\mathbf{U},\Pi^{\perp}\mathbf{U})\,,\end{aligned} \tag{3.4.0.111}$$

where the Fourier coefficients $\hat{\mathcal{Q}}^{\flat}_{3r}(\mathbf{k})$ is given by

$$\hat{\mathcal{Q}}^{\flat}_{3r}(\mathbf{k}) = \sum_{\substack{\mathbf{l},\mathbf{m}\\ (\mathbf{l},\mathbf{m},\mathbf{k})\in \mathrm{R}_3}} \mathcal{Q}^{\perp}_{3r,\mathbf{k},\mathbf{l},\mathbf{m}} \mathbf{U}^{\alpha}_{\mathbf{l}} \mathbf{U}^{\beta}_{\mathbf{m}} \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{\gamma(\gamma-1)}} \\ \frac{1}{\sqrt{\gamma-1}}\frac{\mathbf{k}}{|\mathbf{k}|} \\ \frac{\gamma}{\sqrt{\gamma(\gamma-1)^3}} \end{pmatrix} - \overline{\mathcal{Q}}^{\perp}_{3r,-\mathbf{k},\mathbf{l},\mathbf{m}} \mathbf{U}^{-\alpha}_{\mathbf{l}} \mathbf{U}^{-\beta}_{\mathbf{m}} \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{\gamma(\gamma-1)}} \\ \frac{1}{\sqrt{\gamma-1}}\frac{-\mathbf{k}}{|\mathbf{k}|} \\ \frac{\gamma}{\sqrt{\gamma(\gamma-1)^3}} \end{pmatrix}. \tag{3.4.0.112}$$

Here $\overline{\mathcal{Q}}^{\perp}_{3r,-\mathbf{k},\mathbf{l},\mathbf{m}}$ is the complex conjugate of $\mathcal{Q}^{\perp}_{3r,-\mathbf{k},\mathbf{l},\mathbf{m}}$.

Now we are ready to describe completely (formally) the evolution on the fast mode. It is governed by the following equation:

$$\partial_t(\Pi^{\perp}\mathbf{U}) + \nabla_x\cdot \mathcal{Q}^{\flat}_{2r}(\Pi\mathbf{U},\Pi^{\perp}\mathbf{U}) + \nabla_x\cdot \mathcal{Q}^{\flat}_{3r}(\Pi^{\perp}\mathbf{U},\Pi^{\perp}\mathbf{U}) = \tilde{\mu}\Delta_x\Pi^{\perp}\mathbf{U}\,, \tag{3.4.0.113}$$

where the constant $\tilde{\mu}$ denotes

$$\tilde{\mu} = \frac{1}{2}\left[\frac{\kappa(\gamma-1)^2}{\gamma} + 2\mu + \lambda\right] . \qquad (3.4.0.114)$$

Together with the equation (3.3.0.76) in the slow mode we derived in the last section, we completely in the formal way describe the averaged system of the compressible Navier-Stokes system.

In the above equation

$$\begin{aligned} \mathcal{Q}_{2r}(\Pi\mathbf{U}, \Pi^{\perp}\mathbf{U}) &= \nabla_x \cdot \mathcal{Q}^{\flat}_{2r}(\Pi\mathbf{U}, \Pi^{\perp}\mathbf{U}) , \\ \mathcal{Q}_{3r}(\Pi^{\perp}\mathbf{U}, \Pi^{\perp}\mathbf{U}) &= \nabla_x \cdot \mathcal{Q}^{\flat}_{3r}(\Pi^{\perp}\mathbf{U}, \Pi^{\perp}\mathbf{U}) , \end{aligned} \qquad (3.4.0.115)$$

are nonlocal nonlinear terms with divergence form. So, in applications, say, when we are concerned with the conservation of the energy, it is more convenient to consider the evolution of its coefficients on $\text{Null}(\mathcal{A})^{\perp}$, i.e., $\mathbf{U}_{\mathbf{k}}(t)$.

Test the above equation against $\Phi_{\mathbf{k}}$, notice that $\mathbf{U}_{\mathbf{k}}(t) = \langle \mathbf{U}, \Phi_{\mathbf{k}} \rangle$, $\mathbf{U}_{\mathbf{k}}(t)$ are governed by the following dynamical system:

$$\begin{aligned} &\frac{d}{dt}\mathbf{U}_{\mathbf{k}}(t) + \tilde{\mu}\lambda_{\mathbf{k}}^2 \mathbf{U}_{\mathbf{k}}(t) + i\mathbf{k} \cdot \sum_{\substack{\mathbf{l}\in\mathbb{Z}^3 \\ (\mathbf{l},\mathbf{k})\in \mathrm{R}_2}} \mathcal{Q}^{\perp}_{2r,\mathbf{k},\mathbf{l}}(\Pi\mathbf{U})\mathbf{U}_{\mathbf{l}}(t) \\ &+ i\mathbf{k} \cdot \sum_{\substack{\mathbf{l},\mathbf{m}\in\mathbb{Z}^3 \\ (\mathbf{l},\mathbf{m},\mathbf{k})\in \mathrm{R}_3}} \mathcal{Q}^{\perp}_{3r,\mathbf{k},\mathbf{l},\mathbf{m}} \mathbf{U}^{\alpha}_{\mathbf{l}} \mathbf{U}^{\beta}_{m} = 0 . \end{aligned} \qquad (3.4.0.116)$$

Theorem 5: **(Formal Averaged System)** *The averaged system of the Navier-Stokes system for compressible gas dynamics*

$$\begin{aligned} \partial_t \mathbf{U} + \lim_{T\to\infty}\frac{1}{T}\int_0^T e^{s\mathcal{A}}\mathcal{Q}(e^{-s\mathcal{A}}\mathbf{U}, e^{-s\mathcal{A}}\mathbf{U})ds &= \lim_{T\to\infty}\frac{1}{T}\int_0^T e^{sT}\mathcal{D}(e^{-s\mathcal{A}}\mathbf{U})ds , \\ \mathbf{U}(0,x) &= \mathbf{U}^{in}(x) , \end{aligned} \qquad (3.4.0.117)$$

has the following form:

$$\partial_t \mathbf{U} + \nabla_x \cdot \begin{pmatrix} (\mathrm{PU})(\mathrm{U}_0 - \frac{\gamma-1}{\gamma}\mathrm{U}_4) \\ (\mathrm{PU}\otimes \mathrm{PU}) + pI \\ 0 \end{pmatrix} + \mathcal{Q}_{2r}(\Pi\mathbf{U}, \Pi^{\perp}\mathbf{U}) + \mathcal{Q}_{3r}(\Pi^{\perp}\mathbf{U}, \Pi^{\perp}\mathbf{U})$$

$$= \begin{pmatrix} -\frac{\kappa(\gamma-1)}{\gamma}\Delta_x(\frac{\gamma-1}{\gamma}\mathrm{U}_4 - \mathrm{U}_0) \\ \mu\Delta_x \mathrm{PU} \\ 0 \end{pmatrix} + \tilde{\mu}\Delta_x \begin{pmatrix} \frac{\gamma-1}{\gamma}\mathrm{U}_4 \\ \mathrm{QU} \\ \mathrm{U}_4 \end{pmatrix}, \tag{3.4.0.118}$$

with initial data

$$\mathbf{U}(0,x) = \mathbf{U}^{in}(x)\,, \tag{3.4.0.119}$$

and the nonlocal terms $\mathcal{Q}_{2r}(\Pi\mathbf{U}, \Pi^{\perp}\mathbf{U})$ *and* $\mathcal{Q}_{3r}(\Pi^{\perp}\mathbf{U}, \Pi^{\perp}\mathbf{U})$ *are defined in* (3.4.0.108) *and* (3.4.0.111) *respectively and have divergence form.*

The above averaged system can be understood as the equations satisfied by $\Pi\mathbf{U}$ and $\Pi^{\perp}\mathbf{U}$ respectively. $\Pi\mathbf{U}$ satisfies the incompressible Navier-Stokes system with initial data $\Pi\mathbf{U}^{in}$:

(Equation of $\Pi\mathbf{U}$)

$$\partial_t \begin{pmatrix} \mathrm{U}_0 - \frac{\gamma-1}{\gamma}\mathrm{U}_4 \\ P\mathrm{U} \\ 0 \end{pmatrix} + \nabla_x \cdot \begin{pmatrix} (P\mathrm{U})(\mathrm{U}_0 - \frac{\gamma-1}{\gamma}\mathrm{U}_4) \\ (P\mathrm{U}\otimes P\mathrm{U}) + pI \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\kappa(\gamma-1)}{\gamma}\Delta_x(\mathrm{U}_0 - \frac{\gamma-1}{\gamma}\mathrm{U}_4) \\ \mu\Delta_x P\mathrm{U} \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} \mathrm{U}_0(0,x) - \frac{\gamma-1}{\gamma}\mathrm{U}_4(0,x) \\ P\mathrm{U}(0,x) \\ 0 \end{pmatrix} = \begin{pmatrix} \mathrm{U}_0^{in}(x) - \frac{\gamma-1}{\gamma}\mathrm{U}_4^{in}(x) \\ P\mathrm{U}^{in}(x) \\ 0 \end{pmatrix}. \tag{3.4.0.120}$$

Use a solution of the above incompressible Navier-Stokes system to construct the equations for $\Pi^{\perp}\mathbf{U}$ which is a nonlinear system with two nonlocal terms:

(Equation of $\Pi^{\perp}\mathbf{U}$)

$$\begin{aligned}\partial_t(\Pi^{\perp}\mathbf{U}) + \nabla_x \cdot \mathcal{Q}^{\flat}_{2r}(\Pi\mathbf{U}, \Pi^{\perp}\mathbf{U}) + \nabla_x \cdot \mathcal{Q}^{\flat}_{3r}(\Pi^{\perp}\mathbf{U}, \Pi^{\perp}\mathbf{U}) = \tilde{\mu}\Delta_x \Pi^{\perp}\mathbf{U}\,, \\ \Pi^{\perp}\mathrm{U}(0,x) = \Pi^{\perp}\mathrm{U}^{in}(x)\,.\end{aligned} \tag{3.4.0.121}$$

Remark: As mentioned when considered the general hyperbolic-parabolic system with entropy, an important feature of the averaged system is that the projection on the slow mode is completely decoupled from that on the fast motion, so it can be solved separately. One solution is provided by Leray [47]. But the equation on the fast mode is coupled with slow equation. The coefficient of the nonlocal term $\mathcal{Q}^{\flat}_{2r}(\Pi\mathbf{U}, \Pi^{\perp}\mathbf{U})$ depends on $\Pi\mathbf{U}$. The similar phenomena appeared in many other problems which are related to the motion of fast oscillating waves [4, 5, 63, 50, 51, 68].

3.5 Global Weak Solutions to the Averaged System

In this section, we will state and proof the existence of weak solutions to the averaged system for the compressible Navier-Stokes system. This result is not new, because it is a special case of the global existence of averaged system for general hyperbolic-parabolic system with entropy which was analyzed in Chapter 2. For the compressible Navier-Stokes system, the structure condition stated in Chapter 2 is automatically satisfied. Further more, because the acoustic operator $\mathcal{A}$ is explicitly given for compressible Navier-Stokes system, we can project the averaged system onto Null($\mathcal{A}$) and Null($\mathcal{A}$)$^{\perp}$ respectively. As we derived in last sections, the structures on Null($\mathcal{A}$) and Null$(\mathcal{A})^\perp$ are significantly different. Thus, although the global weak solutions result for compressible Navier-Stokes system is not new, we still write it down in this section. It could be considered as an nontrivial example of the general theory of Chapter 2.

We first present some notation that is necessary for the statement and the proof of the global solution to the averaged system .

Let us define the Hilbert space $\mathbb{H}$

$$\mathbb{H} = \left\{ \mathbf{U} \in \mathbf{L}^2(d\mathbf{x}\,, \mathbb{C}\times\mathbb{C}^3\times\mathbb{C}) : \int_{\mathbb{T}^D} \mathbf{U}\, dx = 0 \right\} \,, \tag{3.5.0.122}$$

with norm $\|\mathbf{U}\|_{\mathbb{H}}^2 = \langle \mathbf{U}\,, \mathbf{U}\rangle$. It is easy to see that $\mathbb{H}$ is the closure of $\mathcal{H}$ under the norm $\langle\,,\rangle$.

Let us define two subspaces $\mathbb{H}_1$ and $\mathbb{H}_2$ as the closure of $\mathcal{H}_1$ and $\mathcal{H}_2$ in $\mathbb{H}$, respectively.

$$\begin{aligned} \mathbb{H}_1 &= \{\mathbf{U} \in \mathbb{H} : \mathcal{A}\mathbf{U} = 0\} \\ &= \{\mathbf{U} \in \mathbb{H} : \mathbf{U} = (\mathrm{U}_0, \mathrm{U}, 0)^T, \nabla_x\cdot \mathrm{U} = 0\} \\ &= \text{the closure of } \mathcal{H}_1 \text{ in } \mathbb{H}\,, \end{aligned} \tag{3.5.0.123}$$

and

$$\begin{aligned} \mathbb{H}_2 &= \{\mathbf{U} \in \mathbb{H} : \langle \mathbf{U}\,, \mathbf{V}\rangle = 0, \forall\, \mathbf{V} \in \mathbb{H}_1\} \\ &= \{\mathbf{U} \in \mathbb{H} : \mathbf{U} = (\mathrm{U}_0, \mathrm{U}, \mathrm{U}_4)^T, \mathrm{U}_4 = \tfrac{\gamma}{\gamma-1}\mathrm{U}_0, \mathrm{U} = \nabla_x\varphi, \quad \varphi\in \mathrm{L}^2, \int_{\mathbb{T}^3}\varphi = 0\} \\ &= \text{the closure of } \mathcal{H}_2 \text{ in } \mathbb{H}\,. \end{aligned} \tag{3.5.0.124}$$

Recall that $\mathcal{H}_1 = $ Null($\mathcal{A}$) and $\mathcal{H}_2 = $ Null$^\perp(\mathcal{A})$.

We need to define more function spaces

$$\begin{aligned}\mathcal{V}_1 &= \{\mathbf{U} \in \mathcal{H}_1 : \int_{\mathbb{T}^D} |\nabla_x \mathrm{U}_0|^2\, dx < \infty\,, \quad \int_{\mathbb{T}^D} |\nabla_x \mathrm{U}|^2\, dx < \infty\}\,,\\ \mathcal{V}_2 &= \{\mathbf{U} \in \mathcal{H}_2 : \int_{\mathbb{T}^D} |\nabla_x \mathrm{U}_0|^2\, dx < \infty\,, \quad \int_{\mathbb{T}^D} |\nabla_x \mathrm{U}|^2\, dx < \infty\}\,,\end{aligned} \tag{3.5.0.125}$$

and define a subspace $\mathcal{V}$ of $\mathcal{H}$ as the direct sum of $\mathcal{V}_1$ and $\mathcal{V}_2$, i.e.,

$$\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2\,. \tag{3.5.0.126}$$

Let us denote by $\mathbb{V}$ the closure of $\mathcal{V}$ in $\mathbb{H}$. Then there is a natural norm on $\mathbb{V}$ induced from that of $\mathbb{H}$,

$$\begin{aligned}\langle \mathbf{U}\,, \mathbf{U}\rangle_{\mathbb{V}} &= \langle \nabla_x \mathbf{U}\,, \nabla_x \mathbf{U}\rangle\\ &= \langle \nabla_x \Pi \mathbf{U}\,, \langle \nabla_x \Pi \mathbf{U}\rangle + \langle \nabla_x \Pi^{\perp} \mathbf{U}\,, \langle \nabla_x \Pi^{\perp} \mathbf{U}\rangle\,.\end{aligned} \tag{3.5.0.127}$$

We can define another norm on $\mathbb{V}$ which is more convenient in our application:

$$\langle \mathbf{U}\,, \mathbf{U}\rangle'_{\mathbb{V}} = \langle \nabla_x \Pi \mathbf{U}\,, \nabla_x \Pi \mathbf{U}\rangle + \langle \mathcal{A}\Pi^{\perp} \mathbf{U}\,, \mathcal{A}\Pi^{\perp} \mathbf{U}\rangle\,. \tag{3.5.0.128}$$

We claim that these two norms are equivalent.

Lemma 7: $\langle \mathbf{U}, \mathbf{U}\rangle_{\mathbb{V}}$ *and* $\langle \mathbf{U}, \mathbf{U}\rangle'_{\mathbb{V}}$ *are equivalent.*

Proof of Lemma:

$$\begin{aligned}\langle \nabla_x \Pi^{\perp} \mathbf{U}\,, \langle \nabla_x \Pi^{\perp} \mathbf{U}\rangle &= \tfrac{(\gamma-1)^3}{\gamma} \int_{\mathbb{T}^D} |\nabla_x \mathrm{U}_4|^2\, dx + (\gamma - 1) \int_{\mathbb{T}^D} |\nabla_x \mathrm{QU}|^2\, dx\,,\\ \langle \mathcal{A}\Pi^{\perp} \mathbf{U}, \mathcal{A}\Pi^{\perp} \mathbf{U}\rangle &= (\gamma-1)^3 \int_{\mathbb{T}^D} |\nabla_x \mathrm{U}_4|^2\, dx + \gamma(\gamma - 1) \int_{\mathbb{T}^D} |\nabla_x \cdot \mathrm{QU}|^2\, dx\,.\end{aligned} \tag{3.5.0.129}$$

Note that

$$\begin{aligned}\int_{\mathbb{T}^D} |\nabla_x \mathrm{QU}|^2\, dx &= \sum_{\mathbf{k}} \sum_{i,j=1}^{3} \left[\frac{\mathbf{k}_i \mathbf{k}_j}{|\mathbf{k}|^2} (\mathbf{k} \cdot \hat{\mathrm{U}})^2 \right]\\ &\le C \sum_{\mathbf{k}} \left[(\mathbf{k} \cdot \hat{\mathrm{U}})^2 \right] = C \int_{\mathbb{T}^D} |\nabla_x \cdot \mathrm{QU}|^2\, dx\,.\end{aligned} \tag{3.5.0.130}$$

The other direction of the inequality is obvious. We finish the proof. □

Now the norm on $\mathbb{V}$ can be written as:

$$\langle \mathbf{U}\,, \mathbf{U}\rangle_{\mathbb{V}} = \langle \nabla_x \Pi \mathbf{U}\,, \nabla_x \Pi \mathbf{U}\rangle + 2\gamma \sum_{\mathbf{k}} |\mathbf{k}|^2 |\mathbf{U}_{\mathbf{k}}(t)|^2\,. \tag{3.5.0.131}$$

Definition 4: **A weak solution** to the averaged system (3.4.0.118) is a vector-valued function $\mathbf{U}(t,x)$ that belongs to $\mathrm{C}([0,\infty); \text{w-}\mathbb{H}) \cap L^2([0,\infty);\mathbb{V})$, satisfies the following conditions:

1. $(\mathrm{U}_0 - \frac{\gamma-1}{\gamma}\mathrm{U}_4, \mathrm{PU})^T$ is a Leray solution to the incompressible Navier-Stokes system

$$\begin{aligned} \partial_t u + u\cdot\nabla_x u + \nabla_x p &= \mu\Delta_x u\,,\\ \partial_t\theta + u\cdot\nabla_x\theta &= \tfrac{\kappa(\gamma-1)}{\gamma}\Delta_x\theta\,, \\ \nabla_x\cdot u &= 0\,, \end{aligned} \tag{3.5.0.132}$$

with initial data

$$(\theta(0,x,)u(0,x))^T = (\mathrm{U}_0^{in}(x) - \tfrac{\gamma-1}{\gamma}\mathrm{U}_4^{in}(x,)\mathrm{PU}^{in}(x))^T\,, \tag{3.5.0.133}$$

2. $\Pi^{\perp}\mathbf{U} = (\frac{\gamma-1}{\gamma}\mathrm{U}_4, \mathrm{QU}, \mathrm{U}_4)^T$ satisfies the averaged system in the weak sense:

$$\begin{aligned} &\langle \Pi^{\perp}\mathbf{U}(t_2,)\chi\rangle - \langle \Pi^{\perp}\mathbf{U}(t_1,)\chi\rangle - \int_{t_1}^{t_2} \langle \mathcal{Q}^{\flat}_{2r}(\Pi\mathbf{U}, \Pi^{\perp}\mathbf{U},)\nabla_x\chi\rangle\,dt \\ &- \int_{t_1}^{t_2} \langle \mathcal{Q}^{\flat}_{3r}(\Pi^{\perp}\mathbf{U}, \Pi^{\perp}\mathbf{U},)\nabla_x\chi\rangle\,dt + \int_{t_1}^{t_2}\sum_{j=1}^{3}\langle\nabla_x\Pi^{\perp}\mathbf{U}, \nabla_x\chi\rangle\,dt = 0\,, \end{aligned} \tag{3.5.0.134}$$

for every $[t_1,t_2]\subset[0,\infty)$, and every $\chi\in\mathbb{V}\cap C^1(\mathbb{T}^D)$.

A weak solution to the averaged system can be understood in the following way: firstly, we solve the incompressible Navier-Stokes equations with the initial data

$\Pi\mathbf{U}^{in}$, the projection of the L^2 initial data on the slow mode Null($\mathcal{A}$). This is provided by Leray's celebrated theorem [47], left uniqueness and global regularity as outstanding wide open problems. Secondly, using Leray's solution gotten from the first step to construct weak solution to the orthogonal part of the averaged system . The main result of this paper is to show that the averaged system on the fast mode Null$(\mathcal{A})^\perp$ shares the similar structure with the incompressible Navier-Stokes equation, and a global weak solution for the L^2 initial data exists. It is an analogue of Leray's result for the Navier-Stokes equation. Because the 2-wave resonant term $\mathcal{Q}_{2r}(\Pi\mathbf{U}, \Pi^\perp\mathbf{U})$ of the averaged system on the fast mode (3.4.0.120) depends on the Leray's solution to (3.5.0.132), it is the best result we can expect.

Theorem 6: *Given a* $\mathbf{U}^{in} = (\mathrm{U}_0^{in}, \mathrm{U}^{in}, \mathrm{U}_4^{in})^T \in \mathbb{H}$, *there exists at least one* $\mathbf{U} \in \mathrm{C}([0,\infty)\,, \text{w-}\mathbb{H}) \cap L^2([0,\infty)\,, \mathbb{V})$ *that is a weak solution to the averaged system* $(3.4.0.118) - (3.4.0.119)$. *Moreover, for every* $t > 0$, $\mathbf{U}$ *satisfies the following dissipation inequality:*

$$\begin{aligned}
&\tfrac{\gamma}{2}\int_{\mathbb{T}^D} |\mathrm{U}_0 - \tfrac{\gamma-1}{\gamma}\mathrm{U}_4|^2\,dx + \tfrac{\gamma-1}{2}\int_{\mathbb{T}^D} |\mathrm{P}\mathrm{U}|^2\,dx \\
&\qquad + \kappa(\gamma-1)\int_0^T\int_{\mathbb{T}^D} |\nabla_x(\mathrm{U}_0 - \tfrac{\gamma-1}{\gamma}\mathrm{U}_4)|^2\,dxds + \mu(\gamma-1)\int_0^T\int_{\mathbb{T}^D} |\nabla_x \mathrm{P}\mathrm{U}|^2\,dxds \\
&\qquad \le \tfrac{\gamma}{2}\int_{\mathbb{T}^D} |\mathrm{U}_0^{in} - \tfrac{\gamma-1}{\gamma}\mathrm{U}_4^{in}|^2\,dx + \tfrac{\gamma-1}{2}\int_{\mathbb{T}^D} |\mathrm{P}\mathrm{U}^{in}|^2\,dx\,,
\end{aligned} \tag{3.5.0.135}$$

and

$$
\begin{aligned}
&\tfrac{(\gamma-1)^3}{2\gamma}\int_{\mathbb{T}^D}|\mathrm{U}_4|^2\,dx+\tfrac{\gamma-1}{2}\int_{\mathbb{T}^D}|\mathrm{QU}|^2\,dx\\
&\qquad+\tfrac{\tilde{\mu}(\gamma-1)^3}{\gamma}\int_0^T\int_{\mathbb{T}^D}|\nabla_x\mathrm{U}_4|^2\,dxds+\mu(\gamma-1)\int_0^T\int_{\mathbb{T}^D}|\nabla_x\mathrm{QU}|^2\,dxds\\
&\qquad\le\tfrac{(\gamma-1)^3}{2\gamma}\int_{\mathbb{T}^D}|\mathrm{U}_4^{in}|^2\,dx+\tfrac{\gamma-1}{2}\int_{\mathbb{T}^D}|\mathrm{QU}^{in}|^2\,dx\,.
\end{aligned}
\tag{3.5.0.136}
$$

proof: The averaged diffusion term is $\tilde{\mu}\Delta\Pi\mathbf{U}$ is strictly dissipated, then from the global existence theorem of Chapter 2, we immediately get the result. □

3.6 Higher Regularity of the Averaged System

3.6.1 Littlewood-Paley Decomposition

We will investigate the global well-posedness of the averaged system in the general Sobolev spaces H^s and the Besov spaces. It is convenient to introduce the Littlewood-Paley decomposition to characterize these spaces. First, we introduce a C^∞ symmetric function φ of one variable supported in $\{r\in\mathbb{R}\,,5/6\le|r|\le 12/5\}$ and such that

$$\sum_{q\in\mathbb{Z}}\varphi(2^{-q}r)=1\quad\text{for}\quad r\neq 0\,. \tag{3.6.1.1}$$

We then define the dyadic blocks as follows:

$$\Delta_q u\triangleq\sum_{\mathbf{k}\in\tilde{\mathbb{Z}^N}}\varphi(2^{-q}|\mathbf{k}|)\hat{u}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{3.6.1.2}$$

and the following low-frequency cut-off:

$$S_q u\triangleq\hat{u}_0+\sum_{p\le q-1}\Delta_p u\,. \tag{3.6.1.3}$$

Obviously, $\Delta_p u = 0$ for negative enough p (depending on the periodic box $\mathbb{T}^N$) and $u = \hat{u}_0 + \sum_1 \Delta_q u$ in $\mathcal{S}'(\mathbb{T}^N)$. The dyadic blocks $\Delta_q u$ are no longer orthogonal in $L^2(\mathbb{T}^N)$ but they still have some properties of quasi-orthogonality: with our choice of φ, we have

$$\Delta_k \Delta_q u \equiv 0 \quad \text{if} \quad |k-q| \geq 2 \quad \text{and} \quad \Delta_k (S_{q-1} u \Delta_q u) \equiv 0 \quad \text{if} \quad |k-q| \geq 4 \,. \tag{3.6.1.4}$$

The Sobolev spaces and Besov spaces can be characterized by means of Littlewood-Paley decomposition:

$$\begin{aligned} H^s(\mathbb{T}^N) &= \left\{ u \in \mathcal{S}'(\mathbb{T}^N) : ||u||_{H^s} \triangleq \left(|\hat{u}_0|^2 + \sum_{q\in\mathbb{Z}} 2^{2sq} ||\Delta_q u||^2_{L^2} \right)^{\frac{1}{2}} < +\infty \right\} , \\ B^s_{2,1}(\mathbb{T}^N) &= \left\{ u \in \mathcal{S}'(\mathbb{T}^N) : ||u||_{B^s} \triangleq |\hat{u}_0| + \sum_{q\in\mathbb{Z}} 2^{sq} ||\Delta_q u||_{L^2} < +\infty \right\} . \end{aligned} \tag{3.6.1.5}$$

In the rest of this chapter, for national simplicity, we use $B^s(\mathbb{T}^N)$ denote $B^s_{2,1}(\mathbb{T}^N)$.

3.6.2 Global Well-posedness

The results of this section closely follow the work of Masmoudi [58] and Danchin [18]. In the last section, we proved global existence in the sense of Leray to the averaged system :

$$\partial_t \mathbf{V} + \mathcal{Q}_1(\mathbf{u}, \mathbf{V}) + \mathcal{Q}_2(\mathbf{V}, \mathbf{V}) = \tilde{\mu} \Delta_x \mathbf{V} \,, \tag{3.6.2.1}$$

where $\mathbf{u} = (u, \theta)$ satisfy the incompressible Navier-Stokes system:

$$\begin{aligned} \partial_t u + u \cdot \nabla_x u + \nabla_x p &= \mu \Delta_x u \,, \\ \nabla_x \cdot u &= 0 \,, \\ \partial_t \theta + u \cdot \nabla_x \theta &= \tfrac{\kappa(\gamma-1)}{\gamma} \Delta_x \theta \,, \end{aligned} \tag{3.6.2.2}$$

with initial data

$$u(0,x) = PU^{in}\,,$$
$$\theta(0,x) = U_0^{in}(x) - \tfrac{\gamma-1}{\gamma}U_4^{in}\,. \tag{3.6.2.3}$$

To be convenient in calculation, we adapt the following notations used by Masmoudi in [58]. The higher regularity of the averaged system heavily depends on the precise geometric structure of the quadratic terms $\mathcal{Q}_1$ and $\mathcal{Q}_2$. To this end, we introduce the following orthogonal basis. (It is basically the same as before, just some light modification which makes the analysis of the structure of the resonant sets clearer.)

$$\Phi_{\mathbf{k}}^{+}(\mathbf{x}) = \tfrac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{\gamma(\gamma-1)}} \\ \mathrm{sg}(\mathbf{k})\frac{1}{\sqrt{\gamma-1}}\frac{\mathbf{k}}{|\mathbf{k}|} \\ \frac{\gamma}{\sqrt{\gamma(\gamma-1)^3}} \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{3.6.2.4}$$

and analysis of the structure of the resonant sets clearer.)

$$\Phi_{\mathbf{k}}^{-}(\mathbf{x}) = \tfrac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{\gamma(\gamma-1)}} \\ -\mathrm{sg}(\mathbf{k})\frac{1}{\sqrt{\gamma-1}}\frac{\mathbf{k}}{|\mathbf{k}|} \\ \frac{\gamma}{\sqrt{\gamma(\gamma-1)^3}} \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{3.6.2.5}$$

where the notation $\mathrm{sg}(\mathbf{k})$ stands for a generalized sign function on $\mathbb{R}^D \setminus \{0\}$: its value is 1 if and only if the first nonzero component of $\mathbf{k}$ is positive, -1 elsewhere. It is very easy to check that $\Phi_{\mathbf{k}}^{\alpha}(\mathbf{x})$ is an eigenvector of $\mathcal{A}$ with the eigenvalue $i\sqrt{\gamma}\alpha\mathrm{sg}(\mathbf{k})|\mathbf{k}| = i\omega_{\mathbf{k}}\alpha\mathrm{sg}(\mathbf{k})|\mathbf{k}|$, where α is $+$ or $-$.

For any vector $\mathbf{V} = \sum_{\alpha,\mathbf{k}} \mathbf{V}_{\mathbf{k}}^{\alpha}\Phi_{\mathbf{k}}^{\alpha}(\mathbf{x}) \in \mathrm{Null}(\mathcal{A})^{\perp}$, after the same calculations as before, we can obtain

$$\begin{aligned}\mathcal{Q}_1(\mathbf{u},\mathbf{V}) =& i\sum_{\delta,\mathbf{m}}\sum_{\substack{\mathbf{k}+\mathbf{l}=\mathbf{m}\\ \alpha\mathrm{sg}(\mathbf{k})=\delta\mathrm{sg}(\mathbf{m})\\ |\mathbf{k}|=|\mathbf{m}|}} \mathbf{V}^\alpha_\mathbf{k}\frac{(\hat{u}_\mathbf{l}\cdot\mathbf{m})(\mathbf{k}\cdot\mathbf{m})}{|\mathbf{k}||\mathbf{m}|}\Phi^\alpha_\mathbf{k}(\mathbf{x})\\ &+ i\tfrac{\sqrt{\gamma(\gamma-1)}}{2} + \sum_{\delta,\mathbf{m}}\sum_{\substack{\mathbf{k}+\mathbf{l}=\mathbf{m}\\ \alpha\mathrm{sg}(\mathbf{k})=\delta\mathrm{sg}(\mathbf{m})\\ |\mathbf{k}|=|\mathbf{m}|}} \mathbf{V}^\alpha_\mathbf{k}\alpha\mathrm{sg}(\mathbf{k})\hat{\theta}_\mathbf{l}\frac{\mathbf{k}\cdot\mathbf{m}}{\mathbf{k}}\Phi^\alpha_\mathbf{k}(\mathbf{x})\,.\end{aligned} \tag{3.6.2.6}$$

where $\hat{u}_\mathbf{l}$ and $\hat{\theta}_\mathbf{l}$ are the Fourier coefficients of u and θ. Note that u is a divergence-free vector, so $\hat{u}_\mathbf{l}\cdot\mathbf{l}$ for all $\mathbf{l}$.

The 3-waves interaction term $\mathcal{Q}_2(\mathbf{V},\mathbf{V})$ can be written as

$$\begin{aligned}\mathcal{Q}_2(\mathbf{V},\mathbf{V}) =& i\tfrac{1}{2\sqrt{2(\gamma-1)}}\sum_{\delta,\mathbf{m}}\sum_{\substack{\mathbf{k}+\mathbf{l}=\mathbf{m}\\ \alpha\mathrm{sg}(\mathbf{k})|\mathbf{k}|+\beta\mathrm{sg}(\mathbf{l})|\mathbf{l}|=\delta\mathrm{sg}(\mathbf{m})|\mathbf{m}|}} \mathbf{V}^\alpha_\mathbf{k}\mathbf{V}^\beta_\mathbf{l}\cdot[(\gamma-1)\alpha\mathrm{sg}(\mathbf{k})\frac{\mathbf{k}\cdot\mathbf{m}}{|\mathbf{k}|}\\ &+ \alpha\mathrm{sg}(\mathbf{k})\beta\mathrm{sg}(\mathbf{l})\gamma\mathrm{sg}(\mathbf{m})\frac{(\mathbf{k}\cdot\mathbf{l})(\mathbf{l}\cdot\mathbf{m})}{|\mathbf{k}||\mathbf{l}||\mathbf{m}|}\\ &+ \tfrac{\gamma-1}{2}\alpha\mathrm{sg}(\mathbf{k})\beta\mathrm{sg}(\mathbf{l})\gamma\mathrm{sg}(\mathbf{m})\frac{(\mathbf{k}\cdot\mathbf{m})(\mathbf{m}\cdot\mathbf{m})}{|\mathbf{k}||\mathbf{l}||\mathbf{m}|}]\Phi^\delta_\mathbf{m}(\mathbf{x})\,.\end{aligned} \tag{3.6.2.7}$$

We apply basically Masmoudi's arguments [58] to analyze the structure of the resonant set. The resonance condition between $((\mathbf{k},\alpha,)(\mathbf{l},\beta,)(\mathbf{m},\delta))$, namely $(\Phi^\alpha_\mathbf{k},\Phi^\beta_\mathbf{l},\Phi^\delta_\mathbf{m})$ is

$$\begin{aligned}\mathbf{k}+\mathbf{l}&=\mathbf{m}\,,\\ \alpha\mathrm{sg}(\mathbf{k})|\mathbf{k}|+\beta\mathrm{sg}(\mathbf{l})|\mathbf{l}|&=\delta\mathrm{sg}(\mathbf{m})|\mathbf{m}|\,.\end{aligned} \tag{3.6.2.8}$$

Hence, $2\mathbf{k}\cdot\mathbf{l} = 2\alpha\mathrm{sg}(\mathbf{k})\beta\mathrm{sg}(\mathbf{l})|\mathbf{k}||\mathbf{l}|$, which means that $\mathbf{k}$ is parallel to $\mathbf{l}$, so is parallel to $\mathbf{m}$, i.e., all the vectors in the 3-waves resonant set are parallel to each others. Rewriting this product again and using that $\mathbf{k}$ is parallel to $\mathbf{l}$, we deduce that $\mathbf{k}\cdot\mathbf{l} = \mathrm{sg}(\mathbf{k})\mathrm{sg}(\mathbf{l})$. This yields that we have $\alpha=\beta$ and then we can see easily that

(3.6.2.8) is equivalent to

$$\begin{aligned} &\mathbf{k}+\mathbf{l}=\mathbf{m}\,, \quad \mathrm{sg}(\mathbf{k})|\mathbf{k}|+\mathrm{sg}(\mathbf{l})|\mathbf{l}|=\mathrm{sg}(\mathbf{m})|\mathbf{m}|\,, \\ &\alpha=\beta=\delta\,. \end{aligned} \tag{3.6.2.9}$$

This means that we can only get resonances between the triplet $(\Phi^+_{\mathbf{k}}, \Phi^+_{\mathbf{l}}, \Phi^+_{\mathbf{m}})$ and $(\Phi^-_{\mathbf{k}}, \Phi^-_{\mathbf{l}}, \Phi^-_{\mathbf{m}})$ separately. As Masmoudi mentioned in [58], that is the reason why we have introduced the notation $\mathrm{sg}(\mathbf{k})$. Applying the above analysis of the resonant sets, we can rewrite $\mathcal{Q}_2(\mathbf{V},\mathbf{V})$ as

$$\mathcal{Q}_2(\mathbf{V},\mathbf{V}) = i\tfrac{1}{2\sqrt{2(\gamma-1)}}\sum_{\delta,\mathbf{m}}\sum_{\substack{\mathbf{k}+\mathbf{l}=\mathbf{m}\,,\alpha=+,-\\ \mathrm{sg}(\mathbf{k})|\mathbf{k}|+\mathrm{sg}(\mathbf{l})|\mathbf{l}|=\mathrm{sg}(\mathbf{m})|\mathbf{m}|}} \mathbf{V}^\alpha_{\mathbf{k}}\mathbf{V}^\alpha_{\mathbf{l}}\chi^\alpha_{\mathbf{klm}}\Phi^\alpha_{\mathbf{m}}(\mathbf{x})\,, \tag{3.6.2.10}$$

where

$$\chi^\alpha_{\mathbf{klm}} = \tfrac{\gamma-1}{2}\alpha\mathrm{sg}(\mathbf{m})|\mathbf{m}|\,. \tag{3.6.2.11}$$

It has very simple form.

To prove the global well-posedness of solutions to the averaged system , we need the following a priori estimates.

Lemma 8: *For all $s \geq 0$, and $\mathbf{V} \in \mathrm{Null}(\mathcal{A})^\perp$ we have*

$$\langle \mathcal{Q}_1(\mathbf{u},\mathbf{V})\,,\mathbf{V}\rangle_{H^s} = 0\,. \tag{3.6.2.12}$$

Proof: The proof employ the symmetry of $\mathcal{Q}_1(\mathbf{u},\mathbf{V})$. Noting that $\Phi^\alpha_{\mathbf{k}} = \overline{\Phi}^\alpha_{-\mathbf{k}}$ and that $\mathbf{V}^\alpha_{\mathbf{k}} = \overline{\mathbf{V}}^\alpha_{-\mathbf{k}}$

$$\begin{aligned} \langle \mathcal{Q}_1(\mathbf{u},\mathbf{V})\,,\mathbf{V}\rangle_{H^s} =& i\sum_{\delta,\mathbf{m}}\sum_{\substack{\mathbf{k}+\mathbf{l}=\mathbf{m}\\ \alpha\mathrm{sg}(\mathbf{k})=\delta\mathrm{sg}(\mathbf{m})\\ |\mathbf{k}|=|\mathbf{m}|}} \mathbf{V}^\alpha_{\mathbf{k}}\mathbf{V}^\delta_{-\mathbf{m}}|\mathbf{m}|^{2s} \\ &\cdot\left[\frac{(\hat{u}_{\mathbf{l}}\cdot\mathbf{m})(\mathbf{k}\cdot\mathbf{m})}{|\mathbf{k}||\mathbf{m}|} + \tfrac{\sqrt{\gamma(\gamma-1)}}{2}\alpha\mathrm{sg}(\mathbf{k})\hat{\theta}_{\mathbf{l}}\frac{\mathbf{k}\cdot\mathbf{m}}{\mathbf{k}}\right]\,. \end{aligned} \tag{3.6.2.13}$$

In above summation, exchange α and δ, and change $\mathbf{k}$ to $-\mathbf{m}$ and change $\mathbf{m}$ to $-\mathbf{k}$. Notice that under this changing index, the relation $\mathbf{l} = \mathbf{m} - \mathbf{k}$ is invariant, so

$$\begin{aligned}\langle \mathcal{Q}_1(\mathbf{u}, \mathbf{V}), \mathbf{V}\rangle_{H^s} = i \sum_{\delta, \mathbf{m}} \sum_{\substack{\mathbf{k}+\mathbf{l}=\mathbf{m} \\ \alpha \mathrm{sg}(\mathbf{k}) = \delta \mathrm{sg}(\mathbf{m}) \\ |\mathbf{k}|=|\mathbf{m}|}} & \mathbf{V}_{\mathbf{k}}^{\alpha} \mathbf{V}_{-\mathbf{m}}^{\delta} |\mathbf{m}|^{2s} \\ & \cdot \left[-\frac{(\hat{u}_{\mathbf{l}} \cdot \mathbf{k})(\mathbf{k} \cdot \mathbf{m})}{|\mathbf{k}||\mathbf{m}|} + \frac{\sqrt{\gamma(\gamma-1)}}{2} \alpha \mathrm{sg}(-\mathbf{m}) \hat{\theta}_{\mathbf{l}} \frac{\mathbf{k} \cdot \mathbf{m}}{\mathbf{k}} \right].\end{aligned} \tag{3.6.2.14}$$

Noting that u is divergence-free, so $\hat{u}_{\mathbf{l}} \cdot \mathbf{m} = \hat{u}_{\mathbf{l}} \cdot \mathbf{k}$, then

$$\langle \mathcal{Q}_1(\mathbf{u}, \mathbf{V}), \mathbf{V}\rangle_{H^s} = -\langle \mathcal{Q}_1(\mathbf{u}, \mathbf{V}), \mathbf{V}\rangle_{H^s}. \tag{3.6.2.15}$$

Then $\langle \mathcal{Q}_1(\mathbf{u}, \mathbf{V}), \mathbf{V}\rangle_{H^s} = 0$. We finish the proof of the lemma. □

We already know that $\langle \mathcal{Q}_2(\mathbf{V}, \mathbf{V}), \mathbf{V}\rangle = 0$, for general $s > 0$,

$$\langle \mathcal{Q}_2(\mathbf{V}, \mathbf{V}), \mathbf{V}\rangle_{H^s} \neq 0. \tag{3.6.2.16}$$

However, we have the following key estimate for $\mathcal{Q}_2$ which closely follows Masmoudi-Danchin's method. We extend to the case including energy equation.

Lemma 9: *For any* $\mathbf{V}, \mathbf{W} \in \mathrm{Null}(\mathcal{A})^{\perp}$,

$$\langle \mathcal{Q}_2(\mathbf{V}, \mathbf{W}), \mathbf{W}\rangle \lesssim ||\mathbf{W}||_{L^2} ||\mathbf{W}||_{B^{\frac{1}{2}}} ||\mathbf{V}||_{H^1}, \tag{3.6.2.17}$$

and

$$||\mathcal{Q}_2(\mathbf{V}, \mathbf{W})||_{H^s} \lesssim ||\mathbf{V}||_{B^{\frac{1}{2}}} ||\mathbf{W}||_{H^{s+1}} + ||\mathbf{W}||_{B^{\frac{1}{2}}} ||\mathbf{V}||_{H^{s+1}}. \tag{3.6.2.18}$$

We leave the proof of this technical lemma to the last of this section. Based on these a priori estimates, we can prove the following global well-posedness (in the sense that solution $\mathbf{u}$ to the incompressible Navier-Stokes equation is defined.)

Theorem 7: **(Global Well-Posedness in Sobolev Spaces H^s, $s \geq 0$)** *Let $s \geq 0$, $T \in (0, +\infty]$, $\mathbf{V}_0 \in H^s \cap \mathrm{Null}(\mathcal{A})$ and $\mathbf{u} \in C([0,T]; H^s) \cap L^2(0,T; H^{s+1})$ is a fixed solution to the incompressible Navier-Stokes equation. Then the averaged system (3.6.2.1) has a solution $\mathbf{V} \in C([0,T]; H^s) \cap L^2(0,T; H^{s+1})$ which remains in $\mathrm{Null}(\mathcal{A})^{\perp}$ for all later time, and uniqueness holds in $C([0,T]; L^2) \cap L^2(0,T; H^1)$. The solution $\mathbf{V}$ satisfies the following energy estimates:*

$$\tfrac{1}{2}||\mathbf{V}(t)||_{L^2}^2 + \tilde{\mu} \int_0^T ||\nabla_x \mathbf{V}(\tau)||_{L^2}^2 \, d\tau \leq \tfrac{1}{2}||\mathbf{V}_0||_{L^2}^2 \,, \tag{3.6.2.19}$$

and

$$||\mathbf{V}(t)||_{H^s}^2 + \tilde{\mu} \int_0^T ||\nabla_x \mathbf{V}||_{H^{s+1}}^2 \leq ||\mathbf{V}_0||_{H^s}^2 \exp\big(C \tfrac{||\mathbf{V}_0||_{L^2}^2}{\tilde{\mu}^2}\big) \,. \tag{3.6.2.20}$$

Proof of the theorem: Given the above technical lemma, the proof of the theorem is standard. We first prove the a priori estimates, i.e, any solutions $\mathbf{V} \in C([0,T]; H^s) \cap L^2(0,T; H^{s+1})$ satisfies the energy estimates (3.6.2.19) and (3.6.2.20). Take the H^s inner product of the averaged system with $\mathbf{V}$. Since, according to (3.6.2.12), $\langle \mathcal{Q}_1(\mathbf{u}, \mathbf{V}) \,, \mathbf{V} \rangle_{H^s} = 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}||\mathbf{V}||_{H^s}^2 + \tilde{\mu}||\nabla_x \mathbf{V}||_{H^s}^2 + \langle \mathcal{Q}_2(\mathbf{V}, \mathbf{V}) \,, \mathbf{V} \rangle_{H^s} = 0 \,. \tag{3.6.2.21}$$

If $s = 0$, the last term vanishes, so one time integration yields (3.6.2.19).

When $s > 0$, the inequality (3.6.2.18) and Young inequality and embedding $H^1 \hookrightarrow B^{\frac{1}{2}}$ yield

$$|\mathcal{Q}_2(\mathbf{V}, \mathbf{V}) \,, \mathbf{V} \rangle_{H^s}| \leq \tfrac{\tilde{\mu}}{2}||\mathbf{V}||_{H^{s+1}}^2 + \tfrac{C}{\tilde{\mu}}||\mathbf{V}||_{H^1}^2 ||\mathbf{V}||_{H^s}^2 \,. \tag{3.6.2.22}$$

Plug this above inequality into (3.6.2.21) and take integration in time, we get

$$\tfrac{1}{2}||\mathbf{V}(t)||_{H^s}^2 + \tfrac{\tilde{\mu}}{2}\int_0^T ||\nabla_x \mathbf{V}(\tau)||_{H^{s+1}}^2 \, d\tau \le \tfrac{1}{2}||\mathbf{V}_0||_{H^s}^2 + \tfrac{C}{\tilde{\mu}}\int_0^T ||\mathbf{V}||_{H^1}^2 ||\mathbf{V}||_{H^s}^2 \, d\tau \, . \tag{3.6.2.23}$$

The Gronwell inequality yields

$$\tfrac{1}{2}||\mathbf{V}(t)||_{H^s}^2 + \tfrac{\tilde{\mu}}{2}\int_0^T ||\nabla_x \mathbf{V}(\tau)||_{H^{s+1}}^2 \, d\tau \le \tfrac{1}{2}||\mathbf{V}_0||_{H^s}^2 e^{\frac{C}{\tilde{\mu}}\int_0^T ||\mathbf{V}(\tau)||_{H^1}^2 \, d\tau} \, . \tag{3.6.2.24}$$

Once the a priori estimates (3.6.2.19) and (3.6.2.20) have been proved, we can use the classical type of regularization, for instance, one can use a Galerkin approximation method as we did in the last section when we proved the global Leray-type solution, to get the existence of a solution to the averaged system (3.6.2.1) in $C([0,T]; H^s) \cap L^2(0,T; H^{s+1})$.

Let us now consider the uniqueness of the weak solutions to averaged system in $C([0,T]; L^2) \cap L^2(0,T; H^1)$. The property is not known for the incompressible Navier-Stokes equations for dimension $D \ge 3$. This means weak solutions to the averaged system in $\text{Null}(\mathcal{A})^{\perp}$ has better properties.

uniqueness of weak solutions: Let $\mathbf{V}_1$ and $\mathbf{V}_2$ be two weak solutions to the averaged system (3.6.2.1) in $C([0,T]; L^2) \cap L^2(0,T; H^1)$. Then $\delta\mathbf{V} = \mathbf{V}_1 - \mathbf{V}_2$ satisfies

$$\partial_t \delta\mathbf{V} - \tilde{\mu}\Delta\delta\mathbf{V} + \mathcal{Q}_1(\mathbf{u}, \delta\mathbf{V}) = -\mathcal{Q}_2(\mathbf{V}_1 + \mathbf{V}_2, \delta\mathbf{V}) \, . \tag{3.6.2.25}$$

Take inner product for the above equation with $\delta\mathbf{V}$, and notice the identity for $\mathcal{Q}_1$, we obtain

$$\frac{1}{2}\frac{d}{dt}||\delta\mathbf{V}||_{L^2}^2 + \tilde{\mu}||\nabla_x \delta\mathbf{V}||_{L^2}^2 = \langle \mathcal{Q}_2(\mathbf{V}_1 + \mathbf{V}_2, \delta\mathbf{V}) \, , \delta\mathbf{V} \rangle \, . \tag{3.6.2.26}$$

Now apply the inequality (3.6.2.17) and Young inequality, we have

$$|\langle \mathcal{Q}_2(\mathbf{V}_1 + \mathbf{V}_2, \delta\mathbf{V})\,, \delta\mathbf{V}\rangle| \leq \tfrac{C}{\tilde{\mu}}(||\mathbf{V}_1 + \mathbf{V}_2||_{H^1})^2||\delta\mathbf{V}||_{L^2}^2 + \tilde{\mu}||\nabla_x\delta\mathbf{V}||_{L^2}^2\,. \quad (3.6.2.27)$$

Then the Gronwell inequality ensures that $\delta\mathbf{V} \equiv 0$. Thus we prove the uniqueness.

□

Now we want to prove the estimates for $\mathcal{Q}_2$. In order to understand more the structure of $\mathcal{Q}_2$,we introduce, following [58], the set $\mathcal{P}$ of prime vectores $\mathbf{p}$, where $\mathbf{p} \in \mathbb{Z}^N$ is such that the N components of $\mathbf{p}$ are prime in their set. This is equivalent to saying that there dose not exit any couple $(n, \mathbf{q}) \in \mathbb{N} \times \mathbb{Z}^N$ such that $\mathbf{q} = n\mathbf{p}$ and $n \geq 2$.

For $\mathbf{V} = \sum\limits_{\alpha,\mathbf{k}} \mathrm{V}_{\mathbf{k}}^{\alpha}\Phi_{\mathbf{k}}^{\alpha}(\mathbf{x})$, a prime vector $\mathbf{p} \in \mathcal{P}$, and $\mathbf{x} \in \mathbb{T}^N$, we define the vector

$$\mathbf{V}_{\mathbf{p}}(\mathbf{x}) \triangleq \sum_{k\in\mathbb{Z}^*,\mathbf{k}=k\mathbf{p}} \mathrm{V}_{k\mathbf{p}}^{\mathrm{sg}(\mathbf{p})}\Phi_{k\mathbf{p}}^{\mathrm{sg}(\mathbf{p})}\,. \quad (3.6.2.28)$$

we can associate to the above vector value function the following real value function defined on the 1-D torus $\mathbb{T}^1$

$$v_{\mathbf{p}}(z) \triangleq \sum_{k\in\mathbb{Z}^*,\mathbf{k}=k\mathbf{p}} \mathrm{V}_{k\mathbf{p}}^{\mathrm{sg}(\mathbf{p})}e^{ikz}\,. \quad (3.6.2.29)$$

We notice that both $\mathbf{V}_{\mathbf{p}}(\mathbf{x})$ and $v_{\mathbf{p}}(z)$ are real. Indeed, this is a consequence of the fact that $\mathbf{V}$ is real. Moreover, we remark that $\mathrm{sg}(\mathbf{p})\mathrm{sg}(\mathbf{k})|\mathbf{k}| = |\mathbf{p}|k$ and for all $s \in \mathbb{R}$, we have

$$||\mathbf{V}_{\mathbf{p}}||_{H^s(\mathbb{T}^N)} = |\mathbf{p}|^s||v_{\mathbf{p}}||_{H^s(\mathbb{T}^1)}\,. \quad (3.6.2.30)$$

Using the set of prime vectors, any vector $\mathbf{V}(\mathbf{x}) = \sum\limits_{\alpha,\mathbf{k}} \mathrm{V}_{\mathbf{k}}^{\alpha}\Phi_{\mathbf{k}}^{\alpha}(\mathbf{x})$ can be represented as

$$\mathbf{V}(\mathbf{x}) = \sum_{\mathbf{p}\in\mathcal{P}} \mathbf{V}_{\mathbf{p}}(\mathbf{x})\,. \quad (3.6.2.31)$$

Vectors $\mathbf{V}'_{\mathbf{p}}s$ have a very good property which is described in the following simple lemma

Lemma 10: *For all* $\mathbf{p}\,,\mathbf{q}\in\mathcal{P}$*, if* $\mathbf{p}\neq\mathbf{q}$*, then*

$$\mathcal{Q}_2(\mathbf{V}_{\mathbf{p}}\,,\mathbf{V}_{\mathbf{q}})=0\,. \tag{3.6.2.32}$$

Proof: From the simple form (3.6.2.10) of $\mathcal{Q}_2$

$$\mathcal{Q}_2(\mathbf{V}_{\mathbf{p}}\,,\mathbf{V}_{\mathbf{q}})=i\tfrac{\sqrt{\gamma-1}}{4\sqrt{2}}\sum_{\delta,\mathbf{m}}\sum_{\substack{k\mathbf{p}+k'\mathbf{q}=\mathbf{m}\,,\alpha=+,-\\ k\mathrm{sg}(\mathbf{p})|\mathbf{k}|+k'\mathrm{sg}(\mathbf{q})|\mathbf{l}|=\mathrm{sg}(\mathbf{m})|\mathbf{m}|}}\mathrm{V}_{k\mathbf{p}}^{\mathrm{sg}(\mathbf{p})}\mathrm{V}_{k'\mathbf{q}}^{\mathrm{sg}(\mathbf{q})}\alpha\mathrm{sg}(\mathbf{m})|\mathbf{m}|\mathbf{V}_{\mathbf{m}}^{\alpha}\,. \tag{3.6.2.33}$$

In the resonant relation, $k\mathbf{p}+k'\mathbf{q}=\mathbf{m}$, but we know $\mathbf{p}$ and $\mathbf{q}$ are parallel, and they are prime vectors, so the only possible case is $\mathbf{p}=\mathbf{q}$.

