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Control and Stabilization of a Class of Nonlinear Systems with Symmetry

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Abstract

Title of Dissertation: Control and Stabilization of a Class of Nonlinear Systems with Symmetry

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The focus of this dissertation is to study issues related to controllability and stabilization of a class of underactuated mechanical systems with symmetry. In particular we look at systems whose configuration can be identified with a Lie group and the reduced equations are of the Lie-Poisson type. Examples of such systems include hovercraft, spacecraft and autonomous underwater vehicles. We present sufficient conditions for the controllability of affine nonlinear control systems where the drift vector field is a Lie-Poisson reduced Hamiltonian vector field. In this setting we show that depending on the existence of a radially unbounded Lyapunov type function, the drift vector field of the reduced system is weakly positively Poisson stable. The weak positive Poisson stability along with the Lie algebra rank condition is used to show controllability. These controllability results are then extended to the unreduced dynamics. Sufficient conditions for controllability are presented in both cases where the symmetry group is compact and noncompact.

We also present a constructive approach to design feedback laws to stabilize relative equilibria of these systems. The approach is based on the observation that, under certain hypotheses the fixed points of the Lie-Poisson dynamics belong to a locally immersed equilibrium submanifold. The existence of such equilibrium manifolds, along with the center manifold theory is used to design stabilizing feedback laws.

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by

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Dedication

To my parents

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Table of Contents

List of Figures

Control and Stabilization of a Class of
Nonlinear Systems with Symmetry

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This comment page is not part of the dissertation.

Chapter 1

Introduction

In the middle of the 19th century Sophius Lie made a far reaching discovery that techniques designed to solve particular unrelated types of ordinary differential equations (ODE's), such as separable, homogeneous and exact equations, were in fact all special cases of a general form of integration procedure based on the invariance of the differential equation under a continuous group of symmetries. Roughly speaking a symmetry group of a system of differential equations is a group that transforms solutions of the system to other solutions. Once the symmetry group has been identified a number of techniques to solve and classify these differential equations becomes possible. In the classical framework of Lie, these groups were local groups and arose locally as groups of transformations on some Euclidean space. The passage from the local Lie group to the present day definition using manifolds was accomplished by Cartan.

These continuous groups, which originally appeared as symmetry groups of differential equations, have over the years had a profound impact on diverse areas such as algebraic topology, differential geometry, numerical analysis, control theory, classical mechanics, quantum mechanics etc. They are now universally known as Lie groups.

One of the main foci of this dissertation is to model and study a class of controlled mechanical systems, whose configuration space can be identified with a finite dimensional Lie group, and whose dynamics can be modeled as Euler-Lagrange or Hamiltonian dynamics. Examples of such systems include hovercraft, spacecraft and underwater vehicles modeled as rigid bodies. To model the dynamics of these mechanical systems we adopt the Hamiltonian formulation. Rather than adopt the canonical Hamiltonian or Euler Lagrange formulation on a Euclidean space, in this dissertation, instead we adopt a more general differential geometric approach to mechanics. We use a non-canonical Hamiltonian formulation, modeling these systems on Poisson manifolds and using the associated Poisson structure to write down the Hamiltonian dynamics.

Hamiltonian mechanics and its relation to the concept of Poisson manifolds has its origins in the original work of Poisson, Hamilton, Liouville and others. The more general notion of a Poisson structure apparently first appears in Lie's theory of "function groups". It was later rediscovered many times under different names in the works of Lie, Dirac, Pauli, Martin, Sudarshan and Mukunda, Hermann, and others. A geometric approach to the study of mechanical systems has had a profound influence in the qualitative analysis of dynamics of mechanical systems. Playing an essential role in this are recent developments in reduction theory which draws its inspiration from Lie's original work (cf. [Marsden and Ratiu, 1994], for further details). Some of the reduction techniques developed include Lagrangian reduction [Marsden and Scheurle, 1993a; R.Yang *et al.*, 1993] which involves dropping the Euler-Lagrange equations to the quotient of the velocity phase space by the symmetry group. Hamiltonian reduction, on the other hand, involves projecting the Poisson bracket to the reduced (quotient) space which also inherits a Poisson structure. In particular, if the configuration space of the system can be identified with a Lie group G, a left invariant Hamiltonian on T^*G gives rise to reduced dynamics on T^*G/G, which is isomorphic to $\mathfrak{g}^*$ the dual of the Lie algebra of G. The Poisson structure on $\mathfrak{g}^*$ is attributed to Lie and Berezin-Kirillov-Kostant-Souriau. (See the work of Weinstein for historical remarks [Weinstein, 1983a]). Apparently Lie was also aware of the Poisson structure on the dual of a Lie algebra, but it was only recently that it became clear that this bracket is obtained by a simple reduction procedure i.e. it is induced from the canonical bracket on T^*G by passing to T^*G/G which is isomorphic to $\mathfrak{g}^*$. This bracket associated with the dual of the Lie algebra is now universally known the Lie-Poisson bracket.

Using this differential geometric approach to mechanics, in Chapter 2 we derive the reduced dynamics of hovercraft, spacecraft, underwater vehicles and surface vehicles. In each case we identify the configuration space with a Lie group, identify the symmetry group of the dynamics, and write down the reduced dynamics on the reduced phase space in terms of the Poisson structure associated with the reduced phase space. The examples discussed in Chapter 2 are of practical interest. For example the amphibious versatility of hovercraft has given them a role in specialized applications including search and rescue, emergency medical services, ice breaking, Arctic off-shore exploration, and recreational activities [Amyot, 1989]. Certain environmental aspects (such as ice-roughness, Arctic rubble fields etc.) also provide a niche for operations by hovercraft. Similarly a growing industry in underwater vehicles for deep sea explorations has lead to the demand for more versatile, robust and high performance autonomous vehicles that can cope with actuator failures, disturbances, exploit sensor based local navigation etc. The design and control of autonomous versions of these vehicles has also been of much recent interest.

Given a particular input to the actuator the approach adopted to write down the dynamics plays a crucial role in providing an insight into the controllability and stability properties of the dynamics of these systems. As we see in Chapter 4 and Chapter 5 the geometric approach to modeling these systems has many advantages over conventional approaches especially in providing insight into controllability and stability properties of these systems. The hovercraft is modeled as a planar rigid body subject to an external force. Its configuration space is identified with the Lie group $SE(2)$, and the reduced dynamics are written on $se(2)^*$. The configuration space of the underwater vehicle and surface vehicle is identified with the Lie group $SE(3)$. The underwater vehicle is modeled as a completely submerged body, in an inviscid, incompressible and irrotational fluid of infinite volume. The study of completely submerged bodies in ideal fluids has a long history dating back to the classic work of Kirchhoff, Lamb and Birkhoff [Lamb, 1945; Birkhoff, 1960]. More recently in [Leonard, 1995] the equations are derived in the geometric framework. We also study the motions of floating bodies (e.g. ships) in quiet water without the consideration of resistance forces. While ship motions arise very rarely in quiet water, there is a great practical value in their study since the characteristics of ships in agitated seas are governed by the characteristics of motion in quiet water. Unlike the case of the completely submerged vehicle, in the case of a tossing vessel as a result of the change in the shape of the submerged volume the force due to buoyancy changes its magnitude, and point of application. In each of the cases we identify the symmetry groups and write down the reduced dynamics on the reduced spaces.

The impact of Lie theory in control theory in the context of nonlinear control became prominent around the early 1970. The fundamental observation that *almost* all the information in the Lie group is contained in its Lie algebra, and questions about systems evolving on Lie groups could be reduced to their Lie algebras, is the cornerstone of the applications of Lie algebras and Lie groups to control theory. In the early 1970's Brockett, Jurdjevic, Sussmann and others exploited this observation and introduced the theory of Lie groups and their associated Lie algebras into the context of nonlinear control to express notions such as controllability, observability and realization theory for right-invariant systems. One of the most notable application of Lie-theoretic techniques in control theory has been in determining controllability of nonlinear systems. Results in this area have inspired many interesting approaches in the designing constructive control laws to steer and stabilize nonlinear control systems.

Some of the early work [Lee and Markus, 1976] (and references therein) on nonlinear controllability was based on linearization of nonlinear systems. It was observed that if the linearization of a nonlinear system at an equilibrium point is controllable, the the system itself is locally controllable. Later a differential geometric approach to the problem was adopted in which a control system was viewed as a family of vector fields. It was observed that (c.f. [Hermann, 1968; Hermann and Krener, 1977; Hermes, 1974; Krener, 1974; Sussmann and Jurdjevic, 1972; Lobry, 1970])) a lot of the interesting control theoretic information was contained in the Lie brackets of these vector fields. It was realized [Hermann and Krener, 1977; Krener, 1974] that Chow's theorem [Hermann, 1968] lead to the characterization of controllability for systems without drift. Chow's theorem provides a Lie algebra rank test, for controllability of nonlinear systems without drift, similar in spirit to that of Kalman's rank condition test for linear systems. In the setting of controlled mechanical systems while drift free dynamics arise when one writes down the kinematics, once dynamics are included the system is no longer drift-free. Chow's theorem can no longer be used to conclude controllability. Studying controllability of systems of general systems with drift is usually a hard problem. Important contributions in this direction have been due to Bonnard, Lobry, Crouch, Byrnes, Jurdjevic and Kupka [Jurdjevic and Kupka, 1981], and others. In [Crouch and Byrnes, 1986] sufficient conditions are given, in terms of a "group action", that a locally accessible system is also locally reachable. In [Lobry, 1974] sufficient conditions for the controllability of a conservative dynamical polysystem on a compact Riemannian manifold are presented. More recently this result was extended by [Lian *et al.*, 1994] to dynamical polysystems where the drift vector field was required to be weakly positively Poisson stable.

A main contribution of this dissertation is discussing controllability of under actuated mechanical systems with symmetry. In this Chapter 4 we present sufficient conditions (Theorem 4.2.2) for controllability of affine nonlinear control systems where the drift vector field is a Lie-Poisson reduced Hamiltonian vector field. We show that depending on the existence of a radially unbounded Lyapunov function, the drift vector field (of the reduced system) is weakly positively Poisson stable. The Weak Positive Poisson stability of the drift vector field along with the Lie algebra rank condition [Lian *et al.*, 1994] is used to show controllability of the reduced system.

Having shown controllability of the reduced dynamics we then present sufficient conditions for controllability of the unreduced dynamics depending on whether the symmetry group is compact or noncompact. In the setting where the symmetry group is compact we show that under assumptions of Theorem 4.2.2, we can conclude that the drift vector field on T^*G is also weakly Positively Poisson stable. This again enables us to conclude controllability on T^*G. In the setting where the symmetry group is noncompact, we show (Theorem 4.4.17) that we can conclude controllability of the unreduced dynamics. The proof relies on that of Theorem 4.2.2. and earlier work by Murray and Lewis [Lewis and Murray, 1996] on configuration controllability. Our result gives a manageable tool to check for controllability of a wide class of mechanical systems with symmetry. These results are then applied to the examples discussed in chapter 3, in each case drawing conclusions on the controllability of the dynamics. Some other results in this chapter are on small time local controllability of these systems.

In Chapter 5 we study stability and feedback stabilization of mechanical system with symmetry. We focus our attention of stability on fixed points of the reduced dynamics. These give rise to relative equilibria, i.e. trajectories that are group orbits in the unreduced phase space. While one can ascertain in a straightforward manner spectral stability of Hamiltonian systems, concluding nonlinear or Lyapunov stability is more difficult as the linearization of a stable Hamiltonian dynamics has eigenvalues on the imaginary axis. For canonical Hamiltonian systems the Lagrange Dirichlet criterion provides sufficient conditions for stability. This result was extended by Arnold [Arnold, 1969], as the method now known as the energy-Casimir method [Bloch *et al.*, 1992a; Bloch and Marsden, 1990; Krishnaprasad and Marsden, 87]. In Chapter 5 we study the stability of the fixed points of the reduced dynamics using the energy Casimir method. We identify the unstable relative equilibria for the example systems discussed in Chapter 3. Having identified the unstable equilibria the main focus of the rest of the chapter is in constructing linear dissipative feedback laws to stabilize the unstable equilibria. We present a general approach (Theorem 5.3.5), based on center manifold theory, to construct stabilizing feedback laws to stabilize relative equilibria of mechanical systems with symmetry. The approach is based on the observation that, under certain hypotheses, the fixed points of the Lie-Poison reduced dynamics can be shown to belong to a locally immersed equilibrium manifold. The existence of this equilibrium manifold is used to construct stable center manifolds. Some other results in this chapter include a discussion and some results on Hamiltonian feedback laws to stabilize relative equilibria of the example systems.

In Chapter 6 we summarize the contributions of this dissertation and present some future topics for research. We also discuss some conjectures on the existence of discontinuous feedback laws to stabilize the origin of the reduced dynamics.

Chapter 2

Preliminaries

In this chapter we review some basic definitions, notations and important theorems in differential geometry and geometric mechanics. Mathematical tools, concepts and results that will be used frequently in the following chapters are collected together in this chapter. As one of our main goals we outline the process of reduction of nonlinear control systems with symmetry. In particular we consider the case when the "free dynamics" are derived from a Hamiltonian that is invariant under the action of a Lie group G. The reduction procedure plays a key role in deriving the reduced dynamics, controllability results and constructive control laws for a large class of mechanical systems discussed in later chapters.

2.1 Differential Geometry and Geometric Mechanics

As mathematical tools and theorems from geometric mechanics and differential geometry will play an important role in the discussions that follow in the later chapters, in this section we introduce some relevant definitions and theorems. [Abraham and Marsden, 1977; Marsden and Ratiu, 1994; Marsden, 1992; Olver, 1993; Arnold, 1989; Crampin and Pirani, 1986; Nomizu, 1956] will serve as our main sources of reference.

2.1.1 Lie Groups and Group Actions

A *Lie group* G is a manifold G that has a group structure consistent with its manifold structure, i.e. the group operations : product and inverse, are differentiable maps. The maps $R_g : G \to G; h \mapsto hg$, and $L_g : G \to G; h \mapsto gh,\ g, h \in G$ are called the *right* and *left* translation maps.

A *Lie algebra* is a vector space V together with an operation $[\cdot, \cdot] : V \times V \to V$ called the Lie bracket for V, satisfying

(i) Bilinearity

$$[cv + c'v', w] = c[v, w] + c'[v, w], \qquad\qquad c, c' \in R$$

(ii) Skew-Symmetry

$$[v, w] = -[w, v]$$

(iii) Jacobi Identity

$$[u,[v,w]] + [w,[u,v]] + [v,[w,u]] = 0$$

$\forall\, u, v, v', w \in V$.

A vector field X on G is called *left invariant* if for every $g \in G$

$$L_g^* X = X \ \text{ i.e. } (T_h L_g) X(h) = X(gh), \quad \forall h \in G \tag{2.1}$$

where $T_h(\cdot)$ is derivative of the map L_g.

Given an element $\xi \in T_e G$, the tangent space at the identity of G, a left invariant vector field X_ξ on G is defined as $X_\xi(g) = T_e L_g(\xi)$. Defining the Lie bracket on $T_e G$ as

$$[\xi, \eta] := [X_\xi, X_\eta](e),$$

$T_e G$ forms a Lie algebra which is isomorphic to the set of left invariant vector fields on G. The vector space $T_e G$ with its Lie algebra structure is called the Lie algebra of G and is denoted by $\mathfrak{g}$. Its dual space is denoted by $\mathfrak{g}^*$.

Let M be a smooth manifold. A *left action* of a Lie group G on M is a smooth mapping $\Phi : G \times M \to M$ such that

(i) $\Phi(e, x) = x$

(ii) $\Phi(g, \Phi(h, x)) = \Phi(gh, x) \quad \forall g, h, \in G, x \in M$.

For every $g \in G$ let $\Phi_g : M \to M$ be given by $\Phi(g, x)$. At various times it will be useful to hold one variable fixed and consider the action Φ as a function of the remaining variable. Hence $\Phi_g : M \to M$ denotes the map $x \mapsto \Phi(g, x)$ and $\Phi_x : G \to M$ denotes the map $g \mapsto \Phi(g, x)$. In the special case where M is a

Banach space V and each $\Phi_g : V \to V$ is a continuous linear transformation, the action Φ of G is called a representation of G on V.

The *orbit* of $x \in M$ under the action Φ is defined by

$$\mathrm{Orb}(x) = \{\Phi_g(x) \mid g \in G\} \subset M.$$

In finite dimensions $\mathrm{Orb}(x)$ is an immersed submanifold of M. An action Φ of G on a manifold M defines an equivalence relation on M, by the relation of belonging to the same orbit. Let M/G (also called the *orbit space*) denotes the set of equivalence classes, $\pi : M \to M/G : x \mapsto \mathrm{Orb}(x)$; then the quotient topology on M/G is given by defining $U \subset M/G$ to be open if and only if $\pi^{-1}(U)$ is open.

An action $\Phi : G \times M \to M$ is said to be *free* if it has no fixed points, i.e. $\Phi_g(x) = x$ implies that $g = e$ or, equivalently, if for each $x \in M$, $g \mapsto \Phi_g(x)$ is one-to-one. An action $\Phi : G \times M \to M$ is *proper* if the mapping $\widetilde{\phi} : G \times M \to M \times M$, defined by $\widetilde{\Phi}(g, x) = (x, \Phi(g, x))$ is proper. (See also Section 4.4)

Remark 2.1.1 In finite dimensions properness means that if $K \subset M \times M$ is compact, then $\widetilde{\phi}^{-1}(K)$ is compact. In general, this means that if $\{x_n\}$ is a convergent sequence in M and if $\phi_{g_n} x_n$ converges in M, then $\{g_n\}$ has a convergent subsequence in G. If G is compact, properness is automatically satisfied.

Proposition 2.1.2 *If $\Phi : G \times M \to M$ is a proper and free action, then M/G is a smooth manifold and $\pi : M \to M/G$ is a smooth submersion.*

Proof: See [Abraham and Marsden, 1977] Proposition 4.1.23, page 266. ∎

Of particular interest to us are the adjoint and the coadjoint actions of G on $\mathfrak{g}$ and $\mathfrak{g}^*$ respectively, and the induced action of G on the tangent bundle TM and the cotangent bundle T^*M of M. The *adjoint action*, denoted by Ad, of G on its Lie algebra $\mathfrak{g}$ is given by

$$Ad : G \times \mathfrak{g} \to \mathfrak{g}, \quad Ad_g(\xi) = T_e(R_{g^{-1}} \circ L_g)\xi, \quad g \in G, \;\; \xi \in \mathfrak{g}. \tag{2.2}$$

Let $Ad_g^* : \mathfrak{g}^* \to \mathfrak{g}^*$ be the dual of Ad_g defined by

$$\langle Ad_g^* \alpha, \xi \rangle \; = < \alpha, Ad_g \xi >, \;\; \alpha \in \mathfrak{g}^*, \;\; \xi \in \mathfrak{g},$$

where $< \cdot, \cdot >$ denoted the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. The *coadjoint action* of G on $\mathfrak{g}^*$ is defined by

$$Ad^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*; \quad (g, \alpha) \mapsto Ad_{g^{-1}}^* \alpha. \tag{2.3}$$

The *tangent lift*, denoted by Φ^T, of the action of G on TM is given by

$$\Phi^T : G \times TM \to TM : \;\; (g, v_q) \mapsto T\Phi_g \cdot v_q \tag{2.4}$$

where $v_q \in T_qM$.

The *cotangent lift*, denoted by Φ^{T^*}, is given by

$$T^*\Phi : G \times T^*M : (g, \alpha_q) \mapsto T^*\Phi_{g^{-1}} \alpha_q, \tag{2.5}$$

where $\alpha_q \in T_q^*M$ and $T_q^*\Phi_{g^{-1}}$ is the dual of $T_q\Phi_{g^{-1}}$

Given an action $\Phi : G \times M \to M$, for each $\xi \in \mathfrak{g}$, the map $\Phi^\xi : \mathbb{R} \times M \to M$, defined by $\Phi^\xi(t, x) = \Phi(\exp t\xi, x)$, is an $\mathbb{R}$-action on M. The corresponding vector field on M given by

$$\xi_M(x) := \frac{d}{dt}\,|_{t=0}\; \Phi_{\exp t\xi}(x), \tag{2.6}$$

is called the *infinitesimal generator* of the action corresponding to ξ.

2.1.2 Hamiltonian Systems

Though, in this dissertation we mainly concern ourselves with Hamiltonian system on Poisson manifolds, understanding the symplectic foliations of Poisson manifolds plays a key role in some of our proofs in later chapters. Hence we start with a description of Hamiltonian systems on symplectic manifolds and then proceed to a description on Poisson manifolds.

A *symplectic manifold* is a pair (P, Ω) where P is an even-dimensional manifold and Ω is a closed non-degenerate two-form on P.

A vector field X on P is called a *Hamiltonian vector field*, if there exists a function $H : P \to \mathbb{R}$ called the *Hamiltonian*, such that

$$\mathbf{i}_X\Omega = \mathbf{d}H \;\Leftrightarrow\; \Omega_z(X(z), v) = dH(z).v \qquad z, v \in P, \tag{2.7}$$

where $\mathbf{i}_X$ is the interior product and $\mathbf{d}\Omega$ is the exterior derivative of Ω.(cf. [Marsden and Ratiu, 1994]). A Hamiltonian vector field is denoted by X_H. If such a function is defined on a neighborhood, we say X is locally Hamiltonian.

Given a manifold M the cotangent bundle T^*M has a natural symplectic structure. When M is the configuration space of a mechanical system, T^*M is called the *momentum phase space*. Choosing $(q^1, \ldots q^n)$ as local coordinates for M, and $(dq^1, \ldots dq^n)$ as a basis for T_q^*M, $\alpha \in T_q^*M$ can then be written as $\alpha = p_i dq^i$.

Hence $(q^1, \dots q^n, p_1, \dots p_n)$ are local coordinates on T^*M. With respect to these local coordinates the symplectic form

$$\Omega_0 = \sum_{i=1}^{n} dq^i \wedge dp_i,$$

defines a two form on T^*M and is called the *canonical symplectic form* on T^*M.

Let $\phi : M \to N$ be a C^∞ map from the manifold M to the manifold N, given a k-form α on N, the *pull back*, $\phi^*\alpha$, of α by ϕ is the k-form on M given by

$$(\phi^*\alpha)_q(v_1, \dots v_m) = \alpha_{\phi(q)}(T_q\phi \cdot v_1, \dots T_q\phi \cdot v_m), \tag{2.8}$$

where $v_1, \dots v_k \in T_qM$.

Given two symplectic manifolds (P_1, Ω_1) and (P_2, Ω_2), a C^∞-mapping $\phi : P_1 \to P_2$ is called *symplectic* or *canonical* if

$$\phi^*\Omega_2 = \Omega_1 \tag{2.9}$$

Proposition 2.1.3 *Let ϕ_t denote the flow of a vector field X. Then ϕ_t consists of symplectic transformations (i.e. for each t, $\phi_t^*\Omega = \Omega$) if and only if X is locally Hamiltonian.*

Proof: See [Marsden and Ratiu, 1994] Proposition 5.4.2, page 141. ■

An n-dimensional manifold M is said to be *orientable* if there is a nonvanishing n-form μ called a *volume form* defined on it. A $2n$-dimensional symplectic manifold is oriented by the *Liouville volume* Ξ which in local coordinates has the expression

$$\Xi = dq^1 \wedge \cdots \wedge dq^n \wedge dp_1 \wedge \cdots dp_n.$$

(P, Ξ) is called a *volume manifold* and the measure associated with Ξ is called the Liouville measure.

The *divergence* of a vector field X relative to a volume form μ, denoted by $\mathrm{div}_\mu(X)$, is given by

$$\mathcal{L}_X\mu = \mathrm{div}_\mu(X)\mu \quad \text{where } \mathcal{L}_X\mu = \frac{d}{dt}\phi_t^*\mu \mid_{t=0} .$$

The flow ϕ_t of X is said to be *volume-preserving* if

$$\mathrm{div}_\mu(X) = 0. \tag{2.10}$$

Hence it follows that

$$\mathrm{div}_\mu(X) = 0 \qquad \textit{iff} \qquad \phi_t^*\mu = \mu. \tag{2.11}$$

Proposition 2.1.4 *The flow ϕ_t of a Hamiltonian vector field X_H defined on a symplectic manifold (P, Ω) is volume preserving and is a local diffeomorphism.*

Proof: The proof follows from Proposition 2.1.3 and Equation 2.11. ■

We now consider Hamiltonian systems on Poisson manifolds. A *Poisson manifold* is a pair $(P, \{\cdot,\cdot\})$ where P is a smooth manifold and $\{\cdot,\cdot\} : \mathcal{C}^\infty(P) \times \mathcal{C}^\infty(P) \to \mathcal{C}^\infty(P)$ is a map called the *Poisson bracket* which satisfies

(i) Bilinearity

$$\{cF + c'F, G\} = c\{F, G\} + c'\{F, G\}, \qquad c, c' \in \mathbb{R}$$

(ii) Skew symmetry

$$\{F, G\} = -\{G, F\}$$

(iii) Jacobi Identity

$$\{\{F,G\},P\}+\{\{P,F\},G\}+\{\{G,P\},F\}=0$$

(iv) Leibniz Rule

$$\{F,G.P\}=\{F,G\}\cdot P+G\cdot\{F,P\}, \qquad\qquad G,F,P\in C^\infty(P),$$

where $\cdot$ denotes the ordinary multiplication of smooth real valued functions on P. Observe that $C^\infty(P)$ forms a Lie algebra under the Poisson bracket. A Poisson structure can be uniquely expressed through a contravariant skew-symmetric two-tensor Λ, called (cf. [Marsden and Ratiu, 1994]) the *Poisson tensor* such that

$$\{F,G\}(z)=\Lambda(z)(\mathbf{d}F(z),\mathbf{d}G(z)) \quad \forall z\in P. \tag{2.12}$$

Given a smooth function $H : P \to \mathbb{R}$ defined on a Poisson manifold P, the *Hamiltonian vector field* associated with H is a unique smooth vector field, denoted by X_H, satisfying

$$X_H(F)=\{F,H\},\ \text{for every } F\in C^\infty(P). \tag{2.13}$$

The equations governing the flow of X_H are referred to as the *Hamilton's equations* for the Hamiltonian function H. Defining the Poisson bracket on a symplectic manifold (P,Ω) as

$$\{F,G\}(z)=\Omega(X_F(z),X_G(z)) \quad z\in P \tag{2.14}$$

one observes that the definition (2.13) agrees with (2.7).

A smooth mapping $f : P_1 \to P_2$ between two Poisson manifolds $(P_1,\{\cdot,\cdot\}_1)$ and $(P_2,\{\cdot,\cdot\}_2)$ is called a *canonical* or *Poisson map* if

$$f^*\{F,G\}_2=\{f^*F,f^*G\}_1 \Leftrightarrow \{F,G\}_2\circ f=\{F\circ f,G\circ f\}_1. \tag{2.15}$$

As in the symplectic case, flows of Hamiltonian vector fields are Poisson maps, and hence preserve the Poisson structure. Further Poisson maps push Hamiltonian flows to Hamiltonian flows (cf. [Marsden and Ratiu, 1994] Prop. 10.5.2).

