ABSTRACT

Title of dissertation:	HIERARCHICAL RECONSTRUCTION METHOD FOR SOLVING ILL-POSED LINEAR INVERSE PROBLEMS
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We present a detailed analysis of the application of a multi-scale Hierarchical Reconstruction method for solving a family of ill-posed linear inverse problems. When the observations on the unknown quantity of interest and the observation operators are known, these inverse problems are concerned with the recovery of the unknown from its observations. Although the observation operators we consider are linear, they are inevitably ill-posed in various ways. We recall in this context the classical Tikhonov regularization method with a stabilizing function which targets the specific ill-posedness from the observation operators and preserves desired features of the unknown. Having studied the mechanism of the Tikhonov regularization, we propose a multi-scale generalization to the Tikhonov regularization method, so-called the Hierarchical Reconstruction (HR) method. First introduction of the HR method can be traced back to the Hierarchical Decomposition method in Image Processing. The HR method *successively* extracts information from the previous hierarchical residual to the current hierarchical term at a *finer* hierarchical *scale*. As the sum of all the hierarchical terms, the hierarchical sum from the HR method provides an reasonable approximate solution to the unknown, when the observation matrix satisfies certain conditions with specific stabilizing functions. When compared to the Tikhonov regularization method on solving the same inverse problems, the HR method is shown to be able to decrease the total number of iterations, reduce the approximation error, and offer self control of the approximation distance between the hierarchical sum and the unknown, thanks to using a ladder of *finitely many* hierarchical scales. We report numerical experiments supporting our claims on these advantages the HR method has over the Tikhonov regularization method.

Hierarchical Reconstruction Method for Solving Ill-posed Linear Inverse Problems

by

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Dedication

To my beloved Parents: Mr. Ruiqin Zhong and Ms. Xiaoli Tang. Without their generous support and selfless love, this thesis would be not possible.

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List of Abbreviations

CS	Compressed Sensing, Compressive Sensing
HR	Hierarchical Reconstruction
LASSO	Least Absolute Shrinkage and Selection Operator
LS	Least Square
LES	Large Eddy Simulation
NSE	Navier Stokes Equations
RIT	Research Interaction Team
TLR	Tikhonov-Lavrentiev Regularization
UMD	University of Maryland, College Park

Chapter 1: Introduction

1.1 The Recovery Problems

We analyze a multi-scale Hierarchical Reconstruction (HR) method on solving a family of ill-posed linear inverse problems. The scenario is as follows. Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hilbert spaces, each equipped with the norms $||\cdot||_{\mathcal{X}}$ and $||\cdot||_{\mathcal{Y}}$ respectively. There is an unknown $\boldsymbol{x}_* \in \mathcal{X}$, to which we do not have direct access. However, we are able to obtain the observation $\boldsymbol{y}_* \in \mathcal{Y}$ by applying the observation operator $A : \mathcal{X} \to \mathcal{Y}$ to $\boldsymbol{x}_*$, i.e., $\boldsymbol{y}_* = A\boldsymbol{x}_*$. With the known $\boldsymbol{y}_*$ and A at hand, these inverse problems are concerned with the recovery of $\boldsymbol{x}_*$ by performing certain extraction of information from $\boldsymbol{y}_*$. Although the observation operator A is linear, the operator A is ill-posed in various ways, making recovery of $\boldsymbol{x}_*$ by direct "solution" of the linear equation $A\boldsymbol{x} = \boldsymbol{y}_*$ impossible. Additional conditions are needed in order to augment the ill-posed linear equation $A\boldsymbol{x} = \boldsymbol{y}_*$, henceforth converting it into well-posed extended problems. The Tikhonov regularization method [57, 58] is a classical tool to address such inverse problems with the help of an extra regularization *parameter* $\lambda > 0$. However, the Tikhonov regularization method is limited by the usage of one such $\lambda > 0$. Since the the regularization *parameter* λ controls the distance between the Tikhonov observation[1] and $\boldsymbol{y}_*$, λ can be understood as a regularization *scale*. Built upon this particular understanding, we propose a *multi-scale* generalization to the Tikhonov regularization method, so-called the HR method. The HR method was first introduced in Image Processing [48, 49] to treate the multi-scale issues in image decomposition and de-noising, and then it was further developed and analyzed in solving linear PDEs [54, 55] to construct uniformly bounded solutions in regularity spaces. We adopt the HR method for its multi-scale approach and feature preserving by the usage of a suitable stabilizing function. The HR method realizes that the information in the residual term can be extracted at a finer regularization scale. Therefore, the HR method performs *successive* extraction of information from the previous hierarchical residual to the current hierarchical term at a *finer* hierarchical *scale*. By utilizing a ladder of *finitely many* hierarchical residuals together with their corresponding hierarchical scales, the HR method can decrease the total number of iterations, reduce approximation error, and offer self control of the distance between the hierarchical sum (sum of all hierarchical terms) and the unknown $\boldsymbol{x}_*$, when compared to the Tikhonov regularization method on solving the same inverse problems.

1.2 Regularization Methods: from Single Scale to Multi Scale

Given the observation $\boldsymbol{y}_*$ and the observation operator A, the Tikhonov regularization method finds an approximate solution $\boldsymbol{x}_{T(\lambda)}$ to $A\boldsymbol{x} = \boldsymbol{y}_*$ from a convex feasible set $\mathcal{C} \subset \mathcal{X}$ with an extra regularization parameter $\lambda > 0$. The approximate solution, $\boldsymbol{x}_{T(\lambda)}$,

[1]The Tikhonov observation is obtained by applying the observation operator to the approximate solution from the Tikhonov regularization method.

also satisfies the following,

$$\boldsymbol{x}_{T(\lambda)} = \arg\min_{\boldsymbol{x}\in\mathcal{C}} \left\{\lambda f(\boldsymbol{x}) + \frac{1}{2}||\boldsymbol{y}_* - A\boldsymbol{x}||_{\mathcal{Y}}^2\right\}. \tag{1.1}$$

Here the non-negative auxiliary function $f : \mathcal{X} \to \mathbb{R}$ is a stabilizing function. The proper choice of the stabilizing function f depends on the desired features from the unknown which we want to preserve. If we plan to recover sparse unknowns, we could set $f(\boldsymbol{x}) = ||\boldsymbol{x}||_{\ell_1}$; if we are interested in preserving the "edges" in images, set $f(\boldsymbol{x}) = ||\boldsymbol{x}||_{TV}$ (Total Variation norm); if we are recovering unknown from the L^2 space of functions, set $f(\boldsymbol{x}) = ||\boldsymbol{x}||_{L^2}^2/2$. The distance between $\boldsymbol{y}_*$ and $A\boldsymbol{x}_{T(\lambda)}$ is controlled by the regularization parameter λ; when $\lambda \downarrow 0$, $A\boldsymbol{x}_{T(\lambda)} \to \boldsymbol{y}_*$. Hence, $A\boldsymbol{x}_{T(\lambda)}$ can be understood as an approximation to $\boldsymbol{y}_*$ at the scale λ. Let $\boldsymbol{x}_\lambda = \boldsymbol{x}_{T(\lambda)}$ and define residual term as, $\boldsymbol{r}_\lambda := \boldsymbol{y}_* - A\boldsymbol{x}_\lambda$. We have a decomposition of $\boldsymbol{y}_*$ at the scale λ as $\boldsymbol{y}_* = A\boldsymbol{x}_\lambda + \boldsymbol{r}_\lambda$. Since $\boldsymbol{r}_\lambda \neq \mathbf{0}$ (or we are done), there is information remaining in $\boldsymbol{r}_\lambda$, which we can extract at a *finer* scale, say $\frac{\lambda}{2}$,

$$\boldsymbol{x}_{\frac{\lambda}{2}} = \arg\min_{\boldsymbol{x}\in\mathcal{C}} \left\{\frac{\lambda}{2} f(\boldsymbol{x}) + \frac{1}{2}||\boldsymbol{r}_\lambda - A\boldsymbol{x}||_{\mathcal{Y}}^2\right\} \quad \text{with} \quad \boldsymbol{r}_{\frac{\lambda}{2}} := \boldsymbol{r}_\lambda - A\boldsymbol{x}_{\frac{\lambda}{2}}. \tag{1.2}$$

We now have a better two-scale representation of $\boldsymbol{y}_*$ given by $\boldsymbol{y}_* = A\boldsymbol{x}_\lambda + A\boldsymbol{x}_{\frac{\lambda}{2}} + \boldsymbol{r}_{\frac{\lambda}{2}}$; information below scale $\frac{\lambda}{2}$ remains intact in $\boldsymbol{r}_{\frac{\lambda}{2}}$. This process in (1.2) can continue. To simplify notations, we start with $\boldsymbol{r}_0 = \boldsymbol{y}_*$ and $\lambda_1 = \lambda$, and find the first hierarchical term $\boldsymbol{x}_{(1)}$ such that the following holds

$$\boldsymbol{x}_{(1)} = \arg\min_{\boldsymbol{x}\in\mathcal{C}} \left\{\lambda_1 f(\boldsymbol{x}) + \frac{1}{2}||\boldsymbol{r}_0 - A\boldsymbol{x}||_{\mathcal{Y}}^2\right\} \quad \text{with} \quad \boldsymbol{r}_1 := \boldsymbol{r}_0 - A\boldsymbol{x}_{(1)},$$

we proceed by iterating at the dyadic scales $\lambda_j = 2^{1-j}\lambda_1$: for $2 \leq j \leq J$, we solve for the other hierarchical terms $\boldsymbol{x}_{(j)}$'s from the following recursive equation,

$$\boldsymbol{x}_{(j)} = \arg\min_{\boldsymbol{x}\in\mathcal{C}} \left\{\lambda_j f(\boldsymbol{x}) + \frac{1}{2}||\boldsymbol{r}_{j-1} - A\boldsymbol{x}||^2_{\mathcal{Y}}\right\} \quad \text{with} \quad \boldsymbol{r}_j := \boldsymbol{r}_{j-1} - A\boldsymbol{x}_{(j)}.$$

Summing the recursive relation, $\boldsymbol{r}_j = \boldsymbol{r}_{j-1} - A\boldsymbol{x}_{(j)}$, we end up with a *hierarchical decomposition* of $\boldsymbol{y}_*$,

$$\boldsymbol{y}_* = A\boldsymbol{x}_{(1)} + A\boldsymbol{x}_{(2)} + \ldots + A\boldsymbol{x}_{(J)} + \boldsymbol{r}_J.$$

In this fashion, we also obtain an approximate solution $\boldsymbol{X}_J$ to the ill-posed linear equation $A\boldsymbol{x} = \boldsymbol{y}_*$, where $\boldsymbol{X}_J$ is the sum of hierarchical terms, i.e., $\boldsymbol{X}_J = \sum_{j=1}^{J}\boldsymbol{x}_{(j)}$. As J increases, the hierarchical term $\boldsymbol{x}_{(j)}$ extracts information from $\boldsymbol{r}_{j-1}$ at a *finer* scale $\sim \lambda_j = 2^{1-j}\lambda_1$. We note that coarser and finer decompositions are available. Different ladder of scales $\lambda_j = \theta^{j-1}\lambda_1$ with $0 < \theta < 1$ can be employed depending on the problems at hand. With additional conditions on the observation operator A and suitable choice of the stabilizing function f, we can show that $\boldsymbol{X}_J$ becomes a reasonable approximation to $\boldsymbol{x}_*$. Thanks to utilizing a ladder of *finitely many* hierarchical residuals $\boldsymbol{r}_j$'s with their corresponding hierarchical scales λ_j's, the HR method shows different advantages over the Tikhonov regularization method, which will be explored in details in the following chapters.

1.3 Thesis Outline

In the following chapters, we will discuss in details the specific application of employing the HR method for three different kinds of ill-posed linear inverse problems. They are structured as follows.

In chapter 2, we investigate the sparse recovery problem from Compressed Sensing. Having known the observation operator $A \in \mathbb{R}^{M \times N}$ and the observation $\hat{\boldsymbol{y}}_* \in \mathbb{R}^M$ from a k-sparse[2] unknown vector $\hat{\boldsymbol{x}}_* \in \mathbb{R}^N$, the original sparse recovery problem is concerned with the recovery of $\hat{\boldsymbol{x}}_*$ by extracting information from $\hat{\boldsymbol{y}}_*$. The ill-posedness is due to an under-determined observation matrix A ($M \ll N$). We discuss some of the reasons why the constrained ℓ_1 method introduced by Candès, et al. and separately by Dohono, is preferred for sparse recovery. We suggest using the unconstrained ℓ_1 method with an extra regularization parameter $\lambda > 0$ to provide an approximate solution $\boldsymbol{x}_\lambda$ to a general unknown vector $\boldsymbol{x}_* \in \mathbb{R}^N$ from its *noisy* observation $\boldsymbol{y}_*^\varepsilon = A\boldsymbol{x}_* + \boldsymbol{\epsilon}$ ($||\boldsymbol{\epsilon}||_2 \le \varepsilon$)[3]. We show that when the observation matrix A satisfies the Robust Null Space Property (RNSP) of order k, $||\boldsymbol{x}_\lambda - \boldsymbol{x}_*||_1 \le \mathcal{O}(\lambda, \varepsilon, \sigma)$[4]. However, it takes larger number of iterations for numerical algorithms to find the solution from the unconstrained ℓ_1 method with an extremely small regularization parameter λ to perform recovery of $\hat{\boldsymbol{x}}_*$ from $\hat{\boldsymbol{y}}_*$. Built upon the understanding of λ being a regularization *scale*, we propose the HR method as a multi-scale generalization to the unconstrained ℓ_1 method in order to reduce the total number of iterations. The hierarchical sum $\boldsymbol{X}_J$, as the sum of hierarchical terms $\boldsymbol{x}_{(j)}$'s, i.e., $\boldsymbol{X}_J = \sum_{j=1}^{J} \boldsymbol{x}_{(j)}$, becomes a reasonable approximate solution to $\boldsymbol{x}_*$ when the observation matrix A satisfies the RNSP of order k. We conduct numerical experiments

[2]A vector $\boldsymbol{x} \in \mathbb{R}^N$ is k-sparse if it has at most k non-zero entries.

[3]For $\boldsymbol{x} \in \mathbb{R}^N$, $||\boldsymbol{x}||_p = \sqrt[p]{\sum_{i=1}^{N} \left|(\boldsymbol{x})_i\right|^p}$.

