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Abstract

We show that solid tori in $\mathbf{R}^n$ satisfy the Pompeiu property. This problem remains open for dimensions $n \neq 4$.

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1. The Pompeiu problem is an old question of harmonic analysis that appears in many different guises and interesting applications. To state it in its original setting, that of the n-dimensional Euclidean space $\mathbf{R}^n, n \geq 2$, let us recall that $M(n)$ denotes the group of orientation preserving rigid motions of $\mathbf{R}^n$, that is, it is generated by all translations and by the rotations in $SO(n)$. Any compact subset K of $\mathbf{R}^n$ induces a *Pompeiu transform* P_K between the spaces of continuous functions on $\mathbf{R}^n$ and $M(n)$:

$$
\begin{aligned}
P_K \ &: \ C(\mathbf{R}^n) \to C(M(n)) \\
P_K f(\sigma) \ &: = \int_{\sigma(K)} f(x)dx, \quad \sigma \in M(n).
\end{aligned} \tag{1}
$$

In order to avoid the triviality of P_K we assume its Lebesgue measure $m(K) > 0$. We say that K has the *Pompeiu property* if P_K is injective. Note that the equation $P_K f = 0$ is really an infinite system of convolution equations

$$\check{\chi}_{\rho(K)} * f = 0 \quad \text{for all } \rho \in SO(n) \tag{2}$$

where $\chi_{\rho(K)}$ represents the characteristic function of the rotated set $\rho(K)$ and, as usual, $\check{g}(x) = g(-x)$. In the particular case K is a disk (or a rotationally symmetric set) this system reduces to a single convolution equation, and thus it will always have exponential solutions $e^{i<\zeta,x>}, \zeta \in \mathbf{C}^n, <\zeta, x>= \zeta_1 x_1 + \cdots + \zeta_n x_n$. The *Pompeiu problem* consists on deciding which sets K have the Pompeiu property.

Let us assume from now on that $K = \bar{\Omega}$, where Ω is a bounded open set, and that the complement K^c is connected. When K is rotationally symmetric this implies that $K = \bar{B}(a, r)$, the closed ball of center a and radius r, for

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some $a \in \mathbf{R}^n, 0 < r < \infty$. If the boundary ∂K is at least Lipschitz, one has the remarkable result that K *fails* to have the Pompeiu property if and only if there is a eigenvalue α of the overdetermined Neumann problem: let $\Delta = \sum_{j=1}^{n} \partial^2/\partial^2 x_j$ be the Laplace operator and the problem is whether there is a function u satisfying the boundary value problem

$$\begin{cases} \Delta u + \alpha u = 0 & \text{in } \Omega \\ u = 1 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

It follows from Green's formula that $\alpha > 0$. There is an old conjecture, usually called *Schiffer's conjecture*, that the only domains Ω for which there is a solution to this overdetermined eigenvalue problem are the balls. One can prove that the existence of a solution to (3) implies that $\partial\Omega$ is real analytic. Thus, any bounded open set Ω with connected exterior $\bar{\Omega}^c$ and boundary which is at least Lipschitz not *everywhere* real analytic has the Pompeiu property. For instance, any polyhedron. In dimension two these domains with the Pompeiu property correspond to Jordan curves which are Lipschitz but not real analytic.

We refer the reader to the two excellent survey and bibliographic articles by Zalcman [Z1], [Z2] for ramifications, generalizations, and a fairly complete bibliography of the Pompeiu problem. (Professor Zalcman has kept updates of [Z2] which are available upon request). Let us mention also [B] and the references therein for suggestions of new directions in the Pompeiu problem as well as references to recent work on the statistics of this property.

Very little is known about the Pompeiu problem in $\mathbf{R}^n$ for $n \geq 3$, among the very simple sets with real analytic boundary that have the Pompeiu property are the proper ellipsoids, this is a result of G. Johnsson [J]. The next simple objects to consider are the solid tori in $\mathbf{R}^n, n \geq 3$. We shall discuss them below but before we conclude this introduction let us remark that most of the positive general answers to the Pompeiu problem in $\mathbf{R}^2$ are due to the work of Garofalo and Segala and of Ebenfelt [GS], [E1-3]. One of the consequences of their work is the following remarkable result.