Applying this last lemma, for any $\mathbf{V}$ and $\mathbf{W}$,

$$\begin{aligned}\mathcal{Q}_2(\mathbf{V}\,,\mathbf{W})&=\sum_{\mathbf{p}\in\mathcal{P}}\mathcal{Q}_2(\mathbf{V}_{\mathbf{p}}\,,\mathbf{W}_{\mathbf{p}})\\&=i\tfrac{\sqrt{\gamma-1}}{4\sqrt{2}}\sum_{\mathbf{p}\in\mathcal{P}}\sum_{k''\neq0}|\mathbf{p}|k''\Phi_{k''\mathbf{p}}^{\mathrm{sg}(\mathbf{p})}\sum_{k+k'=k''}\mathrm{V}_{k\mathbf{p}}^{\mathrm{sg}(\mathbf{p})}\mathrm{W}_{k'\mathbf{p}}^{\mathrm{sg}(\mathbf{p})}\,.\end{aligned} \tag{3.6.2.34}$$

Then

$$\langle\mathcal{Q}_2(\mathbf{V}\,,\mathbf{W})\,,\mathbf{W}\rangle=\tfrac{\sqrt{\gamma-1}}{4\sqrt{2}}\sum_{\mathbf{p}\in\mathcal{P}}\sum_{k''\neq0}|\mathbf{p}|k''\sum_{k+k'=k''}\mathrm{V}_{k\mathbf{p}}^{\mathrm{sg}(\mathbf{p})}\mathrm{W}_{k'\mathbf{p}}^{\mathrm{sg}(\mathbf{p})}\mathrm{W}_{-k''\mathbf{p}}^{\mathrm{sg}(\mathbf{p})}\,. \tag{3.6.2.35}$$

Apply the Parserval identity in $L^2(\mathbb{T}^1)$, we obtain

$$\begin{aligned}\langle\mathcal{Q}_2(\mathbf{V}\,,\mathbf{W})\,,\mathbf{W}\rangle&=\tfrac{\sqrt{\gamma-1}}{4\sqrt{2}}\sum_{\mathbf{p}\in\mathcal{P}}|\mathbf{p}|\left(\partial_z(v_{\mathbf{p}}w_{\mathbf{p}})\,,w_{\mathbf{p}}\right)_{L^2(\mathbb{T}^1)}\,,\\&=\tfrac{\sqrt{\gamma-1}}{8\sqrt{2}}\sum_{\mathbf{p}\in\mathcal{P}}|\mathbf{p}|\int_{\mathbb{T}^1}(w_{\mathbf{p}})^2\partial_z v_{\mathbf{p}}\,dz\,.\end{aligned} \tag{3.6.2.36}$$

From above identity, it is obvious that take $\mathbf{W}=\mathbf{V}$, then

$$\langle\mathcal{Q}_2(\mathbf{V}\,,\mathbf{V})\,,\mathbf{V}\rangle=0\,. \tag{3.6.2.37}$$

which we already know. Otherwise, we obtain

$$\begin{aligned}\int_{\mathbb{T}^1} (w_{\mathbf{p}})^2 \partial_z v_{\mathbf{p}}\, dz &\le ||\partial_z v_{\mathbf{p}}||_{L^2} ||w_{\mathbf{p}}||_{L^2} ||w_{\mathbf{p}}||_{L^\infty} \,, \\ &\le ||\partial_z v_{\mathbf{p}}||_{L^2} ||w_{\mathbf{p}}||_{L^2} ||w_{\mathbf{p}}||_{B^{\frac{1}{2}}} \,.\end{aligned} \tag{3.6.2.38}$$

Here we used the embedding $B^{\frac{1}{2}}(\mathbb{T}^1) \hookrightarrow L^\infty(\mathbb{T}^1)$. To get the estimates on $\mathbf{V}$ and $\mathbf{W}$, we need the lemma (11) which will be proved in the last. Plug (3.6.2.38) in (3.6.2.36) and applying the lemma (11), we obtain that

$$\begin{aligned}|\langle \mathcal{Q}_2(\mathbf{V}, \mathbf{W})\,, \mathbf{W}\rangle| &\lesssim \sum_{\mathbf{p}\in\mathcal{P}} |\mathbf{p}| ||v_{\mathbf{p}}||_{H^1} ||w_{\mathbf{p}}||_{L^2} ||w_{\mathbf{p}}||_{L^\infty} \,, \\ &\lesssim ||\mathbf{V}||_{H^1} ||\mathbf{W}||_{L^2} \sup_{\mathbf{p}\in\mathcal{P}} ||w_{\mathbf{p}}||_{B^{\frac{1}{2}}} \,,\end{aligned} \tag{3.6.2.39}$$

Thus, we proved the fist inequality (3.6.2.17) for $\mathcal{Q}_2$ in the lemma (9).

On the other hand, we have

$$\begin{aligned}||\mathcal{Q}_2(\mathbf{V}, \mathbf{W})||^2_{H^s} &= \tfrac{\gamma-1}{32} \sum_{\mathbf{p}\in\mathcal{P}} \sum_{k''} |k''\mathbf{p}|^{2s} \| \sum_{k+k'=k''} \mathrm{V}^{\mathrm{sg}(\mathbf{p})}_{k\mathbf{p}} \mathrm{W}^{\mathrm{sg}(\mathbf{p})}_{k'\mathbf{p}} \|^2 \,, \\ &= \tfrac{\gamma-1}{32} \sum_{\mathbf{p}\in\mathcal{P}} |\mathbf{p}|^{2s} ||v_{\mathbf{p}} w_{\mathbf{p}}||^2_{H^{s+1}(\mathbb{T}^1)} \,.\end{aligned} \tag{3.6.2.40}$$

Notice that

$$||v_{\mathbf{p}} w_{\mathbf{p}}||_{H^s(\mathbb{T}^1)} \le ||v_{\mathbf{p}}||_{L^\infty(\mathbb{T}^1)} ||w_{\mathbf{p}}||_{H^s(\mathbb{T}^1)} + ||w_{\mathbf{p}}||_{L^\infty(\mathbb{T}^1)} ||v_{\mathbf{p}}||_{H^s(\mathbb{T}^1)} \,. \tag{3.6.2.41}$$

Using the embedding $B^{\frac{1}{2}}(\mathbb{T}^1) \hookrightarrow L^\infty(\mathbb{T}^1)$, and again using the lemma (11) on the Besov spaces, we conclude the proof of the second inequality (3.6.2.18) in (9). □

Finally, we state and prove the following technical lemma on the Besov space.

Lemma 11: *For any $s \in \mathbb{R}$, we have the following inequality on the Besov spaces:*

$$\sum_{\mathbf{p}\in\mathcal{P}} |\mathbf{p}|^{2s} ||v_{\mathbf{p}}||^2_{B^s} \le ||\mathbf{V}||^2_{B^s} \,. \tag{3.6.2.42}$$

Proof: Compute the left hand side using the Littlewood-Paley decomposition:

$$\begin{aligned}
&\left(\sum_{\mathbf{p}\in\mathcal{P}} |\mathbf{p}|^{2s}||v_{\mathbf{p}}||^2_{B^s}\right)^{\frac{1}{2}} \\
&= \left[\sum_{\mathbf{p}\in\mathcal{P}} \left(\sum_{q\in\mathbb{Z}} 2^{(q+\log_2|\mathbf{p}|)s} \left(\sum_{k\in\mathbb{N}^*} |\varphi(2^{-(q+\log_2|\mathbf{p}|)}|k\mathbf{p}|)\mathrm{V}^{\mathrm{sg}(\mathbf{p})}_{k\mathbf{p}}|^2\right)^{\frac{1}{2}}\right)^2\right]^{\frac{1}{2}} .
\end{aligned} \tag{3.6.2.43}$$

Denoting $\psi = \mathbf{1}_{[\frac{1}{2},4]}$, we have

$$\begin{aligned}
&\left(\sum_{\mathbf{p}\in\mathcal{P}} |\mathbf{p}|^{2s}||v_{\mathbf{p}}||^2_{B^s}\right)^{\frac{1}{2}} \\
&\leq 2^{|s|}\left(\sum_{\mathbf{p}\in\mathcal{P}} |\mathbf{p}|^{2s}||v_{\mathbf{p}}||^2_{B^s}\right)^{\frac{1}{2}} \\
&\leq \left[\sum_{\mathbf{p}\in\mathcal{P}} \left(\sum_{q\in\mathbb{Z}} 2^{(q+[\log_2|\mathbf{p}|])s} \left(\sum_{k\in\mathbb{N}^*} |\varphi(2^{-(q+[\log_2|\mathbf{p}|])}|k\mathbf{p}|)\mathrm{V}^{\mathrm{sg}(\mathbf{p})}_{k\mathbf{p}}|^2\right)^{\frac{1}{2}}\right)^2\right]^{\frac{1}{2}} , \\
&\leq 2^{|s|}\left[\sum_{\mathbf{p}\in\mathcal{P}} \left(\sum_{q\in\mathbb{Z}} 2^{qs} \left(\sum_{k\in\mathbb{N}^*} |\psi(2^{-q}|k\mathbf{q}|)\mathrm{V}^{\mathrm{sg}(\mathbf{p})}_{k\mathbf{p}}|^2\right)^{\frac{1}{2}}\right)^2\right]^{\frac{1}{2}} .
\end{aligned} \tag{3.6.2.44}$$

Now apply Minkowski inequality

$$\left(\sum_{\mathbf{p}\in\mathcal{P}}\left(\sum_{q\in\mathbb{Z}} a_{\mathbf{p},q}\right)^2\right)^{\frac{1}{2}} \leq \sum_{q\in\mathbb{Z}}\left(\sum_{\mathbf{p}\in\mathcal{P}} a^2_{\mathbf{p},q}\right)^{\frac{1}{2}} \tag{3.6.2.45}$$

to (3.6.2.44) with

$$a_{\mathbf{p},q} \triangleq 2^{qs}\left(\sum_{k\in\mathbb{N}^*} |\psi(2^{-q}|k\mathbf{q}|)\mathrm{V}^{\mathrm{sg}(\mathbf{p})}_{k\mathbf{p}}|^2\right)^{\frac{1}{2}} . \tag{3.6.2.46}$$

We obtain that

$$\begin{aligned}
\left(\sum_{\mathbf{p}\in\mathcal{P}} |\mathbf{p}|^{2s}||v_{\mathbf{p}}||^2_{B^s}\right)^{\frac{1}{2}} &\lesssim \sum_{q\in\mathbb{Z}} 2^{qs}\left(\sum_{\mathbf{p}\in\mathcal{P}}\sum_{k\in\mathbb{N}^*} \psi(2^{-q}|k\mathbf{q}|)\mathrm{V}^{\mathrm{sg}(\mathbf{p})}_{k\mathbf{p}}|^2\right)^{\frac{1}{2}} , \\
&\lesssim \sum_{q\in\mathbb{Z}} 2^{qs}\left(\sum_{\alpha,\mathbf{m}} |\psi(2^{-q}|\mathbf{m}|)\mathrm{V}^{\alpha}_{\mathbf{m}}|^2\right)^{\frac{1}{2}} , \\
&\lesssim \sum_{q\in\mathbb{Z}} 2^{qs}||\Delta_q\mathbf{V}||_{L^2} = ||\mathbf{V}||_{B^s} .
\end{aligned} \tag{3.6.2.47}$$

Then we finish the proof of the lemma (11). □

4. BOLTZMANN EQUATION PRELIMINARIES

The Boltzmann equation governs the evolution of the distribution of molecules in rarefied gases. Originally, the equation was written by Maxwell for monatomic gases. Various generalizations have been proposed more recently (for polyatomic gases, with exchange of internal energy at the molecular level or chemical reactions.) However, for the sake of simplicity, the present chapter will only address the case of monatomic gases. All of the materials of this chapter are well-known and standard. This chapter does not contain any new results of the author of this dissertation. We follow mostly the presentation in [14, 30, 36], especially the recent survey papers of Golse-Levermore [30] and Golse-Saint-Raymond [36].

In kinetic theory, the state of a (rarefied) gas is adequately described by the distribution of molecules in phase-space (also called the distribution function or the number density,) $F = F(t, x, v)$ which is the density of particles located at the position $x \in \Omega$ with the velocity $v \in \mathbb{R}^D$ at time $t \geq 0$. To remove complications due to boundaries, we take Ω to be the periodic domain $\mathbb{T}^D = \mathbb{R}^D/\mathbb{L}^D$, where $\mathbb{L}^D \subset \mathbb{R}^D$ is any $D-$dimensional lattice. In this chapter, we consider $D = 3$.

In the absence of external forces (such as gravity, Coriolis force, electromagnetic forces in the case of ionized gases,) the number density $F = F(t, x, v)$ satisfies the Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F)\,. \tag{4.0.2.1}$$

where $\mathcal{B}(F, F)$ is the *Boltzmann collision integral*, which will be given in the following section.

The following simple remarks have important consequences on the structure of the Boltzmann equation:

- Because the Boltzmann equation is meant to describe a rarefied gas, molecular collision other than binary are neglected;
- At the kinetic level of description, the molecular radius is neglected everywhere except in the expression giving the mean free path, so that
- In Boltzmann's theory, collision are a purely local and instantaneous process.

In view of these remarks, one anticipates that

- The collision integral is quadratic in the number density F, and
- The collision integral acts only on the v variable in $F(t, x, v)$.

4.1 Boltzmann Collision Integral

The action of the Boltzmann collision integral on a function $f = f(v)$ is

$$\mathcal{B}(f, f) = \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} (f(v_1')f(v') - f(v_1)f(v))b(v_1 - v, \omega)\, d\omega dv_1\,, \tag{4.1.0.2}$$

where the velocities v' and v_1' are defined in terms of v, v_1 and ω by the formulas

$$\begin{aligned} v' &= v'(v, v_1, \omega) = v - (v - v_1) \cdot \omega\omega \,, \\ v_1' &= v_1'(v, v_1, \omega) = v_1 + (v - v_1) \cdot \omega\omega \,. \end{aligned} \tag{4.1.0.3}$$

For simplicity, currently we consider the collision kernel $b(v_1 - v, \omega)$ for a gas of hard spheres with radius r, i.e.,

$$b(v_1 - v, \omega) = 2r^2 |(v_1 - v) \cdot \omega| \,. \tag{4.1.0.4}$$

More general collision kernels will be discussed in the later section.

That the collision integral acts only on the v variable in F means that the right-hand side of the Boltzmann equation (4.0.2.1) is

$$\mathcal{B}(F, F)(t, x, v) = \mathcal{B}(F(t, x, \cdot)F(t, x, \cdot))(v) \tag{4.1.0.5}$$

with the definition (4.1.0.2) above for the collision integral acting on a function of v alone. The following notation may seem unfelicitous; it is however customary in the literature devoted to the Boltzmann equation and must not be ignored.

Notation: One designates $F(t, x, v_1)$, $F(t, x, v')$ and $F(t, x, v_1')$ respectively by F_1,F' and F_1'. with this notation, the collision integral in the right-hand side of (4.0.2.1) can be written as

$$\mathcal{B}(F, F) = \mathcal{B}(F, F) = \iint\limits_{\mathbb{S}^{D-1} \times \mathbb{R}^D} (F_1' F' - F_1 F) b(v_1 - v, \omega) \, d\omega dv_1 \,. \tag{4.1.0.6}$$

Later on, we also designate by $\mathcal{B}$ the symmetric bilinear operator associated to the quadratic expression above:

$$\mathcal{B}(F, G) = \tfrac{1}{2} \left(\mathcal{B}(F + G, F + G) - B(F, F) - B(G, G) \right) \,. \tag{4.1.0.7}$$

Let us discuss the geometrical and mechanical meaning of the relation (4.1.0.3). Observe first that these relations can be equivalently formulated as

$$\begin{aligned} &v' + v_1' = v + v_1 \,, \\ &v' - v_1' = (v - v_1) - 2(v - v_1) \cdot \omega\omega = \mathbb{R}_\omega (v - v_1) \,. \end{aligned} \tag{4.1.0.8}$$

where $\mathbb{R}_\omega$ designates the specular reflection on the plane orthogonal to the vector ω. In particular, one has

$$|v' - v_1'| = |v - v_1| \,. \tag{4.1.0.9}$$

Therefore the 4 points v, v_1, v', and v_1' lie on a same circle, and ω is one of the (external) bisectors of the angle between $v - v_1$ and $v' - v_1'$. From the mechanical viewpoint, the origin $\frac{1}{2}(v + v_1)$ is the velocity of the center of mass for any pair of molecules with velocities v and v_1; in (4.1.0.3), the first equality is the conservation of momentum for any pair of colliding molecules with velocities v, v_1 after collision, and v', v_1' before collision. The equality of relative speeds before and after collision is equivalent to the conservation of kinetic energy by the collision process—-i.e., the collisions considered are purely elastic. In other words, $v'(v, v_1, \omega)$and $v_1'(v, v_1, \omega)$ represent all possible solutions in the unknowns v' and v_1' of the system of equations

$$\begin{aligned} v' + v_1' &= v + v_1 \,, \\ |v'|^2 + |v_1'|^2 &= |v|^2 + |v_1|^2 \,. \end{aligned} \tag{4.1.0.10}$$

Momentum and kinetic energy, together with the number of gas molecules, are the only natural conserved quantities at the microscopic level. The most important properties of the Boltzmann equation, described in the next two sections, are straight foreword consequences of the structure of the collision integral, and more specifically of the conservation laws at the microscopic level established above.

4.2 Local Conservation Laws

First, one expects that the conservation laws (4.1.0.10) should have analogues at the macroscopic (fluid) level. These analogues are formulated according to the general recipe for defining macroscopic observable starting with microscopic quantities.

Proposition 5: *Assume that $f = f(v) \in L^1_{\text{loc}}(\mathbb{R}^D)$ is rapidly decaying at infinity, i.e.,*

$$f(v) = O(|v|^{-n}) \quad as \quad |v| \to +\infty \quad for\ all \quad n \geq 0\,, \tag{4.2.0.11}$$

while $\phi \in C(\mathbb{R}^D)$ has at most polynomial growth at infinity, i.e.,

$$\phi(v) = O(|v|^m) \quad as \quad |v| \to +\infty \quad for\ some \quad m \geq 0\,. \tag{4.2.0.12}$$

Then one has

$$\int_{\mathbb{R}^D} \mathcal{B}(f,f)\phi\, dv = \tfrac{1}{4} \iiint\limits_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} (f'f_1' - ff_1)(\phi + \phi_1 - \phi' - \phi_1')b(v - v_1, \omega)\, d\omega dv dv_1\,. \tag{4.2.0.13}$$

The above identity is called *the Boltzmann identity.* Because the proof of this proposition involves some of the most fundamental tricks in the theory of the Boltzmann collision operator, we give it in detail.

Proof: The assumptions on the decay of f and the growth of ϕ at infinity guarantee that all the integral considered in the course of this proof are absolutely convergent. Start with the obvious equality

$$\int_{\mathbb{R}^D} \mathcal{B}(f,f)\phi\, dv = \iiint\limits_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} (f'f_1' - ff_1)\phi b(v - v_1, \omega)\, d\omega dv_1\,. \tag{4.2.0.14}$$

Noting that we assume the collision kernels are hard sphere (4.1.0.4), in the right-hand side of this equality, for each fixed $\omega \in \mathbb{S}^{D-1}$, apply the change of variables $(v, v_1) \mapsto (v_1, v)$. The formula (4.1.0.3) shows that, under this change of variables $(v', v_1') \mapsto (v_1', v')$, Hence

$$\begin{aligned}
\int_{\mathbb{R}^D} \mathcal{B}(f,f)\phi \, dv &= \iiint\limits_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} (f'f_1' - ff_1)\phi b(v - v_1, \omega)\, d\omega dv dv_1 \\
&= \iiint\limits_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} (f'f_1' - ff_1)\phi_1 b(v - v_1, \omega)\, d\omega dv dv_1 \\
&= \iiint\limits_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} (f'f_1' - ff_1)\tfrac{1}{2}(\phi + \phi_1) b(v - v_1, \omega)\, d\omega dv dv_1 \, .
\end{aligned} \tag{4.2.0.15}$$

Next, apply the change of variables $(v, v_1) \mapsto (v', v_1')$ for each fixed $\omega \in \mathbb{S}^{D-1}$ in the last integral above. In the reference frame of center mass, this change of variables essentially reduces to the specular reflection $\mathbb{R}_\omega$ that exchanges the relative velocities:

$$\mathbb{R}_\omega : v - v_1 \mapsto v' - v_1' \, . \tag{4.2.0.16}$$

Because $\mathbb{R}_\omega$ is an involution (meaning that $\mathbb{R}_\omega^2 - Id$, the change of variables above also is an involution and maps (v', v_1') onto (v, v_1). Moreover, the second relation in (4.1.0.10) implies that this change of variables is an isometry of $\mathbb{R}^D \times \mathbb{R}^D$, and therefore leaves the Lebesgue measure $dvdv_1$ invariant. Since $(v' - v_1') \cdot \omega = -(v - v_1) \cdot \omega$, applying this change of variables in the right-hand side of the above equality implies that

$$\begin{aligned}
\int_{\mathbb{R}^D} \mathcal{B}(f,f)\phi\, dv &= \iiint\limits_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} (f'f_1' - ff_1)\tfrac{1}{2}(\phi+\phi_1)b(v-v_1,\omega)\, d\omega dv dv_1 \\
&= \iiint\limits_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} (f'f_1' - ff_1)\tfrac{1}{2}(\phi'+\phi_1')b(v-v_1,\omega)\, d\omega dv dv_1 \\
&= \iiint\limits_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} (f'f_1' - ff_1)\tfrac{1}{4}(\phi+\phi_1-\phi'-\phi_1')b(v-v_1,\omega)\, d\omega dv dv_1\,.
\end{aligned} \tag{4.2.0.17}$$

as announced. □

In view of the proposition above, the following class of functions is of particular importance.

Definition 5: A collision invariant is a measurable a.e. finite function $\phi \equiv \phi(v)$ such that, for each $(v, v_1) \in \mathbb{R}^D \times \mathbb{R}^D$ and each $\omega \in \mathbb{S}^{D-1}$. one has

$$\phi(v) + \phi(v_1) = \phi(v') + \phi(v_1')\,. \tag{4.2.0.18}$$

Constants are obviously collision invariants. In view of (4.1.0.10), other interesting examples of collision invariants are $\phi(v) = v_j$ for $j = 1, 2, 3$—i.e., the 3 components of v—or $\phi(v) \equiv |v|^2$.

An important result in the theory of the Boltzmann equation asserts that the examples above provide ALL the collision invariants, up to linear combinations.

Proposition 6: *Any collision invariant is a function of the form*

$$\phi(v) = a + b_1v_1 + b_2v_2 + b_3v_3 + c|v|^2\,, \tag{4.2.0.19}$$

where a, b_1,b_2,b_3 *and* c *are arbitrary elements of* $\mathbb{R}$.

The proof of this proposition is far from obvious; see for instance [14] on pp. 36-42.

In any case, whenever ϕ is a collision invariant and f is a measurable, rapidly decaying function, it follows from Proposition (5) that

$$\int_{\mathbb{R}^D} \mathcal{B}(f,f)\phi \, dv = 0\,. \tag{4.2.0.20}$$

This entails in particular the following

Corollary 5: *Let $F \equiv F(t,x,v)$ be a solution to the Boltzmann equation* (4.0.2.1) *that is locally integrable and rapidly decaying in v for each (t,x). Then*

$$\int_{\mathbb{R}^D} \mathcal{B}(F,F)\, dv = \int_{\mathbb{R}^D} \mathcal{B}(F,F) v_k \, dv = \int_{\mathbb{R}^D} \mathcal{B}(F,F)|v|^2 \, dv = 0\,, \tag{4.2.0.21}$$

for $k = 1,2,3$, and the following local conservation laws hold:

$$\begin{aligned} \partial_t \int_{\mathbb{R}^D} F \, dv + \nabla_x \cdot \int_{\mathbb{R}^D} vF \, dv = 0\,, \\ \partial_t \int_{\mathbb{R}^D} vF \, dv + \nabla_x \cdot \int_{\mathbb{R}^D} v \otimes vF \, dv = 0\,, \\ \partial_t \int_{\mathbb{R}^D} \tfrac{1}{2}|v|^2 F \, dv + \nabla_x \cdot \int_{\mathbb{R}^D} v\tfrac{1}{2}|v|^2 F \, dv = 0\,, \end{aligned} \tag{4.2.0.22}$$

respectively the local conservation of mass(or equation of continuity,) momentum and energy.

Define the following fields:

$$\begin{aligned} &\rho = \int_{\mathbb{R}^D} F \, dv\,, \quad u = \tfrac{1}{\rho}\int_{\mathbb{R}^D} vF \, dv\,, \quad P = \int_{\mathbb{R}^D} (u-v)\otimes(u-v)F \, dv\,, \\ &q = \int_{\mathbb{R}^D} (v-u)|v-u|^2 F \, dv\,. \end{aligned} \tag{4.2.0.23}$$

Notice that, by definition of u, one has

$$\begin{aligned} \int_{\mathbb{R}^D} v \otimes vF \, dv = \rho u \otimes u + P\,, \\ \int_{\mathbb{R}^D} |v|^2 F \, dv = \rho|u|^2 + \mathrm{tr}(P)\,, \\ \int_{\mathbb{R}^D} v|v|^2 F \, dv = (\rho|u|^2 + \mathrm{tr}(P))u + 2P\cdot u + q\,. \end{aligned} \tag{4.2.0.24}$$

Therefore, the system of conservation laws above can be put in the form

$$\begin{aligned} \partial_t\rho + \nabla_x \cdot (\rho u) = 0\,, \\ \partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u + P) = 0\,, \\ \partial_t \tfrac{1}{2}(\rho|u|^2 + \mathrm{tr}(P)) + \nabla_x \cdot \tfrac{1}{2}((\rho|u|^2 + \mathrm{tr}(P))u + 2P\cdot u + q) = 0\,. \end{aligned} \tag{4.2.0.25}$$

If we knew that $P = pI$ and $C = 0$,this system is closed and would coincide exactly with the Euler system for compressible fluids, with perfect gas pressure law.

However, one should bear in mind that (4.2.0.25) is satisfied by *any* solution to the Boltzmann equation, and therefore by any perfect gas in a *kinetic* regime. Thus one cannot expect that such a gas in a kinetic regime be in local thermodynamic equilibrium. In other words, one cannot hope that, for a generic solution to the Boltzmann equation, the tensor field P be of the form pI, for instance, or that $C = 0$. As we shall see, deriving the compressible Euler system from the Boltzmann equation requires additional argument.

4.3 Boltzmann's H-Theorem

Undoubtedly, the most important feature of the Boltzmann equation, along with the conservation laws stated in Corollary (5) is Boltzmann's H-Theorem. As in the case of the conservation laws, we begin with a statement that bears exclusively on the collision integral.

Theorem 8: *(Boltzmann's H-Theorem.) Let* $f \equiv f(v) > 0$ *be a locally integrable function that is rapidly decaying and* $\ln f$ *has at most polynomial growth as* $|v| \to$

$+\infty$. *Then*

$$\int_{\mathbb{R}^D} \mathcal{B}(f,f)\ln f\,dv = -\frac{1}{4}\iiint\limits_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} (f'f_1' - ff_1)\ln\left(\frac{f'f_1'}{ff_1}\right) b(v_1 - v,\omega)\,d\omega dv dv_1 \le 0\,. \tag{4.3.0.26}$$

Moreover, the following conditions are equivalent:

1. $\mathcal{B}(f,f) = 0$ *a.e.,*

2. $\int_{\mathbb{R}^D} \mathcal{B}(f,f)\ln f\,dv = 0\,,$

3. f *is a Maxwellian density, i.e.,*

$$f(v) = \mathcal{M}_{(\rho,u,\theta)}(v) := \frac{\rho}{(2\pi\theta)^{\frac{3}{2}}}\exp\left(-\frac{|v-u|^2}{2\theta}\right) \tag{4.3.0.27}$$

for some $\rho,\theta > 0$ *and* $u \in \mathbb{R}^D$.

Proof: Applying Proposition (5) with $\phi = \ln f$ leads to the first equality above; since the logarithm is an increasing function, the right-hand side of this equality is nonnegative.

As for the equality case, observe that (1) obviously implies (2); that (3) implies (1) follow by inspection. The only non-trivial point is that (2) implies (3). If one takes Proposition (6) for granted and assumes that f is continuous, it is immediate. Indeed, $\ln f$ is then a collision invariant, which is clearly equivalent to the fact that f is a Maxwellain.

Since we do not know in general whether f is continuous, the implication (2) $\Rightarrow$ (3) is a consequence of the following

Lemma 12: *(Perthame [64].) Let* $f > 0$ *a.e. be a measurable function such that*

$$\int_{\mathbb{R}^D} (1+|v|^2)f(v)\,dv < +\infty\,. \tag{4.3.0.28}$$

If

$$f(v)f(v_1) = f(v')f(v'_1) \tag{4.3.0.29}$$

for a.e. $(v, v_1, \omega) \in \mathbb{R}^D \times \mathbb{R}^D \times \mathbb{S}^{D-1}$ *with* v' *and* v'_1 *given by* (4.1.0.3), *then* f *is a Maxwellian.*

Perthame's proof uses the Fourier transform of the functional equation on f in a very clever way; see also [10] on pp. 47-48.

From the above statement on the collision integral, we deduce the following important consequence on solutions to the Boltzmann equation.

Corollary 6: *Let* $F \equiv F(t, x, v) > 0$ *be a solution to the Boltzmann equation that is rapidly decaying and such that* $\ln F$ *has at most polynomial growth as* $|v| \to +\infty$. *Then, one has*

$$\begin{aligned} \partial_t \int_{\mathbb{R}^D} F \ln F \, dv + \nabla_x \cdot \int_{\mathbb{R}^D} vF \ln F \, dv \\ = -\frac{1}{4} \iiint\limits_{\mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D} (F'F'_1 - FF_1) \ln \left(\frac{F'F'_1}{FF_1} \right) b(v_1 - v, \omega) \, d\omega dv_1 \le 0 \, . \end{aligned} \tag{4.3.0.30}$$

The above inequality is the so-called entropy inequality. Upon defining

$$S = \frac{1}{\rho} \int_{\mathbb{R}^D} F \ln F \, dv \, , \quad N = - \int_{\mathbb{R}^D} (v - u) F \ln F \, dv \, , \tag{4.3.0.31}$$

we see that the differantial inequality (4.3.0.30) takes the form

$$\partial_t(\rho S) + \nabla_x \cdot (\rho S u + N) \ge 0 \, . \tag{4.3.0.32}$$

Again, this differential inequality is formally reminiscent of the Lax-Friedrichs criterion that selects admissible solutions to hyperbolic systems of conservation laws, of which the Euler equations for compressible fluids are the most famous example. See [17] section 4.3 for a discussion of this criterion. In the case of the Euler equations for perfect gases, $N = 0$, so that the above differential inequality means that the specific entropy S is a nondecreasing quantity along the trajectory of each infinitesimal fluid element.

However, the inequality (4.3.0.32) is satisfied by any solutions to the Boltzmann equation, therefore by any monotomic gas in kinetic regimes.

A considerable difference with the theory of ideal fluids is that Boltzmann's H-Theorem provides an expression for the entropy production rate in terms of the number density that is local in (t, x). In the theory of ideal fluids, one only knows that the entropy is produced across shock waves, but there is no expression of the entropy production there.

4.4 More General Collision Kernels

The Boltzmann collision kernel considered so far involved the collision kernel

$$b(v_1 - v, \omega) = 2r^2|(v_1 - v) \cdot \omega| \tag{4.4.0.33}$$

that corresponds to pairwise elastic collision between hard spheres of radius r. But gas molecules are more complicated objects than just hard spheres, and their pairwise interaction is a rather complex combination of electrostatic potential created by the elementary constituents of the molecules (electrons and protons.)

In general, the collision kernel b is positive almost everywhere, locally integrable. The Galilean invariance of the collisional physics implies that b has the classical form

$$b(v_1 - v, \omega) = |v_1 - v|\Sigma(|v_1 - v|, |\omega \cdot n|)\,. \tag{4.4.0.34}$$

where $n = \frac{v_1 - v}{|v_1 - v|}$ and $\Sigma \geq 0$ is the specific differential cross-section, which has units of area (length^2) over mass.

The specific dependence of b on (v, v_1, ω)—-more specifically the fact that b only depends on $|v_1 - v|$ and $|\omega \cdot n|$—implies that

$$b(v_1 - v, \omega) = b(v - v_1, \omega) = b(v_1' - v', \omega) \tag{4.4.0.35}$$

for each $(v, v_1, \omega) \in \mathbb{R}^D \times \mathbb{R}^D \times \mathbb{S}^{D-1}$. These relations imply that the collision integral (4.1.0.2) satisfies the Boltzmann identity (4.2.0.13) and the resulting conservation laws, as well as Boltzmann's H-Theorem and its consequence.

Of course all these properties are subject to the obvious requirement that the Boltzmann collision integral should converge in some sense. This is however far from obvious whenever the molecular interaction is given by a long-range potential. A typical example of such a situation is the case of an inverse power-law repulsive potential of the form

$$U(r) = \frac{c}{r^k}\,, \tag{4.4.0.36}$$

where c and k are positive constants, and r is the intermolecular distance. Instead of giving a complete derivation of the collision kernel b—-or equivalently of the cross-section Σ—in this case, we refer the interested readers to [13], and summarize the results there.

In this case, one can show that b has the factored form

$$b(v_1 - v, \omega) = |v_1 - v|^\beta \hat{b}(|\omega \cdot n|)\,, \quad \text{with} \quad \beta = 1 - \tfrac{4}{k}\,. \tag{4.4.0.37}$$

This will be locally integrable with respect to dv_1 provided $\beta > -3$, which leads to the constraint

$$k > 1 \tag{4.4.0.38}$$

meaning that the marginal case of the Coulomb potential $\frac{c}{r}$ is excluded. We will not give the function $\hat{b}$ here. We will however remark that $\hat{b}$ is well-behaved except for a singularity at $\omega \cdot n = 0$ of the form

$$\hat{b}(s) \sim s^{-\hat{\beta}} \quad \text{as} \quad s \to 0\,, \quad \text{with} \quad \hat{\beta} = 1 + \tfrac{2}{k}\,. \tag{4.4.0.39}$$

This singularity arises due to the infinite range of the c/r^k potential. It reflects the fact that there are many collisions in which the colliding molecules do not pass very close to each other and are therefore deflected only slightly. This singularity has proved difficult to analyze. For example, the fact that this singularity is not integrable with respect to $d\omega$ means that the gain and loss part of the Boltzmann collision integral, defined respectively as

$$\begin{aligned} \mathcal{B}^+(f,f) &= \iint\limits_{\mathbb{R}^D \times \mathbb{S}^{D-1}} f_1' f' b(v_1 - v, \omega)\, dv_1 d\omega\,, \\ \mathcal{B}^-(f,f) &= \iint\limits_{\mathbb{R}^D \times \mathbb{S}^{D-1}} f_1 f b(v_1 - v, \omega)\, dv_1 d\omega \end{aligned} \tag{4.4.0.40}$$

do not make sense. So-called cut-off collision kernels have therefore been introduced. These replace the exact $\hat{b}$ above with a more regular one, by replacing the angular part of the cross-section, i.e., the function $\hat{b}$ with its truncation for s below some small value s_0, that can be defined as

$$\bar{b}(s) = \inf(\hat{b}(s_0,)\hat{b}(s))\,. \tag{4.4.0.41}$$

Grad [38] argued that this truncation is legitimate on physical grounds for neutral gases, since grazing collisions (which are responsible for the singularity of $\hat{b}$ at $s = 0$) are statistically negligible in that case. In the case of plasmas such a truncation is of course not valid and grazing collisions are important in some variety of physical regimes. In any case, these considerations led Grad to propose the notion of "cut-off potentials"—a slightly improper terminology [37, 38], since in this procedure, it is the collision kernel that is truncated and not the potential. More specifically, we shall say that the collision kernel b comes from a "hard potential" if, for each $(z, \omega) \in \mathbb{R}^D \times \mathbb{S}^{D-1}$

$$0 \leq b(z,\omega) \leq C(1+|z|)^\beta \quad \text{and} \quad \int_{\mathbb{S}^{D-1}} b(z,\omega)\, d\omega \geq \tfrac{1}{C_b}(1+|z|)^\beta \tag{4.4.0.42}$$

for some $\beta \in [0, 1]$ and $C_b > 0$. Instead, we shall say that it comes from a "soft cut-off potential" if b satisfies above conditions with $\beta \in (-3, 0)$.

In addition to the case of hard spheres mentioned above, a notable particular case is that of a "cut-off Maxwellian interaction" corresponding to

$$b(z,\omega) = \hat{b}(|\omega \cdot n|)\,, \tag{4.4.0.43}$$

with $0 < \hat{b} \in C([0, 1])$. This particular case attracted Maxwell's attention since the linearized collision integral can then be reduced to diagonal form explicitly by using Sonine polynomials (a multidimensional variant of Hermite polynomials.) In the sequel, we shall mostly consider hard cut-off potentials, and sometimes only the particular case of hard spheres.

4.5 Linearized Collision Integral

Let ρ and $\theta > 0$, and $u \in \mathbb{R}^D$, $\mathcal{M}_{(\rho,u,\theta)}$ be a uniform Maxwellian; the linearization at $\mathcal{M}_{(\rho,u,\theta)}$ of the collision integral is defined as follows

$$\begin{aligned}\mathcal{L}_{\mathcal{M}_{(\rho,u,\theta)}}\phi &= -2\mathcal{M}_{(\rho,u,\theta)}^{-1}\mathcal{B}(\mathcal{M}_{(\rho,u,\theta)}, \mathcal{M}_{(\rho,u,\theta)}\phi) \\ &= \iint (\phi + \phi_1 - \phi' - \phi_1')b(v_1 - v, \omega)\mathcal{M}_{(\rho,u,\theta)}(v_1)\, dv_1 d\omega\,,\end{aligned} \tag{4.5.0.44}$$

here we have used the relation

$$\mathcal{M}_{(\rho,u,\theta)}(v)\mathcal{M}_{(\rho,u,\theta)}(v_1) = \mathcal{M}_{(\rho,u,\theta)}(v')\mathcal{M}_{(\rho,u,\theta)}(v_1')\,. \tag{4.5.0.45}$$

The dependence on the parameters ρ, u and θ of the linearized collision integral is handed most easily by using the translation and scaling invariance of the collision kernel. So we can actually restrict our discussion to the case where $M = \mathcal{M}_{(1,0,1)}$ is the centered reduced Gaussian.

Translation and the scaling invariance of $\mathcal{L}_{\mathcal{M}_{(\rho,u,\theta)}}$. Indeed, if τ_w and m_λ denote respectively the translation and scaling isometries on $L^1(\mathbb{R}^D)$ defined by

$$\tau_w F(c) = F(v - w)\,, \quad (m_\lambda T)(v) = \lambda^{-3}F(\lambda^{-1}v) \tag{4.5.0.46}$$

One has

$$\mathcal{B}(\tau_w F, \tau_w F) = \tau_w \mathcal{B}(F, F)\,, \quad \mathcal{B}(m_\lambda F, m_\lambda F) = \lambda m_\lambda \mathcal{B}(F, F)\,, \tag{4.5.0.47}$$

We can deduce that

$$\mathcal{L}_{\mathcal{M}_{(\rho,u,\theta)}}(\phi) = (\rho\sqrt{\theta})\tau_u m_{\sqrt{\theta}}\mathcal{L}_M(m_{1/\sqrt{\theta}}\tau_{-u}\phi)\,. \tag{4.5.0.48}$$

This relation shows that it is enough to study the linearization of the collision integral at the centered reduced Gaussian $M = \mathcal{M}_{(1,0,1)}$ with an arbitrary collision kernel b.

Rotational invariance of $\mathcal{L}_{\mathcal{M}_{(1,0,1)}}$. The group $O_3(\mathbb{R})$ of orthogonal 3×3 matrices (i.e., matrices R such that $RR^T = R^T R = I$) acts on functions on $\mathbb{R}^D$ by the formula

$$f_R(v) = f(R^T v)\,, \quad R \in O_3(\mathbb{R}^D)\,, \quad v \in \mathbb{R}^D\,; \tag{4.5.0.49}$$

likewise its action on vector fields is defined by

$$U_R(v) = RU(R^T v)\,, \quad R \in O_3(\mathbb{R}^D)\,, \quad v \in \mathbb{R}^D\,; \tag{4.5.0.50}$$

while its action on symmetric matrix field is given by

$$S_R(v) = RS(R^T v)R^T\,, \quad R \in O_3(\mathbb{R}^D)\,, \quad v \in \mathbb{R}^D\,; \tag{4.5.0.51}$$

The Boltzmann collision integral is obviously invariant under the action of $O_3(\mathbb{R})$—indeed, the microscopic collision process is isotropic. In fact, an elementary change of variables in the collision integral shows that

$$\mathcal{B}(\Phi_R, \Phi_R) = \mathcal{B}(\Phi, \Phi)_R \tag{4.5.0.52}$$

for each continuous, rapidly decaying Φ. since the centered unit Gaussian $M = \mathcal{M}_{(1,0,1)}$ is radial function, this rotation invariance property goes over to $\mathcal{L}_M$:

$$\mathcal{L}_M(\phi_R) = (\mathcal{L}_M \phi)_R\,. \tag{4.5.0.53}$$

Extending $\mathcal{L}_M$ to act componentwise on vector or matrix fields on $\mathbb{R}^D$, one finds that

$$\mathcal{L}_M(U_R) = (\mathcal{L}_M U)_R \tag{4.5.0.54}$$

for continuous, rapidly decaying vector fields U, and

$$\mathcal{L}_M(S_R) = (\mathcal{L}_M S)_R \tag{4.5.0.55}$$

for continuous, rapidly decaying matrix field S and S_R are defined above. As we shall see below, this $O_3(\mathbb{R})-$invariance of $\mathcal{L}_M$ has important consequences: it implies in particular that the viscosity and heat conductivity are scalar quantities (and not matrices.)

The Fredholm property. We assume that the collision kernel b satisfies a hard potential cut-off assumption (4.4.0.42), for some $\beta \in [0,1]$ and $C_b > 0$. Consider $\mathcal{L}$, the linearization of the Boltzmann collision integral at the centered, reduced Gaussian state M above. (Notice that we have discarded the dependence of $\mathcal{L}$ on b and M for notational simplicity.) From (4.5.0.44), we infer that $\mathcal{L}$ can be split as the sum of a local (multiplication) operator and of an integral operator, as follows:

$$\mathcal{L}\phi(v) = a(|v|)\phi(v) - \mathcal{K}\phi(v)\,, \tag{4.5.0.56}$$

where the collision frequency is

$$a(|v|) = \iint\limits_{\mathbb{R}^D\times\mathbb{S}^{D-1}} b(v_1 - v, \omega) M_1 \, dv_1 d\omega\,. \tag{4.5.0.57}$$

The nonlocal operator $\mathcal{K}$ is further split into two parts

$$\mathcal{K}\phi = \mathcal{K}_2\phi - \mathcal{K}_1\phi\,, \tag{4.5.0.58}$$

where

$$\begin{aligned} \mathcal{K}_1\phi &= \iint\limits_{\mathbb{R}^D\times\mathbb{S}^{D-1}} \phi_1 b(v_1 - v, \omega) M_1 \, dv_1 d\omega\,, \\ \mathcal{K}_2\phi &= \iint\limits_{\mathbb{R}^D\times\mathbb{S}^{D-1}} (\phi' + \phi_1') b(v_1 - v, \omega) M_1 \, dv_1 d\omega \\ &= 2 \iint\limits_{\mathbb{R}^D\times\mathbb{S}^{D-1}} \phi' b(v_1 - v, \omega) M_1 \, dv_1 d\omega\,. \end{aligned} \tag{4.5.0.59}$$

It is clear that $\mathcal{K}_1$ is compact operator on $L^2(Mdv)$; that $\mathcal{K}_2$ shares the same property is much less obvious, and was proved by Hilbert [39] in the hard sphere case. Fifty years later, Grad [38] introduced the cut-off assumption which now bears his name and used it in particular to extend Hilbert's result to all cut-off potentials.

Lemma 13: *(Hilbert[39], Grad [38].) Assume that b is a collision kernel that satisfies the hard cut-off assumption* (4.4.0.42)*. Then the operator $\mathcal{K}_2$ is compact on $L^2(Mdv)$.*

With above preliminary results, we establish the main property of the linearized collision operator $\mathcal{L}_M$, i.e., that it satisfies the Fredholm alternative in some weighted L^2 space.

Theorem 9: (Hilbert[39].) *Assume that the collision kernel b satisfies the hard cut-off assumption* (4.4.0.42)*. Then the linear operator $\mathcal{L}$ is a nonnegative unbounded self-adjoint Fredholm with domain $\mathcal{D}(\mathcal{L}) = L^2(a^2Mdv)$ (a being the collision frequency defined in* (4.5.0.57)*.) Its null space is the space of collision invariants:*

$$Null = span\left\{1, v_1, v_2, v_3, |v|^2\right\} . \tag{4.5.0.60}$$

Moreover the following coercivity estimate on $Null^{\perp}(\mathcal{L})$ holds: there exists $C > 0$ such that, for each $\phi \in L^2(aMdv)$, one has

$$\int \phi\mathcal{L}\phi(v)M(v)\,dv \geq C \int_{\mathbb{R}^D} (\phi - \Pi\phi)^2 aM\,dv\,, \tag{4.5.0.61}$$

where Π is the $L^2(Mdv)-$orthogonal projection on $Null(\mathcal{L})$.

An important consequence of Theorem (9) is that the integral equation

$$\mathcal{L}\phi = \psi\,, \quad \psi \in L^2(Mdv) \tag{4.5.0.62}$$

satisfies the Fredholm alternative:

- Either $\psi \perp \mathrm{Null}(\mathcal{L})$, in which case (4.5.0.62) has a unique solution

$$\phi_0 \in L^2(a^2 M dv) \cap \mathrm{Null}^{\perp}(\mathcal{L})\,; \tag{4.5.0.63}$$

  then any solution to (4.5.0.62) is of the form

$$\phi = \phi_0 + \phi_1\,, \quad \text{where} \quad \phi_1 \quad \text{is an arbitrary element of } \mathrm{Null}(\mathcal{L})\,; \tag{4.5.0.64}$$

- Or $\psi \notin \mathrm{Null}(\mathcal{L})$, in which case (4.5.0.62) has no solution.

Example: Consider the matrix field

$$A(v) = v \otimes v - \tfrac{1}{D}|v|^2 I\,, \tag{4.5.0.65}$$

and the vector field

$$B(v) = \tfrac{1}{2} v(|v|^2 - \tfrac{D+2}{2})\,. \tag{4.5.0.66}$$

here we take $D = 3$. Clearly

$$A_{jk} \perp \mathrm{Null}(\mathcal{L})\,, \quad B_l \perp \mathrm{Null}(\mathcal{L})\,, A_{jk} \perp B_l\,, \quad j,k,l = 1,2,\cdots D\,. \tag{4.5.0.67}$$

In fact, more is true:

$$\begin{aligned} &\int_{\mathbb{R}^D} A(v) f(|v|^2) M\, dv = 0\,, \quad \int_{\mathbb{R}^D} A(v) v f(|v|^2) M\, dv = 0 \\ &\int_{\mathbb{R}^D} B(v) f(|v|^2) M\, dv = 0\,, \quad \int_{\mathbb{R}^D} B(v) \cdot v M\, dv = 0\,. \end{aligned} \tag{4.5.0.68}$$

The second and third formulas are obvious since A is even and B odd. As for the first formula, observe that A is an isotropic matrix, in the sense that $A(Rv) =$

$RA(v)R^T$ for each $R \in O_3(\mathbb{R})$ $---$with the notation above for the action of $O_3(\mathbb{R})$ on symmetric matrices, $A_R = A$ for each $R \in O_3(\mathbb{R})$. Hence the matrix

$$\int_{\mathbb{R}^D} A(v)f(|v|^2)M\,dv \tag{4.5.0.69}$$

commutes with any $R \in O_3(\mathbb{R})$ $---$as can be seen by changing v into Rv in the above integral—and is therefore a scalar multiple of the identity matrix. But

$$\mathrm{tr}\int_{\mathbb{R}^D} A(v)f(|v|^2)M\,dv = \int_{\mathbb{R}^D} \mathrm{tr}A(v)f(|v|^2)M\,dv = 0 \tag{4.5.0.70}$$

and hence this scalar multiple of identity matrix is null. The fourth and last formula is based on the following elementary recursion formula for Gaussian integrals

$$\int_{\mathbb{R}^D} |v|^n M\,dv = (n+1)\int_{\mathbb{R}^D} |v|^{n-2}M\,dv\,, \quad n \geq 2\,. \tag{4.5.0.71}$$

(use spherical coordinates and integrate by parts.)

In particular, the Fredholm alternative implies the existence of a matrix field $\hat{A}$ and a vector field $\hat{B}$ such that

$$\begin{aligned} \mathcal{L}\hat{A} = A \quad &\text{and} \quad \hat{A} \perp \mathrm{Null}(\mathcal{L})\,, \\ \mathcal{L}\hat{B} = B \quad &\text{and} \quad \hat{B} \perp \mathrm{Null}(\mathcal{L})\,. \end{aligned} \tag{4.5.0.72}$$

Observe that

$$\begin{aligned} \mathcal{L}(\hat{A}_R) = \hat{A}_R = A\,, \quad &\text{and} \quad \hat{A} \perp \mathrm{Null}(\mathcal{L})\,, \quad \text{for all} \quad R \in O_3(\mathbb{R})\,, \\ \mathcal{L}(\hat{B}_R) = \hat{B}_R = B\,, \quad &\text{and} \quad \hat{B} \perp \mathrm{Null}(\mathcal{L})\,, \quad \text{for all} \quad R \in O_3(\mathbb{R})\,, \end{aligned} \tag{4.5.0.73}$$

so that, by the uniqueness part in the Fredholm alternative

$$\hat{A}_R = \hat{A} \quad \text{and} \quad \hat{B}_R = \hat{B} \quad \text{for all} \quad R \in O_3(\mathbb{R})\,. \tag{4.5.0.74}$$

An elementary geometric argument (considered classical in the literature on the Boltzmann equation, with a complete proof in [19]) shows the existence of two scalar functions

$$a : \mathbb{R}_+ \to \mathbb{R}\,, \quad b : \mathbb{R}_+ \to \mathbb{R} \tag{4.5.0.75}$$

such that

$$\hat{A}(v) = a(|v|^2)A(v)\,, \quad \text{and} \quad \hat{B}(v) = b(|v|^2)B(v)\,. \tag{4.5.0.76}$$

As we shall see in later chapters, the viscosity and heat conductivity of a gas are expressed as Gaussian integrals of the scalar functions a and b—and therefore are scalar quantities themselves.

4.6 Formal Structure of the Boltzmann Equation

In this section we collect some basic facts we will need in later chapters. These will include nondimensionalization, formal conservation and dissipation, and the DiPerna-Lions theory of global solutions to the Cauchy problems of the Boltzmann equation, i.e.,

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F,) \qquad F(0, x, v) = F^{in}(x, v) \geq 0\,. \tag{4.6.0.77}$$

where as introduced in the last section, the Boltzmann collision operator $\mathcal{B}$ acts only on the v arguments of F, namely

$$\mathcal{B}(F, F) = \iint\limits_{\mathbb{S}^{D-1} \times \mathbb{R}^D} (F_1' F' - F_1 F) b(v_1 - v, \omega)\, d\omega dv_1\,, \tag{4.6.0.78}$$

where the collision kernel b satisfies the hard potential cut-off condition (4.4.0.42).

4.6.1 Dimensionless Form

We will work with the nondimensionalized form of the Boltzmann equation that used in Bardos-Golse-Levermore's program [7]. Before entering the subject of hydrodynamic limits, we first describe the Boltzmann equation in dimensionless variables. In these variables, two dimensionless parameters, called the Knudsen and Strouhal numbers naturally appear in the Boltzmann equation. In this section, we consider the Boltzmann equation for general cut-off potential.

Choose a macroscopic length scale L and time scale T, and a reference temperature Θ. This defines two velocity scales:

- One is the speed at which some macroscopic portion of gas is transported over a distance L in time T, i.e.,

$$U = \frac{L}{T}\,; \tag{4.6.1.1}$$

- The other one is the thermal speed of the molecules with energy $\frac{3}{2}\kappa\Theta$; in fact, it is more natural to define this velocity scale as

$$c = \sqrt{\frac{5}{3}\frac{\kappa\Theta}{m}} \tag{4.6.1.2}$$

  with m being the molecular mass, which is the speed of sound in a monatomic gas at the temperature Θ.

Define next the dimensionless variables involved in the Boltzmann equation, i.e., the dimensionless time, space and velocity variables as

$$\hat{t} = \frac{t}{T}\,, \quad \hat{x} = \frac{x}{L}\,, \quad \text{and} \quad \hat{v} = \frac{v}{c}\,. \tag{4.6.1.3}$$

Define also the dimensionless number density

$$\hat{F}(\hat{t}, \hat{x}, \hat{v}) = \frac{L^3 c^3}{\mathcal{N}} F(t, x, v) \,, \tag{4.6.1.4}$$

where $\mathcal{N}$ is the total number of gas molecules in a volume L^3. Finally, we must rescale the collision kernel b. $b(z, \omega)$ is the velocity multiplied by the scattering cross-section of the gas molecules. Define

$$\hat{b}(\hat{z}, \omega) = \frac{1}{c \times \pi r^2} b(z, \omega) \quad \text{with} \quad \hat{z} = \frac{z}{c} \,, \tag{4.6.1.5}$$

where r is the molecular radius.