[4]$\sigma = \sigma(\boldsymbol{x}_*, k, 1)$ is the best k-term approximation error of $\boldsymbol{x}_*$ in ℓ_1 norm, i.e., let $\boldsymbol{x}_*(k)$ be the best k-term approximation of $\boldsymbol{x}_*$, then $\sigma = ||\boldsymbol{x}_* - \boldsymbol{x}_*(k)||_1$. We drop the dependence on $\boldsymbol{x}_*$, k and 1 in order to simplify the notation.

comparing the HR method to the unconstrained ℓ_1 minimization on recovery of a general unknown $\boldsymbol{x}_*$ from $\boldsymbol{y}_*^{\varepsilon}$ with $\lambda = \lambda_J$. We close Chapter 2 with a brief discussion on extending the analysis of the HR method to recovery of vectors in $\mathbb{C}^N$.

In chapter 3, we study the deconvoution problem on the Helmholtz filter for the closure problem in Large Eddy Simulation (LES). The Helmholtz filter $A_\delta : \mathcal{X} \to \mathcal{X}$ with a filtering radius $\delta > 0$ is defined as an convolution with a scaled kernel $K_\delta = (4\pi\delta^2||\boldsymbol{s}||_2)^{-1}\exp(-\delta^{-1}||\boldsymbol{s}||_2)$ for any $\boldsymbol{s} \in \mathbb{R}^3$, i.e., $\boldsymbol{y} = K_\delta * \boldsymbol{x} = A_\delta \boldsymbol{x}$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{X}$. When the filtered output $\boldsymbol{y}_*$ $(= A_\delta \boldsymbol{x}_* \in \mathcal{X})$ is given, we are interested in recovering the unknown $\boldsymbol{x}_*$ by solving $A_\delta \boldsymbol{x} = \boldsymbol{y}_*$. The inverse of A_δ is well-defined as an elliptic differential equation with Dirichlet boundary condition. However direct application of A_δ^{-1} is not possible in the context of LES, when one is only given the access to the numerical output $\boldsymbol{y}_*^h = A^h \boldsymbol{x}_*$, where A^h $(= A_\delta^h)$[5] is the discrete Helmholtz filter and h is the spatial resolution scale from a certain discretization scheme. We will focus on using the Finite Element discretization scheme to provide the discrete Helmholtz filter A^h as a numerical approximation to the Helmholtz filter A_δ. Note that in the discrete setting the Finite Element matrix associated to A^h is a square matrix, i.e., $M = N$. We briefly discuss a family of Tikhonov regularization methods on providing approximate solutions to the the discrete equation $A^h \boldsymbol{x} = \boldsymbol{y}_*^h$. We propose the HR method as an multi-scale generalization of the Tikhonov-Lavrentiev regularizatoin method. In order to show the discrete approximation error by using the HR method, we begin with the analysis of applying the HR method to find an approximate solution to the continuous equation $A_\delta \boldsymbol{x} = \boldsymbol{y}_*$. We then continue our

[5]We drop the dependence on δ to simplify the notation and emphasize the importance of the spatial resolution scale h

analysis on showing the discrete approximation error when we employ the HR method to find an approximation solution to the discrete equation $A^h\boldsymbol{x} = \boldsymbol{y}_*^h$. We also discuss the descent property of using the HR method when we perform de-convolution from a noisy discrete filtered output $\boldsymbol{y}_*^{h,\varepsilon}$ cause by modeling errors when the residual stress tensor is approximated. We also provide a stopping criteria to prevent the HR approximate solution sequence from deviating from its convergence to $\boldsymbol{x}_*$. Numerical experiments, comparing the HR method to other Tikhonov methods on recovery of $\boldsymbol{x}_*$ via de-convolution from $\boldsymbol{y}_*^{h,\varepsilon}$, are reported to support our claim on the improved approximation error provided by the HR method. We conclude the chapter by discussion of possible extension of the approximation error analysis for the HR method to the Finite Difference discretization scheme and other convolution filters.

In chapter 4, we discuss two inverse problems from Linear Regression (LR). In LR, we are given a set of data from M observations: $\{A, \boldsymbol{y}_*^\varepsilon\}$, with the i^{th} row of $A \in \mathbb{R}^{M\times N}$ being the regressors and i^{th} entry of $\boldsymbol{y}_*^\varepsilon \in \mathbb{R}^M$ being the response from the i^{th} observation. However, the regressor matrix A is over-determined ($M \gg N$) and the response $\boldsymbol{y}_*^\varepsilon$ is not in the range of A. Finding a linear model which satisfies $A\boldsymbol{x} = \boldsymbol{y}_*^\varepsilon$ exactly is impossible. Alternatively, we find approximate solutions to $A\boldsymbol{x} = \boldsymbol{y}_*^\varepsilon$. We examine two different scenarios. First, when $\boldsymbol{y}_*^\varepsilon$ is generated as the linear combination of the regressor matrix A with noise, i.e., there is a unique $\boldsymbol{x}_* \in \mathbb{R}^N$ such that $\boldsymbol{y}_*^\varepsilon = A\boldsymbol{x}_* + \boldsymbol{\epsilon}$ with $||\boldsymbol{\epsilon}||_2 \leq \varepsilon$. Second, there is no known relationship between $\boldsymbol{y}_*^\varepsilon$ and A. However, we are interested in performance of using linear models to approximate the relationship between $\boldsymbol{y}_*^\varepsilon$ and A, i.e., $\boldsymbol{y}_*^\varepsilon = A\boldsymbol{x}_* + \boldsymbol{\epsilon}$ for some unknown $\boldsymbol{x}_* \in \mathbb{R}^N$ and a well-controlled[6] modeling error $\boldsymbol{\epsilon}$. We begin our analysis by comparing the Least Square (LS) method to the Least Absolute Shrinkage and Selection Operator (LASSO) method for both scenarios. We note that the regularization parameter λ used in the LASSO method controls the distance between the LASSO linear model and the LS linear model, as well as the distance between the LASSO linear model and the unknown linear model $\boldsymbol{x}_*$. In order to offer better control over the distance between the linear model provided by our linear model to either the LS linear model or $\boldsymbol{x}_*$, we propose the HR method as a multi-scale generalization to the LASSO method. Thanks to using a hierarchy of regularization scales, the HR linear model has its distance to the LS linear model or $\boldsymbol{x}_*$ controlled by the number of HR iterations taken. We report numerical experiments on comparing the HR method to the LASSO method and the LS method. We end Chapter 4 on the extension of using the HR method to Linear Regression on non-linear basis functions.

We conclude the thesis in chapter 5. We survey the advantages which the HR method has over the Tikhonov regularization on solving the three different ill-posed linear inverse problems. We further develop the HR method for solving general ill-posed linear inverse problems. We show necessary assumptions on the stabilizing function f in order to guarantee convergence of the HR approximate solution to $\boldsymbol{x}_*$.

In Appendix A, we discuss individual single scale implementations of the general Tikhonov regularization methods.

[6]By "well-controlled", we mean that the difference, $\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}_*$, is smallest in some ℓ_p norm, or satisfying other conditions.

Chapter 2: Hierarchical Reconstruction Method for Sparse Recovery

2.1 Introduction: Recovery of General $\boldsymbol{x}_*$ from Its Noisy Observation $\boldsymbol{y}_*^\varepsilon$

We study the sparse recovery problem from Compressed Sensing. The k-sparse[1] unknown $\hat{\boldsymbol{x}}_* \in \mathbb{R}^N$ is to be recovered from its indirect observation $\hat{\boldsymbol{y}}_* = A\hat{\boldsymbol{x}}_* \in \mathbb{R}^M$ where $A \in \mathbb{R}^{M \times N}$ is an observation matrix. The observation process is linear, however the observation matrix A is severly under-determined, i.e., $M \ll N$. Hence, recovery of $\hat{\boldsymbol{x}}_*$ by direct "solution" of the linear system $A\boldsymbol{x} = \hat{\boldsymbol{y}}_*$ is impossible. However, the theories from Compressed Sensing show that by knowing sparsity level of $\hat{\boldsymbol{x}}_*$ before hand, namely the constant k, recovery of $\hat{\boldsymbol{x}}_*$ from $\hat{\boldsymbol{y}}_*$ is possible and practical. In 2006, Candès, et al. [7], and separately Donoho [12], introduced the constrained ℓ_1 method[2], which finds a solution from the following

$$\boldsymbol{x}_1 = \underset{\boldsymbol{x} \in \mathbb{R}^N}{\arg\min} \left\{ ||\boldsymbol{x}||_1 \middle| A\boldsymbol{x} = \hat{\boldsymbol{y}}_* \right\}.$$

A major milestone in the theories for Compressed Sensing is that when the observation matrix A satisfies certain recoverability condition, for any k-sparse unknown $\hat{\boldsymbol{x}}_*$, the solution provided by the constrained ℓ_1 method, namely $\boldsymbol{x}_1$, will be the same as $\hat{\boldsymbol{x}}_*$. However,

[1]A vector $\boldsymbol{x} \in \mathbb{R}^N$ is to said to be k-sparse if $\boldsymbol{x}$ has at most k non-zero entries.

[2]For any $\boldsymbol{x} \in \mathbb{R}^N$ and any $p > 0$, $||\boldsymbol{x}||_p = \sqrt[p]{\sum_{i=1}^N |x_i|^p}$; for $p = 0$, $||\boldsymbol{x}||_0$ counts the number of non-zero entries in $\boldsymbol{x}$.

it is not always convenient to encounter sparse quantities, one might be given a task to recover a general unknown vector $\boldsymbol{x}_* \in \mathbb{R}^N$. Moreover, due to inevitable errors in the observation process, one only has access to its noisy observation $\boldsymbol{y}_*^\varepsilon$, i.e., $\boldsymbol{y}_*^\varepsilon = A\boldsymbol{x}_* + \boldsymbol{\epsilon}$. The details of observation error term, $\boldsymbol{\epsilon}$, remain mostly unknown; fortunately, it is bounded above in ℓ_2 norm by some known noise level constant $\varepsilon > 0$, i.e., $||\boldsymbol{\epsilon}||_2 \leq \varepsilon$. Aiming to provide a robust method to recover $\boldsymbol{x}_*$ from $\boldsymbol{y}_*^\varepsilon$, we suggest using the unconstrained ℓ_1 method, which finds an approximate solution to $\boldsymbol{x}_*$ with the help of an extra regularization *parameter* $\lambda > 0$ and the approximate solution satisfies the following,

$$\boldsymbol{x}_\lambda = \underset{\boldsymbol{x} \in \mathbb{R}^N}{\arg\min} \left\{ \lambda ||\boldsymbol{x}||_1 + \frac{1}{2} ||\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}||_2^2 \right\}.$$

We show that when the observation matrix A satisfies the Robust Null Space Property (RNSP) of order k, the approximation error, $\boldsymbol{x}_* - \boldsymbol{x}_\lambda$, is bounded above by $\mathcal{O}(\sigma, \varepsilon, \lambda)$[3]. When the unconstrained ℓ_1 method is used for recovery of $\hat{\boldsymbol{x}}_*$ from $\hat{\boldsymbol{y}}_*$, the approximate solution $\boldsymbol{x}_\lambda$ can be thought of as an approximation to $\hat{\boldsymbol{x}}_*$ at the *scale* λ. Based on the understanding of λ being a regularization *scale*, we adopt the Hierarchical Reconstruction (HR) method, from its first introduction in image processing in [48, 49], as a multi-scale generalization of the unconstrained ℓ_1 method. The HR method *successively* extracts information from the previous hierarchical residual $\boldsymbol{r}_j$ to the current hierarchical term $\boldsymbol{x}_{(j)}$ at a *finer* hierarchical scales λ_j. When the observation matrix A also satisfies RNSP of order k, the hierarchical sum $\boldsymbol{X}_J$, as the sum of all hierarchical terms $\boldsymbol{x}_{(j)}$, i.e., $\boldsymbol{X}_J = \sum_{j=1}^J \boldsymbol{x}_{(j)}$ with a ladder of hierarchical scales $\{\lambda_j\}_{j=1}^J$, approximates $\boldsymbol{x}_*$ with an error bounded above by $\mathcal{O}(\lambda_1, \lambda_J, J, \varepsilon, \sigma)$. Thanks to utilizing a ladder of *finitely many* hi-

[3] $\sigma = \sigma(\boldsymbol{x}_*, k, 1)$ stands for the best k-term approximation error of $\boldsymbol{x}_*$ in ℓ_1 norm. We drop the dependence of $\boldsymbol{x}_*$, k and 1 to simplify the notation.

erarchical residuals together with their corresponding hierarchical scales, the HR method is able to reduce the total number of iterations for numerical solvers when compared to the unconstrained ℓ_1 method on the recovery of $\boldsymbol{x}_*$ from $\boldsymbol{y}_*^\varepsilon$ with $\lambda = \lambda_J$.

The remaining sections of this chapter is structured as follows. In section 2.2, we compare three constrained ℓ_p methods (for $p = 0, 1, 2$) for the recovery of k-sparse unknown $\hat{\boldsymbol{x}}_*$ from its observation $\hat{\boldsymbol{y}}_*$. We conclude that the constrained ℓ_1 method is the most practical convex optimization method to use for recovery of a k-sparse unknown $\hat{\boldsymbol{x}}_*$ from its observation $\hat{\boldsymbol{y}}_*$ due to the presence of ℓ_1 norm. In section 2.3, we introduce the nearby problem: recovery of a general unknown $\boldsymbol{x}_*$ from its noisy observation $\boldsymbol{y}_*^\varepsilon$. We compare the unconstrained ℓ_1 method to the quadratically constrained ℓ_1 method. We show the recoverability condition for both methods such that the approximation error is under controlled. In section 2.3.1, we propose the HR method as a multi-scale generalization to unconstrained ℓ_1 method. We show the recoverability condition for the HR method, so that the hierarchical sum becomes a reasonable approximation to $\boldsymbol{x}_*$. In section 2.4, we present numerical experiments, comparing the HR method to the unconstrained ℓ_1 method on the recovery of $\boldsymbol{x}_*$ from $\boldsymbol{y}_*^\varepsilon$ with $\lambda = \lambda_J$. The HR method uses significantly fewer number of iterations than the unconstrained ℓ_1 method. At the end, we conclude this chapter in section 2.5 by discussion of extending the HR method to the recovery of unknowns in $\mathbb{C}^N$.