Theorem 2.1.5 *Let $f : P_1 \to P_2$ be a Poisson map. If ϕ_t is the flow of X_H and ψ_t is the flow of $X_{h \circ f}$, then $\phi_t \circ f = f \circ \psi_t$ and $Tf \circ X_{H \circ f} = X_H \circ f$*

In finite dimensions one can show that to compute the Poisson bracket of any pair of functions $F, G \in C^\infty(P)$ in some given local set of coordinates, it suffices to know the Poisson bracket between the coordinate functions themselves. Let $x = (x_1, \dots x_m)$ be local coordinates on P. Then

$$\{F, G\} = \sum_{i=1}^{m} \sum_{j=1}^{m} \{x_i, x_j\} \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j},$$

or

$$\{F, G\} = \nabla F \Lambda(x) \nabla G, \text{ where } \quad \Lambda_{ij} = \{x_i, x_j\}. \tag{2.16}$$

Λ is a skew symmetric matrix and is again referred to as the Poisson tensor. For example, on $\mathbb{R}^{2n}$, with coordinates $(q_1, \dots, q_n, p^1, \dots, p^n)$ (for a mechanical system p's would represent momenta and q's positions) the associated *canonical bracket* is given by

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p^i} - \frac{\partial F}{\partial p^i} \frac{\partial H}{\partial q_i} \qquad F, G \in C^\infty(\mathbb{R}^{2n}), \tag{2.17}$$

and the Poisson tensor Λ then takes the form

$$\Lambda = \begin{bmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{bmatrix} \text{ where } \mathrm{I} = n \times n \quad \text{identity matrix.}$$

Hence, given a Hamiltonian $H : P \to \mathbb{R}$ the associated system of Hamilton's equations take the form

$$\begin{aligned} \dot{q} &= \{q, H\} = \frac{\partial H}{\partial p}, \\ \dot{p} &= \{p, H\} = -\frac{\partial H}{\partial q}. \end{aligned}$$

One of the important examples of Poisson structures is the one on the dual of the Lie algebra of a Lie group G. Let us assume that G has dimension r. Let $\{\xi_1, \ldots, \xi_r\}$ and $\{\xi_1^\flat, \ldots, \xi_r^\flat\}$ be a basis for the Lie algebra $\mathfrak{g}$ and a dual basis for the dual space $\mathfrak{g}^*$ respectively, i.e. $< \xi_i^\flat, \xi_j >= \delta_j^i$. Any $\mu \in \mathfrak{g}^*$ can be expressed as $\mu = \sum_{i=1}^r \mu_i \xi_i^\flat$. The minus Lie-Poisson bracket of two differentiable functions $F, G \in C^\infty(\mathfrak{g}^*)$ is given by

$$\{F, G\}_-(\mu) = -\sum_{i,j,k=1}^{r} c_{ij}^k \mu_k \frac{\partial F}{\partial \mu_i} \frac{\partial G}{\partial \mu_j}, \tag{2.18}$$

where $c_{ij}^k, i, j, k = 1, \ldots, r$ are the structure constants of $\mathfrak{g}$ relative to the basis $\{\xi_1, \ldots, \xi_r\}$. Equivalently (2.18) can be written as

$$\{F, H\}_-(\mu) = \nabla F^T \Lambda(\mu) \nabla H \tag{2.19}$$

where

$$[\Lambda(\mu)]_{ij} = -\sum_{k=1}^{r} c_{ij}^k \mu_k. \tag{2.20}$$

The manifold $\mathfrak{g}^*$ together with its minus Lie Poisson bracket is a Poisson manifold and is denoted by $\mathfrak{g}^*_-$. (The manifold $\mathfrak{g}^*$ together with its plus Lie Poisson bracket is a Poisson manifold and is denoted by $\mathfrak{g}^*_+$.) The minus Lie-Poisson bracket can also be defined in its coordinate-free form as

$$\{F, G\}(\mu) = -\langle \mu, [\nabla F(\mu), \nabla G(\mu)] \rangle, \quad \mu \in \mathfrak{g}^* \tag{2.21}$$

where $[\cdot,\cdot]$ is the Lie bracket on $\mathfrak{g}$. Hence if $H : \mathfrak{g}^* \to \mathbb{R}$ is a Hamiltonian, then the Hamilton's equations associated to the minus Poisson bracket are

$$\dot{\mu} = \Lambda(\mu)\nabla H$$

where $\Lambda(\mu)$ is as defined in (2.20).

A function $C \in C^\infty(P)$ is called a *Casimir* function if $\{C, F\} = 0$ for all $F \in C^\infty(P)$. Hence C is a constant along the flow of all Hamiltonian vector fields $X_H \in P$.

In order to gain a complete understanding of the geometry underlying a general Poisson structure on a smooth manifold, we need to look more closely at the Poisson tensor Λ, which determines in local coordinates the Poisson bracket. The most important invariant of this tensor is its rank. If the rank of the Poisson tensor is maximal everywhere, then the manifold is symplectic and we are in the setting of Hamiltonian systems on symplectic manifolds. In the case of variable rank, the Poisson manifold is naturally foliated by symplectic submanifolds or symplectic leaves (see definition below) and any Hamiltonian system on M restricts to one of these leaves.

Let P be a Poisson manifold. Points z_1, z_2 are said to be on the same *symplectic leaf* of P if there is a piecewise smooth curve in P joining z_1, z_2, each segment of which is a trajectory of a locally defined Hamiltonian vector field. This is an equivalence relation and the equivalence class is called a symplectic leaf. The symplectic leaf containing the point z is denoted by Σ_z. The following theorem on symplectic stratification was proved in the finite-dimensional case by [Lie, 1890] and then by [Kirillov, 1976].

Theorem 2.1.6 (Symplectic Stratification Theorem) *Let P be a finite-dimensional Poisson manifold. Then P is the disjoint union of its symplectic leaves. Each symplectic leaf in P is an injectively immersed Poisson submanifold and the induced Poisson structure on the leaf is symplectic. The dimension of the leaf through z equals the rank of the Poisson structure at z.*

The induced symplectic foliation by the Lie-Poisson bracket on $\mathfrak{g}^*_-$ has a particularly nice interpretation in terms of the dual to the adjoint representation of the underlying Lie group G on the Lie algebra $\mathfrak{g}$. This is given by the following theorem, which appears to be due to Kirillov, Arnold, Kostant and Souriau, though similar ideas first appear in the work of Lie, Berezin and Weil. (See [Marsden and Ratiu, 1994] for historical comments and references.) The proof of the following theorem can be found in [Marsden and Ratiu, 1994; Olver, 1993]

Theorem 2.1.7 *Let G be a connected Lie group with coadjoint representation Ad^*G on $\mathfrak{g}^*_-$. Then the orbits of Ad^*G are immersed submanifolds of $\mathfrak{g}^*_-$ and are precisely the leaves of the symplectic foliation induced by the minus Lie-Poisson bracket on $\mathfrak{g}^*_-$. Moreover, for each $g \in G$, the coadjoint map Ad^*G is a Poisson mapping on $\mathfrak{g}^*$ preserving the leaves of the foliation.*

2.1.3 Symmetry and Reduction

Let G be a Lie group and $\Phi_g : M \to M$ denote the action of G on a manifold M. A function $f : M \to N$, where N is a manifold, is called a *G-invariant* function, and G is called the *symmetry group*, if for all $x \in M$ and all $g \in G$

$$f(\Phi_g \cdot x) = f(x).$$

G is a symmetry group of a system of differential equations $\mathcal{S}$, (Φ_g acting on an open subset M of independent and dependent variables of the system), if it has the property that whenever $z = h(x)$ is a solution of $\mathcal{S}$, then $\widetilde{z} = \Phi_g \cdot h(x)$ is also a solution of $\mathcal{S}$. In the setting of ODE's we have the following necessary and sufficient condition. Given a set of ODE's

$$\dot{x} = f(x), \qquad x = (x_1, \ldots, x_n) \in M \tag{2.22}$$

then G is a symmetry group of (2.22) or equivalently (2.22) is G invariant if and only if

$$T_x\Phi_g \cdot f(x) = f(\Phi_g \cdot x). \tag{2.23}$$

In this dissertation we are mainly concerned with dynamics evolving on the cotangent bundle, T^*W, of a Lie group W, and the invariance of Hamiltonian vector fields X_H defined on T^*W to some subgroup G of W. In many cases we will observe that $G = W$. In this setting, Φ_g corresponds to the cotangent lift of L_g, the left action of G on W. Hence we define a function $F : T^*W \to \mathbb{R}$ as *left-invariant* if for all $g \in G$,

$$F \circ T^*L_g = F. \tag{2.24}$$

Here the cotangent lift of L_g on T^*W is denoted by T^*L_g. Similarly a vector field X is *left invariant* if

$$T(T^*L_g) \circ X(x) = X(T^*L_g x), \qquad x \in T^*W. \tag{2.25}$$

Remark 2.1.8 One could have similarly defined right invariance of functions and vector fields with respect to the right action R_g. We will concern ourselves with only left actions and left invariance in this dissertation and depending on the context the reader should interpret G invariance to mean left invariance.

Lemma 2.1.9 *Let G be a subgroup of H. If $H : T^*W \to \mathbb{R}$ is left invariant, then the Hamiltonian vector field X_H is left (G) invariant.*

Proof: The cotangent lift of L_g on T^*W is always symplectic and therefore a Poisson map. Since $T^*L_g \circ H = H$ (by left invariance), substitute for $f = T^*L_g$ and $P_1 = P_2 = T^*W$ in Theorem 2.1.5 and the proof follows. ■

Given a G-invariant vector field X defined on a manifold M, if the action of G is free and proper, then there is an induced vector field $\widetilde{X}(\pi(x)) = T\pi(X(x))$ on the quotient manifold M/G such that

$$\phi_t^{\widetilde{X}}(\pi(x)) = \pi \circ \phi_t^X(x), \tag{2.26}$$

where $\pi : M \to M/G$ is the projection map and $\phi_t^X(\cdot)$ denotes the flow of the vector field X. While in the general setting solving for $\widetilde{X}$ can be quite complicated, in the setting of left-invariant Hamiltonian vector fields defined on Poisson manifolds the geometry can be exploited to solve for $\widetilde{X}$.

If the action $\Phi_g : P \to P$ of a Lie group G on a Poisson manifold $(P, \{\cdot, \cdot\})$ is free and proper and Φ_g is a Poisson map, then there exists a unique Poisson structure on P/G denoted by $\{\cdot, \cdot\}_{P/G}$ such that the projection $\pi : P \to P/G$ is a Poisson map (cf. [Marsden and Ratiu, 1994] (Prop. 10.7.1)). Hence if H

is G invariant Hamiltonian on P, it defines a corresponding function $\widetilde{H}$ on P/G such that $H = \widetilde{H} \circ \pi$. Under the above assumptions, π is a Poisson map and hence from Theorem 2.1.5 $T\pi \circ X_H = X_{\widetilde{H}} \circ \pi$. Hence a G-invariant Hamiltonian vector field X_H reduces to the Hamiltonian vector field $X_{\widetilde{H}}$ on P/G. Further $X_{\widetilde{H}}$ is Hamiltonian with respect to the reduced Hamiltonian $\widetilde{H}$ and the Poisson structure $\{\cdot,\cdot\}_{P/G}$.

In the special case where $P = T^*G$ and $P/G = T^*G/G \cong \mathfrak{g}^*$, the Lie Poisson reduction theorem (cf. [Marsden and Ratiu, 1994; Weinstein, 1983b; Krishnaprasad, 1993]) relates the canonical Poisson bracket on T^*G to the Lie-Poisson bracket on $\mathfrak{g}^*$. We only present the case of left invariance.

Theorem 2.1.10 (Lie-Poisson Reduction Theorem) *Identifying the set of functions on $\mathfrak{g}^*$ with the set of left invariant functions on T^*G endows $\mathfrak{g}^*$ with a Poisson structure given by*

$$\{F, G\}_-(\mu) = -\langle \mu, [\nabla F, \nabla H] \rangle \quad F, G \in C^\infty(T^*G), \mu \in \mathfrak{g}^*.$$

As mentioned earlier, the space $\mathfrak{g}^*$ with this Poisson structure is denoted by $\mathfrak{g}^*_-$. The Poisson map $\pi : T^*G \to \mathfrak{g}_-$ is given by

$$\alpha_g \mapsto T_e^* L_g \cdot \alpha_g, \qquad \alpha_g \in T^*G.$$

Hence the reduced dynamics with respect to coordinates $(\mu_1, \ldots \mu_r)$ defined on $\mathfrak{g}^*_-$ and the reduced Hamiltonian $\widetilde{H}$ are given by

$$\dot{\mu}_i = \{\mu, X_{\widetilde{H}}\}_- \qquad i = 1, \ldots m,$$

or equivalently as

$$\dot{\mu} = X_{\widetilde{H}}(\mu) = \Lambda(\mu)\nabla\widetilde{H}, \qquad \mu \in \mathfrak{g}^*.$$

Remark 2.1.11 The notation $\alpha_g \to T_e^* L_g \cdot \alpha_g$ needs some explanation as it is an abuse of notation. Let $\alpha_g \in T^*G$ have a local coordinate representation (g, p_g), i.e. $p_g \in T_g{}^*G$. Then $T_w^* L_g \cdot \alpha_g := T_w^* L_g \cdot p_g \quad w, g \in G$. Recall that $T_w^* L_g : T_{L_g \cdot w}^* G \to T_w^* G$. Hence $T_e^* L_g \cdot \alpha_g$ maps objects in the fiber of the cotangent bundle over g, to objects in $T_e^* G \cong \mathfrak{g}^*$. This notation is used in literature to avoid the mess of further notation involved in expressing everything in terms of local coordinates.

We now discuss semidirect products and reduction. We state, without proof, the semidirect product reduction theorem. The theorem shows how to reduce a Hamiltonian system on the cotangent bundle of a Lie group to a Hamiltonian system in the dual of the Lie algebra of a semidirect product. Before we state the theorem, we review some basic facts and notation about semidirect products.

Given a Lie group G, let ρ denote the *left* representation of G on a vector space V. Let $\rho_* : g \mapsto [\rho(g^{-1})]^*$, $g \in G$, denote the associated left representation of G on V^*. The right representation of G on V^* is given by $\rho^* : g \mapsto [\rho(g)]^*$. Let G_a denote the stabilizer of $a \in V^*$ under ρ^*, $\mathfrak{g}_a$ its Lie algebra, $S = G \times_\rho V$ the semidirect product, and $\mathfrak{s}$ its Lie algebra. Group multiplication in S is given by

$$(g_1, v_1) \cdot (g_2, v_2) = (g_1 g_2, v_1 + g_1 v_2), \quad g_1, g_2 \in G, v_1, v_2 \in V$$

where the action of g on v is denoted by gv. The Lie algebra $\mathfrak{s}$ of is the semidirect product of the Lie algebras, i.e. $\mathfrak{s} = \mathfrak{g} \times_\rho V$ and the Lie bracket in $\mathfrak{s}$ is defined by

$$[(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], \xi_1 v_2 - \xi_2 v_1), \qquad \xi_1, \xi_2 \in \mathfrak{g}, v_1, v_2 \in V.$$

In [Ratiu, 1980; 1981; 1982] it was shown that reducing T^*G by the left action of G_a, in the sense of [Marsden *et al.*, 1984] leads to coadjoint orbits for S. For historical remarks see [Marsden *et al.*, 1984] where the following theorem is proved.

Theorem 2.1.12 (Semidirect Product Reduction) *The reduction of T^*G by G_a at values $\mu_a = \mu|_{\mathfrak{g}_a}$ gives a space that is isomorphic to the coadjoint orbit through the point $\sigma = (\mu, a) \in \mathfrak{s}^* = \mathfrak{g}^* \times V^*$, the dual of the Lie algebra of S.*

Hence if $H_a : T^*G \to G$ is a left-invariant Hamiltonian under the action on T^*G of the stabilizer G_a, the family of Hamiltonians $\{H_a \mid a \in V^*\}$ induces a Hamiltonian function h on $\mathfrak{s}^*_- \cong T^*G/G_a$, defined by $h((T_eL_g)^*\alpha_g, \rho^*(g)a) = H_a(\alpha_g)$. Hence canonical Hamiltonian dynamics on T^*G, with respect to H_a project to Lie-Poisson dynamics on $\mathfrak{s}^*_-$, with respect to the Lie-Poisson structure defined on $\mathfrak{s}^*_-$ and the reduced Hamiltonian h.

2.2 Hamiltonian Control Systems with Symmetry

The main goal of this section is to define what we mean by a Hamiltonian control system with symmetry. The approach adopted here is in the same spirit as that of [van der Schaft, 1981; Grizzle and Marcus, 1985; de Alvarez, 1986] with some differences.

Definition 2.2.1 A *nonlinear control* system Σ is a 3-tuple (Σ, M, f) where the projection $\pi : B \to M$ (see Figure 2.1)is a smooth fiber bundle and f is a smooth

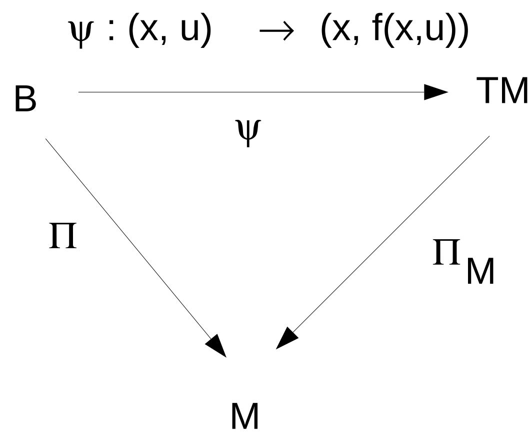

Figure 2.1: Nonlinear Control System

map such that Fig (2.1) commutes.

Here π_m (see Figure 2.1) is the natural projection of TM on M. M is to be interpreted as the state space and the fibers of B as the input spaces. If one chooses fiber-respecting coordinates (x, u) for B, then locally this definition reduces to $\psi : (x, u) \mapsto (x, \psi(x, u))$ i.e.

$$\dot{x} = \psi(x, u).$$

In the problems that we will be studying in this dissertation we make the following assumptions on M,B and ψ.

(A1) $M = T^*W$ is the cotangent bundle of a Lie group W.

(A2) B has a trivial bundle structure $M \times U$ and M is a Poisson manifold $(M, \{\cdot, \cdot\})$.

(A3) ψ is of the following specific form, $\psi : (x, u) \mapsto (x, f(x)+g(x, u)) \quad g(x, 0) = 0$ where $f(\cdot)$ is a Hamiltonian vector field with respect to a Hamiltonian $H : M \to \mathbb{R}$ and the Poisson structure defined on M.

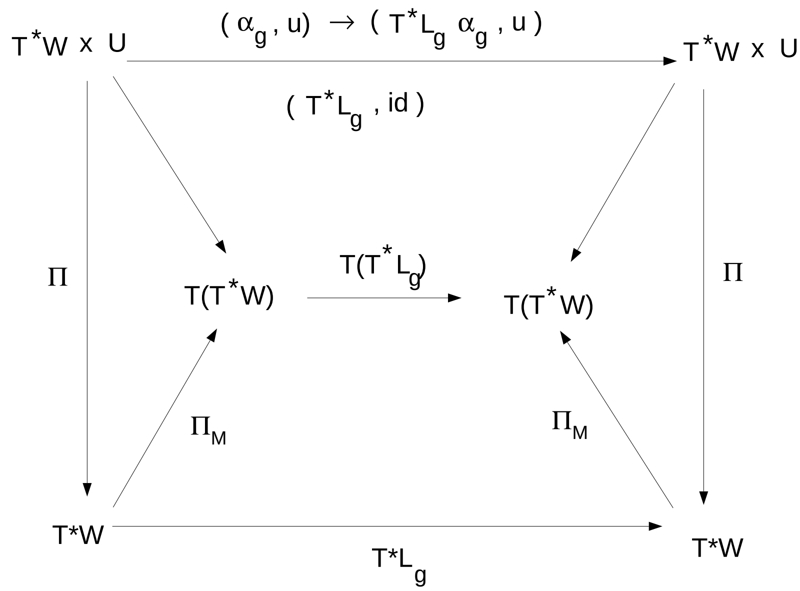

Figure 2.2: Hamiltonian control system with symmetry

Definition 2.2.2 A *nonlinear Hamiltonian control system* Σ_H is a nonlinear control system Σ satisfying (A1)–(A3).

Let G be a subgroup of W that acts on W via left actions and let $T^*L_G : T^*W \to T^*W$ denote its cotangent lift. Assume that the action T^*L_g is free and proper. Then Σ/Σ_H is G invariant or is said to have (G, Φ), $\Phi = T^*L_g$ symmetry if the Figure 2.2 commutes for all $g \in G$.

Let λ, denote the projection $\lambda : T^*W \to T^*W/G$. Then based on the discussions in Section 2.1.3 we have the following proposition :

Proposition 2.2.3 *If Σ_H has (G, Φ) symmetry then Σ_H projects to $\Sigma_{\widetilde{H}}(M/G \times U, M/G, \widetilde{\psi})$. If $X_H = \Sigma_{i=1}^n f_i(x)\frac{\partial}{\partial x_i}$ and $Y = \Sigma_{i=1}^n g_i(x,u)\frac{\partial}{\partial x_i}$ are G invariant vector fields and each of them projects to vector fields $X_{\widetilde{H}} = \lambda_*(X_H) = \Sigma_{i=1}^r \{\mu, \widetilde{H}\}_{M/G}\frac{\partial}{\partial x_i}$ and $\widetilde{Y} = \lambda_*(Y)$ defined on M/G. λ is the projection λ :*

$M \to M/G$ and $\widetilde{H}$ is the reduced Hamiltonian s.t $H = \widetilde{H} \circ \pi$.

In the setting where $M = T^*G$ we know that the projection $\lambda : T^*G \to T^*G/G \cong \mathfrak{g}^*_-$ is given explicitly by $\lambda(\alpha_g) = T^*L_g \cdot \alpha_g$ and hence solving for $\widetilde{Y}$ is not too difficult.

Remark 2.2.4 In the work of [van der Schaft, 1981; de Alvarez, 1986] it was assumed that the vector field Y was of the form $\Sigma_{j=1}^{p} X_{H_j} u_j$, where each $X_{H_j}, \;\; j = 1, \ldots p$. was required to be Hamiltonian with respect to some function $H_j : M \to \mathbb{R}$ and the canonical Poisson bracket on M. As we shall see, it is not always possible for the control vector field Y to have this form.

2.2.1 Reconstruction of Dynamics

The reduced system induced on M/G represents in a sense, the "essential dynamics". The explicitly known dynamics have been factored out in the reduction process. If we know a solution of the reduced nonlinear system $\Sigma_{\widetilde{H}}$ we would like to reconstruct the solutions of the unreduced nonlinear system Σ_H. For the case $u = 0$ this procedure is outlined in [Abraham and Marsden, 1977]. This technique is adopted in [Grizzle and Marcus, 1985] to reconstruct trajectories for the more general case with inputs, and the further assumption that $\lambda : M \to M/G$ admits a smooth cross section. The *reconstruction of trajectories* is outlined below.

Let $x_0 \in M, \;\; u(\cdot)$ be a continuous input, $x(\cdot)$ the integral curve of Σ_H corresponding to $u(\cdot)$ and $\mu(\cdot) = \lambda(x(\cdot)))$, the corresponding integral curve of $\Sigma_{\widetilde{H}}$

having $\mu(0) = \lambda(x(0))$. Assume that $\lambda : M \to M/G$ admits a cross section σ. Define a differentiable curve $d(t)$ in M by $d(t) = \sigma(\mu(t))$. Since $\lambda(x(t)) = \lambda(d(t))$ and Φ is free and proper, one can write $x(t) = \Phi_{g(t)}(d(t))$ for a uniquely defined curve $g(t) \in G$. We now try to find $g(t)$. Now

$$\dot{x} = \frac{d}{dt}\Phi(g(t), d(t)) = T_{d(t)}\Phi_{g(t)}\dot{d}(t) + T_{g(t)}\Phi_{d(t)}\dot{g}(t). \tag{2.27}$$

Note that $\dot{g}(t) \in T_{g(t)}G$ is a left-invariant vector field. But for any left-invariant vector field ξ_g we have $\xi_g = T_e L_g \xi, \;\; \xi \in \mathfrak{g}$. Thus for $m \in M$

$$T_g\Phi_m(\xi_g) \;\; = \;\; T_g\Phi_m \circ T_e L_g(\xi) = T_e(\Phi_m \circ L_g)\xi \tag{2.28}$$

$$= \;\; T_e(\Phi_g \circ \Phi_m)(\xi) = T_m\Phi_g \circ T_e\Phi_m(\xi) \tag{2.29}$$

but

$$(T_e\Phi_m)\xi = \frac{d}{dt}\Phi(\exp t\xi, m) \mid_{t=0} = \xi_M(m) \tag{2.30}$$

is the infinitesimal generator (c.f. 2.6) for Φ corresponding to ξ. Hence

$$\Phi_m(\xi_g) = T_m\Phi_g(T_e L_{g^{-1}}\xi_g)_M(m) \tag{2.31}$$

Substituting (2.31) in (2.27)

$$f(x(t), u(t)) = T_{d(t)}\Phi_{g(t)}\dot{d}(t) + T_{d(t)}\Phi_{g(t)}(T_{g(t)}L_{g^{-1}}\dot{g}(t))_M(d(t)). \tag{2.32}$$

Since Σ_h has (G, Φ) symmetry

$$T_m\Phi_g f(m, u) = f(\Phi_{g(m)}, u). \tag{2.33}$$

Hence we have

$$T_{d(t)}\Phi_{g(t)}f(d(t), u(t)) = T_{d(t)}\Phi_{g(t)}\dot{d}(t) + T_{d(t)}\Phi_{g(t)}(T_e L_{g^{-1}}\dot{g}(t))_M(d(t)).$$

Since $\Phi_g : M \to M$ is a diffeomorphism for all g, $T_{d(t)}\Phi_{g(t)}$ is nonsingular. Hence,

$$f(d(t), u(t)) = \dot{d}(t) + (T_{g(t)}L_{g^{-1}}\dot{g}(t))_M(d(t)). = \dot{d}(t) + \xi_M(d(t))$$

where $\xi_M = (T_{g(t)}L_{g^{-1}}\dot{g}(t))_M(d(t))$. Thus from (2.31)

$$T_e\Phi(d(t)(\xi(t)) = \xi_M(d(t)) = f(d(t), u(t)) - \dot{d}(t). \tag{2.34}$$

Φ being free and proper implies that $\Phi_m : G \to M$ is a diffeomorphism onto is range, and hence (2.34) can be uniquely solved for $\xi(t)$ to give

$$\xi(t) = (T_e\widetilde{\Phi}_{d(t)})^{-1}\xi_M(d(t)) \tag{2.35}$$

or

$$T_g(t)L_{g^{-1}(t)}\dot{g}(t) = (T_e\widetilde{\Phi}_{d(t)})^{-1}\xi_M(d(t)) \tag{2.36}$$

Since L_g is a diffeomorphism for all g and $d(t) = \sigma(y(t))$ we have

$$\dot{g}(t) = (T_eL_{g(t)})(T_e\widetilde{\Phi}_{d(t)}[f(\sigma(y(t)), u(t)) - (T_{y(t)}\sigma)\widetilde{f}(y(t), u(t))]. \tag{2.37}$$

Hence to reconstruct trajectories one solves the algebraic problem (2.35) for $\xi(t) \in \mathfrak{g}$ and then solves (2.37) for $g(t)$. The desired solution $x(t)$ then is

$$x(t) = \Phi_{g(t)}d(t).$$

We end this chapter with some definitions and theorems on controllability of nonlinear systems.

2.3 Accessibility and Controllability

The problem of local and global controllability of nonlinear systems has had a long history. Some of the early work [Lee and Markus, 1976] (and references therein) on nonlinear controllability, is on the theorem that states that if the linearization of a nonlinear system at an equilibrium point x_e is controllable the system itself is locally controllable. More recently a differential geometric approach to the problem was adopted in which a control system was viewed as a family of vector fields. It was observed that (cf. [Hermann, 1968; Hermann and Krener, 1977; Hermes, 1974; Krener, 1974; Sussmann and Jurdjevic, 1972; Lobry, 1970])) a considerable amount of interesting control theoretic information was contained in the Lie brackets of these vector fields. It was realized [Hermann and Krener, 1977; Krener, 1974] that Chow's theorem [Hermann, 1968] led to the characterization of controllability for systems with "symmetry" [1], (systems such that every trajectory run backwards in time is also a trajectory). In this section we introduce some definitions and related theorems on accessibility and controllability. [Nijmeijer and van der Schaft, 1990] will serve as our main source of reference. Further discussions on small time local controllability and controllability of systems with drift can be found in Chapter 3

Consider an affine nonlinear control system given by

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \tag{2.38}$$

where $x = (x_0, \ldots, x_n)$ are local coordinates for a smooth manifold M and $u = (u_1, \ldots u_m) \in U \subset \mathbb{R}^m$. **It is assumed that -**

(i) The input space U is such that the set of associated vector fields of (2.38) $\mathcal{F} = \{f(x) + \sum_{i=1}^{m} g_i(x)u_i \mid (u_1, \ldots u_m) \in U\}$.

(ii) The set of admissible controls consists of piecewise constant functions which

[1] "he .sage wf ehe word "symmetry" an ehis setting should uot be confused with ehat wf 2.23P

are piecewise continuous from the right.

The set $\mathcal{F}$ defines a dynamical polysystem [Jurdjevic and Kupka, 1981; Lobry, 1974]. A *polysystem* on a manifold M is simply a collection of vector fields on M. An integral curve of $\mathcal{F}$ corresponds to a trajectory of (2.38) with piecewise constant inputs.

Definition 2.3.1 The accessibility algebra $\mathcal{L}$ is the smallest subalgebra of $V^\infty(M)$ (the Lie algebra of vector fields on M) that contains $\mathcal{F}$.

Hence the smallest Lie algebra that contains $\mathcal{F}$ is the one generated by $f, g_1, \ldots, g_m$.

The *accessibility distribution* L is the distribution generated by the accessibility algebra $\mathcal{L}$, i.e.,

$$L(x) = \text{span}\{X(x) | X \text{ vector field in } \mathcal{L}\} \quad , x \in M$$

Definition 2.3.2 The system is said to satisfy the *accessibility Lie algebra rank condition (LARC)* if

$$L(x) = T_x M \; \forall x \in M \tag{2.39}$$

Let $x(t, 0, x_0, u)$ denote the solution of (2.38) at time $t \geq 0$ for a particular input function $u(\cdot)$ and initial condition $x(0) = x_0$.