If F satisfies the Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F + \iint\limits_{\mathbb{R}^D \times \mathbb{S}^{D-1}} (F_1' F' - F_1 F) b(v_1 - v, \omega) \, dv_1 d\omega \,, \tag{4.6.1.6}$$

then

$$\frac{L}{cT} \partial_{\hat{t}} \hat{F} + \hat{v} \cdot \nabla_{\hat{x}} \hat{F} = \frac{\mathcal{N} \pi r^2}{L^2} \iint\limits_{\mathbb{R}^D \times \mathbb{S}^{D-1}} (\hat{F}_1' \hat{F}' - \hat{F}_1 \hat{F}) \hat{b}(\hat{v}_1 - \hat{v}, \omega) \, d\hat{v}_1 d\omega \,. \tag{4.6.1.7}$$

The factor multiplying the collision integral is

$$L \times \frac{\mathcal{N} \times \pi r^2}{L^3} = \frac{L}{\text{mean free path}} = \frac{1}{\text{Kn}} \,, \tag{4.6.1.8}$$

where Kn is the Knudsen number defined above. The factor multiplying the time derivative

$$\frac{\frac{1}{T} \times L}{c} =: \text{St} \tag{4.6.1.9}$$

is called the kinetic Strouhal number (by analogy with the notion of Strouhal number used in the dynamics of vortices.) Hence the dimensionless form of the Boltzmann equation is

$$\mathrm{St}\partial_{\hat{t}}\hat{F} + \hat{v}\cdot\nabla_{\hat{x}}\hat{F} = \frac{1}{\mathrm{Kn}}\iint\limits_{\mathbb{R}^D\times\mathbb{S}^{D-1}} (\hat{F}_1'\hat{F}' - \hat{F}_1\hat{F})\hat{b}(\hat{v}_1 - \hat{v},\omega)\, d\hat{v}_1 d\omega\,. \qquad (4.6.1.10)$$

There is some arbitrariness in the way the length, time and temperature scales L, T, Θ are chosen. The most natural thing to do is choose these in a way that is consistent with the geometry of the domain where the gas motion takes place, the time necessary to observe significant gas motion, and the distribution function at the initial instant of time.

All hydrodynamic limits of the Boltzmann equation correspond to situations where the Knudsen number Kn satisfies

$$\mathrm{Kn} = \epsilon \ll 1\,. \qquad (4.6.1.11)$$

But there is no universal prescription for the Strouhal number in the context of the hydrodynamic limit; as we shall see in the next chapter, various hydrodynamic regimes can be derived from the Boltzmann equation by appropriately tuning the Strouhal number.

Fluid models (acoustic system, incompressible Stokes, incompressible Navier-Stokes, etc.) can be formally derived from the Boltzmann equation through a scaling the density F is close to a spatially homogeneous Maxwellian $M = M(v)$ that has the same total mass, momentum, and energy as the initial data F^{in}. By an appropriate choice of a Galilean frame and of mass and velocity units, it can be assumed that this so-called absolute Maxwellian M has the form

$$M(v) \equiv \frac{1}{(2\pi)^{\frac{D}{2}}}\exp\left(-\frac{1}{2}|v|^2\right)\,. \qquad (4.6.1.12)$$

This corresponds to the spatially homogeneous fluid state with density and temperature equal to 1 and bulk velocity equals to 0, and is consistent with the form of both the Compressible Stokes system given by (6.1.0.10).

It is natural to introduce the relative density, $G = G(t, x, v)$, defined by $F = MG$. Recasting the initial-value problem (4.6.0.77) for G yields

$$\partial_t G + v \cdot \nabla_x G = \frac{1}{\epsilon} \mathcal{Q}(G, G,) \qquad G(0, x, v) = G^{in}(x, v) \geq 0 \,, \tag{4.6.1.13}$$

where the collision operator is now given by

$$\mathcal{Q}(G, G) = \iint\limits_{\mathbb{S}^{D-1} \times \mathbb{R}^D} (G_1' G' - G_1 G) b(v_1 - v, \omega) \, d\omega M_1 dv_1 \,, \tag{4.6.1.14}$$

with the non-dimensional collision kernel b being normalized so that

$$\iiint\limits_{\mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D} b(\omega, v_1 - v) \, d\omega M_1 dv_1 M dv = 1 \,. \tag{4.6.1.15}$$

The positive, nondimensional parameter ϵ is the Knudsen number.

This nondimensionalization has the normalizations

$$\int\limits_{\mathbb{S}^{D-1}} d\omega = 1 \,, \quad \int\limits_{\mathbb{R}^D} M \, dv = 1 \,, \quad \int\limits_{\mathbb{T}^D} dx = 1 \,, \tag{4.6.1.16}$$

associated with the domain $\mathbb{S}^{D-1}, \mathbb{R}^D$, and $\mathbb{T}^D$, respectively, (4.6.1.15) associated with the collision kernel b, and

$$\begin{gathered} \iint\limits_{\mathbb{R}^D \times \mathbb{T}^D} G^{in} M \, dv dx = 1 \,, \qquad \iint\limits_{\mathbb{R}^D \times \mathbb{T}^D} v G^{in} M \, dv dx = 0 \,, \\ \iint\limits_{\mathbb{R}^D \times \mathbb{T}^D} \tfrac{1}{2} |v|^2 G^{in} M \, dv dx = \tfrac{D}{2} \,, \end{gathered} \tag{4.6.1.17}$$

associated with the initial data G^{in}.

Because Mdv a positive unit measure on $\mathbb{R}^D$, we denote by $\langle \xi \rangle$ the average over this measure of any integrable function $\xi = \xi(v)$,

$$\langle \xi \rangle = \int_{\mathbb{R}^D} M\, dv\,. \tag{4.6.1.18}$$

because

$$d\mu = b(\omega, v_1 - v)\, d\omega M_1 dv_1 M dv \tag{4.6.1.19}$$

is a positive unit measure on $\mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D$, we denote by $\langle\langle \Xi \rangle\rangle$ the average over this measure of any integrable function $\Xi = \Xi(\omega, v_1, v)$,

$$\langle\langle \Xi \rangle\rangle = \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} \Xi(\omega, v_1, v)\, d\mu\,. \tag{4.6.1.20}$$

the measure $d\mu$ is invariant under the coordinate transformations

$$(\omega, v_1, v) \mapsto (\omega, v, v_1)\,, \quad (\omega, v_1, v) \mapsto (\omega, v_1', v')\,. \tag{4.6.1.21}$$

These, and compositions of these, are called $d\mu-$symmetries.

4.6.2 Formal Conservation and Dissipation Laws

We now list for later reference the basic conservation and entropy dissipation laws that are formally satisfied by solutions to the Boltzmann equation. Derivations of these laws in this nondimensional settings are outlined in [7] and can, up to notational differences, be found in [13], sec. II.6-7, [25], sec. 1.4, or [27].

First, if G solves the Boltzmann equation (4.6.1.13), then G satisfies local conservation laws of mass, momentum, and the energy:

$$\begin{aligned}\partial_t\langle G\rangle + \nabla_x\cdot\langle vG\rangle &= 0\,,\\ \partial_t\langle vG\rangle + \nabla_x\cdot\langle v\otimes vG\rangle &= 0\,,\\ \partial_t\langle \tfrac{1}{2}|v|^2 G\rangle + \nabla_x\cdot\langle v\tfrac{1}{2}|v|^2 G\rangle &= 0\,.\end{aligned} \tag{4.6.2.1}$$

Integrating these over space and time while recalling the normalizations (4.6.1.17) of G^{in} yields the global conservation laws of mass, momentum, and energy:

$$\begin{aligned}\int_{\mathbb{T}^D}\langle G(t)\rangle\,dx &= \int_{\mathbb{T}^D}\langle G^{in}\rangle\,dx = 1\,,\\ \int_{\mathbb{T}^D}\langle vG(t)\rangle\,dx &= \int_{\mathbb{T}^D}\langle vG^{in}\rangle\,dx = 1\,,\\ \int_{\mathbb{T}^D}\langle \tfrac{1}{2}|v|^2 G(t)\rangle\,dx &= \int_{\mathbb{T}^D}\langle \tfrac{1}{2}|v|^2 G^{in}\rangle\,dx = 1\,.\end{aligned} \tag{4.6.2.2}$$

Secondly, if G solves the Boltzmann equation (4.6.1.13), then G satisfies the local entropy dissipation law

$$\begin{aligned}&\partial_t\langle G\log G - G + 1\rangle + \nabla_x\cdot\langle v(G\log G - G + 1)\rangle =\\ &\qquad -\frac{1}{\epsilon}\left\langle\!\!\left\langle \frac{1}{4}\log\left(\frac{G_1'G'}{G_1G}\right)(G_1'G' - G_1G)\right\rangle\!\!\right\rangle \le 0\,.\end{aligned} \tag{4.6.2.3}$$

Integrating this over space and time gives the global entropy equality

$$H(G(t)) + \frac{1}{\epsilon}\int_0^t R(G(s))\,ds = H(G^{in})\,, \tag{4.6.2.4}$$

where $H(G)$ is the relative entropy functional

$$H(G) = \int_{\mathbb{T}^D}\langle G\log G - G + 1\rangle\,dx\,, \tag{4.6.2.5}$$

and $R(G)$ is the entropy dissipation rate functional

$$R(G) = \int_{\mathbb{T}^D}\left\langle\!\!\left\langle \frac{1}{4}\log\left(\frac{G_1'G'}{G_1G}\right)(G_1'G' - G_1G)\right\rangle\!\!\right\rangle dx\,. \tag{4.6.2.6}$$

4.7 DiPerna-Lions Theory for the Boltzmann Equation

In order to mathematically justify the fluid dynamical limits that were derived formally in the last section, two things must be made precise: (1) the notion of solution for the Boltzmann equation and (2) the sense in which the solutions fluctuate about the absolute Maxwellian. Ideally, the solutions should be global in time while the bounds and scalings should be physically natural. We therefore work in the setting of the DiPerna-Lions theory of renormalized solutions. The theory has the virtues of considering the physically natural class of initial data, and consequently, of yielding global solutions. These solutions have been used to study the incompressible Navier-Stokes limit and the incompressible Euler limit, the acoustic limit and the Stokes limit. The works have developed the theory introduced in [8], which uses the relative entropy and the entropy dissipation rate to control the fluctuations about the absolute Maxwellian.

DiPerna and P.-L. Lions defined the following solution:

Definition 6: A nonnegative function $F \in C(\mathbb{R}_+; L^1(\mathbb{R}^D \times \mathbb{R}^D))$ is a renormalized solution to the Boltzmann equation if and only if

$$\frac{\mathcal{B}(F,F)}{\sqrt{1+F}} \in L^1_{\mathrm{loc}}(dtdxdv) \tag{4.7.0.7}$$

and for each $\beta \in C^1(\mathbb{R}_+)$ s.t. $|\beta'(Z)| \leq \frac{C}{\sqrt{1+F}}$ for all $Z \geq 0$, one has

$$(\partial_t + v \cdot \nabla_x)\beta(F) = \beta'(F)\mathcal{B}(F,F) \tag{4.7.0.8}$$

in the sense of distribution on $\mathbb{R}_+^* \times \mathbb{R}^D \times \mathbb{R}^D$.

With this definition (actually a slightly more restrictive one,) DiPerna and P.-L. Lions proved the following remarkable result in [20].

Theorem 10: **DiPerna-Lions** *Let $F^{in} \geq 0$, a.e. satisfying*

$$\iint_{\mathbb{R}^D \times \mathbb{R}^D} (1 + |v|^2 + |\ln F^{in}(t,x,v)|) F^{in}(t,x,v)\, dxdv \leq +\infty\,, \tag{4.7.0.9}$$

and assume that the collision kernel b in Boltzmann's collision integral satisfies the weak cut-off assumption

$$b \in L^1_{\mathrm{loc}}(\mathbb{R}^D \times \mathbb{S}^{D-1},) \quad \frac{1}{1+|z|^2} \int_{|z-w|\leq R} \bar{b}(w)\, dw \to 0 \tag{4.7.0.10}$$

as $|z| \to +\infty$ for each $R > 0$. Then, there exists a renormalized solution to the Boltzmann equation satisfying the initial condition $F|_{t=0} = F^{in}$. Furthermore, this renormalized solution has the following properties

1. *it satisfies the equation of continuity*

$$\partial_t \int_{\mathbb{R}^D} F\, dv + \nabla_x \cdot \int_{\mathbb{R}^D} vF\, dv = 0\,, \tag{4.7.0.11}$$

and the following variant of the local conservation law of momentum:

$$\partial_t \int_{\mathbb{R}^D} vF\, dv + \nabla_x \cdot \int_{\mathbb{R}^D} v \otimes vF\, dv + \nabla_x \cdot \mathfrak{M} = 0\,, \tag{4.7.0.12}$$

where $\mathfrak{M}$ is a nonnegative symmetric matrix whose entries belong to $L^\infty(\mathbb{R}_+; \mathcal{M}(\mathbb{R}^D))$;

2. *it satisfies the total mass and momentum conservation*

$$\begin{aligned} \iint_{\mathbb{R}^D \times \mathbb{R}^D} F(t)\, dxdv &= \iint_{\mathbb{R}^D \times \mathbb{R}^D} F^{in}(t)\, dxdv\,, \\ \iint_{\mathbb{R}^D \times \mathbb{R}^D} vF(t)\, dxdv &= \iint_{\mathbb{R}^D \times \mathbb{R}^D} vF^{in}(t)\, dxdv\,, \end{aligned} \tag{4.7.0.13}$$

together with the following energy inequality: for each $t \geq 0$*,*

$$\iint\limits_{\mathbb{R}^D\times\mathbb{R}^D} \tfrac{1}{2}|v|^2 F(t)\,dxdv \leq \iint\limits_{\mathbb{R}^D\times\mathbb{R}^D} \tfrac{1}{2}|v|^2 F^{in}(t)\,dxdv\,, \tag{4.7.0.14}$$

more precisely, for a.e. $t \geq 0$*, one has*

$$\iint\limits_{\mathbb{R}^D\times\mathbb{R}^D} \tfrac{1}{2}|v|^2 F(t)\,dxdv + \tfrac{1}{2}\int_{\mathbb{R}^D} \operatorname{tr}(\mathfrak{M})(t) = \iint\limits_{\mathbb{R}^D\times\mathbb{R}^D} \tfrac{1}{2}|v|^2 F^{in}(t)\,dxdv\,, \tag{4.7.0.15}$$

3. finally, it satisfies the entropy inequality: for each $t \geq 0$*,*

$$\begin{aligned}\frac{1}{4}\int_0^t ds \int_{\mathbb{R}^D} dx \iiint\limits_{\mathbb{R}^D\times\mathbb{R}^D\mathbb{S}^{D-1}} (F'F_1' - FF_1)\ln\left(\frac{F'F_1'}{FF_1}\right) b(v-v_1,\omega)\,d\omega dv dv_1 \\ \leq \iint\limits_{\mathbb{R}^D\times\mathbb{R}^D} F^{in}\ln F^{in}\,dxdv - \iint\limits_{\mathbb{R}^D\times\mathbb{R}^D} F\ln F(t)\,dxdv\,.\end{aligned} \tag{4.7.0.16}$$

DiPerna-Lions renormanalized solutions are not known to satisfy many properties that one would formally expect to be satisfied by solutions to the Boltzmann equation. In particular, the theory does not assert either the local conservation of momentum, the global conservation of energy, the global entropy equality, or even a local entropy inequality; nor does it assert the uniqueness of the solution. Nevertheless, it provides enough control to establish the limits to incompressible linearized models, for example, Stokes system. Under some assumption on collision kernel, the limit to the weakly nonlinear model can be justified [34, 35].

5. FLUID DYNAMICS FROM BOLTZMANN EQUATIONS

In this chapter, we study the weakly compressible approximations of gas dynamics in kinetic setting. We start from the Boltzmann equation, to derive the fluid dynamical systems. Fluid dynamics regimes are those where the mean free path is small compared to the macroscopic length scales, i.e., where the Knudsen number ϵ is small. Hilbert [39] proposed that at the formal level all derivations of fluid dynamics should be based on a systematic asymptotic expansions in ϵ. A slightly different asymptotic expansion in ϵ, the *Chapman-Enskog expansion*, was proposed later by Enskog [22]. The Chapman-Enskog expansion yields at successive orders the compressible Euler system and the compressible Navier-Stokes system. In this chapter, we use moment-based formal derivations [7, 8, 9], which put fewer demands on the well-posedness and regularity of the solutions to the fluid equations.

In section 5.1, first we present moment-based formal derivations of the acoustic system from the Boltzmann equation. Then in section 5.2 we state the formal limits theorem of the Boussinesq-balanced incompressible models, including incompressible Stokes, incompressible Navier-Stokes, and incompressible Euler systems. In section 5.3, We then review the corresponding convergence theorems from DiPerna-Lions solutions of the Boltzmann equation to solutions of the fluid equations. This program began with Bardos-Golse-Levermore (BGL), later joined by P.-L. Lions, Masmoudi and Saint-Raymond. All the formal and rigorous justification of these fluid limits for Boussinesq-balanced incompressible models are about the well-prepared initial data in the sense that the initial moments of the fluctuation obey the incompressibility and Boussinesq relations.

In the last section of this chapter, we state our new formal derivations of the weakly nonlinear hydrodynamic limits for the general initial data, i.e., the initial data are not necessary to satisfy the incompressibility and Boussenesq relations. In this case, the fast acoustic waves are expected to occur. Even in the formal sense, the derivations are not trivial. we employ our averaging method developed in the chapter 2 and 3 to formally derive that *asymtotically*, the fluid behavior of the Boltzmann equation is governed by linear or weakly nonlinear models, such as weakly compressible Stokes and weakly compressible Navier-Stokes system. The projections of these weakly nonlinear fluids systems on the slow modes are incompressible Stokes, Navier-Stokes, and Euler systems, which are consistent with the formal limits results before. When the initial data are not well-prepared, the projections on the fast modes are nontrivial. We derive the averaged equations which describe the propagations of the fast waves. Thus, we generalize the formal weakly nonlinear hydrodynamic limits before.

The weakly compressible Stokes system and weakly nonlinear Navier-Stokes system can be formally derived from the Boltzmann equation through a scaling in which the density F is close to the absolute Maxwellian M. More precisely, we consider families of solutions parametrized by the Knudsen number ϵ that have the form

$$G_\epsilon^{in} = 1 + \delta_\epsilon g_\epsilon^{in} \,, \quad G_\epsilon = 1 + \delta_\epsilon g_\epsilon \,, \tag{5.0.0.1}$$

where G_ϵ are relative density defined in the last chapter, which satisfies the scaled Boltzmann equation with the initial data G_ϵ^{in}, i.e.,

$$\partial_t G + v \cdot \nabla_x G = \frac{1}{\epsilon} \mathcal{Q}(G, G,) \qquad G(0, x, v) = G^{in}(x, v) \geq 0 \,. \tag{5.0.0.2}$$

where the collision operator is now given by

$$\mathcal{Q}(G, G) = \iint_{\mathbb{S}^{D-1} \times \mathbb{R}^D} (G_1' G' - G_1 G) b(v_1 - v, \omega) \, d\omega M_1 dv_1 \,, \tag{5.0.0.3}$$

We assume formally that the fluctuations g_ϵ^{in} and g_ϵ are bounded while $\delta_\epsilon > 0$ satisfies

$$\delta_\epsilon \to 0 \quad \text{as } \epsilon \to 0 \,. \tag{5.0.0.4}$$

We also assume the normalizations of the collision kernel b, the measures on $\mathbb{S}^{D-1}$, $\mathbb{R}^D$, $\mathbb{T}^D$, and of the initial data, as we did in the last chapter, see (4.6.1.15), (4.6.1.16), and (4.6.1.17).

In these derivations we assume that g_ϵ converges formally to g, where the limiting function g is in $L^\infty(dt; L^2(M dv dx))$, and that all formally small terms vanish.

For example, we express the global conservation laws, which are the same for all of our derivations, in terms of g_ϵ and then formally let $\epsilon \to 0$ to obtain

$$\int_{\mathbb{T}^D} \langle g(t) \rangle \, dx = 0 \,, \quad \int_{\mathbb{T}^D} \langle v g(t) \rangle \, dx = 0 \,, \quad \int_{\mathbb{T}^D} \langle \tfrac{1}{2} |v|^2 g(t) \rangle \, dx = 0 \,. \tag{5.0.0.5}$$

Henceforth, the derivations differ.

5.1 *Formal Derivation of Acoustic System*

Before we formally derive the weakly compressible Stokes system and weakly nonlinear Navier-Stokes system, we derive the acoustic system. All the results in this section we present here belong to Bardos-Golse-Levermore, [9, 29]. Basically, we follow their presentation. The acoustic system is the linearization about the homogeneous state of the compressible Euler system. After a suitable choice of units, int his system the fluid fluctuations (ρ, u, θ) satisfy

$$
\begin{aligned}
\partial_t \rho + \nabla_x \cdot u = 0\,, &\quad \rho(0,x) = \rho^{in}(x)\,, \\
\partial_t u + \nabla_x(\rho + \theta) = 0\,, &\quad u(0,x) = u^{in}(x)\,, \\
\tfrac{D}{2}\partial_t \theta + \nabla_x \cdot u = 0\,, &\quad \theta(0,x) = \theta^{in}(x)\,.
\end{aligned}
\tag{5.1.0.6}
$$

This is one of the simplest system of fluid dynamical equations imaginable, being essentially the wave equation.

Acoustic scaling. It is most natural to derive the acoustic system first because its derivation is simpler and requires no additional assumptions regarding either the scaling or the collision kernel. One sets St $= 1$ and considers a family of formal solutions G_ϵ to the scaled Boltzmann initial-value problem

$$
\partial_t G + v \cdot \nabla_x G = \frac{1}{\epsilon}\mathcal{Q}(G, G,) \qquad G(0,x,v) = G^{in}(x,v)\,. \tag{5.1.0.7}
$$

with $G_\epsilon = 1 + \delta_\epsilon g_\epsilon$ for some δ_ϵ that satisfies

$$
\delta_\epsilon \to 0 \quad \text{as } \epsilon \to 0\,. \tag{5.1.0.8}
$$

The derivation of the acoustic system has two steps [29, 9].

Step 1: Limiting number density fluctuations. We first determine the form of the limiting function g. Observe that the fluctuation g_ϵ satisfy

$$\epsilon(\partial_t g_\epsilon + v \cdot \nabla_x g_\epsilon) + \mathcal{L} g_\epsilon = \delta_\epsilon \mathcal{Q}(g_\epsilon, g_\epsilon)\,, \tag{5.1.0.9}$$

where the linearized collision operator $\mathcal{L}$ is defined formally by

$$\begin{aligned} \mathcal{L}\tilde{g} &= -2\mathcal{Q}(1, \tilde{g}) \\ &= \iint_{\mathbb{R}^D \times \mathbb{S}^{D-1}} (\tilde{g} + \tilde{g}_* - \tilde{g}' - \tilde{g}'_*) b(v - v_*, \omega)\, d\omega M_* dv_* \end{aligned} \tag{5.1.0.10}$$

It can be shown that $\mathcal{L}$ is defined as the unique nonnegative, self-adjoint extension over $L^2(M dv)$ of this formal operator [13]. By letting $\epsilon \to 0$ above one finds that $\mathcal{L} g = 0$. Hence, $g(t, x, \cdot)$ takes values in $\mathrm{Null}(\mathcal{L})$, the null space of $\mathcal{L}$.

We recall that

$$\mathrm{Null}(\mathcal{L}) = \mathrm{span}\{1, v_1, \cdots v_D, |v|^2\}\,. \tag{5.1.0.11}$$

Because the limit $g(t, x, \cdot)$ takes values in $\mathrm{Null}(\mathcal{L})$ and g is assumed to belong to $L^\infty(dt; L^2(M dv dx))$, we conclude that g has the form

$$g(t, x, v) = \rho(t, x) + u(t, x) \cdot v + \theta(t, x)(\tfrac{1}{2}|v|^2 - \tfrac{D}{2})\,, \tag{5.1.0.12}$$

for some (ρ, u, θ) in $L^\infty(dt; L^2(dx; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R}))$.

This form is called an *infinitesimal Maxwellain* because

$$\begin{aligned} \mathcal{M}_{(1+\delta\rho, \delta u, 1+\delta\theta)} &= \frac{1 + \delta\rho}{(2\pi(1 + \delta\theta))^{\frac{D}{2}}} \exp\left(\frac{|v - \delta u|^2}{2(1 + \delta\theta)}\right) \\ &= M(1 + \delta g + O(\delta^2))\,. \end{aligned} \tag{5.1.0.13}$$

Step 2: Conservation Laws. Next we show that the evolution of (ρ, u, θ) is governed by the acoustic system. Observe that the fluctuations g_ϵ formally satisfy the local conservation laws

$$\begin{aligned}
\partial_t\langle g_\epsilon\rangle + \nabla_x \cdot \langle v g_\epsilon\rangle &= 0\,,\\
\partial_t\langle v g_\epsilon\rangle + \nabla_x \cdot \langle v\otimes v g_\epsilon\rangle &= 0\,,\\
\partial_t\langle \tfrac{1}{2}|v|^2 g_\epsilon\rangle + \nabla_x \cdot \langle v\tfrac{1}{2}|v|^2 g_\epsilon\rangle &= 0\,.
\end{aligned} \tag{5.1.0.14}$$

Letting $\epsilon \to 0$ in these equations and using the infinitesimal Maxwellian form of g, one then finds that (ρ, u, θ) solves the acoustic system.

Next, assuming the continuity of the above densities in time, one finds that

$$(\rho^{in}, u^{in}, \theta^{in}) = \lim_{\epsilon\to 0}(\langle g_\epsilon^{in}\rangle, \langle v g_\epsilon^{in}\rangle, \langle(\tfrac{1}{D}|v|^2 - 1)g_\epsilon^{in}\rangle)\,, \tag{5.1.0.15}$$

provided we assume that the limits on the right-hand side exist in the sense of distributions for some $(\rho^{in}, u^{in}, \theta^{in}) \in L^2(dx; \mathbb{R}\times\mathbb{R}^D\times\mathbb{R})$.

Theorem 11: **(The Formal Acoustic Limit Theorem)** *Let G_ϵ be a family of distribution solutions to the scaled Boltzmann initial-value problem* (5.1.0.7) *with initial data G_ϵ^{in} that satisfy the normalization* (4.6.1.17) *above. Let $G_\epsilon^{in} = 1 + \delta_\epsilon g_\epsilon^{in}$ and $G_\epsilon = 1 + \delta_\epsilon g_\epsilon$ where $\delta_\epsilon \to 0$ as $\epsilon \to 0$, and the fluctuations g_ϵ^{in} and g_ϵ are bounded in $L^\infty(dt; L^2(M dv dx))$. Moreover:*

1. *Assume that in the sense of distributions the family g_ϵ^{in} satisfies*

$$\lim_{\epsilon\to 0}(\langle g_\epsilon^{in}\rangle, \langle v g_\epsilon^{in}\rangle, \langle(\tfrac{1}{D}|v|^2 - 1)g_\epsilon^{in}\rangle) = (\rho^{in}, u^{in}, \theta^{in}) \tag{5.1.0.16}$$

   for some $(\rho^{in}, u^{in}, \theta^{in}) \in L^2(dx; \mathbb{R}\times\mathbb{R}^D\times\mathbb{R})$;

2. *Assume that the local conservation laws* (5.1.0.14) *are also satisfied in the sense of distributions for every g_ϵ;*

3. *Assume that the family g_ϵ converges in the sense of distributions as $\epsilon \to 0$ to $g \in C_b(dt; L^2(M dv dx))$; assume furthermore that $\mathcal{L}g_\epsilon \to \mathcal{L}g$, that the moments*

$$\langle g_\epsilon \rangle, \quad \langle v g_\epsilon \rangle, \quad \langle v \otimes v g_\epsilon \rangle, \quad \langle v|v|^2 g_\epsilon \rangle \,, \tag{5.1.0.17}$$

converge to the corresponding moments

$$\langle g \rangle, \quad \langle v g \rangle, \quad \langle v \otimes v g \rangle, \quad \langle v|v|^2 g \rangle \,, \tag{5.1.0.18}$$

and that every formally small term vanishes, all in the sense of distributions as $\epsilon \to 0$.

Then g is the unique local infinitesimal Maxwellian (5.1.0.12) *determined by the solution (ρ, u, θ) of the acoustic system with initial data $(\rho^{in}, u^{in}, \theta^{in})$.*

Complete derivations of the acoustic system from the Boltzmann equation are to be found in [9] in the case of bounded collision kernels, and in [29] for more general kernels.

5.2 Formal Derivation of Incompressible Systems

It is easily seen that any $(\rho, u, \theta) \in L^2(dx; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R})$ such that

$$\nabla_x \cdot u = 0 \,, \quad \nabla_x(\rho + \theta) = 0 \,, \tag{5.2.0.19}$$

is a stationary solution of the acoustic system which will generally vary in space. On the other hand, it can be shown that absolute Maxwellians are only stationary solutions to the Boltzmann equation.

It is clear that the time scale at which the acoustic system was derived was not long enough to see the evolution of these solutions. By considering the Boltzmann equation over a longer time scale. By considering the Boltzmann equation over a longer time scale one can give formal derivations of these incompressible fluid dynamics, depending on the limiting behavior of the ratio $\frac{\delta_\epsilon}{\epsilon}$ as $\epsilon \to 0$. In order to identify how the different regimes arise, we reconsider the Boltzmann initial-value problem on a time scale $\frac{1}{\tau_\epsilon}$, where

$$\tau_\epsilon \to 0 \quad \text{as} \quad \epsilon \to 0\,. \tag{5.2.0.20}$$

Upon setting St $= \tau_\epsilon$, the scaled Boltzmann initial-value problem becomes

$$\tau_\epsilon \partial_t G + v \cdot \nabla_x G = \frac{1}{\epsilon}\mathcal{Q}(G, G,) \qquad G(0, x, v) = G^{in}(x, v)\,. \tag{5.2.0.21}$$

The idea is to identify possible choices for τ_ϵ by seeking different balances between terms as ϵ tends to zero. We will show the following:

- When $\frac{\delta_\epsilon}{\epsilon} \to 0$, one considers time scales of order $\frac{1}{\epsilon}$, i.e., $\tau_\epsilon = \epsilon$, and an incompressible Stokes system is derived.
- when $\frac{\delta_\epsilon}{\epsilon} \to 1$ (or other nonzero number,) one considers time scales of order $\frac{1}{\epsilon}$, i.e., $\tau_\epsilon = \epsilon$, and an incompressible Navier-Stokes system is derived.
- When $\frac{\delta_\epsilon}{\epsilon} \to \infty$, one considers time scales of order $\frac{1}{\delta_\epsilon}$, i.e., $\tau_\epsilon = \delta_\epsilon$, and an incompressible Euler system is derived.

In the previous works [7, 8, 29], the derivations from the Boltzmann equation to the incompressible fluids models work only to the well-prepared initial data in the sense that the initial $(\rho^{in}, u^{in}, \theta^{in})$ satisfies the incompressibility and Boussinesq relation (5.2.0.19). Before we state these formal results, we list the Boussinesq-Balanced incompressible fluid systems: the incompressible Stokes, Navier-Stokes, and Euler systems. They can be derived all govern the fluctuations of mass density, bulk velocity, and temperature about their spatially homogeneous equilibrium values [7, 8, 29]. By suitable choices of Galilean frame and units, one can assume that these equilibrium values are 1, 0, and 1 respectively. We denote the fluctuations about these values by (ρ, u, θ).

For all three systems these fluctuations satisfy the incompressibility and Boussinesq relations

$$\nabla_x \cdot u = 0 \,, \qquad \nabla_x(\rho + \theta) = 0 \,, \tag{5.2.0.22}$$

The systems differ however in the equations that govern the dynamics of these fluctuations.

For the Stokes system the dynamics equations are

$$\begin{aligned} \partial_t u + \nabla_x p = \mu \Delta_x u \,, &\qquad u(x,0) = u^{in}(x) \,, \\ \tfrac{D+2}{2} \partial_t \theta = \kappa \Delta_x \theta \,, &\qquad \theta(x,0) = \theta^{in}(x) \,, \end{aligned} \tag{5.2.0.23}$$

where $\mu > 0$ is the kinetic viscosity and $\kappa > 0$ is the thermal diffusivity. Like the acoustic system, the Stokes system is also one of the simplest systems of fluid dynamical equations imaginable, being essentially a system of linear heat equations.

For the Navier-Stokes system the dynamical equations are

$$\begin{aligned} \partial_t u + u \cdot \nabla_x u + \nabla_x p = \mu \Delta_x u \,, &\qquad u(x,0) = u^{in}(x) \,, \\ \tfrac{D+2}{2} (\partial_t \theta + u \cdot \nabla_x \theta) = \kappa \Delta_x \theta \,, &\qquad \theta(x,0) = \theta^{in}(x) \,, \end{aligned} \tag{5.2.0.24}$$

where the kinematic viscosity μ and the thermal diffusivity κ have the same values as in the Stokes system. Unlike the Stokes system however, the Navier-Stokes system is nonlinear. While this fact does not complicate its formal derivation, it makes the mathematical establishment of its validity much harder.

For the Euler system the dynamical equations are

$$\begin{aligned} \partial_t u + u \cdot \nabla_x u + \nabla_x p = 0\,, \qquad & u(x,0) = u^{in}(x)\,, \\ \partial_t \theta + u \cdot \nabla_x \theta = 0\,, \qquad & \theta(x,0) = \theta^{in}(x)\,. \end{aligned} \tag{5.2.0.25}$$

Like the Navier-Stokes system, the Euler system is nonlinear. The full mathematical establishment of its validity is also an open problem.

As was the case for the acoustic system, the Euler system has stationary solutions that vary in space. It is clear that the time scales at which the Euler system are derived was not long enough to see the evolution of these solutions. Even at a formal level it is unclear how this long-time evolution should be governed.

Now we state the theorem due to Bardos, Golse, and Levermore [7] about the formal derivation of the incompressible limits.

Theorem 12: **(Formal Incompressible Limits Theorem)** *Let G_ϵ be a family of distribution solutions to the scaled Boltzmann initial-value problem* (5.2.0.21) *with initial data G^{in} that satisfy the normalizations* (4.6.1.17). *Let $G_\epsilon^{in} = 1 + \delta_\epsilon g_\epsilon^{in}$ and $G_\epsilon = 1 + \delta_\epsilon g_\epsilon$ where $\delta_\epsilon \to 0$ as $\epsilon \to 0$, and the fluctuations g_ϵ^{in} and g_ϵ are bounded in $L^\infty(dt; L^2(Mdvdx))$. Moreover:*

1. *Assume that in the sense of distributions the family g_ϵ^{in} satisfies*

$$\lim_{\epsilon \to 0} (\mathcal{P}\langle v g_\epsilon^{in} \rangle, \langle (\tfrac{1}{D+2}|v|^2 - 1) g_\epsilon^{in} \rangle) = (u^{in}, \theta^{in}) \tag{5.2.0.26}$$

for some $(u^{in}, \theta^{in}) \in L^2(dx; \mathbb{R}^3 \times \mathbb{R})$ *where* P *denotes the Leray projection onto divergence-free vector fields;*

2. *Assume that the local consideration laws* (5.1.0.14) *are also satisfied in the sense of distributions for every* g_ϵ*;*

3. *Assume that the family* g_ϵ *converges in the sense of distributions as* $\epsilon \to 0$ *to* $g \in L^\infty(dt; L^2(Mdvdx))$*. Furthermore, assume that* $\mathcal{L}g_\epsilon \to \mathcal{L}g$*, while for each* $\xi \in L^2(Mdv)$ *the moments* $\langle \xi g_\epsilon \rangle$ *converges to* $\langle \xi g \rangle$*, and that every formally small term vanishes, all in the sense of distributions as* $\epsilon \to 0$*.*

Then g *is the unique local infinitesimal Maxwellian* (5.1.0.12) *determined by the solutions* (u, θ) *of the Stokes system when* $\frac{\delta_\epsilon}{\epsilon} \to 0$*, the Navier-Stokes system when* $\frac{\delta_\epsilon}{\epsilon} \to 1$*, or the Euler system when* $\frac{\delta_\epsilon}{\epsilon} \to \infty$*, with initial data* (u^{in}, θ^{in})*.*

In the above formal theorem, the initial condition (5.2.0.26) is well-prepared, i.e., it satisfies the incompressibility condition and the Boussinesq relations (note the difference between the initial data (5.1.0.16) and (5.2.0.26). A natural question is: What will happen when the initial data are not well-prepared? In the following section, we will answer this question formally. Our formal derivation automatically covers the above Bardos-Golse-Levermore theorem when the initial data satisfies the incompressibility and Boussinesq relations. So we skip the proof of the Bardos-Golse-Levermore's formal limits theorem, which will be a special case of the formal weakly nonlinear asymptotics theorem.

5.3 Review of Convergence Results

In this section we review some rigorous convergence results for the fluid limits from the Boltzmann equation. We closely followed Golse-Levermore's survey paper [30]. The program laid out by Bardos, Golse, and Levermore (BGL) [8] is to identify those fluid dynamical systems that can be obtained through a moment-based formal derivation and to mathematically justify of those formal derivations.

In order to carry out this program, we must make precise the notion of solution to the Boltzmann equation, and the notion of solution for the fluid dynamical systems. Ideally, these solutions should be global in time, while the estimates should be physically natural.

We therefore work in the setting of DiPerna-Lions renormalized solutions to the Boltzmann equation, in the settings of L^2 solutions to the acoustic and Stokes systems, and in the setting of Leray solutions to the Navier-Stokes equations. These theories have the virtues of considering pahysically natural calsses of initial data, and consequently, of yielding global solutions.

There is no such theory for the Euler system so far, so one must work in the setting of local classical solutions for Euler limits.

One of the central goals for the BGL program is to connect the DiPerna-Lions theory of renormalized solutions to the Boltzmann equation to the Leray theory of weak solutions to the incompressible Navier-Stokes equations in space dimension 3. The main result of [8] for the Navier-Stokes limit is to recover the motion equation for a discrete-time version of the Boltzmann equation assuming the DiPerna-Lions solutions satisfy the local conservation of momentum and with the aid of mild compactness assumption. This result falls short of the goal in five respects.

1. First, the heat equation was not treated because the $|v|^2 v$ terms in the heat flux could not be controlled.

2. Second, local momentum conservation was assumed because all DiPerna-Lions solutions are not known to satisfy the local conservation law of momentum (or energy) that one would formally expect.

3. Third, unnatural technical assumptions were made on the Boltzmann collision kernel.

4. Fourth, the discrete-time case was treated in order to avoid to control the time regularity of the acoustic modes.

5. Finally, a mild compactness assumption was required to pass to the limit in certain nonlinear terms.

In recent works all of these shortcomings have been overcome.

The work of Bardos-Golse-Levermore. First Bardos, Golse, and Levermore [9] recover the acoustic and Stokes limits for the Boltzmann equation for cut-off collision kernels that arise from Maxwell potentials. In doing so, they control the energy flux and establish the local conservation laws of momentum and energy in the limit. The scaling they used was not optimal, essentially requiring

$$\begin{aligned} \frac{\delta_\epsilon}{\epsilon} \to 0 \quad \text{rather than} \quad \delta_\epsilon \to 0 \quad \text{for the acoustic limit}\,, \\ \frac{\delta_\epsilon}{\epsilon^2} \to 0 \quad \text{rather than} \quad \frac{\delta_\epsilon}{\epsilon} \to 0 \quad \text{for the Stokes limit}\,. \end{aligned} \tag{5.3.0.27}$$

The work of Lions-Masmoudi. Lions and Masmoudi [52] recover the Navier-Stokes motion equation with the aid of only the local conservation of momentum assumption and the nonlinear compactness assumption made in [8]. However, they do not recover the heat equation and they retain the same unnatural technical assumptions made in [8] on the collision kernel.

There are two key new ingredients in their work. First, they were able to control the time regularity of acoustic modes. Second, they were able to prove that the contribution of the acoustic modes to the limiting motion equation is just an extra gradient term that can be incorporated into the pressure term.

They also cover the Stokes motion equation in [53] without the local conservation of momentum assumption and with essentially optimal scaling. However, they do not recover the heat equation and they retain the same unnatural technical assumption made in [8] on the collision kernel.

There are two reasons why they do not recover the heat equation. First, it is unknown whether or not DiPerna-Lions solutions satisfy a local energy conservation law. Second, even if local energy conservation were assumed, the techniques they used to control the momentum flux would fail to control the heat flux.

The work of Golse-Levermore. Golse and Levermore [29] recover the acoustic and Stokes systems. They make natural assumptions on the collision kernel that include those classically derived from hard potentials.

For the Stokes limit they recover both the motion and heat equations with a near optimal scaling.

For the acoustic limit the scaling they used was not optimal, essentially requiring

$$\frac{\delta_\epsilon}{\epsilon^{\frac{1}{2}}} \to 0 \quad \text{rather than} \quad \delta_\epsilon \to 0\,. \tag{5.3.0.28}$$

There were two key new ingredients in this work. First, they control the local momentum and energy conservation defects of the DiPerna-Lions solutions with dissipation rate estimates that allowed them to recover these local conservation laws in the limit. Second, they also control the heat flux with dissipation rate estimates.

Because they treat the linear Stokes case, they do not face the need either to control the acoustic modes or for a compactness assumption, both of which are used to pass to the limit in the nonlinear terms in [52].

The Work of Golse-Saint Raymond. Without making any nonlinear compactness hypothesis, Golse-Saint Raymond [34] recover the Navier-Stokes system for the Boltzmann equation with cut-off collision kernels that arise from Maxwell potentials. Their majot breakthrough was the development of a new L^1 averaging lemma [32] to prove the compactness assumption. This was extracted from Saint-Raymond [65] where she recovered the Navier-Stokes limit for the BGK model. Their proof also employs key elements from [52] and [29]. Recently they have extended their result to the hard sphere collision kernel.

The Work of Levermore-Masmoudi. This extends the work of Golse and Saint-Raymond. It recovers the Navier-Stokes system for the Boltzmann equation with weakly cut-off collision kernels that arise from a wide range of hard and soft potentials.

Using the L^1 averaging lemma of Golse-Saint Raymond, they show that this nonlinear compactness hypothesis is satisfied for soft potential. New estimates allow one to extend the analysis beyond Grad cut-off collision kernels. These new estimates also allow one to carry out the acoustic and Stokes limits for soft potentials.

5.4 *Formal Derivation of the Weakly Compressible Navier-Stokes System*

In this section, we state our new formal derivations of the weakly nonlinear hydrodynamic limits for the general initial data, i.e., the initial data are not necessary to satisfy the incompressibility and Boussenesq relations. In this case, the fast acoustic waves occur. we use averaging method developed in the chapter 2 and 3 to formally derive that *asymtotically*, the fluid behavior of the Boltzmann equation is governed by linear or weakly nonlinear models, such as weakly compressible Stokes and weakly compressible Navier-Stokes system. The projections of these weakly nonlinear fluids systems on the incompressible modes are incompressible Stokes, Navier-Stokes, and Euler systems, which are consistent with the formal limits results before. When the initial data are not well-prepared, the projections on the fast modes are nontrivial. We derive the averaged equations which describe the propagations of the fast waves. Thus, we generalize the formal weakly nonlinear hydrodynamic limits before.

We start from considering a family of formal solutions G_ϵ to the scaled Boltzmann initial-value problem

$$\tau_\epsilon \partial_t G + v \cdot \nabla_x G = \frac{1}{\epsilon} \mathcal{Q}(G, G,) \qquad G(0, x, v) = G^{in}(x, v) \, . \tag{5.4.0.29}$$

whose fluctuations g_ϵ are given by (5.0.0.1) for some δ_ϵ that satisfies

$$\frac{\delta_\epsilon}{\epsilon} \to 0, \quad \text{or} \quad 1, \quad \text{or} \quad \infty \quad \text{as } \epsilon \to 0 \, . \tag{5.4.0.30}$$

the time scale $\tau_\epsilon = \epsilon$ for slow time (Stokes) scale, $\tau_\epsilon = 1$ for fast time (acoustic) scale.

The family of the fluctuations g_ϵ formally satisfied the local conservation laws

$$\begin{aligned} \partial_t \langle g_\epsilon \rangle + \frac{1}{\tau_\epsilon} \nabla_x \cdot \langle v g_\epsilon \rangle &= 0 \, , \\ \partial_t \langle v g_\epsilon \rangle + \frac{1}{\tau_\epsilon} \nabla_x \cdot \langle v \otimes v g_\epsilon \rangle &= 0 \, , \\ \partial_t \langle \tfrac{1}{2} |v|^2 g_\epsilon \rangle + \frac{1}{\tau_\epsilon} \nabla_x \cdot \langle v \tfrac{1}{2} |v|^2 g_\epsilon \rangle &= 0 \, . \end{aligned} \tag{5.4.0.31}$$

From the formal derivations of acoustic system and incompressible systems, in the limit $\epsilon \to 0$, the *limit* of the fluctuations g_ϵ will be in the null space of the linearized collision operator $\mathcal{L}$. So it is very natural to consider the projection of the fluctuations g_ϵ on the $\mathrm{Null}(\mathcal{L})$. If we denote these projection as $\mathcal{P} g_\epsilon$, then $\mathcal{P}^\perp g_\epsilon = (I - \mathcal{P}) g_\epsilon \in \mathrm{Null}(\mathcal{L})^\perp$ will be very small *asymptotically* as $\epsilon \to 0$, because in the limit, it will vanish.

From simple linear algebra, the orthogonal projection from $L^2(M dv)$ onto $\mathrm{Null}(\mathcal{L})$, which is spanned by $1, v_1, \cdots, v_D, |v|^2$, is: for every $\tilde{g} \in L^2(M dv)$

$$\mathcal{P} \tilde{g} = \langle \tilde{g} \rangle + \langle v \tilde{g} \rangle \cdot v + \left\langle \left(\tfrac{1}{D} |v|^2 - 1 \right) \tilde{g} \right\rangle \left(\tfrac{1}{2} |v|^2 - \tfrac{D}{2} \right) \, . \tag{5.4.0.32}$$

Comparing with the form of the infinitesimal Maxwellian (5.1.0.12), it is very natural to define the fluid variables associated with the fluctuation of the number density $\tilde{g}$:

$$\hat{\rho} = \langle \tilde{g} \rangle\,, \quad \hat{u} = \langle v\tilde{g} \rangle\,, \quad \hat{\theta} = \tfrac{2}{D} \left\langle \left(\tfrac{1}{2}|v|^2 - \tfrac{D}{2}\right) \tilde{g} \right\rangle\,. \tag{5.4.0.33}$$

Using these notations, the local conservation laws (5.4.0.31) can be written as:

$$\begin{aligned} \partial_t \hat{\rho}_\epsilon + \frac{1}{\tau_\epsilon} \nabla_x \cdot \hat{u}_\epsilon = 0\,, \\ \partial_t \hat{u}_\epsilon + \frac{1}{\tau_\epsilon} \nabla_x (\hat{\rho}_\epsilon + \hat{\theta}_\epsilon) + \frac{1}{\tau_\epsilon} \nabla_x \cdot \langle A(v) g_\epsilon \rangle = 0\,, \\ \tfrac{D}{2} \partial_t \hat{\theta}_\epsilon + \frac{1}{\tau_\epsilon} \nabla_x \cdot \hat{u}_\epsilon + \frac{1}{\tau_\epsilon} \nabla_x \cdot \langle B(v) g_\epsilon \rangle = 0\,. \end{aligned} \tag{5.4.0.34}$$

where the matrix-valued function $A(v)$ and the vector-valued function $B(v)$ are defined by

$$A(v) = v \otimes v - \tfrac{1}{D}|v|^2 I\,, \qquad B(v) = \tfrac{1}{2}|v|^2 v - \tfrac{D+2}{2} v\,. \tag{5.4.0.35}$$

As we discuss in the last chapter, the entries of $A(v)$, and the components of $B(v)$ are elements in $\mathrm{Null}(\mathcal{L})^\perp$. Furthermore, $A_{ij}(v)$ and $B_k(v)$ are mutually perpendicular.

We assume that for some $l > 0$ the operator $\mathcal{L}$ satisfies the coercivity estimate

$$l\langle \xi^2 \rangle \leq \langle \xi \mathcal{L} \xi \rangle \quad \text{for every}\, \xi \in \mathrm{Dom}(\mathcal{L}) \cap \mathrm{Null}(\mathcal{L})^\perp\,. \tag{5.4.0.36}$$

This estimate hold for every linearized collision operator that arises from a classical hard potential with a small deflection cutoff. This assumption is equivalent to assuming that the Fredholm alternative holds for $\mathcal{L}$, namely, that $\mathrm{Range}(\mathcal{L}) = \mathrm{Null}(\mathcal{L})^\perp$. In particular, it implies that unique $\hat{A} \in L^2(Mdv; \mathbb{R}^{D\times D})$ and $\hat{B} \in L^2(Mdv; \mathbb{R}^D)$ exist which solve

$$\begin{aligned} \mathcal{L}\hat{A} = A\,, \quad \hat{A} \in \mathrm{Null}^\perp(\mathcal{L})\text{entrywise}\,, \\ \mathcal{L}\hat{B} = B\,, \quad \hat{B} \in \mathrm{Null}^\perp(\mathcal{L})\text{entrywise}\,. \end{aligned} \tag{5.4.0.37}$$

There are some deeper properties of the pseudo-inverses of $A(v)$ and $B(v)$: there exits two scalar functions α and β such that

$$\hat{A}(v) = \alpha(|v|)A(v)\,, \quad \hat{B}(v) = \alpha(|v|)B(v)\,. \tag{5.4.0.38}$$

Applying the self-adjoint property of the linearized collision operator $\mathcal{L}$, the unknown terms in the local conservation laws (5.4.0.34) are

$$\langle A(v)g_\epsilon\rangle = \langle \hat{A}(v)\mathcal{L}g_\epsilon\rangle\,, \quad \langle B(v)g_\epsilon\rangle = \langle \hat{B}(v)\mathcal{L}g_\epsilon\rangle\,, \tag{5.4.0.39}$$

Observe that by the original scaled Boltzmann equation (5.2.0.21) the fluctuations g_ϵ satisfy

$$\epsilon\partial_t g_\epsilon + \frac{\epsilon}{\tau_\epsilon} v\cdot\nabla_x g_\epsilon + \frac{1}{\tau_\epsilon}\mathcal{L}g_\epsilon = \frac{\delta_\epsilon}{\tau_\epsilon}\mathcal{Q}(g_\epsilon, g_\epsilon)\,. \tag{5.4.0.40}$$

We need to calculate the terms in (5.4.0.40). Notice that no matter the relative sides of δ_ϵ and ϵ, the term $\epsilon\partial_t g_\epsilon$ should be small asymptotically, as $\epsilon \to 0$. So we need only consider the convection term $v\cdot\nabla_x g_\epsilon$ and the quadratic term $\mathcal{Q}(g_\epsilon, g_\epsilon)$.

Let's denote by

$$\mathcal{P}g_\epsilon = \hat{\rho}_\epsilon + \hat{u}_\epsilon\cdot v + \hat{\theta}_\epsilon(\tfrac{1}{2}|v|^2 - \tfrac{D}{2}) \tag{5.4.0.41}$$

the infinitesimal Maxwellian of g_ϵ, then

$$v\cdot\nabla_x g_\epsilon = v\cdot\nabla_x\mathcal{P}g_\epsilon + v\cdot\nabla_x\mathcal{P}^\perp g_\epsilon\,, \tag{5.4.0.42}$$

and

$$\mathcal{Q}(g_\epsilon, g_\epsilon) = \mathcal{Q}(\mathcal{P}g_\epsilon, \mathcal{P}g_\epsilon) + \mathcal{Q}(\mathcal{P}^\perp g_\epsilon, \mathcal{P}g_\epsilon + \mathcal{P}^\perp g_\epsilon) \tag{5.4.0.43}$$

where $\mathcal{P}^\perp = I - \mathcal{P}$. Then from (5.4.0.40), we obtain

$$\frac{1}{\tau_\epsilon}\mathcal{L}g_\epsilon = -\frac{\epsilon}{\tau_\epsilon} v\cdot\nabla_x\mathcal{P}g_\epsilon + \frac{\delta_\epsilon}{\tau_\epsilon}\mathcal{Q}(\mathcal{P}g_\epsilon, \mathcal{P}g_\epsilon) - r_\epsilon\,, \tag{5.4.0.44}$$

where

$$r_\epsilon = \frac{\epsilon}{\tau_\epsilon} v \cdot \nabla_x \mathcal{P}^\perp g_\epsilon - \frac{\delta_\epsilon}{\tau_\epsilon} \mathcal{Q}(\mathcal{P}^\perp g_\epsilon, \mathcal{P} g_\epsilon + \mathcal{P}^\perp g_\epsilon) + \epsilon \partial_t g_\epsilon \,. \tag{5.4.0.45}$$

Thus, the local conservation laws (5.4.0.34) becomes

$$\begin{aligned}
\partial_t \hat{\rho}_\epsilon + \frac{1}{\tau_\epsilon} \nabla_x \cdot \hat{u}_\epsilon &= 0 \,, \\
\partial_t \hat{u}_\epsilon + \frac{1}{\tau_\epsilon} \nabla_x (\hat{\rho}_\epsilon + \hat{\theta}_\epsilon) &= \frac{\epsilon}{\tau_\epsilon} \nabla_x \cdot \langle \hat{A} v \cdot \nabla_x \mathcal{P} g_\epsilon \rangle - \frac{\delta_\epsilon}{\tau_\epsilon} \nabla_x \cdot \langle \hat{A} \mathcal{Q}(\mathcal{P} g_\epsilon, \mathcal{P} g_\epsilon) \rangle + R^1_\epsilon \,, \\
\tfrac{D}{2} \partial_t \hat{\theta}_\epsilon + \frac{1}{\tau_\epsilon} \nabla_x \cdot \hat{u}_\epsilon &= \frac{\epsilon}{\tau_\epsilon} \nabla_x \cdot \langle \hat{B} v \cdot \nabla_x \mathcal{P} g_\epsilon \rangle - \frac{\delta_\epsilon}{\tau_\epsilon} \nabla_x \cdot \langle \hat{B} \mathcal{Q}(\mathcal{P} g_\epsilon, \mathcal{P} g_\epsilon) \rangle + R^2_\epsilon \,,
\end{aligned} \tag{5.4.0.46}$$

where

$$R^1_\epsilon = \nabla_x \cdot \langle \hat{A} r_\epsilon \rangle \,, \qquad R^2_\epsilon = \nabla_x \cdot \langle \hat{B} r_\epsilon \rangle \,, \tag{5.4.0.47}$$

The first term on the right-hand sides are diffusion terms in the fluid equation, more precisely:

Lemma 14:

$$\begin{aligned}
\frac{\epsilon}{\tau_\epsilon} \nabla_x \cdot \langle \hat{A}(v) v \cdot \nabla_x \mathcal{P} g_\epsilon \rangle &= \frac{\epsilon}{\tau_\epsilon} \nabla_x \cdot [\mu(\nabla_x \hat{u}_\epsilon + \nabla_x \hat{u}^\tau_\epsilon - \tfrac{2}{D} \nabla_x \cdot \hat{u}_\epsilon)] \,, \\
\frac{\epsilon}{\tau_\epsilon} \nabla_x \cdot \langle \hat{B}(v) v \cdot \nabla_x \mathcal{P} g_\epsilon \rangle &= \frac{\epsilon}{\tau_\epsilon} \nabla_x \cdot (\tfrac{D+2}{2} \kappa \nabla_x \hat{\theta}_\epsilon) \,.
\end{aligned} \tag{5.4.0.48}$$

Proof: After simple calculations, we obtain

$$\begin{aligned}
v \cdot \nabla_x (\mathcal{P} g_\epsilon) =& A(v) : \nabla_x \hat{u}_\epsilon + B(v) \cdot \nabla_x \hat{\theta}_\epsilon \\
& + v \cdot \nabla_x (\hat{\rho}_\epsilon + \hat{\theta}_\epsilon) + \tfrac{1}{D} |v|^2 \nabla_x \cdot \hat{u}_\epsilon
\end{aligned} \tag{5.4.0.49}$$

Let $\zeta(v)$ denote $A(v)$ or $B(v)$, then $\hat{\zeta}(v) \in \mathrm{Null}(\mathcal{L})^\perp$. Thus the inner product of $\hat{\zeta}(v)$ with the last two terms in (5.4.0.49) vanish because they are in the null space of $\mathcal{L}$.