2.2 The Original Problem: Recovery of k-sparse $\hat{\boldsymbol{x}}_*$ from Observation $\hat{\boldsymbol{y}}_*$

The sparse recovery problem, recovery of the k-sparse unknown $\hat{\boldsymbol{x}}_*$ from its observation $\hat{\boldsymbol{y}}_*$, has the following setup. There is the k-sparse unknown vector $\hat{\boldsymbol{x}}_* \in \mathbb{R}^N$, one is given the indirect observations stored in $\hat{\boldsymbol{y}}_* \in \mathbb{R}^M$ with the i^{th} entry $(\hat{\boldsymbol{y}}_*)_i$ being the observation result taken as an usual Euclidean inner product of $\hat{\boldsymbol{x}}_*$ with the observing vector $\boldsymbol{a}_i \in \mathbb{R}^N$, i.e., $(\hat{\boldsymbol{y}}_*)_i = \langle \hat{\boldsymbol{x}}_*, \boldsymbol{a}_i \rangle$. We define the observation matrix $A \in \mathbb{R}^{M \times N}$ as the concatenation of $\boldsymbol{a}_i$'s, i.e., the i^{th} row of A is $\boldsymbol{a}_i^\top$. Knowing the observation $\hat{\boldsymbol{y}}_*$ and the observation matrix A, one is entrusted with the task of recovering the k-sparse unknown $\hat{\boldsymbol{x}}_*$ from $\hat{\boldsymbol{y}}_*$. However, the observation matrix A is severally under-determined ($M \ll N$), recovery of $\hat{\boldsymbol{x}}_*$ is impossible by direct "solution" of the linear equation $A\boldsymbol{x} = \hat{\boldsymbol{y}}_*$. However, when the sparsity level of $\hat{\boldsymbol{x}}_*$, namely the constant k, is known before hand, the recovery of $\hat{\boldsymbol{x}}_*$ becomes possible. The question on an effective and efficient procedure to recover $\hat{\boldsymbol{x}}_*$ remains unanswered. We have mentioned in chapter 1 that the general Tikhonov regularization method can be employed to solve ill-posed linear systems. In the sparse recovery setting, the general Tikhonov regularization method finds an approximate solution $\boldsymbol{x}_{T(\lambda)}$ to $A\boldsymbol{x} = \hat{\boldsymbol{y}}_*$ from a convex feasible set $\mathcal{C} \subset \mathbb{R}^N$ with an extra regularization parameter $\lambda > 0$. The approximate solution $\boldsymbol{x}_{T(\lambda)}$ also satisfies the following,

$$\boldsymbol{x}_{T(\lambda)} = \arg\min_{\boldsymbol{x} \in \mathcal{C}} \left\{ \lambda f(\boldsymbol{x}) + \frac{1}{2} ||\hat{\boldsymbol{y}}_* - A\boldsymbol{x}||_2^2 \right\}. \tag{2.1}$$

The stabilizing function f is chosen to retain the desired feature of $\hat{\boldsymbol{x}}_*$, namely the sparsity level, in other words, to recover the compact support[4] of $\hat{\boldsymbol{x}}_*$. The ℓ_0 functional, which counts the number of non-zero entries of a vector, is a natural candidate for the stabilizing function f. The constrained ℓ_0 method finds to an approximate solution $\boldsymbol{x}_0$ from within a convex feasibility set $\mathcal{B} = \{\boldsymbol{x} \in \mathbb{R}^N \Big| A\boldsymbol{x} = \hat{\boldsymbol{y}}_*\}$ such that the following holds,

$$\boldsymbol{x}_0 = \underset{\boldsymbol{x}\in\mathcal{B}}{\arg\min} \Big\{ ||\boldsymbol{x}||_0 \Big\}. \tag{P_0}$$

Remark 2.2.1. *We set the stabilizing function $f(\boldsymbol{x}) = ||\boldsymbol{x}||_0$ (for other constrained ℓ_p methods, we set $f(\boldsymbol{x}) = ||\boldsymbol{x}||_p$ for $p > 0$) and set the convex feasible set as $\mathcal{C} = \mathcal{B}$ in* (2.1). *When we let the regularization parameter λ in* (2.1) *approach infinity, the general Tikhonov regularization method becomes the constrained ℓ_0 method (or the constrained ℓ_p method respectively).*

The recoverability condition for the constrained ℓ_0 method to recover the k-sparse $\hat{\boldsymbol{x}}_*$ from $\hat{\boldsymbol{y}}_*$ is that $\mathrm{sp}(A) > 2k$ (see [13] for the definition of the spark of a matrix, i.e., $\mathrm{sp}(A)$.). It is relatively easy to construct an observation matrix A with $M = 2k$ and $\mathrm{sp}(A) > M$, e.g., the Vandermonde matrix. However, for large k, the constrained ℓ_0 method is considered NP-hard [43]. A straightforward alternative would be to consider the following constrained ℓ_p problem,

$$\boldsymbol{x}_p = \underset{\boldsymbol{x}\in\mathcal{B}}{\arg\min} \Big\{ ||\boldsymbol{x}||_p \Big\}, \quad \text{for } 0 < p < 1. \tag{2.2}$$

The ℓ_p functional $||\cdot||_p$ (for $0 < p < 1$) becomes a quasi-norm and sparsity inducing, however solving (2.2) is nevertheless NP-hard [25]. There are three major alterna-

[4]The support of a vector $\boldsymbol{x} \in \mathbb{R}^N$ is a set of indices corresponding to the non-zero entries of $\boldsymbol{x}$.

tives to remedy the numerical inefficiency of the constraiend ℓ_0 method: greedy methods, thresholding methods, and convex optimization methods. The greedy approach [8, 10, 11, 24, 32, 39, 45–47, 51, 61, 65] estimates the support of $\hat{\boldsymbol{x}}_*$ by adding to a possible support set one suitable index at a time. The thresholding-based methods [5, 22] tries to estimate the whole support of $\hat{\boldsymbol{x}}_*$ by finding a set of k possible indices via special selections. Both approaches are discrete and sensitive to small perturbation in $\hat{\boldsymbol{y}}_*$. Especially the greedy approach, slight change in one entry of $\hat{\boldsymbol{y}}_*$ might result in choosing a totally different index. On the contrary, the convex optimization method offers a continuous approach for seeking the support of $\hat{\boldsymbol{x}}_*$ and robust against small perturbation in $\hat{\boldsymbol{y}}_*$. The question of choosing a suitable convex objective function remains unanswered. We begin our search by probing into the relationship between ℓ_0 functional and the ℓ_p functional in the following limit,

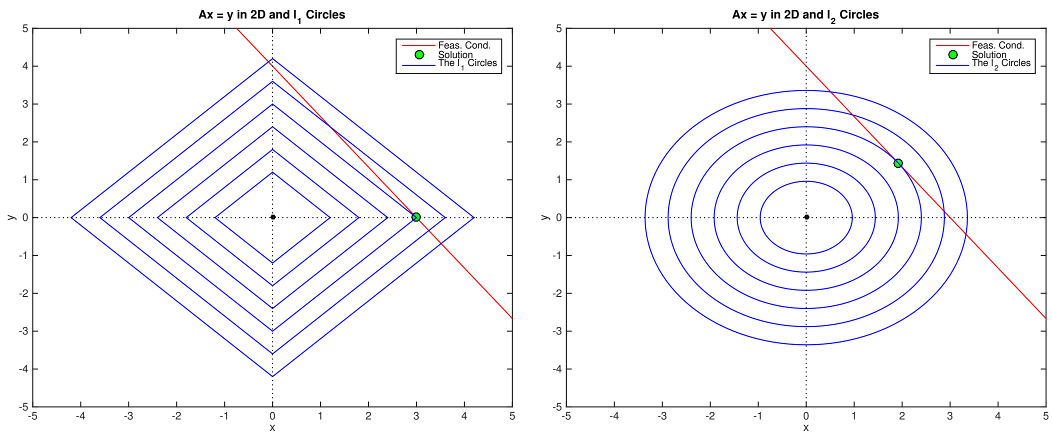
$$||\boldsymbol{x}||_0 = \lim_{p\to 0} ||\boldsymbol{x}||_p = \lim_{p\to 0} \sqrt[p]{\sum_{i=1}^{N} \left|(\boldsymbol{x})_i\right|^p}.$$

Only the non-zero terms survive the limit, therefore the ℓ_0 functional counts the number of non-zero entries in $\boldsymbol{x}$. The ℓ_p functional becomes a norm when $p \geq 1$ and therefore convex. However, when $p > 1$, the ℓ_p norm is strictly convex[5]. We choose the ℓ_2 norm as the representative case, since the vector space $\mathbb{R}^N$ with ℓ_2 norm becomes a Hilbert space. The constrained ℓ_2 method finds an approximate solution $\boldsymbol{x}_2$ from $\mathcal{B}$ such that $\boldsymbol{x}_2$ satisfies the following

$$\boldsymbol{x}_2 = \underset{\boldsymbol{x}\in\mathcal{B}}{\arg\min} \left\{||\boldsymbol{x}||_2\right\}. \tag{P_2}$$

[5]A function $g : \mathbb{R}^N \to \mathbb{R}$ is strictly convex if for any $\boldsymbol{x}_1$ and $\boldsymbol{x}_2 \in \text{dom}(g)$ ($\boldsymbol{x}_1 \neq \boldsymbol{x}_2$) and any $t \in (0, 1)$, when $t\boldsymbol{x}_1 + (1-t)\boldsymbol{x}_2 \in \text{dom}(g)$, then $g(t\boldsymbol{x}_1 + (1-t)\boldsymbol{x}_2) < tg(\boldsymbol{x}_1) + (1-t)g(\boldsymbol{x}_2)$.

The approximate solution $\boldsymbol{x}_2$ from (P_2), despite of having a closed form expression (see section A.1 for details) is rarely sparse. In figure 2.1, we compare the solutions from (P_1) and (P_2) in $\mathbb{R}^2$.



(a) $\boldsymbol{x}_1$ (b) $\boldsymbol{x}_2$

Figure 2.1: $\boldsymbol{x}_1$ versus $\boldsymbol{x}_2$

The strict convexity from the ℓ_2 norm is one of the reasons why the approximate solution $\boldsymbol{x}_2$ is rarely sparse. We need a convex (but not strictly convex) ℓ_p functional to induce sparsity on its solutions. Therefore, the ℓ_1 norm becomes a logical choice to make. Moreover, thanks to the well established history of using the ℓ_1 minimization in geophysical and statistical applications, there exist a great number of polished and well-tested algorithms to solve ℓ_1 related minimization problems. In fact, Candès, et al., and separately by Donoho, suggested using the the constrained ℓ_1 method to recover k-sparse $\hat{\boldsymbol{x}}_*$ from $\hat{\boldsymbol{y}}_*$. The constrained ℓ_1 method finds an approximate solution $\boldsymbol{x}_1 \in \mathcal{B}$ such that the approximate solution satisfies the following,

$$\boldsymbol{x}_1 = \arg\min_{\boldsymbol{x} \in \mathcal{B}} \Big\{ ||\boldsymbol{x}||_1 \Big\}. \tag{P_1}$$

The solution from the constrained ℓ_1 method, namely $\boldsymbol{x}_1$, will be the same as $\hat{\boldsymbol{x}}_*$ when the observation matrix A satisfies one of the following recoverability conditions: Restricted Isometry Propertiy (RIP) in [7], ℓ_1-Coherence in [13, 14, 29, 61], or Null space Property (NSP) in [9]. We present a recoverability condition, known as the Robust Null Space Property (RNSP) in [23]. Such recoveryability condition will be used to show the approximation errors between the approximate solutions (provided by three different methods which will be discussed in later sections) and $\hat{\boldsymbol{x}}_*$. Before we present the definition of RNSP, we give the following clarification: the restriction of a vector $\boldsymbol{x} \in \mathbb{R}^N$ on an index set $K \subset [N] = \{1, 2, \ldots, N\}$ is denoted as $(\boldsymbol{x})_K$, and it is defined as follows,

$$((\boldsymbol{x})_K)_i = \begin{cases} (\boldsymbol{x})_i, & i \in K \\ 0, & i \notin K \end{cases}, \quad \text{for } i \in [N].$$

Definition 2.2.2 (Definition 4.17 in [23])**.** *A matrix $A \in \mathbb{R}^{M \times N}$ is said to satisfy the RNSP (with respect to $||\cdot||_2$) with constants $0 < \rho < 1$ and $\tau > 0$ relative to a set $K \subset [N]$ if*

$$||\boldsymbol{v}_K||_1 \leq \rho ||(\boldsymbol{v})_{K^c}||_1 + \tau ||A\boldsymbol{v}||_2, \quad \forall \boldsymbol{v} \in \mathbb{R}^N.$$

The matrix A is said to satisfy the RNSP of order k (with respect to $||\cdot||_2$) with constants $0 < \rho < 1$ and $\tau > 0$ if it satisfies the RNSP with the same constants $0 < \rho < 1$ and $\tau > 0$ relative to any set $K \subset [N]$ with card$(K) \leq k$.

Remark 2.2.3. *The RNSP of order k implies NSP of order k. Let $\boldsymbol{v} \in$ Null(A) (the null space of A), when A satisfies RNSP of order k with constants $\tau > 0$ and $0 < \rho < 1$, we have for any set $K \subset [N]$ with card$(K) \leq k$, the following holds*

$$||(\boldsymbol{v})_K||_1 \leq \rho ||(\boldsymbol{v})_{K^c}||_1 + \tau ||A\boldsymbol{v}||_2 = \rho ||(\boldsymbol{v})_{K^c}||_1 < ||(\boldsymbol{v})_{K^c}||_1.$$

When a matrix A satisfies RIP of order $2k$, the matrix A also satisfies RNSP of order k (see theorem 6.13 in [23]).

The following theorem provides an equivalent description of definition 2.2.2, such description will be used throughout this chapter instead of the original definition of RNSP.