Let $R^V(\mathcal{F}, x_0, T)$ denote the reachable set, defined as

$$\begin{aligned} R^V(\mathcal{F}, x_0, T) \;\; = \;\; & \{x \in M \mid \text{ there exists an admissible input } u : [0, T] \to U \\ & \text{such that } x(t, 0, x_0, u) \in V, 0 \leq t \leq T \text{ and } x(T) = x\} \end{aligned}$$

Let

$$R^V(\mathcal{F}, x_0, \leq T) = \bigcup_{\tau \leq T} R^V(\mathcal{F}, x_0, \tau) \quad \text{and}$$

$$R(\mathcal{F}, x_0) = \bigcup_{0 \leq T \leq \infty} R^M(\mathcal{F}, x_0, \leq T).$$

Definition 2.3.3 The system (2.38) is *locally accessible* from x_0 if $R^V(\mathcal{F}, x_0, \leq T)$ contains a non-empty open set of M for all neighborhoods V of x_0 and all $T > 0$. If this holds for any $x_0 \in M$, the the system is called locally accessible.

Theorem 2.3.4 *[Lobry, 1970; Sussmann and Jurdjevic, 1972] The system (2.38) is locally accessible iff* $dim L(x) = n \ \ \forall x \in M$

Definition 2.3.5 The system (2.38) is said to be *locally strongly accessible* from x_0 if for any neighborhood V of x_0 the set $R^V(\mathcal{F}, x_0, T)$ contains a non-empty open set for any $T > 0$ sufficiently small.

Let $\mathcal{L}_0$ be the smallest Lie subalgebra which contains $g_1, \ldots, g_m$ and satisfies $[f, X] \in \mathcal{L}_0, \forall X \in \mathcal{L}_0$ and $L_0(x) = \text{span}\{X(x) \mid X \text{ vector field in } \mathcal{L}_0\} \ \ , x \in M$.

Definition 2.3.6 The system is said to satisfy the *strong accessibility Lie algebra rank condition* if

$$L_0(x) = T_x M, \ \forall x \in M \tag{2.40}$$

Theorem 2.3.7 *If* $dim L_0(x_0) = n$*, then the system (2.38) is locally strongly accessible from* x_0*.*

The system (2.38) is called *controllable* if for any two points x_1, x_2 in M there exists a finite time T and an admissible function $u : [0, T] \to U$ such that $x(t, 0, x_1, u) = x_2$.

In terms of reachable sets the controllability definitions can be stated as follows.

Definition 2.3.8 The dynamical polysystem $\mathcal{F}$ is controllable if $R(\mathcal{F}, x_0) = M$

Definition 2.3.9 A dynamical polysystem is said to be *symmetric* if for every $X \in \mathcal{F}$, $-X \in \mathcal{F}$.

For systems without drift, i.e. $f = 0$, or equivalently a symmetric polysystem LARC implies controllability.

Theorem 2.3.10 *(Chow, c.f. [Hermann, 1968]) The nonlinear system*

$$\dot{x} = \sum_{i=1}^{m} g_i(x) u_i, \quad u = (u_1, \ldots, u_m) \subset U \tag{2.41}$$

is controllable if the accessibility LARC is satisfied.

Chapter 3

Left-Invariant Hamiltonian Systems: Examples, Dynamics and Reduction

The configuration space of a large class of mechanical systems can be identified with Lie groups G. Often the dynamics of such systems are G-invariant and hence they can be reduced to obtain a set of reduced dynamics on T^*G/G. Examples of such systems include hovercraft, spacecraft and underwater vehicles modeled as rigid bodies. See also [Bloch *et al.*, 1992a; Krishnaprasad and Tsakiris, 1994; 1995; Krishnaprasad, 1995; Wang, 1990] for some more examples. The design and control of autonomous versions of these vehicles has been of much recent interest. For example the amphibious versatility of hovercraft has given them a role in specialized applications including search and rescue, emergency medical services, ice breaking, Arctic off-shore exploration, and recreational activities [Amyot, 1989]. Certain environmental aspects (such as ice-roughness, Arctic rubble fields etc.) also provide a niche for operations by hovercraft. Similarly a growing industry in underwater vehicles for deep sea explorations has led to the demand for more versatile, robust and high performance autonomous vehicles that can cope with actuator failures, disturbances, exploit sensor based local navigation etc. In this chapter we discuss the modeling and reduction of the dynamics of hovercraft, spacecraft and underwater vehicles subject to external forces.

3.1 Hovercaft: Planar Rigid Body with a Vectored Thrust

A configuration of the system is shown in Figure 3.1. Let $\{e_1^r, e_2^r, e_3^r\}$ be an inertial frame of reference fixed at **O** and $\{e_1^b, e_2^b, e_3^b\}$ be a body frame fixed on the rigid body $\mathcal{B}$ at its center of mass. Since the rigid body is restricted to move in the $e_1^r e_2^r$ plane e_3^r is parallel to e_3^b. A typical material point q^b in the rigid body is then represented in the inertial frame as $q^r = Rq^b + r$ where R is an element of $SO(2)$, the special orthogonal group of 2×2 matrices and $r = (x, y)$ is a vector from **O**, the origin of the inertial coordinate system, to the center of mass of $\mathcal{B}$. Hence at any instant, the configuration $X(t)$ of $\mathcal{B}$ can be uniquely identified by the pair (R, r) or equivalently as an element of $SE(2)$, the Special Euclidean group of 3×3 matrices. Recall

$$SE(2) \triangleq \{ \begin{pmatrix} R & r \\ 0 & 1 \end{pmatrix} \mid R \in SO(2), r \in \mathbb{R}^2 \}$$

Let us assume that the thruster is mounted at the point C defined by the vector d^b in body coordinates and d^r in the inertial frame of reference. The thrusters exert a force f^r in inertial coordinates such that the line of action of the force

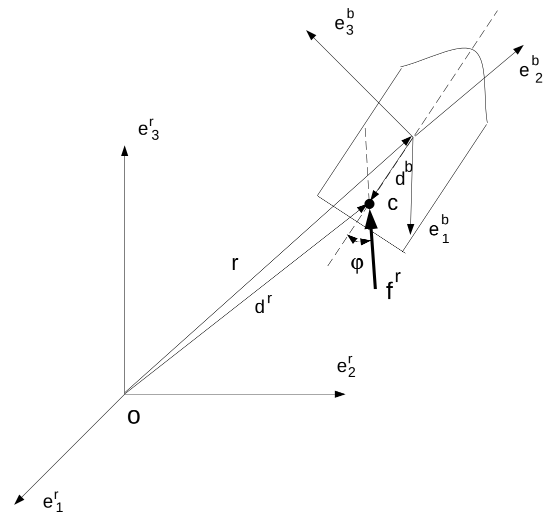

Figure 3.1: Planar rigid body with thruster

passes through C and makes an angle ϕ with the vector d^b. We now derive the equations of motion of a rigid body subject to a force f^r along a specified line of action.

Let $\Omega = \dot{\theta}$ denote the angular velocity and $v = (v_1, v_2)$ denote the linear components of the translational velocity along the body fixed frame. The kinematics are defined by

$$\dot{g} = g\xi,$$

or

$$\begin{bmatrix} \dot{R} & \dot{r} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R & r \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \widehat{\Omega} & v \\ 0 & 1 \end{bmatrix}$$

where $g \in SE(2)$ and ξ is a curve in $se(2)$, the Lie algebra of $SE(2)$ and

$$\widehat{\Omega} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Omega.$$

Equivalently the kinematics could be written as

$$\dot{R} = R\widehat{\Omega} \tag{3.1-a}$$

$$\dot{r} = Rv \tag{3.1-b}$$

3.1.1 Newton-Euler Description of Dynamics

Let $p = m\dot{r}$ denote the linear momentum. Then from Newton-Euler balance laws we know that

$$\dot{p} = f^r \tag{3.2}$$

Let $\Pi = I\Omega$ denote the scalar angular momentum (about an axis through the center of mass and perpendicular to the lamina). Here I is the moment of inertia about this axis. Then

$$\dot{\Pi} = d^b \times F \tag{3.3}$$

$$= d^b \|F\| \sin\phi \tag{3.4}$$

where $\|F\|$ denotes the magnitude of the force. We now express (3.2) in convected or body variables. Define

$$P = R^T p$$

then

$$\begin{aligned}\dot{P} &= \dot{R}^T p + R^T \dot{p} \\ &= -\widehat{\Omega} R^T p + F \\ &= -\widehat{\Omega} P + F\end{aligned}$$

thus

$$\dot{P}_1 = \frac{\Pi}{I} P_2 + F_1$$

$$\dot{P}_2 = -\frac{\Pi}{I}P_1 + F_2$$

where we have substituted, $\Omega = \Pi/I$. Collecting together Newton-Euler balance laws the dynamics can be written as

$$\dot{r} = Rv \tag{3.5}$$

$$\dot{R} = R\widehat{\Omega} \tag{3.6}$$

$$\dot{P}_1 = \frac{\Pi}{I}P_2 + F_1 \tag{3.7}$$

$$\dot{P}_2 = -\frac{\Pi}{I}P_1 + F_2 \tag{3.8}$$

$$\dot{\Pi} = d^b \times F \tag{3.9}$$

As we shall see in the following section equations (3.7)-(3.9) are the reduced equations, defined on $se(2)^*$, corresponding to that of a Hamiltonian control system defined on $T^*SE(2)$.

3.1.2 Lie-Poisson Reduction and Reduced Dynamics

The kinetic energy of the rigid body relative to the inertial frame is

$$\begin{aligned} T &= \frac{1}{2}\int_{\mathcal{B}} \|\dot{q}^r\|^2 dm(q^b) \\ &= \frac{1}{2}I\Omega^2 + \frac{m}{2}\|\dot{r}\|^2 \end{aligned}$$

where m is the total mass. We assume for now that the rigid body has sufficient lift and glides on the surface with no friction. Models with lift and friction will be studied later. Hence the Lagrangian $L : TSE(2) \to \mathbb{R}$ for this case is simply the kinetic energy, i.e.

$$L(R, r, \dot{R}, \dot{r}) = \frac{1}{2}I\Omega^2 + \frac{m}{2}\|\dot{r}\|^2 \tag{3.10}$$

The corresponding Hamiltonian on $T^*SE(2)$ is given by

$$H = \frac{1}{2} < \Pi, I^{-1}\Pi > + \frac{\|p\|^2}{2m}. \tag{3.11}$$

The dynamics on $T^*SE(2)$, written as a Hamiltonian control system, Σ_H, takes the form

$$\begin{aligned}
\dot{x} &= p_1/m \\
\dot{y} &= p_2/m \\
\dot{\theta} &= \Pi/I \\
\dot{p_1} &= (\cos(\theta + \phi))u \\
\dot{p_2} &= (\sin(\theta + \phi))u \\
\dot{\Pi} &= (d \sin \phi)u.
\end{aligned} \tag{3.12}$$

In (3.12) u is the magnitude of the force F. Observe that X_H is a Hamiltonian vector field with respect to the Hamiltonian (3.11) and the canonical Poisson bracket on $T^*SE(2)$.

Let L_g denote the left action of $SE(2)$ on itself. Hence given $\bar{g} = (\bar{R}, \bar{r}), L_{\bar{g}}g = \bar{g} \cdot g = (\bar{R}R, \bar{R}r + \bar{r})$.

Proposition 3.1.1 *The Hamiltonian control system Σ_H defined by (3.12) has $(T^*SE(2), T^*L_g)$ symmetry.*

Proof: Commutativity of Figure 2.2 is equivalent to showing that

$$T_q(T^*L_g) \cdot \psi(q, u) = \psi(T^*L_g \cdot q, u)$$

where $q = (x, y, \theta, p_1, p_2, \Pi)^T$,

$$\psi(q,u) = \frac{p_1}{m}\frac{\partial}{\partial x} + \frac{p_2}{m}\frac{\partial}{\partial y} + \frac{\Pi}{I}\frac{\partial}{\partial \theta} + u\cos(\theta+\phi)\frac{\partial}{\partial p_1} + u\sin(\theta+\phi)\frac{\partial}{\partial p_2} + ud\sin\phi\frac{\partial}{\partial \pi}$$

and $T^*L_g : T^*SE(2) \to T^*SE(2), \quad g = (\bar{x}, \bar{y}, \bar{\theta})$ such that

$(x, y, \theta, p_1, p_2, \Pi) \mapsto (x\cos\bar{\theta} - y\sin\bar{\theta} + \bar{x}, x\sin\bar{\theta} + y\cos\bar{\theta} + \bar{y}, \theta + \bar{\theta},\ p_1\cos\bar{\theta} - p_2\sin\bar{\theta}, p_1\sin\bar{\theta} + p_2\cos\bar{\theta}, \Pi)$.

Hence

$$\begin{aligned}
T_q(T^*L_g)\cdot\psi(q) &= \begin{bmatrix} \cos\bar{\theta} & -\sin\bar{\theta} & 0 & 0 & 0 & 0 \\ \sin\bar{\theta} & \cos\bar{\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\bar{\theta} & -\sin\bar{\theta} & 0 \\ 0 & 0 & 0 & \sin\bar{\theta} & \cos\bar{\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \psi(q,u) \\
&= \begin{bmatrix} (p_1\cos\bar{\theta} - p_2\sin\bar{\theta})/m \\ (p_1\sin\bar{\theta} - p_2\cos\bar{\theta})/m \\ \Pi/I \\ u\cos(\theta+\phi+\bar{\theta}) \\ u\sin(\theta+\phi+\bar{\theta}) \\ d\sin\phi \end{bmatrix} \\
&= \psi(T^*L_g\cdot q, u)
\end{aligned}$$

■

Hence from Proposition 2.2.3 it follows that (3.12) projects to a Hamiltonian control system $\Sigma_{\widetilde{H}}(se(2)^* \times U, se(2)^*, \widetilde{\psi})$. We now solve for $\Sigma_{\widetilde{H}}$. Since

$$(TL_{(\bar{R},\bar{r})})L(R, r, \dot{R}, \dot{r}) = L(\bar{R}R, \bar{R}r + \bar{r}, \bar{R}\dot{R}, \bar{R}\dot{r}) = \frac{1}{2}I\Omega^2 + \frac{m}{2}\|\dot{r}\|^2 \qquad (3.13)$$

the Lagrangian is $SE(2)$ invariant and hence the Hamiltonian (3.11) is also $SE(2)$ invariant. Hence from Theorem 2.1.10 X_H projects to Lie-Poisson reduced dynamics on $\mathfrak{g}^*$. The projection $\lambda : T^*G \to \mathfrak{g}^*$ is given by $\lambda : \alpha_g \mapsto (TL_G)^*\alpha_g$. Hence we chose convected variables $P = R^T p$ and Π as coordinates for $\mathfrak{g}^*$.

The reduced Hamiltonian $\widetilde{H}$ is given by

$$\widetilde{H} = \frac{1}{2I}\Pi^2 + \frac{\|P\|^2}{2m}. \tag{3.14}$$

Choosing

$$X_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ X_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ X_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

as a basis for $se(2)$ we have the commutation relations: $[X_1, X_2] = 0$, $[X_1, X_3] = -X_2$ and $[X_2, X_3] = X_1$. The Lie-Poisson bracket of two differentiable functions G, H on $se(2)^*$ is then given by

$$\{G, H\}_-(\mu) = \nabla G^T \Lambda(\mu) \nabla H \tag{3.15}$$

where $\mu = (P_1, P_2, \Pi) \in se(2)^*$ and

$$\Lambda = \begin{bmatrix} 0 & 0 & P_2 \\ 0 & 0 & -P_1 \\ -P_2 & P_1 & 0 \end{bmatrix}.$$

The reduced Hamiltonian System $\Sigma_{\widetilde{H}}$ takes the form

$$\dot{\mu} = X_{\widetilde{H}}(\mu) + \widetilde{g}u \tag{3.16}$$

where $X_{\widetilde{H}} = \Lambda(\mu)\nabla\widetilde{H}$ and $\widetilde{g}u$ is the external force projected appropriately. In the present setting

$$\widetilde{g} = (\cos\phi, \sin\phi, |d|\sin\phi)^T.$$

Depending on the control authority we distinguish two versions of the problem.

Case 1: The Jet-Puck Problem: Here we assume that the line of action of the force is fixed (i.e. ϕ is fixed) but its direction can be reversed. Equation 3.16 take the form

$$\begin{aligned} \dot{P}_1 &= P_2\Pi/I + \alpha u \\ \dot{P}_2 &= -P_1\Pi/I + \beta u \\ \dot{\Pi} &= d\beta u \end{aligned} \tag{3.17}$$

where $\alpha = \cos\phi, \beta = \sin\phi$ and $u \in [1, -1]$.

Case 2: The Hovercraft Problem: Here we assume that we now have control over both the magnitude of the thrust and ϕ. The equations now take the form

$$\begin{aligned} \dot{P}_1 &= P_2\Pi/I + u_1\cos(u_2) \\ \dot{P}_2 &= -P_1\Pi/I + u_1\sin(u_2) \\ \dot{\Pi} &= du_1\sin(u_2) \end{aligned} \tag{3.18}$$

where $u_1 \in [-1, 1]$ and $u_2 \in [\phi_{min}, \phi_{max}]$

Remark 3.1.2 If the actuation (forces and torques) on the a rigid body are due to body fixed thrusters/actuators then these forces are obviously invariant to translations and rotations, i.e. invariant to the left action of $SE(3)$, or any subgroup of it. Let us assume that in addition the Hamiltonian defined on T^*G is G invariant. Then the Hamiltonian control system (where the drift vector field is a Hamiltonian vector field with respect to the canonical Poisson bracket

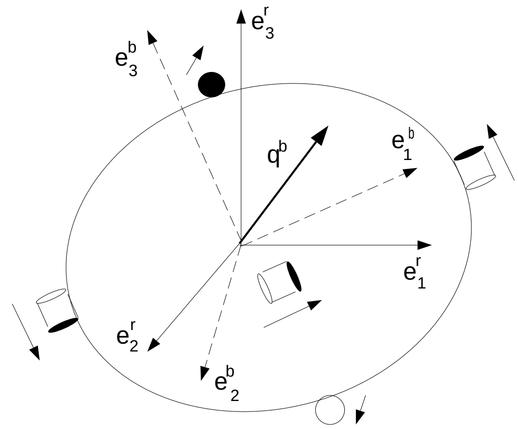

Figure 3.2: Spacecraft with gas jets

on T^*G, and the control vector field is due to body fixed thrusters/actuators) is G invariant. This follows from Lemma 2.1.9 and the fact that control vector fields are G invariant. In the rest of the examples where the dynamics of a rigid body with G invariant Hamiltonian with body fixed actuators/thrusters shows up, we will directly proceed to write down the reduced dynamics.

3.2 Attitude Control of Spacecraft with Gas Jets

We now discuss the dynamics describing spacecraft attitude control with gas jet actuators. Let $\{e^b_1, e^b_2, e^b_3\}$ be a body frame fixed on the rigid body (spacecraft) at its center of mass and let $\{e^r_1, e^r_2, e^r_3\}$ be an inertial frame of reference with origin coincident with the origin of the body fixed frame (see Fig 3.2). A typical material point q^b in the rigid body is then represented in the inertial frame as $q^r = Rq^b$ where R is an element of $SO(3)$, the special orthogonal group of 3×3 matrices. Hence the configuration space of the rigid body may be identified with $SO(3)$, the velocity space with the tangent bundle $TSO(3)$ and the momentum phase space with the cotangent bundle $T^*SO(3)$. Let $b_1, \ldots b_m$ be the axis about which the corresponding control torque of magnitude $\|b_i\|u_i$ is applied by means of opposing pairs of gas jets. The dynamical equations for the controlled spacecraft are then given by

$$\dot{R} = R\widehat{\Omega} \tag{3.19-a}$$

$$I\dot{\Omega} = I\Omega \times \Omega + \sum_{i=1}^{m} b_i u_i \tag{3.19-b}$$

where $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ is the body angular velocity, $\widehat{\Omega}$ is a 3×3 skew symmetric matrix given by

$$\widehat{\Omega} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}$$

and $I = \mathrm{diag}(I_1, I_2, I_3)$ is the inertia matrix. In the rest of the discussion $\widehat{\ }$ defines a map $\widehat{\ }: \mathbb{R}^3 \to so(3)$, such that $\widehat{\alpha}\beta = \alpha \times \beta,\ \alpha, \beta \in \mathbb{R}^3$. Thus

$$\widehat{\alpha} = \begin{bmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{bmatrix}$$

3.2.1 Symmetry and Reduction

The Lagrangian $L : TSO(3) \to \mathbb{R}$ is again simply the kinetic energy and is given by

$$L(R, \dot{R}) = \frac{1}{2} < \Omega, I\Omega >$$

and the corresponding Hamiltonian $H : T^*SO(3) \to \mathbb{R}$ is given by

$$\frac{1}{2} < \Pi, I^{-1}\Pi >$$

where $\Pi = I\Omega$ is the body angular momentum. Observe that the tangent lift of $g = \bar{R} \in SO(3)$ on $TSO(3)$ defined as

$$TL_g : TSO(3) \to TSO(3)$$
$$(R, \dot{R}) \mapsto (\bar{R}R, \bar{R}R\widehat{\Omega})$$

leaves the Lagrangian (and hence also the Hamiltonian) invariant. Hence one can induce a Hamiltonian on the quotient space, $T^*SO(3)/SO(3)$, and express the dynamics in terms of the appropriate reduced variables. The quotient space $T^*SO(3)/SO(3)$ is isomorphic to $so(3)^*$, the dual of the Lie algebra of $SO(3)$ and the reduced variables are $\Pi = (\Pi_1, \Pi_2, \Pi_3)$ corresponding to the body angular momentum. Choosing

$$X_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \ X_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \ X_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

as a basis of $so(3)$ and with the commutation relations $[X_1, X_2] = X_3$, $[X_3, X_1] = X_2$ and $[X_3, X_2] = -X_1$, the Lie-Poisson bracket of two differentiable functions G, H on $so(3)^*$ is given by

$$\{G, H\}_-(\mu) = \nabla G^T \Lambda(\mu) \nabla H \tag{3.20}$$

where $\mu = (\Pi_1, \Pi_2, \Pi_3) \in so(3)^*$ and

$$\Lambda = \begin{bmatrix} 0 & -\Pi_3 & \Pi_2 \\ \Pi_3 & 0 & -\Pi_1 \\ -\Pi_2 & \Pi_1 & 0 \end{bmatrix}.$$

The reduced equations take the form

$$\dot{\Pi} = f(\Pi) + + \sum_{i=1}^{m} \widetilde{b_i} u_i \tag{3.21}$$

where $f(\Pi) = (\frac{I_2 - I_3}{I_2 I_3}\Pi_2\Pi_3, \frac{I_3 - I_1}{I_1 I_3}\Pi_2\Pi_3, \frac{I_1 - I_2}{I_1 I_2}\Pi_1\Pi_2)^T$, $\widetilde{b_i} = R^T b_i$. Depending on the control authority and material symmetry (with respect to the principal axes) we distinguish between the following two versions

Case 1: Axisymmetric Spacecraft: Assuming that we have only one control and $I_1 = I_2$, then (3.21) can be written as

$$\begin{aligned} \dot{\Pi}_1 &= \frac{(I_1 - I_3)}{I_1 I_3}\Pi_2\Pi_3 + \alpha u \\ \dot{\Pi}_2 &= -\frac{(I_1 - I_3)}{I_1 I_3}\Pi_1\Pi_3 + \beta u \\ \dot{\Pi}_3 &= \gamma u \end{aligned} \tag{3.22}$$

Case 2: Asymmetric Spacecraft: Assuming that $I_1 \neq I_2 \neq I_3$ and two pure torques are available as controls, the reduced dynamics are given by

$$\begin{aligned} \dot{\Pi}_1 &= \frac{I_2 - I_3}{I_2 I_3}\Pi_2\Pi_3 + u_1 \\ \dot{\Pi}_2 &= \frac{I_3 - I_1}{I_1 I_3}\Pi_2\Pi_3 + u_2 \\ \dot{\Pi}_3 &= \frac{I_1 - I_2}{I_1 I_2}\Pi_1\Pi_2 \end{aligned} \tag{3.20}$$

3.3 Autonomous Underwater Vehicle

In this section we discuss the reduced space dynamics of a neutrally buoyant underwater vehicle. We distinguish between the cases of coincident and noncoincident centers of buoyancy and gravity. The Lie-Poisson dynamics for these cases have been derived in [Lamb, 1945; Birkhoff, 1960; Leonard, 1995]. We only present a brief overview of the Lie-Poisson dynamics.

3.3.1 Non Coincident Center of Mass and Center of Buoyancy

Let $\{e_1^r, e_2^r, e_3^r\}$ be an inertial frame of reference (see Figure 3.3) fixed at **O** and $\{e_1^b, e_2^b, e_3^b\}$ be a body frame fixed on the vehicle at its center of buoyancy (CB). A material point q^b in the underwater vehicle is then represented in the inertial frame as $q^r = Rq^b + r$ where R is an element of $SO(3)$, the special orthogonal group of 3×3 matrices and $r = (x, y, z)$ is a vector from **O** to the center of buoyancy (CB). Hence at any instant, the configuration $X(t)$ of the underwater vehicle can be uniquely identified by the pair (R, r) or equivalently as an element of $SE(3)$, the Special Euclidean group of 3×3 matrices. Recall

$$SE(3) \triangleq \{ \begin{pmatrix} R & r \\ 0 & 1 \end{pmatrix} \mid R \in SO(3), r \in \mathbb{R}^3 \}.$$

While deriving the dynamics we assume that the underwater vehicle is submerged in an infinitely large mass of incompressible, inviscid fluid. Further, we assume that the flow is irrotational (the motion of the fluid is entirely due to that of

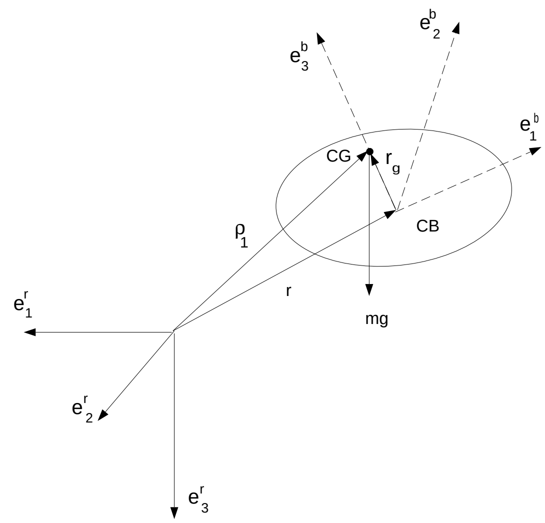

Figure 3.3: Autonomous Underwater Vehicle

the underwater vehicle). Under these assumptions the motion of the fluid can be characterized by the existence of a single-valued potential ϕ which satisfies

$$\begin{aligned} \nabla^2\phi &= 0 \\ \nabla\phi &= 0 \text{ at infinity} \\ -\frac{\partial\phi}{\partial n} &= n\cdot(v+\Omega\times r^b) \quad \text{at body surface,} \end{aligned}$$

where r^b is a vector from the CB to the vehicle's surface, n is the unit outward normal vector of the vehicle, $\Omega = (\Omega_1, \Omega_2, \Omega_3)^T$ are the body angular velocities, and $v = (v_1, v_2, v_3)^T$ are the linear velocity components along the body-fixed frame. Under these assumptions Kirchhoff showed that

$$\phi = v_1\phi_1 + v_2\phi_2 + v_3\phi_3 + \Omega_1\chi_1 + \Omega_2\chi_2 + \Omega_3\chi_3 \tag{3.21}$$

where $\phi_1, \phi_2, \phi_3, \chi_1, \chi_2, \chi_3$ are functions of x, y, z determined by the configuration of the surface of the solid. Using the form of ϕ as expressed in (3.21, the kinetic energy of the fluid

$$T_f = \frac{1}{2}\rho_0 \iiint (\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2) dxdydz,$$

where ρ_0 is the fluid density, can be expressed as a quadratic form

$$T_f = \frac{1}{2} W^T \Theta W, \quad \Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}.$$

$W = (v^T, \Omega^T)^T$, Θ_{11} is referred to as the added mass matrix, Θ_{22} as the added inertia matrix, Θ_{12} and Θ_{21} account for cross terms (c.f. [Lamb, 1945; Birkhoff, 1960]).

The kinetic energy of the vehicle alone can be expressed as

$$T_b = \frac{1}{2} W^T \mathbb{I} W \qquad \mathbb{I} = \begin{bmatrix} mI & -m\widehat{r}_g \\ m\widehat{r}_g & J_b \end{bmatrix},$$

where I is the 3×3 identity matrix, m is the mass and J_b is the inertia matrix. Hence the total kinetic energy can be expressed in a quadratic form as

$$T = \frac{1}{2} W^T (\mathbb{I} + \Theta) W = \frac{1}{2} (\Omega^T J \Omega + 2\Omega^T D v + v^T M v)$$

$J = J_b + \Theta_{11}$, $D = ml\widehat{r}_g + \Theta_{12}$ and $M = mI + \Theta_{22}$ (I is the 3×3 identity matrix).