Then

$$\frac{\epsilon}{\tau_\epsilon}\langle\hat{\zeta}(v)v\cdot\nabla_x\mathcal{P}g_\epsilon\rangle = \frac{\epsilon}{\tau_\epsilon}\langle\hat{\zeta}A\rangle : \nabla_x u_\epsilon + \frac{\epsilon}{\tau_\epsilon}\langle\hat{\zeta}B\rangle\cdot\nabla_x\theta_\epsilon\,. \tag{5.4.0.50}$$

Notice that the relations (5.4.0.38), $\alpha(|v|)$ and $\beta(|v|)$ are even in v, $A(v)$ so as $\hat{A}(v)$ is even in v, and $B(v)$ so as $\hat{B}(v)$ is odd in v, then one obtain

$$\langle\hat{A}B\rangle = 0\,,\qquad \langle\hat{B}A\rangle = 0 \tag{5.4.0.51}$$

Thus

$$\frac{\epsilon}{\tau_\epsilon}\langle\hat{A}(v)v\cdot\nabla_x\mathcal{P}g_\epsilon\rangle = \frac{\epsilon}{\tau_\epsilon}\langle\hat{A}\otimes A\rangle : \nabla_x u_\epsilon\,, \tag{5.4.0.52}$$

and

$$\frac{\epsilon}{\tau_\epsilon}\langle\hat{B}(v)v\cdot\nabla_x\mathcal{P}g_\epsilon\rangle = \frac{\epsilon}{\tau_\epsilon}\langle\hat{B}\otimes B\rangle\cdot\nabla_x\theta_\epsilon\,. \tag{5.4.0.53}$$

To finish the proof of Lemma 14, we state the following lemma which was proved in [8] (Lemma 4.4,)

Lemma 15: *The components of $\hat{A}\otimes A$ and $\hat{B}\otimes B$ satisfy the following identities:*

$$\begin{aligned}\langle A_{ij}\otimes\hat{A}_{kl}\rangle &= \mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-\tfrac{2}{D}\delta_{ij}\delta_{kl})\,,\\ \langle B_i\otimes\hat{B}_j\rangle &= \tfrac{D+2}{2}\kappa\delta_{ij}\,,\end{aligned} \tag{5.4.0.54}$$

where μ and κ are given by

$$\mu = \tfrac{1}{(D-1)(D+2)}\langle A:\hat{A}\rangle\,,\quad \kappa = \tfrac{2}{D(D+2)}\langle B\cdot\hat{B}\rangle\,. \tag{5.4.0.55}$$

Applying Lemma 15 to (5.4.0.52) and (5.4.0.53), we finish the proof of Lemma 14. □

The derivation of the convection terms which are stated in the following lemma are more difficult.

Lemma 16:

$$\begin{aligned} \frac{\delta_\epsilon}{\tau_\epsilon}\nabla_x \cdot \langle \hat{A}(v)\mathcal{Q}(\mathcal{P}g_\epsilon, \mathcal{P}g_\epsilon)\rangle &= \frac{\delta_\epsilon}{\tau_\epsilon}\nabla_x \cdot (\hat{u}_\epsilon \otimes \hat{u}_\epsilon - \tfrac{1}{D}|\hat{u}_\epsilon|^2 I)\,, \\ \frac{\delta_\epsilon}{\tau_\epsilon}\nabla_x \cdot \langle \hat{B}(v)\mathcal{Q}(\mathcal{P}g_\epsilon, \mathcal{P}g_\epsilon)\rangle &= \frac{\delta_\epsilon}{\tau_\epsilon}\tfrac{D+2}{2}\nabla_x \cdot (\hat{u}_\epsilon\hat{\theta}_\epsilon)\,. \end{aligned} \tag{5.4.0.56}$$

Proof: The nonlinear term is simplified as follows.

For each $\phi \in \mathrm{Null}(\mathcal{L})$, one has

$$\mathcal{Q}(\phi, \phi) = \tfrac{1}{2}\mathcal{L}(\phi^2)\,. \tag{5.4.0.57}$$

To prove the above identity one simply takes the second derivative of the relation

$$\mathcal{B}(\mathcal{M}_{(\rho,u,\theta)}, \mathcal{M}_{(\rho,u,\theta)}) = 0 \tag{5.4.0.58}$$

with respect to the parameters $(\rho,\ u,\ \theta)$, and evaluates it at $(1, 0, 1)$. See [7] for a complete argument. □

Applying the above lemma, we obtain

$$\langle \hat{\zeta}(v)\mathcal{Q}(\mathcal{P}g_\epsilon, \mathcal{P}g_\epsilon)\rangle = \tfrac{1}{2}\langle \zeta(v)(\mathcal{P}g_\epsilon)^2\rangle\,. \tag{5.4.0.59}$$

where $(\mathcal{P}g_\epsilon)^2$ is given by

$$\begin{aligned} (\mathcal{P}g_\epsilon)^2 =& \hat{\rho}_\epsilon^2 + 2\hat{\rho}_\epsilon\hat{u}_\epsilon \cdot v + 2\hat{\rho}_\epsilon\hat{\theta}_\epsilon(\tfrac{1}{2}|v|^2\tfrac{D}{2}) + \hat{\theta}_\epsilon^2(\tfrac{D}{2}|v|^2 + \tfrac{D^2}{4}) \\ &+ (\hat{u}_\epsilon \cdot v)^2 + \hat{\theta}_\epsilon^2(\tfrac{1}{4}|v|^4) + \hat{\theta}_\epsilon\hat{u}_\epsilon \cdot v(|v|^2 - D)\,. \end{aligned} \tag{5.4.0.60}$$

The first four terms above are in the null space of $\mathcal{L}$, so their inner products with either A or B vanish. Furthermore, the last term is odd in v, and $A(v)$ is even in v, so their inner product is zero. Thus

$$\langle A_{ij}(v)(\mathcal{P}g_\epsilon)^2\rangle = \langle A_{ij}(v)(\hat{u}_\epsilon \cdot v)^2\rangle + \tfrac{1}{4}\langle |v|^4 A_{ij}(v)\rangle\hat{\theta}_\epsilon^2\,. \tag{5.4.0.61}$$

For a fixed pair (i,j), if $i \neq j$,

$$\langle A_{ij}(v)(u\cdot v)^2\rangle = 2\langle v_i^2 v_j^2\rangle u_i u_j = 2(u\otimes u)_{ij} \tag{5.4.0.62}$$

if $i=j$,

$$\begin{aligned}\langle v_i^2(u\cdot v)^2\rangle &= \langle v_i^4\rangle |u_i|^2 + \sum_{j\neq i}\langle v_i^2 v_j^2\rangle |u_j|^2\,,\\ & 3|u_i|^2 + \sum_{j\neq i}|u_j|^2\,,\\ &= |u|^2 + 2|u_i|^2\end{aligned} \tag{5.4.0.63}$$

thus

$$\begin{aligned}\langle A_{ii}(v)(u\cdot v)^2\rangle &= \langle v_i^2(u\cdot v)^2\rangle - \tfrac{1}{D}\langle |v|^2(u\cdot v)^2\rangle\\ &= |u|^2 + 2|u_i|^2 - \tfrac{1}{D}\sum_{j=1}^{D}\langle v_j^2(u\cdot v)^2\rangle\\ &= |u|^2 + 2|u_i|^2 - \tfrac{1}{D}(D|u|^2 + 2|u|^2)\\ &= 2|u|^2 - \tfrac{2}{D}|u|^2\,.\end{aligned} \tag{5.4.0.64}$$

Then we proved

$$\tfrac{1}{2}\langle A_{ij}(v)(\hat{u}_\epsilon\cdot v)^2\rangle = (\hat{u}_\epsilon\otimes\hat{u}_\epsilon)_{ij} - \tfrac{1}{D}|\hat{u}_\epsilon|^2\delta_{ij}\,. \tag{5.4.0.65}$$

Observe that

$$\langle \tfrac{1}{4}|v|^4 A_{ij}(v)\rangle = \tfrac{1}{4}\langle v_i v_j |v|^4\rangle - \tfrac{1}{4D}\langle |v|^6\rangle\delta_{ij} \tag{5.4.0.66}$$

If $i\neq j$, then $\langle v_i v_j|v|^4\rangle = 0$, so $\langle \frac{1}{4}|v|^4 A_{ij}(v)\rangle = 0$

If $i=j$, then $\frac{1}{4}\langle v_i^2|v|^4\rangle = \frac{1}{4D}\langle |v|^6\rangle$, we also obtain $\langle \frac{1}{4}|v|^4 A_{ij}(v)\rangle = 0$. Combine with (5.4.0.61), we proved the first identity in (5.4.0.56). Notice that $B(v)$ is in $\mathrm{Null}(\mathcal{L})^\perp$ and it is odd in v, after taking inner product with (5.4.0.60), what is left is

$$\langle B_i(v)(\mathcal{P}g_\epsilon)^2\rangle = \langle B_i(v)v_j(|v|^2 - D)\rangle \hat{u}_{\epsilon j}\hat{\theta}_\epsilon\,. \tag{5.4.0.67}$$

The coefficient $\langle B_i(v)v_j(|v|^2 - D)\rangle$ is

$$\langle B_i(v)v_j(|v|^2 - D)\rangle = \tfrac{1}{2}\langle v_i v_j |v|^4\rangle - (D+1)\langle v_i v_j |v|^2\rangle + \tfrac{D(D+2)}{2}\delta_{ij}\,. \tag{5.4.0.68}$$

After some simple calculations, we get

$$\begin{aligned} \tfrac{1}{2}\langle v_i v_j |v|^4\rangle &= \tfrac{1}{2}[15 + (D-1)(D+7)]\delta_{ij}\,, \\ (D+1)\langle v_i v_j |v|^2\rangle &= (D+1)(D+2)\delta_{ij} \end{aligned} \tag{5.4.0.69}$$

Then

$$\tfrac{1}{2}\langle B_i(v)v_j(|v|^2 - D)\rangle = \tfrac{D+2}{2}\,. \tag{5.4.0.70}$$

Thus we proved the second identity in (5.4.0.56).

Denote by $\hat{U}_\epsilon = (\hat{\rho}_\epsilon, \hat{u}_\epsilon, \hat{\theta}_\epsilon)$, combining the lemma (14) and lemma (5.4.0.56) the local conservation laws (5.4.0.34) has the form of

$$\partial_t \hat{U}_\epsilon + \frac{1}{\tau_\epsilon}\mathcal{A}\hat{U}_\epsilon + \frac{\delta_\epsilon}{\tau_\epsilon}\mathcal{Q}(\hat{U}_\epsilon, \hat{U}_\epsilon) = \frac{\epsilon}{\tau_\epsilon}\mathcal{D}\hat{U}_\epsilon + \hat{R}_\epsilon \tag{5.4.0.71}$$

where the first order linear differential operator $\mathcal{A}$ is

$$\mathcal{A}U = \mathcal{A}\begin{pmatrix} \rho \\ u \\ \theta \end{pmatrix} = \begin{pmatrix} \nabla_x \cdot u \\ \nabla_x(\rho + \theta) \\ \frac{2}{D}\nabla_x \cdot u \end{pmatrix}\,, \tag{5.4.0.72}$$

and the quadratic term $\mathcal{Q}(U, U)$ is

$$\mathcal{Q}(U, U) = \begin{pmatrix} 0 \\ \nabla_x \cdot (u \otimes u) - \frac{1}{D}\nabla_x |u|^2 \\ \frac{D+2}{D}\nabla_x \cdot (u\theta) \end{pmatrix}\,, \tag{5.4.0.73}$$

and the second order linear diffusion operator $\mathcal{D}U$ is

$$\mathcal{D}U = \mathcal{D}\begin{pmatrix}\rho \\ u \\ \theta\end{pmatrix} = \begin{pmatrix} 0 \\ \nabla_x \cdot (\mu\sigma(u) \\ \frac{D+2}{D}\nabla_x \cdot (\kappa \nabla_x \theta)\end{pmatrix}, \tag{5.4.0.74}$$

where $\sigma(u)$ is the strain tensor

$$\sigma(u) = \nabla_x u + (\nabla_x u)^\tau - \tfrac{2}{D}\nabla_x \cdot uI) \,. \tag{5.4.0.75}$$

The linear operator $\mathcal{A}$ is a skew-symmetric under the inner product

$$\langle U \,, V\rangle = \int_\Omega (\rho\tilde{\rho} + u \cdot \tilde{u} + \tfrac{D}{2}\theta\tilde{\theta})\, dx \tag{5.4.0.76}$$

for $U = (\rho, u, \theta)$ and $V = (\tilde{\rho}, \tilde{u}, \tilde{\theta})$, i.e.,

$$\langle \mathcal{A}U \,, V\rangle = -\langle U \,, \mathcal{A}V\rangle \,. \tag{5.4.0.77}$$

Then the semi-group $e^{\tau\mathcal{A}}$ preserves the norm defined by this inner product, i.e.,

$$||e^{\tau\mathcal{A}}U|| = ||U|| \,, \tag{5.4.0.78}$$

where $||U|| = \langle U \,, U\rangle$.

Now define $\hat{V}_\epsilon = e^{\frac{t}{\tau_\epsilon}\mathcal{A}}\hat{U}_\epsilon$, applying the semi-group $e^{\frac{t}{\tau_\epsilon}\mathcal{A}}$ to the identity (5.4.0.71), we obtain,

$$\partial_t \hat{V}_\epsilon + \frac{\delta_\epsilon}{\tau_\epsilon} e^{\frac{t}{\tau_\epsilon}\mathcal{A}} \mathcal{Q}(e^{-\frac{t}{\tau_\epsilon}\mathcal{A}}\hat{V}_\epsilon \,, e^{-\frac{t}{\tau_\epsilon}\mathcal{A}}\hat{V}_\epsilon) = \frac{\epsilon}{\tau_\epsilon} e^{\frac{t}{\tau_\epsilon}\mathcal{A}} \mathcal{D} e^{-\frac{t}{\tau_\epsilon}\mathcal{A}} \hat{V}_\epsilon + e^{\frac{t}{\tau_\epsilon}\mathcal{A}} \hat{R}_\epsilon \,, \tag{5.4.0.79}$$

where the remainder $\hat{R}_\epsilon$ is

$$\hat{R}_\epsilon = (0, \nabla_x \cdot \langle \hat{A}(v) r_\epsilon\rangle \,, \nabla_x \cdot \langle \hat{A}(v) r_\epsilon\rangle) \tag{5.4.0.80}$$

Here r_ϵ is the given by (5.4.0.45),

$$r_\epsilon = \frac{\epsilon}{\tau_\epsilon} v \cdot \nabla_x \mathcal{P}^\perp g_\epsilon - \frac{\delta_\epsilon}{\tau_\epsilon} \mathcal{Q}(\mathcal{P}^\perp g_\epsilon, \mathcal{P} g_\epsilon - \mathcal{P}^\perp g_\epsilon) + \epsilon \partial_t g_\epsilon \,. \tag{5.4.0.81}$$

This r_ϵ should be going to vanishing as $\epsilon \to 0$, because $\mathcal{P}^\perp g_\epsilon \to 0$ so as the first two terms, and the third term $\epsilon \partial_t g_\epsilon \to 0$, as $\epsilon \to 0$. Noting that the semi-group $e^{\frac{t}{\tau_\epsilon}\mathcal{A}}$ preserves the norm, so

$$\hat{R}_\epsilon \to 0 \,, \quad \text{as} \quad \epsilon \to 0 \,. \tag{5.4.0.82}$$

Before we derive the limiting behavior of the terms in (5.4.0.71), let us introduce some basic properties about almost-periodic functions, which were introduced by Bohr [3] in the case of complex functions and then extendedto Banach spaces by Bochner and others. We also refer to [1] for the case of almost periodic functions in Banach spaces. One can find a rather extensive bibliography there. We begin by giving a classical definition.

Definition 7: Let $F \in C(\mathbb{R}\,, \mathbf{B})$, where $\mathbf{B}$ is a Banach space. F is said to be *almost−periodic* if and only if, given an $\epsilon > 0$, there exists a length L such that each interval of $\mathbb{R}$ of length L contains an almost-period p associated to ϵ, namely,

$$\sup_{\tau \in \mathbb{R}} ||f(\tau + p) - f(\tau)||_{\mathbf{B}} \leq \epsilon \,. \tag{5.4.0.83}$$

We then denote by $AP(\mathbb{R}\,, \mathbf{B})$ the set of all such functions F.

In the sequel, we will use the following proposition, which could have been given as an equivalent definition:

Proposition 7: *Let $F \in C(\mathbb{R}\,, \mathbf{B})$, F is almost-periodic if and only if it can be ap-*

proximated uniformly by trigonometric polynomials

$$\begin{aligned} &\forall \alpha > 0, \exists N\,, a_n \in \mathbf{B}, w_n \in \mathbb{R}, 0 \le n \le N, \textit{such that} \\ &\qquad ||F - \sum_{n=0}^{N} a_n e^{i w_n \tau}||_{L^\infty(\mathbb{R},\mathbf{B})} \le \alpha\,. \end{aligned} \tag{5.4.0.84}$$

The lemma stated below is one of the most important properties of the almost-periodic functions, which has wide applications in multiple time scales problems.

Lemma 17: *Let* $F \in AP(\mathbb{R}\,, \mathbf{B})$ *with* $\mathbf{B} = L^\infty([0,T]\,, H^s)$*. Then*

$$F(\tfrac{t}{\epsilon}, t) \rightharpoonup \bar{F}(t) \quad \textit{weakly-star in} \quad \mathbf{B}\,, \tag{5.4.0.85}$$

where

$$\bar{F}(t) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau F(s,t)\, ds\,. \tag{5.4.0.86}$$

The existence of $\bar{F}$ *is a classical consequence of the definition and is called the mean value of* F *(see [1].)*

Applying the characterization of the almost-periodic function, see the Proposition (7), it is easy to see $e^{\tau\mathcal{A}}\mathcal{Q}(e^{-\tau\mathcal{A}}U\, e^{-\tau\mathcal{A}}U)$ and $e^{\tau\mathcal{A}}\mathcal{D}e^{-\tau\mathcal{A}}U$ are almost-periodic in τ.

Suppose $\hat{V}_\epsilon \to \hat{V}$ as $\epsilon \to 0$, noting that $\tau_\epsilon \to 0$ when we consider the Stokes, Navier-Stokes and Euler limits, then by the lemma (17), we obtain

$$\begin{aligned} \lim_{\epsilon \to 0} e^{\frac{t}{\tau_\epsilon}\mathcal{A}} \mathcal{Q}(e^{-\frac{t}{\tau_\epsilon}\mathcal{A}}\hat{V}_\epsilon\,, e^{-\frac{t}{\tau_\epsilon}\mathcal{A}}\hat{V}_\epsilon) &= \overline{\mathcal{Q}}(\hat{V}\,, \hat{V})\,, \\ \lim_{\epsilon \to 0} e^{\frac{t}{\tau_\epsilon}\mathcal{A}} \mathcal{D} e^{-\frac{t}{\tau_\epsilon}}\hat{V}_\epsilon &= \overline{\mathcal{D}}\hat{V}\,, \end{aligned} \tag{5.4.0.87}$$

where the averaged operators $\overline{\mathcal{Q}}$ and $\overline{\mathcal{D}}$ are defined as

$$\begin{aligned} \overline{\mathcal{Q}}(V\,, V) &= \lim_{\tau \to \infty} \int_0^\tau e^{s\mathcal{A}} \mathcal{Q}(e^{-s\mathcal{A}}V\,, e^{-s\mathcal{A}}V)\,, \\ \overline{\mathcal{D}}V &= \lim_{\tau \to \infty} \int_0^\tau e^{s\mathcal{A}} \mathcal{D} e^{-s\mathcal{A}} V\,. \end{aligned} \tag{5.4.0.88}$$

Applying the same method used in Chapter 2, we can calculate $\overline{\mathcal{Q}}$ and $\overline{\mathcal{D}}$. We are interested in the 3-dimensional case, so with out lose of generality, we take the spacial dimension $D = 3$. Exact the same as we did in Chapter 2, we can characterize the null space of $\mathcal{A}$, which is nontrivial and its orthogonal complement space with respect to the inner product $\langle \cdot\,, \cdot \rangle$ defined above (5.4.0.76) as follows

$$\mathrm{Null}(\mathcal{A}) = \{V = (\rho, u, \theta) \quad | \quad \rho + \theta = 0\,, \quad \nabla_x \cdot u = 0\,, \quad \int_\Omega V\,dx = 0\} \tag{5.4.0.89}$$

and

$$\mathrm{Null}(\mathcal{A}) = \{V = (\rho, u, \theta) \quad | \quad \theta = \tfrac{2}{D}\rho\,, \quad u = \nabla_x \phi\,, \quad \int_\Omega V\,dx = 0\} \tag{5.4.0.90}$$

Then we have the following orthogonal decomposition

$$V = \Pi V + \Pi^{\perp} V = \begin{pmatrix} \frac{2}{D+2}\rho - \frac{D}{D+2}\theta \\ \\ Pu \\ \\ -\frac{2}{D+2}\rho + \frac{D}{D+2}\theta \end{pmatrix} + \begin{pmatrix} \frac{D}{D+2}(\rho + \theta) \\ \\ Qu \\ \\ \frac{2}{D+2}(\rho + \theta) \end{pmatrix}, \tag{5.4.0.91}$$

where P is the usual Leray projection onto the divergence-free vector space, and $Q = I - P$.

The calculations in Chapter 2 and Chapter 3 for more general hyperbolic-parabolic system with entropy and application to the general Navier-Stokes system show that the projections of the averaged operators $\overline{\mathcal{Q}}$ and $\overline{\mathcal{D}}$ are the convection and diffusion terms in fluid system respectively:

$$\Pi\overline{\mathcal{Q}}(V\,, V) = \begin{pmatrix} \frac{2}{D+2} u \cdot \nabla_x (\frac{2}{D+2}\rho - \frac{D}{D+2}\theta) \\ \\ Pu \cdot \nabla_x Pu + \nabla_x p \\ \\ \frac{2}{D+2} u \cdot \nabla_x (-\frac{2}{D+2}\rho + \frac{D}{D+2}\theta) \end{pmatrix}, \tag{5.4.0.92}$$

and

$$\Pi\overline{\mathcal{D}}V = \begin{pmatrix} \frac{2}{D+2}\nabla_x \cdot [\kappa\nabla_x(\frac{2}{D+2}\rho - \frac{D}{D+2}\theta)] \\ Pu \cdot \nabla_x Pu + \nabla_x p \\ \frac{2}{D+2}\nabla_x \cdot [\kappa\nabla_x(-\frac{2}{D+2}\rho + \frac{D}{D+2}\theta)] \end{pmatrix} . \tag{5.4.0.93}$$

The projections on the orthogonal complement of the null space $\mathrm{Null}(\mathcal{A})^{\perp}$ are

$$\Pi^{\perp}\overline{\mathcal{Q}}(V\,,V) = Q_1(\Pi V\,,\Pi^{\perp}V) + Q_1(\Pi^{\perp}\,,\Pi^{\perp})\,, \tag{5.4.0.94}$$

and

$$\Pi^{\perp}\overline{\mathcal{D}}V = \tilde{\mu}\Delta_x\Pi^{\perp}V\,, \tag{5.4.0.95}$$

where Q_1 and Q_2 are non-local 2-wave and 3-wave resonant terms respectively, and $\tilde{\mu} = c_1\mu + c_2\kappa$, the linear combination of the viscosity μ and the heat conductivity κ.

Recall that the notations $\hat{\rho} = \langle\tilde{g}\rangle$, and $\hat{\theta} = \langle(\frac{1}{D}|v|^2 - 1)\tilde{g}\rangle$, then

$$\Pi\hat{V}_\epsilon = \left(\langle(1 - \tfrac{1}{D+2}|v|^2)g_\epsilon\rangle\,, P\langle vg_\epsilon\rangle\,, \langle(\tfrac{1}{D+2}|v|^2 - 1)g_\epsilon\rangle\right)\,, \tag{5.4.0.96}$$

and

$$\Pi^{\perp}\hat{V}_\epsilon = \left(\langle\tfrac{1}{D+2}|v|^2 g_\epsilon\rangle\,, Q\langle vg_\epsilon\rangle\,, \langle\tfrac{2}{D(D+2)}|v|^2 g_\epsilon\rangle\right)\,. \tag{5.4.0.97}$$

Now we can state our theorem on the formal derivation of the weakly nonlinear approximations of the Boltzmann equation with the general initial data, which is a generalization of the Bardos-Golse-Levermore theorem.

Theorem 13: **The Formal Weakly Compressible Approximations Theorem** *Let G_ϵ be a family of distribution solutions to the scaled Boltzmann initial-value problem* (5.2.0.21) *with initial data G^{in} that satisfy the normalizations* (4.6.1.17).

Let $G_\epsilon^{in} = 1 + \delta_\epsilon g_\epsilon^{in}$ *and* $G_\epsilon = 1 + \delta_\epsilon g_\epsilon$ *where* $\delta_\epsilon \to 0$ *as* $\epsilon \to 0$*, and the fluctuations* g_ϵ^{in} *and* g_ϵ *are bounded in* $L^\infty(dt; L^2(Mdvdx))$*. Moreover:*

1. *Assume that in the sense of distributions the family* g_ϵ^{in} *satisfies*

$$\lim_{\epsilon\to 0}(\langle g_\epsilon\rangle\,, \langle v g_\epsilon^{in}\rangle, \langle (\tfrac{1}{D}|v|^2 - 1) g_\epsilon^{in}\rangle) = (\rho^{in}\,, u^{in}\,, \theta^{in}) = U^{in} \tag{5.4.0.98}$$

   for some $(\rho^{in}\,, u^{in}\,, \theta^{in}) \in L^2(dx; \mathbb{R}\times\mathbb{R}^3\times\mathbb{R})$*;*

2. *Assume that the local consideration laws* (5.1.0.14) *are also satisfied in the sense of distributions for every* g_ϵ*;*

3. *For the family of the fluctuations* g_ϵ*, assume that*

$$\mathcal{P}^\perp g_\epsilon = (I - \mathcal{P}) g_\epsilon \to 0, \quad as \quad \epsilon \to 0 \tag{5.4.0.99}$$

   and the following moments with $\hat{\zeta} =$ *either* $\hat{A}$ *or* $\hat{B}$

$$\langle \hat{\zeta}(v) v\cdot\nabla_x \mathcal{P}^\perp g_\epsilon\rangle\,, \quad \langle \hat{\zeta}(v)\mathcal{Q}(\mathcal{P}^\perp g_\epsilon, \mathcal{P} g_\epsilon + \mathcal{P}^\perp g_\epsilon)\rangle\,, \tag{5.4.0.100}$$

   go to zero, as $\epsilon \to 0$*; and*

$$\epsilon\langle \hat{\zeta}\partial_t g_\epsilon\rangle \to 0\,, \quad as \quad \epsilon \to 0; \tag{5.4.0.101}$$

Then the family of the moments

$$\hat{U}_\epsilon = \big(\langle g_\epsilon\rangle\,, \langle v g_\epsilon\rangle\,, \langle \tfrac{1}{D}|v|^2 g_\epsilon\rangle\big) \tag{5.4.0.102}$$

satisfy the asymptotics

$$\hat{U}_\epsilon - \Pi\hat{V} - e^{-\frac{t}{\tau_\epsilon}\mathcal{A}}(\Pi^\perp\hat{V}) \to 0\,, \quad as \quad \epsilon \to 0\,, \tag{5.4.0.103}$$

where $\Pi\hat{V}$ *and* $\Pi^\perp\hat{V}$ *satisfy the equations:*

1. *when* $\tau_\epsilon = \epsilon$, *and* $\frac{\delta_\epsilon}{\epsilon} \to 0$, $\Pi\hat{V}$ *satisfies the incompressible Stokes system with initial data* ΠU^{in}*; and* $\Pi^\perp \hat{V}$ *satisfies the averaged equation*

$$\begin{aligned} \partial_t \Pi^\perp \hat{V} &= \tilde{\mu}\Delta_x \Pi^\perp \hat{V}\,, \\ \Pi^\perp \hat{V}(0,x) &= \Pi^\perp U^{in}(x)\,; \end{aligned} \tag{5.4.0.104}$$

2. *when* $\tau_\epsilon = \epsilon$, *and* $\frac{\delta_\epsilon}{\epsilon} \to 1$, $\Pi\hat{V}$ *satisfies the incompressible Navier-Stokes system with initial data* ΠU^{in}*; and* $\Pi^\perp \hat{V}$ *satisfies the averaged equation*

$$\begin{aligned} \partial_t \Pi^\perp \hat{V} + Q_1(\Pi\hat{V}, \Pi^\perp \hat{V}) + Q_2(\Pi^\perp \hat{V}, \Pi^\perp \hat{V}) &= \tilde{\mu}\Delta_x \Pi^\perp \hat{V}\,, \\ \Pi^\perp \hat{V}(0,x) &= \Pi^\perp U^{in}(x)\,; \end{aligned} \tag{5.4.0.105}$$

   here $\Pi\hat{V}$ *is a solution to the incompressible Navier-Stokes system with initial data* ΠU^{in}*;*

3. *when* $\tau_\epsilon = \tau_\epsilon$, *and* $\frac{\delta_\epsilon}{\epsilon} \to \infty$, $\Pi\hat{V}$ *satisfies the incompressible Euler system with initial data* ΠU^{in}*; and* $\Pi^\perp \hat{V}$ *satisfies the averaged equation*

$$\begin{aligned} \partial_t \Pi^\perp \hat{V} + Q_1(\Pi\hat{V}, \Pi^\perp \hat{V}) + Q_2(\Pi^\perp \hat{V}, \Pi^\perp \hat{V}) &= 0\,, \\ \Pi^\perp \hat{V}(0,x) &= \Pi^\perp U^{in}(x)\,; \end{aligned} \tag{5.4.0.106}$$

   here $\Pi\hat{V}$ *is a solution to the incompressible Euler system with initial data* ΠU^{in}*;*

Remark: when the initial data are well-prepared, i.e., $\Pi^\perp U^{in} = 0$, the solutions to the averaged equation vanish, the above theorem is exactly matches with Bardos-Golse-Levermore's theorem on the formal incompressible limits. For the Stokes dynamics, the averaged equation is completely decoupled from the projection on the incompressible regime. But for the Navier-Stokes and Euler dynamics, the averaged equations are coupled with the corresponding incompressible regime. This is because of the propagation of the fast acoustic waves which prevent the strong limits in the rigorous justification of the incompressible limits. For the linear Stokes dynamics, we can rigorously justify the asymptotics with the non-well-prepared initial data so that we can generalize the Golse-Levermore's Stokes-Fourier limits to the general initial data. Furthermore, we can provide a uniform proof of Acoustic-Stokes-Fourier limits, provided some more restrictive assumptions on the DiPerna-Golse solutions, using the so-called relative entropy method.

6. WEAKLY COMPRESSIBLE STOKES APPROXIMATION FROM BOLTZMANN EQUATIONS

In this chapter, we shall consider the hydrodynamics of the Boltzmann equations in the Stokes scaling, i.e., when the order of the fluctuation δ_ϵ is smaller than the Kudsen number ϵ: $\frac{\delta_\epsilon}{\epsilon} \to 0$, as $\epsilon \to 0$. We shall show that if from the initial data $U^{in} = (\rho^{in}, u^{in}, \theta^{in})$, we construct a local Maxwellian M_ϵ^{in}, (after some mollification,) such that it is close to the initial number density F_ϵ^{in} in the sense of the scaled relative entropy, i.e., $\frac{1}{\delta_\epsilon^2} H(F_\epsilon^{in} | M_\epsilon^{in}) \to 0$, as $\epsilon \to 0$, then for the later time $t > 0$, we can also construct a family of Maxwellians from the solutions to the weakly compressible Stokes system with initial data $U^{in} = (\rho^{in}, u^{in}, \theta^{in})$, i.e., $M_\epsilon(t) = \mathcal{M}_{(1+\delta_\epsilon \rho_\epsilon(t,)\delta_\epsilon u_\epsilon(t,)1+\delta_\epsilon \theta_\epsilon(t))}$, and furthermore, this local Maxwellian governs the behavior of Diperna-Lions solutions $F_\epsilon(t)$ to the scaled Boltzmann equation, in the sense that $\frac{1}{\delta_\epsilon^2} H(F_\epsilon(t) | M_\epsilon(t)) \to 0$, as $\epsilon \to 0$. This means that the long time behavior of the relative entropy is stable. Furthermore, we also will show that the fluctuation g_ϵ around the absolute Maxwellian M, is governed by the infinitesimal constructed from the solution to the weakly compressible Stokes in an appropriate sense. We also show that at the fluid dynamics level, in the short time scale, the weakly compressible Stokes system asymptotically close to acoustic system. In the longer time scale, say $\tau \sim \frac{1}{\epsilon}$, the weakly compressible Stokes system has singular behavior. The projection onto the null space of the acoustic operator $\mathcal{A}$, which we called the slow mode, tends to the incompressible Stokes equations, while the projection on the fast mode, i.e., Null($\mathcal{A}$), propagate in the evolution of an "averaged equation" which is a diffusion equation, with diffusive coefficient the linear combination of the viscosity and heat conductivity. So we unit the previous work of Golse-Levermore on the acoustic and Stokes-Fourier limits in one fluid model, compressible Stokes system, under slightly restrictive initial condition.

In the justification of the asymptotics from the Boltzmann equation to the weakly compressible Stokes system, we used a key fact: the construction of the local Maxwellian $M_\epsilon(t)$ comes from the solution to the weakly compressible Stokes system, which has a good properties, say, existence, regularities, etc. We didn't use the averaged equation in a direct way. This is because of some deeper reasons which will be explained below.

Formally, we start from the scaled Boltzmann equation with initial data:

$$\tau_\epsilon \partial_t G + v \cdot \nabla_x G = \frac{1}{\epsilon}\mathcal{Q}(G, G,) \qquad G(0, x, v) = G^{in}(x, v)\,. \tag{6.0.0.1}$$

then the family of the fluctuation g_ϵ formally satisfies the local conservation laws

$$\begin{aligned} \partial_t \langle g_\epsilon \rangle + \frac{1}{\tau_\epsilon}\nabla_x \cdot \langle v g_\epsilon \rangle = 0\,, \\ \partial_t \langle v g_\epsilon \rangle + \frac{1}{\tau_\epsilon}\nabla_x \cdot \langle v \otimes v g_\epsilon \rangle = 0\,, \\ \partial_t \langle \tfrac{1}{2}|v|^2 g_\epsilon \rangle + \frac{1}{\tau_\epsilon}\nabla_x \cdot \langle v \tfrac{1}{2}|v|^2 g_\epsilon \rangle = 0\,. \end{aligned} \tag{6.0.0.2}$$

We define the fluid variables associated with the fluctuation of the number density

g_ϵ:

$$\tilde{\rho}_\epsilon = \langle g_\epsilon \rangle \,, \quad \tilde{u}_\epsilon = \langle v g_\epsilon \rangle \,, \quad \tilde{\theta}_\epsilon = \tfrac{2}{D} \left\langle \left(\tfrac{1}{2}|v|^2 - \tfrac{D}{2} \right) g_\epsilon \right\rangle \,. \tag{6.0.0.3}$$

After some tedious algebraic calculations, we derive that $\tilde{U}_\epsilon = (\tilde{\rho}_\epsilon, \tilde{u}_\epsilon, \tilde{\theta}_\epsilon)$ satisfies the local conservation laws

$$\partial_t \tilde{U}_\epsilon + \frac{1}{\epsilon} \mathcal{A} \tilde{U}_\epsilon + \frac{\delta_\epsilon}{\epsilon} \mathcal{Q}(\tilde{U}_\epsilon, \tilde{U}_\epsilon) = \mathcal{D} \tilde{U}_\epsilon + \tilde{R}_\epsilon \tag{6.0.0.4}$$

where the first order linear differential operator $\mathcal{A}$ is

$$\mathcal{A}U = \mathcal{A} \begin{pmatrix} \rho \\ u \\ \theta \end{pmatrix} = \begin{pmatrix} \nabla_x \cdot u \\ \nabla_x (\rho + \theta) \\ \frac{2}{D} \nabla_x \cdot u \end{pmatrix} , \tag{6.0.0.5}$$

and the quadratic term $\mathcal{Q}(U, U)$ is

$$\mathcal{Q}(U, U) = \begin{pmatrix} 0 \\ \nabla_x \cdot (u \otimes u) - \frac{1}{D} \nabla_x |u|^2 \\ \frac{D+2}{D} \nabla_x \cdot (u\theta) \end{pmatrix} , \tag{6.0.0.6}$$

and the second order linear diffusion operator $\mathcal{D}U$ is

$$\mathcal{D}U = \mathcal{D} \begin{pmatrix} \rho \\ u \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ \nabla_x \cdot (\mu \sigma(u) \\ \frac{D+2}{D} \nabla_x \cdot (\kappa \nabla_x \theta) \end{pmatrix} , \tag{6.0.0.7}$$

where $\sigma(u)$ is the strain-rate tensor

$$\sigma(u) = \nabla_x u + (\nabla_x u)^\tau - \tfrac{2}{D} \nabla_x \cdot u I \,. \tag{6.0.0.8}$$

From the asymptotic local conservation law (6.0.0.4), in the Stokes scaling, $\frac{\delta_\epsilon}{\epsilon} \to 0$, it is natural to guess that the fluid dynamics which governs the evolution of the moments of the number density g_ϵ is:

$$\begin{aligned} \partial_t U_\epsilon + \mathcal{A} U_\epsilon = \mathcal{D} U_\epsilon \,, \\ U_\epsilon(0,x) = U_\epsilon^{in}(x)\,. \end{aligned} \tag{6.0.0.9}$$

This is exactly the weakly compressible Stokes system, about which we know a lot: global existence, uniqueness, regularity,$\cdots$. So we can construct the local Maxwellian from the solutions to this scaled compressible Stokes system, and expect it is close to DiPerna-Lions renormalized solution in the sense of relative entropy, provided that initially it is. Another technical reason is that in our proof , because in the Stokes scaling both the local conservations laws (with defects which will vanish in the limit) and the weakly compressible Stokes system are linear. Hence, taking convolution did not change the equations and linearity.

This chapter include two sections. In the first section, we shall investigate the asymptotic behavior of the weakly compressible Stokes system (6.1.0.10) in short time scale, i.e $\tau_\epsilon = 1$ and in longer time scale $\tau_\epsilon = \epsilon$. We shall show in Theorem (14) that in the short time scale, the solutions to the weakly compressible Stokes will converges uniformly in time to the acoustic system. In the Stokes time scaling, the behavior as $\epsilon \to 0$ is singular. We will shall in Theorem (15) that the solutions to the weakly compressible Stokes system will asymptotically governed by summation of the solution to the incompressible Stokes system, which is in the null space of the acoustic operator $\mathcal{A}$, and a correction term, which is the solution to the diffusion equation operated by the semigroup $e^{\frac{t}{\epsilon}\mathcal{A}}$.

In the second section of this chapter, we construct a family of local Maxwellian $M_\epsilon = \mathcal{M}_{(1+\rho_\epsilon, u_\epsilon, 1+\theta_\epsilon)}$, where $(\rho_\epsilon, u_\epsilon, \theta_\epsilon)$ are solutions to the weakly compressible

Stokes system. We shall employ the method of relative entropy, DiPerna-Lions solutions to the Boltzmann equation in the hard sphere case, in the Stokes scaling, are close to the local Maxwellian M_ϵ in the sense that relative entropy goes to zero if initially it does. Our main assumption is the local conservation law of the energy, which is not satisfied by DiPerna-Lions solutions. Our main theorem will be stated in Theorem (16).

6.1 Weakly Compressible Stokes System

The compressible Stokes system is the linearization about the zero state of the compressible Navier-Stokes system. It governs (ρ, u, θ) , the fluctuations of mass density, bulk velocity, and temperature about their spatially homogeneous equilibrium values. After a suitable choice of units, in this model the fluid fluctuations satisfy

$$\begin{aligned} &\tau_\epsilon \partial_t \rho_\epsilon + \nabla_x \cdot u_\epsilon = 0 \,, \\ &\tau_\epsilon \partial_t u_\epsilon + \nabla_x(\rho_\epsilon + \theta_\epsilon) = \epsilon \nabla_x \cdot \mu \left[\nabla_x u_\epsilon + (\nabla_x u_\epsilon)^T - \tfrac{2}{D} \nabla_x \cdot u_\epsilon I \right] \,, \\ &\tau_\epsilon \tfrac{D}{2} \theta_\epsilon + \nabla_x \cdot u_\epsilon = \epsilon \kappa \Delta_x \theta_\epsilon \,. \end{aligned} \tag{6.1.0.10}$$

with initial data $(\rho^{in}, u^{in}, \theta^{in}) \in L^2(dx\,; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R})$.

The acoustic (5.1.0.13) and the incompressible Stokes-Fourier system (5.2.0.23) are considered in different time scales. Then can be connected by above compressible Stokes system (6.1.0.10). Notice unlike the Stokes and acoustic system, in the weakly compressible Stokes system the Knudsen number ϵ appears explicitly, where the solutions also depend on ϵ even though the initial data does not. There is no way by some clever scaling to get rid of the ϵ in front of the dissipation terms, see explanation in Golse's lecture in [10].

The compressible Stokes system is linear, but it has very interesting multiple time scale properties even in the formal sense. When we consider the short Euler time scale, i.e., $\tau_\epsilon = 1$ Obviously in formal sense, the solutions to this system converges to those of the acoustic system (5.1.0.13) with the same initial data as $\epsilon \to 0$.

However, when we consider the times scale of order $\frac{1}{\epsilon}$, i.e., taking $\tau_\epsilon = \epsilon$ solutions to this system obviously depend on two different time scales, because if divided by ϵ on both sides, the first order derivative term formally has side $\frac{1}{\epsilon}$. The key is that the null space of this first order differential operator $\mathcal{A}$ is nontrivial, which is exactly the incompressibility and Boussinesq relations. We call Null($\mathcal{A}$) the slow mode, and Null$(\mathcal{A})^\perp$ the fast mode. We will show that the projection on the slow mode converges to solutions of the Stokes system (5.2.0.23) with initial data

$$\left(\mathrm{P}u^{in}\,, \tfrac{D}{D+2}\theta^{in} - \tfrac{2}{D+2}\rho^{in}\right) \quad \text{as } \epsilon \to 0\,. \tag{6.1.0.11}$$

where P is the usual Leray projection to the space of divergence free vectors. The projection on the fast mode, after the action of the semigroup generated by the acoustic system, converges to a diffusion equation.

6.1.1 Acoustic Approximation

In this section, we rigorously justify the formally obvious limit from compressible Stokes to acoustic system, which is stated in the following theorem.

Theorem 14: *Let* $U_\epsilon = (\rho_\epsilon, u_\epsilon, \theta_\epsilon) \in C([0,\infty); L^2(d\mathbf{x}; \mathbb{R}\times\mathbb{R}^D\times\mathbb{R}))$ *be the weak solution to the linearized Navier-Stokes system:*

$$\begin{aligned} &\partial_t\rho_\epsilon + \nabla_x\cdot u_\epsilon = 0\,,\\ &\partial_t u_\epsilon + \nabla_x(\rho_\epsilon+\theta_\epsilon) = \epsilon\nabla_x\cdot\mu\left[\nabla_x u_\epsilon + (\nabla_x u_\epsilon)^T - \tfrac{2}{D}\nabla_x\cdot u_\epsilon I\right]\,,\\ &\tfrac{D}{2}\partial_t\theta_\epsilon + \nabla_x\cdot u_\epsilon = \epsilon\kappa\Delta_x\theta_\epsilon\,. \end{aligned} \tag{6.1.1.1}$$

with initial data $U^{in} = (\rho^{in}, u^{in}, \theta^{in}) \in L^2(dx\,;\mathbb{R}\times\mathbb{R}^D\times\mathbb{R})$. *Let* $U_{ac} = (\rho_{ac}, u_{ac}, \theta_{ac}) \in C([0,\infty); L^2(d\mathbf{x}; \mathbb{R}\times\mathbb{R}^D\times\mathbb{R}))$ *be the weak solution to the acoustic system*

$$\begin{aligned} &\partial_t U + \mathcal{A}U = 0\\ &U(0,\mathbf{x}) = U^{in}(\mathbf{x})\,, \end{aligned} \tag{6.1.1.2}$$

where the initial data $U^{in} \in L^2(dx\,;\mathbb{R}\times\mathbb{R}^D\times\mathbb{R})$ *is the same as that of the the linearized Navier-Stokes system* (6.1.1.1)*, and the linear acoustic operator* $\mathcal{A}$ *is defined as*

$$\mathcal{A}\begin{pmatrix}\rho\\ u\\ \theta\end{pmatrix} = \begin{pmatrix}\nabla_x\cdot u\\ \nabla_x(\rho+\theta)\\ \tfrac{2}{D}\nabla_x\cdot u\end{pmatrix} \tag{6.1.1.3}$$

Then, $U_\epsilon \to U_{ac}$, *in* $C([0,\infty)\,; L^2(dx; \mathbb{R}\times\mathbb{R}^D\times\mathbb{R}))$ *as* $\epsilon\to 0\,;$

Proof: Our proof includes 3 steps:

Step 1: Relative compactness in w-$L^2(dx)$ pointwise in time.

The weak solutions U_ϵ to the weakly compressible Stokes system (6.1.1.1) satisfy the energy identity:

$$\begin{aligned} &\tfrac{1}{2}\int_\Omega |U_\epsilon(t_2)|^2\,dx + \epsilon\int_{t_1}^{t_2}\int_\Omega \mu\tfrac{1}{2}\sigma(u_\epsilon)(s) : \sigma(u_\epsilon)(s) + \kappa|\nabla_x\theta_\epsilon(s)|^2\,dxds\\ &\qquad = \tfrac{1}{2}\int_\Omega |U_\epsilon(t_1)|^2\,dx\,. \end{aligned} \tag{6.1.1.4}$$

where $\sigma(u):\sigma(u) = \sigma_{ij}(u)\sigma_{ji}(u) = \sum_{i,j=1}^{D} \sigma_{ij}(u)^2 \geq 0$. Thus, $\forall 0 \leq t < \infty$,

$$\int_\Omega |U_\epsilon(t)|^2\,dx \leq \int_\Omega |U_\epsilon^{in}|^2\,dx \leq C^{in} \tag{6.1.1.5}$$

then $\{U_\epsilon(t)\}_{\epsilon>0}$ is relatively compact in w-$L^2(dx)$, $\forall t > 0$. i.e., there exits a $U = (\rho, u, \theta) \in L^2(dx)$, such that $(\rho_\epsilon, u_\epsilon, \theta_\epsilon) \to (\rho, u, \theta)$ in w-$L^2(dx)$ pointwisely in time $t \geq 0$.

Step2: Relative compactness in $C([0,\infty); \text{w-}L^2(dx))$.

We firstly state the well-known Arzelà-Ascoli Theorem:

Arzelà-Ascoli: A subset $\mathcal{K}$ in $C([0,\infty); \text{w-}X)$ is relatively compact if and only if the following two conditions are satisfied:

- For any $0 < T < \infty$, there exits a dense subset D in $[0,T]$, such that, for each $t \in D$,

$$\mathcal{K}(t) = \{f(t) \| f \in \mathcal{K}\} \quad \text{is relatively compact in} \quad \text{w-}X\,; \tag{6.1.1.6}$$

- $\mathcal{K}$ is equi-continuous on $[0,T]$, i.e., for each $\eta > 0$, $t_1 \in [0,T]$, and $\varphi \in X^*$, there exits $\delta = \delta(\eta, t_1, \varphi) > 0$, such that, for each $t \in [0,T]$, with $|t - t_1| < \delta$, we have

$$|\langle f(t) - f(t_1,)\varphi\rangle| < \eta\,, \quad \text{uniformly in} \quad \mathcal{K}\,. \tag{6.1.1.7}$$

From the step 1, we already know that for each $t > 0$, $\{U_\epsilon(t)\}_{\epsilon>0}$ is relatively compact in w-$L^2(dx)$. We need only to verify the equi-continuity in $C([0,T], \text{w-}L^2(dx))$. This can be proven from the following weak formulation of U_ϵ.

$$\int_\Omega (\rho_\epsilon(t_2) - \rho_\epsilon(t_1))\varphi \, dx = \int_{t_1}^{t_2} u_\epsilon(s) \cdot \nabla_x \varphi \, dxds \,,$$
$$\int_\Omega (u_\epsilon(t_2) - u_\epsilon(t_1)) \cdot \phi \, dx = \int_{t_1}^{t_2} \int_\Omega (\rho_\epsilon + \theta_\epsilon)(s) \nabla_x \cdot \phi \, dxds$$
$$- \epsilon\mu \int_{t_1}^{t_2} \int_\Omega \sigma(u_\epsilon) : \sigma(\phi) \, dxds \,,$$
$$\tfrac{D}{2} \int_\Omega (\theta_\epsilon(t_2) - \theta_\epsilon(t_1))\chi \, dx = \int_{t_1}^{t_2} \int_\Omega u_\epsilon \cdot \nabla_x \chi \, dxds - \epsilon\kappa \int_{t_1}^{t_2} \int_\Omega \nabla_x \theta_\epsilon \cdot \nabla_x \chi \,, \tag{6.1.1.8}$$

for any $(\varphi, \phi, \chi) \in C^1(dx) \times (C^1(dx))^D \times C^1(dx)$.

From the first identity above, it is easy to see

$$\begin{aligned} \left| \int_\Omega (\rho_\epsilon(t_2) - \rho_\epsilon(t_1))\varphi \, dx \right| &\leq \int_{t_1}^{t_2} \|u_\epsilon\|_{L^2} \|\nabla_x \varphi\|_{L^2} \, ds \\ &\leq C|t_2 - t_1| \,. \end{aligned} \tag{6.1.1.9}$$

The weak formulation for u_ϵ yields

$$\begin{aligned} \left| \int_\Omega (u_\epsilon(t_2) - u_\epsilon(t_1)) \cdot \phi \, dx \right| &\leq \|\rho_\epsilon + \theta_\epsilon\|_{L^2} \|\phi\|_{L^2} |t_2 - t_1| \\ &+ \sqrt{\epsilon\mu}\sqrt{t_2 - t_1} \left(\int_{t_1}^{t_2} \int_\Omega \mu \tfrac{1}{2} \sigma(u_\epsilon)(s) : \sigma(\phi) \, dxds \right)^{\frac{1}{2}} \\ &\leq C \|\nabla_x \phi\|_{L^2} |t_2 - t_1| + C\sqrt{\epsilon\mu}\sqrt{t_2 - t_1} \|\sigma(\phi)\|_{L^2} \,. \end{aligned} \tag{6.1.1.10}$$

and the weak formulation for u_ϵ yields

$$\begin{aligned} \tfrac{D}{2} \left| \int_\Omega (\theta_\epsilon(t_2) - \theta_\epsilon(t_1))\chi \, dx \right| &\leq C \|\nabla_x \chi\|_{L^2} |t_2 - t_1| \\ &+ \sqrt{\epsilon\kappa}\sqrt{t_2 - t_1} \|\nabla_x \chi\|_{L^2} \left(\epsilon\kappa \int_{t_1}^{t_2} \int_\Omega |\nabla_x \theta_\epsilon|^2 \, dxds \right)^{\frac{1}{2}} \\ &\leq C|t_2 - t_1| + C\sqrt{t_2 - t_1} \end{aligned} \tag{6.1.1.11}$$

Then we proved the equi-continuity of U_ϵ in $C([0, \infty); \text{w-}L^2)$, so have relative compactness. Now we can take any subsequence ϵ_n such that U_ϵ convergent to $\tilde{U} =$

$(\tilde{\rho}, \tilde{u}, \tilde{\theta})$. Let $\epsilon_n \to 0$ in the weak formulation (6.1.1.8), we obtain that $\tilde{U}$ is a solution of the acoustic system. By the uniqueness of the acoustic system, any limit of the convergent subsequence of U_ϵ is the solution of the acoustic system in $C([0,\infty); \text{w-}L^2(dx))$, then $\tilde{U} = U$, and $\int_\Omega |U(t)|^2\, dx = \int_\Omega |U^{in}|^2\, dx$. Thus we finish step 2.

Step 3: Relative compactness in $C([0,\infty); L^2(dx))$.

From the Arzelà-Ascoli theorem, we need to verify the following 2 conditions:

- For each $t \geq 0$, $\{U_\epsilon(t)\}_{\epsilon>0}$ is relatively compact in $L^2(dx)$.
- $\{U_\epsilon(t)\}_{\epsilon>0}$ is equi-continuous in $C([0,\infty); L^2(dx))$.