Theorem 2.2.4 (Theorem 4.20 in [23])**.** *A matrix $A \in \mathbb{R}^{M\times N}$ satisfies the RNSP with constants $0<\rho<1$ and $\tau>0$ relative to $K \subset [N]$ if and only if for any $\boldsymbol{u}$, $\boldsymbol{v} \in \mathbb{R}^N$, the following holds*

$$||\boldsymbol{u}-\boldsymbol{v}||_1 \leq \frac{1+\rho}{1-\rho}(||\boldsymbol{u}||_1 - ||\boldsymbol{v}||_1 + 2||(\boldsymbol{v})_{K^c}||_1) + \frac{2\tau}{1-\rho}||A(\boldsymbol{u}-\boldsymbol{v})||_2.$$

Remark 2.2.5. *The inequality in theorem 2.2.4 is asymmetric. When we interchange $\boldsymbol{u}$ and $\boldsymbol{v}$, we obtain*

$$||\boldsymbol{v}-\boldsymbol{u}||_1 \leq \frac{1+\rho}{1-\rho}(||\boldsymbol{v}||_1 - ||\boldsymbol{u}||_1 + 2||(\boldsymbol{u})_{K^c}||_1) + \frac{2\tau}{1-\rho}||A(\boldsymbol{v}-\boldsymbol{u})||_2.$$

Adding the two inequalities together, we derive an symmetric inequality

$$||\boldsymbol{u}-\boldsymbol{v}||_1 \leq \frac{1+\rho}{1-\rho}(||(\boldsymbol{u})_{K^c}||_1 + ||(\boldsymbol{v})_{K^c}||_1) + \frac{2\tau}{1-\rho}||A(\boldsymbol{v}-\boldsymbol{u})||_2. \tag{2.3}$$

The new inequality (2.3) *sheds some light into another explanation of the RNSP property: when both $\boldsymbol{u}$ and $\boldsymbol{v}$ are supported on the set K and they give the same observation, i.e., $A\boldsymbol{u} = A\boldsymbol{v}$, then $\boldsymbol{u} = \boldsymbol{v}$.*

2.3 The Nearby Problem: Recovery of General $\boldsymbol{x}_*$ from Noisy Observation $\boldsymbol{y}_*^\varepsilon$

Beside recovery of sparse unknowns, we also invest special interests in recovery of general unknowns, since sparsity is rather a rare property to encounter. Let $\boldsymbol{x}_*$ be any general unknown, we plan to investigate the recovery of $\boldsymbol{x}_*$ from its noisy observation $\boldsymbol{y}_*^\varepsilon = A\boldsymbol{x}_* + \boldsymbol{\epsilon}$ instead of its observation $\boldsymbol{y}_*$, since the observation process is inevitably corrupted by errors: human errors, machine precision, etc. The error term $\boldsymbol{\epsilon}$ is caused by a number of factors and its details remains untraceable. Fortunately, the error term $\boldsymbol{\epsilon}$ is bounded above in ℓ_2 norm, i.e., $||\boldsymbol{\epsilon}||_2 \leq \varepsilon$ for some known noise level $\varepsilon > 0$. Before we introduce the robust methods to recover $\boldsymbol{x}_*$, we introduce the following remark on measuring the distance from $\boldsymbol{x}_*$ to a set of k-sparse vectors.

Remark 2.3.1. *Let $\mathcal{S}_k$ be the set of all k-sparse vectors in $\mathbb{R}^N$: $\mathcal{S}_k = \{\boldsymbol{x} \in \mathbb{R}^N \Big| ||\boldsymbol{x}||_0 \leq k\}$. We define the function $\sigma(\boldsymbol{x}_*, k, 1)$ as the best k-term approximation error of $\boldsymbol{x}_*$ in ℓ_1 norm, i.e.,*

$$\sigma(\boldsymbol{x}_*, k, 1) = \inf_{\boldsymbol{x} \in \mathcal{S}_k} \{||\boldsymbol{x}_* - \boldsymbol{x}||_1\}.$$

Thanks to the usage of the ℓ_1 norm (or any other ℓ_p norm see [9]), such distance be realized by a k-sparse vector in $\mathcal{S}_k$, which is obtained by keeping the k largest entries (in magnitude) of $\boldsymbol{x}_*$; we denote this vector as $\boldsymbol{x}_*(k)$ (Note that such vector might not be unique), and name it the best k-term approximation of $\boldsymbol{x}_*$ in ℓ_1 norm. It follows that, $\sigma(\boldsymbol{x}_*, k, 1) = ||\boldsymbol{x}_* - \boldsymbol{x}_*(k)||_1$. Unless otherwise redefined, we use $\sigma = \sigma(\boldsymbol{x}_*, k, 1)$ to simplify the notation.

The question of an effective and efficient procedure to recover $\boldsymbol{x}_*$ from $\boldsymbol{y}_*^\varepsilon$ remains unanswered. One can tweak the constrained ℓ_1 method by replacing the original feasible set $\mathcal{B}$ with a new feasible set $\mathcal{B}_\eta = \{\boldsymbol{x} \in \mathbb{R}^N \Big| ||\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}||_2 \leq \eta\}$ with an extra parameter $\eta > 0$ to obtain the following quadratically constrained ℓ_1 method,

$$\boldsymbol{x}_{1+\eta} = \underset{\boldsymbol{x} \in \mathcal{B}_\eta}{\arg\min} \Big\{||\boldsymbol{x}||_1\Big\}. \tag{2.4}$$

Remark 2.3.2. *We set the stabilizing function* $f(\boldsymbol{x}) = ||\boldsymbol{x}||_1$ *and set the convex feasible set as* $\mathcal{C} = \mathcal{B}_\eta$ *in* (2.1)*. When we let the regularization parameter* λ *in* (2.1) *approach infinity, the general Tikhonov regularization method becomes the quadratically constrained* ℓ_1 *method.*

We present the following theorem from [23] on the recoverability condition for the quadratically constrained ℓ_1 method to recover $\boldsymbol{x}_*$ from $\boldsymbol{y}_*^\varepsilon$.

Theorem 2.3.3 (Theorem 4.19 in [23])**.** *Assume that a matrix* $A \in \mathbb{R}^{M\times N}$ *satisfies the RNSP of order* k *with constants* $0 < \rho < 1$ *and* $\tau > 0$*. For any general unknown* $\boldsymbol{x}_* \in \mathbb{R}^N$*, a solution,* $\boldsymbol{x}_{1+\eta}$ *of* (2.4) *where the input* $\boldsymbol{y}_*^\varepsilon = A\boldsymbol{x}_* + \boldsymbol{\epsilon}$ *(*$||\boldsymbol{\epsilon}||_2 \leq \varepsilon$*) and the parameter* $\eta \geq \varepsilon$ *are used, approximates* $\boldsymbol{x}_*$ *with the following* ℓ_1 *error*

$$||\boldsymbol{x}_{1+\eta} - \boldsymbol{x}_*||_1 \leq \frac{2(1+\rho)}{1-\rho}\sigma + \frac{2\tau}{1-\rho}(\eta+\varepsilon).$$

If $\eta = \varepsilon$*, then* $\boldsymbol{x}_{1+\eta} = \boldsymbol{x}_{1+\varepsilon}$ *and*

$$||\boldsymbol{x}_{1+\varepsilon} - \boldsymbol{x}_*||_1 \leq \frac{2(1+\rho)}{1-\rho}\sigma + \frac{4\tau}{1-\rho}\varepsilon.$$

Recall that $\sigma = \sigma(\boldsymbol{x}_*, k, 1)$*.*

Proof. By the optimality of $\boldsymbol{x}_{1+\eta}$, we have $||\boldsymbol{x}_{1+\eta}||_1 \leq ||\boldsymbol{x}_*||_1$. Set $\boldsymbol{u} = \boldsymbol{x}_{1+\eta}$, $\boldsymbol{v} = \boldsymbol{x}_*$, $K = \text{supp}(\boldsymbol{x}_*(k))$ (recall that $\boldsymbol{x}_*^\sigma(k)$ is the best k-term approximation of $\boldsymbol{x}_*^\sigma$ in ℓ_1 norm). It follows from theorem 2.2.4,

$$\begin{aligned}
||\boldsymbol{x}_{1+\eta} - \boldsymbol{x}_*||_1 &\leq \frac{1+\rho}{1-\rho}\Big(||\boldsymbol{x}_{1+\eta}||_1 - ||\boldsymbol{x}_*||_1 + 2||(\boldsymbol{x}_*)_{K^c}||_1\Big) + \frac{2\tau}{1-\rho}||A(\boldsymbol{x}_{1+\eta} - \boldsymbol{x}_*)||_2 \\
&\leq \frac{2(1+\rho)}{1-\rho}\sigma + \frac{2\tau}{1-\rho}(||A(\boldsymbol{x}_{1+\eta} - \boldsymbol{y}_*^\varepsilon||_2 + ||\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}_*||_2) \\
&\leq \frac{2(1+\rho)}{1-\rho}\sigma + \frac{2\tau}{1-\rho}(\eta+\varepsilon).
\end{aligned}$$

Here, we used the equality: $(\boldsymbol{x}_*)_{K^c} = \boldsymbol{x}_* - \boldsymbol{x}_*(k)$ and $\sigma = ||(\boldsymbol{x}_*)_{K^c}||_1$. $\square$

Remark 2.3.4. *When we apply* (2.4) *to the recovery a k-sparse unknown $\hat{\boldsymbol{x}}_*$ from its clean observation $\hat{\boldsymbol{y}}_*$, we should be able to recover exactly $\hat{\boldsymbol{x}}_*$ when we set the parameter $\eta = 0$. In fact, given the error bound in theorem 2.3.3, we have $||\boldsymbol{x}_{1+0} - \hat{\boldsymbol{x}}_*||_1 \leq 0 \Rightarrow \boldsymbol{x}_{1+0} = \hat{\boldsymbol{x}}_*$.*

We also have interests in another method to recover a general unknown $\boldsymbol{x}_*$ from its noisy observation $\boldsymbol{y}_*^\varepsilon$, so-called the unconstrained ℓ_1 method, which finds an approximate solution $\boldsymbol{x}_\lambda$ with a regularization parameter $\lambda > 0$. The approximate solution $\boldsymbol{x}_\lambda$ satisfies the following,

$$\boldsymbol{x}_\lambda = \underset{\boldsymbol{x} \in \mathbb{R}^N}{\arg\min} \left\{ \lambda ||\boldsymbol{x}||_1 + \frac{1}{2} ||\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}||_2^2 \right\}. \tag{P_U}$$

Remark 2.3.5. *We set the stabilizing function $f(\boldsymbol{x}) = ||\boldsymbol{x}||_1$, set the convex feasible set as $\mathcal{C} = \mathbb{R}^N$, and use $\boldsymbol{y}_*^\varepsilon$ instead of $\hat{\boldsymbol{y}}_*$ in* (2.1). *The general Tikhonov regularization method becomes the unconstrained ℓ_1 method.*

The following theorem discusses the approximation error when the unconstrained ℓ_1 method is used to recover $\boldsymbol{x}_*$ from $\boldsymbol{y}_*^\varepsilon$.

Theorem 2.3.6. *Assume that a matrix $A \in \mathbb{R}^{M \times N}$ satisfies the RNSP of order k with constants $0 < \rho < 1$ and $\tau > 0$. For any general unknown $\boldsymbol{x}_* \in \mathbb{R}^N$, a solution, $\boldsymbol{x}_\lambda$ of (P_U) where the input $\boldsymbol{y}_*^\varepsilon = A\boldsymbol{x}_* + \boldsymbol{\epsilon}$ with $||\boldsymbol{\epsilon}||_2 \leq \varepsilon$ is used, approximates $\boldsymbol{x}_*$ with the following ℓ_1 error*

$$||\boldsymbol{x}_\lambda - \boldsymbol{x}_*||_1 \leq \frac{1+\rho}{1-\rho}\Big(\frac{\varepsilon^2}{2\lambda} - \frac{\lambda}{2\beta^2} + 2\sigma\Big) + \frac{2\tau}{1-\rho}(\lambda\alpha + \varepsilon).$$

Recall that $\sigma = \sigma(\boldsymbol{x}_*, k, 1)$. We also define the constants β and α as $\beta := ||A^\top||_2$ and $\alpha := \sqrt{N}||(AA^\top)^{-1}A||_2$.

Proof. First, we use the optimality condition of $\boldsymbol{x}_\lambda$,

$$\lambda||\boldsymbol{x}_\lambda||_1 + \frac{1}{2}||\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}_\lambda||_2^2 \leq \lambda||\boldsymbol{x}_*||_1 + \frac{1}{2}||\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}_*||_2^2.$$

It follows from lemma A.3.6,

$$||\boldsymbol{x}_\lambda||_1 - ||\boldsymbol{x}_*||_1 \leq \frac{1}{2\lambda}||\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}_*||_2^2 - \frac{1}{2\lambda}||\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}_\lambda||_2^2 \leq \frac{\varepsilon^2}{2\lambda} - \frac{\lambda}{2\beta^2}.$$

Second, let $\boldsymbol{u} = \boldsymbol{x}_\lambda$, $\boldsymbol{v} = \boldsymbol{x}_*$, and set $K = \text{supp}(\boldsymbol{x}_*(k))$ (recall that $\boldsymbol{x}_*(k)$ is the best k-term approximation of $\boldsymbol{x}_*$ in ℓ_1 norm). By theorem 2.2.4, we have

$$\begin{aligned}
||\boldsymbol{x}_\lambda - \boldsymbol{x}_*||_1 &\leq \frac{1+\rho}{1-\rho}(||\boldsymbol{x}_\lambda||_1 - ||\boldsymbol{x}_*||_1 + 2||(\boldsymbol{x}_*)_{K^c}||_1) + \frac{2\tau}{1-\rho}||A(\boldsymbol{x}_\lambda - \boldsymbol{x}_*)||_2 \\
&\leq \frac{1+\rho}{1-\rho}\Big(\frac{\varepsilon^2}{2\lambda} - \frac{\lambda}{2\beta^2} + 2\sigma\Big) + \frac{2\tau}{1-\rho}(||A\boldsymbol{x}_\lambda - \boldsymbol{y}_*^\varepsilon||_2 + ||\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}_*||_2) \\
&\leq \frac{1+\rho}{1-\rho}\Big(\frac{\varepsilon^2}{2\lambda} - \frac{\lambda}{2\beta^2} + 2\sigma\Big) + \frac{2\tau}{1-\rho}(\alpha\lambda + \varepsilon).
\end{aligned}$$

□

Remark 2.3.7. *When we apply the unconstrained ℓ_1 method to the recovery of k-sparse unknown $\hat{\boldsymbol{x}}_*$ from its clean observation $\hat{\boldsymbol{y}}_*$, we obtain the following error bound,*

$$||\boldsymbol{x}_\lambda - \hat{\boldsymbol{x}}_*||_1 \leq \frac{4\tau\alpha\beta^2 - (1+\rho)}{(1-\rho)\beta^2}\lambda.$$

In other words, the approximate solution $\boldsymbol{x}_\lambda$ of (P_U) is within the $\mathcal{O}(\lambda)$ neighborhood of $\hat{\boldsymbol{x}}_*$. When we use the unconstrained ℓ_1 method to recover a general $\boldsymbol{x}_*$ from its noisy observation $\boldsymbol{y}_*^\varepsilon$, we have to be careful at choosing a proper λ. In order to provide the optimal error bound, we have to minimize the following term

$$\frac{1+\rho}{1-\rho}\Big(\frac{\varepsilon^2}{2\lambda} - \frac{\lambda}{2\beta^2}\Big) + \frac{2\tau\alpha\lambda}{1-\rho}, \quad \forall\lambda > 0.$$

We have the optimal λ_{opt} calculated from the following expression,

$$\lambda_{opt} = \varepsilon\beta\sqrt{\frac{1+\rho}{4\beta^2\alpha\tau - (1+\rho)}}. \tag{2.5}$$

Such optimal λ_{opt} is obtainable, when $4\tau\mu_{\max}^2\sqrt{N} > (1+\rho)\mu_{\min}$, where $\mu_{\max}$ and $\mu_{\min}$ are the maximum and minimum singular values of the matrix A respectively.