Assume that the center of gravity (CG) does not coincide with the center of buoyancy (CB) and lies on the e_3^b axis at a distance $l > 0$ (bottom heavy) from the CB, i.e, $r_g = l i_3$ where i_3 denotes a unit vector (in body coordinates) along the e_3^b axis. Also let i_g denote a unit vector (in inertial coordinates) in the direction of gravity, i.e. along the e_3^r axis. Let m be the mass of the vehicle and J_b the inertia matrix for the vehicle. The moment applied to the body due to gravity, expressed in body coordinates is given by

$$r_g \times R^T mgi_g = -mgl(\Gamma \times i_3)$$

where $\Gamma = R^T i_g$

The Lagrangian $L : TSE(3) \to \mathbb{R}$ is then given by

$$L(R, r, \dot{R}, \dot{r}) = \frac{1}{2}(\Omega^T J\Omega + 2\Omega^T Dv + v^T Mv + 2mgl(i_g.Ri_3))$$

The potential $2mgl(i_g.Ri_3)$ accounts for the moment contribution due to noncoincident center of mass and center of buoyancy. In the rest of the discussion the underwater vehicle is approximated as an ellipsoid and hence $\Theta_{12} = \Theta_{21} = 0$.

It can be shown that the impulse of the body-fluid system varies, in consequence of extraneous forces acting on the solid, in exactly the same way as the momentum of a finite dynamical system. In the case of coincident center of mass and center of gravity these equations were derived by [Lamb, 1945] and in the Lie group setting as early as 1943 by Birkhoff [Birkhoff, 1960]. The observation that the reduced dynamics for the the noncoincident center of mass and buoyancy are of the "Lie-Poisson" type was made in [Leonard, 1995]. The reduction procedure discussed in [Leonard, 1995] is briefly outlined here. This system has a sufficient amount of complexity, and serves as a challenging example for application of controllability and stabilization results derived in later chapters.

3.3.2 Newton-Euler Balance Laws

Let p and π be the linear and angular components of the impulse with respect to the inertial coordinates. Again let P and Π, the convected variables, denote the components along the body fixed frame. Then

$$p = RP \tag{3.22}$$

$$\pi = R\Pi + r \times p \tag{3.23}$$

Let us assume that an external force, f_i^r and torque τ_i , given in inertial coordinates, are applied to the body. Let ρ_i be the vector, in inertial coordinates, from the origin of the inertial frame to the point on the line of action of the force f_i. Then from Newton-Euler balance laws we have

$$\dot{p} = f \tag{3.24}$$

$$\dot{\pi} = \tau + \Sigma_i^k \rho_i \times f_i \tag{3.25}$$

Differentiating p and π and expressing (3.24 - 3.25) in terms of convected variables we have

$$\dot{P} = P \times \Omega + R \sum_{i=1}^{k} f_i \tag{3.26}$$

$$\begin{aligned}\dot{\Pi} &= \Pi \times \Omega + P \times v - mgl(\Gamma \times i_3) \\ &\quad + \sum_{i=1}^{k} (R^T(\rho_i - r)) \times R^T f_i(t) + R^T \tau\end{aligned} \tag{3.27}$$

$$\dot{\Gamma} = \Gamma \times \Omega \tag{3.28}$$

where P and Π can be computed from the total energy T as

$$P = \frac{\partial T}{\partial v} = Mv + D^T \Omega \tag{3.29}$$

$$\Pi = \frac{\partial T}{\partial \Omega} = J\Omega + Dv \tag{3.30}$$

3.3.3 Symmetry and Reduction

Observe that the Lagrangian is invariant under the action of the group

$$G = \{(R, r) \in SE(3) \mid R^T i_g = i_g\} = SE(2) \times \mathbb{R}.$$

and hence the Hamiltonian system on $T^*SE(3)$ (which is also left-invariant under the action of $SE(2)\times\mathbb{R}$) can be reduced to a Hamiltonian system on $\mathfrak{s}^*$, the dual of the Lie algebra of the semi-direct product $S = SE(3) \times_\rho \mathbb{R}^3$ (see [Leonard, 1995] for details). The reduced Hamiltonian on $\mathfrak{s}^*$ is

$$\widetilde{H}(\Pi, P, \Gamma) = \frac{1}{2}(\Pi^T A \Pi + 2\Pi^T B^T P + P^T C P - 2mgl(\Gamma \cdot i_3)),$$

where

$$A = (J - DM^{-1}D^T)^{-1}, \quad B = -CD^T J^{-1}, \quad C = (M - D^T J^{-1} D)^{-1},$$
$$\Pi = J\Omega + Dv, \quad P = Mv + D^T\Omega, \quad \text{and } \Gamma = R^T i_g.$$

Choosing

$$B_i = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}, \quad i = 1, \ldots, 6, \quad B_i = \begin{bmatrix} 0 & e_{i-6} \\ 0 & 0 \end{bmatrix}, \quad i = 7, 8, 9.$$

where

$$A_i = \begin{bmatrix} \widehat{e}_i & 0 \\ 0 & 0 \end{bmatrix}, \quad i = 1, 2, 3 \quad A_i = \begin{bmatrix} 0 & \widehat{e}_i \\ 0 & 0 \end{bmatrix}, \text{ and } \quad i = 4, 5, 6$$

as a basis for $\mathcal{S}$ the Lie algebra of S. The Lie-Poisson bracket of two differentiable functions G, H on $\mathfrak{s}^*$ is given by

$$\{G, H\}_-(\mu) = \nabla G^T \Lambda(\mu) \nabla H$$

where $\mu = (\Pi, P, \Gamma)$ and

$$\Lambda = \begin{bmatrix} \widehat{\Pi} & \widehat{P} & \widehat{\Gamma} \\ \widehat{P} & 0 & 0 \\ \widehat{\Gamma} & 0 & 0 \end{bmatrix}.$$

The Lie-Poisson reduced equations (see [Leonard, 1995] for a complete description of reduction procedure) are then given by

$$\dot{\mu}_i = \{\mu_i, \widetilde{H}\}_-(\mu)$$

or explicitly as

$$\begin{aligned} \dot{\Pi} &= \Pi \times (A\Pi + B^T P) + P \times (CP + B\Pi) - mgl\Gamma \times i_3 \\ \dot{P} &= P \times (A\Pi + B^T P) \\ \dot{\Gamma} &= \Gamma \times (A\Pi + B^T P) \end{aligned} \tag{3.31}$$

3.3.4 Coincident Center of Mass and Center of Buoyancy

In the case of coincident center of gravity and center of buoyancy (i.e. $l = 0$), $D = 0$ and hence the Lagrangian is given by

$$L = \frac{1}{2}(\Omega^T T\Omega + v^T M v).$$

Hence the Hamiltonian system on $T^*SE(3)$ is left invariant under the $SE(3)$ action of rotations and translations, and we can derive a set of reduced Lie-Poisson equations on $se(3)^*$. Choosing

$$A_i = \begin{pmatrix} \widehat{e}_i & 0 \\ 0 & 0 \end{pmatrix}, \quad i = 1, 2, 3 \quad A_i = \begin{pmatrix} 0 & e_i \\ 0 & 0 \end{pmatrix}, \quad i = 4, 5, 6$$

as the basis for $se(3)$ the structure matrix $[\Lambda(\mu)]_{ij} = -\sum_{k=1}^{6} c_{ij}^k \mu_k$ is given by

$$\Lambda(\mu) = \Lambda(\Pi, P) = \begin{bmatrix} \widehat{\Pi} & \widehat{P} \\ \widehat{P} & 0 \end{bmatrix}$$

where $\Pi = J\Omega$ and $P = Mv$. The Lie-Poisson reduced equations are given by

$$\dot{\Pi} = \Pi \times (A\Pi) + P \times CP \qquad (3.32\text{-a})$$

$$\dot{P} = P \times A\Pi \qquad (3.32\text{-b})$$

3.4 Ships: Partially Submerged Floating Bodies

In this section we study motions of ships in incompressible, and inviscid fluid in the absence of waves. While ship motions arise very rarely in quiet water, there is a great practical value in their study since the characteristics of ships in agitated seas are governed by the characteristics of motion in quiet water.

As in the case of the underwater vehicle we identify the configuration space of a ship with the Lie group $SE(3)$. Let $\{e_1^r, e_2^r, e_3^r\}$ denote the inertial frame of reference and let $\{e_3^b, e_2^b, e_3^b\}$ denote the body-fixed frame attached to the center of mass as shown in Figure 3.4. Note that unlike the underwater vehicle the body-fixed frame is attached to the center of mass of the vessel as opposed to the center of buoyancy. Any material point with body coordinates $q^b = (x_b, y_b, z_b)$ is then represented in inertial coordinates by $q^r = Rq^b + r$, where $R \in SO(3)$ and $r = (x, y, z)$ describes the position of the center of mass in inertial coordinates.

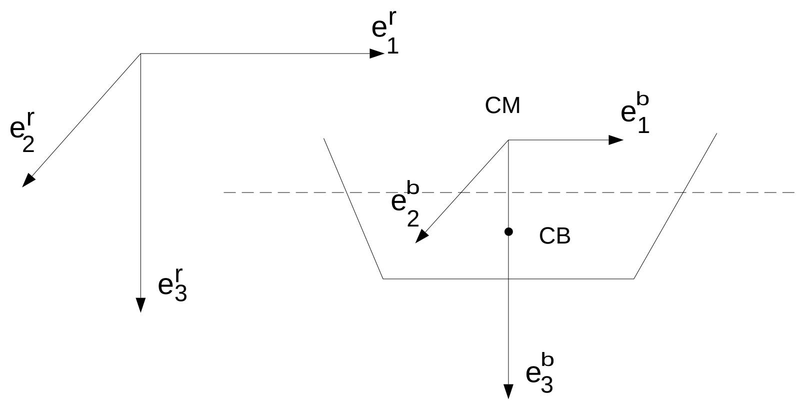

Figure 3.4: A floating body in equilibrium

Remark 3.4.1 To simplify calculation one normally makes the following assumptions -

(i) There is a vertical plane of symmetry about which the vessel is symmetric with respect to its shape and mass distribution.

(ii) The longitudinal axis of symmetry is directed horizontally.

These assumptions completely determine the principal axes of inertia of the vessel. Further the body fixed frame is chosen along the principal axis, such that in equilibrium e_1^b is directed towards the bow, e_2^b to starboard and e_3^b downwards.

A floating vessel in equilibrium experiences only two forces: gravitational force, acting vertically downwards at the center of mass, and a buoyant force, equal to the weight of the volume of water displaced by the submerged part, acting vertically upward at the center of buoyancy (centroid of the of the submerged volume). These forces are equal and opposite in direction and their points of application lie on a single vertical line. Let us assume that the center of buoyancy of the vessel in equilibrium lies along e_3^b at a distance a from the center of mass.

Hence in body fixed coordinates, the center of buoyancy in equilibrium is given by $q^0_{cb} = (0, 0, a)$.

Let $(\Omega_1, \Omega_2, \Omega_3)$ denote the body angular velocities and $v = (v_1, v_2, v_3)^T$ denote the linear velocity components along the body fixed frame, i.e. $v = R^T\dot{r}$. The kinematics are then given by

$$\begin{aligned} \dot{R} &= R\widehat{\Omega} \\ \dot{r} &= Rv \end{aligned}$$

As in the case of the underwater vehicle the kinetic energy (KE) of the body plus fluid is given by

$$KE = KE_{body} + KE_{fluid} = \frac{1}{2}W^T\mathbb{I}_{body}W + \frac{1}{2}W^T\mathbb{I}_{fluid}W^T. \tag{3.33}$$

In (3.33)

$$\mathbb{I}_{body} = \begin{bmatrix} mI & 0 \\ 0 & J_b \end{bmatrix} \tag{3.34}$$

and

$$\mathbb{I}_{fluid} = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \tag{3.35}$$

Θ_{11} is referred to as the added mass matrix, Θ_{22} the added inertia matrix. $\Theta_{12} = \Theta_{22}$ account for cross terms. Hence the total kinetic energy can be expressed in quadratic form as

$$T = \frac{1}{2}W^T(\mathbb{I} + \Theta)W = \frac{1}{2}(\Omega^T J\Omega + 2\Omega^T Dv + v^T Mv)$$

$J = J_b + \Theta_{11}$, $D = \Theta_{12}$ and $M = mI + \Theta_{22}$ (I is the 3×3 identity matrix).

Remark 3.4.2 In the case of the underwater vehicle $\mathbb{I}_{fluid}$ is calculated with respect to the body frame of reference attached to the center of buoyancy. Assuming an incompressible, irrotational fluid, at rest at infinity it was shown that

$$[\mathbb{I}_{fluid}]_{ij} = \epsilon_{ij} = \gamma \iint_{A_{sub}} \phi_i \nabla(\phi_j.n) ds \tag{3.36}$$

where the velocity potential ϕ was given by Kirchhoff as

$$\phi = v_1\phi_1 + v_1\phi_2 + v_1\phi_3 + \Omega_1\phi_4 + \Omega_2\phi_5 + \Omega_3\phi_6$$

and n is the normal vector at any point on the surface directed into the body. Since in the case of a partially submerged fluid only a part of the body is below the surface this approach to calculating the the fluid inertia matrix is not entirely valid. However is has been shown [Newman, 1992; Fossen, 1994; Balgoveshchensky, 1962] that using imaging methods a similar approach can be adopted to calculate $\mathbb{I}_{fluid}$. This approach assumes that the the added mass for a vessel floating on the surface of water equals half that of a body entirely submerged in a fluid of infinite extent, and having the shape of the the submerged portion doubled. $\mathbb{I}_{fluid}$ is calculated w.r.t the point of intersection of the waterline, the longitudinal plane of symmetry and the middle frame and transformed to body coordinates attached to the center of mass. Secondly it is assumed that the flow over the sides of the vessel is two dimensional, the so called plane of flow hypothesis. According to these hypotheses, the computation of the added mass per unit length of the vessel may be performed for each section as for an infinitely long cylinder moving in a direction perpendicular to its axis and having the same cross-sectional shape as the doubled frame. The results of these computations, performed for each frame, independent to its neighbor are integrated over the length of the vessel.

3.4.1 Newton-Euler Balance Laws

The gravitational force (expressed in body coordinates) acting on the vessel is given by

$$F_g = R^T(mg\mathbf{k}), \tag{3.37}$$

where $\mathbf{k}$ is a unit vector in the direction of gravity i.e along positive e_3^r.

The hydrostatic force expressed in body coordinates is given by

$$F_{hydro} = -R^T(\rho g \iint\limits_{A_{sub}} n(z-\kappa)ds) \tag{3.38}$$

where $z-\kappa$ denotes the depth, in inertial coordinates, of a point on the surface of the submerged part. Here, the normal vector n is taken to be positive when pointing out of the fluid volume and hence into the body, and ρ is the fluid density. Applying Gauss's theorem to (3.38) yields

$$F_{hydro} = -\rho g \iiint\limits_{V_{sub}} \nabla(z-\kappa)dV = -\gamma V_{sub}\mathbf{k} \tag{3.39}$$

where V_{sub} denoted the instantaneous volume enclosed by the water plane and the submerged body surface. Hence the net external force acting on the vessel is given by

$$F_{ext} = R^T(mg - \gamma V_{sub})\mathbf{k} \tag{3.40}$$

To calculate the moment due to the external force observe that the moment about the center of mass due to the force of gravity is zero. The moment due to the hydrostatic force about the center of mass is given by

$$\mathcal{M}_{hydro} = \iiint\limits_{V_{sub}} q_b \times R^T k \gamma dV(q_b) \tag{3.41}$$

Let $V_{sub} = V_0 + \Delta V(R, r)$, where V_0 denotes the volume of the submerged part, when the vessel in equilibrium. Since in equilibrium the net force on the body is zero, i.e. $mg\mathbf{k} = \gamma V_0 \mathbf{k}$, (3.40) and (3.41) can be written as follows.

$$F_{ext} = R^T(mg - \gamma V_{sub})\mathbf{k} \tag{3.42}$$

$$= R^T(mg - \gamma V_0 - \gamma \Delta V)\mathbf{k} \tag{3.43}$$

$$= -(\gamma \Delta V) R^T \cdot \mathbf{k} \tag{3.44}$$

and

$$\mathcal{M}_{hydro} = -\gamma R^T \mathbf{k} \times \iiint\limits_{V_{sub}} q_b dV(q_b) \tag{3.45}$$

$$= -\gamma R^T \mathbf{k} \times \iiint\limits_{V_0} q_b dV(q_b) - \gamma R^T \mathbf{k} \times \iiint\limits_{\Delta V} q_b dV(q_b) \tag{3.46}$$

$$= -\gamma V_0 R^T \mathbf{k} \times q_{cb}^0 - \gamma R^T \mathbf{k} \times \iiint\limits_{\Delta V} q_b dV(q_b). \tag{3.47}$$

Hence from the Newton-Euler balance laws we have

$$\dot{r} = Rv \tag{3.48}$$

$$\dot{\Omega} = R\widehat{\Omega} \tag{3.49}$$

$$\dot{P} = P \times \Omega - (\gamma \Delta V) R^T \cdot \mathbf{k} \tag{3.50}$$

$$\dot{\Pi} = \pi \times \Omega + P \times v + \tag{3.51}$$

$$-\gamma V_0 R^T \mathbf{k} \times q_{cb}^0 - \gamma R^T \mathbf{k} \times \iiint_{\Delta V} q_b dV(q_b) \tag{3.52}$$

where $\Pi = \frac{\partial T}{\partial \Omega} =$ and $P = \frac{\partial T}{\partial v}$.

3.5 Symmetry and Reduction

Observe that the the kinetic energy, hydrostatic forces and the forces due to gravitation are $SE(2)$ invariant, i.e. are invariant to translations in the $e_1^r e_2^r$ plane, and rotations about an axis perpendicular to this plane. Hence the dynamics can be reduced from $T^*SE(3)$ to $\mathfrak{s}^* = T^*SE(3)/SE(2)$. While writing down the reduced dynamics in this case we observed that unlike the autonomous underwater vehicle, we could not express the buoyant force as a potential and hence the reduced dynamics could not be written down as a Hamiltonian system on $\mathfrak{s}^*$.

Choosing coordinates (z, Γ, P, Π), with $\Gamma = R^T\mathbf{k}$ we now write down the reduced dynamics.

Observe that, $\|\Gamma\| = 1$,

$$\dot{\Gamma} = -\widehat{\Omega}\Gamma \tag{3.53}$$

$$= \Gamma \times \Omega \tag{3.54}$$

and

$$\dot{z} = Rv \cdot \mathbf{k} \tag{3.55}$$

$$= \mathbf{k}^T Rv \tag{3.56}$$

$$= \Gamma^t v \tag{3.57}$$

$$= v \cdot \Gamma \tag{3.58}$$

The reduced dynamics on $T^*SE(3)/SE(2)$ are then given by

$$\dot{z} = v \cdot \Gamma \tag{3.59}$$

$$\dot{\Gamma} = \Gamma \times \Omega \tag{3.60}$$

$$\dot{P} = P \times \Omega + \gamma \Delta\bar{V}\Gamma \tag{3.61}$$

$$\dot{\Pi} = \Pi \times \Omega + P \times V + V_0(\Gamma \times q_{cb}^0) + \gamma \iiint_{\Delta\bar{V}} \Gamma \times q_b dq_b \tag{3.62}$$

Where $\Delta V = \Delta\bar{V}(\Gamma, z)$ denotes the change in submerged volume expressed as a function of Γ and z.

Chapter 4

Controllability of Lie Poisson Reduced Dynamics

As seen in Chapter 3 the state space of a large class of mechanical systems such as hovercraft, spacecraft underwater vehicle etc. can be identified with a Lie group G. The Hamiltonian dynamics (defined on T^*G) of these systems subject to external forces can be written in the form of a control system as

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i \tag{4.1}$$

where $x \in T^*G$, $f(x) = X_H$ and $u = (u_1, \dots u_m)$. (H is the Hamiltonian defined on T^*G). The G-invariance of (4.1) allows us to drop the the vector fields f and $g_i's$ from T^*G to $T^*G/G \cong \mathfrak{g}^*$ and the reduced dynamics take the form

$$\dot{\mu} = \widetilde{f}(\mu) + \sum_{i=1}^{m} \widetilde{g}_i(\mu)\widetilde{u}_i \tag{4.2}$$

where $\mu \in \mathfrak{g}^*$, $\widetilde{f}$ and $\widetilde{g}_i$ are the projections of f and g on T^*G/G. From the discussion in chapter 2 (cf. Proposition 2.2.3) we know that $\widetilde{f} = X_{\widetilde{H}}$ where $\widetilde{H}$ is the reduced Hamiltonian and $X_{\widetilde{H}}$ is Hamiltonian with respect to the Lie-Poisson structure defined on $\mathfrak{g}^*$. Studying controllability of systems of the form (4.2) or of more general systems of the form

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \quad x \in \mathbb{R}^n, \quad u = (u_1, \ldots u_m) \in U \tag{4.3}$$

is usually a hard problem. We know that if a system of the form (4.3) satisfies the Lie algebra rank condition (LARC) then it is locally accessible, and in addition if $f = 0$ then LARC implies that the system is controllable. (cf. Theorem 2.3.10). While the kinematic equations of motion can often be written as a drift-free system, once dynamics are included LARC does not imply controllability. Proving controllability is usually much harder than proving accessibility. In [Crouch and Byrnes, 1986] sufficient conditions are given, in terms of a "group action", that a locally accessible system is also locally reachable. In [Lobry, 1974] sufficient conditions for the controllability of a conservative dynamical polysystem on a compact Riemannian manifold are presented. More recently this result was extended by [Lian *et al.*, 1994] to dynamical polysystems where the drift vector field was required to be weakly positively Poisson stable. We extend this result to reduced dynamics where the drift vector field is Lie-Poisson. We prove conditions under which the reduced dynamics are controllable. Before we present our results we introduce some definitions and related theorems regarding Poisson stable systems. We follow the development in [Lian *et al.*, 1994; Nijmeijer and van der Schaft, 1990; Dayawansa, 1994; Arnold, 1989; Brockett, 1976; Nemytskii and Stepanov, 1960].

4.1 Poisson Stability and Controllability

Let X be a smooth complete vector field on M and let $\phi_t^X(\cdot)$ denote its flow.

Definition 4.1.1 A point $p \in M$ is called *positively Poisson stable* for X if for all $T > 0$ and any neighborhood V_p of p, there exists a time $t > T$, such that $\phi_t^X(p) \in V_p$. The vector field X is called positively Poisson stable if the set of Poisson stable points for X is dense in M.

Definition 4.1.2 A point $p \in M$ is called nonwandering point of X if for all $T > 0$ and for any neighborhood V_p of p, there exists a time $t > T$ such that $\phi_t^X(V_p) \bigcap V_p \neq \emptyset$, where $\phi_t^X(V_p) = \{\phi_t^X(q) \mid q \in V_p\}$.

One should observe here that though the positive Poisson stability is a sufficient condition that the nonwandering set of a positively Poisson stable vector field is the entire manifold M, there could exist weaker conditions under which the nonwandering set is M. This gives rise to the definition of a weakly positively Poisson stable (WPPS) vector field.

Definition 4.1.3 The vector field X is called weakly positively Poisson stable if the associated nonwandering set is M.

The following theorem on controllability of nonlinear affine control systems where the drift vector field is WPPS is due to [Lian *et al.*, 1994]. Earlier versions of this theorem and the corollary that follows, where the hypothesis required f to be only Poisson Stable, are due to Lobry [Lobry, 1974], Bonnard and Crouch [Crouch *et al.*, 1980].

Theorem 4.1.4 *If the system*

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \quad u = (u_1, \ldots u_m) \in U \subset \mathbb{R}^m$$

where U contains $\{u \mid |u_i| \leq M_i \neq 0, i, \ldots, m\}$ is such that f is a weakly positively Poisson stable vector field, then the system is controllable if the accessibility LARC is satisfied.

Before we present the proof of the theorem (presented in [Lian *et al.*, 1994]) we present the following theorem by [Jurdjevic and Kupka, 1981; Hermes and LaSalle, 1969] which will be used in the proof.

Theorem 4.1.5 *Let $\mathcal{F}$ be a dynamical polysystem. Then*

$$cl(R(\mathcal{F}, p)) = cl(R(conv(\mathcal{F}), p)), \qquad \forall p \in M$$

Here conv$(\cdot)$ and cl$(\cdot)$ denote the convex hull and closure respectively.

Proof: **(of Theorem 4.1.4)** If the dynamical system is controllable then LARC is satisfied. This follows from Theorem 2.3.4. The "if" part (WPPS + LARC $\Rightarrow$ Controllability) is proved as follows. Let $\mathcal{E} = \mathcal{F} \bigcup \{-f\}$ and let $\mathcal{L}(\mathcal{E})$ denote the Lie algebra generated by $\mathcal{E}$. Since LARC is satisfied, span$\mathcal{L}(\mathcal{E})(p)$ = span$\mathcal{L}(\text{conv}(\mathcal{E}))(p) = T_pM$. While conv$(\mathcal{E})$ is not symmetric, in the sense of Definition 2.3.9 it satisfies the property that for every $X_i \in \mathcal{E}, -\alpha_i X_i \in \mathcal{E} \quad \alpha_i \in (0, 1]$. From a slight modification of the controllability proof [1] for symmetric systems presented in [Nijmeijer and van der Schaft,

[1] "he entuition being yhat yhe tffect lf ilowing tlong yhe vector iield X ior yime $t = T$ can be tchieved by ilowing tlong yhe vector iield αX ior yime $t = T/\alpha$P

1990] it follows that $R(\text{conv}(\mathcal{E}, p)) = M, \forall p \in M$. From Theorem 4.1.5 it follows that $\text{cl}(R(\mathcal{E}, p)) = M, \forall p \in M$. Since $\text{span}\mathcal{L}(\mathcal{E})(p) = \text{span}\mathcal{L}(-\mathcal{E})(p) = T_p(M), R(-\mathcal{E}, y)$ has a non empty interior for all $y \in M$. Hence for any $q \in M$ there exists some point in $\text{int}(R(-\mathcal{E}, q)) \bigcap R(\mathcal{E}, p)$. This implies $q \in R(\mathcal{E}, p)$. Hence $R(\mathcal{E}, p) = M \ \forall p \in M$.

Let $p \in \text{int}(R(-\mathcal{F}, q))$ and $w \in \text{int}(R(\mathcal{F}, p))$. Since $R(\mathcal{E}, w) = M, \forall w \in M$, there exists an integral curve of $\mathcal{E}$ joining w to z, i.e.

$$\exists t_1, \ldots, t_k > 0 \text{ and } X_1, \ldots, X_k \in \mathcal{E}$$

such that

$$z = \Phi_{t_1}^{X_1} \circ \cdots \circ \Phi_{t_k}^{X_k}(w).$$

If $X_i, i = 1, \ldots, k$ belong to $\mathcal{F}$, then $q \in R(\mathcal{F}, p)$. If there are some X_i's such that $X_i = -f \in \mathcal{E}$, then one exploits the WPPS property of f to correct this. Without loss of generality assume that $X_1 = -f$. A neighborhood U_z of z, can be found in the interior of $R(-\mathcal{F}, q)$ such that

$$(\Phi_{t_1}^{X_1} \circ \cdots \circ \Phi_{t_k}^{X_k})^{-1}(U_z) \subset \text{int}(R(\mathcal{F}, p)) \tag{4.4}$$

with $w \in (\phi_{t_1}^{X_1} \circ \cdots \circ \phi_{t_k}^{X_k})^{-1}(U_z)$.

The WPPS of the vector field f implies that for U_z and t_1 there exists $T_1 > t_1$ such that $\phi_{T_1}^{f}(U_z) \bigcap U_z \neq \emptyset$. Accordingly there exist $\xi, \bar{\xi} \in U_z \subset \text{int}(R(-\mathcal{F}, q))$, such that $\xi = \phi_{T_1}^{f}(\bar{\xi})$. From (4.4) we can find $s \in \text{int}(R(\mathcal{F}, p))$, with $\bar{\xi} = \phi_{t_1}^{-f} \circ \cdots \circ \phi_{t_k}^{X_k}(s)$. Thus

$$\xi = \phi_{T_1}^{f} \circ \phi_{t_1}^{-f} \circ \cdots \circ \phi_{t_k}^{X_k}(s) = \phi_{T_1 - t_1}^{f} \circ \cdots \circ \phi_{t_k}^{X_k}(s).$$

Since $\xi \in \text{int}(R(-\mathcal{F}, q))$ and $s \in \text{int}(R(\mathcal{F}, p))$ it follows that

$$q = \phi^{\mathcal{F}} \circ \phi_{T_1 - t_1}^{f} \circ \cdots \circ \phi_{t_k}^{X_k} \circ \phi^{\mathcal{F}}(p)$$

where $\phi^{\mathcal{F}}$'s denote some flow of $\mathcal{F}$. Other possible $-f$ can be treated in a similar way. Hence arbitrary q can be reached from any p by some integral curve of $\mathcal{F}$.

■

As shown in [Crouch *et al.*, 1980] controllability can be achieved by restricting the controls to the discrete set $U = \{-1, 1\}\}$.