It is well known that the strong convergence in $L^2(dx)$ is the weak convergence which has been proven in step 1, combine with the convergence in norm. So we need to prove: for any sequence $\epsilon_n \to 0$,

$$\lim_{\epsilon_n \to 0} \|U_\epsilon(t)\| = \|U(t)\|\,. \tag{6.1.1.12}$$

Suppose $U_{\epsilon_n} \to U$ in $C([0,\infty); \text{w-}L^2(dx))$, where U is the unique solution to the acoustic system with initial data U^{in}, then from the energy identity of the acoustic system, compressible Stokes system and the Fatou's lemma:

$$\begin{aligned} \tfrac{1}{2}\|U^{in}\|_{L^2}^2 = \tfrac{1}{2}\|U(t)\|_{L^2}^2 &\leq \tfrac{1}{2}\liminf_{\epsilon_n \to 0} \|U_{\epsilon_n}\|_{L^2} \\ &\leq \tfrac{1}{2}\limsup_{\epsilon_n \to 0} \|U_{\epsilon_n}\|_{L^2} + \liminf_{\epsilon_n \to 0} \epsilon_n \int_0^t D_{\epsilon_n}(s)\, ds \\ &\leq \tfrac{1}{2}\|U^{in}\|_{L^2}^2\,, \end{aligned} \tag{6.1.1.13}$$

where

$$D_\epsilon(t) = \int_\Omega \mu \tfrac{1}{2}\sigma(u_\epsilon)(t) : \sigma(u_\epsilon)(t) + \kappa |\nabla_x \theta_\epsilon(t)|^2\, dx\,. \tag{6.1.1.14}$$

Then,

$$\tfrac{1}{2}\liminf_{\epsilon_n}\|U_{\epsilon_n}\|_{L^2}^2 = \tfrac{1}{2}\limsup_{\epsilon_n}\|U_{\epsilon_n}\|_{L^2}^2 + \limsup_{\epsilon_n}\epsilon_n\int_0^t D_{\epsilon_n}(s)\,ds$$
$$= \tfrac{1}{2}\|U(t)\|_{L^2}^2 \leq \tfrac{1}{2}\liminf_{\epsilon_n}\|U_{\epsilon_n}\|_{L^2}^2\,. \tag{6.1.1.15}$$

Thus we conclude that for each $t \geq 0$:

$$\lim_{\epsilon_n\to 0}\|U_{\epsilon_n}(t)\|_{L^2} = \|U(t)\|_{L^2}\,, \tag{6.1.1.16}$$

and

$$\lim_{\epsilon_n\to 0}\epsilon_n\int_0^t D_{\epsilon_n}(s)\,ds = 0\,. \tag{6.1.1.17}$$

To prove the relative compactness in $C([0,\infty);L^2(dx))$, by the the Arzelà-Ascoli theorem, we need to prove the following equi-continuity:

- For any fixed $T > 0$, $\forall \eta > 0$, and $t \in [0,T]$, there exists a $\delta = \delta(t,\eta) > 0$, such that $|\tilde{t} - t| < \delta$ implies

$$\|U_{\epsilon_n}(\tilde{t}) - U_{\epsilon_n}(t)\|_{L^2}^2 < \eta\,. \tag{6.1.1.18}$$

Claim: For the solution of the weakly compressible Stokes system U_{ϵ_n}, the following inequality implies (6.1.1.18).

$$\|U_{\epsilon_n}(\tilde{t})\|_{L^2}^2 - \|U_{\epsilon_n}(t)\|_{L^2}^2 < \eta\,. \tag{6.1.1.19}$$

Proof of the claim: Denote $\langle\cdot,\cdot\rangle$ the L^2 inner product.

$$\begin{aligned}\|U_{\epsilon_n}(t) - U_{\epsilon_n}(\tilde{t})\|_{L^2}^2 &= \langle U_{\epsilon_n}(t) - U_{\epsilon_n}(t,)U_{\epsilon_n}(\tilde{t}) - U_{\epsilon_n}(\tilde{t})\rangle\\ &= \|U_{\epsilon_n}(\tilde{t})\|_{L^2}^2 - \|U_{\epsilon_n}(t)\|_{L^2}^2\\ &+ 2\langle U_{\epsilon_n}(t,)U_{\epsilon_n}(t) - U_{\epsilon_n}(\tilde{t})\rangle + 2\langle U_{\epsilon_n}(t) - U(t,)U_{\epsilon_n}(t) - U_{\epsilon_n}(\tilde{t})\rangle\\ &= I_{\epsilon_n}^1 + I_{\epsilon_n}^2 + I_{\epsilon_n}^3\,.\end{aligned} \tag{6.1.1.20}$$

Applying the basic L^2-estimate,

$$I^3_{\epsilon_n} \leq 4\sqrt{C^{in}}\|U_{\epsilon_n}(t) - U(t)\|_{L^2} \,. \tag{6.1.1.21}$$

Pointwisely strong convergence of U_{ϵ_n} in L^2 implies that there exists a $N_1 = N_1(t,\eta)$, such that when $n > N_1$,

$$\|U_{\epsilon_n}(t) - U(t)\|_{L^2} < \frac{\eta}{12\sqrt{C^{in}}} \,, \tag{6.1.1.22}$$

so that when $n > N_1$,

$$I^3_{\epsilon_n} < \frac{\eta}{3} \,. \tag{6.1.1.23}$$

For $I^2_{\epsilon_n}$, we have shown in step 2 that $\{U_{\epsilon_n}\}_{\epsilon_n > 0}$ is equi-continuous in $C([0,\infty); \text{w-}L^2(dx))$. So, select $U(t) \in L^2(dx)$ as our test function, there exists a $\delta_1 = \delta_1(t, U(t,)\eta)$, such that $|\tilde{t} - t| < \delta_1$ implies that

$$I^2_{\epsilon_n} < \frac{\eta}{3} \,. \tag{6.1.1.24}$$

Now we estimate $I^1_{\epsilon_n}$, from the energy identity of the weakly compressible Stokes system:

$$I^1_{\epsilon_n} = \epsilon_n \int_t^{\tilde{t}} D_{\epsilon_n}(s)\, ds \,. \tag{6.1.1.25}$$

Fix a $0 < T < \infty$, notice that $[t,\tilde{t}] \subset [0,T]$,, then

$$|\epsilon_n \int_t^{\tilde{t}} D_{\epsilon_n}(s)\, ds| \leq \epsilon_n \int_0^T D_{\epsilon_n}(s)\, ds \,. \tag{6.1.1.26}$$

From the limit (6.1.1.17), we pick a $N_2 = N_2(\eta)$, such that when $n > N_2$,

$$\epsilon_n \int_0^T D_{\epsilon_n}(s)\, ds < \frac{\eta}{3} \,. \tag{6.1.1.27}$$

Now, denote $N_0 = \max(N_1, N_2)$, consider the finite terms

$$\epsilon_1 \int_t^{\tilde{t}} D_{\epsilon_1}(s)\, ds, \cdots, \epsilon_{N_0} \int_t^{\tilde{t}} D_{\epsilon_{N_0}}(s)\, ds \tag{6.1.1.28}$$

The absolute continuity of the Lebesgue integral ensures that that exists $\delta_j = \delta_j(\eta, T, \epsilon_j)$, where $j = 1, \cdots, N_0$, such that when $|\tilde{t} - t| < \delta_j$,

$$|\epsilon_j \int_t^{\tilde{t}} D_{\epsilon_j}(s)\, ds| < \frac{\eta}{3}. \tag{6.1.1.29}$$

Take $\delta_0 = \min(\delta_1, \cdots, \delta_{N_0})$, we conclude that when $|\tilde{t} - t| < \delta_0$,

$$I^1_{\epsilon_n} < \frac{\eta}{3}. \tag{6.1.1.30}$$

Then we proved that $\|U_{\epsilon_n}(t) - U_{\epsilon_n}(\tilde{t})\|^2_{L^2} < \eta$, thus $\{U_\epsilon\}_{\epsilon>0}$ is relatively compact in $C([0,\infty); L^2(dx))$, which implies that the solutions to the weakly compressible Stokes system U_ϵ,

$$U_\epsilon \to U \quad \text{in} \quad C([0,\infty); L^2(dx)) \tag{6.1.1.31}$$

where U is the solution to the acoustic system with the same initial data. We finish the proof of the theorem. □

6.1.2 Incompressible Stokes Approximation

In last section, we considered the short time scale, i.e., $\tau_\epsilon = 1$. To see the evolution of the incompressible flow, we have to consider the longer time scale, say, $\tau_\epsilon = \epsilon$. In this time scale, the weakly compressible Stokes system has singular behavior as $\epsilon \to 0$. We consider 3-dimensional case:

$$\begin{aligned} \partial_t \rho_\epsilon + \frac{1}{\epsilon}\nabla_x \cdot u_\epsilon &= 0\,, \\ \partial_t u_\epsilon + \frac{1}{\epsilon}\nabla_x(\rho_\epsilon + \theta_\epsilon) &= \mu\left[\nabla_x u_\epsilon + (\nabla_x u_\epsilon)^T - \tfrac{2}{3}\nabla_x \cdot u_\epsilon I\right]\,, \\ \tfrac{3}{2}\partial_t \theta_\epsilon + \frac{1}{\epsilon}\nabla_x \cdot u_\epsilon &= \kappa \Delta_x \theta_\epsilon\,. \end{aligned} \tag{6.1.2.1}$$

Let $U_\epsilon = (\rho_\epsilon, u_\epsilon, \theta_\epsilon)$. We can rewrite the weakly compressible Stokes system in the form of

$$
\begin{aligned}
\partial_t U_\epsilon + \frac{1}{\epsilon}\mathcal{A}U_\epsilon &= \mathcal{D}U_\epsilon \,, \\
U(0,x) &= U^{in}(x)\,.
\end{aligned}
\tag{6.1.2.2}
$$

Multiplying by ϵ on both sides above, and using the weak formulation of the weakly compressible Stokes system, we can easily see that for every $t > 0$,

$$
\rho_\epsilon + \theta_\epsilon \to 0\,, \quad \text{and} \quad \nabla_x \cdot u_\epsilon \to 0 \quad \text{as} \quad \epsilon \to 0\,. \tag{6.1.2.3}
$$

As we formally analyze before, the elements in the null space of $\mathcal{A}$ satisfy the incompressibility and Boussinesq relations. We decompose the solution to the weakly compressible Stokes system into two orthogonal parts in Null($\mathcal{A}$) and Null$(\mathcal{A})^\perp$ respectively:

$$
U_\epsilon = \Pi U_\epsilon + \Pi^\perp U_\epsilon = \begin{pmatrix} \frac{2}{5}\rho_\epsilon - \frac{3}{5}\theta_\epsilon \\ P u_\epsilon \\ -\frac{2}{5}\rho_\epsilon + \frac{3}{5}\theta_\epsilon \end{pmatrix} + \begin{pmatrix} \frac{3}{5}(\rho_\epsilon + \theta_\epsilon) \\ Q u_\epsilon \\ \frac{2}{5}(\rho_\epsilon + \theta_\epsilon) \end{pmatrix} \tag{6.1.2.4}
$$

Project the weakly compressible Stokes system onto Null($\mathcal{A}$):

$$
\partial_t \begin{pmatrix} -(\frac{3}{5}\theta_\epsilon - \frac{2}{5}\rho_\epsilon) \\ P u_\epsilon \\ \frac{3}{5}\theta_\epsilon - \frac{2}{5}\rho_\epsilon \end{pmatrix} = \begin{pmatrix} -\frac{2}{5}\kappa\Delta_x(\frac{3}{5}\theta_\epsilon - \frac{2}{5}\rho_\epsilon) \\ \mu P u_\epsilon \\ \frac{2}{5}\kappa\Delta_x(\frac{3}{5}\theta_\epsilon - \frac{2}{5}\rho_\epsilon) \end{pmatrix} + \begin{pmatrix} -\frac{4}{15}\kappa\Delta_x(\rho_\epsilon + \theta_\epsilon) \\ 0 \\ \frac{4}{15}\kappa\Delta_x(\rho_\epsilon + \theta_\epsilon) \end{pmatrix}. \tag{6.1.2.5}
$$

Formally let $\rho_\epsilon + \theta_\epsilon \to 0$ in the above equations, we see the limit $\Pi U_\epsilon \to U_s = (-\theta_s, u_s, \theta_s)$, where U_s is the solution to the incompressible Stokes system and satisfies the Boussinesq relation. However, as we know, the projection on Null$(\mathcal{A})^\perp$ goes to 0 only *weakly*. If we project the weakly compressible Stokes system onto

Null$(\mathcal{A})^\perp$: let $\Pi^\perp U_\epsilon = (\frac{2}{5}(\rho_\epsilon + \theta_\epsilon,)Qu_\epsilon, \frac{3}{5}(\rho_\epsilon + \theta_\epsilon))$, which satisfies

$$\partial_t \Pi^\perp U_\epsilon + \frac{1}{\epsilon}\mathcal{A}\Pi^\perp U_\epsilon = \begin{pmatrix} \frac{2\kappa}{5}\Delta_x \frac{2}{5}(\rho_\epsilon + \theta_\epsilon) \\ \\ \frac{4\mu}{3}\Delta_x Qu_\epsilon \\ \\ \frac{4\kappa}{15}\Delta_x \frac{2}{5}(\rho_\epsilon + \theta_\epsilon) \end{pmatrix} + \begin{pmatrix} \frac{2}{5}\kappa\Delta_x(\frac{2}{5}\rho_\epsilon - \frac{3}{5}\theta_\epsilon) \\ \\ 0 \\ \\ \frac{4}{15}\kappa\Delta_x(\frac{2}{5}\rho_\epsilon - \frac{3}{5}\theta_\epsilon) \end{pmatrix}. \qquad (6.1.2.6)$$

As we will analyze in details later, this weakly convergent to 0 sequence in Null$(\mathcal{A})^\perp$ propagates in very fast speed and carries energy in the asymptotic process. This is the so-called fast acoustic wave. It will prevent the strong convergence in Null$(\mathcal{A})$, even in the linear system. A natural question is: if the initial data are well-prepared, i.e., $\Pi^\perp U^{in}$, is there fast acoustic wave? The answer is "no". It depends on the precise description that how the fast waves propagates, which is governed by the averaged equation. We will derive that for the weakly compressible Stokes system, the averaged equation is a strictly dissipative diffusion equation. So, if initially zero, then vanishes in later time. Then the convergence in the null space will be strong.

Now we can state our main theorem this section:

Theorem 15: **From compressible to incompressible Stokes dynamics** *Let* $U_\epsilon = (\rho_\epsilon, u_\epsilon, \theta_\epsilon) \in C([0,\infty); L^2(d\mathbf{x}; \mathbb{R}\times\mathbb{R}^D\times\mathbb{R}))$ *be the weak solution to the linearized Navier-Stokes system. Then*

1. *We have convergence*

$$U_\epsilon - U_s - e^{-\frac{t}{\epsilon}\mathcal{A}}V^0 \rightharpoonup 0\,, \quad in \quad L^2([0,\infty); \text{w-}L^2(dx; \mathbb{R}\times\mathbb{R}^D\times\mathbb{R}))\,; \quad (6.1.2.7)$$

where V^0 *satisfies the averaged equation (diffusion equation)* (6.1.2.12), *and*

$U_s = (-\theta_s, u_s, \theta_s)$ *is the solution to the incompressible Stokes system:*

$$\partial_t \begin{pmatrix} -\theta_s \\ u_s \\ \theta_s \end{pmatrix} + \begin{pmatrix} 0 \\ \nabla_x p \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{2}{5}\kappa\Delta_x\theta_s \\ \mu\Delta_x u_s \\ \frac{2}{5}\kappa\Delta_x\theta_s \end{pmatrix}, \tag{6.1.2.8}$$

$$\nabla_x \cdot u_s = 0\,,$$

$$U_s(0,x) = \Pi U^{in}\,.$$

2. *and*

$$\Pi U_\epsilon \rightharpoonup U_s\,, \quad in \quad L^\infty([0,\infty); \text{w-}L^2(dx; \mathbb{R}\times\mathbb{R}^D\times\mathbb{R}))\,; \tag{6.1.2.9}$$

3. *when the initial data are well-prepared, i.e.,* $\Pi^\perp U^{in} = 0$,

$$\Pi U_\epsilon \to U_s\,, \quad in \quad L^\infty([0,\infty); L^2(dx; \mathbb{R}\times\mathbb{R}^D\times\mathbb{R}))\,; \tag{6.1.2.10}$$

Proof: The crucial part in the proof is to understand how $\Pi^\perp U_\epsilon$ propagate in the evolution. The equation satisfied by $\Pi^\perp U_\epsilon$ (6.1.2.6) has a singular term $\frac{1}{\epsilon}\mathcal{A}\Pi^\perp U_\epsilon$. So $\Pi^\perp U_\epsilon$ itself is not convergent. We apply Schochet's technique, see [68], we act a semi-group $e^{\frac{t}{\epsilon}\mathcal{A}}$ on the equation (6.1.2.6). We denote by $W_\epsilon(t) = e^{\frac{t}{\epsilon}\mathcal{A}}\Pi^\perp U_\epsilon$, which is convergent. This property is stated in the following simple lemma:

Lemma 18: *We have convergence*

$$W_\epsilon \to V^0\,, \quad in \quad L^2([0,T]; H^{-m}) \quad for\ some\ m \in (0,1)\,. \tag{6.1.2.11}$$

where V^0 *satisfies the averaged equation:*

$$\partial_t V^0 = \tilde{\mu}\Delta_x V^0\,, \qquad V^0(0,x) = \Pi^\perp U^{in}(x)\,, \tag{6.1.2.12}$$

where $\tilde{\mu} = \frac{2}{3}\mu + \frac{1}{5}\kappa$ *is a linear combination of the viscosity and heat conductivity.*

Proof: W_ϵ satisfies a system without singular term.

$$\partial_t W_\epsilon = e^{\frac{t}{\epsilon}\mathcal{A}} \begin{pmatrix} \frac{2\kappa}{5}\Delta_x \frac{2}{5}(\rho_\epsilon + \theta_\epsilon) \\ \frac{4\mu}{3}\nabla_x \nabla_x \cdot u_\epsilon \\ \frac{4\kappa}{15}\Delta_x \frac{2}{5}(\rho_\epsilon + \theta_\epsilon) \end{pmatrix} + e^{\frac{t}{\epsilon}\mathcal{A}} \begin{pmatrix} \frac{2}{5}\kappa\Delta_x(\frac{2}{5}\rho_\epsilon - \frac{3}{5}\theta_\epsilon) \\ 0 \\ \frac{4}{15}\kappa\Delta_x(\frac{2}{5}\rho_\epsilon - \frac{3}{5}\theta_\epsilon) \end{pmatrix} . \tag{6.1.2.13}$$

We are going to prove first that $\Pi^\perp U\epsilon$ (and thus $W_\epsilon = e^{\frac{t}{\epsilon}\mathcal{A}}\Pi^\perp U_\epsilon$ since $e^{t\mathcal{A}}$ is an isometry) is bounded in $L^\infty([0,T];L^2(dx))$. It is straight forward from the energy identity of the weakly compressible Stokes system. Secondly, $\partial_t W_\epsilon$ is bounded in $L^2([0,T];H^{-1})$. It again comes from the energy identity of Stokes system. Then the relative compactness of W_ϵ follows the classical Aubin-Lions compactness theorem choosing m in $(0,1)$. Suppose limit point is V^0.

To derive the averaged equation (6.1.2.12) obeyed by the limit point V^0, we employ the standard almost-periodic function theory as we discussed in the previous section. Notice that the second term in (6.1.2.13) convergent weakly in $L^2([0,T];H^{-1})$ to $(-\frac{2}{5}\kappa\Delta_x\theta_s, 0, \frac{4}{15}\kappa\Delta_x\theta_s)$ which does not has any effect in the process of the time averaging. Then V^0 satisfies

$$\begin{aligned} \partial_t V^0 &= \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau e^{s\mathcal{A}}\mathcal{D}_1 e^{s\mathcal{A}} V^0 \, ds \\ &= \tilde{\mu}\Delta_x V^0 \, . \end{aligned} \tag{6.1.2.14}$$

It is a heat equation. We can conclude immediately that if initially $V^0(0,x) = \Pi^\perp U^{in}(x) = 0$, i.e., the initial data satisfies the incompressibility and Boussinesq relations, from the maximum principle of the heat equation, in any layer time $t > 0$, $V^0(t) \equiv 0$. Then there would be no fast waves.

Now define $V_\epsilon(t) = e^{-\frac{t}{\epsilon}\mathcal{A}}V^0 = (\psi_\epsilon, m_\epsilon, \frac{2}{3}\psi_\epsilon)$, where $m_\epsilon = \nabla_x q_\epsilon$, then V_ϵ satisfies the equation:

$$\partial_t V_\epsilon + \frac{1}{\epsilon}\mathcal{A}V_\epsilon = \tilde{\mu}\Delta V_\epsilon\,,$$
$$V_\epsilon(0,x) = \Pi^\perp U^{in}(x)\,. \tag{6.1.2.15}$$

The semi-group $e^{-\frac{t}{\epsilon}\mathcal{A}}$ preserve the Sobolev norm, so $\|V_\epsilon(t)\|_{L^2} = \|V^0\|_{L^2}$. Then although the equation obeyed by V_ϵ is singular, $V_\epsilon(t) = 0$, if $\Pi^\perp U^{in} = 0$.

We define $U_\epsilon(t) - U_s(t) - V_\epsilon(t) = (\alpha_\epsilon, w_\epsilon, \beta_\epsilon)$, i.e.,

$$\alpha_\epsilon = \rho_\epsilon - (-\theta_s) - \psi_\epsilon\,,$$
$$w_\epsilon = u_\epsilon - u_s - m_\epsilon\,, \tag{6.1.2.16}$$
$$\beta_\epsilon = \theta_\epsilon - \theta_s - \tfrac{2}{3}\psi_\epsilon\,,$$

and

$$E_\epsilon(t) \triangleq \tfrac{1}{2}\|U_\epsilon(t) - U_s(t) - V_\epsilon(t)\|^2_{L^2}\,. \tag{6.1.2.17}$$

We will calculate the evolution of $E_\epsilon(t)$.

$$\begin{aligned}\frac{d}{dt}E_\epsilon(t) = &\frac{1}{2}\frac{d}{dt}\|U_\epsilon(t)\|^2_{L^2} + \frac{1}{2}\frac{d}{dt}\|U_s(t)\|^2_{L^2} + \frac{1}{2}\frac{d}{dt}\|V_\epsilon(t)\|^2_{L^2}\\ &- \frac{d}{dt}\langle U_\epsilon\,, U_s\rangle(t) - \frac{d}{dt}\langle U_\epsilon\,, V_\epsilon\rangle(t)\,.\end{aligned} \tag{6.1.2.18}$$

We recall the energy identity for the weakly compressible Stokes system:

$$\frac{1}{2}\frac{d}{dt}\|U_\epsilon(t)\|^2_{L^2} = -\int_\Omega \mu|\nabla_x u_\epsilon|^2\,dx - \int_\Omega \tfrac{\mu}{3}|\nabla_x\cdot u_\epsilon|^2\,dx - \int_\Omega \kappa|\nabla_x\theta_\epsilon|^2\,dx\,. \tag{6.1.2.19}$$

the energy identity for the incompressible Stokes equations:

$$\frac{1}{2}\frac{d}{dt}\|U_s(t)\|^2_{L^2} = -\int_\Omega \mu|\nabla_x u_s|^2\,dx - \int_\Omega \kappa|\nabla_x\theta_s|^2\,dx\,. \tag{6.1.2.20}$$

The energy identity for V_ϵ:

$$\frac{1}{2}\frac{d}{dt}\|V_\epsilon(t)\|^2_{L^2} = \frac{1}{2}\frac{d}{dt}\|V^0\|^2_{L^2} = -\tilde{\mu}\int_\Omega |\nabla_x V^0|^2\,dx \tag{6.1.2.21}$$

Next, using the weak formulation of the weakly compressible Stokes and incompressible Stokes system, we obtain that

$$\begin{aligned}\frac{d}{dt}\langle U_\epsilon\,, U_s\rangle = -2\int_\Omega \mu\nabla_x u_\epsilon\cdot\nabla_x u_s\,dx - \int_\Omega \kappa\nabla_x\theta_\epsilon\cdot\nabla_x\theta_s\,dx \\ - \int_\Omega \nabla_x p\cdot u_\epsilon - \int_\Omega \kappa\nabla_x\theta_s\cdot\nabla_x(\tfrac{3}{5}\theta_\epsilon - \tfrac{2}{5}\rho_\epsilon)\,dx\,.\end{aligned} \tag{6.1.2.22}$$

while the weak formulation of (6.1.2.15), yields as before the following identity

$$\begin{aligned}\frac{d}{dt}\langle U_\epsilon\,, V_\epsilon\rangle = -\int_\Omega \mu\nabla_x u_\epsilon\cdot\nabla_x m_\epsilon\,dx - \int_\Omega \tfrac{\mu}{3}(\nabla_x\cdot u_\epsilon)(\nabla_x\cdot m_\epsilon)\,dx \\ - \int_\Omega \tfrac{2}{3}\kappa\nabla_x\theta_\epsilon\cdot\nabla_x\psi_\epsilon\,dx + \langle e^{-\frac{t}{\epsilon}\mathcal{A}}\partial_t V^0\,, U_\epsilon\rangle\,,\end{aligned} \tag{6.1.2.23}$$

where

$$\begin{aligned}\langle e^{-\frac{t}{\epsilon}\mathcal{A}}\partial_t V^0\,, U_\epsilon\rangle &= \langle \partial_t V^0\,, e^{-\frac{t}{\epsilon}\mathcal{A}}\Pi U_\epsilon + e^{-\frac{t}{\epsilon}\mathcal{A}}\Pi^\perp U_\epsilon\rangle \\ &= \langle \tilde{\mu}\Delta_x V^0\,, U_s\rangle + \langle \tilde{\mu}\Delta_x V^0\,, V^0\rangle + r_\epsilon\,, \\ &= -\tilde{\mu}\langle|\nabla_x V^0|^2\,dx + r_\epsilon\,,\end{aligned} \tag{6.1.2.24}$$

here $r_\epsilon \to 0$ uniformly.

Next add up (6.1.2.19), (6.1.2.20), (6.1.2.21) and subtract (6.1.2.22), (6.1.2.23), and take integral from 0 to t. Notice that

$$\begin{aligned}&\tfrac{1}{2}\|U_\epsilon(0)\|_{L^2}^2 + \tfrac{1}{2}\|U_s(0)\|_{L^2}^2 + \tfrac{1}{2}\|V^0(0)\|_{L^2}^2 \\ &= \langle U_\epsilon(0)\,, U_s(0)\rangle + \langle U_\epsilon(0)\,, V^0(0)\rangle\,.\end{aligned} \tag{6.1.2.25}$$

and

$$\begin{aligned}&\int_\Omega \kappa\nabla_x\theta_\epsilon\cdot\nabla_x\theta_s\,dx + \int_\Omega \kappa\nabla_x\theta_s\cdot\nabla_x(\tfrac{3}{5}\theta_\epsilon - \tfrac{2}{5}\rho_\epsilon)\,dx\,, \\ &= 2\int_\Omega \kappa\nabla_x\theta_\epsilon\cdot\nabla_x\theta_s\,dx - \int_\Omega \kappa\nabla_x\theta_s\cdot\nabla_x\tfrac{2}{5}(\rho_\epsilon + \theta_\epsilon)\,, \\ &= 2\int_\Omega \kappa\nabla_x\theta_\epsilon\cdot\nabla_x\theta_s\,dx + r_\epsilon^1\,.\end{aligned} \tag{6.1.2.26}$$

and

$$\int_\Omega \nabla_x p\cdot u_\epsilon = \int_\Omega \nabla_x p\cdot Q u_\epsilon = r_\epsilon^2\,, \tag{6.1.2.27}$$

where $r_\epsilon^1, r_\epsilon^2 \to 0$ as $\epsilon \to 0$, because $\rho_\epsilon + \theta_\epsilon, Qu_\epsilon \to 0$ weakly in L^2.

Thus,

$$\begin{aligned}
&\tfrac{1}{2}\int_\Omega \left(\alpha_\epsilon^2(t) + |w_\epsilon(t)|^2 + \beta_\epsilon^2(t)\right)\,dx + \int_0^t\int_\Omega \kappa|\nabla_x(\theta_\epsilon - \theta_s)|^2\,dxds \\
&+ \int_0^t\int_\Omega \mu|\nabla_x w_\epsilon(s)|^2\,dxds + \int_0^t\int_\Omega \tfrac{\mu}{3}|\nabla_x\cdot w_\epsilon(s)|^2\,dxds \\
&= \int_0^t\int_\Omega \mu|\nabla_x m_\epsilon|^2\,dxds + \int_0^t\int_\Omega \tfrac{\mu}{3}|\nabla_x\cdot m_\epsilon|^2\,dxds \\
&- \int_0^t\int_\Omega \mu(\nabla_x u_\epsilon)\cdot(\nabla_x m_\epsilon) - \int_0^t\int_\Omega \tfrac{\mu}{3}(\nabla_x\cdot u_\epsilon)(\nabla_x\cdot m_\epsilon) \\
&- \int_0^t\int_\Omega \tfrac{2}{3}\kappa\nabla_x\theta_\epsilon\cdot\nabla_x\psi_\epsilon\,dx + R_\epsilon\,.
\end{aligned} \tag{6.1.2.28}$$

Recall that $m_\epsilon = \nabla_x q_\epsilon$, so

$$\int_\Omega |\nabla_x m_\epsilon|^2\,dx = \int_\Omega |\nabla_x\cdot m_\epsilon|^2\,dx = -\int_\Omega \Delta_x m_\epsilon\cdot m_\epsilon\,dx \tag{6.1.2.29}$$

and

$$\int_\Omega (\nabla_x u_\epsilon)\cdot(\nabla_x m_\epsilon)\,dx = \int_\Omega (\nabla_x\cdot u_\epsilon)(\nabla_x\cdot m_\epsilon)\,dx = -\int_\Omega \Delta_x u_\epsilon\cdot m_\epsilon\,dx \tag{6.1.2.30}$$

Then

$$\begin{aligned}
&- \int_0^t\int_\Omega \mu(\nabla_x u_\epsilon)\cdot(\nabla_x m_\epsilon) - \int_0^t\int_\Omega \tfrac{\mu}{3}(\nabla_x\cdot u_\epsilon)(\nabla_x\cdot m_\epsilon) \\
&- \int_0^t\int_\Omega \tfrac{2}{3}\kappa\nabla_x\theta_\epsilon\cdot\nabla_x\psi_\epsilon\,dx \\
&= \tfrac{4\mu}{3}\int_0^t\int_\Omega \Delta_x u_\epsilon\cdot m_\epsilon\,dxds + \tfrac{2\kappa}{3}\int_0^t\int_\Omega \Delta_x\theta_\epsilon\cdot m_\epsilon\,dxds \\
&= \int_0^t \left\langle \begin{pmatrix} 0 \\ \mu\Delta_x u_\epsilon + \frac{\mu}{3}\nabla_x(\nabla_x\cdot u_\epsilon) \\ \frac{2\kappa}{3}\Delta_x\theta_\epsilon \end{pmatrix}, \begin{pmatrix} \psi_\epsilon \\ m_\epsilon \\ \frac{2}{3}\psi_\epsilon \end{pmatrix} \right\rangle ds\,, \\
&= \int_0^t \left\langle e^{\frac{t}{\epsilon}\mathcal{A}}\mathcal{D}e^{-\frac{t}{\epsilon}\mathcal{A}}\tilde{V}_\epsilon, V^0 \right\rangle ds\,,
\end{aligned} \tag{6.1.2.31}$$

where $\tilde{V}_\epsilon = e^{\frac{t}{\epsilon}\mathcal{A}} U_\epsilon$. As we did for W_ϵ, we have

$$\tilde{V}_\epsilon \to U_s + V^0\,, \quad \text{in} \quad L^2([0,T]; H^{-m}) \quad \text{for some m} \in (0,1)\,. \tag{6.1.2.32}$$

And from the theory of the almost-periodic function, we obtain:

$$\begin{aligned}
&\lim_{\epsilon\to 0} e^{\frac{t}{\epsilon}\mathcal{A}} \mathcal{D} e^{-\frac{t}{\epsilon}\mathcal{A}} \tilde{V}_\epsilon \\
&= \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau e^{s\mathcal{A}} \mathcal{D} e^{-s\mathcal{A}} (U_s + V^0)\, ds \\
&= \begin{pmatrix} -\frac{2}{5}\kappa \Delta_x \theta_s \\ \\ \mu \Delta_x u_s \\ \\ \frac{2}{5}\kappa \Delta_x \theta_s \end{pmatrix} + \tilde{\mu} \Delta_x V^0\,,
\end{aligned} \tag{6.1.2.33}$$

in the sense of distributions. Recall that V^0 is the solution to the heat equation. It has good regularity, so that we can take limit in (6.1.2.31), then, we obtain

$$\int_0^t \langle e^{\frac{t}{\epsilon}\mathcal{A}} \mathcal{D} e^{-\frac{t}{\epsilon}\mathcal{A}} \tilde{V}_\epsilon\,, V^0 \rangle\, ds = -\int_0^t \int_\Omega \tilde{\mu} |\nabla_x V^0|^2 \, dx ds + r_\epsilon(t) \tag{6.1.2.34}$$

Combine the above identity with (6.1.2.28), it is easy to see that when the initial data are "well-prepared", we have the following strong convergence:

$$\Pi U_\epsilon \to U_s \quad , \text{in} \quad L^\infty([0,\infty); L^2(dx; \mathbb{R}\times\mathbb{R}^2\times\mathbb{R}))\,, \tag{6.1.2.35}$$

as $\epsilon \to 0$. When the initial data are general, i.e., $\Pi^\perp U^{in}$ is nonzero, we would have only weak convergence. Then we finish the proof of the theorem. □

6.2 From Boltzmann Equation to Weakly Compressible Stokes System: Relative Entropy Method

In [29], Golse and Levermore established a Stokes-Fourier limit for the Boltzmann equation considered over any periodic spatial domain for dimension 2 or more.

Appropriately scaled family of DiPerna-Lions renormalized solutions are shown to have fluctuations that globally in time converge weakly to a unique limit governed by a solution to Stokes-Fourier motion and heat equations provided that the fluid moments of their initial fluctuations converge to appropriate L^2 initial data of the Stokes-Fourier equations. It was the first time that both the motion and heat equations are recovered in the limit by controlling the fluxes and the local conservation defects of the DiPerna-Lions solutions with dissipation rate estimates. The scaling of the fluctuations with respect to Knudsen number is essentially optimal. The assumptions on the collition kernel are little more than required for the DiPerna-Lions theory and that the viscosity and heat conduction are finite. For the acoustic limit, these techniques also remove restrictions to bounded collision kernels and improve the scaling of the fluctuations. Both weak limits become strong when the initial fluctuations converge entropically to appropriate L^2 initial data.

As we showed in the last section, the weakly compressible Stokes system (6.1.0.10) governs both acoustic system and incompressible Stokes dynamics, depending on the time scales considering. When the time scale $\tau_\epsilon = 1$, the limit is acoustic system, while $\tau_\epsilon = \epsilon$, the asymptotics of the weakly compressible Stokes system is the incompressible Stokes equations with a correct term which describe the fast oscillating acoustic waves. When the initial data are well-prepared, i.e., satisfying the incompressibility and Boussinesq relations, this fast wave will vanish. This case has been treated in Golse-Levermore's work [29]. Then, a natural question is to derive the limiting behavior from the Boltzmann equation to the weakly compressible Stokes system. A significant difference is that this process is not a *limit*, but an *asymptotics*. Because from the formal derivation from the Boltzmann equation, (see section 3,) the weakly compressible Stokes system depends on the Knudsen number ϵ even though the initial data does not. The basic idea is that starting from the solution $(\rho_\epsilon, u_\epsilon, \theta_\epsilon)$ to the weakly compressible Stokes system, we construct a local Maxwellian $M_\epsilon = \mathcal{M}_{(1+\rho_\epsilon, u_\epsilon, 1+\theta_\epsilon)}$ such that the fluid dynamics associated to this Maxwellian has fluctuation $(\rho_\epsilon, u_\epsilon, \theta_\epsilon)$. We will show that the stability of DiPerna-Lions solutions to the scaled Boltzmann equation around this local Maxwellian. The functional which measures the stability is obtained naturally from the relative entropy $H(F_\epsilon|M)$ that is a nonnegative Lyapunov functional for the Boltzmann equation, and controls the size of the fluctuation in incompressible regimes.

The modulated entropy is then defined as

$$H(F_\epsilon|M_\epsilon)(t) = \int_\Omega \int_{\mathbb{R}^D} F_\epsilon(t) \ln\left(\frac{F_\epsilon(t)}{M_\epsilon(t)}\right) - F_\epsilon(t) + M_\epsilon(t)\, dvdx\,. \qquad (6.2.0.36)$$

The core of the proof is therefore to establish a stability inequality on the modulated entropy. This will provide the convergence of the modulated entropy to zero as $\epsilon \to 0$. Finally, we conclude by providing that the relative entropy $H(F|G)$ controls the L^1 norm of the difference $F - G$.

The idea of using the notion of relative entropy for this kind of problems comes from the notion of entropic convergence developed by Bardos, Golse and Levermore in [8], and on the other hand from Yau's elegant derivation of the hydrodynamics limit of the Ginzburg-Landau lattice model [75].

Applying the relative entropy method to the case of the Boltzmann equation, the convergence of renormalized solutions to the scaled Boltzmann equation to solutions of the incompressible Euler equations for well-prepared data is established in [10] and [66].

In this section, we will apply the method of relative entropy to justify the asymptotic behavior of DiPerna-Lions solutions to the scaled Boltzmann equation:

$$\begin{aligned} \epsilon\partial_t F_\epsilon + v\cdot\nabla_x F_\epsilon &= \frac{1}{\epsilon}\mathcal{B}(F_\epsilon\,, F_\epsilon)\,, \\ F_\epsilon(0,x,v) &= F_\epsilon^{in}(x,v)\,. \end{aligned} \tag{6.2.0.37}$$

For any pair of measurable functions f and g defined a.e. on $\Omega\times\mathbb{R}^D$ and satisfying $f\geq 0$ and $g>0$ a.e., we use the following notation for the relative entropy

$$H(f|g) = \int_\Omega\int_{\mathbb{R}^D}\left[f\ln\left(\frac{f}{g}\right) - f + g\right]\,dvdx\,. \tag{6.2.0.38}$$

We claim that the relative entropy defined above is always nonnegative based on the following argument. Define the usual entropy function $\mathfrak{h}(f) = f\ln(f)$, thus $\mathfrak{h}''(f) = \frac{1}{f}$. So $h(f)$ is a strictly convex function on $(0,+\infty)$, then

$$\mathfrak{h}(f) - \mathfrak{h}(g) - \mathfrak{h}'(g)(f-g) \geq 0\,. \tag{6.2.0.39}$$

Simple calculation shows that

$$\mathfrak{h}(f) - \mathfrak{h}(g) - \mathfrak{h}'(g)(f-g) = f\ln\left(\frac{f}{g}\right) - f + g\,. \tag{6.2.0.40}$$

Thus we proved our claim.

Because we will work in the context of the DiPerna-Lions solution, we state the DiPerna-Lions theorem [20], including some improvement late by Lions-Masmoudi [53].

DiPerna-Lions-(Masmoudi): For each $\epsilon > 0$, given any initial data F_ϵ^{in} in the entropy class

$$E = \{F^{in} \geq 0 : H(F^{in}|M) < +\infty\}, \tag{6.2.0.41}$$

there exists at least one $F_\epsilon \geq 0$, in $C([0,\infty); L^1(dvdx))$, that satisfies (6.2.0.37) in renormalized sense, with $F_\epsilon(0,x,v) = F_\epsilon^{in}(x,v)$. Moreover, F_ϵ satisfies:

- the global entropy inequality

$$\int_\Omega \int_{\mathbb{R}^D} F_\epsilon \ln F_\epsilon \, dvdx + \frac{1}{4\epsilon^2} \int_0^t \int_\Omega D(F_\epsilon)(s,x)\, dxds \leq \int_\Omega \int_{\mathbb{R}^D} F_\epsilon^{in} \ln F_\epsilon^{in} \, dvdx \,, \tag{6.2.0.42}$$

  where the entropy dissipation $D(F)$ is defined as

$$D(F) = \iiint\limits_{\mathbb{R}^D \times \mathbb{R}^D \times \mathbb{S}^{D-1}} (F_1' F' - F_1 F) \ln \left(\frac{F_1' F'}{F_1 F} \right) b(v_1 - v, \omega)\, d\omega dv_1 dv \geq 0\,; \tag{6.2.0.43}$$

- the local conservation law of mass

$$\partial_t \int\limits_{\mathbb{R}^D} F_\epsilon(t,x,\cdot)\, dv + \frac{1}{\epsilon} \nabla_x \cdot \int\limits_{\mathbb{R}^D} v F_\epsilon(t,x,\cdot)\, dv = 0\,; \tag{6.2.0.44}$$

- local conservation law of momentum with defect

$$\partial_t \int\limits_{\mathbb{R}^D} v F_\epsilon(t,x,\cdot)\, dv + \frac{1}{\epsilon} \nabla_x \cdot \int\limits_{\mathbb{R}^D} v \otimes v F_\epsilon(t,x,\cdot)\, dv + \frac{1}{\epsilon} \nabla_x \cdot \mathfrak{M}_\epsilon = 0\,, \tag{6.2.0.45}$$

  for some matrix-valued nonnegative bounded measure

$$\mathfrak{M}_\epsilon \in L^\infty([0,\infty); \mathbb{M}(\Omega; M_{N\times N}),) \tag{6.2.0.46}$$

  where $\mathbb{M}$ denotes space of bounded measure, and $M_{N\times N}$ the space of $N \times N$ matrices;

- global conservation of energy with defect measure

$$\int_\Omega \int_{\mathbb{R}^D} |v|^2 F_\epsilon \, dvdx + \int_\Omega \mathrm{tr}(\mathfrak{M}_\epsilon) \, dx = \int_\Omega \int_{\mathbb{R}^D} |v|^2 F_\epsilon^{in} \, dvdx \,. \tag{6.2.0.47}$$

Assume that

$$F_\epsilon^{in} = M(1 + \delta_\epsilon g_\epsilon^{in}) \quad \text{and} \quad F_\epsilon = M(1 + \delta_\epsilon g_\epsilon) \tag{6.2.0.48}$$

where in the case of hard sphere,

$$\frac{\delta_\epsilon}{\epsilon} \to 0 \quad \text{as} \quad \epsilon \to 0 \,, \tag{6.2.0.49}$$

Denote by $\mathfrak{m}_\epsilon = \frac{1}{\delta_\epsilon}\mathfrak{M}_\epsilon$, we can rewrite conservation laws as:

$$\begin{gathered} \partial_t \langle v g_\epsilon \rangle + \frac{1}{\epsilon} \nabla_x \cdot \langle v \otimes v g_\epsilon \rangle + \frac{1}{\epsilon} \nabla_x \cdot \mathfrak{m}_\epsilon = 0 \,, \\ \int_\Omega \langle \tfrac{1}{2}|v|^2 g_\epsilon \rangle + \tfrac{1}{2} \int_\Omega \mathrm{tr}(\mathfrak{m}_\epsilon) \, dx - \int_\Omega \langle \tfrac{1}{2}|v|^2 g_\epsilon^{in} \rangle = 0 \end{gathered} \tag{6.2.0.50}$$

If we normalize the initial data, such that

$$\begin{gathered} \int_\Omega \int_{\mathbb{R}^D} F_\epsilon^{in} \, dvdx = 1 \,, \\ \int_\Omega \int_{\mathbb{R}^D} v F_\epsilon^{in} \, dvdx = 0 \,, \\ \int_\Omega \int_{\mathbb{R}^D} \tfrac{1}{2}|v|^2 F_\epsilon^{in} \, dvdx = \tfrac{D}{2} \,. \end{gathered} \tag{6.2.0.51}$$

and the entropy bound

$$\int_\Omega \int_{\mathbb{R}^D} F_\epsilon^{in} \ln F_\epsilon^{in} \, dvdx \leq -\tfrac{D}{2} + C^{in} \delta_\epsilon^2 \,, \tag{6.2.0.52}$$

These bounds can be written as the bound for the relative entropy

$$H(F_\epsilon^{in} | M) \leq C^{in} \delta_\epsilon^2 \,. \tag{6.2.0.53}$$

Now we can rewrite the global inequality for the relative entropy with respect to the absolute Maxwellian M:

$$H(F_\epsilon|M)(t) + \delta_\epsilon \int_\Omega \frac{1}{2}\mathrm{tr}(\mathfrak{m}_\epsilon)\,dx + \frac{1}{4\epsilon^2}\int_0^t \int_\Omega D(F_\epsilon)(s,x)\,dxds \leq H(F_\epsilon^{in}|M)\,. \tag{6.2.0.54}$$

The crucial part in our proof of the asymptotic behavior from the Boltzmann equation to the weakly compressible Stokes system is to estimate the evolution in time the following relative entropy:

$$H(F_\epsilon|M_\epsilon)(t) = \int_\Omega \int_{\mathbb{R}^D} F_\epsilon(t)\ln\left(\frac{F_\epsilon(t)}{M_\epsilon(t)}\right) - F_\epsilon(t) + M_\epsilon(t)\,dvdx\,. \tag{6.2.0.55}$$

where the local Maxwellian M_ϵ:

$$M_\epsilon = \mathcal{M}_{(1+\delta_\epsilon\rho_\epsilon, \delta_\epsilon u_\epsilon, 1+\delta_\epsilon\theta_\epsilon)} = \frac{1+\delta_\epsilon\rho_\epsilon}{(2\pi(1+\delta_\epsilon\theta_\epsilon))^{\frac{D}{2}}}\exp\left(\frac{|v-\delta_\epsilon u_\epsilon|^2}{2(1+\delta_\epsilon\theta_\epsilon)}\right)\,, \tag{6.2.0.56}$$

where $(\rho_\epsilon, u_\epsilon, \theta_\epsilon)$ is the solution to the weakly compressible Stokes system:

$$\begin{aligned}
\partial_t\rho_\epsilon + \frac{1}{\epsilon}\nabla_x\cdot u_\epsilon &= 0\,,\\
\partial_t u_\epsilon + \frac{1}{\epsilon}\nabla_x(\rho_\epsilon+\theta_\epsilon) &= \mu\left[\nabla_x u_\epsilon + (\nabla_x u_\epsilon)^T - \tfrac{2}{3}\nabla_x\cdot u_\epsilon I\right]\,,\\
\tfrac{D}{2}\partial_t\theta_\epsilon + \frac{1}{\epsilon}\nabla_x\cdot u_\epsilon &= \kappa\Delta_x\theta_\epsilon\,.
\end{aligned} \tag{6.2.0.57}$$

with initial data $(\rho^{in}, u^{in}, \theta^{in}) \in L^2(dx)$.

Before we state our main theorems, we make some remarks on the local Maxwellain (6.2.0.56). Our construction, $M_\epsilon = \mathcal{M}_{(1+\delta_\epsilon\rho_\epsilon, \delta_\epsilon u_\epsilon, 1+\delta_\epsilon\theta_\epsilon)}$, requires that the positivity of the mass and temperature, i.e., $1+\delta_\epsilon\rho_\epsilon > 0$ and $1+\delta_\epsilon\theta_\epsilon > 0$, so that it is physical. Then the sole L^2 bound can not guarantee this positivity. So we need the following modification of the fluctuations which originates in the Bardos-Golse-Levermore's "realizability of the initial data lemma", see [9].

Lemma 19: **(Realizability of the Initial Data.)** *Let* $\delta_\epsilon > 0$*, and* $\delta_\epsilon \to 0$ *as* $\epsilon \to 0$*, and let* $(\rho^{in}, u^{in}, \theta^{in}) \in L^2(dx; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R})$ *Then there exits a family of physical local Maxwellian* $M_\epsilon^{in} = \mathcal{M}_{(1+\delta_\epsilon \rho_\epsilon^{in}, \delta_\epsilon u_\epsilon^{in}, 1+\delta_\epsilon \theta_\epsilon^{in})}$*, i.e.,* $1 + \delta_\epsilon \rho_\epsilon^{in} > 0$ *and* $1 + \delta_\epsilon \theta_\epsilon^{in} > 0$*. Furthermore, the fluctuations* $g_\epsilon^{in} = \frac{1}{\delta_\epsilon}(F_\epsilon/M - 1)$ *converges entropically at order* δ_ϵ *as* $\epsilon \to 0$ *to the infinitesimal Maxwellian* $g^{in} = \rho^{in} + u^{in} \cdot v + \theta^{in}(\frac{1}{2}|v|^2 - \frac{D}{2})$*.*

Proof: Let $j \in C_c^\infty(\mathbb{R}^D)$ be a mollifying function:

$$j \geq 0, \quad \mathrm{supp}(j) \subset B_{\frac{1}{2}}(0,) \quad \int_{\mathbb{R}^D} j(x)\, dx = 1\,. \tag{6.2.0.58}$$

For every $\epsilon \in (0, 1]$, define $j_\epsilon \in C^\infty(\mathbb{T}^D)$ by

$$j_\epsilon(x) = \frac{1}{\delta_\epsilon} \sum_{z \in \mathbb{Z}^D} j\left(\frac{x+z}{\delta_\epsilon^{\frac{1}{D}}}\right)\,. \tag{6.2.0.59}$$

The assumption on the support of j guarantees that the supports of the various terms in the above sum never overlap for $0 < \epsilon \leq 1$. Then j_ϵ is a mollifying family over $\mathbb{T}^D$. Define

$$\rho_\epsilon^{in} = j_\epsilon \star \rho^{in}\,, \tag{6.2.0.60}$$

where the symbol " $\star$ " designates the convolution over $\mathbb{T}^D$. The Cauchy-Schwarz inequality gives

$$\|\rho^{in}\|_{L^\infty} \leq \|j_\epsilon\|_{L^2} \|\rho^{in}\|_{L^2} = \frac{1}{\delta_\epsilon^{\frac{1}{2}}} \|j_1\|_{L^2} \|\rho^{in}\|_{L^2}\,, \tag{6.2.0.61}$$

whereby it is clear that for all $\epsilon \in (0, 1]$ sufficiently small one has $1 + \delta_\epsilon \rho_\epsilon^{in} > \frac{1}{2}$. For all such ϵ define

$$u_\epsilon^{in} = \frac{j_\epsilon \star u^{in}}{1 + \delta_\epsilon \rho_\epsilon^{in}}, \quad \theta_\epsilon^{in} = \frac{j_\epsilon \star \theta^{in}}{1 + \delta_\epsilon \rho_\epsilon^{in}} - \delta_\epsilon \tfrac{1}{D} |u_\epsilon^{in}|^2\,. \tag{6.2.0.62}$$

The Cauchy-Schwarz gives

$$\|u_\epsilon^{in}\|_{L^\infty} \le 2\|j_\epsilon\|_{L^2}\|u^{in}\|_{L^2} = \frac{2}{\delta_\epsilon^{\frac{1}{2}}}\|j_1\|_{L^2}\|u^{in}\|_{L^2}\,,$$

$$\|\theta_\epsilon^{in}\|_{L^\infty} \le 2\|j_\epsilon\|_{L^2}\|\theta^{in}\|_{L^2} + \delta_\epsilon\frac{1}{D}\|u_\epsilon^{in}\|_{L^\infty}^2 \qquad (6.2.0.63)$$

$$\le \frac{2}{\delta_\epsilon^{\frac{1}{2}}}\|j_1\|_{L^2}\|\theta^{in}\|_{L^2} + \frac{4}{D}\|j_1\|_{L^2}^2\|u^{in}\|_{L^2}^2\,.$$

It therefore clear that for all $\epsilon \in (0,1]$ sufficiently small one has $1+\delta_\epsilon\theta_\epsilon^{in} > \frac{1}{2}$. Now for all such ϵ define

$$M_\epsilon^{in} \triangleq \mathcal{M}_{(1+\delta_\epsilon\rho_\epsilon^{in},\delta_\epsilon u_\epsilon^{in},1+\delta_\epsilon\theta_\epsilon^{in})} \qquad (6.2.0.64)$$

It is a physical Maxwellian. It also easy to check that the associated fluctuations converge entropically at order δ_ϵ as $\epsilon \to 0$ to the infinitesimal Maxwellian g^{in}. Thus we prove the lemma.

Now we modify our construction of the local Maxwellian. We kept definition of M_ϵ by (6.2.0.56), where $(\rho_\epsilon, u_\epsilon, \theta_\epsilon)$ is the solution to the weakly compressible Stokes system (6.2.0.57) with initial data not $(\rho^{in}, u^{in}, \theta^{in}) \in L^2(dx)$, but $(\rho_\epsilon^{in}, u_\epsilon^{in}, \theta_\epsilon^{in})$. The advantage of this new construction is that with this mollified initial data, the results of Matsumura-Nishida and Ponce [60, 62]on the regularity of the linearized Navier-Stokes system (compressible Stokes) provides the L^∞ bounds of the solution and their derivatives. Under this construction, the local Maxwellian $M_\epsilon(t,x,v)$ is physical.

Now we state our main theorem in this chapter, which implies the stability of the relative entropy with respect to the local Maxwellian constructed from the solution to the weakly compressible Stokes system.