The following remark is concerned with the performance between the quadratically constrained ℓ_1 method and the unconstrained ℓ_1 method on recovery of $\boldsymbol{x}_*$ from $\boldsymbol{y}_*^\varepsilon$.

Remark 2.3.8. *We consider the approximate solution $\boldsymbol{x}_\lambda$ obtained from solving (P_U) with input $\boldsymbol{y}_*^\varepsilon$ and the regularization parameter λ is set at the optimal λ_{opt} in* (2.5)*, and the approximate solution $\boldsymbol{x}_{1+\eta}$ obtained from solving* (2.4) *with input $\boldsymbol{y}_*^\varepsilon$ and the parameter η being set at $\eta = \varepsilon$. When the following inequality is satisfied, i.e.,*

$$4\tau^2\beta^2 + (1+\rho)^2 > 4(1+\rho)\tau\alpha\beta^2,$$

the difference, $\boldsymbol{x}_\lambda - \boldsymbol{x}_*$, would have a smaller upper bound than $\boldsymbol{x}_{1+\eta} - \boldsymbol{x}_*$.

We refer the readers to Proposition 3.2 in [23] for connection between the quadratically constrained ℓ_1 method and the unconstrained ℓ_1 method.

2.3.1 The HR Method for Recovery of General $\boldsymbol{x}_*$ from Noisy Observation $\boldsymbol{y}_*^\varepsilon$

We have shown that when using the unconstrained ℓ_1 method to recover a k-sparse $\hat{\boldsymbol{x}}_*$ from its clean observation $\hat{\boldsymbol{y}}_*$ with an observation matrix A satisfying RNSP of order k, the approximate solution $\boldsymbol{x}_\lambda$ in (P_U) is within an $\mathcal{O}(\lambda)$-neighborhood of $\hat{\boldsymbol{x}}_*$. The regularization *parameter* λ controls the distance between $\boldsymbol{x}_\lambda$ and $\hat{\boldsymbol{x}}_*$. When $\lambda \downarrow 0$, $\boldsymbol{x}_\lambda \to \hat{\boldsymbol{x}}_*$.

It is advantageous to use an extremely small λ in order to obtain a highly accurate approximate solution; however, smaller λ usually leads to larger number of iterations taken by a numerical solver. Recall in section 1.2, we proposed the HR method as a multi-scale generalization to the general Tikhonov regularization in order to improve the regularization procedure. Along the same line of reasoning in section 1.2, we propose the HR method as a multi-scale generalization to the unconstrained ℓ_1 method in order to decrease the total number of iterations, via a ladder of gradually decreasing regularization *parameters* λ_j's, which are understood as hierarchical *scales* in the HR setting. In the setting of recovery of a general unknown $\boldsymbol{x}_*$ from its noisy observation $\boldsymbol{y}_*^\varepsilon$, the HR method is used for an under-determined matrix A. We begin with a general $\lambda > 0$ for the unconstrained ℓ_1 method,

$$\boldsymbol{x}_\lambda = \underset{\boldsymbol{x}\in\mathbb{R}^N}{\arg\min} \left\{ \lambda||\boldsymbol{x}||_1 + \frac{1}{2}||\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}||_2^2 \right\},$$

since there is information left in the residual, $\boldsymbol{r}_\lambda = \boldsymbol{y}_*^\varepsilon - \boldsymbol{x}_\lambda$, i.e., $\boldsymbol{r}_\lambda \neq \mathbf{0}$ (or we are done), we can extract further information from $\boldsymbol{r}_\lambda$ at a *finer* scale, say $\frac{\lambda}{2}$,

$$\boldsymbol{x}_{\frac{\lambda}{2}} := \underset{\boldsymbol{x}\in\mathbb{R}^N}{\arg\min} \left\{ \frac{\lambda}{2}||\boldsymbol{x}||_1 + \frac{1}{2}||\boldsymbol{r}_\lambda - A\boldsymbol{x}||_2^2 \right\} \quad \text{with} \quad \boldsymbol{r}_{\frac{\lambda}{2}} := \boldsymbol{r}_\lambda - A\boldsymbol{x}_{\frac{\lambda}{2}}.$$

We obtain with a two scale decomposition of $\boldsymbol{y}_*^\varepsilon$, i.e., $\boldsymbol{y}_*^\varepsilon = A\boldsymbol{x}_\lambda + A\boldsymbol{x}_{\frac{\lambda}{2}} + \boldsymbol{r}_{\frac{\lambda}{2}}$. The previous extraction process can continue. To simplify the notations, we will use numbered subscripts from now on. We start from setting $\boldsymbol{x}_{(1)} = \boldsymbol{x}_\lambda$, $\boldsymbol{r}_0 = \boldsymbol{y}_*^\varepsilon$, and choose a ladder of hierarchical scales designed as $\lambda_j = 2^{1-j}\lambda$. The HR method will solve the following,

$$\boldsymbol{x}_{(j)} = \underset{\boldsymbol{x}\in\mathbb{R}^N}{\arg\min} \left\{ \lambda_j||\boldsymbol{x}||_1 + \frac{1}{2}||\boldsymbol{r}_{j-1} - A\boldsymbol{x}||_2^2 \right\}, \quad \text{for } 1 \leq j \leq J.$$

The hierarchical residual $\boldsymbol{r}_j$ satisfies a recursive relation: $\boldsymbol{r}_j := \boldsymbol{r}_{j-1} - A\boldsymbol{x}_{(j)}$. We sum up the hierarchical terms $\boldsymbol{x}_{(j)}$, and obtain an approximate solution in the form of a hierarchical sum, i.e., $\boldsymbol{X}_J = \sum_{j=1}^{J} \boldsymbol{x}_{(j)}$. The hierarchical observation $A\boldsymbol{X}_J$ will provide a multi-scale approximate description of $\boldsymbol{y}_*^\varepsilon$ as follows,

$$\boldsymbol{y}_*^\varepsilon = A\boldsymbol{x}_{(1)} + A\boldsymbol{x}_{(2)} + \ldots + A\boldsymbol{x}_{(J)} + \boldsymbol{r}_J.$$

When the observation matrix satisfies certain recoverability condition, the hierarchical sum $\boldsymbol{X}_J$, as the sum of the hierarchical terms $\boldsymbol{x}_{(j)}$, i.e., $\boldsymbol{X}_J = \sum_{j=1}^{J} \boldsymbol{x}_{(j)}$, would provide a multi-scale approximation of $\boldsymbol{x}_*$, as follows

$$\boldsymbol{X}_J = \boldsymbol{x}_{(1)} + \boldsymbol{x}_{(2)} + \ldots + \boldsymbol{x}_{(J)} \approx \boldsymbol{x}_*.$$

We note that different ladders of the hierarchical scales can be employed, e.g., $\lambda_j = \theta^{j-1}\lambda_1$ for $0 < \theta < 1$ and a carefully chosen λ_1. Such *finitely many* scales are important in optimizing the approximation error, $\boldsymbol{X}_J - \boldsymbol{x}_*$. We summarize the HR method with a general ladder of scales $\{\lambda_j\}_{j=1}^{J}$ designed as $\lambda_j = \theta^{j-1}\lambda_1$. Given the initial hierarchical residual $\boldsymbol{r}_0 = \boldsymbol{y}_*^\varepsilon$, the HR method chooses a suitable starting hierarchical scale $\lambda_1 > 0$ and finds the first hierarchical term $\boldsymbol{x}_{(1)}$ from the following

$$\boldsymbol{x}_{(1)} = \operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{R}^N} \left\{\lambda_1||\boldsymbol{x}||_1 + \frac{1}{2}||\boldsymbol{r}_0 - A\boldsymbol{x}||_2^2\right\} \quad \text{and} \quad \boldsymbol{r}_1 := \boldsymbol{r}_0 - A\boldsymbol{x}_{(1)}.$$

Note that $\boldsymbol{x}_{(1)} \neq \boldsymbol{x}_1$. Next, the HR method finds other hierarchical terms $\boldsymbol{x}_{(j)}$ for $2 \leq j \leq J$ with $\lambda_j = \theta^{j-1}\lambda_1$ from the following recursive relationship,

$$\boldsymbol{x}_{(j)} = \operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{R}^N} \left\{\lambda_j||\boldsymbol{x}||_1 + \frac{1}{2}||\boldsymbol{r}_{j-1} - A\boldsymbol{x}||_2^2\right\} \quad \text{and} \quad \boldsymbol{r}_j := \boldsymbol{r}_{j-1} - A\boldsymbol{x}_{(j)}. \qquad (P_{HR})$$

We begin the approximation error analysis of the hierarchical sum $\boldsymbol{X}_J$ with the following lemma on the bounds of the hierarchical residual $\boldsymbol{r}_j$'s.

Lemma 2.3.9. *Assume that the matrix A has linearly independent rows. The hierarchical residuals, $\boldsymbol{r}_j = \boldsymbol{r}_{j-1} - A\boldsymbol{x}_{(j)}$, satisfy the following bounds,*

$$\lambda_j ||A^\top||_p^{-1} \le ||\boldsymbol{r}_j||_p \le \lambda_j \sqrt[p]{N} ||(AA^\top)^{-1}A||_p, \quad \textit{for } 1 \le p \le \infty, 1 \le j \le J.$$

Hence $\boldsymbol{r}_{(j)} \to \mathbf{0}$ as $\lambda_j \to 0$.

Proof. Since the hierarchical residual $\boldsymbol{r}_j$ and the hierarchical term $\boldsymbol{x}_{(j)}$ satisfy the signum equation: $A^\top \boldsymbol{r}_j = \lambda_j \mathbf{sgn}(\boldsymbol{x}_{(j)})$. Following the proof presented in lemma A.3.6, we have

$$\lambda_j ||A^\top||_p^{-1} \le ||\boldsymbol{r}_j||_p \le \lambda_j \sqrt[p]{N} ||(AA^\top)^{-1}A||_p.$$

Here we used the fact that the matrix A has linearly independent rows, hence $A^\top A$ is non-singular. Thus, $||\boldsymbol{r}_j||_p \to 0$ as $\lambda_j \to 0$. □

We present the recoverability condition in the following theorem for the HR method to provide reasonable approximate solution to $\boldsymbol{x}_*^\sigma$.

Theorem 2.3.10. *Assume that the matrix $A \in \mathbb{R}^{M \times N}$ satisfies the RNSP of order k with constants $0 < \rho < 1$ and $\tau > 0$. For any unknown $\boldsymbol{x}_* \in \mathbb{R}^N$, a solution, $\boldsymbol{X}_J$ as sum of the hierarchical terms in (P_{HR}) where the input $\boldsymbol{r}_0 = \boldsymbol{y}_*^\varepsilon = A\boldsymbol{x}_* + \boldsymbol{\epsilon}$ ($||\boldsymbol{\epsilon}||_2 \le \varepsilon$) is used, approximates $\boldsymbol{x}_*$ with the following ℓ_1 error,*

$$||\boldsymbol{X}_J - \boldsymbol{x}_*||_1 \le \frac{1+\rho}{1-\rho}\Big(\frac{\varepsilon^2}{2\lambda_1} - \frac{\lambda_1}{2\beta^2} + \frac{\lambda_1}{2}(\alpha\theta^{-2} - \beta^{-2})\frac{\theta - \theta^J}{1-\theta} + 2\sigma\Big) + \frac{2\tau}{1-\rho}\Big(\lambda_J \alpha + \varepsilon\Big).$$

Recall that $\sigma = \sigma(\boldsymbol{x}_*, k, 1)$ and the constants $\beta := ||A^\top||_2$ and $\alpha := \sqrt{N}||(A^\top A)^{-1}A||_2$.

Proof. We begin with finding the upper bound on $||\boldsymbol{X}_J||_1 - ||\boldsymbol{x}_*||_1$. We start from the first hierarchical term $\boldsymbol{x}_{(1)}$. By the optimality of $\boldsymbol{x}_{(1)}$, we have

$$\lambda_1||\boldsymbol{x}_{(1)}||_1 + \frac{1}{2}||\boldsymbol{r}_0 - A\boldsymbol{x}_{(1)}||_2^2 \leq \lambda_1||\boldsymbol{x}_*||_1 + \frac{1}{2}||\boldsymbol{r}_0 - A\boldsymbol{x}_*||_2^2.$$

It follows that,

$$||\boldsymbol{x}_{(1)}||_1 - ||\boldsymbol{x}_*||_1 \leq \frac{1}{2\lambda_1}(||\boldsymbol{r}_0 - A\boldsymbol{x}_*||_2^2 - ||\boldsymbol{r}_0 - A\boldsymbol{x}_{(1)}||_2^2) \leq \frac{\varepsilon^2}{2\lambda_1} - \frac{\lambda_1}{2\beta^2}.$$

Regarding the bound on other hierarchical terms $\boldsymbol{x}_{(j)}$ for $j \geq 2$, we follow the energy estimate proof in [54]. By the optimality of $\boldsymbol{x}_{(j)}$, we have

$$\lambda_j||\boldsymbol{x}_{(j)}||_1 + \frac{1}{2}||\boldsymbol{r}_{j-1} - A\boldsymbol{x}_{(j)}||_2^2 \leq \frac{1}{2}||\boldsymbol{r}_{j-1}||_2^2,$$

then we derive,

$$||\boldsymbol{x}_{(j)}||_1 \leq \frac{1}{2\lambda_j}(||\boldsymbol{r}_{j-1}||_2^2 - ||\boldsymbol{r}_j||_2^2) \leq \frac{1}{2\lambda_j}(\alpha^2\lambda_{j-1}^2 - \beta^{-2}\lambda_j^2) \leq \frac{\lambda_j}{2}(\alpha^2\theta^{-2} - \beta^{-2}).$$