Corollary 4.1.6 *If the system*

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \quad u = (u_1, \ldots u_m) \in U$$

is such that f is a weakly positively Poisson stable vector field, and accessibility LARC is satisfied, then the system with controls constrained by $u_i \in \{-M_i, M_i\}, M_i > 0, \ i = 1, \ldots, m$ is controllable.

4.2 Controllability of Reduced Dynamics

A natural question that arises is whether is there is a sufficiently large class of vector fields that are $WPPS$. In the setting of Hamiltonian vector fields on bounded symplectic manifolds this question is answered by the Poincaré recurrence theorem [Arnold, 1989; Nemytskii and Stepanov, 1960] stated below.

Theorem 4.2.1 *Let ψ be a volume-preserving continuous bijective map on a bounded region D onto itself. Then in any neighborhood U of any point in D, there exists a point $x \in U$ which returns to U after a repeated application of the mapping, i.e. $\psi^n(x) \in D$.*

Theorem 2.1.4 shows that Hamiltonian vector fields on symplectic manifolds are volume-preserving. Hence, if in addition, the flows are restricted to a bounded set, or live on a bounded manifold, then it follows that a time-independent Hamiltonian vector field on a bounded symplectic manifold is WPPS.

As is easily observed from the dynamics of the hovercraft and underwater vehicles the state space of these systems is not a bounded manifold and hence one cannot easily conclude the WPPS nature of the drift vector field in these cases.

In the setting of reduced dynamics where the drift vector field is Lie-Poisson we can make the following observation.

Theorem 4.2.2 *Let G be a Lie group that acts on itself by left (right) translations. Let $H : T^*G \to \mathbb{R}$ be a left (right) invariant Hamiltonian. Then,*

(i) If G is a compact group, the coadjoint orbits of $\mathfrak{g}^* = T^*G/G$ are bounded and the Lie-Poisson reduced Hamiltonian vector field $X_{\widetilde{H}}$ is WPPS.

(ii) If G is a noncompact group then the Lie-Poisson reduced Hamiltonian vector field $X_{\widetilde{H}}$ is WPPS if there exists a function $V : \mathfrak{g}^* \to \mathbb{R}$ such that $V(\mu)$ is bounded below, $V(\mu) \to \infty$ as $\|\mu\| \to \infty$ and $\dot{V} = 0$ along trajectories of the system.

Here $\widetilde{H}$ is the induced Hamiltonian on the quotient manifold $\mathfrak{g}^* = T^*G/G$ and $\{\cdot,\cdot\}_{-(+)}$ is the induced minus (plus) Lie-Poisson bracket on the quotient manifold $\mathfrak{g}^* = T^*G/G$.

Proof: (i) The projection $\lambda : T^*G \to \mathfrak{g}^*_-$ is a Poisson map, and the Poisson manifold $\mathfrak{g}^*_-$ is symplectically foliated by coadjoint orbits, i.e. it is a disjoint union of symplectic leaves that are just the coadjoint orbits. Any Hamiltonian system on $\mathfrak{g}^*_-$ leaves invariant the symplectic leaves and hence restricts to a canonical Hamiltonian system on a leaf. To study the dynamics of a particular system with initial condition $\mu(0) \in \mathfrak{g}^*_-$, we therefore restrict attention to the coadjoint orbit through $\mu(0)$. By hypothesis, each coadjoint orbit is compact. The flow starting at $\mu(0)$ preserves the symplectic volume measure on the orbit. Hence by the Poincaré Recurrence Theorem, we know that for almost every point $p \in \mathfrak{g}^*_-$ and any neighborhood V_p of p there exists a time $t > T$ such that $\phi^X_t(p)$ returns to V_p i.e. $X_{\widetilde{H}}$ is WPPS.

(ii) Let $D = \{\mu \mid V(\mu) \leq E\}$, and let $\text{Orb}(\cdot)$ denote the coadjoint orbit through $\mu(0)$ in $\mathfrak{g}^*_-$. Then the integral curve of $X_{\widetilde{H}}$ starting at $\mu(0) \in D$ lies entirely in the set $S = D \cap \text{Orb}(\cdot)$. Since S closed and bounded in $\mathfrak{g}^*_-$, it is compact in $\text{Orb}(\cdot)$, and hence as before $X_{\widetilde{H}}$ is WPPS. ■

In many situations the function $H_\phi = \widetilde{H} + \phi(C_i)$ where $\widetilde{H}$ is the reduced Hamiltonian and C_i a Casimir is a good choice for $V(\cdot)$.

Remark 4.2.3 In our present setting of Lie-Poisson reduced dynamics, WPPS conditions in Theorem 4.1.4 can be verified whenever the hypotheses of Theorem 4.2.2 hold. Once WPPS of the drift vector field has been established Theorem 4.1.4 can be used to conclude controllability.

Remark 4.2.4 As mentioned in Chapter 2, often the dynamics on T^*G are not invariant under the whole group G, but some subgroup of it. In such situations it might be possible to write down the reduced dynamics, using the semidirect product reduction theorem on the dual of the Lie algebra of a different group S which is a semidirect product. As these dynamics on $\mathfrak{s}^*$ are still Lie-Poisson (cf. Theorem 2.1.12 and the reduced dynamics of AUV with coincident center of mass and center of buoyancy discussed in Section 3.3) Theorem 4.2.2 still applies.

Applying the above results to the examples discussed in Chapter 2 we have the following results.

Proposition 4.2.5 *The jet-puck dynamics defined by (3.17) are controllable if* $\sin\phi \neq 0$.

Proof: We first show that LARC is satisfied. To show that

$$\dim(\mathrm{span}\mathcal{L}_{\{f,g\}})(p) = 3, \quad \forall p \in se(2)^*$$

where $f = (P_2\Pi/I, -P_1\Pi/I, 0)^T$ and $g = (\alpha, \beta, d\beta)^T$, observe that

$$\begin{aligned} &\det(g, [[f,g],g], [[f,g],[[f,g],g]]) \\ &= det \begin{bmatrix} \alpha & 2\frac{d}{I}\beta^2 & -2\frac{d^2}{I^2}\beta^2\alpha \\ \beta & -2\frac{d}{I}\beta\alpha & -2\frac{d^2}{I^2}\beta^3 \\ d\beta & 0 & 0 \end{bmatrix} \\ &= -4\tfrac{(d\beta)^4}{I^3}(\beta^2+\alpha^2) \\ &= -4\tfrac{(d\beta)^4}{I^3} \ \ (\text{since } \alpha^2+\beta^2=1) \end{aligned}$$

Hence $\dim(\mathrm{span}\mathcal{L}_{\{f,g\}})(p) = 3\ \forall p \in se(2)^*$ as long as $\beta = \sin\phi \neq 0$, i.e. as long as the line of action of F does not pass through the center of mass.

We observe that the reduced Hamiltonian

$$\widetilde{H} = \frac{1}{2I}\Pi^2 + \frac{\|P\|^2}{2m} \tag{4.5}$$

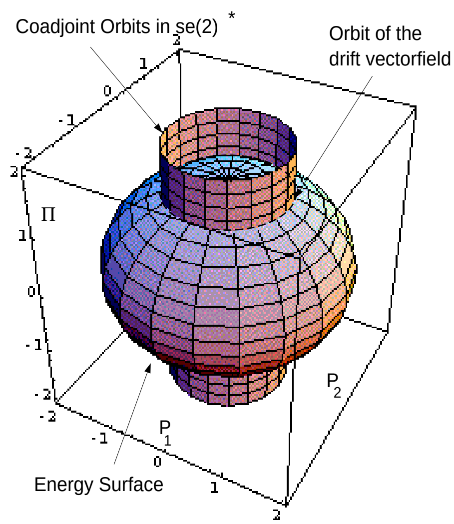

Figure 4.1: Energy surface and coadjoint orbits in $se(2)^*$

is bounded below, radially unbounded and is such that $\dot{\tilde{H}} = 0$. Hence it follows from Theorem 4.2.2 that f is WPPS and hence from Theorem 4.1.4 we conclude that the jet-puck dynamics are controllable. ■

In fact one observes that every orbit of f is periodic and hence trivially Poisson stable.

Remark 4.2.6 Observe that the coadjoint orbits in $se(2)^*$ are cylinders

$$\{(P_1, P_2, \Pi) \in \mathbb{R}^3 \mid P_1^2 + P_2^2 = constant \neq 0\}.$$

The surfaces defined by $D = \{(P_1, P_2, \Pi) \mid \frac{P_1^2}{2m} + \frac{P_2^2}{2m} + \frac{\Pi^2}{2I} = const\}$ are ellipsoids. From Theorem 4.2.2 the integral curves of the the vector field $\frac{P_2\Pi}{I}\frac{\partial}{\partial P_1} - \frac{P_1\Pi}{I}\frac{\partial}{\partial P_2}$ are restricted to a connected component of the set $S = D \cap \mathrm{Orb}(\cdot)$, which in this case is simply S^1 (see Fig. (4.1)).

Proposition 4.2.7 *The hovercraft dynamics defined by Equation (3.18) are controllable.*

Proof: In (3.18) setting $u_2 = k$, where k is some constant not equal to zero, the equations reduce to those of the jet-puck and hence from Proposition 4.2.5 the dynamics are controllable. ■

Observe the similar structure of base space equations for the jet-puck and those of the controlled Euler equations for an axisymmetric spacecraft (Equation 3.22) with one control vector. Hence similar claims regarding controllability can be made. (see [Crouch, 1984; Baillieul, 1981] where these results originally appeared). Proofs are omitted as they are similar to those of the jet-puck dynamics.

Proposition 4.2.8 *The spacecraft dynamics of an axisymmetric spacecraft defined by (3.22) are controllable if* $\alpha^2 + \beta^2 \neq 0$ *and* $\gamma \neq 0$.

Remark 4.2.9 The coadjoint orbits in $so(3)^*$ are spheres (see Fig 4.2)

$$\{(\Pi_1, \Pi_2, \Pi_3) \in \mathbb{R}^3 \mid \Pi_1^2 + \Pi_2^2 + \Pi_3^2 = const\}.$$

In this case since the coadjoint orbits are compact manifolds one can conclude from Theorem 4.2.2 that the drift vector field is WPPS. Fig (4.2) shows the intersection of the coadjoint orbits and the energy surface.

In the setting of the autonomous underwater vehicle with coincident center of mass and center of buoyancy we assume that the vehicle is an ellipsoid with semiaxes l_1, l_2 and l_3 where l_i lies along the e_i^b axis. Assuming that the principal

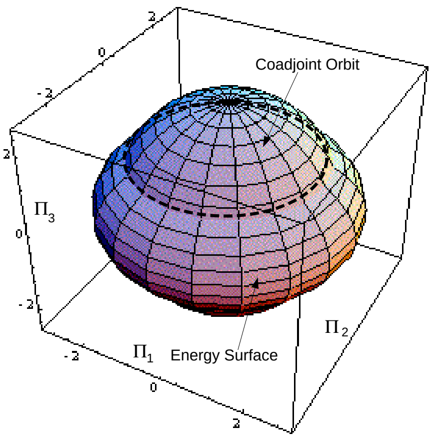

Figure 4.2: Energy surface and coadjoint orbits in $se(2)^*$

axes of the vehicle and the principal axes of the displaced fluid are the same we have

$$J = diag(I_1, I_2, I_3) \text{ and } M = diag(m_1, m_2, m_3) \tag{4.6}$$

Lets further assume that that we have three controls u_1, u_2, u_3 such that u_1 and u_2 provide pure torques and u_3 provides a pure force. The reduced dynamics are

$$\begin{aligned}
\dot{\Pi}_1 &= \frac{I_2 - I_3}{I_2 I_3}\Pi_2\Pi_3 + \frac{m_2 - m_3}{m_2 m_3}P_2 P_3 + u_1 \\
\dot{\Pi}_2 &= \frac{I_3 - I_1}{I_3 I_1}\Pi_3\Pi_1 + \frac{m_3 - m_1}{m_3 m_1}P_3 P_1 + u_2 \\
\dot{\Pi}_3 &= \frac{I_1 - I_2}{I_1 I_2}\Pi_1\Pi_2 + \frac{m_1 - m_2}{m_1 m_2}P_1 P_2 \\
\dot{P}_1 &= \frac{P_2\Pi_3}{I_3} - \frac{P_3\Pi_2}{I_2} + u_3 \\
\dot{P}_2 &= \frac{P_3\Pi_1}{I_1} - \frac{P_1\Pi_3}{I_3} \\
\dot{P}_3 &= \frac{P_1\Pi_2}{I_2} - \frac{P_2\Pi_1}{I_1}
\end{aligned} \tag{4.7}$$

Proposition 4.2.10 *The Lie-Poisson reduced dynamics, defined by (4.7), of the underwater vehicle with coincident center of buoyancy and center of gravity are* *controllable if* $I_1 \neq I_2$*.*

Proof: Let

$$\begin{aligned} f &= (\frac{I_2-I_3}{I_2I_3}\Pi_2\Pi_3 + \frac{m_2-m_3}{m_2m_3}P_2P_3)\frac{\partial}{\partial\Pi_1} + (\frac{I_3-I_1}{I_3I_1}\Pi_3\Pi_1 + \frac{m_3-m_1}{m_3m_1}P_3P_1)\frac{\partial}{\partial\Pi_2} \\ &+(\frac{I_1-I_2}{I_1I_2}\Pi_1\Pi_2 + \frac{m_1-m_2}{m_1m_2}P_1P_2)\frac{\partial}{\partial\Pi_3} + (\frac{P_2\Pi_1}{I_3} - \frac{P_3\Pi_2}{I_2})\frac{\partial}{\partial P_1} \\ &+(\frac{P_3\Pi_1}{I_1} - \frac{P_1\Pi_3}{I_3})\frac{\partial}{\partial P_2} + (\frac{P_1\Pi_2}{I_2} - \frac{P_2\Pi_1}{I_1})\frac{\partial}{\partial P_3}, \end{aligned}$$

$$g_1 = \frac{\partial}{\partial\Pi_1}, \quad g_2 = \frac{\partial}{\partial\Pi_2}, \quad g_3 = \frac{\partial}{\partial P_1}$$

Choose $V = (\Pi, P) = \frac{1}{2}(\Pi^T A\Pi + P^T CP)$, where $A = J^{-1}$ and $C = M^{-1}$ are positive definite symmetric matrices. Observing that V is radially bounded and $\dot{V} = 0$ along trajectories of (4.7), we can conclude that f is WPPS. Further we have,

$$[[f, g_1], g_2] = \frac{(I_1 - I_2)}{I_1I_2}\frac{\partial}{\partial\Pi_3}, \qquad [[f, g_2], g_3] = \frac{1}{I_2}\frac{\partial}{\partial P_3}$$

$$[[[f, g_2], [f, g_3]], g_1] = \frac{(I_1 - I_2)}{I_1I_2I_3}\frac{\partial}{\partial P_2}$$

Treating vector fields f and g_i's as coloumn vectors and observing that

$$det(g_1, g_2, g_3, [[f, g_1], g_2], [[f, g_2], g_3], [[[f, g_2], [f, g_3]], g_1]) = \frac{(I_1 - I_2)^2}{I_1^2 I_3^3 I_3} \neq 0 \quad (4.8)$$

if $I_1 \neq I_2$, i.e. $\dim(\text{span}\mathcal{L}_{\{f,g_1,g_2,g_3\}})(p) = 6, \ \ \forall p \in se(3)^*$, and that f is WPPS the result follows from Theorem 4.2.2. ∎

Proposition 4.2.11 *The Lie-Poisson reduced Hamiltonian vector field (given by the right hand side of Equation (3.31)) defined on* $\mathfrak{s}^*$ *is WPPS.*

Proof: Choose $V(\Pi, P, \Gamma) = \widetilde{H}(\Pi, P, \Gamma) + \Gamma^T\Gamma$. Observing that V is radially unbounded and that $\dot{V} = 0$ along trajectories of (3.31) the result follows from Theorem 4.2.2. ∎

4.3 Small-Time Local Controllability

Whereas showing controllability in systems can be quite difficult one can often show that the system is small-time locally controllable (STLC) [Sussmann, 1983; 1987]

Definition 4.3.1 The control system (4.3) is said to be *small-time locally controllable* (STLC) from $x_0 \in M$ if it is locally accessible from x_0, and x_0 is in the interior of $R^V(x_0, \leq T)$ for all $T > 0$ and each neighborhood V of x_0. If this holds for any $x_0 \in M$ then the system is called small-time locally controllable.

Let $X = \{X_0, \ldots X_m\}$. Let $Br(X)$ denote the set of all possible "brackets" of elements of X. Let $\delta_i(B)$ denote the number of occurrence of X_i in $B \in Br(X)$. An element $B \in Br(X)$ is said to be *bad* if $\delta_0(B)$ is odd and $\delta_i(B)$ is even for each $i = 1, \ldots, m$. A bracket is *good* if it is not bad. Let S_m denote the permutation group on m symbols. For $\pi \in S_m$ and $B \in Br(X)$, define $\bar{\pi}(B)$ to be the bracket obtained by fixing X_0 and sending X_i to $X_{\pi(i)}$ for $a = 1, \ldots, m.$. Now define

$$\beta(B) = \sum_{\pi \in S_m} \bar{\pi}(B).$$

Consider the bijection $\phi : X \to \{f, g_1, \ldots, g_m\}$ which sends X_0 to f and X_i to g_i for $i = 1, \ldots m$ define the evaluation map

$$Ev(\psi) : \mathcal{L}(X) \to \mathcal{L}(\mathcal{F}) : \sum_I \alpha_i X_i \mapsto \sum_I \alpha_i \psi X \quad \alpha_i \in \mathbb{R}$$

In [Sussmann, 1987] the following sufficient condition for STLC in terms of the Lie brackets and Lie algebra generated by the the vector fields $\{f, g_1, \ldots, g_m\}$ was given.

Theorem 4.3.2 *Consider the bijection $\psi : X \to \{f, g_1, \dots, g_m\}$ which sends X_0 to f and X_i to g_i for $i = 1, \dots m$. Suppose that the systems (4.3) is such that every bad bracket $B \in Br(X)$ has the property that*

$$Ev_x(\psi)(\beta(B)) = \sum_{i=1}^{m} \alpha^i Ev_x(\psi)(C_i)$$

where C_i are good brackets in $Br(X)$ of lower degree than B and $\alpha_i \in \mathbb{R}, i = 1, \dots, m$. Also suppose that (4.3) satisfies the LARC at x. Then (4.3) is STLC from x.

Hence if all bad brackets can be "neutralized" or can be expressed as a linear combination of good bracket of a lower degree then the system is STLC. In the case of a single input system [Sussmann, 1983] showed the following necessary condition for single input systems.

Theorem 4.3.3 *Consider an analytic system*

$$\dot{x} = f_0(x) + f_1(x)u, \quad |u(t)| \leq A \tag{4.9}$$

and a point x_0 such that

$$[f_1, [f_0, f_1]](x_o) \notin \mathcal{S}^1(f_0 + \widetilde{u} f_1, f_1)(x_0)$$

where $\mathcal{S}^1(X_1, X_2)$ is the linear span of X_1, X_2, and the brackets $(adX_1)^j X_2$ for $j \geq 1$ and $\widetilde{u}$ is such that $f_0(x_0) + \widetilde{u} f_1(x_0) = 0$. Then (4.9) is not STLC from x_0.

We use the above result to show that the unreduced jet-puck dynamics are not STLC.

$$
\begin{aligned}
\dot{x} &= \frac{\cos\theta P_1}{m} - \frac{\sin\theta P_2}{m} \\
\dot{y} &= \frac{\sin\theta P_1}{m} + \frac{\cos\theta P_2}{m} \\
\dot{\theta} &= \frac{\Pi}{I} \\
\dot{P}_1 &= \frac{P_2\Pi}{I} + u\cos\phi \\
\dot{P}_2 &= -\frac{P_1\Pi}{I} + u\sin\phi \\
\dot{\Pi} &= d\sin\phi
\end{aligned}
\tag{4.10}
$$

Proposition 4.3.4 *The unreduced dynamics (4.10) are locally strongly accessible if* $\sin\phi \neq 0$.

Proof: Given

$$
f = (\frac{P_1\cos\theta}{m} - \frac{P_2\sin\theta}{m})\frac{\partial}{\partial x} + (\frac{P_2\cos\theta}{m} + \frac{P_1\sin\theta}{m})\frac{\partial}{\partial y} + \frac{\Pi}{I}\frac{\partial}{\partial\theta}
$$

and

$$
g = \cos\phi\frac{\partial}{\partial P_1} + \sin\phi\frac{\partial}{\partial P_2} + d\sin\phi\frac{\partial}{\partial\Pi}
$$

we calculate the following brackets

$$
\begin{aligned}
\xi_1 &= [f,g] \\
&= \frac{\cos(\theta+\phi)}{m}\frac{\partial}{\partial x} + \frac{\sin(\theta+\phi)}{m}\frac{\partial}{\partial y} - \frac{d\sin\phi}{I}\frac{\partial}{\partial\theta} - \frac{(\Pi + dP_2)\sin\phi}{I}\frac{\partial}{\partial P_1} \\
&\quad + \frac{\Pi\cos\phi + dP_1\sin\phi}{I}\frac{\partial}{\partial P_2} \\
\xi_2 &= [[f,g],g] \\
&= \frac{2d\sin^2\phi}{I}\frac{\partial}{\partial P_1} - \frac{2d\cos\phi\sin\phi}{I}\frac{\partial}{\partial P_2} \\
\xi_3 &= [f,[[f,g],g]] \\
&= -\frac{2d\sin\phi\sin(\phi+\theta)}{Im}\frac{\partial}{\partial x} + \frac{2d\cos(\phi+\theta)\sin\phi}{Im}\frac{\partial}{\partial y} + \frac{\Pi d\sin 2\phi}{I^2}\frac{\partial}{\partial P_1}
\end{aligned}
$$

$$\frac{2\Pi d \sin^2\phi}{I^2}\frac{\partial}{\partial P_2}$$

$$\begin{aligned}
\xi_4 &= [[f,g],[[f,g],g]] \\
&= \frac{-2d^2\cos\phi\sin^2\phi}{I^2}\frac{\partial}{\partial P_1} - \frac{2d^2\sin^3\phi}{I^2}\frac{\partial}{\partial P_2} \\
\xi_5 &= [f,[[f,g],[[f,g],g]]] \\
&= \frac{2d^2\cos(\phi+\theta)\sin^2\phi}{I^2 m}\frac{\partial}{\partial x} + \frac{2d^2\sin^2\phi\sin(\phi+\theta)}{I^2 m}\frac{\partial}{\partial y} + \frac{2\Pi d^2\sin^3\phi}{I^3}\frac{\partial}{\partial P_1} \\
&\quad -\frac{2\Pi d^2\cos\phi\sin^2\phi}{I^3}\frac{\partial}{\partial P_2}
\end{aligned}$$

Again treating g and ξ_i's as coloumn vectors, observe that

$$det[g,\xi_1,\xi_2,\xi_3,\xi_4,\xi_5] = -\frac{16d^8\sin^8\phi}{I^7m^2}$$

Hence again if $\sin\phi \neq 0$, $\dim(\text{span}\mathcal{L}_{\{f,g\}})(p) = 6\ \forall p \in T^*SE(2)$. Also $[f,X] \in \text{span}(g,\xi_1,\xi_2,\xi_3,\xi_4,\xi_5)\ \forall X \in \{g,\xi_1,\xi_2,\xi_3,\xi_4,\xi_5\}$. Hence the complete system is locally strongly accessible. ∎

Proposition 4.3.5 *The unreduced hovercraft-dynamics dynamics defined (4.10) are not STLC from the origin.*

Proof: It is sufficient to consider STLC of the reduced dynamics (3.17). With $\widetilde{u} = 0$ observe that $\mathcal{S}^1(f,g)(0)$ is a one-dimensional space spanned by $\alpha\frac{\partial}{\partial P_1} + \beta\frac{\partial}{\partial P_2} + d\beta\frac{\partial}{\partial \Pi}$ while $[g,[f,g]](0) = -2\frac{d}{I}\beta^2\frac{\partial}{\partial P_1} + 2\frac{d}{I}\beta\alpha\frac{\partial}{\partial P_2}$.
Hence $[g,[f,g]](0) \notin \mathcal{S}^1(f,g)(0)$ ∎

Again, as the equations of the axisymmetric spacecraft with a single control (cf. 3.22) are similar in structure to those of the jet puck we can make similar comclusions about STLC.

Proposition 4.3.6 *The unreduced axisymmetric spacecraft dynamics defined (3.22) are not STLC from the origin.*

The dynamics of the asymmetric spacecraft with two gas jet actuators has been shown to be STLC in [Krishnan *et al.*, 1992; 1994].

We now study the STLC of the underwater vehicle with coincident center of mass and center of buoyancy. Again assuming the the principal axes of the vehicle and the principal axes of the displaced fluid are the same we have

$$J = diag(I_1, I_2, I_3) \text{ and } M = diag(m_1, m_2, m_3).$$

The unreduced dynamics on T^*G are given by

$$\dot{r} = RM^{-1}P \tag{4.11}$$

$$\dot{R} = R\widehat{J^{-1}\Pi} \tag{4.12}$$

$$\dot{\Pi} = \Pi \times J^{-1}\Pi + P \times M^{-1}P + U_1 \tag{4.13}$$

$$\dot{P} = P \times J^{-1}\Pi + U_2 \tag{4.14}$$

where $U_1 = (u_1, u_2, 0)^T$ and $U_2 = (u_3, 0, 0)^T$.

Proposition 4.3.7 *The reduced AUV dynamics defined by (4.13-4.14) are small-time locally controllable if* $I_1 \neq I_2$

Proof:

In Proposition 4.2.10 we already showed that the LARC was satisfied. Hence we need to verify that all bad brackets can be expressed as a linear combination of good brackets of lower degree. One first observes that from Theorem 4.3.2 all bad brackets are of odd degree. From (4.8) it follows that all brackets of degree 6 or higher can be expressed a linear combination of lower order brackets. Further all brackets in (4.8) are good. Hence we need to only check for brackets of order 1, 3, and 5. The degree 1 bracket is f which is equal to 0 at the equilibrium $(\Pi, P) = (0, 0)$. The degree 3 brackets are $[[f, g_i], g_i],\ i = 1, 2, 3$ which are equal to 0 for all (Π, P). The degree 5 brackets can be broken into three sets (i) $[[[[f, g_i], g_i], g_i], g_i],\ i = 1, 2, 3$ which are again equal to zero since $[[f, g_i], g_i] = 0,\ \ i = 1, 2, 3$, (ii) $[[[f, g_i], f], [f, g_i]],\ i = 1, 2, 3$ and (iii) $[[[f, g_i], [f, [f, g_i]],\ i = 1, 2, 3$. The brackets (ii) and (iii) are equal to zero at $(\Pi, P) = (0, 0)$. (The verification was done using Mathematica). Hence we conclude that the reduced dynamics are STLC. ∎

While calculating the LARC and verifying STLC conditions for the unreduced dynamics (4.11-4.14) of the autonomous underwater vehicle can be very tedious and messy, we conjecture that the unreduced dynamics are STLC.

4.4 Cotangent Space Controllability

In this section we exploit the reduction procedure to gain some insight into the controllability properties of the unreduced dynamics. Before we present our results we recall a few definitions.

Definition 4.4.1 A map $\psi : M \to N$ is called a proper map if $\psi^{-1}(V)$ is compact for all compact $V_n \subset N$.

Definition 4.4.2 *An action $\Phi : G \times M \to M$ is proper if the mapping $\widetilde{\phi}$: $G \times M \to M \times M$, defined by $\widetilde{\Phi}(g, x) = (x, \Phi(g, x))$ is proper*

Lemma 4.4.3 *Let G be a compact Lie group whose action $\Phi : G \times M \to M$ on a manifold M is free. Let $\pi : M \to M/G$ denote the projection map. Then $D = \pi^{-1}(\widetilde{D})$ is compact iff $\widetilde{D} \subset M/G$ is compact i.e. the projection map π is a proper map.*

Proof: Assume that D is compact. Since G is compact Φ is proper (see previous remark). Hence from Proposition 2.1.2, π is a smooth submersion. Since π is a smooth submersion, if D is compact, then $\widetilde{D} = \pi(D)$ is compact.