Let Ω be a Jordan domain in the complex plane, $\partial\Omega$ a real analytic curve which is not a circle. Let $\varphi : B(0,1) \to \Omega, \psi : \Omega \to B(0,1)$ be the conformal mappings, guaranteed to exist by the Riemann mapping theorem. If either φ or ψ is a rational function, then Ω has the Pompeiu property.

Therefore, the first class of simple domains with real analytic boundaries for which one has difficulty in deciding whether they have the Pompeiu property or not are the solid tori in $\mathbf{R}^n, n \geq 3$. (Ebenfelt's 3-dimensional examples in [E1, §3] do no include the torus). We show below that certain tori in $\mathbf{R}^4$ do have the Pompeiu property.

2. Let us now consider a special kind of tori in $\mathbf{R}^4$. A typical point in $\mathbf{R}^4$ will have coordinates (x, y, z, w). Let $R > 1$ and let $D(R, 1)$ denote the disk of center $(R, 0)$ and radius 1 in the plane $y = z = 0$ in $\mathbf{R}^4$. By rotating this disk about the w-axis in $\mathbf{R}^4$ we obtain a torus Ω of equation

$$(\sqrt{x^2 + y^2 + z^2} - R)^2 + w^2 < 1$$

In fact, each point of $D(R, 1)$ traces a sphere S^2 in this process. Assume there is an eigenvalue $\alpha > 0$ for the boundary value problem (3) and corresponding eigenfunction. Introducing the function $v = (1 - u)/\alpha$ we see that it satisfies the overdetermined equation

$$\begin{cases} \Delta v + \alpha v = 1 & \text{in } \Omega \\ v = \nabla v = 0 & \text{on } \partial\Omega \end{cases} \tag{4}$$

The domain Ω and the equation (4) with the given boundary conditions are invariant under the action of $SO(3)$ in the coordinates (x, y, z), thus the solution v of the boundary value problem (4) must also be invariant under this action. Hence, if we denote

$$\rho = \sqrt{x^2 + y^2 + z^2},$$

we have that

$$v(x, y, z, w) = V(\rho, w) \tag{5}$$

for a function V defined in $\bar{D}(R, 1)$. From the expression of the radial part of the Laplace operator on $\mathbf{R}^3$ we conclude that

$$\Delta v = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial w^2}\right) v = \frac{\partial^2 V}{\partial w^2} + \frac{\partial^2 V}{\partial \rho^2} + \frac{2}{\rho}\frac{2}{\partial \rho} V = \Box V + \frac{2}{\rho}\frac{\partial V}{\partial \rho},$$

where we have denoted by $\Box$ the Laplace operator in the Euclidean variables ρ, w. Thus, the function V satisfies the boundary value problem

$$\begin{cases} \Box V + \frac{2}{\rho}\frac{\partial V}{\partial \rho} + \alpha V = 1 & \text{in } D(R,1) \\ V = \text{grad } V = 0 & \text{on } \partial D(R,1) \end{cases} \tag{6}$$

Let us introduce the auxiliary function

$$U = \rho(V - \frac{1}{\alpha}) \tag{7}$$

which satisfies now

$$\begin{cases} \Box U + \alpha U = 0 & \text{in } D(R,1) \\ U = -\rho/\alpha & \text{on } \partial D(R,1) \\ \text{grad } U = -\text{grad}(\rho/\alpha) & \text{on } \partial D(R,1) \end{cases} \tag{8}$$

where the gradient in (8) is in the coordinates (ρ, w).