Combining all the bounds together, we obtain

$$\begin{aligned}
||\boldsymbol{X}_J||_1 - ||\boldsymbol{x}_*||_1 &\leq (\sum_{j=1}^{J}||\boldsymbol{x}_{(j)}||_1) - ||\boldsymbol{x}_*^{\sigma}||_1 \leq \frac{\varepsilon^2}{2\lambda_1} - \frac{\lambda_1}{2\beta^2} + (\alpha^2\theta^{-2} - \beta^{-2})\sum_{j=2}^{J}\frac{\lambda_j}{2} \\
&\leq \frac{\varepsilon^2}{2\lambda_1} - \frac{\lambda_1}{2\beta^2} + \frac{\lambda_1}{2}(\alpha^2\theta^{-2} - \beta^{-2})\sum_{j=2}^{J}\theta^{j-1} \\
&\leq \frac{\varepsilon^2}{2\lambda_1} - \frac{\lambda_1}{2\beta^2} + \frac{\lambda_1}{2}(\alpha^2\theta^{-2} - \beta^{-2})\frac{\theta - \theta^J}{1-\theta}.
\end{aligned}$$

Using lemma 2.3.9 and theorem 2.2.4, we let $\boldsymbol{u} = \boldsymbol{X}_J$, $\boldsymbol{v} = \boldsymbol{x}_*$, and set $K = \text{supp}(\boldsymbol{x}_*(k))$ (recall $\boldsymbol{x}_*(k)$ is the best k-term approximation of $\boldsymbol{x}_*$ in ℓ_1 norm), then we have

$$\begin{aligned}
||\boldsymbol{X}_J - \boldsymbol{x}_*||_1 &\leq \frac{1+\rho}{1-\rho}\Big(||\boldsymbol{X}_J||_1 - ||\boldsymbol{x}_*||_1 + 2||(\boldsymbol{x}_*)_{K^c}||_1\Big) + \frac{2\tau}{1-\rho}||A(\boldsymbol{X}_J - \boldsymbol{x}_*)||_2 \\
&\leq \frac{1+\rho}{1-\rho}\Big(\frac{\varepsilon^2}{2\lambda_1} - \frac{\lambda_1}{2\beta^2} + \frac{\lambda_1}{2}(\alpha^2\theta^{-2} - \beta^{-2})\frac{\theta-\theta^J}{1-\theta} + 2\sigma\Big) \\
&\quad + \frac{2\tau}{1-\rho}\Big(\lambda_J\alpha + \varepsilon\Big)
\end{aligned}$$

□

Corollary 2.3.11. *Assume that the matrix $A \in \mathbb{R}^{M\times N}$ satisfies the RNSP of order k with constants $0 < \rho < 1$ and $\tau > 0$; in fact, we can lessen the restriction on A, requiring it to be only having linear independent rows. The sequence of hierarchical partial sums, $\{\boldsymbol{X}_J\}_{J=1}^{\infty}$, where $\boldsymbol{X}_J = \sum_{j=1}^{J} \boldsymbol{x}_{(j)}$ with $\boldsymbol{x}_{(j)}$ in (P_{HR}), forms a Cauchy sequence. Thus, $\boldsymbol{X}_J$ converges as $J \to \infty$.*

Proof. We let $2 \le J_1 < J_2$, and define $\boldsymbol{X}_{J_i} = \sum_{j=1}^{J_i} \boldsymbol{x}_{(j)}$ for $i = 1, 2$ respectively. Using the bounds on the hierarchical term $\boldsymbol{x}_{(j)}$ shown in the first half of the proof for theorem 2.3.10, we have

$$||\boldsymbol{X}_{J_2} - \boldsymbol{X}_{J_1}||_1 \le \frac{\lambda_1}{2}(\alpha^2\theta^{-2} - \beta^{-2}) \sum_{j=J_1}^{J_2} \theta^{j-1} = \frac{\lambda_1}{2}(\alpha^2\theta^{-2} - \beta^{-2}) \frac{\theta^{J_1-1} - \theta^{J_2}}{1-\theta}.$$

It follows that the sequence of hierarchical partial sums, $\{\boldsymbol{X}_J\}_{J=1}^{\infty}$, forms a Cauchy sequence, thus it converges. □

The following remark is concerned with the asymptotic behavior of the hierarchical sum.

Remark 2.3.12. *When we use the HR method to the recovery of a k-sparse unknown $\hat{\boldsymbol{x}}_*$ from its clean observation $\hat{\boldsymbol{y}}_*$, the HR method with the input $\boldsymbol{r}_0 = \hat{\boldsymbol{y}}_*$ give the following error bound*

$$||\boldsymbol{X}_J - \hat{\boldsymbol{x}}_*||_1 \le \frac{1+\rho}{1-\rho}\Big(\frac{\lambda_1}{2}(\alpha^2\theta^{-2} - \beta^{-2})\frac{\theta - \theta^J}{1-\theta} - \frac{\lambda_1}{2\beta^2}\Big) + \frac{2\tau}{1-\rho}\alpha\lambda_J.$$

We have $\theta^J \to 0$ as $J \to \infty$. From corollary 2.3.11, we learn that $\boldsymbol{X}_J$ is convergent when $J \to \infty$. We let $\boldsymbol{X}_J \to \boldsymbol{X}_{HR}$ as $J \to \infty$, it follows that

$$||\boldsymbol{X}_{HR} - \hat{\boldsymbol{x}}_*||_1 \le \frac{1+\rho}{1-\rho}\frac{\lambda_1}{2(1-\theta)}\Big(\alpha^2\theta^{-1} - \beta^{-2}\Big).$$

Consider the function $g : (0,1) \to \mathbb{R}$ defined as,

$$g(\theta) = \frac{\alpha^2\theta^{-1} - \beta^{-2}}{1-\theta}.$$

The function g has only one root

$$\theta_1 = (\alpha\beta)^2 = (\sqrt{N}\frac{\mu_{\max}}{\mu_{\min}})^2 > 1.$$

Recall that $\mu_{\max}$ and $\mu_{\min}$ are maximum and minimum singular values of the matrix A respectively. Thus, the function g is never zero in $(0,1)$. The function g has two critical points:

$$\theta_{cp} = 1 \pm \sqrt{1-(\alpha\beta)^{-1}}.$$

Since we require that the hierarchical multiplier $0 < \theta < 1$, we set $\theta = 1 - \sqrt{1-(\alpha\beta)^{-1}}$ in order to obtain the optimal error bound for $\boldsymbol{X}_{HR}$.

We will conclude this section by the following remark regarding the performance of the HR method when compared to the unconstrained ℓ_1 method on recovery of a general unknown $\boldsymbol{x}_*$ from its noisy observation $\boldsymbol{y}_*^\varepsilon$.

Remark 2.3.13. *We consider the hierarchical sum, $\boldsymbol{X}_J$ as a sum of hierarchical terms $\boldsymbol{x}_{(j)}$'s in (P_{HR}) with the input $\boldsymbol{r}_0 = \boldsymbol{y}_*^\varepsilon$ and a ladder of hierarchical scales $\{\lambda_j\}_{j=1}^J$, and the solution, $\boldsymbol{x}_\lambda$ in (P_U) with the input $\boldsymbol{y}_*^\varepsilon$ and regularization parameter λ. When $\boldsymbol{y}_*^\varepsilon = A\boldsymbol{x}_* + \boldsymbol{\epsilon}$ ($||\boldsymbol{\epsilon}||_2 \le \varepsilon$) for some general unknown $\boldsymbol{x}_*$ and $\lambda_J = \lambda$, the approximation error from using the HR method, namely $\boldsymbol{X}_J - \boldsymbol{x}_*$, would have a smaller upper bound than the approximation error from using the unconstrained ℓ_1 method, namely $\boldsymbol{x}_\lambda - \boldsymbol{x}_*$, when the following inequality is satisfied,*

$$\lambda_1\theta^{J-1} - (\alpha\beta\lambda_1)^2\theta^{J-2} - (\beta\varepsilon)^2\theta + (\beta\varepsilon)^2 > 0.$$

We require implicitly that $J \geq 2$; otherwise when $J = 1$, we have $\boldsymbol{X}_1 = \boldsymbol{x}_\lambda$.

We are also interested in the optimal performance of the HR method for recovery of $\boldsymbol{x}_*$ from $\boldsymbol{y}_*^\varepsilon$, when we have the freedom to choose λ_1, θ and J. The optimal 3-tuple, (λ_1, θ, J) is chosen such that the following term,

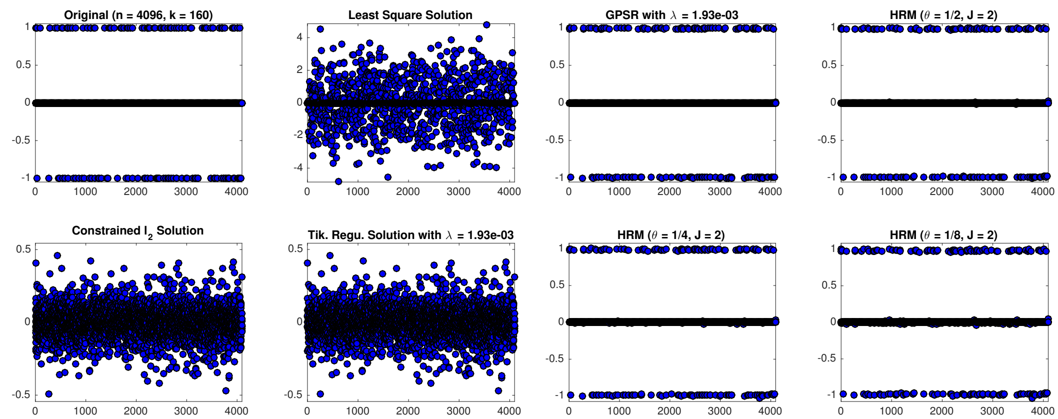
$$\frac{1+\rho}{1-\rho}\Big(\frac{\varepsilon^2}{2\lambda_1} - \frac{\lambda_1}{2\beta^2} + \frac{\lambda_1}{2}(\alpha^2\theta^{-2} - \beta^{-2})\frac{\theta - \theta^J}{1-\theta}\Big) + \frac{2\tau}{1-\rho}\alpha\lambda_1\theta^{J-1},$$

is at its minimum. We also require that $\lambda_1 > 0$, $0 < \theta < 1$ and $J \geq 2$.

2.4 Numerical Experiements

We run the numerical simulations by comparing the unconstrained ℓ_1 method and the HR method on recovery of a k-sparse $\hat{\boldsymbol{x}}_*$ from its noisy observation $\hat{\boldsymbol{y}}_*^\varepsilon = A\hat{\boldsymbol{x}}_* + \boldsymbol{\epsilon}$ ($||\boldsymbol{\epsilon}||_2 \leq \varepsilon$) with $\lambda = \lambda_J$. We implement the Gradient Projection for Sparse Reconstruction (GPSR) algorithm [20] for solving (P_U) in MATLAB. The k-sparse unknown $\hat{\boldsymbol{x}}_* \in \mathbb{R}^N$ is generated with $k = 160$ non-zero entries, and the values of those entries are randomly picked as ± 1, and the location of the non-zero entries is also randomly picked. We set the number of observations M to 1024, and the dimension of the unknown is set at $N = 4096$. The entries of $A \in \mathbb{R}^{M\times N}$ are identically distributed standard normal variables (satisfying the Gaussian distribution with mean 0 and standard deviation 1), i.e., $A = \text{randn}(M, N)$ in MATLAB. We orthonormalize the rows of A by doing $A = (\text{orth}(A.')).'$. The noise level is set as $\varepsilon = \gamma * ||A\boldsymbol{x}_*||_2$, where γ (the noise to signal ratio) changes from 0 to 0.04. Other parameters for the GPSR algorithm is set as follows: $\beta = 0.5$, $\mu = 0.1$, $\bar{\alpha} = 0.1$, the maximum number of iterations is allowed at 500, we do not de-bias the solution, and the stopping criteria is set as $\big|\min(\boldsymbol{z}, \nabla F(\boldsymbol{z}))\big|$, where

$\boldsymbol{z}$ is the approximate solution of GPSR at the j^{th} iterate, and $\nabla F(\cdot)$ is the gradient of the objective function F defined in equation (8) in [20], and the tolerance for stopping criteria is set at 10^{-3}. First, we show the comparison between the ℓ_2 based methods and the ℓ_1 based methods in figure 2.2.



(a) ℓ_2 Regularization (b) ℓ_1 Regularization

Figure 2.2: ℓ_1 Methods versus ℓ_2 Methods

For the ℓ_2 based methods, we employ the Least Square (LS) method, the constrained ℓ_2 method, and the $\ell_2^2 - \ell_2^2$ Tikhonov regularization method (also known as Ridge Regression, see section A.2 for details). For the ℓ_1 based methods, we employ the GPSR and the HR method (based on GPSR) with $\theta = 1/2, 1/4, 1/8$ and $J = 2$. The regularization parameter is fixed at $\lambda_J = \lambda \approx 0.0019$. All three ℓ_2 methods produce approximate solutions which are not sparse at all; whereas the ℓ_1 methods are able to produce sparse approximate solutions to $\hat{\boldsymbol{x}}_*$. Table 2.1 shows the comparison between the GPSR and the HR method on sparse recovery from $\hat{\boldsymbol{y}}_*^{\varepsilon}$. Each cell in table (2.1) has two values: the integer value represents the total number of iterations used for each algorithm, the real value inside parenthesis represents the approximation error.

Algorithms	$\gamma = 0$	$\gamma = 0.01$	$\gamma = 0.02$	$\gamma = 0.04$
GPSR	283(1.49)	287(2.76)	301(6.24)	329(14.3)
HR ($\theta = 1/2, J = 2$)	160(1.47)	164(2.59)	172(5.77)	189(13.5)
HR ($\theta = 1/4, J = 2$)	86(1.51)	88(2.66)	92(6.07)	135(13.5)
HR ($\theta = 1/8, J = 2$)	75(1.64)	82(2.90)	94(6.25)	116(14.1)

Table 2.1: (P_U) versus (P_{HR})

As show in the table 2.1, the HR method is able to reduce the total number of iterations to half of which is used by GPSR when we use $\theta = 1/2$. When $\theta = 1/4$, the number of iterations is further reduced to a quarter of the total number from GPSR. However, we note that by simply decreasing θ does not always significantly reduces the number of iterations, notice the change from $\theta = 1/4$ to $\theta = 1/8$. It signifies that the need to find an optimal θ which we should choose.