(<=) Now assume that $\widetilde{D}$ is compact. Let $\{y_k\}$ be an sequence in $D = \pi^{-1}(\widetilde{D})$. Let $\{x_k\} = \{\pi(y_k)\}$. $\{x_k\} \in \widetilde{D}$, and since $\widetilde{D}$ is compact $\{x_k\}$ has convergent subsequence $\{x_{k_j}\}$ that converges to $x^* \in \widetilde{D}$. Now consider the subsequence $\{y_{k_j}\}$ such that $y_{k_j} \in \pi^{-1}(x_{k_j})$. Since $\{x_{k_j}\}$ converges to x^*, $\{y_{k_j}\}$ converges to $\pi^{-1}(x^*)$ i.e. given any ϵ there exists N and $\{y'_{k_j}\} \in \pi^{-1}(x^*)$ s.t $\|y_{k_j} - y'_{k_j}\| < \epsilon/2$ for all $k_j > N$. Since G is compact, $\pi^{-1}(x^*)$ is compact and hence there exists a convergent subsequence $\{y'_{k_{jm}}\}$ that converges to some $y^* \in \pi^{-1}(x^*)$, i.e. given any ϵ there exists N' s.t. $\|y'_{k_{jm}} - y^*\| < \epsilon/2$ for all $n_i > N'$. Thus there exists a subsubsequence $\{y_{k_{jm}}\}$ and N'' such that $\|y_{k_{jm}} - y'_{k_{jm}}\| < \epsilon/2$ and hence $\|y_{k_{jm}} - y^*\| < \epsilon$ for all $k_{jm} > N''$, i.e the subsubsequence $\{y_{k_{jm}}\}$ converges to $y^* \in \pi^{-1}(x^*) \subset D$. Hence we conclude that D is compact. ∎

Theorem 4.4.4 *Let G be a compact Lie group whose action on a Poisson manifold M is free and proper. A G-invariant Hamiltonian vector field X_H defined on a manifold M is WPPS if there exists a function $V : M/G \to \mathbb{R}$ that is* *proper and $\dot{V} = 0$ along trajectories of the projected vector field $X_{\widetilde{H}}$ defined on M/G*

Proof: Let $\widetilde{D} = \{\mu \mid V(\mu) \leq E, \mu in M/G\}$ Then the integral curve of $X_{\widetilde{H}}$ starting at $\mu_0 \in \widetilde{D}$, denoted by $\phi_t^{X_{\tilde{H}}}(\mu_0)$, lies entirely in $\widetilde{D}$. Since $\widetilde{D}$ is closed and bounded in M/G, it is compact. Let $\phi_t^{X_H}(x_0)$ be the integral curve of the Hamiltonian vector field X_H starting at x_0, at $t = 0$. At any given time $t' > 0$, $\phi_{t'}^{X_H}(x_0) \in \pi^{-1}(\phi_t^{X_{\tilde{H}}}(\mu_0))$ where $\mu_0 = \pi(x_0)$. But $\phi_t^{X_{\tilde{H}}}(\mu_0)) \in \widetilde{D}, \forall\, t > 0$. Hence $\phi_{t'}^{X_H}(x_0) \in \pi^{-1}(\widetilde{D})$. Since $\widetilde{D}$ is compact from Lemma 4.4.3 $\pi^{-1}(\widetilde{D})$ is compact and the integral curve of the Hamiltonian vector field X_H starting at x_0 is restricted to the compact set $\pi^{-1}(\widetilde{D})$. To study the dynamics of X_H through x_0 we restrict ourselves to the symplectic leaf, induced by the Poisson bracket on M, passing through x_0. Let Σ be the symplectic leaf passing through x_0. (If the Poisson bracket on M is the Lie-Poisson bracket then Σ is the coadjoint orbit through x_0.) The integral curve $\phi_t^{X_H}(x_0)$ lies entirely in $W = \pi^{-1}(\widetilde{D}) \bigcap \Sigma$, which is compact in Σ, and hence as in the proof of Theorem 4.2.2 X_H is WPPS. ■

Again having concluded WPPS nature of the Hamiltonian vector field, if the Hamiltonian control system on M and M/G satisfy the LARC, then from Theorem 4.1.4 controllability can be concluded.

Remark 4.4.5 See also result on controllability on principal fiber bundles with compact structure group [Martin and Crouch, 1984].

While Theorem 4.4.4 gives a sufficient condition to check for WPPS of drift vector field and hence for controllability of systems where the symmetry group was compact it is not of too much help in the noncompact case. In the present setting of the hovercraft and the underwater vehicles we observe that though $SE(n),\ n = 2, 3$ is not a compact group, it is a semidirect product, i.e. $G = H \times_\rho V$ where $H = SO(n)$ is compact and $V = \mathbb{R}^n$ is a vector space. In the setting of semidirect products one can make the following observation (See [Rose, 1978] for the proof).

Theorem 4.4.6 *Let H act on V, with the action ρ. Let $G = H \times_\rho V$. Then*

1. *H is a subgroup of G.*

2. *V is a normal subgroup of G*

3. *$G/V \cong H$*

Hence if the dynamics are G invariant and G is a semidirect product, then one can perform reduction of dynamics in two stages. First by V, to obtain dynamics on $H \times \mathfrak{g}^*$, and then by H to obtain the reduced dynamics on $\mathfrak{g}^*$. Exploiting this reduction by stages we have the following controllability result.

Theorem 4.4.7 *Let $G = H \times_\rho V$, H compact and V a vector space. Then the Lie-Poisson reduced dynamics defined on T^*G/G, corresponding to G-invariant dynamics are controllable iff the reduced dynamics on $H \times T^*G$ are controllable.*

Applying these results to the examples discussed earlier we have the following results.

Proposition 4.4.8 *The reduced dynamics on $SO(2) \times se(2)^*$ given by*

$$\dot{\theta} \ = \ \frac{\Pi}{I}$$

$$\begin{aligned}\dot{P}_1 &= \frac{P_2\Pi}{I} + u\cos\phi \\ \dot{P}_2 &= -\frac{P_1\Pi}{I} + u\sin\phi \\ \dot{\Pi} &= d\sin\phi\end{aligned}$$

are controllable if $\sin\phi \neq 0$.

Proof: Let $f = (\frac{\Pi}{I}, \frac{P_2\Pi}{I}, -\frac{P_1\Pi}{I}, 0)^T$ and $g = (0, \cos\phi, \sin\phi, d\sin\phi)^T$. Observe that

$$\det(g[f,g]\ [[f,g],g]\ [[f,g]\ [[f,g],g]]) = -\frac{4d^5\sin^5\phi}{I^4}.$$

Hence LARC is satisfied iff $sin\phi \neq 0$ The proof follows from Proposition 4.2.5 Theorem 4.4.7.

■

Remark 4.4.9 In the case of the spacecraft since G is compact one can now conclude controllability of the complete dynamics in the case of axisymmetric spacecraft with one control and the asymmetric spacecraft with 2 pure torques. While this is an old result we have provided what we think as a more elegant proof to the problem as compared with that of [Crouch, 1984].

Proposition 4.4.10 *The reduced dynamics (4.12-4.14) of the underwater vehicle with coincident center of mass and center of buoyancy, defined on* $SO(3) \times \mathfrak{se}(3)^*$ *are controllable if* $I_1 \neq I_2$.

Proof:

The proof follows from Proposition 4.2.10, theorem 4.4.4 and the LARC. The LARC was verified using Mathematica.

■

We end this chapter with a sufficient condition for controllability of the unreduced dynamics defined on T^*G where G need not be compact. We use the results of [Lewis and Murray, 1996; Lewis, 1995; Bullo and Lewis, 1996] in the proof of our theorem. Here sufficient conditions for configuration controllability (see definitions below) of mechanical systems on a Lie group, with a left invariant Lagrangian and left-invariant forces are presented. The results presented are derived assuming that the dynamics are written on TG as opposed to T^*G, as has been the case in our work. But in class of problems that we are considering (see below for precise statement on assumptions of the class of mechanical systems) the two formulations are equivalent and are related via the fiber derivative, or the Legendre transform. Their results on configuration controllability coupled with our results on controllability of reduced dynamics will be used to prove a sufficient condition for controllability of the complete (unreduced) dynamics. Before presenting the result we briefly discuss definitions and previous result on configuration controllability as is applicable to the present setting.

Consider a mechanical system, whose configuration space can be identified with a Lie group G. Let $L : TG \to \mathbb{R}$ be a left (G) invariant Lagrangian. Let $\widetilde{L} : \mathfrak{g} \to \mathbb{R}$ be the restriction of the Lagrangian to the identity. Let $f^i u_i(t),\ i = \{1, \ldots, m\}$ denote left invariant forces. Let $\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g}; \eta \mapsto [\xi, \eta]$ denote the adjoint map and ad^*_ξ denote its dual.

In terms of the configuration $g \in G$ and body fixed velocities $\xi \in \mathfrak{g}$ the motion of the system can now be defined by

$$\dot{g} \;=\; g \cdot \xi \tag{4.15}$$

$$\frac{d}{dt}\frac{\delta \widetilde{L}}{\delta \xi} = \mathrm{ad}_{\xi}\frac{\delta \widetilde{L}}{\delta \xi} + \sum_{i=1}^{m} f^i u_i \tag{4.16}$$

Equation 4.16 is called the Euler-Poincaré equation (cf. [Marsden and Scheurle, 1993b; 1993a; Bloch *et al.*,] and references therein.) The Euler-Poincare equations represent the reduced dynamics on $TG/G \cong \mathfrak{g}$. The equivalence between the Euler-Poincaré and Lie-Poisson dynamics can be seen by making the following Legendre transformation for $\mathfrak{g}$ to $\mathfrak{g}^*$.

$$\mu = \frac{\delta \widetilde{L}}{\delta \xi}, \quad \widetilde{H} =< \mu, \xi > -\widetilde{L}(\xi).$$

Observing that

$$\frac{\delta \widetilde{H}}{\delta \mu} = \xi + \langle \mu, \frac{\delta \xi}{\delta \mu} \rangle - \langle \frac{\delta \widetilde{L}}{\delta \xi}, \frac{\delta \xi}{\delta \mu} \rangle = \xi$$

if follows that the Euler-Poincaré equations are equivalent to

$$\dot{\mu} = \mathrm{ad}^*_{\frac{\delta \widetilde{H}}{\delta \mu}} \mu + \sum_{i=1}^{m} f^i u_i$$

where the drift term is nothing but the Lie-Poisson reduced dynamics (cf. [Marsden and Ratiu, 1994] Theorem 13.6.2). In the setting where the Lagrangian is the kinetic energy of the system and $\widetilde{L} = \xi^T \mathbb{I} \xi$, where $\mathbb{I} : \mathfrak{g} \to \mathfrak{g}^*$ is the inertia tensor, (4.15-4.16) can be written as

$$\dot{g} = g\xi \tag{4.17}$$

$$\mathbb{I}\dot{\xi} = \mathrm{ad}^*_{\xi} \mathbb{I} \xi + \sum_{i=1}^{m} f^i u_i \tag{4.18}$$

Having shown the equivalence between the two formulation we now define configuration controllability on (4.15-4.16), and present results by [Lewis and Murray, 1996; Lewis, 1995; Bullo and Lewis, 1996] on configuration controllability before we present our result.

Let $g_0 \in G$ and let V be a neighborhood of G. Define

$$\begin{aligned} R_G^V(g_0, T) = \{g \in G \mid & \text{ there exists an admissible input } u : [0, T] \to U \\ & \text{such that the evolution for } (4.17\text{-}4.18) \text{ with initial conditions} \\ & g(0) = g_0,\ \xi(0) = 0 \text{satisfies } g(t) \in V,\ 0 \leq t \leq T \text{ and } g(T) = g\} \end{aligned}$$

Let

$$R_G^V(g_0, \leq T) = \bigcup_{0 \leq t \leq T} R_G^V(g_0, t)$$

Observe that in the above definition of a reachable set, the set of initial conditions is restricted to the set with zero initial velocity and further the final velocity is not relevant. Further the reachability set is defined in terms of a neighborhood V of $g_0 \in G$ and not of $(g_0, \xi_0) \in TG$.

Definition 4.4.11 The system is *small time locally configuration controllable* at g_0 if there exists a $T > 0$ such that $g_0 \in int(R_G^V(g_0, \leq t))$ for every neighborhood V of g_0 and $0 < t \leq T$

Definition 4.4.12 The system (4.17-4.18) is *equilibrium controllable* if for any $(g_1, 0)$, $(g_2, 0)$ there exist at $T > 0$ and an admissible input $u : [0, T] \to U$ such that the solution $(g(t), \xi(t))$ of (4.17-4.18) with initial conditions $(g(0), \xi(0)) = (g_1, 0)$ satisfy $(g(T), \xi(T)) = (g_2, 0)$.

Remark 4.4.13 The set of points $\mathcal{E} = \{(g, 0) : g \in G\}$ defines the set of all equilibrium points of (4.17-4.18).

Definition 4.4.14 The *symmetric product* of $\langle \cdot : \cdot \rangle : \mathfrak{g} \to G : \xi, \eta \mapsto \langle \xi : \eta \rangle$ is defined as

$$\langle \xi : \eta \rangle = -\mathbb{I}^{-1}(\mathrm{ad}_\xi^* \mathbb{I} \eta + \mathrm{ad}_\eta^* \mathbb{I} \xi) \tag{4.19}$$

Let $B = \{b_1, \ldots, b_m\} \subset \mathfrak{g}$ (a left invariant distribution on G) denote the input subspace. In the present setting $b_i = \mathbb{I}^{-1} f^i$. Let $\overline{Lie}_{\mathfrak{g}}(\mathcal{B})$ and $\overline{Sym}_{\mathfrak{g}}(\mathcal{B})$ denote the involutive and symmetric closure of $\mathcal{B}$ in $\mathfrak{g}$. As in section 3.2 on small time local controllability, a symmetric product is *bad* if it contains an even number of each of the vectors in $\mathcal{B}$. A symmetric bracket is *good* if it is not bad.

Theorem 4.4.15 *The system (4.17-4.18) is*

(i) locally configuration accessible if rank$(\overline{Lie}_{\mathfrak{g}}(\overline{Sym}_{\mathfrak{g}})(\mathcal{B})) = dim(G)$ *and*

(ii) equilibrium controllable if it is locally configuration accessible and if every bad symmetric product can be written as a linear combination of good symmetric products of lower degree.

Proof: [Lewis, 1995] ■

We now present a sufficient condition for controllability of (4.17-4.18).

Theorem 4.4.16 *If the dynamics of the mechanical systems given by (4.17 - 4.18) are such that*

(i) The system is equilibrium controllable, and

(ii) the reduced dynamics (4.18) are controllable,

then the system is controllable.

Proof: To show controllability [2] we need to show that there exists a $T > 0$ and an admissible control $u : [0, T] \to U$ such that given any (g_1, ξ_1) and (g_f, ξ_f) the

[2] "he euthor thanks Herbert Struemper Yor ciscussions ln this croof.

solution $(g(t), \xi(t))$ satisfies $(g(0), \xi(0)) = (g_1, \xi_1)$ and $(g(T), \xi(T)) = (g_f, \xi_f)$. Using the properties (i) and (ii) we construct such a control.

Assume that there exists a $(g_3, 0)$ such that there exists an admissible control u' that will steer the system from $(g_3, 0)$ to (g_f, ξ_f).(The existence of such a $(g_3, 0)$ an u' is shown later.) The problem is now reduced to finding a control to steer the system form (g_1, ξ_1) to (g_3, ξ_3) which is done as follows.

Let $g(t, 0, g_0, \xi(t))$ denote the the solution of (4.17) at $t > 0$ for a particular curve $\xi(t) \in \mathfrak{g}$ and initial condition g_0. Similarly let $\xi(t, 0, \xi_0, \xi(t))$ denote the the solution of (4.18) at $t > 0$ for a particular input u and initial condition ξ_0 and $\zeta(t, 0, (g_0, \xi_0), u)$ denote the solution of (4.17-4.18) at $t > 0$ for a particular input u and initial condition (g_0, ξ_0).

1. Since the reduced dynamics are controllable there exists a control u_1 such that

$\zeta(T_1, 0, (g_1, \xi_1), u_1) = (g_2, 0)$.

2. Since the dynamics are equilibrium controllable there exists a control u_2 such that $\zeta(T_2, 0, (g_2, 0), u_1) = (g_3, 0)$.

3. Finally applying u_3 we have $\zeta(T_3, 0, (g_3, 0), u_3) = (g_f, x_f)$.

The existence of $(g_3, 0)$ and u_3 is shown as follows. Find u_3, such that $\xi(T_3, 0, 0, u_3) = \xi_f$. Existence of such a control follows from the reduced space controllability of 4.18. Apply the control u_3 to (4.17-4.18) with initial condition $\xi(0) = 0$ and arbitrary $g(0) = g_3'$. Then $\zeta(T_3, 0, (g_3', 0), u_3) = (g_4, \xi_f)$ where g_4 need not be equal to g_f. Let $g(t, 0, g_3', \xi(t))$ denote the solution to (4.17) where $\xi(t) = \xi(t, 0, 0, u_3)$. Let $R \in G$. Then by left invariance $\bar{g} = Rg(t, 0, g_3', \xi(t))$ is a solution to (4.17-4.18). Choose R such that $\bar{g}(T_3) = Rg(T_3, 0, g_3', \xi(t)) = g_f$, i.e $R = g_4^{-1} g_f$ and hence $\bar{g}(t) = g_4^{-1} g_f g(t, 0, g_3', \xi(t))$. Again left-invariance implies that $\bar{g}(T_3, 0, g_4^{-1} g_f g_3', \xi(t)) = g_f$ or equivalently $\zeta(T_3, 0, (g_4^{-1} g_f g_3', 0), u_3) = (g_f, \xi_f)$. Hence choose $g_3 = g_4^{-1} g_f g_3'$ and $u' = u_3$. ∎

Remark 4.4.17 Given a mechanical system with symmetry i.e. G-invariant dynamics, configuration controllability and controllability of reduced space can be verified using Theorem 4.4.15 and Theorem 4.2.2.

We now apply Theorem 4.4.16 to the autonomous underwater vehicle with coincident center of mass and center of buoyancy.

Proposition 4.4.18 *The unreduced dynamics (4.11-4.14) of the autonomous underwater vehicle with coincident center of mass and center of buoyancy, defined on $T^*SE(3)$ (or equivalently $TSE(3)$) are controllable if $I_1 \neq I_2$*

Proof: As shown in Theorem 4.4.16, controllability of (4.11-4.14) can be shown if controllability of reduced dynamics (4.13-4.14) and equilibrium controllability of (4.11-4.14) can be shown. In Proposition 4.2.10 controllability of reduced dynamics has already been show. We now show that the dynamics are equilibrium controllable. Defining J and M as in (4.6). Rewriting the reduced dynamics on $\mathfrak{se}(3)$ we have.

$$\dot{\Omega} = J^{-1}(J\Omega \times \Omega + Mv \times v) + J^{-1}U_1 \quad (4.20)$$

$$\dot{v} = M^{-1}(Mv \times \Omega) + M^{-1}U_2 \quad (4.21)$$

Where Ω and V are as defined in Section 3.3, $U_1 = (u_1, u_2, 0)$ and $U_2 = (u_3, 0, 0)$. Thus the input space is spanned by the vectors

$$b_1 = \begin{bmatrix} \frac{1}{I_1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 \\ \frac{1}{I_2} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad b_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix}$$

Calculating the symmetric brackets we have

$$\langle b_1 : b_2 \rangle = \begin{bmatrix} 0 \\ 0 \\ \frac{I_1 - I_2}{I_1 I_2 I_3} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \langle b_2 : b_3 \rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{I_2 m_3} \end{bmatrix}, \quad \langle b1 : \langle b_2 : b_3 \rangle \rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{-1}{I_1 I_2 m_2} \\ 0 \end{bmatrix}$$

Observing that

(i) $\mathcal{D} = \{b_1, b_2, b_3, \langle b_1 : b_2 \rangle, \langle b_2 : b_3 \rangle, \langle b1 : \langle b_2 : b_3 \rangle \rangle\}$ spans $\mathbb{R}^6$ if $I_1 \neq I_2$.

(ii) Every symmetric bracket in $\mathcal{D}$ is good, and from (i) every bracket of degree 4 or higher degree can be expressed a combination of good lower-degree good brackets.

(iii) Every bad symmetric bracket of degree 2 is of the form $\langle b_i : b_i \rangle$ $i = 1, 2, 3$ and is equal to 0, It follows that the dynamics are equilibrium controllable. Hence controllability follows from theorem 4.4.16.

■

Remark 4.4.19 in the case of the unreduced jet-puck dynamics as we have only one input, every non-trivial second-order symmetric bracket is bad. Hence sufficient conditions for equilibrium controllability are not satisfied and hence we can not conclude controllability of unreduced dynamics. In [Lynch and Mason, 1997] the problem of controlling a hovercraft (planar rigid body) is discussed in connection with dynamcis prehensile manipulation. Here is it shown, using similar ideas that the unreduced dynamcis of a hovercraft are controllable with two unidirectional thrusters providing opposite torques.

Chapter 5

Control and Stabilization

Having shown controllability of the reduced and in some case of the unreduced dynamics of a class of mechanical systems in Chapter 4, in this chapter we study the stability, stabilization and control of the dynamics of these systems.

When studying the class of mechanical systems discussed in Chapter 2, the stability of certain trajectories of the free dynamics is of practical interest. For example in the rigid body example, the stability of a motion corresponding to spinning about a certain axis, is of crucial interest in satellite control. Similarly engineers are interested in the stability of motions corresponding to rotations and translation about a principal axis in the case of the underwater vehicle. These trajectories correspond to group orbits and hence project to equilibrium points of the reduced dynamics and are called relative equilibria. One of the main goals of this chapter is to study the stability and stabilization of these relative equilibria.

5.1 Stability of Relative Equilibria

Let M be a differentiable manifold and G be a Lie group. Let $\Phi_g : x \mapsto \Phi_g(x),\ x \in M,\ g \in G$ denote the action of G on M. Let X be a G invariant vector field defined on M.

Definition 5.1.1 A *relative equilibrium* of X is a point $x_e \in M$ such that for some $\xi_e \in \mathfrak{g}$ called the generator of the relative equilibrium, the curve

$$t \mapsto exp(\xi_e t)x_e$$

is an integral curve of X starting at x_e.

In other words x_e is a relative equilibrium if the flow of X starting at x_e is an orbit of x_e corresponding to the action of the one-parameter subgroup $exp(\xi_e t)$. If G acts regularly and freely on M, then M/G is a manifold and X projects to $\widetilde{X}$ on M/G (c.f. Proposition 2.1.2). Since the dynamical orbit starting at x_e is a group orbit, it projects to an equilibrium of the vector field $\widetilde{X}$. Hence one may alternately define a relative equilibrium as:

Definition 5.1.2 A point x_e is called a *relative equilibrium* of X iff

$$\widetilde{X}(\pi(x_e)) = 0,$$

where $\pi : M \to M/G$ is a smooth submersion.

Before we present any tools to study stability of relative equilibria it becomes essential to define stability as the word "stable dynamics" has been interpreted in various ways in literature. Different interpretations of stability can lead to different stability criteria. To avoid any confusion we present definitions and results on stability of autonomous systems.

Consider the autonomous system

$$\dot{x} = X(x) \tag{5.1}$$

where $x = (x_1, \dots x_n)$ are local coordinates for a smooth manifold M and X is a smooth vector field. Let x_e be an equilibrium point of (5.1) i.e.

$$X(x_e) = 0.$$

Definition 5.1.3 The equilibrium point x_e is said to be

- *locally stable* if for any neighborhood V_0 of x_e there exists a neighborhood $\widetilde{V}$ of x_e such that for all $x(0) = x_0 \in \widetilde{V}$ the solution $x(t, 0, x_0)$ of (5.1) belongs to V for all $t \geq 0$. Or equivalently, for each $\epsilon > 0$

  $$\exists \ \delta = \delta(\epsilon) > 0, \ suchthat \ \|x_0 - x_e\| < \delta \Rightarrow \ \|x(t, 0, x_0) - x_e\| < \epsilon \ \ \forall t \geq 0$$

- *locally asymptotically stable* if for any neighborhood V_0 of x_e there exists a neighborhood $\widetilde{V}$ of x_e such that for all $x(0) = x_0 \in \widetilde{V}$ the solution $x(t, 0, x_0)$ of (5.1) converges to x_e as $t \to \infty$. Or equivalently, x_e is stable and

  $$\|x(0) - x_e\| < \delta \Rightarrow \lim_{t \to \infty} \|x(t, 0, x_0) - x_e\| = 0$$

- *linearly stable* if the linearized equation

  $$\dot{x} = Ax \qquad \text{where} \qquad A = \frac{\partial f}{\partial x}(x) \mid_{x_e} \tag{5.2}$$

  is locally stable.

- *spectrally stable* if the all the eigenvalues of A have non-positive real parts.

Remark 5.1.4 If all the eigenvalues of the A lie in the open left half plane then the know that the linear system is (5.2) is asymptotically stable and we can conclude from Lyapunov's first method that the equilibrium point x_e of (5.1) is locally asymptotically stable. If $X(x)$ is a conservative system then the eigenvalues of the linearized system are symmetrically distributed under reflection about the real and imaginary axis. Hence in this setting one can conclude spectral stability.

We now recall Lyapunov's direct method (also know as the second method of Lyapunov) that allows us to determine the stability of a system without explicitly integrating the system.

Theorem 5.1.5 *Let $x = 0$ be an equilibrium point of (5.1). Let $V : D \to \mathbb{R}$ be a continuously differentiable function on a neighborhood D of $x = 0$, such that*

$$V(0) = 0 \quad \text{and } V(x) > 0 \ \text{ in } D - \{0\} \quad \text{and}$$

$$L_X(V) \leq 0 \quad \text{in } D.$$

Then, $x_e = 0$ is stable. Moreover if

$$L_X(V) < 0 \quad \text{in } D - \{0\}$$

then $x = 0$ is asymptotically stable.

To study the stability of relative equilibria, $x_e \in M$ of a G-invariant vector field, X we study the study the the stability of its projection $\mu_e = \pi(x_e)$ with respect to the reduced dynamics $X_{\widetilde{H}}$ defined on M/G.

Definition 5.1.6 A relative equilibrium $x_e \in M$ of a vector field $X \in \mathcal{X}(M)$ is *relatively stable modulo G,* or simply stable if the equilibrium $\mu_e = \pi(x_e)$ is stable with respect to the reduced dynamics $\widetilde{X} \in \widetilde{\mathcal{X}}(M/G)$.

In the setting of Lie-Poisson reduced dynamics on $\mathfrak{g}^*$ the autonomous system of interest to us, setting $u = 0$, takes the form (c.f. Theorem 2.1.10)

$$\dot{\mu} = X_{\widetilde{H}}(\mu) = \Lambda(\mu)\nabla\widetilde{H}(\mu). \tag{5.3}$$

Since $X_{\widetilde{H}}$ is a Hamiltonian vector field, $L_{X_{\widetilde{H}}}(\widetilde{H}) = X_{\widetilde{H}}(\widetilde{H}) = \{\widetilde{H}, \widetilde{H}\} = 0$. Hence $\widetilde{H}$ is trivially a conserved quantity. Lets further assume that the null space of the Poisson tensor $\Lambda(\cdot)$ is not empty and is spanned by the Casimirs $C_i(\mu)\ \ i = 1, \ldots, m$ i.e. $\Lambda(\mu)\nabla C_i(\mu) = 0$. Then μ_e is an equilibrium of (5.3) if and only if

$$\nabla\widetilde{H}(\mu_e) = \sum_{i=1}^{m} \lambda_i \nabla C_i(\mu_e), \qquad \lambda_i \in \mathbb{R},$$

or

$$\nabla(\widetilde{H} - \sum_{1=1}^{m} \lambda_i \nabla C_i)(\mu_e) = 0$$

Now (5.3) can be rewritten as

$$\dot{\mu} = \Lambda(\mu)\nabla(\widetilde{H} - \sum_{1=1}^{m} \lambda_i \nabla C_i)(\mu)$$

Hence it follows that $\nabla(\widetilde{H} - \sum_{i=1}^{m} \lambda_i \nabla C_i)$ are conserved quantities along trajectories of (5.3).

The Casimirs and the reduced Hamiltonian $\widetilde{H}$ can be exploited to come up with a suitable choice of a Lyapunov function. This approach is known as the energy Casimir method [Arnold, 1969] and is a generalization of the Lagrange-Dirichlet theorem.

5.1.1 Energy-Casimir Method

Theorem 5.1.7 *If there exists a Casimir function C (or in some examples a Casimir plus other conserved quantities) such that*

$$\nabla(H+C)(\mu_e) \quad = \quad 0 \tag{5.4}$$

$$\nabla^2(H+C)(\mu_e) \quad > \quad 0, \qquad (\text{ or } < 0) \tag{5.5}$$

then μ_e is a stable equilibrium of (5.3).

Proof: (cf. [Wang, 1990]) Choose

$$V(\mu) = (H+C)(\mu) - (H+C)(\mu_e).$$

By assumption $\nabla^2(H+C)(\mu_e)$ is positive-definite and hence μ_e is a strict local minimum. Thus there exists a neighborhood U of μ_e such that $V(\mu_e) = 0$ and $V(\mu) > 0 \;\; \forall \; \mu \in U - \{x_e\}$. Further since $H + C$ is a conserved quantity along trajectories of the system

$$L_{X_{\tilde{H}}}(V) = 0 \;\; \forall \mu \in U - \{x_e\}.$$

Hence from Lyapunov's direct method it follows that x_e is a stable equilibrium.