We can now introduce polar coordinates in the disk $D(R,1)$, so that

$$\begin{cases} \rho = R + r \cos \theta \\ w = r \sin \theta \end{cases} \tag{9}$$

$0 \le r < 1, \theta \in [-\pi, \pi]$. It is natural to split the solution U of the equation (7) as a sum of two functions

$$U = U_1 + U_2,$$

which satisfy the same differential equation but the boundary values are determined in polar coordinates by

$$\begin{cases} U_1(1,\theta) = -\frac{\cos\theta}{\alpha} \\ \frac{\partial U_1}{\partial r}(1,\theta) = -\frac{\cos\theta}{\alpha} \end{cases} \tag{10'}$$

$$\begin{cases} U_2(1,\theta) = -\frac{R}{\alpha} \\ \frac{\partial U_2}{\partial r}(1,\theta) = 0 \end{cases} \tag{10''}$$

which correspond to the two terms appearing in ρ from the equation in polar coordinates (9). One can solve explicitly the two new boundary value problems for U_1 and U_2 by separation of variables. Namely, from [CH, vol. I, Chapter 7]

$$U_1(r,\theta) = (A_1 J_1(\sqrt{\alpha} r) + B_1 Y_1(\sqrt{\alpha} r)) \cos \theta$$

where A_1, B_1 are constants to be determined and J_1, Y_1 are the Bessel functions of order 1 and first and second kind, respectively.

Similarly, in terms of Bessel functions of order 0 we have

$$U_2(r,\theta) = A_0 J_0(\sqrt{\alpha} r) + B_0 Y_0(\sqrt{\alpha} r).$$

Recall that the function U is smooth even for $r = 0$, while the functions Y_0 and Y_1 blow up at $r = 0$. Thus, it must be the case that the coefficients $A_0 = B_0 = 0$. On the other hand, the coefficients A_1, B_1, A_0, B_0 are uniquely determined by the boundary conditions (9′) and (9″), which can be expressed in terms of the two systems of linear equations

$$\begin{cases} A_1 J_1(\sqrt{\alpha}) + B_1 Y_1(\sqrt{\alpha}) = -\frac{1}{\alpha} \\ A_1 J_1'(\sqrt{\alpha}) + B_1 Y_1'(\sqrt{\alpha}) = -\frac{1}{\alpha^{3/2}} \end{cases} \tag{11'}$$

and

$$\begin{cases} A_0 J_0(\sqrt{\alpha}) + B_0 Y_0(\sqrt{\alpha}) = -\frac{R}{\alpha} \\ A_0 J_0'(\sqrt{\alpha}) + B_0 Y_0'(\sqrt{\alpha}) = 0 \end{cases} \tag{11''}$$

Both systems also determine A_0, B_0, A_1, B_1 uniquely since the Wronskian determinants $W(J_0, Y_0)(\sqrt{\alpha}) \neq 0, W(J_1, Y_1)(\sqrt{\alpha}) \neq 0$. Equation (11″) implies that, unless $A_0 = 0, J_0'(\sqrt{\alpha}) = 0$. On the other hand, the Bessel functions are related by the identity

$$J_0' = -J_1$$

thus $J_1(\sqrt{\alpha}) = 0$, and since $B_1 = 0$, the first equation of (11′) could not be satisfied. The remaining possibility $A_0 = 0$ contradicts the first equation of (11″) since $B_0 = 0$.

Therefore, we have shown that the boundary value problem cannot have a solution and so, neither can (3) in the torus Ω. In view of the considerations of the previous section we have obtained the following result.

Theorem. *Let $R > 1$ and let Ω be the solid torus in $\mathbf{R}^4$ defined by*

$$(\sqrt{x^2 + y^2 + z^2} - R)^2 + w^2 < 1,$$

then Ω has the Pompeiu property in $\mathbf{R}^4$.

In $\mathbf{R}^n$ the only difference is that in equation (5) one has to replace the coefficient 2 by $n-2$. Already for $n=3$ this small difference makes the rest of the proof break down. One can introduce the polar coordinates (9) and represent V as a series $\sum_{n=0}^{\infty} U_n(r)\cos n\theta$, but this time instead of two separate problems (10′) and (10″) one has an infinite number of (finitely) coupled differential equations for the coefficients U_n. We believe that for any dimension these solid tori have the Pompeui property but so far the proof has escaped us.

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