2.5 Conclusion

We started the chapter by discussing various constrained ℓ_p methods for the recovery of k-sparse unknown $\hat{\boldsymbol{x}}_*$ from its clean observation $\hat{\boldsymbol{y}}_*$. Next, we suggested using two different regularization methods for the recovery of general unknown $\boldsymbol{x}_*$'s from its noisy observation $\boldsymbol{y}_*^\varepsilon$. One of the suggestions is the unconstrained ℓ_1 method, which is the preferred method discussed in details in this chapter. We then proposed a multi-scale generalization of the unconstrained ℓ_1 method, the HR method, and discussed its convergence property and its performance for recovery of $\boldsymbol{x}_*$ from $\boldsymbol{y}_*^\varepsilon$. At the end, we note that the HR method can be extended to recovery of unknowns in $\mathbb{C}^N$, since the RNSP property used in this chapter is also defined in $\mathbb{C}^N$ [23].

Chapter 3: Hierarchical Reconstruction Method for Deconvolution

3.1 Introduction: Deconvolution on the Helmholtz Filter

We analyze the de-convolution problem on the Helmholtz filter for the closure problem in Large Eddy Simulation (LES), since LES has gained more and more popularity in simulating fluid flows, especially turbulent flows, due to its capability of reducing computational cost by appropriate usage of certain LES filter. The closure problem arises in LES when a low pass spatial and linear filter, with filtering radius $\delta > 0$ (also known as LES filter), is applied to the Navier-Stokes equations (NSE). The setup is as follows. Let $\Omega \subset \mathbb{R}^3$ be the physical domain with a Lipschitz boundary $\partial\Omega$, ρ be the constant density and ν the viscosity. The velocity $\boldsymbol{u} = (u_1, u_2, u_3)^\top$ and pressure p are functions of time $t > 0$ and space $\boldsymbol{s} \in \Omega$. The filtered velocity $\bar{\boldsymbol{u}} = (\bar{u}_1, \bar{u}_2, \bar{u}_3)^\top$and the filtered pressure $\bar{p}$ satisfy the following filtered NSE (after application of a LES filter to the incompresible flow),

$$\begin{gathered} \frac{\partial \bar{u}_i}{\partial s_i} = 0, \quad \text{(Mass)} \\ \frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial s_j}\overline{(u_i u_j)} + \frac{1}{\rho}\frac{\partial \bar{p}}{\partial s_i} = \nu \frac{\partial}{\partial s_j}\left(\frac{\partial \bar{u}_i}{\partial s_j} + \frac{\partial \bar{u}_j}{\partial s_i}\right), \quad \text{(Momentum)} \end{gathered} \tag{3.1}$$

Here, the low pass filter (also known as LES filter) commutes with temporal partial derivative $\partial/\partial t$ and all spatial derivatives $\partial/\partial s_i$. The filtered equations in (3.1) are almost closed with respect to $\bar{\boldsymbol{u}}$ and $\bar{p}$ except the second order term $\overline{(u_i u_j)}$. When the term $\overline{(u_i u_j)}$

is replaced by $\bar{u}_i\bar{u}_j$, the difference, $\overline{(u_iu_j)} - \bar{u}_i\bar{u}_j$, is not zero. The residual stress tensor, $\tau_{ij} = \overline{(u_iu_j)} - \bar{u}_i\bar{u}_j$, contains the interaction between the unfiltered velocity and the filtered velocity, and it has to be modeled in terms of only the filtered velocity, henceforth requiring de-convolution from the filtered velocity. The study on modeling of residual stress tensor is beyond the scope of this thesis. Out of many possible LES filters, we choose the Helmholtz filter [26,27]. Because when the Helmholtz filter is used, the residual stress tensor can be expressed exactly in terms of the filtered velocity

$$\tau_{ij} = \overline{2\delta^2 \nabla \bar{u}_i \cdot \nabla \bar{u}_j + \delta^4 (\Delta \bar{u}_i)(\Delta \bar{u}_j)}. \tag{3.2}$$

Therefore, we focus ourselves on the study of de-convolution on the Helmholtz filter. We consider a more general setup for the de-convolution problem as follows. Let $\mathcal{X}$ be a Hilbert space of functions defined over the physical domain Ω. The space $\mathcal{X}$ is equipped with the standard L^2 norm and the standard inner product, i.e., for any $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{X}$,

$$||\boldsymbol{x}||_{L^2(\Omega)} = \int_{\boldsymbol{s}\in\Omega} \left|\boldsymbol{x}(\boldsymbol{s})\right|^2 d\boldsymbol{s} \quad \text{and} \quad \langle \boldsymbol{x}, \boldsymbol{y} \rangle = \int_{\boldsymbol{s}\in\Omega} \boldsymbol{x}(\boldsymbol{s}) \cdot \boldsymbol{y}(\boldsymbol{s})\, d\boldsymbol{s}. \tag{3.3}$$

For the unknown function $\boldsymbol{x}_* \in \mathcal{X}$, its filtered output $\boldsymbol{y}_* \in \mathcal{X}$ is obtained by applying the Helmholtz filter to $\boldsymbol{x}_*$ in the following convolution setting,

$$\boldsymbol{x}_* \mapsto \boldsymbol{y}_* : \quad \boldsymbol{y}_*(\boldsymbol{s}) := \int_{\boldsymbol{s}'\in\Omega} K_\delta(\boldsymbol{s} - \boldsymbol{s}')\boldsymbol{x}_*(\boldsymbol{s}')\, d\boldsymbol{s}' = K_\delta * \boldsymbol{x}_*. \tag{3.4}$$

Here the integral kernel K_δ is given by the standard integral kernel K scaled by δ, i.e., $K_\delta(\boldsymbol{s}) = \delta^{-3}K(\delta^{-1}\boldsymbol{s})$. The standard integral kernel K is defined as follows,

$$K(\boldsymbol{s}) = \frac{1}{4\pi}\frac{\exp(-||\boldsymbol{s}||_2)}{||\boldsymbol{s}||_2}, \quad \text{for } \boldsymbol{s} \in \mathbb{R}^3. \tag{3.5}$$

We denote the convolution action with this kernel K_δ as a multiplication with the operator A_δ, i.e., $\boldsymbol{y}_* = K_\delta * x_* = A_\delta \boldsymbol{x}_*$. When the filtered output $\boldsymbol{y}_*$ is given, the de-convolution problem is inquiring about the possibility of recovery of the unknown $\boldsymbol{x}_*$ via de-convolution from $\boldsymbol{y}_*$. In the case of the Helmholtz filter, the unknown $\boldsymbol{x}_*$ can be exactly recovered by applying the inverse of the Helmholtz filter A_δ, namely A_δ^{-1}, to $\boldsymbol{y}_*$. The equation $\boldsymbol{x}_* = A_\delta^{-1}\boldsymbol{y}_*$ gives the following elliptic differential equation with a Dirichlet boundary condition,

$$\boldsymbol{y}_* \mapsto \boldsymbol{x}_* : \quad \begin{cases} \boldsymbol{x}_*(\boldsymbol{s}) & = -\delta^2 \Delta \boldsymbol{y}_*(\boldsymbol{s}) + \boldsymbol{y}_*(\boldsymbol{s}), \qquad \boldsymbol{s} \in \Omega \\ \boldsymbol{x}_*(\boldsymbol{s}) & = \boldsymbol{y}_*(\boldsymbol{s}), \qquad \boldsymbol{s} \in \partial\Omega \end{cases} . \tag{3.6}$$

Despite having a well-defined inverse, recovery of $\boldsymbol{x}_*$ by direct application of A_δ^{-1} to $\boldsymbol{y}_*$ is not always possible. In actual implementations, one only has access to the *approximate* numerical output $\boldsymbol{y}_*^h$ (where h represents the spatial resolution for certain discretization scheme), which is obtained by solving LES numerically. In this chapter, we will focus on the Finite Element Method (FEM) and use it to define the discrete Helmholtz filter, $A^h (= A_\delta^h)$[1]: $\mathcal{X} \to \mathcal{X}^h$ ($\mathcal{X}^h$ is a finite dimensional subspace of $\mathcal{X}$), as a numerical approximation to A_δ. The discrete equation $\boldsymbol{y}_*^h = A^h \boldsymbol{x}_*$ is defined in the weak formulation setting: for the unknown $\boldsymbol{x}_* \in \mathcal{X}$, there is a unique FEM solution $\boldsymbol{y}_*^h \in \mathcal{X}^h$ such that for all $\boldsymbol{v}^h \in \mathcal{X}^h$, the following holds

$$\boldsymbol{x}_* \mapsto \boldsymbol{y}_*^h : \quad \delta^2 \langle \nabla \boldsymbol{y}_*^h, \nabla \boldsymbol{v}^h \rangle + \langle \boldsymbol{y}_*^h, \boldsymbol{v}^h \rangle = \langle \boldsymbol{x}_*, \boldsymbol{v}^h \rangle. \tag{3.7}$$

The discrete operator A^h is Symmetric Positive Definite (SPD) over $\mathcal{X}^h$ but Symmetric Positive *Semi-Definite* (SPSD) over $\mathcal{X}$ [40]. Therefore, when given the discrete filtered output $\boldsymbol{y}_*^h$, it is not possible to recover $\boldsymbol{x}_*$ by directly solving the discrete equation $A^h \boldsymbol{x} = \boldsymbol{y}_*^h$. To counter this particular kind of ill-posedness of A^h and considering that A^h is SPD

[1] We drop the dependence of δ for A_δ^h to emphasize the presence of the numerical resolution scale h.

over $\mathcal{X}^h$, one can employ the Tikhonov-Lavrentiev regularization (TLR) method. The TLR method finds a unique FEM solution $\boldsymbol{x}^h_{L(\lambda)} \in \mathcal{X}^h$ such that for all $\boldsymbol{v}^h \in \mathcal{X}^h$ the following holds,

$$\lambda\delta^2\langle\nabla\boldsymbol{x}^h_{L(\lambda)}, \nabla\boldsymbol{v}^h\rangle + (1+\lambda)\langle\boldsymbol{x}^h_{L(\lambda)}, \boldsymbol{v}^h\rangle = \delta^2\langle\nabla\boldsymbol{y}^h_*, \nabla\boldsymbol{v}^h\rangle + \langle\boldsymbol{y}^h_*, \boldsymbol{v}^h\rangle.$$

Based on the understanding that the regularization *parameter* λ represents an regularization *scale* and further exploiting the filter property of A^h, we propose the HR method as a multi-scale generalization to the TLR method. By setting the initial discrete hierarchical residual $\boldsymbol{r}^h_0 = \boldsymbol{y}^h_*$, the HR method finds the unique FEM solution $\boldsymbol{x}^h_{(j)} \in \mathcal{X}^h$ such that for all $\boldsymbol{v}^h \in \mathcal{X}^h$ the following holds,

$$\lambda\delta^2\langle\nabla\boldsymbol{x}^h_{(j)}, \nabla\boldsymbol{v}^h\rangle + \langle\boldsymbol{x}^h_{(j)}, \boldsymbol{v}^h\rangle = \delta^2\langle\nabla\boldsymbol{r}^h_{j-1}, \nabla\boldsymbol{v}^h\rangle + \langle\boldsymbol{r}^h_{j-1}, \boldsymbol{v}^h\rangle, \quad 1 \leq j \leq J.$$

Here the discrete hierarchical residual $\boldsymbol{r}^h_j \in \mathcal{X}^h$ is found as the unique FEM solution such that for all $\boldsymbol{v}^h \in \mathcal{X}^h$, the following recursive relationship holds,

$$\delta^2\langle\nabla\boldsymbol{r}^h_j, \nabla\boldsymbol{v}^h\rangle + \langle\boldsymbol{r}^h_j, \boldsymbol{v}^h\rangle = \delta^2\langle\nabla\boldsymbol{r}^h_{j-1}, \nabla\boldsymbol{v}^h\rangle + \langle\boldsymbol{r}^h_{j-1} - \boldsymbol{x}^h_{(j)}, \boldsymbol{v}^h\rangle.$$

The HR method utilizes a ladder of discrete hierarchical residuals with their corresponding hierarchical scales in a recursive manner, i.e., it *successively* extracts information from the previous discrete hierarchical residuals $\boldsymbol{r}^h_{j-1}$ to the discrete hierarchical term $\boldsymbol{x}^h_{(j)}$ at a *finer* hierarchical scale λ_j. After J steps of hierarchical extraction, we sum up the discrete hierarchical terms to obtain the discrete hierarchical sum, i.e., $\boldsymbol{X}^h_J = \sum_{j=1}^J \boldsymbol{x}^h_{(j)}$. We show that the discrete hierarchical sum $\boldsymbol{X}^h_J$ can approximate $\boldsymbol{x}_*$ with the following error,

$$\begin{aligned}||\boldsymbol{x}_* - \boldsymbol{X}^h_J||_{L^2(\Omega)} \leq \delta^{2J}\Big(\prod_{j=1}^J \lambda_j\Big)||\Delta^J\boldsymbol{x}_*||_{L^2(\Omega)} \\ + Ch^k \max_{1\leq j\leq J}(\sqrt{\delta^2\lambda_j} + h)||D_jA\boldsymbol{x}_*||_{H^{k+1}(\Omega)}.\end{aligned}$$

Here D_j is the total hierarchical de-convolution operator after j steps of hierarchical iterations (see definition 3.4.2) and k represents the degree of basis functions used in a Finite Element scheme. When the residual stress tensor is approximated, one is given the noisy discrete filtered output $\boldsymbol{y}_*^{h,\varepsilon}$. We show an (near) optimal stopping criteria for the HR method to provide an approximate solution to $\boldsymbol{x}_*$ via de-convolution from $\boldsymbol{y}_*^{h,\varepsilon}$.

The remaining sections of this chapter are structured as follows. In section 3.2, we introduce the necessary inequalities needed for the proofs of the approximation errors for the HR method, then we expand more on the discussion about properties of the Helmholtz filter. In section 3.3, we introduce the nearby problem: recovery of $\boldsymbol{x}_*$ via de-convolution from the discrete filtered output $\boldsymbol{y}_*^h$. We discuss a couple Tikhonov regularization methods on solving the nearby problem. Next, in section 3.4, we first provide the continuous approximation error analysis for the HR method on recovery of $\boldsymbol{x}_*$ via de-convolution from $\boldsymbol{y}_*$. Such continuous approximation error bound will be used in the discrete approximation error analysis later. We then continue our analysis about the HR method on solving the nearby problem and show the discrete approximation error analysis. Due to the fact that the residual stress tensor might be approximated, one is given the noisy filtered output $\boldsymbol{y}_*^{h,\varepsilon}$. In section 3.5, we present the analysis of the HR method on providing an approximate solution to $\boldsymbol{x}_*$ via de-convolution from $\boldsymbol{y}_*^{h,\varepsilon}$ and we also provide an (near) optimal stopping criteria to enforce the convergence of the HR approximate solutions to $\boldsymbol{x}_*$. We conduct numerical experiments by comparing a family of Tikhonov regularization methods to the HR method on solving the nearby problem, testing the (near) optimal stopping criteria on solving the noisy nearby problem, and show the convergence rate of the HR method on the nearby problem in $\mathbb{R}^2$. At the end, we conclude the chapter by discussing the possibility of extending the analysis of using the HR method with other discretization scheme, such as the Finite Difference Method (FDM), and applying the HR method to other type of LES filters.