Theorem 16: **From Boltzmann to weakly compressible Stokes** *Let* $b(v_1 - v, \omega)$

be a collision kernel satisfies the hard sphere potential with a small deflection cutoff condition which means: there exits $C_b > 0$, such that for each $z \in \mathbb{R}^D$ and $\omega \in \mathbb{S}^{D-1}$, one has

$$0 \le b(z,\omega) \le C_b(1+|z|) \quad \text{and} \quad \int_{\mathbb{S}^{D-1}} b(z,\omega) \ge \tfrac{1}{C_b}(1+|z|)\,. \qquad (6.2.0.65)$$

We define initial local Maxwellian M_ϵ^{in} as in (6.2.0.64)*, and for $t > 0$, the local Maxwellian M_ϵ as in* (6.2.0.56)*, where $(\rho_\epsilon(t,)u_\epsilon(t,)\theta_\epsilon(t))$ is the solution to the weakly compressible Stokes system* (6.2.0.57) *with initial data $(\rho_\epsilon^{in}, u_\epsilon^{in}, \theta_\epsilon^{in})$ in lemma* (19)*. Let $F_\epsilon^{in}(x,v) \ge 0$ is a family of measurable function a.e. on $\mathbb{T}^D \times \mathbb{R}^D$, satisfying the condition*

$$\frac{1}{\delta_\epsilon^2} H(F_\epsilon^{in}|M_\epsilon^{in}) \to 0, \quad as \quad \epsilon \to 0\,. \qquad (6.2.0.66)$$

Let $F_\epsilon(t,x,v)$ be a family of DiPerna-Lions renormalized solutions to the Boltzmann equation (6.2.0.37) *that have F_ϵ^{in} as initial values. Assume furthermore that F_ϵ satisfies the local conservation law of energy:*

$$\partial_t\Big(\int_{\mathbb{R}^D} \tfrac{1}{2}|v|^2 F_\epsilon\, dv\Big) + \frac{1}{\epsilon}\nabla_x \cdot \Big(\int_{\mathbb{R}^D} v\tfrac{1}{2}|v|^2 F_\epsilon\, dv\Big) = 0\,, \qquad (6.2.0.67)$$

Then as $\epsilon \to 0$, the relative entropy

$$\frac{1}{\delta_\epsilon^2} H(F_\epsilon(t)|M_\epsilon(t)) \to 0\,; \qquad (6.2.0.68)$$

The family of fluctuations g_ϵ given by (6.2.0.48) *satisfies*

$$g_\epsilon - g_\epsilon^S \to 0\,; \quad in \quad L^1([0,\infty); L^1(Mdvdx))\,; \qquad (6.2.0.69)$$

where g_ϵ^S is the infinitesimal Maxwellian associated with the solution to the weakly compressible Stokes system,

$$g_\epsilon^S(t,x,v) = \rho_\epsilon(t,x) + u_\epsilon(t,x)\cdot v + \theta_\epsilon(t,x)(\tfrac{1}{2}|v|^2 - \tfrac{D}{2})\,. \qquad (6.2.0.70)$$

and

$$\Pi g_\epsilon - g_\epsilon^S \to 0\,; \quad in \quad C([0,\infty); L^1(Mdvdx))\,; \tag{6.2.0.71}$$

where Π *is the orthogonal projection from* $L^2(Mdvdx)$ *onto Null*$\mathcal{L}$*. In addition, one has that*

$$\begin{aligned} \langle g_\epsilon\rangle - \rho_\epsilon \to 0 \quad & in \quad C([0,\infty); L^1(dx;\mathbb{R}))\,, \\ \langle v g_\epsilon\rangle - u_\epsilon \to 0 \quad & in \quad C([0,\infty); L^1(dx;\mathbb{R}^D))\,, \\ \left\langle \left(\tfrac{1}{D}|v|^2 - 1\right) g_\epsilon \right\rangle - \theta_\epsilon \to 0 \quad & in \quad C([0,\infty); L^1(dx;\mathbb{R}))\,. \end{aligned} \tag{6.2.0.72}$$

Proof of the Theorem: We start our proof with an relative entropy identity. After simple calculation, we obtain the following identity:

$$H(F_\epsilon|M) = H(F_\epsilon|M_\epsilon) + \int_\Omega \int_{\mathbb{R}^D} \left[F_\epsilon \ln(\frac{M_\epsilon}{M}) - M_\epsilon + M \right] dvdx\,. \tag{6.2.0.73}$$

which implies that the evolution of the relative entropy $H(F_\epsilon|M_\epsilon)$ depends on that of $H(F_\epsilon|M)$. From the DiPerna-Lions theory, $H(F_\epsilon|M)$ obeys the global relative entropy inequality (6.2.0.54) provided the initial entropy bound

$$\frac{1}{\delta_\epsilon^2} H(F_\epsilon^{in}|M) \le C^{in}\,. \tag{6.2.0.74}$$

We claim that our initial relative entropy condition (6.2.0.66) implies the entropy bound (6.2.0.74) with respect to the absolute Maxwellian.

Proof of (6.2.0.66) $\Rightarrow$ (6.2.0.74): We start with the identity

$$\begin{aligned} &\frac{1}{\delta_\epsilon^2} \iint \left[F_\epsilon^{in} \ln\left(\frac{M_\epsilon^{in}}{M}\right) - M_\epsilon^{in} + M \right] dvdx \\ &= \frac{1}{\delta_\epsilon^2} \iint (F_\epsilon^{in} - M_\epsilon^{in}) \ln\left(\frac{M_\epsilon^{in}}{M}\right) dvdx + \frac{1}{\delta_\epsilon^2} \iint M_\epsilon^{in} \ln\left(\frac{M_\epsilon^{in}}{M}\right) dvdx \\ &- \frac{1}{\delta_\epsilon} \int \rho_\epsilon^{in}\, dx\,. \end{aligned} \tag{6.2.0.75}$$

We employ the formula which will be proved latter lemma (20) on the expression of $\ln\left(\frac{M_\epsilon^{in}}{M}\right)$. We obtain that

$$
\begin{aligned}
&\frac{1}{\delta_\epsilon^2}\iint M_\epsilon^{in}\ln\left(\frac{M_\epsilon^{in}}{M}\right)\,dvdx - \frac{1}{\delta_\epsilon}\int \rho_\epsilon^{in}\,dx \\
&= \frac{1}{\delta_\epsilon}\int \rho_\epsilon^{in}\,dx + \frac{1}{2}\int_\Omega \left(|\rho_\epsilon^{in}|^2 + |u_\epsilon^{in}|^2 + \tfrac{D}{2}|\theta_\epsilon^{in}|^2\right)\,dx + \tilde{C}(\rho_\epsilon^{in}, u_\epsilon^{in}, \theta_\epsilon^{in}) - \frac{1}{\delta_\epsilon}\int \rho_\epsilon^{in}\,dx \\
&\leq \tfrac{1}{2}\int_\Omega \left(|\rho_\epsilon^{in}|^2 + |u_\epsilon^{in}|^2 + \tfrac{D}{2}|\theta_\epsilon^{in}|^2\right)\,dx + C\,,
\end{aligned}
\tag{6.2.0.76}
$$

where $\tilde{C}$ in the last inequality above includes the higher order terms related to $(\rho_\epsilon^{in}, u_\epsilon^{in}, \theta_\epsilon^{in})$. Its boundedness is provided by the L^∞ boundedness of $(\rho_\epsilon^{in}, u_\epsilon^{in}, \theta_\epsilon^{in})$ and their derivatives. □

The next, we use again the lemma (20)

$$
\begin{aligned}
&\frac{1}{\delta_\epsilon^2}\iint (F_\epsilon^{in} - M_\epsilon^{in})\ln\left(\frac{M_\epsilon^{in}}{M}\right)\,dvdx \\
&\leq \frac{1}{\delta_\epsilon}\iint (F_\epsilon^{in} - M_\epsilon^{in})\left[\rho_\epsilon^{in} + u_\epsilon^{in}\cdot v + \theta_\epsilon^{in}(\tfrac{1}{2}|v|^2 - \tfrac{3}{2})\right] + \tilde{C} \\
&\leq \frac{C}{\delta_\epsilon}\iint (F_\epsilon^{in} - M_\epsilon^{in})(1+|v|^2)\,dvdx + \tilde{C}\,.
\end{aligned}
\tag{6.2.0.77}
$$

Now consider the convex function $h = h(z)$ defined over $z > -1$ by

$$
h(z) = (1+z)\ln(1+z) - z\,. \tag{6.2.0.78}
$$

We will use the Young inequality. Generally stated, if h and h^* are strictly convex function defined over the convex domain $\mathbf{D}$ and $\mathbf{D}^*$ in the dual linear spaces $\mathbf{E}$ and $\mathbf{E}^*$ respectively that are dual under the Legendre transformation, then for all $\eta \in [0,1]$, they satisfy the inequality

$$
y|z \leq \eta h^*(y) + \tfrac{1}{\eta}h(z)\,, \tag{6.2.0.79}
$$

for every $z \in \mathbf{E}$ and $y \in \mathbf{E}^*$. The Legendre transformation of h is explicitly given by

$$h^*(y) = \exp(y) - 1 - y\,. \tag{6.2.0.80}$$

Now apply the Young inequality with $y = (1 + |v|^2)/4$, $z = (F_\epsilon^{in} - M_\epsilon^{in})/M_\epsilon^{in}$ and $\eta = 4\epsilon/\alpha$ we have

$$\frac{1}{\epsilon}|F_\epsilon^{in} - M_\epsilon^{in}|(1 + |v|^2) \leq C M_\epsilon^{in} h\left(\frac{F_\epsilon^{in}}{M_\epsilon^{in}} - 1\right) + \frac{16}{\alpha} M_\epsilon^{in} e^{\frac{1}{4}(1+|v|^2)}\,. \tag{6.2.0.81}$$

Thus, we prove that

$$\begin{aligned} &\frac{1}{\delta_\epsilon^2}\iint \left[F_\epsilon^{in}\ln\left(\frac{M_\epsilon^{in}}{M}\right) - M_\epsilon^{in} + M\right] dvdx \\ &\leq \frac{C}{\delta_\epsilon^2} H(F_\epsilon^{in}|M_\epsilon^{in}) + C\|U^{in}\|_{L^2(dx)}^2\,, \end{aligned} \tag{6.2.0.82}$$

where $U_\epsilon^{in} = (\rho_\epsilon^{in}, u_\epsilon^{in}, \theta_\epsilon^{in})$. Combine with the relative entropy identity (6.2.0.73), we obtain that

$$\frac{1}{\delta_\epsilon^2} H(F^{in}|M) \leq \tilde{C}\frac{1}{\delta_\epsilon^2} H(F^{in}|M^{in}) + C\|U^{in}\|_{L^2(dx)}^2\,. \tag{6.2.0.83}$$

Thus we finish the proof of the claim (6.2.0.66) $\Rightarrow$ (6.2.0.74). □

Now under the initial bound (6.2.0.74), the entropy inequality (6.2.0.54) is satisfied. Then the relative entropy identity (6.2.0.73) combining with the entropy inequality (6.2.0.54) yields the inequality:

$$\begin{aligned} &H_\epsilon(t) + \delta_\epsilon \int_\Omega \mathrm{tr}(\mathfrak{m}_\epsilon)\,dx + \frac{1}{4\epsilon^2}\int_0^t \int_\Omega D(F_\epsilon)(s,x)\,dxds \\ &\int_\Omega \int_{\mathbb{R}^D} \left[F_\epsilon \ln\left(\frac{M_\epsilon}{M}\right) - M_\epsilon\right](t)\,dvdx \\ &\leq H_\epsilon(0) + \int_\Omega \int_{\mathbb{R}^D} \left[F_\epsilon^{in} \ln\left(\frac{M_\epsilon^{in}}{M}\right) - M_\epsilon^{in}\right](t)\,dvdx\,, \end{aligned} \tag{6.2.0.84}$$

where $H_\epsilon(t)$ denotes $H(F_\epsilon|M_\epsilon)$ and $H_\epsilon(0)$ denotes $H(F_\epsilon^{in}|M_\epsilon^{in})$.

Simple calculation yields

$$\begin{aligned}\int_\Omega \int_{\mathbb{R}^D} M_\epsilon \, dv dx &= 1 + \delta_\epsilon \int_\Omega \rho_\epsilon \, dx \,, \\ \int_\Omega \int_{\mathbb{R}^D} M_\epsilon^{in} \, dv dx &= 1 + \delta_\epsilon \int_\Omega \rho_\epsilon^{in} \, dx\end{aligned} \tag{6.2.0.85}$$

From the first equation in the weakly compressible Stokes system

$$\partial_t \rho_\epsilon + \frac{1}{\epsilon} \nabla_x \cdot u_\epsilon = 0 \,, \tag{6.2.0.86}$$

we know

$$\int_\Omega \rho_\epsilon \, dx = \int_\Omega \rho_\epsilon^{in} \, dx \,, \tag{6.2.0.87}$$

whereby

$$\int_\Omega \int_{\mathbb{R}^D} M_\epsilon \, dv dx = \int_\Omega \int_{\mathbb{R}^D} M_\epsilon^{in} \, dv dx \,. \tag{6.2.0.88}$$

From the inequality (6.2.0.84), we know that to estimate the evolution of the scaled relative entropy $\frac{1}{\delta_\epsilon^2} H_\epsilon(t)$, the key is to estimate the quantity:

$$\int_\Omega \int_{\mathbb{R}^D} \left[F_\epsilon(t) \ln \left(\frac{M_\epsilon(t)}{M} \right) - F_\epsilon^{in} \ln (\frac{M_\epsilon^{in}}{M}) \right] dv dx \tag{6.2.0.89}$$

To this goal, firstly we need to calculate $\ln \frac{M_\epsilon}{M}$, which is stated in the following lemma:

Lemma 20: *We have expansion*

$$\ln \left(\frac{M_\epsilon}{M} \right) = \alpha_\epsilon + \beta_\epsilon \cdot v + \gamma_\epsilon (\frac{|v|^2}{2} - \frac{D}{2}) \,, \tag{6.2.0.90}$$

where $\alpha_\epsilon, \beta_\epsilon, \gamma_\epsilon$ *are expressions respectively:*

$$\alpha_\epsilon(t,x) = \delta_\epsilon \rho_\epsilon - \tfrac{1}{2} \delta_\epsilon^2 [\rho_\epsilon^2 + |u_\epsilon|^2 + \tfrac{D}{2} \theta_\epsilon^2] + \sum_{j=3}^{\infty} (-1)^{j-1} \delta_\epsilon^j f^j(\rho_\epsilon, u_\epsilon, \theta_\epsilon) \,, \tag{6.2.0.91}$$

where

$$f^j(\rho,u,\theta) = \tfrac{1}{j}\rho^j + \tfrac{1}{2}\theta^{j-2}|u|^2 + \tfrac{D}{2}(1-\tfrac{1}{j})\theta^j\,. \tag{6.2.0.92}$$

and

$$\beta_\epsilon(t,x) = \delta_\epsilon u_\epsilon - \delta_\epsilon^2\theta_\epsilon u_\epsilon + \sum_{j=3}^{\infty}(-1)^{j-1}\delta_\epsilon^j\theta_\epsilon^{j-1}u_\epsilon\,, \tag{6.2.0.93}$$

and

$$\gamma_\epsilon = \delta_\epsilon - \delta_\epsilon^2\theta_\epsilon^2 + \sum_{j=3}^{\infty}(-1)^{j-1}\delta_\epsilon^j\theta^j\,. \tag{6.2.0.94}$$

From the above expression, we know that $\ln\left(\frac{M_\epsilon}{M}\right) \in$Null($\mathcal{A}$), and alternatively

$$\ln\left(\frac{M_\epsilon}{M}\right) = \delta_\epsilon\tilde{g}_\epsilon^1 + \delta_\epsilon^2\tilde{g}_\epsilon^2 + \text{higher order terms}\,, \tag{6.2.0.95}$$

where $\tilde{g}_\epsilon^1$ is the infinitesimal Maxwellian with respect to the local Maxwellian M_ϵ:

$$\tilde{g}_\epsilon^1 = \rho_\epsilon + u_\epsilon\cdot v + \theta_\epsilon(\tfrac{1}{2}|v|^2 - \tfrac{D}{2})\,, \tag{6.2.0.96}$$

and $\tilde{g}_\epsilon^2$ is

$$\tilde{g}_\epsilon^2 = \tfrac{1}{2}\rho_\epsilon^2 + |u_\epsilon|^2 + \tfrac{D}{2}\theta_\epsilon^2 + \theta_\epsilon u_\epsilon\cdot v + \theta_\epsilon^2(\tfrac{1}{2}|v|^2 - \tfrac{D}{2})\,, \tag{6.2.0.97}$$

$\tilde{g}_\epsilon^2$ has a good property that

$$\int_\Omega \langle\tilde{g}_\epsilon^2\rangle\,dx = \tfrac{1}{2}\int_\Omega \rho_\epsilon^2 + |u_\epsilon|^2 + \tfrac{D}{2}\theta_\epsilon^2\,dx\,, \tag{6.2.0.98}$$

which is the energy of the solution to the weakly compressible Stokes system. This property will be useful in the evolution of the scaled relative entropy $\frac{1}{\delta_\epsilon^2}H_\epsilon(t)$.

Proof of the lemma:

$$\ln\left(\frac{M_\epsilon}{M}\right) = \ln(1+\delta_\epsilon\rho_\epsilon)\frac{D}{2}\ln(1+\delta_\epsilon\theta_\epsilon) - \frac{|v-\delta_\epsilon u_\epsilon|^2}{2(1+\delta_\epsilon)} + \frac{|v|^2}{2}\,. \tag{6.2.0.99}$$

Then the Taylor expansion yields the proof. □

From the lemma above, we obtain that

$$\int_\Omega \int_{\mathbb{R}^D} \left[F_\epsilon(t) \ln \left(\frac{M_\epsilon(t)}{M} \right) - F_\epsilon^{in} \ln \left(\frac{M_\epsilon^{in}}{M} \right) \right] dvdx = \mathrm{I}_\epsilon + \mathrm{II}_\epsilon + \mathrm{III}_\epsilon \tag{6.2.0.100}$$

where

$$\begin{aligned} \mathrm{I}_\epsilon &= \int_\Omega \int_{\mathbb{R}^D} \left[F_\epsilon(t)\alpha_\epsilon(t) - F_\epsilon^{in}\alpha_\epsilon(0) \right] dvdx \\ \mathrm{II}_\epsilon &= \int_\Omega \int_{\mathbb{R}^D} \left[vF_\epsilon(t) \cdot \beta_\epsilon(t) - vF_\epsilon^{in} \cdot \beta_\epsilon(0) \right] dvdx \\ \mathrm{III}_\epsilon &= \int_\Omega \int_{\mathbb{R}^D} \left[(\tfrac{1}{2}|v|^2 - \tfrac{D}{2})F_\epsilon(t)\gamma_\epsilon(t) - (\tfrac{1}{2}|v|^2 - \tfrac{D}{2})F_\epsilon^{in}\gamma_\epsilon(0) \right] dvdx\,. \end{aligned} \tag{6.2.0.101}$$

Calculations of I_ϵ: Now, let's calculate I_ϵ. From the local conservation law of mass, taking α_ϵ as test function, we obtain

$$\mathrm{I}_\epsilon = \int_0^t \int_\Omega (\partial_s \alpha_\epsilon)(\int_{\mathbb{R}^D} F_\epsilon\, dv) dxds + \int_0^t \int_\Omega \frac{1}{\epsilon}(\int_{\mathbb{R}^D} vF_\epsilon\, dv) \cdot \nabla_x \alpha_\epsilon\, dxds\,. \tag{6.2.0.102}$$

Notice that

$$\begin{aligned} \int_{\mathbb{R}^D} F_\epsilon\, dv &= 1 + \delta_\epsilon \langle g_\epsilon \rangle\,, \\ \int_{\mathbb{R}^D} vF_\epsilon\, dv &= \delta_\epsilon \langle v g_\epsilon \rangle\,. \end{aligned} \tag{6.2.0.103}$$

Then

$$\begin{aligned} \mathrm{I}_\epsilon = &\int_0^t \int_\Omega (\partial_s \alpha_\epsilon)\, dxds + \delta_\epsilon \int_0^t \int_\Omega (\partial_s \alpha_\epsilon) \langle g_\epsilon \rangle\, dxds \\ &+ \delta_\epsilon \int_0^t \int_\Omega \frac{1}{\epsilon} \langle v g_\epsilon \rangle \cdot \nabla_x \alpha_\epsilon\, dxds\,. \end{aligned} \tag{6.2.0.104}$$

From (6.2.0.91), we know

$$\begin{aligned} \int_0^t \int_\Omega (\partial_s \alpha_\epsilon)\, dxds = \delta_\epsilon \int_\Omega [\rho_\epsilon(t) - \rho^{in}]\, dx - \tfrac{1}{2}\delta_\epsilon^2 \int_\Omega (|U_\epsilon(t)|^2 - |U^{in}|^2)\, dx \\ + \int_\Omega \sum_{j=3}^{\infty} (-1)^{j-1} \delta_\epsilon^j (f^j(t) - f^j(0))\, dx\,. \end{aligned} \tag{6.2.0.105}$$

The first term above is zero, because of the conservation of mass. From the energy identity of the weakly compressible Stokes system

$$-\tfrac{1}{2}\int_\Omega (|U_\epsilon(t)|^2 - |U^{in}|^2)\,dx = \int_0^t\int_\Omega \tfrac{\mu}{2}\sigma(u_\epsilon):\sigma(u_\epsilon) + \kappa|\nabla_x\theta_\epsilon|^2\,dxds\,. \qquad (6.2.0.106)$$

Then we obtain that

$$\begin{aligned} \mathrm{I}_\epsilon = {} & \delta_\epsilon^2\int_0^t\int_\Omega \tfrac{\mu}{2}\sigma(u_\epsilon):\sigma(u_\epsilon) + \kappa|\nabla_x\theta_\epsilon|^2\,dxds \\ & + \delta_\epsilon\int_0^t\int_\Omega (\partial_s\alpha_\epsilon)\langle g_\epsilon\rangle\,dxds + \delta_\epsilon\int_0^t\int_\Omega \frac{1}{\epsilon}\langle v g_\epsilon\rangle\cdot\nabla_x\alpha_\epsilon\,dxds \\ & + \int_\Omega \sum_{j=3}^\infty (-1)^{j-1}\delta_\epsilon^j(f^j(t) - f^j(0))\,dx\,. \end{aligned} \qquad (6.2.0.107)$$

Calculations of II_ϵ: Next, to calculate II_ϵ, from the local conservation law of the momentum with defect measure, taking β_ϵ as test function, we have

$$\begin{aligned} \mathrm{II}_\epsilon = {} & \int_0^t\int_\Omega (\partial_s\beta_\epsilon)\cdot(\int_{\mathbb{R}^D} vF_\epsilon\,dv)dxds + \int_0^t\int_\Omega \frac{1}{\epsilon}(\int_{\mathbb{R}^D} v\otimes vF_\epsilon\,dv):\nabla_x\beta_\epsilon\,dxds \\ & + \int_0^t\int_\Omega \frac{1}{\epsilon}\mathfrak{M}_\epsilon:\nabla_x\beta_\epsilon\,dxds\,. \end{aligned} \qquad (6.2.0.108)$$

Denote by $A(v) = v\otimes v - \frac{1}{D}|v|^2 I$, then

$$\begin{aligned} \mathrm{II}_\epsilon = {} & \delta_\epsilon\int_0^t\int_\Omega (\partial_s\beta_\epsilon)\cdot(\int_{\mathbb{R}^D} vg_\epsilon\,dv)dxds \\ & + \delta_\epsilon\int_0^t\int_\Omega \frac{1}{\epsilon}\int_{\mathbb{R}^D}\langle A(v)g_\epsilon\rangle:\nabla_x\beta_\epsilon\,dxds + \delta_\epsilon\int_0^t\int_\Omega \frac{1}{\epsilon}\langle \tfrac{1}{D}|v|^2 g_\epsilon\rangle(\nabla_x\cdot\beta_\epsilon)\,dxds \\ & + \int_0^t\int_\Omega \frac{1}{\epsilon}\mathfrak{M}_\epsilon:\nabla_x\beta_\epsilon\,dxds\,. \end{aligned} \qquad (6.2.0.109)$$

Calculations of III_ϵ: Now, applying the main assumption on the local conservation law of the energy which is not satisfied by the DiPerna-Lions solutions:

$$\partial_t(\int_{\mathbb{R}^D}\tfrac{1}{2}|v|^2F_\epsilon\,dv) + \frac{1}{\epsilon}\nabla_x\cdot(\int_{\mathbb{R}^D} v\otimes v\tfrac{1}{2}|v|^2F_\epsilon\,dv) = 0\,, \qquad (6.2.0.110)$$

Combining with local conservation of mass, taking γ_ϵ as test functions,

$$
\begin{aligned}
\mathrm{III}_\epsilon = & \int_0^t \int_\Omega (\partial_s \gamma_\epsilon) \cdot \Big(\int_{\mathbb{R}^D} (\tfrac{1}{2}|v|^2 - \tfrac{D}{2}) F_\epsilon \, dv \Big) \\
& + \int_0^t \int_\Omega \frac{1}{\epsilon} \int_{\mathbb{R}^D} (v \tfrac{1}{2}|v|^2 - \tfrac{D}{2}) F_\epsilon \, dv \cdot \nabla_x \gamma_\epsilon \, dx ds \,.
\end{aligned}
\tag{6.2.0.111}
$$

Denote by $B(v) = v(\frac{1}{2}|v|^2 - \frac{D+2}{2})$, then

$$
\begin{aligned}
\mathrm{III}_\epsilon = & \, \delta_\epsilon \int_0^t \int_\Omega (\partial_s \gamma_\epsilon) \cdot \langle (\tfrac{1}{2}|v|^2 - \tfrac{D}{2}) g_\epsilon \rangle \, dx ds \\
& + \int_0^t \int_\Omega \frac{1}{\epsilon} \langle v g_\epsilon \rangle \cdot \nabla_x \gamma_\epsilon \, dx ds + \delta_\epsilon \int_0^t \int_\Omega \frac{1}{\epsilon} \langle B(v) g_\epsilon \rangle \cdot \nabla_x \gamma_\epsilon \, dx ds \,.
\end{aligned}
\tag{6.2.0.112}
$$

We use the notations

$$
\alpha_\epsilon = \delta_\epsilon \rho_\epsilon + \delta_\epsilon^2 \tilde{\alpha}_\epsilon \,, \quad \beta_\epsilon = \delta_\epsilon u_\epsilon + \delta_\epsilon^2 \tilde{\beta}_\epsilon \,, \quad \gamma_\epsilon = \theta_\epsilon + \delta_\epsilon^2 \tilde{\gamma}_\epsilon \,. \tag{6.2.0.113}
$$

and use the fact $(\rho_\epsilon, u_\epsilon, \theta_\epsilon)$ is the solution to the weakly compressible Stokes system (6.2.0.57), we obtain that

$$
\begin{aligned}
\frac{1}{\delta_\epsilon^2}(\mathrm{I}_\epsilon + \mathrm{II}_\epsilon + \mathrm{III}_\epsilon) = & \int_0^t \int_\Omega \tfrac{\mu}{2} \sigma(u_\epsilon) : \sigma(u_\epsilon) + \kappa |\nabla_x \theta_\epsilon|^2 \, dx ds \\
& + \int_0^t \int_\Omega \mu (\nabla_x \cdot \sigma(u_\epsilon)) \cdot \langle v g_\epsilon \rangle + \kappa (\Delta_x \theta_\epsilon) \langle (\tfrac{1}{D}|v|^2 - 1) g_\epsilon \rangle \, dx ds \\
& + \int_0^t \int_\Omega \frac{1}{\epsilon} \langle A(v) g_\epsilon \rangle : \nabla_x u_\epsilon + \frac{1}{\epsilon} \langle B(v) g_\epsilon \rangle \cdot \nabla_x \theta_\epsilon \, dx ds \\
& + \int_0^t \int_\Omega \frac{1}{\epsilon} \mathfrak{m}_\epsilon : (u_\epsilon + \delta_\epsilon \tilde{\beta}_\epsilon) \, dx ds \\
& + R_\epsilon^1 \,,
\end{aligned}
\tag{6.2.0.114}
$$

where the remainder R_ϵ is

$$
\begin{aligned}
R_\epsilon^1 = & \int_\Omega \sum_{j=3}^{\infty} (-1)^{j-1} \delta_\epsilon^{j-2} (f^j(t) - f^j(0))\, dx \\
& + \delta_\epsilon \int_0^t \int_\Omega \frac{1}{\epsilon} \tilde{u}_\epsilon \cdot \nabla_x (\tilde{\alpha}_\epsilon + \tilde{\gamma}_\epsilon) + \frac{1}{\epsilon} (\tilde{\rho}_\epsilon + \tilde{\theta}_\epsilon) \nabla_x \cdot \tilde{\beta}_\epsilon \, dx ds \\
& + \delta_\epsilon \int_0^t \int_\Omega \tilde{\rho}_\epsilon (\partial_s \tilde{\alpha}_\epsilon) + \tilde{u}_\epsilon \cdot (\partial_s \tilde{\beta}_\epsilon) + \tfrac{D}{2} \tilde{\theta}_\epsilon (\partial_s \tilde{\gamma}_\epsilon) \\
& + \delta_\epsilon \int_0^t \int_\Omega \frac{1}{\epsilon} \langle A(v) g_\epsilon \rangle : \nabla_x \tilde{\beta}_\epsilon + \frac{1}{\epsilon} \langle B(v) g_\epsilon \rangle : \nabla_x \tilde{\gamma}_\epsilon \, dx ds \,.
\end{aligned}
\tag{6.2.0.115}
$$

In the equation (6.2.0.114), we have to relate the terms involving the moments of g_ϵ, particularly $\frac{1}{\epsilon}\langle A(v) g_\epsilon \rangle$ and $\frac{1}{\epsilon}\langle B(v) g_\epsilon \rangle$ to the associated fluid variables. We did this in the formal derivation of the weakly nonlinear approximation. The main difficulty in the rigorous justification comes from the fact that $F_\epsilon = M(1+\delta_\epsilon g_\epsilon)$ are a family of weak solutions not in the usual sense, but in the sense of renormalization. So the weak formulation of the DiPerna-Lions solutions give the equation of the renormalization of g_ϵ, not g_ϵ. A deeper reason is most of the estimates were based on the fundamental global entropy inequality (6.2.0.54), which includes the relative entropy control

$$
\frac{1}{\delta_\epsilon^2} \int_\Omega \langle h(\delta_\epsilon g_\epsilon(t, x, \cdot)) \rangle \, dx \le C^{in} \tag{6.2.0.116}
$$

We keep here the notations from [8] and denote the nonlinearity involved in the relative entropy by

$$
h(z) = (1 + z) \ln(1 + z) - z \,, \quad z > -1 \,. \tag{6.2.0.117}
$$

Since $h(z) \sim \frac{1}{2} z^2$ near $z = 0$, the entropy control (6.2.0.116) is more or less equivalent to the L^2 estimate of the type

$$
\iint |g_\epsilon(t, x, v)|^2 M dv dx \le 2C^{in} \,. \tag{6.2.0.118}
$$

However, this is not entirely correct, since g_ϵ can take values $\gg \frac{1}{\epsilon}$, for which replacing $h(z)$ by $\frac{1}{2}z^2$ is not justified. For this reason, we propose to consider the following renormalized fluctuation

$$\hat{g}_\epsilon = \frac{2}{\delta_\epsilon}(\sqrt{G_\epsilon} - 1)\,. \tag{6.2.0.119}$$

The advantage of this renormalized fluctuation over the original one is explained that from the relative entropy bound (6.2.0.116),

$$\int \langle (\tfrac{2}{\delta_\epsilon}(\sqrt{G_\epsilon} - 1))^2 \rangle \, dx = \int \langle \hat{g}_\epsilon^2 \rangle \, dx \leq \tfrac{1}{4} C^{in}\,; \tag{6.2.0.120}$$

A natural application of this refined a priori estimate is to decompose

$$g_\epsilon = \hat{g}_\epsilon + \tfrac{1}{4}\delta_\epsilon \hat{g}_\epsilon^2\,. \tag{6.2.0.121}$$

Therefore, we see that the fluctuation g_ϵ is bounded in $L^2(Mdvdx)$, up to a remainder of order ϵ in $L^1(Mdvdx)$, uniformly in $t \geq 0$.

As explained in our description of the DiPerna-Lions existence theorem, the Boltzmann equation can be equivalently renormalized with *any* admissible nonlinearity whose derivative saturates the quadratic growth of the collision integral. Throughout the proof of the theorem, we shall essentially employ two kinds of normalizing nonlinearity used by Golse and Saint-Raymond in their work of incompressible Navier-Stokes limit:

- compactly supported nonlinearities that coincide with the identity near reference Maxwellian state; and
- variants of the maximal, i.e., squre-root renormalization.

Nonlinearities of the first kind are used to define the renormalized form of the Boltzmann equation in which one passes to the limit as $\epsilon \to 0$, while the square-root normalization is used to establish compactness properties of family of solution to the scaled Boltzamnn equation.

The first kind of normalizing nonlinearities is defined through the class of bump function $\gamma \in C^{\infty}(\mathbb{R}_+)$ such that

$$\gamma\big|_{[0,\frac{3}{2}]} \equiv 1, \quad \gamma|_{[2,\infty)} \equiv 0, \quad \gamma \quad \text{is nonincreasing on} \quad R_+ . \tag{6.2.0.122}$$

The Boltzmann equation (6.2.0.37) is then renormalized with the nonlinearity

$$\Gamma(Z) = (Z-1)\gamma(Z) ; \tag{6.2.0.123}$$

Later on, we denote

$$\hat{\gamma}(Z) = \gamma(Z) + (Z-1)\frac{d\gamma}{dz}(Z) = \Gamma'(Z) . \tag{6.2.0.124}$$

The scaled Boltzmann equation renormalized with Γ us put in the form

$$\begin{aligned}&(\partial_t + \frac{1}{\epsilon} v \cdot \nabla_x)(g_\epsilon \gamma_\epsilon) \\ &= \frac{1}{\epsilon} \iint_{\mathbb{R}^3 \times \mathbb{S}^2} \frac{G'_{\epsilon 1} G'_{\epsilon} - G_{\epsilon 1} G_{\epsilon}}{\epsilon \delta_\epsilon} \hat{\gamma}_\epsilon b(v_1 - v, \omega)\, d\omega M_1 dv_1 ,\end{aligned} \tag{6.2.0.125}$$

where we have denoted

$$\gamma_\epsilon = \gamma(G_\epsilon,) \quad \hat{\gamma}_\epsilon = \hat{\gamma}(G_\epsilon) . \tag{6.2.0.126}$$

The second class of normalizing nonlinearities that we shall use to establish compactness properties of the number density fluctuations G_ϵ is defined as

$$\Gamma_\zeta(Z) = \sqrt{\zeta + Z}, \quad \zeta > 0 \tag{6.2.0.127}$$

where the parameter ζ will be adapted to ϵ.

We shall need truncations in the velocity variable at a level that is tied to ϵ. For each function $\zeta \equiv \zeta(v)$, and each $K > 6$, we define

$$\zeta_{K_\epsilon}(v) = \zeta(v)\mathbf{1}_{|v|^2 \le K|\ln\epsilon|} \,. \tag{6.2.0.128}$$

Multiplying each side of the scaled, renormalized Bolztmann equation (6.2.0.125) the truncated collision invariants $\zeta_{K_\epsilon}(v) = 1_{K_\epsilon}, v_{K_\epsilon}, |v|^2_{K_\epsilon}$ we deduce that

$$\begin{aligned}
\partial_t \langle 1_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle + \frac{1}{\epsilon}\nabla_x \cdot \langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle &= J^1_\epsilon \,, \\
\partial_t \langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle + \frac{1}{\epsilon}\nabla_x \langle \tfrac{1}{D}|v|^2_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle + \nabla_x \cdot \mathbf{F}_\epsilon(A) &= J^2_\epsilon \,, \\
\tfrac{D}{2}\partial_t \langle (\tfrac{1}{D}|v|^2_{K_\epsilon} - 1) g_\epsilon \gamma_\epsilon \rangle + \frac{1}{\epsilon}\nabla_x \cdot \langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle + \nabla_x \cdot \mathbf{F}_\epsilon(B) &= J^3_\epsilon \,,
\end{aligned} \tag{6.2.0.129}$$

where $\mathbf{F}_\epsilon(A)$ is the truncated, renormalized traceless part of the momentum flux

$$\mathbf{F}_\epsilon(A) = \frac{1}{\epsilon}\langle A_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle \,, \tag{6.2.0.130}$$

and $\mathbf{F}_\epsilon(B)$ is truncated, renormalized energy flux

$$\mathbf{F}_\epsilon(B) = \frac{1}{\epsilon}\langle B_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle \,, \tag{6.2.0.131}$$

The above equations (6.2.0.129) is satisfied in the weak sense with the conservation defects $\mathbf{J}_\epsilon$

$$\mathbf{J}_\epsilon = \begin{pmatrix} J^1_\epsilon \\ J^2_\epsilon \\ J^3_\epsilon \end{pmatrix} = \begin{pmatrix} \langle 1_{K_\epsilon} q_\epsilon \hat{\gamma}_\epsilon \rangle \\ \langle v_{K_\epsilon} q_\epsilon \hat{\gamma}_\epsilon \rangle \\ \langle (\tfrac{1}{2}|v|^2_{K_\epsilon} - \tfrac{D}{2}) q_\epsilon \hat{\gamma}_\epsilon \rangle \end{pmatrix} \,, \tag{6.2.0.132}$$

where the scaled entropy dissipation rate

$$q_\epsilon = \frac{G'_{\epsilon 1} G'_\epsilon - G_{\epsilon 1} G_\epsilon}{\epsilon \delta_\epsilon} \,. \tag{6.2.0.133}$$

Notice that truncating large velocities in the number density, or large values thereof (which is what the renormalization procedure does) break the symmetries in the collision integral leading to the local conservation laws of mass, momentum, energy: this accounts for the defect $\mathbf{J}_\epsilon$ on the right-hand side of (6.2.0.129). As $\epsilon \to 0$. $\zeta_{K_\epsilon}(v) \to \zeta(v)$ while $G_\epsilon \to 1$ so that $\hat{\gamma}_\epsilon \to 1$: hence the missing symmetries are restored in the integrand defining $\mathbf{J}_\epsilon$. Hence one can hope that $\mathbf{J}_\epsilon \to 0$ as $\epsilon \to 0$.

In their work of incompressible Navier-Stokes limit, Golse and Saint-Raymond proved the following nonlinear compactness estimate, based on which they can derive the vanishing of the momentum conservation defect for the hard sphere collision kernel. (in [48], Levrmore and Masmoudi treated the very general collision kernel under some nonlinear compactness assumption.)

Proposition 8: **Nonlinear compactness estimate:**
$(1+|v|)\left(\frac{\sqrt{G_\epsilon}-1}{\delta_\epsilon}\right)^2$ *is uniformly integrable on* $[0,T]\times K\times\mathbb{R}^D$ *for the measure* $Mdvdxdt$*, for each* $T>0$ *and each compact* $K\subset\mathbb{R}^D$*.*

Follow the line of Golse-Levermore [29], Golse and Saint-Raymond proved the vanishing of conservation defects

Proposition 9: *Under the assumption as in the Main Theorem*

$$\mathbf{J}_\epsilon \to 0\,, \quad \text{in} \quad L^1_{\text{loc}}(dt; L^1(dx)) \quad \text{as} \quad \epsilon \to 0\,. \tag{6.2.0.134}$$

To derive the hydrodynamics of (6.2.0.129), we need the deduce the momentum and energy flux to some asymptotic normal form, based on which we can describe the evolution of the relative entropy $H_\epsilon(t)$:

Lemma 21: *Let* Π *be the* $L^2(Mdv)-$ *orthogonal projection on* $\mathrm{Null}(\mathcal{L})$*, then under the same assumption as in the Main Theorem*

$$\mathbf{F}_\epsilon(\zeta) = 2\frac{\delta_\epsilon}{\epsilon}\left\langle \zeta\left(\Pi\frac{\sqrt{G_\epsilon}-1}{\delta_\epsilon}\right)^2\right\rangle - 2\left\langle \hat{\zeta}\frac{1}{\delta_\epsilon\epsilon}\mathcal{Q}(\sqrt{G_\epsilon},\sqrt{G_\epsilon})\right\rangle + o(1)_{L^1_{\mathrm{loc}}(dxdt)}\,, \tag{6.2.0.135}$$

where ζ *denotes either* A *or* B*.*

The proof of this lemma is based on splitting the momentum flux as

$$\begin{aligned}\mathbf{F}_\epsilon(A) &= \frac{1}{\epsilon}\left\langle A_{K_\epsilon}\gamma_\epsilon\frac{G_\epsilon-1}{\delta_\epsilon}\right\rangle\\ &= \left\langle A_{K_\epsilon}\gamma_\epsilon\left(\frac{\sqrt{G_\epsilon}-1}{\delta_\epsilon}\right)^2\right\rangle + \frac{2}{\epsilon}\left\langle A_{K_\epsilon}\gamma_\epsilon\frac{\sqrt{G_\epsilon}-1}{\delta_\epsilon}\right\rangle\\ &= \mathbf{F}^1_\epsilon(A) + \mathbf{F}^2_\epsilon(A)\,,\end{aligned} \tag{6.2.0.136}$$

as a consequence of the elementary identity

$$\begin{aligned}\frac{1}{\epsilon}\frac{G_\epsilon-1}{\delta_\epsilon} &= \frac{1}{\epsilon}(\sqrt{G_\epsilon}-1)(\sqrt{G_\epsilon}+1)\\ &= \frac{1}{\epsilon}(\sqrt{G_\epsilon}-1)^2 + \frac{2}{\delta_\epsilon}(\sqrt{G_\epsilon}-1)\,.\end{aligned} \tag{6.2.0.137}$$

Then, one applies the following corollary of the nonlinear compactness estimate

Corollary 7:

$$\frac{\sqrt{G_\epsilon}-1}{\delta_\epsilon} - \Pi\frac{\sqrt{G_\epsilon}-1}{\delta_\epsilon} \longrightarrow 0\,, \quad \mathit{in} \quad L^2_{\mathrm{loc}}(dt;L^2(Mdvdx)) \tag{6.2.0.138}$$

as $\epsilon \longrightarrow 0$.

With the corollary above, one can show that the term $\mathbf{F}^1_\epsilon(A)$ in the decomposition of the momentum flux is asymptotically close to

$$\frac{\delta_\epsilon}{\epsilon}\left\langle A\left(\Pi\frac{\sqrt{G_\epsilon}-1}{\delta_\epsilon}\right)^2\right\rangle \tag{6.2.0.139}$$

Notice that the high velocity truncation is disposed of since $\Pi\frac{\sqrt{G_\epsilon}-1}{\delta_\epsilon}$ has at most polynomial growth in v as $|v| \to \infty$. In order to deal with the second term $\mathbf{F}^2_\epsilon(A)$, we introduce the following decomposition

$$\frac{1}{\delta_\epsilon}\left\langle A\frac{\sqrt{G_\epsilon}-1}{\delta_\epsilon}\right\rangle = \frac{1}{\delta_\epsilon}\left\langle \hat{A}\mathcal{L}\frac{\sqrt{G_\epsilon}-1}{\delta_\epsilon}\right\rangle$$
$$= 2\frac{\delta_\epsilon}{\epsilon}\left\langle \hat{A}\mathcal{Q}\left(\frac{\sqrt{G_\epsilon}-1}{\delta_\epsilon}, \frac{\sqrt{G_\epsilon}-1}{\delta_\epsilon}\right)\right\rangle - 2\left\langle \hat{A}\frac{1}{\delta_\epsilon \epsilon}\mathcal{Q}(\sqrt{G_\epsilon}, \sqrt{G_\epsilon})\right\rangle , \tag{6.2.0.140}$$

from which we deduce with the corollary above that $\mathbf{F}^2_\epsilon(A)$ is close to

$$\frac{\delta_\epsilon}{\epsilon}\left\langle A\left(\Pi\frac{\sqrt{G_\epsilon}-1}{\delta_\epsilon}\right)^2\right\rangle - 2\left\langle \hat{A}\frac{1}{\delta_\epsilon \epsilon}\mathcal{Q}(\sqrt{G_\epsilon}, \sqrt{G_\epsilon})\right\rangle \tag{6.2.0.141}$$

Next we use lemma (21) and its corollary to derive the hydrodynamic asymptotics of (6.2.0.129).

More generally, we can show the following Navier-Stokes "Asymptotes":

Proposition 10: **Navier-Stokes Asymptotes:**

$$\partial_t \begin{pmatrix} \rho^\flat_\epsilon \\ u^\flat_\epsilon \\ \frac{3}{2}\theta^\flat_\epsilon \end{pmatrix} + \frac{1}{\epsilon}\begin{pmatrix} \nabla_x \cdot u^\flat_\epsilon \\ \nabla_x(\rho^\flat_\epsilon + \theta^\flat_\epsilon) \\ \nabla_x \cdot u^\flat_\epsilon \end{pmatrix} + \frac{\delta_\epsilon}{\epsilon}\begin{pmatrix} 0 \\ \nabla_x \cdot (u^\flat_\epsilon \otimes u^\flat_\epsilon) - \frac{1}{3}\nabla_x |u^\flat_\epsilon|^2 \\ u^\flat_\epsilon \cdot \nabla_x \theta^\flat_\epsilon \end{pmatrix} - \begin{pmatrix} 0 \\ \mu\nabla_x \cdot \sigma(u^\flat_\epsilon) \\ \kappa\Delta_x \theta^\flat_\epsilon \end{pmatrix} \to 0 , \tag{6.2.0.142}$$

in $\text{w-}L^1_{\text{loc}}(dt; \text{w-}L^1(dx))$ as $\epsilon \to 0$. *where*

$$\rho^\flat_\epsilon = \langle 1_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle , \quad u^\flat_\epsilon = \langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle , \quad \theta^\flat_\epsilon = \langle (\tfrac{1}{D}|v|^2_{K_\epsilon} - 1) g_\epsilon \gamma_\epsilon\rangle \tag{6.2.0.143}$$

Proof of the proposition: The proof basically follow the line of [34], the only difference is that [34] treat the limit, we consider the asymptotic behavior.

In our calculations above, we kept the $\frac{\delta_\epsilon}{\epsilon}$ term which will vanish in the Stokes scaling. We considered the general case since it will be useful in the nonlinear Navier-Stokes asymptotics.

To begin with, we notice that the following asymptotic equivalence of the two renormalizations:

$$\frac{\sqrt{G_\epsilon}-1}{\delta_\epsilon} \simeq \frac{1}{2} g_\epsilon \gamma_\epsilon \tag{6.2.0.144}$$

The asymptotics $F_\epsilon(A)$ and $F_\epsilon(B)$ are obtained by an argument that closely follows [29] and [34]. The proposition follows directly from the following lemma:

Lemma 22: *Define*

$$\mu = \tfrac{1}{(D-1)(D+2)} \langle A : \hat{A} \rangle \,, \quad \kappa = \tfrac{1}{D} \langle B \cdot \hat{B} \rangle \,. \tag{6.2.0.145}$$

Then, as $\epsilon \to 0$, *the diffusion part*

$$\begin{aligned} &2\langle \hat{A} \tfrac{1}{\epsilon \delta_\epsilon} \mathcal{Q}(\sqrt{G_\epsilon}, \sqrt{G_\epsilon}) \rangle + \mu \sigma(u_\epsilon^\flat) \to 0 \,; \\ &2\langle \hat{B} \tfrac{1}{\epsilon \delta_\epsilon} \mathcal{Q}(\sqrt{G_\epsilon}, \sqrt{G_\epsilon}) \rangle + \kappa \nabla_x \theta_\epsilon^\flat \to 0 \end{aligned} \tag{6.2.0.146}$$

in w-$L^1_{\text{loc}}(dt; \text{w-}L^1(dx))$; *and the convection part*

$$\begin{aligned} \langle A (\Pi \tfrac{\sqrt{G_\epsilon}-1}{\delta_\epsilon})^2 \rangle - (u_\epsilon^\flat \otimes u_\epsilon^\flat - \tfrac{1}{D} |u_\epsilon^\flat|^2 I) \to 0 \,, \\ \langle B (\Pi \tfrac{\sqrt{G_\epsilon}-1}{\delta_\epsilon})^2 \rangle - u_\epsilon^\flat \theta_\epsilon^\flat \to 0 \,, \end{aligned} \tag{6.2.0.147}$$

in $L^1_{\text{loc}}(dt; L^1(dx).)$

Proof of the lemma: We start from the elementary formula

$$\begin{aligned} G'_{\epsilon 1} G'_\epsilon - G_{\epsilon 1} G_\epsilon &= \left(\sqrt{G'_{\epsilon 1} G'_\epsilon} - \sqrt{G_{\epsilon 1} G_\epsilon} \right) \left(\sqrt{G'_{\epsilon 1} G'_\epsilon} + \sqrt{G_{\epsilon 1} G_\epsilon} \right) \\ &= \left(\sqrt{G'_{\epsilon 1} G'_\epsilon} - \sqrt{G_{\epsilon 1} G_\epsilon} \right)^2 + 2\sqrt{G_{\epsilon 1} G_\epsilon} \left(\sqrt{G'_{\epsilon 1} G'_\epsilon} - \sqrt{G_{\epsilon 1} G_\epsilon} \right) . \end{aligned} \tag{6.2.0.148}$$

If we denote by

$$\tilde{q}_\epsilon = \frac{1}{\epsilon\delta_\epsilon}\left(\sqrt{G'_{\epsilon 1}G'_\epsilon} - \sqrt{G_{\epsilon 1}G_\epsilon}\right)\,, \tag{6.2.0.149}$$

then from above identity we have the relation between $\tilde{q}_\epsilon$ and the original scaled entropy dissipation rate q_ϵ,

$$q_\epsilon = \epsilon\delta_\epsilon\left(\tilde{q}_\epsilon\right)^2 + 2\sqrt{G_{\epsilon 1}G_\epsilon}\,\tilde{q}_\epsilon\,. \tag{6.2.0.150}$$

The global entropy inequality (6.2.0.116) provides the entropy dissipation rate bound which implies that

$$\begin{aligned}&\int_0^\infty\int\left\langle\!\!\left\langle\frac{1}{(\epsilon\delta_\epsilon)^2}\left(\sqrt{G'_{\epsilon 1}G'_\epsilon} - \sqrt{G_{\epsilon 1}G_\epsilon}\right)^2\right\rangle\!\!\right\rangle dxdt\\ &= \int_0^\infty\int\left\langle\!\left\langle(\tilde{q}_\epsilon)^2\right\rangle\!\right\rangle dxdt \le C^{in}\,,\end{aligned} \tag{6.2.0.151}$$

The scaled, renormalized Boltzmann equation (6.2.0.125) can be written in the form

$$\begin{aligned}&v\cdot\nabla_x(g_\epsilon\gamma_\epsilon) - \iint\limits_{\mathbb{S}^2\times\mathbb{R}^3} q_\epsilon\hat{\gamma}_\epsilon b(v_1 - v,\omega)\,d\omega M_1 dv_1\\ &= -\epsilon\partial_t(g_\epsilon\gamma_\epsilon) + v\cdot\nabla_x(\Pi g_\epsilon\gamma_\epsilon - g_\epsilon\gamma_\epsilon)\,.\end{aligned} \tag{6.2.0.152}$$

This means that for every $\chi \in L^\infty(Mdv; C^1(\Omega))$ and every $0 \le t_1 < t_2 < \infty$

$$\begin{aligned}&-\int_{t_1}^{t_2}\int\langle g_\epsilon\gamma_\epsilon v\cdot\nabla_x\chi\rangle\,dxdt + \int_{t_1}^{t_2}\int\langle\!\langle q_\epsilon\gamma_\epsilon\gamma_{\epsilon 1}\gamma'_\epsilon\gamma'_{\epsilon 1}\rangle\!\rangle\,dxdt\\ &= \epsilon\int\langle g_\epsilon\gamma_\epsilon(t_1)\chi\rangle - \epsilon\int\langle g_\epsilon\gamma_\epsilon(t_2)\chi\rangle - \int_{t_1}^{t_2}\int\langle(g_\epsilon\gamma_\epsilon - \Pi g_\epsilon\gamma_\epsilon)v\cdot\nabla_x\chi\rangle\,dxdt\,.\end{aligned} \tag{6.2.0.153}$$

Using the asymptotic equivalence (6.2.0.144) between $g_\epsilon\gamma_\epsilon$ and $\frac{\sqrt{G_\epsilon}-1}{\delta_\epsilon}$ while the later is relatively compact in $w\!-\!L^2_{\mathrm{loc}}(dt; w\!-\!L^2(Mdvdx))$,

$$\epsilon\int\langle g_\epsilon\gamma_\epsilon(t_1)\chi\rangle - \epsilon\int\langle g_\epsilon\gamma_\epsilon(t_2)\chi\rangle \to 0\,, \quad \text{as} \quad \epsilon \to 0\,. \tag{6.2.0.154}$$

It is easily to derive that

$$\int_{t_1}^{t_2} \int \langle (g_\epsilon \gamma_\epsilon - \Pi g_\epsilon \gamma_\epsilon) v \cdot \nabla_x \chi \rangle \, dxdt \to 0 \,, \quad \text{as} \quad \epsilon \to 0 \,. \tag{6.2.0.155}$$

Actually, weak convergence is enough here.

Furthermore, apply the similar arguments of [29] and [34], we know both $q_\epsilon \hat{\gamma}_\epsilon$ and $\tilde{q}_\epsilon$ are relatively compact in w-$L^2_{\text{loc}}(dt; \text{w-}L^2(d\mu dx))$, and if the limit point of $q_\epsilon \hat{\gamma}_\epsilon$ is q which satisfies the limiting Boltzmann equation, see 4.3 in [8],

$$v \cdot \nabla_x g = \iint q b(v_1 - v, \omega) \, d\omega M_1 dv_1 \tag{6.2.0.156}$$

then, from the relation (6.2.0.150), the limit point of $\tilde{q}_\epsilon$ is $\frac{q}{2}$, and

$$\int_{t_1}^{t_2} \int \langle\langle q_\epsilon \hat{\gamma}_\epsilon - 2\tilde{q}_\epsilon \rangle\rangle \, dxdt \to 0 \quad \text{as} \quad \epsilon \to 0 \,. \tag{6.2.0.157}$$

recall that

$$\langle \hat{\zeta} \frac{1}{\epsilon \delta_\epsilon} \mathcal{Q}(\sqrt{G_\epsilon}, \sqrt{G_\epsilon}) \rangle = \langle\langle \zeta \tilde{q}_\epsilon \rangle\rangle \,, \tag{6.2.0.158}$$

Thus, we proved the "asymptotic" Boltzmann equation, which is an analogue of the "limiting" Boltzmann equation, see Proposition 4.1 in [8],

Lemma 23:

$$v \cdot \nabla_x \Pi g_\epsilon \gamma_\epsilon - \iint q_\epsilon \hat{\gamma}_\epsilon b(v_1 - v, \omega) \, d\omega M_1 dv_1 \to 0 \quad as \quad \epsilon \to 0 \,. \tag{6.2.0.159}$$

in w-$L^1_{\text{loc}}(dt; \text{w-}L^1(M dv dx))$.