■

The approach to study the stability of a relative equilibria using the energy Casimir method can be summarized as follows:

(1) Consider a function $H_{\Phi,\Psi} = H + \Phi(C_1, \ldots, C_n) + \Psi(K_1, \ldots, K_n)$, where H is the Hamiltonian, $C_1, \ldots, C_n$ are Casimirs such that ∇C_i span the null space of the Poisson tensor $\Lambda(\mu)$, and K_i are other conserved quantities.

(2) Choose Φ, Ψ such that $H_{\Phi,\Psi}$ has a critical point at the relative equilibrium of interest.

(3) Definiteness of the second variation of $H_{\Phi,\Psi}$ at the critical point is sufficient for Lyapunov stability.

The energy-Casimir method provides a systematic method to determine the stability of the equilibrium of the reduced dynamics. As an example application of the energy Casimir we apply it to study the the stability of relative equilibria of the jet puck dynamics. Recall that the jet puck dynamics are given by

$$\begin{aligned}\dot{P}_1 &= P_2\Pi/I + \alpha u \\ \dot{P}_2 &= -P_1\Pi/I + \beta u \\ \dot{\Pi} &= \gamma u\end{aligned} \tag{5.6}$$

where $\alpha = \sin\phi$, $\beta = \sin\phi$ and $\gamma = d\sin\phi$. The reduced Hamiltonian is given by

$$\widetilde{H} = \frac{1}{2I}\Pi^2 + \frac{\|P\|^2}{2m} \tag{5.7}$$

The dynamics (5.6) are Hamiltonian with respect to the Hamiltonian $\widetilde{H}$ and the Lie-Poisson structure on $\mathfrak{se}(2)^*$, given by

$$\Lambda = \begin{bmatrix} 0 & 0 & P_2 \\ 0 & 0 & -P_1 \\ -P_2 & P_1 & 0 \end{bmatrix}. \tag{5.8}$$

The equilibria of (5.6), the reduced dynamics, are given by $E = \{P_1, P_2, \Pi \mid P_1 = P_2 = 0\} \bigcup \{P_1, P_2, \Pi \mid \Pi = 0\}$

The Casimir for this Poisson structure is given by

$$C = P \cdot P = \|P\|^2 \tag{5.9}$$

where $P = (P_1, P_2)^T$. Any function $\Phi(P \cdot P)$ is also a Casimir. Since Π is a constant of motion any function $\Psi(\Pi)$ too is constant along flows of (5.6).

We now study the stability of a particular equilibrium $\mu_e = (\Pi^0, 0, 0),\ \ \Pi^0 \neq 0$. Choose

$$H_{\Phi,\Psi} = \widetilde{H} + \Psi(P \cdot P) + \Psi(\Pi) \tag{5.10}$$

The first variation is given by

$$\begin{aligned} DH_{\Phi,\Psi}(\delta\Pi, \delta P_1, \delta P_2) \;\; = \;\; & (\frac{\Pi}{I} + \Psi'(\Pi)) \cdot \delta\Pi + (\frac{P_1}{m} + \Phi'(P \cdot P)2P_1) \cdot \delta P_1 \\ & (\frac{P_2}{m} + \Phi'(P \cdot P)2P_2) \cdot \delta P_2 \end{aligned} \tag{5.11}$$

where

$$\Psi' = \frac{\partial \Psi}{\partial \Pi} \qquad \text{and} \qquad \Phi' = \frac{\partial \Phi}{\partial (P \cdot P)}$$

Now $(DH_{\Phi,\Psi}) \mid_{\mu_e} = 0$ implies

$$\Psi'(\Pi) = -\frac{\Pi^0}{I}, \text{ and } \Phi'(0) \text{ can be chosen arbitrarily} \tag{5.12}$$

The second variation $D^2 H_{\Phi,\Psi}$ evaluated at μ_e is

$$\begin{bmatrix} \frac{1}{I} + \Psi''(\Pi^0) & 0 & 0 \\ 0 & \frac{1}{m} + 2\Phi'(0) & 0 \\ 0 & 0 & \frac{1}{m} + 2\Phi'(0) \end{bmatrix} \tag{5.13}$$

Definiteness of the second variation implies

$$\begin{aligned} \frac{1}{I} + \Psi''(\Pi^0) &> 0, \\ (\frac{1}{I} + \Psi'(\Pi^0))(\frac{1}{m} + 2\Phi'(0)) &> 0, \\ (\frac{1}{I} + \Psi''(\Pi^0))(\frac{1}{m} + 2\Phi''(0))^2 &> 0 \end{aligned} \tag{5.14}$$

We can obviously choose Φ, Ψ to satisfy (5.12) and (5.14). For example choose $\Psi = -\frac{\Pi\Pi^0}{I}$ and $\Phi = 0$. Hence we can conclude that $(\Pi^0, 0, 0)$ is a stable relative equilibrium.

We now study the stability of the equilibrium $\mu_e = (0, P_1^0, P_2^0)$. Again $(DH_{\Phi,\Psi}) |_{\mu_e} = 0$ implies

$$\Psi'(\Pi) = 0, \text{ and } \Phi'(P^0 \cdot P^0) = -\frac{1}{m} \tag{5.15}$$

The second variation $D^2 H_{\Phi,\Psi}$ evaluated at $\mu_e = (0, P_1^0, P_2^0)$ is

$$\begin{bmatrix} \frac{1}{I} + \Psi'(0) & 0 & 0 \\ 0 & 4\Phi''(P^0 \cdot P^0)(P_1^0)^2 & 4\Phi''(P^0 \cdot P^0)(P_1^0 P_2^0) \\ 0 & 4\Phi''(P^0 \cdot P^0)(P_1^0 P_2^0) & 4\Phi''(P^0 \cdot P^0)(P_1^0)^2 \end{bmatrix} \tag{5.16}$$

Since $\det(D^2 H_{\Phi,\Psi}) |_{(0,P_1^0,P_2^0)} = 0$, it implies that the second variation is semidefinite and the energy Casimir method in this case is inconclusive in determining the stability of the equilibrium point $\mu_e = (0, P_1^0, P_2^0)$.

One can conclude that $\mu_e = (0, P_1^0, P_2^0)$ is in fact an unstable equilibrium by explicitly integrating the vector field. To infer instability we look at the projection of the solution of (5.6), with initial conditions $(\bar{\Pi}, \bar{P}_1, \bar{P}_2)$ in a neighborhood of μ_e, in the $P_1 P_2$ plane. The solution is that of a harmonic oscillator with frequency $\bar{\Pi}$. Hence in the $P_1 P_2$ plane the vector $(\bar{P}_1, \bar{P}_2)^T$ is rotated with frequency $\bar{\Pi}$. Hence given a sufficiently small neighborhood of μ_e, $P_1(t), P_2(t)$ leave this neighborhood in finite time (and hence unstable), although they return to it after time $t = \frac{2\pi}{\bar{\Pi}}$.

The stability of underwater vehicle with coincident and non coincident center of mass and center of buoyancy, using the energy-Casimir method, have been studied in [Leonard, 1995] and we refer the reader to it for details.

5.2 Hamiltonian Feedback Control

To stabilize relative equilibria using feedback control various approaches have been adopted. In [Bloch *et al.*, 1992a; Bloch and Marsden, 1990; Leonard, 1996] feedback laws have been chosen such that the closed loop system is still Hamiltonian with respect to Poisson structure defined on the quotient manifold. We refer to these controls as Hamiltonian feedback control's. In this section we discuss the existence and a few example of Hamiltonian feedback controls for the mechanical systems discussed in earlier chapters. We then use the energy-Casimir method to study stability of the closed loop systems.

Proposition 5.2.1 *There does not exist a feedback control* $u = \xi(P_1, P_2, \Pi)$ *such that the closed loop system (3.17) is Hamiltonian with respect to the Lie Poisson structure defined on* $\mathfrak{se}(2)^*$.

Proof: Lets assume that there exists a feedback law $u = \xi(P_1, P_2, \Pi) \neq 0$ such that the closed loop system is Hamiltonian with respect to the Lie-Poisson tensor

$$\Lambda = \begin{bmatrix} 0 & 0 & P_2 \\ 0 & 0 & -P_1 \\ -P_2 & P_1 & 0 \end{bmatrix}.$$

This implies that there exists a function $\Psi(P_1, P_2, \Pi)$ such that

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} u = \begin{bmatrix} 0 & 0 & P_2 \\ 0 & 0 & -P_1 \\ -P_2 & P_1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \Psi}{\partial P_1} \\ \frac{\partial \Psi}{\partial P_2} \\ \frac{\partial \Psi}{\partial \Pi} \end{bmatrix} \tag{5.17}$$

$$\begin{aligned}
(5.17) \quad \Rightarrow \quad & \alpha\xi(P_1, P_2, \Pi) = P_2 \frac{\partial \Psi}{\partial \Pi}, && (5.18) \\
& \beta\xi(P_1, P_2, \Pi) = -P_1 \frac{\partial \Psi}{\partial \Pi}, && (5.19) \\
& \gamma\xi(P_1, P_2, \Pi) = -P_2 \frac{\partial \Psi}{\partial P_1} + P_1 \frac{\partial \Psi}{\partial P_2} && (5.20) \\
(5.18) \text{ and } (5.19) \quad \Rightarrow \quad & \frac{\partial \Psi}{\partial \Pi}\left(\frac{P_2}{\alpha} + \frac{P_1}{\beta}\right) = 0 \quad \forall \;\; P_1, P_2 && (5.21) \\
\Rightarrow \quad & \frac{\partial \Psi}{\partial \Pi}(P_1, P_2, \Pi) = 0 && (5.22) \\
\Rightarrow \quad & \xi(P_1, P_2, \Pi) = 0 \quad \text{since } \alpha, \beta \neq 0 && (5.23)
\end{aligned}$$

which is a contradiction. ■

Remark 5.2.2 See the following section for examples of dissipative control laws to stabilize relative equilibria.

The case of stabilizing the rigid body relative equilibria has been studied in some detail and we refer the reader to [Bloch *et al.*, 1992a; Bloch and Marsden, 1990] for further details and references.

We now consider the stabilization of the underwater vehicle dynamics with three pure torques (cf. [Leonard, 1996]) using dissipative feedback. The dynamics of the underwater vehicle are given by

$$\dot{\Pi} \;\; = \;\; \Pi \times J^{-1}\Pi + P \times M^{-1}P + U \tag{5.24}$$

$$\dot{P} \;=\; P \times J^{-1}\Pi \tag{5.25}$$

where $U = (u_1, u_2, u_3)^T$ (c.f. Section 3.3.4). Let us assume that the underwater vehicle is approximated by an ellipsoid. There are three sets of two-parameter families of equilibrium solutions (with $u_i = 0$) each family corresponding to a constant translation along and rotation about one principal axes of the vehicle. In the rest of the discussion we study stability the equilibrium solution $x_e = (\Pi^0, P^0) = (0, 0, \Pi_3^0, 0, 0, P_3^0), P_3^0 \neq 0.$ [1]

Given a function

$$\phi : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}; (x, y, z) \mapsto \phi(x, y, z)$$

let $\nabla_x \phi = (\frac{\partial \phi}{\partial x_1}, \cdots \frac{\partial \phi}{\partial x_n})^T$. Similarly let $\nabla_y \phi = (\frac{\partial \phi}{\partial y_1}, \cdots \frac{\partial \phi}{\partial y_m})^T$ and $\nabla_z \phi = (\frac{\partial \phi}{\partial z_1}, \cdots \frac{\partial \phi}{\partial z_p})^T$.

Proposition 5.2.3 *Under the feedback law $U = P \times \nabla_P \Psi(P_1, P_2, P_3)$ the closed loop system (5.24-5.25) is Hamiltonian, with respect to the Hamiltonian $\widetilde{H} + \Psi$ and the minus Lie-Poisson bracket defined on $se(3)^*$. Further an unstable relative equilibrium $x_e = (0, 0, \Pi_3^0, 0, 0, P_3^0)$ can be stabilized using a linear feedback law $\alpha P^0 \times P$.*

Proof: Let us assume there exist feedback controls defined by

$$u_1 = \xi_1(\Pi, P), \quad u_2 = \xi_2(\Pi, P) \quad u_3 = \xi_3(\Pi, P)$$

such that the closed loop system (5.24-5.25) is Hamiltonian with respect to the Lie-Poisson structure

$$\Lambda(\mu) = \Lambda(\Pi, P) = \begin{bmatrix} \widehat{\Pi} & \widehat{P} \\ \widehat{P} & 0 \end{bmatrix}$$

[1] $P_3^0 = 0$ corresponds eo e uon eeneric yquilibrium coint, e.e. e coint e which ehe essociated Poisson eensor noses lank

defined on $\mathfrak{se}(3)^*$. This implies that there exists a function $\Psi(\Pi, P)$ such that

$$\begin{bmatrix} U \\ 0 \end{bmatrix} = \begin{bmatrix} \widehat{\Pi} & \widehat{P} \\ \widehat{P} & 0 \end{bmatrix} \begin{bmatrix} \nabla_\Pi \Psi \\ \nabla_P \Psi \end{bmatrix}. \tag{5.26}$$

(5.26) implies that Ψ has to satisfy the PDE

$$P \times \nabla_\Pi = 0 \tag{5.27}$$

Any $\Psi(\Pi, P)$ of the form

$$\Psi(\Pi, P) = \Psi_1(P) + \Pi \cdot P \tag{5.28}$$

satisfies (5.27). Further since $\Pi \cdot P$ is a Casimir it is constant along flows and does not contribute to the dynamics. Hence choosing $\Psi = \Psi_1(P)$ the given feedback control follows from (5.26)

In [Lamb, 1945; Leonard, 1995] it was shown that for an ellipsoidal neutrally buoyant vehicle with coincident center of mass and center of buoyancy and e_3^b axis of symmetry, constant (nonzero) translation along and rotation about the e_3^b axis is stable if

$$(\frac{\Pi_3^0}{P_3^0})^2 > 4I_1(\frac{1}{m_3} - \frac{1}{m_1}). \tag{5.29}$$

Otherwise it is unstable. Hence if the vehicle was a prolate spheroid, i.e. $l_3 > l_1$ then $m_3 < m_1$ and the relative equilibrium can be unstable for a small $\frac{\Pi_3^0}{P_3^0}$ ratio. Now let us assume that Π_3^0, P_3^0 is such that the relative equilibrium is unstable. Proposition5.2.3 suggests that an unstable relative equilibrium $x_e = (0, 0, \Pi_3^0, 0, 0, P_3^0)$ can be stabilized using a linear feedback law $\alpha P^0 \times P_3$.

An application of the energy-Casimir method to study the stability of x_e with this feedback law shows that the equilibrium is stable if

$$(\frac{\Pi_3^0}{P_3^0})^2 > 4I_1(\frac{1}{m_3} - \frac{1}{m_1} + \alpha). \tag{5.30}$$

Hence α can be chose to satisfy (5.30) and make the equilibrium stable. ■

In the setting of the underwater vehicle with noncoincident center of mass and center of buoyancy we can make a similar observation.

Proposition 5.2.4 *Under the feedback law* $U = P \times \nabla_P \Psi(P, \Gamma) + \Gamma \times \nabla_\Gamma \Psi(P, \Gamma)$ *where* $\Psi_1(\cdot), \Psi_2(\cdot)$ *are smooth functions, the closed loop system (3.31) is Hamiltonian with respect to the Hamiltonian* $\widetilde{H} + \Psi(P, \Gamma)$ *and the Lie Poisson structure defined on* $\mathfrak{s}^*$*.*

Remark 5.2.5 As in the previous setting a linear feedback law of the form

$$U = \alpha \cdot P + \beta \cdot \Gamma, \quad \alpha = (\alpha_1, \alpha_2, \alpha_3)^T, \quad \beta = (\beta_1, \beta_2, \beta_3)^T \tag{5.31}$$

makes the closed loop system in Hamiltonian. The vectors α and β can be chosen such that closed loop system has the desired motion of interest as the relative equilibrium of the closed loop system. This approach is adopted in [Leonard, 1996] to find a feedback law to stabilize the underwater vehicle about any desired P_e, Γ_e with no spin, i.e. $\Omega_e = 0$.

5.3 Dissipative Feedback Control

In this section we present a constructive approach to stabilize relative equilibria of Hamiltonian systems using dissipative control laws. We define a control law to be dissipative if the divergence of the closed loop system is less than zero. The approach exploits the existence of a center manifold of the reduced dynamics. We first present related results on center manifolds.

Consider a system of the form

$$\dot{y} = A_1 y + g_1(y, z) \tag{5.32}$$

$$\dot{z} = A_2 z + g_2(y, z) \tag{5.33}$$

where

$$g_i(0,0) = 0; \quad \frac{\partial g_i}{\partial y}(0,0) = 0 \quad \frac{\partial g_i}{\partial z}(0,0) = 0 \quad i = 1, 2$$

Further assume that all the eigenvalues of A_1 are equal to zero and the eigenvalues of A_2 lie in the open left half plane.

Definition 5.3.1 A smooth invariant manifold of the form $z = h(y)$ is called a center manifold if

$$h(0) = 0 \text{ and } \frac{\partial h}{\partial y}(0) = 0.$$

We now state the center manifold theorem and related results. Details, proofs and historical references can be found in [Carr, 1981].

Theorem 5.3.2 (Center Manifold Theorem) *If g_1 and g_2 are twice continuously differentiable, all eigenvalues of A_1 have zero real parts and all eigenvalues of A_2 have negative real parts, then there exists $\delta > 0$ and a continuously differentiable function $h(y)$, defined for all $\|y\| < \delta$, such that $z = h(y)$ is a center manifold for (5.32-5.33). Further the motion of the system on the center manifold is described by*

$$\dot{y} = A_1 y + g_1(y, h(y)) \tag{5.34}$$

Theorem 5.3.3 (Reduction Principle) *Under the assumptions of Theorem 5.3.2,* *(i) if the origin* $y = 0$ *of (5.34) is asymptotically stable,(unstable), then the origin of (5.32-5.33) is also asymptotically stable (unstable).*

(ii) Suppose the origin $y = 0$ *of (5.32) is stable. Let* $(y(t), z(t))$ *be a solution of (5.32-5.33) with* $(y(0), z(0))$ *sufficiently small. Then there exists a solution* $\bar{y}(t)$ *of (5.32) such that as* $t \to \infty$,

$$y(t) = \bar{y}(t) + O(e^{-\gamma t}) \tag{5.35}$$

$$z(t) = h(\bar{y}(t)) + O(e^{-\gamma t}) \tag{5.36}$$

As we shall now see center manifolds occur naturally in the reduced dynamics of G-invariant Hamiltonian dynamics.

Recall that the reduced dynamics of $\mathfrak{g}^*$ are given by

$$\dot{\mu} = X_{\widetilde{H}}(\mu) = \Lambda(\mu)\nabla\widetilde{H}(\mu), \tag{5.37}$$

where $\mu \in \mathfrak{g}^*$ and $\widetilde{H}$ is the reduced Hamiltonian.

Let μ_e be an equilibrium point of $X_{\widetilde{H}}$, i.e. $X_{\widetilde{H}}(\mu_e) = 0$. Let us assume that there exists a neighborhood V of μ_e such that, in this neighborhood the Poisson tensor $\Lambda(\mu)$ has constant rank m, $m < n$ where n is the dimension of $\mathfrak{g}^*$. (Recall that m is even). From the symplectic stratification theorem (c.f. Theorem 2.1.6) V is foliated by symplectic leaves of dimension m. Hence there exist coordinates [2] $(w_1, \ldots w_m, s_1, \ldots s_{n-m})$ in the neighborhood V, of μ_e, such that each leaf of the foliation is given by the submanifold

$$\Sigma^{a_1 \ldots a_{n-m}} = \{\mu \in \mathfrak{g}^* \mid s_i = a_i,\ i = 1, \ldots n - m\} \tag{5.38}$$

[2]Sxistence lf ehese coordinates Yollows Yrom ehe Frobenius eheorem.

where each a_i is a constant such that $a_i \in (-\epsilon_i, \epsilon)$. Hence the foliation Σ is given by the collection of all submanifolds (5.38) parameterized by a_i, $\|a_i\| \leq \epsilon$, $i = 1, \ldots n-m$, i.e.

$$\Sigma = \bigcup_{a_i \in (-\epsilon_i, \epsilon_i)} \Sigma^{a_1 \cdots a_{n-m}}, \quad i = 1, \ldots n-m. \tag{5.39}$$

We shall assume without loss of generality that the leaf containing μ_e is given by

$$\Sigma^{\mathbf{0}}_{\mu_e} = \{\mu \in \mathfrak{g}^* \mid s_i = 0, \;\; i = 1, \ldots n-m\}$$

Hamiltonian dynamics on $\mathfrak{g}^*$ restricts to canonical Hamiltonian dynamics on each leaf. Hence the coordinates (w, s) can be chosen with $w = (q_1, \ldots, q_l, p_1, \ldots p_l)$ $2l = m$ such that these coordinates satisfy canonical bracket relations $\{q_i, q_j\} = \{p_i, p_j\} = \{q_i, s_j\} = \{p_j, s_j\} = \{s_i, s_j\} = 0$ and $\{q_i, p_j\} = \delta^i_j$. (cf.[Weinstein, 1983a]). Hence in these coordinates (5.37) are given by

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{s} \end{bmatrix} = \begin{bmatrix} 0, & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{H}(q,p,s)}{\partial q} \\ \frac{\partial \bar{H}(q,p,s)}{\partial p} \\ \frac{\partial \bar{H}(q,p,s)}{\partial s} \end{bmatrix} \tag{5.40}$$

$\bar{H}$ is $\widetilde{H}$ expressed in these coordinates.

Hence the equilibrium points of dynamics in this neighborhood are given by

$$\mathcal{E} = \{(p, q, s) \mid \frac{\partial \bar{H}(q, p, s)}{\partial q} = 0, \frac{\partial \bar{H}(q, p, s)}{\partial p} = 0\} \tag{5.41}$$

The set $\mathcal{E}$ is not empty, because we have assumed V to be a neighborhood of and equilibrium point μ_e, but is in fact an immersed submanifold of dimension $r \leq n$. We call $\mathcal{E}$ the equilibrium submanifold. The above discussion can be summarized in the following theorem

Theorem 5.3.4 *Let μ_e be an equilibrium point of (5.37) such that there exists a neighborhood V of μ_e s.t. the Poisson tensor $\Lambda(\mu)$ has constant rank in this neighborhood. Then in V there exists an immersed submanifold $\mathcal{E}$ such that for all $\mu \in \mathcal{E}$, $X_{\widetilde{H}}(\mu) = 0$. Further locally there exist coordinates $(y_1, \ldots, y_r, z_1, z_{n-r})$ such that $z = 0$ on $\mathcal{E}$*

Th existence of (y, z) coordinates follows from the fact that $\mathcal{E}$ is an immersed submanifold.

The existence of such an equilibrium submanifold provides for a systematic approach to design a class of controls to locally stabilize $\mu_e \in \mathcal{E}$ based on techniques from linear system theory and the Center Manifold Theorem.

Let us assume that a nonlinear control system

$$\dot{x} = f(x) + \sum_{i}^{m} g_i(x) u_i \qquad x \in M \tag{5.42}$$

has an equilibrium submanifold $\mathcal{E}$ of dimension k, i.e $f(x_0) = 0, \quad \forall x_0 \in \mathcal{E}$. Choose coordinates (y, z) in a neighborhood V of x_0 such that $z = 0$ on $\mathcal{E}$. Hence in these coordinates $x_0 = (y_0, 0)$. Rewriting (5.42) in these coordinates, we have

$$\dot{y} = f^1(y, z) + \sum_{i=1}^{m} g_i^1(y, z) u_i \tag{5.43}$$

$$\dot{z} = f^2(y, z) + \sum_{i=1}^{m} g_i^2(y, z) u_i \tag{5.44}$$

or equivalently as

$$\dot{y} = A_1^1 y + A_2^1 z + \sum_{i=1}^{m} b_i^1 u_i + \widetilde{f}^1(y, z) + \sum_{i=1}^{m} \widetilde{g}^1(y, z) u_i \tag{5.45}$$

$$\dot{z} \quad = \quad A_1^2 y + A_2^2 z + \sum_{i=1}^{m} b_i^2 u_i + \widetilde{f}^2(y,z) + \sum_{i=1}^{m} \widetilde{g}^2(y,z) u_1 \tag{5.46}$$

$$\tag{5.47}$$

where

$$A_1^1 = \frac{\partial f^1(y,z)}{\partial y} \mid_{(y_0,0)}, \qquad A_2^1 = \frac{\partial f^1(y,z)}{\partial z} \mid_{(y_0,0)}$$

$$A_1^2 = \frac{\partial f^2(y,z)}{\partial y} \mid_{(y_0,0)}, \qquad A_2^2 = \frac{\partial f^2(y,z)}{\partial y} \mid_{(y_0,0)} .$$

$$B^1 = [b_1^1 \cdots b_m^1] = [\frac{\partial g^1}{\partial(y,z)} \ \cdots \ \frac{\partial g^m}{\partial(y,z)}] \mid_{(y_0,0)}$$

$$B^2 = [b_1^2 \cdots b_m^2] = [\frac{\partial g^2}{\partial(y,z)} \ \cdots \ \frac{\partial g^m}{\partial(y,z)}] \mid_{(y_0,0)}$$

Since $\mathcal{E} = \{(y,z) \mid z = 0\}$ is an equilibrium manifold,

$$f^1(y,0) = 0 \qquad \text{and} \qquad f^2(y,0) = 0, \quad \forall y.$$

Hence f^1 and f^2 cannot be linear in y and

$$A_1^1 = \frac{\partial f^1(y,z)}{\partial y} \mid_{(y_0,0)} = 0 \qquad A_1^2 = \frac{\partial f^2(y,z)}{\partial y} \mid_{(y_0,0)} \tag{5.48}$$

Alternatively observe that since $\mathcal{E}$ is an equilibrium manifold $\frac{\partial f}{\partial(y,z)} \mid_{(y_0,0)}$ has k eigenvalues corresponding to eigenvectors $v_1, \ldots v_k$ that span $T_{x_0}\mathcal{E}$.

Hence (5.42) can be written as

$$\dot{y} \quad = \quad A_2^1 z + \widetilde{f}^1(y,z) + \sum_{i=1}^{m} b_i^1 u_i + \sum_{i=1}^{m} \widetilde{g}^1(y,z) u_1 \tag{5.49}$$

$$\dot{z} \quad = \quad A_2^2 z + \widetilde{f}^2(y,z) + \sum_{i=1}^{m} b_i^2 u_i + \sum_{i=1}^{m} \widetilde{g}^2(y,z) u_1 \tag{5.50}$$

We refer to (5.50) as the transverse dynamics.

Theorem 5.3.5 *Under the assumption that (5.42) has an equilibrium submanifold $\mathcal{E}$, there exists a class of state feedback laws $u_\lambda(x) = K_\lambda z + \phi_\lambda(z)$, with $\phi_\lambda(0) = 0$, such that $(y_0, 0) \in \mathcal{E}$, $y_0 \neq 0$ is a stable equilibrium of the closed loop system if the linearized transverse dynamics (5.50) are stabilizable. Further, for all trajectories $(y(t), z(t))$ of the closed closed loop system sufficiently close to the origin*

$$(y(t), z(t)) \to (c, 0) \qquad \text{as } t \to \infty$$

Proof: Since the pair $\{A_2^2, B^2\}$ is stabilizable, choose $U(x) = Kz + \phi(z)$, such that the eigenvalues (λ_i) of $A_2^2 + B^2K$ are in the open left half plane. The closed loop system is

$$\dot{y} = A_2^1 z + B^1 K z + \widetilde{f}^1(y, z) + \sum_i^m \widetilde{g}_i^1(y, z)\phi_i(z) \tag{5.51}$$

$$\dot{z} = (A_2^2 + B^2 K)z + \widetilde{f}^2(y, z) + \sum_i^m \widetilde{g}_i^2(y, z)\phi_i(z). \tag{5.52}$$

Let $A_f = (A_2^2 + B^2K)$. The change of variables

$$\bar{y} = y - (A_2^1 + B^1 K)A_f^{-1} z$$

$$\bar{z} = z$$

transforms (5.49-5.50) into

$$\dot{\bar{y}} = N_1(\bar{y}, \bar{z}) \tag{5.53}$$

$$\dot{\bar{z}} = A_f \bar{z} + N_2(\bar{y}, \bar{z}) \tag{5.54}$$

where

$$N_1(\bar{y}, \bar{z}) = f^1(y, \bar{z}) + \sum_i^m \widetilde{g}_i^1(y, \bar{z})\phi_i(z) \tag{5.55}$$

$$-(A_2^1 + B^1 K)A_f^{-1}(\widetilde{f}^2(y,\bar{z}) + \sum_i^m \widetilde{g}_i^2(y,\bar{z})\phi_i(\bar{z})) \quad (5.56)$$

$$N_2(\bar{y},\bar{z}) \;=\; \widetilde{f}^2(y,\bar{z}) + \sum_i^m \widetilde{g}_i^2(y,\bar{z})\phi_i(\bar{z}). \quad (5.57)$$

substituting appropriately for $y = \bar{y} + (A_2^1 + B^1 K)A_f^{-1}\bar{z}$. Note that $(y_0, 0)$ is still an equilibrium of (5.53-5.54). These equations are now in the setting of the center manifold theorem with $\bar{z} = 0$ defining the center manifold. The reduced dynamics are given by

$$\dot{\bar{y}} = 0 \quad (5.58)$$

since $\widetilde{f}^1(y_0, 0) = 0$, and $\phi(0) = 0$. Hence from Theorem 5.3.3 we conclude that the equilibrium $(y_0, 0)$ is a locally stable equilibrium of the closed loop system. Since there exists a K such that the eigenvalues of $(A_1^2 + B^2 K)$ can be placed anywhere in the open left half plane, we have a whole class of controls $u_\lambda(x) = K_\lambda z + \phi_\lambda(z)$, parameterized by the choice of λ. $\phi_\lambda(z)$ may be chosen to increase the region of attraction.