Using the project Π simple calculation yields that

$$\begin{aligned} v \cdot \nabla_x \Pi g_\epsilon \gamma_\epsilon &= A(v) : \nabla_x \hat{u}^\flat_\epsilon + B(v) \cdot \nabla_x \hat{\theta}^\flat_\epsilon \\ &\quad + v \cdot \nabla_x (\hat{\rho}^\flat_\epsilon + \hat{\theta}^\flat_\epsilon) + \tfrac{1}{D} |v|^2 \nabla_x \cdot \hat{u}^\flat_\epsilon \,. \end{aligned} \tag{6.2.0.160}$$

Then, we can obtain from lemma 4.1 in [8]

$$\begin{aligned}\langle \hat{A}v\cdot\nabla_x\Pi g_\epsilon\gamma_\epsilon\rangle &= \mu\sigma(\hat{u}_\epsilon^\flat)\,,\\ \langle \hat{B}v\cdot\nabla_x\Pi g_\epsilon\gamma_\epsilon\rangle &= \kappa\nabla_x\hat{\theta}_\epsilon^\flat\,.\end{aligned} \tag{6.2.0.161}$$

Thus we finish the proof of the Lemma (22). □

To get the asymptotics of the convection terms, we again use the asymptotic equivalence relation (6.2.0.144), Recalling the definition of the projection Π

$$\Pi g = \langle g\rangle + \langle vg\rangle\cdot v + \langle(\tfrac{1}{D}|v|^2-1)g\rangle(\tfrac{1}{2}|v|^2-\tfrac{D}{2})\,, \tag{6.2.0.162}$$

One has

$$\begin{aligned}\langle A_{K_\epsilon}(\Pi\tfrac{\sqrt{G_\epsilon}-1}{\delta_\epsilon})^2\rangle &\simeq \langle v_{K_\epsilon}g_\epsilon\gamma_\epsilon\rangle\otimes\langle v_{K_\epsilon}g_\epsilon\gamma_\epsilon\rangle - \tfrac{1}{D}|\langle v_{K_\epsilon}g_\epsilon\gamma_\epsilon\rangle|^2 I\\ &= u_\epsilon^\flat\otimes u_\epsilon^\flat - \tfrac{1}{D}|u_\epsilon^\flat|^2 I\,,\end{aligned} \tag{6.2.0.163}$$

and

$$\begin{aligned}\langle B_{K_\epsilon}(\Pi\tfrac{\sqrt{G_\epsilon}-1}{\delta_\epsilon})^2\rangle &\simeq \langle v_{K_\epsilon}g_\epsilon\gamma_\epsilon\rangle\langle(\tfrac{|v|^2_{K_\epsilon}}{D}-1)g_\epsilon\gamma_\epsilon\rangle\\ &= u_\epsilon^\flat\theta_\epsilon^\flat\end{aligned} \tag{6.2.0.164}$$

Now it is ready to estimate the terms involving the moments of g_ϵ in (6.2.0.114), namely the following four terms

$$\begin{aligned}\Theta_\epsilon = &\int_0^t\int_\Omega \mu(\nabla_x\cdot\sigma(u_\epsilon))\cdot\langle vg_\epsilon\rangle + \kappa(\Delta_x\theta_\epsilon)\langle(\tfrac{1}{D}|v|^2-1)g_\epsilon\rangle\,dxds\\ &+\int_0^t\int_\Omega \frac{1}{\epsilon}\langle A(v)g_\epsilon\rangle:\nabla_x u_\epsilon + \frac{1}{\epsilon}\langle B(v)g_\epsilon\rangle\cdot\nabla_x\theta_\epsilon\,dxds\end{aligned} \tag{6.2.0.165}$$

Now let $\xi(v)$ denote v or $\frac{1}{D}|v|^2-1$, then we decompose $\langle\xi(v)g_\epsilon\rangle$ into

$$\begin{aligned}\langle\xi(v)g_\epsilon\rangle &= \langle\xi_{K_\epsilon}(v)g_\epsilon\rangle + \langle\xi(v)\mathbf{1}_{|v|^2>K|\ln\epsilon|}g_\epsilon\rangle\\ &= \langle\xi_{K_\epsilon}(v)\tfrac{2(\sqrt{G_\epsilon}-1)}{\delta_\epsilon}\rangle + \tfrac{1}{4}\delta_\epsilon\langle\xi_{K_\epsilon}(v)(\tfrac{2(\sqrt{G_\epsilon}-1)}{\delta_\epsilon})^2\rangle\\ &+\langle\xi(v)\mathbf{1}_{|v|^2>K|\ln\epsilon|}g_\epsilon\rangle\\ &\simeq \langle\xi_{K_\epsilon}(v)g_\epsilon\gamma_\epsilon\rangle + \tfrac{1}{4}\delta_\epsilon\langle\xi_{K_\epsilon}(v)(\tfrac{2(\sqrt{G_\epsilon}-1)}{\delta_\epsilon})^2\rangle + \langle\xi(v)\mathbf{1}_{|v|^2>K|\ln\epsilon|}g_\epsilon\rangle\,.\end{aligned} \tag{6.2.0.166}$$

By our definitions, the first term above is either $u^\flat_\epsilon$ or $\theta^\flat_\epsilon$. We claim that the last two terms will be vanishing as $\epsilon \to 0$ based on the following argument.

From the following classical estimate on the tail of Gaussian integrals

$$\int_{\mathbb{R}^N} e^{-|v|^2/2}|v|^\alpha \mathbf{1}_{|v|^2>R}\, dv = O(R^{\frac{\alpha+N}{2}-1}e^{-R/2}) \quad \text{as} \quad R \to +\infty\,. \tag{6.2.0.167}$$

Take K large enough, the third term above will go to zero. From the nonlinear compactness estimate, see Proposition (8), we know that $(1+|v|)\left(\frac{\sqrt{G_\epsilon}-1}{\delta_\epsilon}\right)^2$ is uniformly integrable on $[0,T]\times K\times\mathbb{R}^D$ for the measure $Mdvdxdt$, for each $T>0$ and each compact $K\subset\mathbb{R}^D$. Noting that in the second term above the large velocity has already been truncated, the nonlinear compactness estimate easily implies that the second term above goes to zero as $\epsilon\to 0$.

Let $\frac{1}{\epsilon}\langle\zeta(v)\rangle$ denote $\frac{1}{\epsilon}\langle A(v)\rangle$ or $\frac{1}{\epsilon}\langle B(v)\rangle$, similarly as above we decompose them into

$$\begin{aligned}
\frac{1}{\epsilon}\langle\zeta(v)g_\epsilon\rangle &= \frac{1}{\epsilon}\langle\zeta_{K_\epsilon}(v)g_\epsilon\rangle + \frac{1}{\epsilon}\langle\zeta(v)\mathbf{1}_{|v|^2>K|\ln\epsilon|}g_\epsilon\rangle \\
&= \frac{1}{\epsilon}\langle\zeta_{K_\epsilon}(v)\tfrac{2(\sqrt{G_\epsilon}-1)}{\delta_\epsilon}\rangle + \frac{\delta_\epsilon}{4\epsilon}\langle\zeta_{K_\epsilon}(v)(\tfrac{2(\sqrt{G_\epsilon}-1)}{\delta_\epsilon})^2\rangle + \frac{1}{\epsilon}\langle\zeta(v)\mathbf{1}_{|v|^2>K|\ln\epsilon|}g_\epsilon\rangle \\
&\simeq \frac{1}{\epsilon}\langle\zeta_{K_\epsilon}(v)g_\epsilon\gamma_\epsilon\rangle + \frac{\delta_\epsilon}{4\epsilon}\langle\zeta_{K_\epsilon}(v)(\tfrac{2(\sqrt{G_\epsilon}-1)}{\delta_\epsilon})^2\rangle + \frac{1}{\epsilon}\langle\zeta(v)\mathbf{1}_{|v|^2>K|\ln\epsilon|}g_\epsilon\rangle\,.
\end{aligned} \tag{6.2.0.168}$$

By definitions, the first term above is either $\mathbf{F}_\epsilon(A)$ or $\mathbf{F}_\epsilon(B)$. Very much similar to our analysis above, and noting that in the Stokes scaling, the assumption $\frac{\delta_\epsilon}{\epsilon}\to 0$, the last two terms above are vanishing as $\epsilon\to 0$.

Combine the above arguments and the Proposition (10) on the Navier-Stokes asymptotics, dropping the quadratic terms because their coefficients are $\frac{\delta_\epsilon}{\epsilon}$ which goes to zero in the Stokes scaling. (It will be kept in the Navier-Stokes scaling, this case will be treated in the next chapter.) We obtain that

$$\begin{aligned}\Theta_\epsilon = &\int_0^t \int_\Omega \mu \left[\nabla_x \cdot \sigma(u_\epsilon)\right] \cdot u_\epsilon^\flat + \kappa(\Delta_x \theta_\epsilon)\theta_\epsilon^\flat \, dxds \\ &+ \int_0^t \int_\Omega \mu\sigma(u_\epsilon^\flat) : \nabla_x u_\epsilon + \kappa \nabla_x \theta_\epsilon^\flat \cdot \nabla_x \theta_\epsilon \, dxds + r_\epsilon\end{aligned} \tag{6.2.0.169}$$

Finally, we obtain the evolution of the relative entropy $H_\epsilon(t)$:

Proposition 11: **the evolution of the relative entropy** *For every $t \geq 0$,*

$$\begin{aligned}&\frac{1}{\delta_\epsilon^2} H(F_\epsilon|M_\epsilon)(t) + \left[\frac{1}{4(\epsilon\delta_\epsilon)^2}\int_0^t \int_\Omega D(F_\epsilon)\, dx\, ds - \int_0^t \int_\Omega \tfrac{\mu}{2}\sigma(u_\epsilon^\flat) : \sigma(u_\epsilon^\flat) + \kappa|\nabla_x \theta_\epsilon^\flat|^2 \, dx\, ds\right] \\ &\qquad + \int_0^t \int_\Omega \tfrac{\mu}{2}\sigma(u_\epsilon - u_\epsilon^\flat) : \sigma(u_\epsilon - u_\epsilon^\flat) + \kappa|\nabla_x(\theta_\epsilon - \theta_\epsilon^\flat)|^2 \, dx\, ds \\ &\qquad + R_\epsilon^1 + R_\epsilon^2 \leq \frac{1}{\delta_\epsilon^2} H(F_\epsilon^0|M_\epsilon^0)\,,\end{aligned} \tag{6.2.0.170}$$

where

$$R_\epsilon^1 = \frac{1}{\delta_\epsilon}\int_\Omega \tfrac{1}{2}\mathrm{tr}(\mathfrak{m}_\epsilon) dx + \int_0^t \int_\Omega \frac{1}{\epsilon}\mathfrak{m}_\epsilon : (u_\epsilon + \delta_\epsilon \tilde{\beta}_\epsilon)\, dx\, ds \tag{6.2.0.171}$$

and

$$\begin{aligned}R_\epsilon^2 = &\delta_\epsilon \int_0^t \int_\Omega \mu \nabla \cdot \sigma(u_\epsilon) \cdot \langle v g_\epsilon^\sharp\rangle + \kappa \Delta_x \theta_\epsilon \langle (\tfrac{|v|^2}{3} - 1) g_\epsilon^\sharp\rangle \, dx\, ds \\ &+ \frac{\delta_\epsilon}{\epsilon}\int_0^t \int_\Omega \langle A(v) g_\epsilon^\sharp\rangle : \nabla_x (u_\epsilon + \delta_\epsilon \tilde{\beta}_\epsilon)\, dx\, ds \\ &+ \frac{\delta_\epsilon}{\epsilon}\int_0^t \int_\Omega \langle B(v) g_\epsilon^\sharp\rangle : \nabla_x (\theta_\epsilon + \delta_\epsilon \tilde{\gamma}_\epsilon)\, dx\, ds \\ &+ \frac{\delta_\epsilon}{\epsilon}\int_0^t \int_\Omega \langle v g_\epsilon\rangle \cdot \nabla_x (\tilde{\alpha}_\epsilon + \tilde{\gamma}_\epsilon) + \langle \tfrac{|v|^2}{3} g_\epsilon\rangle \nabla_x \cdot \tilde{\beta}_\epsilon \, dx\, ds \\ &+ \frac{\delta_\epsilon}{\epsilon}\int_0^t \int_\Omega \langle A(v) g_\epsilon^\flat\rangle : \nabla_x \tilde{\beta}_\epsilon + \langle B(v) g_\epsilon^\flat\rangle \cdot \nabla_x \tilde{\gamma}_\epsilon \, dx\, ds \\ &+ \delta_\epsilon \int_0^t \int_\Omega \langle g_\epsilon\rangle(\partial_s \tilde{\alpha}_\epsilon) + \langle v g_\epsilon\rangle \cdot (\partial_s \tilde{\beta}_\epsilon) + \tfrac{3}{2}\langle (\tfrac{1}{2}|v|^2 - 1) g_\epsilon\rangle (\partial_s \tilde{\gamma}_\epsilon)\, dx\, ds \\ &+ \sum_3^\infty (-1)^{j-1}(\delta_\epsilon)^{j-2}\int_\Omega (f_j(\rho_\epsilon, u_\epsilon, \theta_\epsilon)(t) - f_j(\rho_\epsilon, u_\epsilon, \theta_\epsilon)(0))\, dx\end{aligned} \tag{6.2.0.172}$$

where f_j are higher order terms in $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$.

Let us analyze the remainder terms. Firstly we claim that in R^2_ϵ, because $\mathfrak{m}_\epsilon$ is a non-negative matrix-value measure, then

$$\frac{1}{\delta_\epsilon}\int_\Omega \mathrm{tr}(\mathfrak{m}_\epsilon) dx \geq 0\,. \tag{6.2.0.173}$$

To analyze the second term in R^1_ϵ, we recall the global entropy inequality (6.2.0.54), we have:

$$\begin{aligned}\frac{1}{\delta_\epsilon^2}\int_\Omega h(\delta_\epsilon g_\epsilon(t))\,dx + \frac{1}{\delta_\epsilon}\int_\Omega \mathrm{tr}(\mathfrak{m}_\epsilon)(t)\,dx &\leq \frac{1}{\delta_\epsilon^2}\int_\Omega h(\delta_\epsilon g^{in}_\epsilon(t))\,dx \\ &\leq C^{in}\,.\end{aligned} \tag{6.2.0.174}$$

Then

$$\frac{1}{\epsilon}\int_\Omega \mathrm{tr}(\mathfrak{m}_\epsilon)(t)\,dx \leq C^{in}\frac{\delta_\epsilon}{\epsilon}\,. \tag{6.2.0.175}$$

In the Stokes scaling, $\frac{\delta_\epsilon}{\epsilon} \to 0$, thus

$$\frac{1}{\epsilon}\mathrm{tr}(\mathfrak{m}_\epsilon) \to 0 \quad \text{in} \quad L^\infty([0,\infty); L^1(dx)) \tag{6.2.0.176}$$

Note that $\mathfrak{m}_\epsilon$ is non-negatively definite matrix-value measure, then

$$\frac{1}{\epsilon}\mathfrak{m}_\epsilon \to 0 \quad \text{in} \quad L^\infty([0,\infty); L^1(dx)) \tag{6.2.0.177}$$

From the regularity results of Matsumura-Nishida and Ponce [60, 62]for the linearized Navier-Stokes system, we have

$$\|u_\epsilon + \delta_\epsilon \tilde{\beta}_\epsilon\|_{L^1_{\mathrm{loc}}(dt; L^\infty(dx))} \leq C\,. \tag{6.2.0.178}$$

Then we have shown that

$$\int_0^t \int_\Omega \frac{1}{\epsilon}\mathfrak{m}_\epsilon : (u_\epsilon + \delta_\epsilon\tilde{\beta}_\epsilon)\,dx\,ds \to 0 \quad \text{as} \quad \epsilon \to 0\,. \tag{6.2.0.179}$$

We apply the regularity results of the linearized Navier-Stokes system again combing with the arguments we made before the Proposition, it is easily to derive that the remainder $R_\epsilon^2 \to 0$, as $\epsilon \to 0$.

Next, we shall show the following asymptotic inequality:

Lemma 24: *For the entropy dissipation rate $D(F_\epsilon)$ and the fluid variables $u_\epsilon^\flat, \theta_\epsilon^\flat$ associated with a family of fluctuations of DiPerna-Lions solutions F_ϵ to the Boltzmann equation* (6.2.0.125),

$$\liminf_{\epsilon\to 0}\left[\frac{1}{4(\epsilon\delta_\epsilon)^2}\int_0^t\int_\Omega D(F_\epsilon)\,dx\,ds - \int_0^t\int_\Omega \tfrac{\mu}{2}\sigma(u_\epsilon^\flat):\sigma(u_\epsilon^\flat)+\kappa|\nabla_x\theta_\epsilon^\flat|^2\,dx\,ds\right]\geq 0\,. \tag{6.2.0.180}$$

Proof of the lemma: First, recall the scaled entropy dissipation rate $\tilde{q}_\epsilon$:

$$\tilde{q}_\epsilon = \frac{1}{\epsilon\delta_\epsilon}\left(\sqrt{G'_{\epsilon 1}G'_\epsilon} - \sqrt{G_{\epsilon 1}G_\epsilon}\right) \tag{6.2.0.181}$$

The elementary inequality

$$(\sqrt{\xi}-\sqrt{\eta})^2 \leq \tfrac{1}{4}(\xi-\eta)(\ln\xi - \ln\eta) \tag{6.2.0.182}$$

yields that

$$(\tilde{q}_\epsilon)^2 \leq \tfrac{1}{4}D(G_\epsilon)\,. \tag{6.2.0.183}$$

Next, we claim that $\tilde{q}_\epsilon$ satisfies the inequality

$$\tfrac{1}{2}\tfrac{1}{\mu}\langle\!\langle \hat{A}\tilde{q}_\epsilon\rangle\!\rangle : \langle\!\langle \hat{A}\tilde{q}_\epsilon\rangle\!\rangle + \tfrac{1}{\kappa}\langle\!\langle \hat{B}\tilde{q}_\epsilon\rangle\!\rangle\cdot\langle\!\langle \hat{B}\tilde{q}_\epsilon\rangle\!\rangle \leq \tfrac{1}{4}\langle\!\langle \tilde{q}_\epsilon^2\rangle\!\rangle\,. \tag{6.2.0.184}$$

where μ and κ are defined in (6.2.0.145).

Proof of the claim: Introduce

$$\begin{aligned}\Phi &= \tfrac{1}{4}(\hat{A}_1 + \hat{A} - \hat{A}'_1 - \hat{A}_1)\\ \Psi &= \tfrac{1}{4}(\hat{B}_1 + \hat{B} - \hat{B}'_1 - \hat{B}_1)\,.\end{aligned} \tag{6.2.0.185}$$

First, notice that the symmetries of $\tilde{q}_\epsilon$ under the $d\mu-$symmetry imply

$$\langle\!\langle \hat{A}\tilde{q}_\epsilon \rangle\!\rangle = \langle\!\langle \Phi\tilde{q}_\epsilon \rangle\!\rangle, \quad \langle\!\langle \hat{B}\tilde{q}_\epsilon \rangle\!\rangle = \langle\!\langle \Psi\tilde{q}_\epsilon \rangle\!\rangle \,. \tag{6.2.0.186}$$

Next repeated application of the $d\mu-$symmetries shows

$$\langle \hat{A} \otimes A \rangle = 4\langle\!\langle \Phi \otimes \Phi \rangle\!\rangle \,, \quad \langle \hat{B} \otimes B \rangle = 4\langle\!\langle \Psi \otimes \Phi \rangle\!\rangle \,. \tag{6.2.0.187}$$

Then any vector $a \in \mathbb{R}^D$ and any traceless symmetric matrix $M \in \mathbb{R}^D \times \mathbb{R}^D$ satisfy the identities

$$\langle\!\langle \Phi \otimes \Phi \rangle\!\rangle : M = \tfrac{1}{2}\mu M \,, \quad \langle\!\langle \Psi \otimes \Phi \rangle\!\rangle \cdot a = \tfrac{D+2}{8}\kappa a \,. \tag{6.2.0.188}$$

Applying the Cauchy-Schwarz inequality and then identities (6.2.0.187) shows that

$$\begin{aligned}
(\langle\!\langle \Phi\tilde{q}_\epsilon \rangle\!\rangle : M + \langle\!\langle \Psi\tilde{q}_\epsilon \rangle\!\rangle \cdot a)^2 &= \langle\!\langle (\Phi : M + \Psi \cdot a)\tilde{q}_\epsilon \rangle\!\rangle^2 \\
&\leq \langle\!\langle (\Phi : M + \Psi \cdot a)\rangle\!\rangle^2 \langle\!\langle \tilde{q}_\epsilon^2 \rangle\!\rangle \\
&= (M : \langle\!\langle \Phi \otimes \Phi \rangle\!\rangle : M + a \cdot \langle\!\langle \Psi \otimes \Phi \rangle\!\rangle \cdot a)\langle\!\langle \tilde{q}_\epsilon^2 \rangle\!\rangle \\
&= \left(\tfrac{1}{2}\mu M : M + \tfrac{D+2}{8}\kappa a \cdot a\right) \langle\!\langle \tilde{q}_\epsilon^2 \rangle\!\rangle \,.
\end{aligned} \tag{6.2.0.189}$$

Now the result follows by using (6.2.0.186) and setting

$$\begin{aligned}
M &= \tfrac{1}{2}\tfrac{1}{\mu}\langle\!\langle \Phi\tilde{q}_\epsilon \rangle\!\rangle = \tfrac{1}{2}\tfrac{1}{\mu}\langle\!\langle \hat{A}\tilde{q}_\epsilon \rangle\!\rangle \,, \\
a &= \tfrac{1}{\kappa}\langle\!\langle \Psi\tilde{q}_\epsilon \rangle\!\rangle = \tfrac{1}{\kappa}\langle\!\langle \hat{B}\tilde{q}_\epsilon \rangle\!\rangle \,,
\end{aligned} \tag{6.2.0.190}$$

in the inequality (6.2.0.184). Thus we proved the claim.

Then apply the "asymptotic" Boltzmann equation, and lemma (22), we finish the proof of the lemma. □

Now from the evolution of the relative entropy, we obtain that under the assumption of the vanishing of the initial relative entropy

$$\frac{1}{\delta_\epsilon^2} H(F_\epsilon^0 | M_\epsilon^0) \to 0\,, \quad \text{as} \quad \epsilon \to 0\,, \tag{6.2.0.191}$$

for each $t > 0$, we have the following limits:

- asymptotics of the entropy dissipation rate

$$\lim_{\epsilon\to 0}\left[\frac{1}{4(\epsilon\delta_\epsilon)^2}\int_0^t\int_\Omega D(F_\epsilon)\,dx\,ds - \int_0^t\int_\Omega \tfrac{\mu}{2}\sigma(u_\epsilon^\flat):\sigma(u_\epsilon^\flat) + \kappa|\nabla_x\theta_\epsilon^\flat|^2\,dx\,ds\right] = 0\,; \tag{6.2.0.192}$$

- asymptote of the momentum and energy flux:

$$\lim_{\epsilon\to 0}\int_0^t\int_\Omega \tfrac{\mu}{2}\sigma(u_\epsilon - u_\epsilon^\flat):\sigma(u_\epsilon - u_\epsilon^\flat) + \kappa|\nabla_x(\theta_\epsilon - \theta_\epsilon^\flat)|^2\,dx\,ds = 0\,; \tag{6.2.0.193}$$

- and finally, the relative entropy:

$$\lim_{\epsilon\to 0}\frac{1}{\delta_\epsilon^2} H(F_\epsilon | M_\epsilon)(t) = 0\,. \tag{6.2.0.194}$$

Thus, we finish the proof of the main theorem. □

From the Navier-Stokes asymptotics, see proposition (10), and recalling the Stokes scaling $\frac{\delta_\epsilon}{\epsilon} \to 0$, as $\epsilon \to 0$, we have the weakly compressible Stokes asymptotics which is dropping the $\frac{\delta_\epsilon}{\epsilon}$ terms in (10) with the new remainder terms

$$\tilde{\mathbf{J}} = (\tilde{J}_{1\epsilon}, \tilde{J}_{2\epsilon}, \tilde{J}_{3\epsilon}) \to 0\,, \quad \text{as} \quad \epsilon \to 0\,, \tag{6.2.0.195}$$

in w-$L^1_{\mathrm{loc}}(dt; \text{w-}L^1(dx))$.

In the later proof, especially in the equi-continuity argument, we will multiply the above asymptotics $U_\epsilon^\flat = (\rho_\epsilon^\flat, u_\epsilon^\flat, \theta_\epsilon^\flat)$ to derive the asymptotic energy identity.

However, $U_\epsilon^\flat$ belongs to $L^2_{\rm loc}(dt; L^2(dx))$ only, so the remainder terms $\tilde{\mathbf{J}}_\epsilon \cdot U_\epsilon^\flat$ would not vanish as $\epsilon \to 0$. To overcome this difficulty, we employ a mollifier over the periodic space variable. Recall that $\mathbb{T}^D = \mathbb{R}^D/\mathbb{L}^D$, where $\mathbb{L}^D \subset \mathbb{R}^D$ is some $D-$dimensional lattice. Let $\xi \in C^\infty(\mathbb{R}^D)$ be such that $\xi \geq 0$, $\int_{\mathbb{R}^D} \xi(x)\,dx = 1$, and $\xi(x) = 0$ for $|x| > 1$. We then define $\xi_\delta \in C^\infty(\mathbb{T}^D)$ by

$$\xi_\delta(x) = \frac{1}{\delta^D} \sum_{l \in \mathbb{L}^D} \xi\left(\frac{x+l}{\delta}\right) . \tag{6.2.0.196}$$

In this section all convolutions are taken only in the x variable.

Now taking convolution with the asymptotic compressible Stokes system ((10) without the $\frac{\delta_\epsilon}{\epsilon}$ terms,) we obtain

$$\begin{aligned} \partial_t \rho_\epsilon^{\flat,\delta} + \frac{1}{\epsilon}\nabla_x \cdot u_\epsilon^{\flat,\delta} &= \tilde{J}_{1\epsilon}^\delta \,, \\ \partial_t u_\epsilon^{\flat,\delta} + \frac{1}{\epsilon}\nabla_x(\rho_\epsilon^{\flat,\delta} + \theta_\epsilon^{\flat,\delta}) &= \mu \nabla_x \cdot \sigma(u_\epsilon^{\flat,\delta}) + \tilde{J}_{2\epsilon}^\delta \,, \\ \tfrac{D}{2}\partial_t \theta_\epsilon^{\flat,\delta} + \frac{1}{\epsilon}\nabla_x \cdot u_\epsilon^{\flat,\delta} &= \kappa \Delta_x \theta_\epsilon^{\flat,\delta} + \tilde{J}_{3\epsilon}^\delta \,. \end{aligned} \tag{6.2.0.197}$$

with initial data $U_{0\epsilon}^{\flat,\delta} = (\rho_{0\epsilon}^{\flat,\delta}, u_{0\epsilon}^{\flat,\delta}, \theta_{0\epsilon}^{\flat,\delta})$ which is defined as

$$\rho_{0\epsilon}^{\flat,\delta} = \langle 1_{K_\epsilon} g_\epsilon^{in} \gamma_\epsilon \rangle \star \xi_\delta \,, \quad u_{0\epsilon}^{\flat,\delta} = \langle v_{K_\epsilon} g_\epsilon^{in} \gamma_\epsilon \rangle \star \xi_\delta \,, \quad \theta_{0\epsilon}^{\flat,\delta} = \langle (\tfrac{1}{D}|v|^2_{K_\epsilon} - 1) g_\epsilon^{in} \gamma_\epsilon \rangle \star \xi_\delta \,. \tag{6.2.0.198}$$

Now define $\tilde{U}_\epsilon^{\flat,\delta} = U_\epsilon^{\flat,\delta} - U_\epsilon$ where $U_\epsilon = (\rho_\epsilon, u_\epsilon, \theta_\epsilon)$ is the solution of the weakly compressible Stokes system (6.2.0.57). Then because of the linearity of the system, $\tilde{U}_\epsilon^{\flat,\delta}$ satisfies the asymptotic compressible Stokes system:

$$\partial_t \tilde{U}_\epsilon^{\flat,\delta} + \mathcal{A}\tilde{U}_\epsilon^{\flat,\delta} = \mathcal{D}\tilde{U}_\epsilon^{\flat,\delta} + \tilde{\mathbf{J}}_\epsilon^\delta \,. \tag{6.2.0.199}$$

with initial data $U_{0\epsilon}^{\flat,\delta} - U_\epsilon^{in}$. The first order operator $\mathcal{A}$ and the second order operator $\mathcal{D}$ are defined as before, see (5.4.0.72) and (5.4.0.74). We will show next the following key lemma:

Lemma 25: *For each fixed $\delta > 0$, we have the following continuous in time L^2-strong convergence:*

$$\tilde{U}_\epsilon^{\flat,\delta} \to 0\,, \quad as \quad \epsilon \to 0\,, \tag{6.2.0.200}$$

in $C([0,\infty); L^2(dx))$.

Proof of the lemma: From the Arzelà-Ascoli theorem, we need to show that $\tilde{U}_\epsilon^{\flat,\delta}$ are

- equi-continuous in $C([0,\infty); L^2(dx))$;
- pointwisely convergent for every $t \geq 0$.

The idea of the proof of the lemma based on the evolution of the relative entropy. We will show that the equi-continuity is a straightforward consequence of the asymptotics of the momentum and energy flux (6.2.0.193), while the point-wise L^2 norm of $\tilde{U}_\epsilon^{\flat,\delta}$ can be controlled by the relative entropy $\frac{1}{\delta_\epsilon^2} H(F_\epsilon | M_\epsilon)$.

First, we want to show that: for each $t > 0$,

$$\frac{1}{2}\|U_\epsilon^{\flat,\delta}(t) - U_\epsilon(t)\|_{L^2}^2 \leq \frac{1}{\delta_\epsilon} H(F_\epsilon(t)|M_\epsilon(t))\,. \tag{6.2.0.201}$$

Multiplying the asymptotic compressible Stokes system (6.2.0.199) with $\tilde{U}_\epsilon^{\flat,\delta}$, then integrating in time t, we obtain the energy identity:

$$\begin{aligned} &\tfrac{1}{2}\|\tilde{U}_\epsilon^{\flat,\delta}(t)\|_{L^2}^2 + \int_0^t \int_\Omega \tfrac{\mu}{2}\sigma(u_\epsilon^{\flat,\delta} - u_\epsilon) : \sigma(u_\epsilon^{\flat,\delta} - u_\epsilon) + \kappa |\nabla_x(\theta_\epsilon^{\flat,\delta} - \theta_\epsilon)|^2 \, dx\, ds \\ &= \tfrac{1}{2}\|\tilde{U}_{0\epsilon}^{\flat,\delta}\|_{L^2}^2 + \int_0^t \langle \tilde{\mathbf{J}}_\epsilon^\delta, \tilde{U}_\epsilon^{\flat,\delta}\rangle \, ds\,, \end{aligned} \tag{6.2.0.202}$$

where $\int_0^t \langle \tilde{\mathbf{J}}_\epsilon, \tilde{U}_\epsilon^{\flat,\delta}\rangle \, ds \to 0$, as $\epsilon \to 0$, uniformly in ϵ.

Then, the inequality

$$\frac{1}{2}\|U^{\flat,\delta}_{0\epsilon} - U^{in}_{\epsilon}\|^2_{L^2} \leq \frac{1}{\delta_\epsilon^2} H(F^{in}_\epsilon | M^{in}_\epsilon) \tag{6.2.0.203}$$

will imply the inequality (6.2.0.201).

For notational simplicity, we drop δ and in in the following computation and estimates. We introduce the following two infinitesimal Maxwellian associated to $U^\flat_\epsilon$ and U_ϵ:

$$\begin{aligned} g^\flat_\epsilon &= \rho^\flat_\epsilon + u^\flat_\epsilon \cdot v + \theta^\flat_\epsilon(\tfrac{1}{2}|v|^2 - \tfrac{D}{2})\,, \\ &= \langle 1_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle + \langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle \cdot v + \langle (\tfrac{1}{D}|v|^2_{K_\epsilon} - 1) g_\epsilon \gamma_\epsilon \rangle (\tfrac{1}{2}|v|^2 - \tfrac{D}{2})\,, \\ g^S_\epsilon &= \rho_\epsilon + u_\epsilon \cdot v + \theta_\epsilon(\tfrac{1}{2}|v|^2 - \tfrac{D}{2})\,. \end{aligned} \tag{6.2.0.204}$$

Some simple calculations yield that

$$\tfrac{1}{2}\|U^\flat_\epsilon - U_\epsilon\|^2_{L^2} = \tfrac{1}{2}\int_\Omega \langle (g^\flat_\epsilon)^2 \rangle\, dx + \tfrac{1}{2}\int_\Omega \langle (g^S_\epsilon)^2 \rangle\, dx - \int_\Omega \langle g^\flat_\epsilon, g^S_\epsilon \rangle\, dx\,. \tag{6.2.0.205}$$

We claim the following two statements which imply the the inequality (6.2.0.203):

Claim 1:

$$\begin{aligned} &\int_\Omega \int_{\mathbb{R}^D} \left[F_\epsilon \ln(\tfrac{M_\epsilon}{M}) - M_\epsilon + M \right] dv dx \\ &= -\tfrac{1}{2}\int_\Omega \langle (g^S_\epsilon)^2 \rangle\, dx + \int_\Omega \langle g^\flat_\epsilon, g^S_\epsilon \rangle\, dx + \int_\Omega r_\epsilon\, dx\,, \end{aligned} \tag{6.2.0.206}$$

where $\int_\Omega r_\epsilon\, dx \to 0$, as $\epsilon \to 0$, uniformly in ϵ.

Claim 2:

$$\liminf_{\epsilon \to 0} \frac{1}{\delta_\epsilon^2} H(F_\epsilon | M) - \frac{1}{2}\int_\Omega \langle (g^\flat_\epsilon)^2 \rangle\, dx \geq 0\,. \tag{6.2.0.207}$$

Proof of the Claim 1: We apply the lemma (20) which gives the formula on $\ln(\frac{M_\epsilon}{M})$,

$$\begin{aligned}
&\int_\Omega \int_{\mathbb{R}^D} \left[F_\epsilon \ln(\tfrac{M_\epsilon}{M}) - M_\epsilon + M\right] \, dvdx \\
&= \int_\Omega \int_{\mathbb{R}^D} (1+\delta_\epsilon g_\epsilon) \left[\alpha_\epsilon + \beta_\epsilon \cdot v + \gamma_\epsilon(\tfrac{1}{2}|v|^2 - \tfrac{D}{2})\right] M \, dvdx - \frac{1}{\delta_\epsilon} \int_\Omega \rho_\epsilon \, dx \\
&= \int_\Omega \int_{\mathbb{R}^D} g_\epsilon \left[\rho_\epsilon + u_\epsilon \cdot v + \theta_\epsilon(\tfrac{1}{2}|v|^2 - \tfrac{D}{2})\right] M \, dvdx + \delta_\epsilon \int_\Omega \int_{\mathbb{R}^D} g_\epsilon \tilde{r}_\epsilon M \, dvdx \\
&- \tfrac{1}{2} \int_\Omega (\rho_\epsilon^2 + |u_\epsilon|^2 + \tfrac{D}{2}\theta_\epsilon^2) \, dx \,,
\end{aligned} \tag{6.2.0.208}$$

where $\tilde{r}_\epsilon$ is the higher order term in the expansion of $\ln(\frac{M_\epsilon}{M})$ which is bounded in L^∞. If we denote by $\xi(v) = 1, v$ or $|v|^2$, then decompose $\xi(v) g_\epsilon$ as in (6.2.0.166):

$$\begin{aligned}
\langle \xi(v) g_\epsilon \rangle &= \langle \xi_{K_\epsilon}(v) g_\epsilon \rangle + \langle \xi(v) \mathbf{1}_{|v|^2 > K|\ln\epsilon|} g_\epsilon \rangle \\
&\simeq \langle \xi_{K_\epsilon}(v) g_\epsilon \gamma_\epsilon \rangle + \tfrac{1}{4}\delta_\epsilon \left\langle \xi_{K_\epsilon}(v) \left(\tfrac{2(\sqrt{G_\epsilon}-1)}{\delta_\epsilon}\right)^2 \right\rangle + \left\langle \xi(v) \mathbf{1}_{|v|^2 > K|\ln\epsilon|} g_\epsilon \right\rangle \,.
\end{aligned} \tag{6.2.0.209}$$

Then we obtain

$$\begin{aligned}
&(6.2.0.208) \\
&= -\tfrac{1}{2} \int_\Omega \langle (g_\epsilon^S)^2 \rangle \, dx + \int_\Omega \langle g_\epsilon^\flat, g_\epsilon^S \rangle \, dx + \int_\Omega r_\epsilon \, dx \,,
\end{aligned} \tag{6.2.0.210}$$

The classical estimate on the tail of the Gaussian integrals and the nonlinear compactness estimate Proposition (8), and L^∞ boundedness of the remainder in $\ln(\frac{M_\epsilon}{M})$, yields the remainder

$$\int_\Omega r_\epsilon \, dx \to 0 \quad \text{as} \quad \epsilon \to 0 \tag{6.2.0.211}$$

uniformly in ϵ. Thus we finish the proof of the claim 1. □

Now we turn to claim 2.

Proof of the claim 2:

$$
\begin{aligned}
&\frac{1}{\delta_\epsilon^2}\int_\Omega \langle G_\epsilon \ln G_\epsilon - G_\epsilon + 1\rangle \, dx - \tfrac{1}{2}\int_\Omega \langle (g_\epsilon^\flat)^2\rangle \, dx \\
&= \int_\Omega \left\langle \left(\frac{1}{\delta_\epsilon^2} h(\delta_\epsilon g_\epsilon) - \tfrac{1}{2}(g_\epsilon^\flat)^2\right) \mathbf{1}_{|g_\epsilon^\flat|\le K} \right\rangle dx + \tfrac{1}{2}\int_\Omega \left\langle (g_\epsilon^\flat)^2 \mathbf{1}_{|g_\epsilon^\flat|\ge K}\right\rangle dx \qquad (6.2.0.212)\\
&= \mathrm{I}_K^\epsilon + \mathrm{II}_K^\epsilon \,.
\end{aligned}
$$

We will show that

$$
\liminf_{\epsilon\to 0} \mathrm{I}_K^\epsilon \ge 0\,, \quad \text{and} \quad \lim_{K\to\infty} \mathrm{II}_K^\epsilon = 0\,. \qquad (6.2.0.213)
$$

then the claim 2 follows.

The convexity of h gives the inequality

$$
\frac{1}{\delta_\epsilon^2} h(\delta_\epsilon g_\epsilon^\flat) + \frac{1}{\delta_\epsilon} h'(\delta_\epsilon g_\epsilon^\flat)(g_\epsilon - g_\epsilon^\flat) \le \frac{1}{\delta_\epsilon^2} h(\delta_\epsilon g_\epsilon)\,. \qquad (6.2.0.214)
$$

Fix $K > 0$ and multiply this inequality by the indicator function $\mathbf{1}_{|g_\epsilon^\flat|\le K}$; the non-negativity of h then implies

$$
\frac{1}{\delta_\epsilon^2} h(\delta_\epsilon g_\epsilon) - \frac{1}{\delta_\epsilon^2} h(\delta_\epsilon g_\epsilon^\flat)\mathbf{1}_{|g_\epsilon^\flat|\le K} \ge \frac{1}{\delta_\epsilon} h'(\delta_\epsilon g_\epsilon^\flat)\mathbf{1}_{|g_\epsilon^\flat|\le K}(g_\epsilon - g_\epsilon^\flat)\,. \qquad (6.2.0.215)
$$

In above inequality

$$
\frac{1}{\delta_\epsilon^2} h(\delta_\epsilon g_\epsilon^\flat)\mathbf{1}_{|g_\epsilon^\flat|\le K} \simeq \tfrac{1}{2}(g_\epsilon^\flat)^2\,. \qquad (6.2.0.216)
$$

Note that

$$
g_\epsilon - g_\epsilon^\flat = (\Pi g_\epsilon - g_\epsilon^\flat) + (g_\epsilon - \Pi g_\epsilon) \qquad (6.2.0.217)
$$

where we know the latter goes to 0 strongly in $L^2_{\mathrm{loc}}(dt; L^2(Mdvdx))$ as $\epsilon \to 0$, see the corollary 6. Now we use again the decomposition (6.2.0.166), we get

$$
\Pi g_\epsilon - g_\epsilon^\flat \simeq \delta_\epsilon g_\epsilon^\sharp + r_\epsilon\,, \qquad (6.2.0.218)
$$

where

$$g_\epsilon^\sharp = \tfrac{1}{4}\langle \mathbf{1}_{K_\epsilon}(\tfrac{2(\sqrt{G_\epsilon}-1)}{\delta_\epsilon})^2\rangle + \tfrac{1}{4}\langle v_{K_\epsilon}(\tfrac{2(\sqrt{G_\epsilon}-1)}{\delta_\epsilon})^2\rangle \cdot v$$
$$+ \tfrac{1}{4}\langle(\tfrac{|v|^2_{K_\epsilon}}{D} - 1)(\tfrac{2(\sqrt{G_\epsilon}-1)}{\delta_\epsilon})^2\rangle(\tfrac{1}{2}|v|^2 - \tfrac{D}{2})\,, \tag{6.2.0.219}$$

and

$$r_\epsilon = \langle \mathbf{1}_{|v|^2>K|\ln\epsilon|} g_\epsilon\rangle + \langle v\mathbf{1}_{|v|^2>K|\ln\epsilon|} g_\epsilon\rangle \cdot v$$
$$+ \langle(\tfrac{1}{D}|v|^2 - 1)\mathbf{1}_{|v|^2>K|\ln\epsilon|} g_\epsilon\rangle(\tfrac{1}{2}|v|^2 - \tfrac{D}{2})\,. \tag{6.2.0.220}$$

Turn back to inequality (6.2.0.215), and note that $h'(\delta_\epsilon g_\epsilon^\flat) = \ln(1 + \delta_\epsilon g_\epsilon^\flat)$. Then

$$\frac{1}{\delta_\epsilon} h'(\delta_\epsilon g_\epsilon^\flat)\mathbf{1}_{|g_\epsilon^\flat|\le K} \quad \text{is bounded}\,. \tag{6.2.0.221}$$

As before we use again the nonlinear compactness Proposition (8) and the classical estimate on the tail of the Gaussian integrals,

$$g_\epsilon - g_\epsilon^\flat \to 0\,, \quad \text{in} \quad \text{w-}L^1_{\text{loc}}(dt; \text{w-}L^1(Mdvdx),) \quad \text{as} \quad \epsilon \to 0\,. \tag{6.2.0.222}$$

Average the inequality (6.2.0.215) over $[t_1, t_2] \times \Omega \times \mathbb{R}^D$ for arbitrary time interval $[t_1, t_2]$ and then consider its asymptotics as ϵ tends to zero, the asymptote (6.2.0.217) and the limit (6.2.0.222) yield

$$\frac{1}{t_2 - t_1}\int_\Omega \tfrac{1}{2}(g_\epsilon^\flat)^2\mathbf{1}_{|g_\epsilon^\flat|\le K}\, dxdt$$
$$\le \liminf_{\epsilon\to 0} \frac{1}{t_2 - t_1}\int_{t_1}^{t_2}\int_\Omega \langle \frac{1}{\delta_\epsilon^2} h(\delta_\epsilon g_\epsilon)\rangle\, dxdt\,. \tag{6.2.0.223}$$

Let t_1 approach to t_2, then we prove

$$\liminf_{\epsilon\to 0} \mathrm{I}_K^\epsilon \ge 0\,. \tag{6.2.0.224}$$

Remark: In the above argument, we proved that generally, the asymptotic inequality $\liminf\limits_{\epsilon\to 0} \mathrm{I}_K^\epsilon \ge 0$ for $t \ge 0$ almost everywhere, but not all $t \ge 0$. This is the very reason that we would not prove the inequality (6.2.0.203) for every $t \ge 0$ directly, but only prove it at $t = 0$, then extend to all $t \geq$ through the asymptotic energy identity (6.2.0.202).

Now we turn to II_K^ϵ. From the nonlinear compactness estimate Proposition (8), and the asymptotic equivalence of the two renormalizations (6.2.0.144), we can easily derive that

$$(1+|v|)(g_\epsilon^\flat)^2 \quad \text{is relatively compact in w-}L^1_{\mathrm{loc}}(dt;\text{w-}L^1(Mdvdx))\,. \qquad (6.2.0.225)$$

Then by the Dunford-Pettis criteria of relatively compactness of w-L^1, it follows that

$$\lim_{K\to\infty} \mathrm{II}_K^\epsilon = 0\,. \qquad (6.2.0.226)$$

Then we proved the statements in (6.2.0.213), then the claim 2 follows. Combining claim 1 and claim 2, we proved the pointwise L^2-estimate (6.2.0.201): for each $t > 0$,

$$\frac{1}{2}\|U_\epsilon^{\flat,\delta}(t) - U_\epsilon(t)\|_{L^2}^2 \leq \frac{1}{\delta_\epsilon^2} H(F_\epsilon(t)|M_\epsilon(t)) \to 0\,, \qquad (6.2.0.227)$$

As $\epsilon \to 0$. The last limitcomes from the evolution of the relative entropy (6.2.0.194).

Our final step is to prove the equi-continuity of $\tilde{U}_\epsilon^{\flat,\delta}$ in $C([0,\infty);L^2(dx))$ which is a consequence of the energy identity (6.2.0.202): for every $[t_1,t_2] \subset [0,T]$,

$$\begin{aligned}
&\tfrac{1}{2}\|\tilde{U}_\epsilon^{\flat,\delta}(t_2)\|_{L^2}^2 - \tfrac{1}{2}\|\tilde{U}_\epsilon^{\flat,\delta}(t_1)\|_{L^2}^2 \\
&= -\int_{t_1}^{t_2}\int_\Omega \tfrac{\mu}{2}\sigma(u_\epsilon^{\flat,\delta} - u_\epsilon) : \sigma(u_\epsilon^{\flat,\delta} - u_\epsilon) + \kappa|\nabla_x(\theta_\epsilon^{\flat,\delta} - \theta_\epsilon)|^2\,dx\,dt \qquad (6.2.0.228)\\
&= +\int_{t_1}^{t_2} \langle \tilde{\mathbf{J}}_\epsilon^\delta, \tilde{U}_\epsilon^{\flat,\delta}\rangle\,ds\,,
\end{aligned}$$

From the asymptotics of the momentum and energy flux (6.2.0.193), we know the right-hand side in the above identity will be sufficiently small when $|t_2 - t_1|$ small enough.

Similar to our arguments in the limit from compressible Stokes to acoustic system, we use the following identity:

$$\begin{aligned}&\tfrac{1}{2}\|\tilde{U}_\epsilon^{\flat,\delta}(t_2)-\tilde{U}_\epsilon^{\flat,\delta}(t_1)\|_{L^2}^2\\&=\tfrac{1}{2}\|\tilde{U}_\epsilon^{\flat,\delta}(t_2)\|_{L^2}^2-\tfrac{1}{2}\|\tilde{U}_\epsilon^{\flat,\delta}(t_1)\|_{L^2}^2+\langle\tilde{U}_\epsilon^{\flat,\delta}(t_1,)\tilde{U}_\epsilon^{\flat,\delta}(t_1)-\tilde{U}_\epsilon^{\flat,\delta}(t_2)\rangle\,.\end{aligned}\tag{6.2.0.229}$$

If we fix t_1, then we already proved the pointwise strong L^2 convergence to 0 which implies the second term above will vanish. Then the energy identity (6.2.0.229) gives the equi-continuity. Thus, we proved that for each fixed $\delta > 0$, the continuous in time L^2 strong convergence:

$$U_\epsilon^{\flat,\delta}-U_\epsilon\to 0\,,\quad \text{as}\quad \epsilon\to 0\,,\tag{6.2.0.230}$$

in $C([0,\infty);L^2(dx))$.

It remains to remove the mollifier ξ_δ in (6.2.0.230). In order to do so, we need the compactness in the spacial variable x for the kinetic equations. Velocity averaging is the natural way. For the purpose of studying the compactness of $U_\epsilon^\flat = \langle\zeta_{K_\epsilon}g_\epsilon\gamma_\epsilon\rangle$, we use the nonlinear compactness estimate Proposition (8) coupled with the following variant of the L^2 case of the velocity averaging theorem.

Lemma 26: *Let ϕ_ϵ be a bounded family in $L^2_{\text{loc}}(dt;L^2(Mdvdx))$ such that $|\phi_\epsilon|^2$ is locally uniformly integrable on $\mathbb{R}_+^*\times\Omega\times\mathbb{R}^D$ for the Lebesgue measure. Assume that*

$$(\epsilon\partial_t+v\cdot\nabla_x)\phi_\epsilon\quad \textit{is bounded in}\quad L^1_{\text{loc}}(dt;L^1(dvdx))\,.\tag{6.2.0.231}$$

Then, for each $\psi\in L^2(Mdv)$, the family $\langle\phi_\epsilon\psi\rangle$ is relatively compact in $L^2_{\text{loc}}(dt;L^2(dx))$ with respect to the $x-$variable, meaning that, for each $T>0$ and each compact

$K \subset \Omega$, *one has*

$$\iint_{[0,T]\times K} |\langle \phi_\epsilon \psi \rangle(t, x+y) - \langle \phi_\epsilon \psi \rangle(t,x)|^2 \, dxdt \to 0 \,, \tag{6.2.0.232}$$

as $y \to 0$ *uniformly in* ϵ.

See [34] for the proof.

Now we apply the lemma above to the second type of the normalization

$$\phi_\epsilon = \frac{\sqrt{\epsilon^c + G_\epsilon} - 1}{\epsilon} \tag{6.2.0.233}$$

since

$$(\epsilon \partial_t + v \cdot \nabla_x) \phi_\epsilon = \frac{1}{\epsilon^2} \frac{\mathcal{Q}(G_\epsilon, G_\epsilon)}{2\sqrt{\epsilon^c + G_\epsilon}} = O(1)_{L^1_{\mathrm{loc}}(dtdxdv)} \tag{6.2.0.234}$$

for $c \in (1,2)$, by the entropy production estimate (6.2.0.151). Since

$$\phi_\epsilon = \frac{\sqrt{\epsilon^c + G_\epsilon} - 1}{\epsilon} \simeq \frac{1}{2} g_\epsilon \,, \tag{6.2.0.235}$$

Applying the velocity averaging lemma above leads to the following compactness in the x variable results

Proposition 12: *On the same assumption on the collision kernel* $b(z,\omega)$ *as in the main theorem, for each* $T > 0$ *and* $K \subset \Omega$ *compact, one has*

$$\iint_{[0,T]\times K} |\langle \zeta_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle(t, x+y) - \langle \zeta_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle(t,x)|^2 \, dxdt \to 0 \,, \tag{6.2.0.236}$$

as $y \to 0$ *uniformly in* ϵ*, where* $\zeta(v)$ *denotes* $1, v, |v|^2$.

Noting the definition of $U^\flat_\epsilon = \langle \zeta_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle$, and $U^{\flat,\delta}_\epsilon$ is the convolution with the mollifier

ξ_ϵ, the above proposition immediately implies that

$$\begin{aligned}
&\|\rho^{\flat,\delta} - \rho^{\flat}\|_{L^2(dxdt)} \to 0\,,\\
&\|u^{\flat,\delta} - u^{\flat}\|_{L^2(dxdt)} \to 0\,, \qquad (6.2.0.237)\\
&\|\theta^{\flat,\delta} - \theta^{\flat}\|_{L^2(dxdt)} \to 0\,,
\end{aligned}$$

uniformly in ϵ, as $\delta \to 0$. Thus, we showed the strong L^2 limit without mollifier:

$$U^{\flat}_\epsilon - U_\epsilon \to 0\,, \quad \text{as} \quad \epsilon \to 0\,, \qquad (6.2.0.238)$$

in $C([0,\infty); L^2(dx))$.

We define the macroscopic variables associated with the fluctuation without truncation g_ϵ. Let $\hat{U}_\epsilon = \langle \zeta(v) g_\epsilon \rangle$, where $\zeta(v) = (1, v, \frac{1}{D}|v|^2 - 1)$, then again we use the decomposition

$$\langle \zeta(v) g_\epsilon \rangle \simeq \langle \zeta_{K_\epsilon}(v) g_\epsilon \gamma_\epsilon \rangle + \tfrac{1}{4}\delta_\epsilon \langle \zeta_{K_\epsilon}(v) (\tfrac{2(\sqrt{G_\epsilon}-1)}{\delta_\epsilon})^2 \rangle + \left\langle \zeta(v) \mathbf{1}_{|v|^2 > K|\ln \epsilon|} g_\epsilon \right\rangle\,. \qquad (6.2.0.239)$$

Note that the key nonlinear compactness estimate Proposition (8), we have the following L^1 convergence:

$$\hat{U}_\epsilon - U_\epsilon \to 0\,, \quad \text{as} \quad \epsilon \to 0\,, \qquad (6.2.0.240)$$

in $C([0,\infty); L^1(dx))$.

Furthermore, we have decomposition

$$g_\epsilon = \Pi g_\epsilon + (g_\epsilon - \Pi g_\epsilon) \qquad (6.2.0.241)$$

and for the L^2 part relaxation limit

$$\frac{\sqrt{G_\epsilon} - 1}{\delta_\epsilon} - \Pi \frac{\sqrt{G_\epsilon} - 1}{\delta_\epsilon} \to 0\,, \quad \text{in} \quad L^2_{\text{loc}}(dt; L^2(M dv dx)) \qquad (6.2.0.242)$$

as $\epsilon \to 0$. Then the decomposition (6.2.0.239) and the nonlinear compactness estimate Proposition (8) implies

$$g_\epsilon - \Pi g_\epsilon \to 0\,, \quad \text{in} \quad L^1_{\mathrm{loc}}(dt; L^1(Mdvdx)) \tag{6.2.0.243}$$

as $\epsilon \to 0$. Thus

$$\Pi g_\epsilon - g_\epsilon^S \to 0\,, \quad \text{in} \quad C([0,\infty); L^1(Mdvdx))\,, \tag{6.2.0.244}$$

as $\epsilon \to 0$, where g_ϵ^S is the infinitesimal Maxwellian associated with the solution to the weakly compressible Stokes system,

$$g_\epsilon^S(t,x,v) = \rho_\epsilon(t,x) + u_\epsilon(t,x)\cdot v + \theta_\epsilon(t,x)(\tfrac{1}{2}|v|^2 - \tfrac{D}{2})\,. \tag{6.2.0.245}$$

and finally

$$g_\epsilon - g_\epsilon^S \to 0\,, \quad \text{in} \quad L^1([0,\infty); L^1(Mdvdx))\,, \tag{6.2.0.246}$$

as $\epsilon \to 0$. Then we finish the proof of the theorem. □

7. WEAKLY COMPRESSIBLE NAVIER-STOKES APPROXIMATION FROM BOLTZMANN EQUATIONS

In the chapter, we shall consider the weakly compressible Navier-Stokes approximation. We concern the Navier-Stokes scaling, i.e., $\frac{\delta_\epsilon}{\epsilon} \to 1$, as $\epsilon \to 0$, the long time hydrodynamics of the Boltzmann equation. From the initial data $U^{in} = (\rho^{in}, u^{in}, \theta^{in})$, we construct an initial local Maxwellian M_ϵ^{in}, such that it is close to the initial number density F_ϵ^{in} in the sense that the scaled relative entropy, i.e., $\frac{1}{\epsilon^2} H(F_\epsilon^{in} | M_\epsilon^{in}) \to 0$, as $\epsilon \to 0$, then for the later time $t > 0$, we shall construct a family of local Maxwellian $M_\epsilon(t)$, such that $\frac{1}{\epsilon^2} H(F_\epsilon(t) | M_\epsilon(t)) \to 0$, as $\epsilon \to 0$. Thus, this family of local Maxwellians governs the long time behavior of solutions to the Boltzmann equation. Unfortunately, mainly because of the lack of good regularity and compactness of DiPerna-Lions solutions which are the only global solutions available, this weakly compressible Navier-Stokes approximation has not been rigorously justified. Our main theorem in this chapter will be stated in Theorem (19) under assumptions about passing to the limit in certain relative entropy dissipation terms. In the final section, we shall state some future plans after this dissertation.

7.1 Formal Derivation of Weakly Compressible Navier-Stokes Approximation

We start from the scaled Boltzmann equation with initial data:

$$\epsilon\partial_t G_\epsilon + v\cdot\nabla_x G_\epsilon = \frac{1}{\epsilon}\mathcal{Q}(G_\epsilon, G_\epsilon,) \qquad G_\epsilon(0,x,v) = G_\epsilon^{in}(x,v)\,. \tag{7.1.0.1}$$

The family of the fluctuation g_ϵ which is defined as

$$g_\epsilon = \frac{1}{\epsilon}(G_\epsilon - 1)\,, \tag{7.1.0.2}$$

formally satisfies the local conservation laws

$$\begin{aligned}\partial_t\langle g_\epsilon\rangle + \frac{1}{\tau_\epsilon}\nabla_x\cdot\langle v g_\epsilon\rangle = 0\,,\\ \partial_t\langle v g_\epsilon\rangle + \frac{1}{\tau_\epsilon}\nabla_x\cdot\langle v\otimes v g_\epsilon\rangle = 0\,,\\ \partial_t\langle \tfrac{1}{2}|v|^2 g_\epsilon\rangle + \frac{1}{\tau_\epsilon}\nabla_x\cdot\langle v\tfrac{1}{2}|v|^2 g_\epsilon\rangle = 0\,.\end{aligned} \tag{7.1.0.3}$$

As we did in Chapter 5 and 6, we define the fluid variables $\hat{U}_\epsilon = (\hat{\rho}_\epsilon, \hat{u}_\epsilon, \hat{\theta}_\epsilon)$ associated with the fluctuation of the number density g_ϵ:

$$\hat{\rho}_\epsilon = \langle g_\epsilon\rangle\,, \quad \hat{u}_\epsilon = \langle v g_\epsilon\rangle\,, \quad \hat{\theta}_\epsilon = \tfrac{2}{D}\left\langle \left(\tfrac{1}{2}|v|^2 - \tfrac{D}{2}\right) g_\epsilon\right\rangle\,, \tag{7.1.0.4}$$

and initial data $\hat{U}_\epsilon^{in}$ as the corresponding moments of g_ϵ^{in}. We assume that for some $U^{in}\in L^2(dx)$,

$$\hat{U}_\epsilon^{in} \to U^{in}\,, \tag{7.1.0.5}$$

in the sense of distribution. After some tedious algebraic calculations, we derive that $\hat{U}_\epsilon$ satisfies the local conservation laws

$$\begin{aligned}\partial_t\hat{U}_\epsilon + \frac{1}{\epsilon}\mathcal{A}\hat{U}_\epsilon + \frac{\delta_\epsilon}{\epsilon}\mathcal{Q}(\hat{U}_\epsilon, \hat{U}_\epsilon) = \mathcal{D}\hat{U}_\epsilon + \hat{R}_\epsilon\,,\\ \hat{U}_\epsilon(0,x) = \hat{U}_\epsilon^{in}\,,\end{aligned} \tag{7.1.0.6}$$

where $\hat{R}_\epsilon$ is the remainder term in the formal derivation which will vanish, as $\epsilon \to 0$, formally. When G_ϵ are DiPerna-Lions solutions to (7.1.0.1), $\hat{R}_\epsilon \to 0$ in $L^1_{\rm loc}(dt, L^1(dx))$. This was shown by Golse-Saint-Raymond for hard sphere collision kernel [35] and Levermore-Masmoudi for more general collision kernels [48].

In the Navier-Stokes scaling, i.e., $\frac{\delta_\epsilon}{\epsilon} \sim 1$, the quadratic term $\frac{\delta_\epsilon}{\epsilon}\mathcal{Q}(\hat{U}_\epsilon, \hat{U}_\epsilon)$ can not be ignored. From the asymptotic local conservation law (7.1.0.6), it is natural to guess that the corresponding fluid system which captures the long time behavior of the Boltzmann equation should be

$$\begin{aligned} \partial_t U_\epsilon + \frac{1}{\epsilon}\mathcal{A}U_\epsilon + \mathcal{Q}(U_\epsilon, U_\epsilon) = \mathcal{D}U_\epsilon \,, \\ U_\epsilon(0,x) = U^{in}(x)\,. \end{aligned} \tag{7.1.0.7}$$

Unfortunately, this system is not a good choice. Actually, this system does not even satisfy formally the energy identity, because generally

$$\langle \mathcal{Q}(U,U,)U\rangle = \tfrac{D+2}{D}\int |u|^2(\nabla_x\!\cdot u)\,dx + \tfrac{1}{2}\int \theta^2(\nabla_x\!\cdot u)\,dx \neq 0\,, \tag{7.1.0.8}$$

unless the velocity $u(t,x)$ is divergence free, which is not the case when we consider the general initial data, i.e., either incompressibility, or Bousinesq relation is not satisfied. Furthermore, it does even not have the definite sign. Another difficulty is that the diffusion term $\mathcal{D}$ is not strictly dissipative, i.e., the strict dissipation

$$\langle \mathcal{D}U, U\rangle \geq C\langle \nabla_x U, \nabla_x U\rangle \tag{7.1.0.9}$$

is not true. Further more the first and the second order differential operators $\mathcal{A}$ and $\mathcal{D}$ are not commutable, i.e.,

$$\mathcal{A}\mathcal{D} \neq \mathcal{D}\mathcal{A}\,. \tag{7.1.0.10}$$

For these reasons, the fluid system (7.1.0.7) is not well-posed. We don't know even the global solutions exist. So, it is not a good fluid model from which we construct local Maxwellian. To overcome this difficulty, we notice that the asymptotic local conservation laws (7.1.0.6) depends on at the same time two time variables, t and $\tau = \frac{t}{\epsilon}$, we apply the method of multiple time scale, or the averaging method to derive the following averaged system:

$$\begin{aligned} \partial_t U_\epsilon + \frac{1}{\epsilon}\mathcal{A}U_\epsilon + \overline{\mathcal{Q}}(U_\epsilon, U_\epsilon) = \overline{\mathcal{D}}U_\epsilon\,, \\ U_\epsilon(0,x) = U^{in}(x)\,. \end{aligned} \tag{7.1.0.11}$$

where the averaged operator $\overline{\mathcal{Q}}$ and $\overline{\mathcal{D}}$ are defined as:

$$\overline{\mathcal{Q}}(U,U) = \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau e^{s\mathcal{A}}\mathcal{Q}(e^{-s\mathcal{A}}U, e^{-s\mathcal{A}}U)\,ds\,, \tag{7.1.0.12}$$

and

$$\overline{\mathcal{D}}U = \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau e^{s\mathcal{A}}\mathcal{D}(e^{-s\mathcal{A}}U)\,ds\,. \tag{7.1.0.13}$$

The averaged equation (7.1.0.11) has some good properties:

- The averaged quadratic term $\overline{\mathcal{Q}}$ does not contribute in the energy estimate:

$$\langle\overline{\mathcal{Q}}(U,U,)U\rangle = 0\,; \tag{7.1.0.14}$$

- There exists a positive constant $C > 0$, such that

$$\langle\mathcal{D}U, U\rangle \geq C\langle\nabla_x U, \nabla_x U\rangle; \tag{7.1.0.15}$$

- $\mathcal{A}$ and $\overline{\mathcal{D}}$ are commutable, i.e.,

$$\mathcal{A}\overline{\mathcal{D}} = \overline{\mathcal{D}}\mathcal{A}\,; \tag{7.1.0.16}$$

- The averaged quadratic term $\overline{\mathcal{Q}}$ is commutable with $\mathcal{A}$ in the following sense

$$\overline{\mathcal{Q}}(e^{s\mathcal{A}}U, e^{s\mathcal{A}}U) = e^{s\mathcal{A}}\overline{\mathcal{Q}}(U,U)\,. \tag{7.1.0.17}$$

Based on the above properties, we proved in Chapter 3 the global existence of weak solutions:

Theorem 17: *For any initial data $U^{in} \in L^2(dx)$, there exists at least one weak solution $U_\epsilon \in C([0,\infty); \text{w-}L^2(dx)) \cap L^2([0,\infty); H^1(dx))$ to (7.1.0.11) in the sense of Leray.*

The averaged equation (7.1.0.11) has more structures. If we denote by Π and $\Pi^\perp$ the orthogonal projection onto $\text{Null}(\mathcal{A})$ and $\text{Null}(\mathcal{A})^\perp$:

$$\Pi U_\epsilon = \begin{pmatrix} \frac{2}{D+2}\rho_\epsilon - \frac{D}{D+2}\theta_\epsilon \\ Pu_\epsilon \\ -\frac{2}{D+2}\rho_\epsilon + \frac{D}{D+2}\theta_\epsilon \end{pmatrix} \tag{7.1.0.18}$$

and

$$\Pi^\perp U_\epsilon = \begin{pmatrix} \frac{D}{D+2}(\rho_\epsilon + \theta_\epsilon) \\ Qu_\epsilon \\ \frac{2}{D+2}(\rho_\epsilon + \theta_\epsilon) \end{pmatrix} \tag{7.1.0.19}$$

If we project the averaged equation (7.1.0.11) onto slow mode $\text{Null}(\mathcal{A})$, we will have formally $\Pi U_\epsilon \to U$, where U satisfies the Navier-Stokes system for the incompressible flow with the initial data ΠU^{in}, the projection of U^{in} onto $\text{Null}(\mathcal{A})$:

$$\begin{aligned} &\partial_t u + u\cdot\nabla_x u + \nabla_x p = \mu\Delta_x u\,, \\ &\partial_t\theta + u\cdot\nabla_x = \kappa\Delta_x\theta\,, \\ &\rho+\theta = 0\,, \quad \nabla_x\cdot u = 0\,, \end{aligned} \tag{7.1.0.20}$$

with initial data

$$u(0,x) = Pu^{in}\,, \qquad \theta(0,x) = -\tfrac{2}{D+2}\rho^{in} + \tfrac{D}{D+2}\theta^{in}\,. \tag{7.1.0.21}$$

If we define $V_\epsilon = e^{\frac{t}{\epsilon}\mathcal{A}}\Pi^\perp U_\epsilon$, then formally $V_\epsilon \to V$, where V satisfies the averaged equation on the fast mode $\mathrm{Null}(\mathcal{A})^\perp$ with initial data $\Pi^\perp U^{in}$:

$$\partial_t V + \overline{\mathcal{Q}}(V\,,V) = \overline{\mathcal{D}}V\,, \qquad V(0,x) = \begin{pmatrix} \frac{D}{D+2}(\rho^{in}+\theta^{in}) \\ Qu^{in} \\ \frac{2}{D+2}(\rho^{in}+\theta^{in}) \end{pmatrix}. \tag{7.1.0.22}$$

The above averaged equation has better regularity in the existence interval of the incompressible Navier-Stokes system. More importantly, it is a strictly parabolic system, so if the initial data vanish, i.e the initial data satisfies the incompressibility and Boussinesq relations, the solutions will vanish in the later time. This case has been considered in previous works done by Bardos-Golse-levermore [8], Golse-Levermore [29], and Golse-Saint-Raymond [34].

Before we state our work for the general initial data, let us review the previous work in this direction. In [52], [53], P.-L. Lions and Masmoudi applied the relative entropy method in justification of the Euler limit under some compactness assumption on the remainder of the fluctuation of the number density. In [66], Saint-Raymond applied the "Flat-Sharp" decomposition used in [32] to derive Euler limit. All of these work were done for the well-prepared initial data. In [31], Golse, Saint-Rayond and Levermore used the relative entropy method to formally derive incompressible Navier-Stokes limit. They still considered the well-prepared initial data. Their relative entropy was constructed from solutions to the target incompressible Navier-Stokes equations. More precisely, they showed the following theorem on the formal level:

Theorem 18: **(Golse-Levermore-Saint-Raymond.)** *Let*

$$M_\epsilon^{in}(x,v) = \mathcal{M}_{(1-\epsilon\theta^{in}(x,)\frac{\epsilon u^{in}(x)}{1-\epsilon\theta^{in}(x)}, \frac{1}{1-\epsilon\theta^{in}(x)})}(v) \tag{7.1.0.23}$$

be the initial relative entropy. For $t > 0$, Let

$$M_\epsilon(t,x,v) = \mathcal{M}_{(1-\epsilon\theta(t,x,)\frac{\epsilon u(t,x)}{1-\epsilon\theta(t,x)}, \frac{1}{1-\epsilon\theta(t,x)})}(v)\,. \tag{7.1.0.24}$$

where $(\rho(t,x,)\theta(t,x))$ is a solution to the incompressible Navier-Stokes equations with initial data $(\rho^{in}(x,)\theta^{in}(x))$:

$$\begin{aligned} \nabla_x \cdot u = 0\,, \\ \partial_t u + u\cdot\nabla u + \nabla_x p = \mu\Delta_x u\,, \\ \partial_t\theta + u\cdot\nabla\theta = \kappa\Delta_x\theta\,. \end{aligned} \tag{7.1.0.25}$$

If $\lim\limits_{\epsilon\to 0}\frac{1}{\epsilon^2}H(F_\epsilon^{in}|M_\epsilon^{in}) \to 0$, then, under some assumptions that the remainder terms in the formal calculations vanish as $\epsilon \to 0$, in the later time $t > 0$,

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}H(F_\epsilon(t)_\epsilon|M_\epsilon(t)) \to 0\,, \quad \text{as} \quad \epsilon \to 0\,. \tag{7.1.0.26}$$

In our work, we extend Golse-Levermore-Saint-Raymond's result to general initial data. In this case, the fast acoustic waves permanently exist, their propagation is described by the averaged equation (7.1.0.22). To construct a local Maxwellian from the fluid models with general initial data, an appropriate choice is the averaged system (7.1.0.11) which includes both incompressible Navier-Stokes system

(7.1.0.20) and the orthogonal complement (7.1.0.22). We expect that this averaged system captures the long time behavior of the Boltzmann equation scaled in the Navier-Stokes scaling in the sense that if initially the number density is close to the local Maxwellian, then they are close in the later time. More precise statement is in the following formal theorem.

Theorem 19: **(Formal Weakly Compressible Navier-Stokes Approximation.)** *Let M_ϵ be a family of local Maxwellian constructed from solutions to the averaged equation* (7.1.0.11)*, i.e.,*

$$M_\epsilon = \mathcal{M}_{(1+\epsilon\rho_\epsilon, \epsilon u_\epsilon, 1+\epsilon\theta_\epsilon)} \,. \tag{7.1.0.27}$$

where $(\rho_\epsilon, u_\epsilon, \theta_\epsilon)$ are solutions to the averaged equation (7.1.0.11)*, with general initial data $(\rho^{in}, u^{in}, \theta^{in})$. If initially, the relative entropy*

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon^2} H(F_\epsilon^{in} | M_\epsilon^{in}) \to 0\,, \quad \text{as} \quad \epsilon \to 0\,. \tag{7.1.0.28}$$

Let $F_\epsilon(t,x,v)$ be a family of solutions to the scaled Boltzmann equation with initial data F^{in}, i.e.,

$$\begin{aligned} \epsilon\partial_t F_\epsilon + v\cdot\nabla_x F_\epsilon &= \frac{1}{\epsilon}\mathcal{B}(F_\epsilon, F_\epsilon)\,, \\ F_\epsilon(0,x,v) &= F^{in}(x,v)\,. \end{aligned} \tag{7.1.0.29}$$

Assume that all the remainders in the formal calculations vanish as $\epsilon \to 0$. Then, in the later time $t > 0$,

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon^2} H(F_\epsilon(t) | M_\epsilon(t)) \to 0\,, \quad \text{as} \quad \epsilon \to 0\,. \tag{7.1.0.30}$$

Remark: Strictly speaking, the above result is not a "theorem" before rigorous justification, and leaves passing to the limit of some terms as assumptions, (we will point them out when appear.) The difficulties are mainly due to the lack of regularity and compactness of DiPerna-Lions solutions to the Boltzmann equation. Another reason is that the averaged equation (7.1.0.11) is nonlinear and its projection on Null$\mathcal{A}$ is incompressible Navier-Stokes equation whose regularity is an outstanding open problem. The following proof shows that even under these assumptions, it is still nontrivial. Our calculations illustrates that the ideas of using method of relative entropy, constructing local Maxwellians from solutions to the averaged equation (7.1.0.11), are promising. Its rigorous justification will be our future work.

Proof of the theorem: First we introduce our notations:

$$\begin{aligned} H_\epsilon(t) &= \frac{1}{\epsilon^2} H(F_\epsilon(t)|M_\epsilon(t))\,, \\ \tilde{H}_\epsilon(t) &= \frac{1}{\epsilon^2} H(F_\epsilon(t)|M)\,. \end{aligned} \tag{7.1.0.31}$$

Then from the relative entropy identity (6.2.0.73),

$$\frac{d}{dt} H_\epsilon(t) = \frac{d}{dt}\tilde{H}_\epsilon(t) - \frac{1}{\epsilon^4}\frac{d}{dt}\iint_{\Omega\times\mathbb{R}^D}\left[F_\epsilon \ln\left(\frac{M_\epsilon}{M}\right) - M_\epsilon + M\right] dvdx\,. \tag{7.1.0.32}$$

From the entropy identity relative to the absolute Maxwellian M,

$$\frac{d}{dt}\tilde{H}_\epsilon(t) = -\frac{1}{4\epsilon^2} D(F_\epsilon)\,, \tag{7.1.0.33}$$

where $R(F)$ is the entropy dissipation rate defined as

$$D(F) = \iiiint_{\Omega\times\mathbb{S}^{D-1}\times\mathbb{R}^{2D}} (F_1'F' - F_1F)\ln\left(\frac{F_1'F'}{F_1F}\right) dv_1 f v d\omega dx\,. \tag{7.1.0.34}$$

As we calculated in the last chapter the expression of $\ln(\frac{M_\epsilon}{M})$, see Lemma (20),

$$\ln\left(\frac{M_\epsilon}{M}\right) = \epsilon\tilde{g}_\epsilon - \epsilon^2\tilde{h}_\epsilon\,, \tag{7.1.0.35}$$

where

$$\tilde{g}_\epsilon = \rho_\epsilon + u_\epsilon \cdot v + \theta_\epsilon(\tfrac{1}{2}|v|^2 - \tfrac{D}{2}) ,$$
$$\tilde{h}_\epsilon = \tfrac{1}{2}[\rho_\epsilon^2 + |u_\epsilon|^2 + \tfrac{D}{2}\theta_\epsilon^2] + \theta_\epsilon u_\epsilon \cdot v + \theta_\epsilon^2(\tfrac{1}{2}|v|^2 - \tfrac{D}{2}) + R_\epsilon^1 . \tag{7.1.0.36}$$

We turn to

$$\begin{aligned} &-\frac{1}{\epsilon^2}\frac{d}{dt}\iint\limits_{\Omega\times\mathbb{R}^D}\left[F_\epsilon \ln\left(\frac{M_\epsilon}{M}\right) - M_\epsilon + M\right] dvdx \\ &= -\frac{1}{\epsilon^2}\iint\limits_{\Omega\times\mathbb{R}^D}(\partial_t F_\epsilon)\ln\left(\frac{M_\epsilon}{M}\right) dvdx + \frac{1}{\epsilon^2}\int_\Omega \frac{d}{dt}\int_{\mathbb{R}^D} M_\epsilon \, dvdx \\ &-\frac{1}{\epsilon^2}\iint\limits_{\Omega\times\mathbb{R}^D} F_\epsilon \partial_t \ln\left(\frac{M_\epsilon}{M}\right) dvdx \\ &= \mathrm{I}_\epsilon + \mathrm{II}_\epsilon + \mathrm{III}_\epsilon . \end{aligned} \tag{7.1.0.37}$$

Simple calculation shows that

$$\mathrm{II}_\epsilon = \frac{1}{\epsilon}\frac{d}{dt}\int_\Omega \rho_\epsilon \, dx . \tag{7.1.0.38}$$

Let $F_\epsilon = M(1+\epsilon g_\epsilon)$, and define the fluid variables associated the fluctuation of the number density as before (5.4.0.33) :

$$\hat{\rho}_\epsilon = \langle g_\epsilon \rangle , \quad \hat{u}_\epsilon = \langle v g_\epsilon \rangle , \quad \hat{\theta}_\epsilon = \tfrac{2}{D}\left\langle \left(\tfrac{1}{2}|v|^2 - \tfrac{D}{2}\right) g_\epsilon \right\rangle . \tag{7.1.0.39}$$

we can obtain that

$$\begin{aligned} \mathrm{I}_\epsilon = &-\int_\Omega \left[(\partial_t \hat{\rho}_\epsilon)\rho_\epsilon + \partial_t \hat{u}_\epsilon \cdot u_\epsilon + \tfrac{D}{2}(\partial_t \hat{\theta}_\epsilon)\theta_\epsilon\right] dx \\ &+ \epsilon\int_\Omega \left[\partial_t \hat{\rho}_\epsilon \tfrac{1}{2}(\rho_\epsilon^2 + |u_\epsilon|^2 + \tfrac{D}{2}\theta_\epsilon^2) + \partial_t \hat{u}_\epsilon \cdot u_\epsilon \theta_\epsilon + \tfrac{D}{2}\partial_t \hat{\theta}_\epsilon \theta_\epsilon^2\right] dx \\ &+ R_\epsilon^2 . \end{aligned} \tag{7.1.0.40}$$

Now we turn to III_ϵ,

$$\begin{aligned} \mathrm{III}_\epsilon &= -\frac{1}{\epsilon^2}\int_\Omega \left\langle \partial_t \ln\left(\frac{M_\epsilon}{M}\right)\right\rangle dx \quad -\frac{1}{\epsilon}\int_\Omega \left\langle g_\epsilon \partial_t \ln\left(\frac{M_\epsilon}{M}\right)\right\rangle dx \\ &= \mathrm{III}_\epsilon^1 + \mathrm{III}_\epsilon^2 . \end{aligned} \tag{7.1.0.41}$$

Simple calculation yields that

$$\mathrm{III}_\epsilon^1 = \frac{1}{2}\frac{d}{dt}\int_\Omega \left[\rho_\epsilon^2 + |u_\epsilon|^2 + \tfrac{D}{2}\theta_\epsilon^2\right]\, dx - \frac{1}{\epsilon}\frac{d}{dt}\int_\Omega \rho_\epsilon \, dx + R_\epsilon^3 \,. \tag{7.1.0.42}$$

Then

$$\mathrm{II}_\epsilon + \mathrm{III}_\epsilon^1 = \frac{1}{2}\frac{d}{dt}\int_\Omega \left[\rho_\epsilon^2 + |u_\epsilon|^2 + \tfrac{D}{2}\theta_\epsilon^2\right]\, dx \,, \tag{7.1.0.43}$$

up to a small term R_ϵ^3 which could be ignored formally. Finally, III_ϵ^2 is

$$\begin{aligned}\mathrm{III}_\epsilon^2 = &-\int_\Omega \left[\hat\rho_\epsilon(\partial_t\rho_\epsilon) + \hat u_\epsilon\cdot\partial_t u_\epsilon + \tfrac{D}{2}\hat\theta_\epsilon\partial_t\theta_\epsilon\right]\, dx \\ &+\epsilon\int_\Omega \left[\hat\rho_\epsilon\tfrac{1}{2}\partial_t(\rho_\epsilon^2 + |u_\epsilon|^2 + \tfrac{D}{2}\theta_\epsilon^2) + \hat u_\epsilon\cdot\partial_t(u_\epsilon\theta_\epsilon) + \tfrac{D}{2}\hat\theta_\epsilon\partial_t(\theta_\epsilon^2)\right]\, dx \\ &+ R_\epsilon^4 \,.\end{aligned} \tag{7.1.0.44}$$

We use notations $\hat U_\epsilon = (\hat\rho_\epsilon, \hat u_\epsilon, \hat\theta_\epsilon)$ and $U_\epsilon = (\rho_\epsilon, u_\epsilon, \theta_\epsilon)$. Combining (7.1.0.40), (7.1.0.43) and (7.1.0.44), we obtain that

$$\begin{aligned}&-\frac{1}{\epsilon^2}\frac{d}{dt}\iint\limits_{\Omega\times\mathbb{R}^D}\left[F_\epsilon \ln\left(\frac{M_\epsilon}{M}\right) - M_\epsilon + M\right]\, dvdx \\ &= -\langle\partial_t\hat U_\epsilon \,, U_\epsilon\rangle - \langle\hat U_\epsilon \,, \partial_t U_\epsilon\rangle \\ &+ \frac{1}{2}\frac{d}{dt}\int_\Omega |U_\epsilon|^2\, dx + R_\epsilon \,,\end{aligned} \tag{7.1.0.45}$$

where R_ϵ is the summation of the remainder term in above calculations. In the formal calculations, R_ϵ could be ignored, while in the rigorous justification, it needs hard work to estimate.

Starting from the local conservation laws of the Boltzmann equation, we know "asymtotically", $\hat U_\epsilon$ obeys

$$\begin{aligned}&\partial_t\hat U_\epsilon + \frac{1}{\epsilon}\mathcal{A}\hat U_\epsilon + \mathcal{Q}(\hat U_\epsilon, \hat U_\epsilon) = \mathcal{D}\hat U_\epsilon + r_\epsilon \,, \\ &\hat U_\epsilon(0,x) = (\langle g_\epsilon^{in}\rangle, \langle v g_\epsilon^{in}\rangle, \langle(\tfrac{1}{D}|v|^2 - 1)g_\epsilon^{in}\rangle) \,.\end{aligned} \tag{7.1.0.46}$$

while the averaged equations satisfied by U_ϵ are

$$\partial_t U_\epsilon + \frac{1}{\epsilon}\mathcal{A}U_\epsilon + \overline{\mathcal{Q}}(U_\epsilon, U_\epsilon) = \overline{\mathcal{D}}U_\epsilon \,, \qquad U_\epsilon(0,x) = (\rho^{in}(x,)u^{in}(x,)\theta^{in}(x))\,. \tag{7.1.0.47}$$

Thus, the evolution of the relative entropy is

$$\frac{d}{dt}H_\epsilon(t) = -\frac{1}{\epsilon^4}\mathcal{D}(F_\epsilon) + \frac{1}{2}\frac{d}{dt}\langle U_\epsilon\,, U_\epsilon\rangle - \langle \partial_t \hat{U}_\epsilon, U_\epsilon\rangle - \langle \hat{U}_\epsilon, \partial_t U_\epsilon\rangle \tag{7.1.0.48}$$

In [8], Bardos-Golse-Levermore showed the following inequality

$$\frac{1}{\epsilon^2}D(F_\epsilon) + \langle \mathcal{D}\hat{U}_\epsilon, \hat{U}_\epsilon\rangle \geq 0\,. \tag{7.1.0.49}$$

Hence the evolution of the relative entropy inequality could be written as

$$\frac{d}{dt}H_\epsilon(t) + \frac{1}{4\epsilon^4}D(F_\epsilon) + \frac{1}{2}\frac{d}{dt}\langle \hat{U}_\epsilon, \hat{U}_\epsilon\rangle = \langle \partial_t \hat{U}_\epsilon, U_\epsilon\rangle + \langle \hat{U}_\epsilon, \partial_t U_\epsilon\rangle + \frac{1}{2}\frac{d}{dt}\langle U_\epsilon, U_\epsilon\rangle + \frac{1}{2}\frac{d}{dt}\langle \hat{U}_\epsilon, \hat{U}_\epsilon\rangle\,. \tag{7.1.0.50}$$

We applied S. Schochet's technique to $\hat{U}_\epsilon$ to eliminate the singular term. Define $\hat{V}_\epsilon = e^{\frac{t}{\epsilon}\mathcal{A}}\hat{U}_\epsilon$, then $\hat{V}_\epsilon$ satisfies the equations

$$\partial_t \hat{V}_\epsilon + e^{\frac{t}{\epsilon}\mathcal{A}}\mathcal{Q}(e^{-\frac{t}{\epsilon}\mathcal{A}}\hat{V}_\epsilon, e^{-\frac{t}{\epsilon}\mathcal{A}}\hat{V}_\epsilon) = e^{\frac{t}{\epsilon}\mathcal{A}}\mathcal{D}e^{-\frac{t}{\epsilon}\mathcal{A}}\hat{V}_\epsilon + \tilde{r}_\epsilon\,, \qquad \hat{V}_\epsilon(0) = \hat{U}^{in}_\epsilon\,. \tag{7.1.0.51}$$

As we did in the last chapter, the standard almost periodic function theory yields that

$$\lim_{\epsilon\to 0} e^{\frac{t}{\epsilon}\mathcal{A}}\mathcal{Q}(e^{-\frac{t}{\epsilon}\mathcal{A}}V, e^{-\frac{t}{\epsilon}\mathcal{A}}V) = \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau e^{s\mathcal{A}}\mathcal{Q}(e^{-s\mathcal{A}}V, e^{-s\mathcal{A}}V)\,ds = \overline{\mathcal{Q}}(V,V)\,. \tag{7.1.0.52}$$

and

$$
\begin{aligned}
\lim_{\epsilon\to 0} e^{\frac{t}{\epsilon}\mathcal{A}}\mathcal{D}(e^{-\frac{t}{\epsilon}\mathcal{A}}V) &= \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^{\tau} e^{s\mathcal{A}}\mathcal{D}(e^{-s\mathcal{A}}V) \\
&= \overline{\mathcal{D}}V\,.
\end{aligned}
\tag{7.1.0.53}
$$

$\hat{V}_\epsilon$ satisfies the non-singular equation (7.1.0.51), then in some appropriate sense, we have the limit

$$
\hat{V}_\epsilon \to \hat{V}\,, \tag{7.1.0.54}
$$

where $\hat{V}$ satisfies the averaged equation

$$
\begin{aligned}
\partial_t \hat{V} + \overline{\mathcal{Q}}(\hat{V},\hat{V}) &= \overline{\mathcal{D}}\hat{V}\,, \\
\hat{V}(0) &= \hat{U}^{in}_\epsilon\,.
\end{aligned}
\tag{7.1.0.55}
$$

We derived in Chapter 2 that $\hat{V}$ could be decompose onto Null($\mathcal{A}$) and Null$(\mathcal{A})^\perp$:

$$
\begin{aligned}
\hat{V} &= \Pi\hat{V} + \Pi^\perp\hat{V} \\
&= (\rho, u, \theta) + (\rho^\perp, u^\perp, \theta^\perp)\,.
\end{aligned}
\tag{7.1.0.56}
$$

Here, (ρ, u, θ) satisfies the incompressible Navier-Stokes system and Bousinesq relation, with initial data

$$
\Pi\hat{V}(0) = (\tfrac{2}{D+2}\hat{\rho}^{in} - \tfrac{D}{D+2}\hat{\theta}^{in}, P\hat{u}^{in}, -\tfrac{2}{D+2}\hat{\rho}^{in} + \tfrac{D}{D+2}\hat{\theta}^{in}) \tag{7.1.0.57}
$$

where $\Pi^\perp\hat{V}$ obeys the averaged equations on Null$(\mathcal{A})^\perp$,

$$
\begin{aligned}
\partial_t\Pi^\perp\hat{V} + \mathcal{Q}_1(\Pi\hat{V}, \Pi^\perp\hat{V}) + \mathcal{Q}_2(\Pi^\perp\hat{V}, \Pi^\perp\hat{V}) &= \tilde{\mu}\Delta_x\Pi^\perp\hat{V}\,, \\
\Pi^\perp\hat{V}(0) &= \Pi^\perp\hat{U}^{in}\,.
\end{aligned}
\tag{7.1.0.58}
$$

where the 2-waves and 3-waves resonant terms $\overline{\mathcal{Q}}_1$, $\overline{\mathcal{Q}}_2$ respectively and the averaged coefficient $\tilde{\mu}$ are defined in Chapter 3, see (3.6.2.6), (3.6.2.10), and (3.4.0.114). Hence $\hat{U}_\epsilon$ can be represented as

$$
\hat{U}_\epsilon = \Pi\hat{V} + e^{-\frac{t}{\epsilon}\mathcal{A}}\Pi^\perp\hat{V} + \hat{R}_\epsilon\,, \tag{7.1.0.59}
$$

where the remainder $\hat{R}_\epsilon$ comes from the limiting process $\hat{V}_\epsilon \to \hat{V}$. It vanishes as $\epsilon \to 0$. But the rigorous justification is so far not available because it involves the estimate of the so-called "small divisor". We will leave this problem in the future work.

Now let us turn to U_ϵ. From its definition, we know $U_\epsilon = e^{-\frac{t}{\epsilon}\mathcal{A}}V$ where $V(t,x)$ satisfies the same equation as $\hat{V}$ with initial data U^{in}. Similarly, we can decompose V orthogonally

$$\begin{aligned} V &= \Pi V + \Pi^\perp V \\ &= (\rho, u, \theta) + (\rho^\perp, u^\perp, \theta^\perp)\,, \end{aligned} \tag{7.1.0.60}$$

where ΠV and $\Pi^\perp V$ satisfy the same equations with $\Pi\hat{V}$ and $\Pi^\perp\hat{V}$, the only difference difference is the initial datum, the former are ΠU^{in} and $\Pi^\perp U^{in}$ respectively. So, similar to $\hat{U}_\epsilon$, U_ϵ could be represented as

$$U_\epsilon = \Pi V + e^{-\frac{t}{\epsilon}\mathcal{A}}\Pi^\perp V\,. \tag{7.1.0.61}$$

Now turn to the evolution of the relative entropy (7.1.0.50).

$$\begin{aligned} \langle \partial_t \hat{U}_\epsilon, U_\epsilon\rangle &= \langle \partial_t \Pi\hat{V}, \Pi V\rangle + \langle \partial_t \Pi^\perp \hat{V}, \Pi^\perp V\rangle - \frac{1}{\epsilon}\langle \mathcal{A}\Pi^\perp\hat{V}, \Pi^\perp V\rangle \\ &+ \langle \partial_t \hat{R}_\epsilon, \Pi V\rangle + \langle \partial_t \hat{R}_\epsilon, e^{-\frac{t}{\epsilon}\mathcal{A}}\Pi^\perp V\rangle\,. \end{aligned} \tag{7.1.0.62}$$

Here we used the property that the semigroup $e^{-\frac{t}{\epsilon}\mathcal{A}}$ preserve the norm, i.e.,

$$\langle e^{t\mathcal{A}}U, e^{t\mathcal{A}}V\rangle = \langle U, V\rangle\,, \tag{7.1.0.63}$$

because of the skew-symmetry of $\mathcal{A}$. Similarly

$$\begin{aligned} \langle \partial_t U_\epsilon, \hat{U}_\epsilon\rangle &= \langle \partial_t \Pi V, \Pi\hat{V}\rangle + \langle \partial_t \Pi^\perp V, \Pi^\perp \hat{V}\rangle - \frac{1}{\epsilon}\langle \mathcal{A}\Pi^\perp V, \Pi^\perp \hat{V}\rangle \\ &+ \langle \partial_t U_\epsilon, \hat{R}_\epsilon\rangle\,. \end{aligned} \tag{7.1.0.64}$$

Using the skew-symmetry of the linear operator $\mathcal{A}$, when we add (7.1.0.62) and (7.1.0.64) together, the "bad" $\frac{1}{\epsilon}$ terms cancel out. Thus,

$$\begin{aligned}&\langle \partial_t \hat{U}_\epsilon, U_\epsilon\rangle + \langle \hat{U}_\epsilon, \partial_t U_\epsilon\rangle \\ &= \langle \partial_t \Pi \hat{V}, \Pi V\rangle + \langle \partial_t \Pi V, \Pi \hat{V}\rangle \\ &+ \langle \partial_t \Pi^\perp \hat{V}, \Pi^\perp V\rangle + \langle \partial_t \Pi^\perp V, \Pi^\perp \hat{V}\rangle + \tilde{R}_\epsilon \,.\end{aligned} \tag{7.1.0.65}$$

Use the incompressible Navier-Stokes equations satisfied by $\Pi\hat{V}$ and ΠV, we obtain

$$\begin{aligned}&-\langle \partial_t \Pi \hat{V}, \Pi V\rangle - \langle \partial_t \Pi V, \Pi \hat{V}\rangle \\ &= 2\int_\Omega \tfrac{D+2}{2}\kappa \nabla_x \theta \cdot \nabla_x \hat{\theta}\, dx + 2\int_\Omega \mu \nabla_x u : \nabla_x \hat{u}\, dx \\ &+ \int_\Omega \tfrac{D+2}{2}[(\hat{u}\cdot\nabla_x\hat{\theta})\theta + (u\cdot\nabla_x\theta)\hat{\theta}]\, dx + \int_\Omega [(\hat{u}\cdot\nabla_x\hat{u})\cdot u + (u\cdot\nabla_x u)\cdot\hat{u}]\, dx\end{aligned} \tag{7.1.0.66}$$

Use the averaged equations of $\Pi^\perp\hat{V}$ and $\Pi^\perp V$,

$$\begin{aligned}&-\langle \partial_t \Pi^\perp \hat{V}, \Pi^\perp V\rangle - \langle \partial_t \Pi^\perp V, \Pi^\perp \hat{V}\rangle \\ &= 2\int_\Omega \tilde{\mu} \nabla_x \Pi^\perp \hat{V} : \nabla_x \Pi^\perp V\, dx \\ &+ \langle \overline{\mathcal{Q}}_1(\Pi\hat{V}, \Pi^\perp\hat{V},)\Pi^\perp V\rangle + \langle \overline{\mathcal{Q}}_1(\Pi V, \Pi^\perp V,)\Pi^\perp \hat{V}\rangle \\ &+ \langle \overline{\mathcal{Q}}_2(\Pi^\perp\hat{V}, \Pi^\perp\hat{V},)\Pi^\perp V\rangle + \langle \overline{\mathcal{Q}}_2(\Pi^\perp V, \Pi^\perp V,)\Pi^\perp \hat{V}\rangle \,.\end{aligned} \tag{7.1.0.67}$$

Note that

$$\begin{aligned}\langle U_\epsilon\,, U_\epsilon\rangle &= \langle \Pi V\,, \Pi V\rangle + \langle \Pi^\perp V\,, \Pi^\perp V\rangle\,, \\ \langle \hat{U}_\epsilon\,, \hat{U}_\epsilon\rangle &= \langle \Pi \hat{V}\,, \Pi \hat{V}\rangle + \langle \Pi^\perp \hat{V}\,, \Pi^\perp \hat{V}\rangle + R_\epsilon\,,\end{aligned} \tag{7.1.0.68}$$

Then use the formal energy identity of incompressible Navier-Stokes system and the averaged equations on $\mathrm{Null}(\mathcal{A})^\perp$, we have

$$\frac{1}{2}\frac{d}{dt}\langle U_\epsilon\,, U_\epsilon\rangle = -\int_\Omega \tfrac{D+2}{2}\kappa|\nabla_x\theta|^2\, dx - \int_\Omega \mu|\nabla_x u|^2\, dx - \int_\Omega \tilde{\mu}|\nabla_x \Pi^\perp V|^2\, dx\,. \tag{7.1.0.69}$$

and

$$\frac{1}{2}\frac{d}{dt}\langle \hat{U}_\epsilon\,, \hat{U}_\epsilon\rangle = -\int_\Omega \tfrac{D+2}{2}\kappa|\nabla_x\hat{\theta}|^2\,dx - \int_\Omega \mu|\nabla_x\hat{u}|^2\,dx - \int_\Omega \tilde{\mu}|\nabla_x\Pi^\perp\hat{V}|^2\,dx + R_\epsilon\,. \tag{7.1.0.70}$$

Furthermore

$$\begin{aligned}&\int_\Omega \tfrac{D+2}{2}[(\hat{u}\cdot\nabla_x\hat{\theta})\theta + (u\cdot\nabla_x\theta)\hat{\theta}]\,dx + \int_\Omega[(\hat{u}\cdot\nabla_x\hat{u})\cdot u + (u\cdot\nabla_x u)\cdot\hat{u}]\,dx\\ &= \int_\Omega \tfrac{D+2}{2}(u-\hat{u})\cdot\nabla_x\theta\hat{\theta}\,dx + \int_\Omega[(u-\hat{u})\cdot\nabla_x u]\cdot\hat{u}\,dx\,.\end{aligned} \tag{7.1.0.71}$$

Now we estimate the resonant terms. Applying the relations

$$\begin{aligned}\langle\overline{\mathcal{Q}}_1(\Pi U\,, \Pi^\perp U)\,, \Pi^\perp U\rangle &= 0\,,\\ \langle\overline{\mathcal{Q}}_2(\Pi^\perp U\,, \Pi^\perp U)\,, \Pi^\perp U\rangle &= 0\,.\end{aligned} \tag{7.1.0.72}$$

we can obtain that

$$\begin{aligned}&\langle\overline{\mathcal{Q}}_1(\Pi\hat{V}, \Pi^\perp\hat{V},)\Pi^\perp V\rangle + \langle\overline{\mathcal{Q}}_1(\Pi V, \Pi^\perp V,)\Pi^\perp\hat{V}\rangle\\ &= \langle\overline{\mathcal{Q}}_1(\Pi\hat{V}, \Pi^\perp\hat{V},)\Pi^\perp\hat{V}\rangle + \langle\overline{\mathcal{Q}}_1(\Pi\hat{V}, \Pi^\perp\hat{V},)\Pi^\perp V - \Pi^\perp\hat{V}\rangle\\ &+ \langle\overline{\mathcal{Q}}_1(\Pi V, \Pi^\perp V,)\Pi^\perp V\rangle + \langle\overline{\mathcal{Q}}_1(\Pi V, \Pi^\perp V,)\Pi^\perp\hat{V} - \Pi^\perp V\rangle\\ &= \langle\overline{\mathcal{Q}}_1(\Pi\hat{V}, \Pi^\perp\hat{V}) - \overline{\mathcal{Q}}_1(\Pi V, \Pi^\perp V)\,, \Pi^\perp V - \Pi^\perp\hat{V}\rangle\,.\end{aligned} \tag{7.1.0.73}$$

Similarly,

$$\begin{aligned}&\langle\overline{\mathcal{Q}}_2(\Pi^\perp\hat{V}, \Pi^\perp\hat{V},)\Pi^\perp V\rangle + \langle\overline{\mathcal{Q}}_2(\Pi^\perp V, \Pi^\perp V,)\Pi^\perp\hat{V}\rangle\\ &= \langle\overline{\mathcal{Q}}_2(\Pi^\perp\hat{V}, \Pi^\perp\hat{V}) - \overline{\mathcal{Q}}_2(\Pi^\perp V, \Pi^\perp V)\,, \Pi^\perp V - \Pi^\perp\hat{V}\rangle\,.\end{aligned} \tag{7.1.0.74}$$

To estimate (7.1.0.73) and (7.1.0.74), we use the notations of Chapter 3, see see (3.6.2.6) and (3.6.2.10). For $\Pi^\perp\hat{V} = \sum_{\alpha,\mathbf{k}} \hat{\mathbf{V}}^\alpha_\mathbf{k}\Phi^\alpha_\mathbf{k}(\mathbf{x})$ and $\Pi^\perp V = \sum_{\alpha,\mathbf{k}} \mathbf{V}^\alpha_\mathbf{k}\Phi^\alpha_\mathbf{k}(\mathbf{x})$, both are in $\mathrm{Null}(\mathcal{A})^{\perp}$,

$$\langle \overline{\mathcal{Q}}_1(\Pi\hat{V},\Pi^{\perp}\hat{V}) - \overline{\mathcal{Q}}_1(\Pi V,\Pi^{\perp}V)\,,\Pi^{\perp}V-\Pi^{\perp}\hat{V}\rangle$$

$$= \sum_{\delta,\mathbf{m}} \sum_{\substack{\mathbf{k}+\mathbf{l}=\mathbf{m}\\ \alpha sg(\mathbf{k})=\delta sg(\mathbf{m})\\ |\mathbf{k}|=|\mathbf{m}|}} \mathbf{V}^{\alpha}_{\mathbf{k}} \frac{[(\hat{\mathbf{V}}^{\alpha}_{\mathbf{k}}\hat{u}_{\mathbf{l}} - \mathbf{V}^{\alpha}_{\mathbf{k}}u_{\mathbf{l}})\cdot\mathbf{m}](\mathbf{k}\cdot\mathbf{m})}{|\mathbf{k}||\mathbf{m}|}(\mathbf{V}^{\delta}_{-\mathbf{m}} - \hat{\mathbf{V}}^{\delta}_{-\mathbf{m}})$$

$$+ \tfrac{\sqrt{\gamma(\gamma-1)}}{2}\sum_{\delta,\mathbf{m}} \sum_{\substack{\mathbf{k}+\mathbf{l}=\mathbf{m}\\ \alpha sg(\mathbf{k})=\delta sg(\mathbf{m})\\ |\mathbf{k}|=|\mathbf{m}|}} sg(\mathbf{k})[\hat{\mathbf{V}}^{\alpha}_{\mathbf{k}}\alpha\hat{\theta}_{\mathbf{l}} - \mathbf{V}^{\alpha}_{\mathbf{k}}\alpha\theta_{\mathbf{l}}]\frac{\mathbf{k}\cdot\mathbf{m}}{|\mathbf{k}|}(\mathbf{V}^{\delta}_{-\mathbf{m}} - \hat{\mathbf{V}}^{\delta}_{-\mathbf{m}})\,. \tag{7.1.0.75}$$

Then

$$|\langle \overline{\mathcal{Q}}_1(\Pi\hat{V},\Pi^{\perp}\hat{V}) - \overline{\mathcal{Q}}_1(\Pi V,\Pi^{\perp}V)\,,\Pi^{\perp}V-\Pi^{\perp}\hat{V}\rangle|$$

$$\leq (\|\nabla_x\Pi\hat{V}\cdot\Pi^{\perp}\hat{V} - \nabla_x\Pi V\cdot\Pi^{\perp}V\|_{L^2} + \|\Pi\hat{V}\cdot\nabla_x\Pi^{\perp}\hat{V} - \Pi V\cdot\nabla_x\Pi^{\perp}V\|_{L^2})$$

$$\cdot\,\|\Pi^{\perp}V - \Pi^{\perp}\hat{V}\|_{L^2}$$

$$= C_1(V,\hat{V})\|\Pi^{\perp}V - \Pi^{\perp}\hat{V}\|_{L^2}\,. \tag{7.1.0.76}$$

Similarly

$$\langle \overline{\mathcal{Q}}_2(\Pi^{\perp}\hat{V},\Pi^{\perp}\hat{V}) - \overline{\mathcal{Q}}_2(\Pi^{\perp}V,\Pi^{\perp}V)\,,\Pi^{\perp}V-\Pi^{\perp}\hat{V}\rangle$$

$$= \tfrac{\sqrt{c_v}}{2\sqrt{2}}\sum_{\delta,\mathbf{m}} \sum_{\substack{\mathbf{k}+\mathbf{l}=\mathbf{m}\,,\alpha=+,-\\ sg(\mathbf{k})|\mathbf{k}|+sg(\mathbf{l})|\mathbf{l}|=sg(\mathbf{m})|\mathbf{m}|}} (\hat{\mathbf{V}}^{\alpha}_{\mathbf{k}}\hat{\mathbf{V}}^{\alpha}_{\mathbf{l}} - \mathbf{V}^{\alpha}_{\mathbf{k}}\mathbf{V}^{\alpha}_{\mathbf{l}})\chi^{\alpha}_{\mathbf{klm}}(\mathbf{V}^{\alpha}_{-\mathbf{m}} - \hat{\mathbf{V}}^{\alpha}_{-\mathbf{m}})\,, \tag{7.1.0.77}$$

where

$$\chi^{\alpha}_{\mathbf{klm}} = \tfrac{\gamma-1}{2}\alpha sg(\mathbf{m})|\mathbf{m}|\,. \tag{7.1.0.78}$$

Then

$$
\begin{aligned}
&|\langle \overline{\mathcal{Q}}_2(\Pi^\perp \hat{V}, \Pi^\perp \hat{V}) - \overline{\mathcal{Q}}_2(\Pi^\perp V, \Pi^\perp V)\,, \Pi^\perp V - \Pi^\perp \hat{V}\rangle| \\
&\leq (\|\nabla_x \Pi^\perp \hat{V}\cdot \Pi^\perp \hat{V} - \nabla_x \Pi^\perp V \cdot \Pi^\perp V\|_{L^2} + \|\Pi^\perp \hat{V}\cdot \nabla_x \Pi^\perp \hat{V} - \Pi^\perp V \cdot \nabla_x \Pi^\perp V\|_{L^2}) \\
&\cdot \|\Pi^\perp V - \Pi^\perp \hat{V}\|_{L^2} \\
&= C_2(V, \hat{V})\|\Pi^\perp V - \Pi^\perp \hat{V}\|_{L^2}\,.
\end{aligned}
\tag{7.1.0.79}
$$

Combine all the above estimate together, and integrate over time, we obtain the evolution of the relative entropy

$$
\begin{aligned}
&H_\epsilon + \int_0^t \left[\tfrac{1}{4\epsilon^4} D(F_\epsilon)(s) + \tfrac{1}{2}\langle \hat{U}_\epsilon\,, \hat{U}_\epsilon\rangle\right]\, ds \\
&+ \int_0^t \int_\Omega \tfrac{D+2}{2}\kappa |\nabla_x(\hat{\theta} - \theta)|^2 + \mu |\nabla_x(\hat{u} - u)|^2\, dxds \\
&+ \int_0^t \int_\Omega \tilde{\mu} |\nabla_x(\Pi \hat{V} - \Pi V)|^2\, dxds \\
&\leq \int_0^t \left(C_1(V, \hat{V}) + C_2(V, \hat{V})\right) \|\Pi^\perp V - \Pi^\perp \hat{V}\|_{L^2} + H_\epsilon(0) + R_\epsilon\,.
\end{aligned}
\tag{7.1.0.80}
$$

where the remainder R_ϵ will formally vanish as $\epsilon \to 0$. Furthermore, note that the inequality proved in the last chapter, see also Bardos-Golse-Levermore [8],

$$
\int_0^t \left[\tfrac{1}{4\epsilon^4} D(F_\epsilon)(s) + \tfrac{1}{2}\langle \hat{U}_\epsilon\,, \hat{U}_\epsilon\rangle\right]\, ds \geq 0\,, \tag{7.1.0.81}
$$

and the relative entropy control

$$
\|\Pi^\perp V(t) - \Pi^\perp \hat{V}(t)\|_{L^2} \leq H_\epsilon(t)\,. \tag{7.1.0.82}
$$

Then, the Gronwell inequality yields that when initially the relative entropy $H_\epsilon \to 0$, then for the later time $t > 0$,

$$
\lim_{\epsilon \to 0} H_\epsilon(t) \to 0\,, \quad \text{as} \quad \epsilon \to 0\,. \tag{7.1.0.83}
$$

Thus, we finish the proof of the main theorem of this chapter, which generalize the formal theorem of Golse-Levermore-Saint-Raymond [31]. □

7.2 Future Work

In this final section, we briefly state some future directions we could work.

1. An immediate work we have to do is the rigorous justification of the formal results this chapter, which will generalize the Golse-Saint Raymond's work [34] and [35] to the general initial data. Recently, Levermore-Masmoudi [48] considered more general collision kernel, including the soft potential case. There some technical difficulties we have to overcome. First, the averaged equation under the Navier-Stokes scaling has much worse properties than the weakly compressible Stokes system. Because it includes the incompressible Navier-Stokes equations as the projection onto the slow mode Null($\mathcal{A}$), the only global in time solutions are Leray's solution. As we see in our proof of the weakly compressible Stokes approximation, the sole H^1 bound is not enough to control the remainder term in the expansion of the local Maxwellian $M_\epsilon(t)$; Secondly, even in the time interval in which the regular solutions to the incompressible Navier-Stokes equations exist, and the averaged equation has more regular solutions, the justification is not trivial, because we have to face a "small divisor problem" to control the remainders come out from the difference between the quadratic terms $\mathcal{Q}$ and $\overline{\mathcal{Q}}$, the diffusion term $\mathcal{D}$ and $\overline{\mathcal{D}}$. The nonlocal feature of the averaged operators $\overline{\mathcal{Q}}$ and $\overline{\mathcal{D}}$ make the problem even harder.

2. Another question is that we know the equation before averaging

$$\begin{aligned} \partial_t U_\epsilon + \mathcal{A} U_\epsilon + \mathcal{Q}(U_\epsilon, U_\epsilon) &= \mathcal{D} U_\epsilon \,, \\ U_\epsilon(0,x) &= U_\epsilon^{in}(x) \,. \end{aligned} \tag{7.2.0.84}$$

does not has global solutions. We can consider the following "partially averaged equation"

$$\begin{aligned} \partial_t U_\epsilon + \mathcal{A} U_\epsilon + \overline{\mathcal{Q}}(U_\epsilon, U_\epsilon) &= \mathcal{D} U_\epsilon \,, \\ U_\epsilon(0,x) &= U_\epsilon^{in}(x) \,. \end{aligned} \tag{7.2.0.85}$$

i.e., we take time averaging only on the quadratic term. The new system (7.2.0.85) is much easier than the full averaged system (7.1.0.11), but still capture the asymptotic properties of the full averaged system. Its projection onto the slow mode Null($\mathcal{A}$) is still the incompressible Navier-Stokes equations, but the "small divisor problem" in this new model will be easier. The first question to be answered is: what is global existence and regularity of (7.2.0.85)? Can we use this easier partially averaged system to construct family of local Maxwellians to capture the long time behavior of the solutions to the Boltzmann equation?

3. Another even more fundamental problem is that current research in the kinetic equations and their macroscopic limits treat case without boundary, which is not physically realistic in many applications where the boundary effect is not negligible. For example, if we want to investigate a physical situation where a gas flows past a solid body or is contained in a region bounded by one or many solid bodies, the Boltzmann equation must be accompanied by boundary conditions, which describe the interaction of gas molecules with solid walls. At the

macroscale, the fluid equations are complemented with boundary conditions, which can be derived from these of the kinetic equations by a hydrodynamic limit (for example, see the work of Masmoudi and Saint-Raymond [59].) Near the boundary, large gradient and the formation of boundary layers can be expected. Usually the physical scales in boundary layers are of multiscale nature, which are different with those of interior region. It has a multiscale nature. The regularity, stability theory of boundary layers and their nonlinear interaction with fast waves is far from being well understood. To investigate the boundary layers in the macroscopic limits of the Boltzmann equation with boundary conditions is a new challenging problem. It will be deeply related to the initial-boundary problem of hyperbolic system, which is another attractive field.

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