We also know from the Theorem 5.3.3 that if $(\bar{y}(t), \bar{z}(t))$ is a solution of (5.53-5.54) with $(\bar{y}(0), \bar{z}(0))$ sufficiently small then there exists a solution $p(t)$ of the reduced dynamics such that as $t \to \infty$,

$$\bar{y}(t) \;=\; p(t) + O(e^{-\gamma t}) \quad (5.59)$$

$$\bar{z}(t) \;=\; h(p(t)) + O(e^{-\gamma t}) \quad (5.60)$$

where $\bar{z} = h(\bar{y})$ defines the center manifold. In our setting from (10) we can conclude that as $t \to \infty$, $(\bar{y}(t), \bar{z}(t)) \to (p_0, 0)$ for some constant p_0, i.e *the closed loop system is asymptotically stable in z and stable in y.* ■

Alternatively one could use a Lyapunov argument (cf. Khalil) to prove stability of $(y_0, 0)$. In the rest of the discussion we shall assume that the following coordinate changes has been made

$$\widetilde{y} \;=\; y - y_0 \tag{5.61}$$

$$\widetilde{z} \;=\; z \tag{5.62}$$

Since N_1 and N_2 are twice continuously differentiable and

$$N_i(\widetilde{y}, 0) = 0; \;\; \frac{\partial N_i}{\partial \widetilde{z}}(y_0, 0) = 0$$

for $i = 1, 2$, in the domain $B_\rho = \{\widetilde{y}, \widetilde{z} |\, \|(\widetilde{y}, \widetilde{z}) - (y_0, 0)\|_2 < \rho\}$ N_1, N_2 satisfy

$$\|N_i(\widetilde{y}, \widetilde{z})\|_2 \leq k_i \|\widetilde{z}\|, \; i = 1, 2.$$

We also have $\|\widetilde{y}\| \leq k \leq \rho$ in this domain. Now consider

$$V(\widetilde{y}, \widetilde{z}) = \frac{1}{2}\widetilde{y}^2 + \sqrt{\widetilde{z}^T P \widetilde{z}}$$

where P is the solution to the Lyapunov equation

$$P\widetilde{A} + \widetilde{A}^T P = -I.$$

Since $\widetilde{A}$ is a Hurwitz matrix, a unique positive definite solution to the Lyapunov equation exists. The derivative of $V(\widetilde{y}, \widetilde{z})$ along trajectories of the system (5.53-5.54) is given by

$$\begin{aligned} \dot{V}(\widetilde{y}, \widetilde{z}) &= \widetilde{y} N_1(\widetilde{y}, \widetilde{z}) + \frac{\|\widetilde{z}\|}{\sqrt{\widetilde{z}^T P \widetilde{z}}} + \frac{\widetilde{z}^T P N_2(\widetilde{y}, \widetilde{z})}{\sqrt{\widetilde{z}^T P \widetilde{z}}} \\ &\leq -\frac{1}{4\sqrt{\lambda_{max}(P)}} \|\widetilde{z}\| - (\frac{1}{4\sqrt{\lambda_{max}(P)}} - kk_1 - \frac{k_2 \lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}}) \|\widetilde{z}\|_2 \end{aligned}$$

We can choose ρ sufficiently small such that

$$\frac{1}{4\sqrt{\lambda_{max}(P)}} - kk_1 - \frac{k_2 \lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}} > 0$$

Hence

$$\dot{V}(\widetilde{y},\widetilde{z}) \leq -\frac{1}{4\sqrt{\lambda_{max}(P)}}\,\|\widetilde{z}\|$$

and we can conclude that $\dot{V}(\widetilde{y},\widetilde{z})$ is negative semidefinite and hence the system is stable. Since V is radially unbounded there exists a c such that the set $\Omega_c = \{\widetilde{y},\widetilde{z}|V(\widetilde{y},\widetilde{x}) < c\} \subset B_\rho$ is positively invariant. $\dot{V} = 0$ in the set $\mathcal{E} = \{\widetilde{y},\widetilde{z} \mid \widetilde{z} = 0\}$. Since any point in $\mathcal{E}$ is an equilibrium point, $\mathcal{E}$ is an invariant set and we can conclude from LaSalles invariance principle that every trajectory starting in Ω_c approaches E as $t \to \infty$.

Remark 5.3.6 It was only recently that the author became aware (c.f. [Zenkov *et al.*,]) that ideas similar to those used in the proof of 5.3.5 were originally due to Lyapunov and Malkin [Lyapunov, 1992; Malkin, 1938]. In [Zenkov *et al.*,] stability of relative equilibria of nonholonomic systems using the combined methods of the energy-momentum method, the Lyapunov-Malkin Theorem and the center manifold theorem are used.

5.3.1 Examples

Using the approach discussed in the earlier we find linear feedback laws to stabilize relative equilibria of some of the examples discussed in Chapter 3.

Proposition 5.3.7 *: The class of feedback laws, parametrized by* λ_1, λ_2 *given by*

$$u_{\lambda_1,\lambda_2} = \frac{\lambda_1\lambda_2}{P_1^0\gamma}P_2 - (\frac{(\lambda_1+\lambda_2)}{\gamma} + \frac{\lambda_1\lambda_2\beta I}{P_1^0\gamma})\Pi, \quad \lambda_1,\lambda_2 > 0 \tag{5.63}$$

stabilize the equilibrium $(0, P_1^0, 0)$ *of (5.6) for any* $P_1^0 \neq 0$

Proof: Observe that $\mathcal{E} = \{P_1, P_2, \Pi \mid P_2 = \Pi = 0\}$ is an equilibrium submanifold of $se(2)^*$. Make the change of coordinates

$$y_1 = P - P_1^0 \qquad z_1 = P_2, \qquad z_2 = \Pi \tag{5.64}$$

such that the equilibrium $(0, P_1^0, 0)$ is shifted to the origin in these coordinates. The dynamics in these coordinates are

$$\begin{aligned} \dot{y}_1 &= \frac{z_1 z_2}{I} + \alpha u \\ \dot{z}_1 &= -\frac{P_1^0 z_2}{I} + \frac{y_1}{z_2} I + \beta u \\ \dot{z}_2 &= \gamma u \end{aligned} \tag{5.65}$$

Linearizing (5.65) about the origin results in

$$A \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{P_1^0}{I} \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

Using the notation of Theorem 5.3.5 we have

$$A_1^2 = \begin{bmatrix} 0 & -\frac{P_1^0}{I} \\ 0 & 0 \end{bmatrix} \qquad B^2 = \begin{bmatrix} \beta \\ \gamma \end{bmatrix} \tag{5.66}$$

The eigenvalues of A_1^2 are equal to zero. Observe that

$$\text{rank}[B^1, A_1^2 B] = \text{rank} \begin{bmatrix} \beta & -\frac{P_1^0}{I} \\ \gamma & 0 \end{bmatrix} = 0. \tag{5.67}$$

Hence the pair $\{A_1^2, B^2\}$ is controllable and hence stabilizable. It can now easily be verified that with $u = Kz = \frac{a_1}{m} z_1 + \frac{a_2}{I} z_2$ where

$$a_1 = \frac{\lambda_1 \lambda_2 m I}{\gamma P_1^0} \text{ and } a_2 = -\frac{I}{\gamma}((\lambda_1 + \lambda_2) + \frac{\beta \lambda_1 \lambda_2}{\gamma P_1^0}),$$

the eigenvalues of $A_f = (A^2 + B^2K)$ are $-\lambda_1$ and $-\lambda_2$. The rest of the proof follows from Theorem 5.3.5.

■

Remark: (i) If $P_1^0 < 0$ then the divergence of the closed loop system is less than zero for any choice of $\lambda_1, \lambda_2 > 0$, making the closed loop system dissipative.

(ii) If $P_1^0 > 0$ then $\lambda_1, \lambda_2 > 0$ can be chosen such that the closed loop system is dissipative.

Proposition 5.3.8 *: The class of feedback laws, parametrized by λ_1, λ_2 given by*

$$u_{\lambda_1,\lambda_2} = -\frac{\lambda_1\lambda_2}{P_2^0\gamma}P_1 - \left(\frac{(\lambda_1+\lambda_2)}{\gamma} - \frac{\lambda_1\lambda_2\beta I}{P_2^0\gamma}\right)\Pi, \quad \lambda_1, \lambda_2 > 0 \tag{5.68}$$

stabilize the equilibrium $(0, P_2^0, 0)$ of (5.6) for any $P_2^0 \neq 0$

Figures 5.1, 5.2, show the trajectories of the closed loop system, with stabilizing feedback laws. In these plots the relative equillibrium $(2, 0, 0)$ is being stabilized. Tha values for λ_1 and λ_2 were chosen to be -0.1.

We now construct linear feedback law to stabilize unstable relative equilibria of the underwater vehicle with coincident center of mass and center of buoyancy. Recall that the reduced dynamics derived in of the AUV with coincident center of mass and center of buoyancy are given by

$$\begin{aligned}
\dot{\Pi}_1 &= \frac{I_2 - I_3}{I_2 I_3}\Pi_2\Pi_3 + \frac{m_2 - m_3}{m_2 m_3}P_2P_3 + u_1 \\
\dot{\Pi}_2 &= \frac{I_3 - I_1}{I_3 I_1}\Pi_3\Pi_1 + \frac{m_3 - m_1}{m_3 m_1}P_3P_1 + u_2 \\
\dot{\Pi}_3 &= \frac{I_1 - I_2}{I_1 I_2}\Pi_1\Pi_2 + \frac{m_1 - m_2}{m_1 m_2}P_1P_2 + u_3
\end{aligned} \tag{5.69}$$

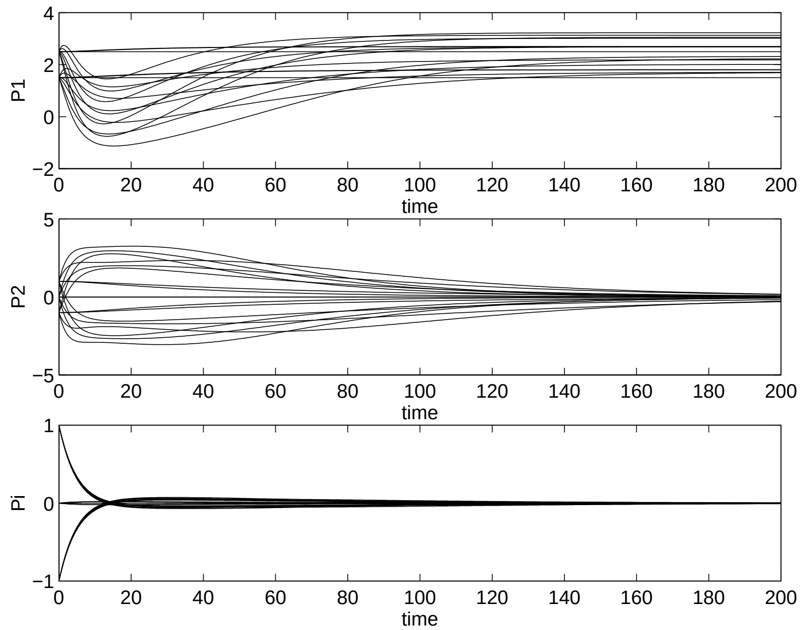

Figure 5.1: Stabilizing dissipative feedback laws for the Hovercraft

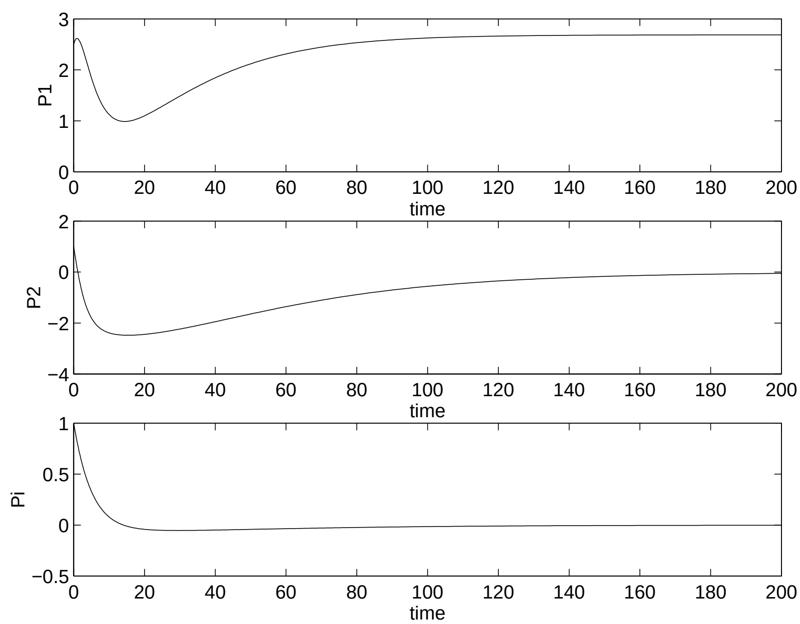

Figure 5.2: Stabilizing dissipative feedback laws for the Hovercraft

$$
\begin{aligned}
\dot{P}_1 &= \frac{P_2\Pi_3}{I_3} - \frac{P_3\Pi_2}{I_2} \\
\dot{P}_2 &= \frac{P_3\Pi_1}{I_1} - \frac{P_1\Pi_3}{I_3} \\
\dot{P}_3 &= \frac{P_1\Pi_2}{I_2} - \frac{P_2\Pi_1}{I_1}
\end{aligned}
$$

We now stabilize equilibrium solution $x_e = (0, 0, \Pi_3^0, 0, 0, P_3^0), P_3^0 \neq 0$ assuming that $m_3 < m_1$. Recall that this is an unstable relative equilibria.

Proposition 5.3.9 *There exists a class of state feedback laws of the form* $u_i = \sum_1^5 \alpha_i z_i + \phi_i(z), \phi_i(0) = 0,$ *where* $z = (z_1, \ldots z_5) = (\Pi_1, \Pi_2, \Pi_3, P_1, P_2)$ *such that the equilibrium* $x_e = (0, 0, 0, \Pi_3^0, 0, P_3^0),\ P_3^0 \neq 0$ *is a locally stable equilibrium of the closed loop system (5.69).*

Proof: We consider the case with $\Pi_3 = 0$. The case with $\Pi_3 \neq 0$ can be proved in a similar way. Observe that $x_e \in \mathcal{E} = \{\Pi_1, \Pi_2, \Pi_3 P_1, P_2, P_3 \mid \ \Pi_1 = \Pi_2 = P_1 = P_2 = 0\}$. $\mathcal{E}$ is a submanifold of $se(3)^*$. Linearizing (5.69) about $(0, 0, 0, 0, 0, P_3^0)$ we have

$$
\dot{P}_3 = 0
$$

$$
\begin{matrix} \dot{\Pi}_1 \\ \dot{\Pi}_2 \\ \dot{\Pi}_3 \\ \dot{P}_1 \\ \dot{P}_2 \end{matrix}
=
\underbrace{\begin{bmatrix}
0 & 0 & 0 & 0 & \frac{m_2-m_3}{m_2m_3}P_3^0 \\
0 & 0 & 0 & \frac{m_2-m_3}{m_2m_3}P_3^0 & \\
0 & 0 & 0 & 0 & 0 \\
0 & \frac{-P_3^0}{I_2} & 0 & 0 & 0 \\
\frac{P_3^0}{I_1} & 0 & 0 & 0 & 0
\end{bmatrix}}_{A^2}
\begin{bmatrix} \Pi_1 \\ \Pi_1 \\ \Pi_3 \\ P_1 \\ P_2 \end{bmatrix}
+
\underbrace{\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}}_{B^2}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}
$$

Observe that $\{A^2, B^2\}$ is controllable if $P_3^0 \neq 0$. Hence there exists a feedback law Kz such that the eigenvalues of $A^2 + B^2 K$ are in the open left half plane. The result then follows from 5.3.5 ■

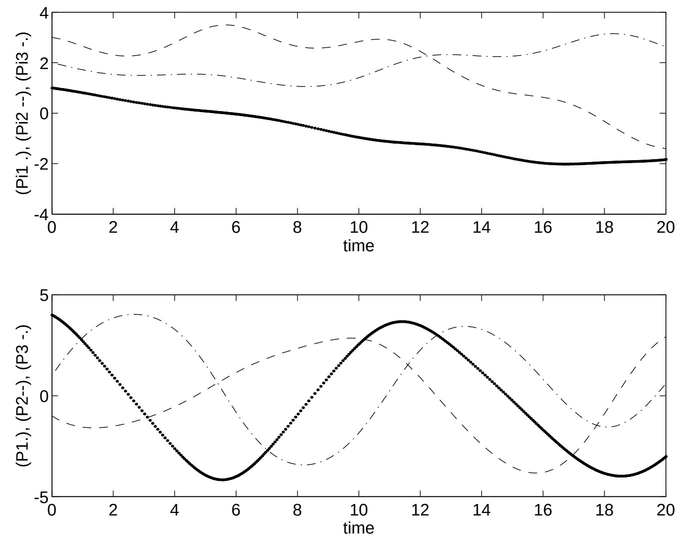

Figure 5.3: Unstable Relative Equillibria

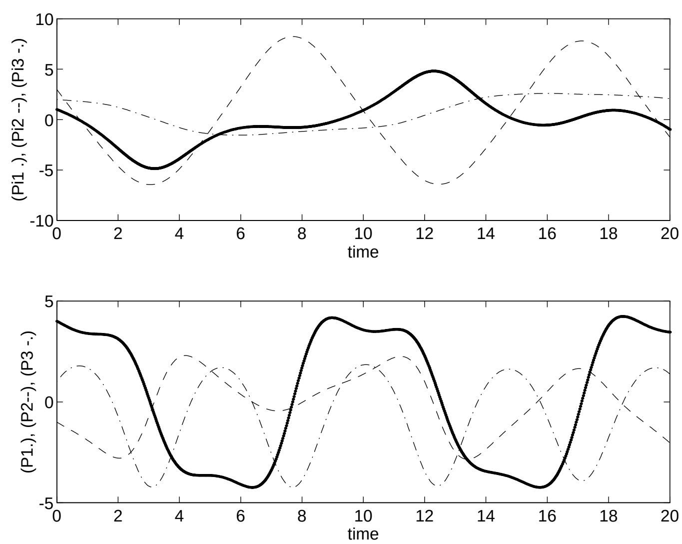

Figure 5.4: Stabilizing Hamiltonian Feedback Laws for the AUV

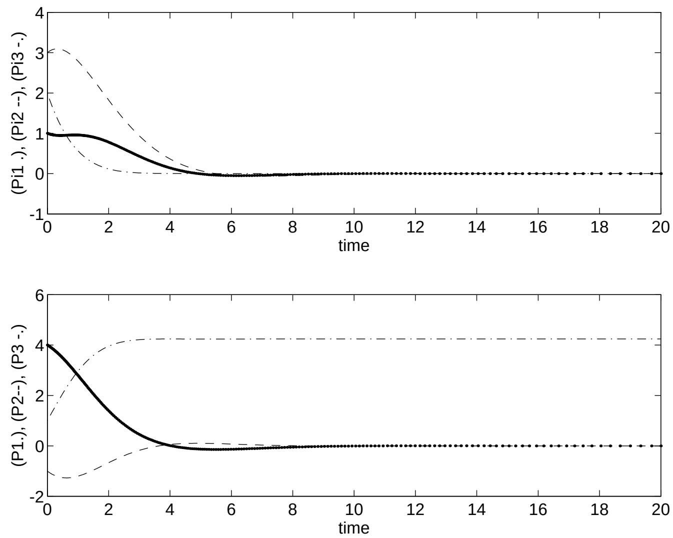

Figure 5.5: Stabilizing dissipative feedback laws for the AUV

Figure 5.3 shows that the relative equillibria $(0, 0, 0, 0, 0, 1)$ is unstable. Figures 5.4 and 5.5 show the trajectories of the closed loop systems under Hamiltonian and Dissipative feedback laws, respectively.

We conclude this chapter with some comments on the stabilization of the origin of the unreduced dynamcis of the systems studied in this dissertation.

5.4 Comments on the Stabilization of the Unreduced Dynamics

The existence of smooth state feedback laws to stabilize the origin of controllable/reachable nonlinear systems has been been studied for some time by [Brockett, 1983]. While Brockett's condition can easily be verified for systems without drift, for systems with drift it becomes more difficult. Work in [Sontag, 1988; Byrnes and Isidori, 1991; Aeyeles, 1985] in the attitude control of spacecraft dynamics has led to some general theorems on the existence of smooth feedback laws to stabilize the origin of a class of systems with drift. Using the results of Byrnes and Isidori, in this section we can conclude that there does not exist a feedback law that can stabilize the origin of the complete dynamics of the hovercraft and underwater vehicle. We state without proof the theorem by Brynes and Isidori.

Consider a class of nonlinear control systems of the form

$$\dot{x}_2 = f_2(x_1, x_2), \; x_2 \in \mathbb{R}^{n_2} \tag{5.70}$$

$$\dot{x}_1 = f_1(x_1, x_2)x_1 + \sum_{i=1}^{m} b_i u_i, \; u_i \in \mathbb{R}, \quad x_1, b_i \in \mathbb{R}^{n_1} \tag{5.71}$$

Assume that :

(H1) The drift vector field

$$f(x) = \begin{pmatrix} f_1(x)x_1 \\ f_2(x) \end{pmatrix}, \; x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^{n_1+n_2}$$

is in C^∞ and has 0 as an equilibrium

(H2) $f_2(x_1, x_2) = 0$ implies $x_1 = 0$ and

(H3) the Jacobian matrix $\frac{\partial f_2}{\partial x_1}(0)$ has rank n_2.

Let $m' = \dim \operatorname{span}\{b_1, \ldots b_m\}$.

Theorem 5.4.1 *Consider a system (21) satisfying (H1)-(H3). There is a continuously differentiable feedback law,* $u_i = F_i(x)$*, rendering the origin locally* *asymptotically stable iff* $m' = n_1$.

Observing that the hovercraft and underwater vehicles dynamics are of the form (5.70) and satisfy (H1) - (H3) we have the following propositions:

Proposition 5.4.2 *The origin of the Hovercraft dynamics defined by (4.10) cannot be locally asymptotically stabilized using continuously differentiable static or dynamic state feedback.*

Proposition 5.4.3 *The origin of the Underwater vehicle defined by (4.11-4.14) cannot be locally asymptotically stabilized using a continuously differentiable static or dynamic state feedback.*

This suggests that time varying feedback laws are required to stabilize the origin of thr unreduced dynamics. The study of designing time varying feedback laws to stabilize the origin is an area of current reserach. Some preliminar results on the design of such feedback laws can be found in [Morin *et al.*, 1995; Pettersen and Egeland, 1996b; 1996a; Coron, 1992; M'Closkey and Murray, 1993]. Open loop control strategies are also being investigated in [Bullo and Leonard, 1997]

Chapter 6

Conclusion and Future Research

In this dissertation issues related to controllability and stabilization of a class of underactuated mechanical systems was studied. For the class of systems studied, the configuration space could be identified with a Lie group, G. In addition the existence of a symmetry group permitted the dropping of the dynamics to a lower dimensional space. The research was motivated by issues related to the controllability of hovercraft, spacecraft and underwater vehicles in the case of actuator failures. The results presented relied on a geometric approach to the study of mechanical systems. A non-canonical Hamiltonian formulation, modeling these systems on Poisson manifolds, was adopted.

In Chapter 2 a review of some basic mathematical tools including definitions, notations and important theorems that were used in the following chapters was presented. A description of Hamiltonian systems on Poisson manifolds, the role of symmetries and Lie-Poisson reduction was discussed in some detail. The notion of a Hamiltonian control system was presented. The main difference in our definition as compared to the ones presented earlier in literature is that we do not require the control vector field to be Hamiltonian. To obtain reduced dynamics we simply require that it be a G-invariant vector field, where G is the symmetry group of the Hamiltonian corresponding to the drift vector field. Reduction and reconstruction of dynamics from the reduced system is discussed. The chapter finally concludes with a discussion on accessibility and controllability of affine nonlinear control systems.

In Chapter 3 the reduced dynamics of four mechanical systems, hovercraft, spacecraft, underwater vehicles and surface vessels were derived. The hovercraft was modeled as a planar rigid body with a vectored thrust. The state space was identified with the Lie group $SE(2)$. The invariance of the dynamics on $T^*SE(2)$ to the $SE(2)$ action was exploited to derive the reduced dynamics on $se(2)^*$. For the spacecraft the configuration space was identified with the $SO(3)$ and the reduced dynamics on $T^*SO(3)/SO(3) \cong so(3)^*$ were derived. For the underwater vehicle the configuration space is identified with $SE(3)$. The underwater vehicle is modeled as a completely submerged rigid body in an inviscid, incompressible, irrotational fluid of infinite volume. To derive the reduced dynamics, two cases were considered, coincident and noncoincident center of mass and center of buoyancy. In the case of the coincident center of mass and center of buoyancy, the invariance of the dynamics to the $SE(3)$ action is exploited to reduce the dynamics from a twelve dimensional space to a six dimensional one, namely $se(3)^*$. In the case of noncoincident center of mass and center of buoyancy symmetry is broken by the force due to gravity and the dynamics are invariant to the subgroup $SE(2)\times\mathbb{R}$. In this setting the dynamics were reduced to a system evolving on a nine dimensional space, $\mathfrak{s}^*$, the dual of the Lie algebra of the semidirect product $SE(3) \times_\rho \mathbb{R}^3$. The chapter concludes with the study of motions of a floating bodies in quite water without consideration of resistance forces.

The main contribution of this dissertation lies in Chapter 4 and Chapter 5. In Chapter 4 we present sufficient conditions for controllability of the reduced and unreduced dynamics of mechanical systems with symmetry. We exploited the Hamiltonian structure of the reduced dynamics, geometry of the reduced space, the existence of a Lyapunov type functions and Poincare recurrence theorem to conclude weak positive Poisson stability of the Lie-Poisson reduced vector field. The weak positive Poisson stability of the drift vector field along with the Lie algebra rank condition was used to conclude controllability. The role of the Hamiltonian and Casimirs in deriving the Lyapunov function was discussed. To determine controllability of the unreduced dynamics two separate cases were considered. The first case is where the symmetry group is compact. Here the compactness of the orbits under the action of the group along with the weak positive poisson structure of the reduced dynamics was again used to conclude controllability. In the noncompact case we showed that under additional conditions of equilibrium controllability, controllability of the unreduced dynamics can be concluded. These results were then applied to examples discussed in Chapter 3 making appropriate conclusions about controllability in each case. We also presented results on small time controllability for these examples.

In Chapter 5 stabilization of relative equilibria of mechanical systems with symmetry was discussed. Stabilization using "Hamiltonian" feedback laws was discussed. We then presented a constructive approach to design dissipative feedback laws using centermanifold theorem like techniques. The approach exploited the observation that the relative equilibria, or equivalently the fixed points of the reduced dynamics, belonged locally to an embedded equilibrium manifold. We then showed that as opposed to obtaining stable (in the sense of Lyapunov) solutions using Hamiltonian feedback laws, our approach guarantees asymptotic convergence in the directions transverse to the center manifold, and stability in directions along the center manifold.

There are several directions for future research related to the work presented in this dissertation. One of them is to design a constructive open loop control strategy to steer the unreduced system. Except in special cases where the unforced dynamics can be explicitly integrated, the weak positive Poisson stability does not offer much insight into the design of controllers. But since we know that the system is controllable one can possibly formulate optimal control problems which may provide more insight about feasible controllers. In addition one could also attempt to use periodic controls in the base space, and thereby steer in the fiber.

Another promising direction is in showing global stability of the closed loop system under the dissipative feedback laws designed in Chapter 5. Since the divergence of the closed loop system is less than zero and we were able to show that the trajectories converge to the stable manifold ("attractor") one might conjecture that under assumptions of boundedness of solutions and absence of limit cycles the closed loop system is globally stable. Analytical results for the examples discussed did seem to indicate this. Current research includes efforts in this direction.

In addition current and future research includes design of hybrid control laws and architectures for the generation of behaviors for obstacle avoidance and path planning for the hovercraft and autonomous underwater vehicle.

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