3.2 The Original Problem: Recovery of $\boldsymbol{x}_*$ from Continuous Filtered Output $\boldsymbol{y}_*$

Before we discuss more about the Helmholtz filter, we plan to introduce some of the notations and inequalities used through out this chapter. First, we use the standard notation for Lebesgue and Sobolev spaces and their norms. Meanwhile, the physical domain Ω is a regular, bounded, polyhedral domain in $\mathbb{R}^3$. We define the following space,

$$\mathcal{X} = H_0^1(\Omega)^n = \Big\{ \boldsymbol{x} \in L^2(\Omega)^n : \nabla \boldsymbol{x} \in L^2(\Omega)^{n\times n} \quad \text{and } \boldsymbol{x} = \mathbf{0} \text{ on } \partial\Omega \Big\}. \tag{3.8}$$

The dimensional parameter n can be either 1 (for the pressure) or 3 (for the velocity). We mentioned that $\mathcal{X}^h$ is a finite dimensional subspace of $\mathcal{X}$. When using FEM, an example $\mathcal{X}^h$ is the set of continuous polynomials of degree k. We also assume that we have homogeneous boundary data throughout. We use the following approximation inequalities from [6],

$$\begin{aligned} &\inf_{\boldsymbol{v}^h \in \mathcal{X}^h} ||\boldsymbol{x} - \boldsymbol{v}^h||_{L^2(\Omega)} \le Ch^{k+1} ||\boldsymbol{x}||_{H^{k+1}(\Omega)}, \qquad \boldsymbol{x} \in H^{k+1}(\Omega)^n, \\ &\inf_{\boldsymbol{v}^h \in \mathcal{X}^h} ||\boldsymbol{x} - \boldsymbol{v}^h||_{H^1(\Omega)} \le Ch^{k} ||\boldsymbol{x}||_{H^{k+1}(\Omega)}, \qquad \boldsymbol{x} \in H^{k+1}(\Omega)^n. \end{aligned} \tag{3.9}$$

We will also employ the following inequalities for our proofs:

- Cauchy-Schwartz inequality: $\big|\langle \boldsymbol{x}, \boldsymbol{y}\rangle\big| \le ||\boldsymbol{x}||_{L^2(\Omega)} ||\boldsymbol{y}||_{L^2(\Omega)}, \forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{X}$;

- Young's inequality: given $\epsilon > 0$, $a, b \geq 0$, and $1 < p, q < \infty$ with $p^{-1} + q^{-1} = 1$, then

$$ab \leq \frac{\epsilon}{p} a^p + \frac{\epsilon^{1-q}}{q} b^q;$$

- Triangle inequality: $||\boldsymbol{x} + \boldsymbol{y}||_{L^2(\Omega)} \leq ||\boldsymbol{x}||_{L^2(\Omega)} + ||\boldsymbol{y}||_{L^2(\Omega)}, \forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{X}$.

The de-convolution problem on a general linear filter is considered an important inverse problem [26, 34, 40, 41, 53]. Such problem occurse in many applications including parameter identification [18, 19], the de-convolution problem of image processing [4], and the closure problem in turbulence modeling [3, 28, 36, 41]. We invest special interests in the de-convolution problem on the Helmholtz filter, since the Helmholtz filter has a well-defined inverse. The Helmholtz filter, also known as the Helmholtz differential filter (since its differential form is used more often), is used in several Large Eddy Simulation models [3,26–28,36,40,41]. It is equivalent to the Pao filter used in image processing [36]. The original problem in the de-convolution on the Helmholtz filter is the recovery $\boldsymbol{x}_*$ via de-convolution from $\boldsymbol{y}_*$. We mentioned that the original problem is well-posed, since recovery of $\boldsymbol{x}_*$ can be obtained via $\boldsymbol{x}_* = A_\delta^{-1} \boldsymbol{y}_*$ and A_δ^{-1} is well-defined.

Lemma 3.2.1. *The Helmholtz filter A_δ defined in* (3.4) *with the convolution kernel defined in* (3.5) *has the inverse operator A_δ^{-1} defined in terms of the elliptic differential equation with Dirichlet boundary equation in* (3.6).

Proof. We start from the differential equation. For any $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{X}$, $\boldsymbol{x} = A_\delta^{-1} \boldsymbol{y} \Rightarrow \boldsymbol{x} = -\delta^2 \Delta \boldsymbol{y} + \boldsymbol{y}$. Let $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{y}}$ be the Fourier transform of $\boldsymbol{x}$ and $\boldsymbol{y}$ respectively, and $\boldsymbol{\xi}$ be the variable in the Fourier space. Apply the Fourier transform to the differential equation, we obtain $\tilde{\boldsymbol{x}} = (1+\delta^2||\boldsymbol{\xi}||_2^2)\tilde{\boldsymbol{y}}$. Hence,

$$\tilde{\boldsymbol{y}} = \frac{1}{1+\delta^2||\boldsymbol{\xi}||_2^2}\tilde{\boldsymbol{x}} \quad \text{and} \quad \tilde{\boldsymbol{y}} = \tilde{K}_\delta \tilde{\boldsymbol{x}} \Rightarrow \tilde{K}_\delta = \frac{1}{1+\delta^2||\boldsymbol{\xi}||_2^2}.$$

Here $\tilde{K}_\delta$ represents the Fourier symbol of the Helmholtz convolution kernel K_δ. Let $\mathcal{F}^{-1}$ be the inverse Fourier operator, we have $\boldsymbol{y} = \mathcal{F}^{-1}(\tilde{K}_\delta) * \boldsymbol{x}$. Next, we focus on find the inverse Fourier transform of the standard kernel (since $\delta^2||\boldsymbol{\xi}||_2^2 = ||\delta\boldsymbol{\xi}||_2^2$), i.e.,

$$\mathcal{F}^{-1}\left(\frac{1}{1+||\boldsymbol{\xi}||_2^2}\right) = \int_{\boldsymbol{\xi}} \frac{\exp(-i\boldsymbol{s}\cdot\boldsymbol{\xi})}{1+||\boldsymbol{\xi}||_2^2}\, d\boldsymbol{\xi}. \tag{3.10}$$

After a change of coordinates: $\boldsymbol{\xi} = r\boldsymbol{\xi}'$ with $r > 0$ and $\boldsymbol{\xi}' \in S^2$ (the unite sphere in $\mathbb{R}^3$), (3.10) becomes

$$\begin{aligned}\int_{r=0}^{\infty}\int_{\boldsymbol{\xi}'\in S^2} \frac{\exp(-ir\boldsymbol{s}\cdot\boldsymbol{\xi}')}{1+r^2}\, r^2 d\sigma(\boldsymbol{\xi}')dr &= \int_{r=0}^{\infty} \frac{r^2}{1+r^2}\frac{\sin(r||\boldsymbol{s}||_2)}{r||\boldsymbol{s}||_2}\, dr \\ &= \int_{r=0}^{\infty} \frac{r\sin(r||\boldsymbol{s}||_2)}{(1+r^2)||\boldsymbol{s}||_2}\, dr = \frac{1}{4\pi}\frac{\exp(-||\boldsymbol{s}||_2)}{||\boldsymbol{s}||_2}.\end{aligned}$$

Here $\sigma(\xi')$ is the surface measure on S^2 (see [21]). Therefore

$$K_\delta = \exp(-||\boldsymbol{s}||_2/\delta)/(4\pi\delta^2||\boldsymbol{s}||_2).$$

□

We end this section with the following remark.

Remark 3.2.2. *The Helmholtz filter A_δ is symmetric. To see that, consider $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{X}$,*

$$\begin{aligned}\langle A_\delta \boldsymbol{u}, \boldsymbol{v}\rangle &= \int_{\boldsymbol{s}}\Big(\int_{\boldsymbol{s}'} K_\delta(\boldsymbol{s}-\boldsymbol{s}')\boldsymbol{u}(\boldsymbol{s}')d\boldsymbol{s}'\Big)\cdot \boldsymbol{v}(\boldsymbol{s})d\boldsymbol{s} \\ &= \int_{\boldsymbol{s}}\int_{\boldsymbol{s}'} K_\delta(\boldsymbol{s}-\boldsymbol{s}')\boldsymbol{u}(\boldsymbol{s}')\cdot\boldsymbol{v}(\boldsymbol{s})d\boldsymbol{s}'d\boldsymbol{s} \\ &= \int_{\boldsymbol{s}'}\int_{\boldsymbol{s}} \boldsymbol{u}(\boldsymbol{s}')\cdot\Big(K_\delta(\boldsymbol{s}'-\boldsymbol{s})\boldsymbol{v}(\boldsymbol{s})\Big)d\boldsymbol{s}d\boldsymbol{s}' \\ &= \langle \boldsymbol{u}, A_\delta \boldsymbol{v}\rangle.\end{aligned}$$

Next, we investigate the upper bound on A_δ. By basic Harmonic Analysis and theories of Fourier multipliers, we have for any $\boldsymbol{x} \in \mathcal{X}$

$$||A_\delta \boldsymbol{x}||_{L^2(\Omega)} = ||K_\delta * \boldsymbol{x}||_{L^2(\Omega)} = ||\mathcal{F}(K_\delta * \boldsymbol{x})||_{L^2(\Omega)} = ||\tilde{K}_\delta \tilde{\boldsymbol{x}}||_{L^2(\Omega)}$$

$$\leq ||\tilde{K}_\delta||_{L^\infty(\Omega)} ||\tilde{\boldsymbol{x}}||_{L^2(\Omega)} \leq ||\tilde{\boldsymbol{x}}||_{L^2(\Omega)}.$$

Here, we used the fact $\tilde{K}_\delta(\boldsymbol{\xi}) = (1 + \delta^2 ||\boldsymbol{\xi}||_2^2)^{-1} \leq 1$ for any $\boldsymbol{\xi}$ in the Fourier space.

We conclude this section by a figure showing the Fourier symbol of the Helmholtz filter A for various δ's to demonstrate the scale "cut-off" effect of the filtering radius.

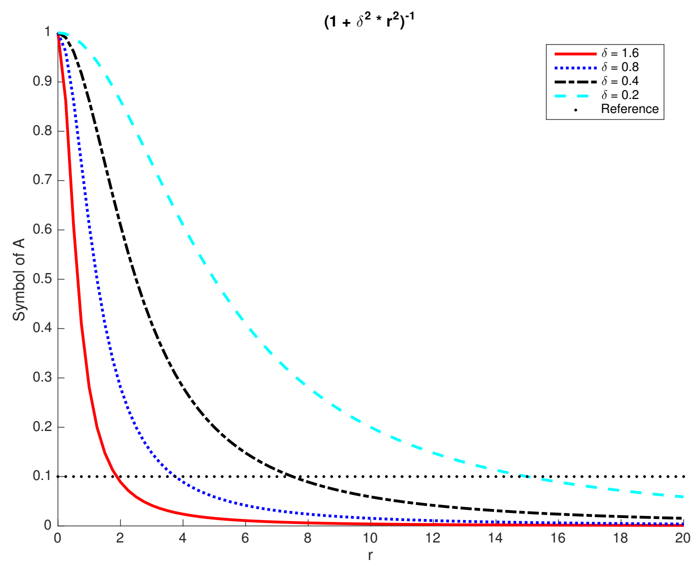


Figure 3.1: Fourier Symbol: $(1 + \delta^2 r^2)^{-1}$

3.3 The Nearby Problem: Recovery of $\boldsymbol{x}_*$ from Discrete Filtered Output $\boldsymbol{y}_*^h$

We mentioned in the section 3.1 that the discrete Helmholtz operator A^h is ill-posed, hence recovery of $\boldsymbol{x}_*$ by directly solving the discrete equation $A^h \boldsymbol{x} = \boldsymbol{y}_*^h$ is impossible.

We consider a linear operator $A : \mathcal{X} \to \mathcal{Y}$, which is compact and its range is finite dimensional, inverting the linear equation $A\boldsymbol{x} = \boldsymbol{y}$ is considered ill-posed [1,2,30,44,52, 62,64]. In order to find a suitable approximate solution to $A\boldsymbol{x} = \boldsymbol{y}$, one can consider the general Tikhonov regularization method, which finds an approximate solution from

$$\boldsymbol{x}_{T(\lambda)} = \arg\min_{\boldsymbol{x}\in\mathcal{C}} \Big\{\lambda f(\boldsymbol{x}) + \frac{1}{2}||\boldsymbol{y} - A\boldsymbol{x}||_{\mathcal{Y}}^2\Big\}. \tag{3.11}$$

In the case of $A = A^h$, solving the linear equation $A^h\boldsymbol{x} = \boldsymbol{y}_*^h$ to recover $\boldsymbol{x}_*$ is ill-posed in exactly the same manner. We will present two specific implementations of (3.11) to address the ill-posed equation $A^h\boldsymbol{x} = \boldsymbol{y}_*$. First, we consider the $\ell_2^2 - \ell_2^2$ Tikhonov regularization method [19, 59, 60], which finds the FEM solution $\boldsymbol{x}_{T(\lambda)}^h \in \mathcal{X}^h$ (with a regularization parameter $\lambda > 0$) such that for all $\boldsymbol{v}^h \in \mathcal{X}^h$, the following holds

$$\begin{aligned} &\lambda\delta^4\langle\Delta\boldsymbol{x}_{T(\lambda)}^h, \Delta\boldsymbol{v}^h\rangle + 2\lambda\delta^2\langle\nabla\boldsymbol{x}_{T(\lambda)}^h, \nabla\boldsymbol{v}^h\rangle + (1+\lambda)\langle\boldsymbol{x}_{T(\lambda)}^h, \boldsymbol{v}^h\rangle \\ &\quad = \delta^2\langle\nabla\boldsymbol{y}_*^h, \nabla\boldsymbol{v}^h\rangle + \langle\boldsymbol{y}_*^h, \boldsymbol{v}^h\rangle. \end{aligned} \tag{3.12}$$

Remark 3.3.1. *Denoted in the operator form, the $\ell_2^2 - \ell_2^2$ Tikhonov regularization (ℓ_2TR) method finds the FEM solution $\boldsymbol{x}_{T(\lambda)}^h$ satisfying the following equation*

$$((A^h)^2 + \lambda I)\boldsymbol{x}_{T(\lambda)}^h = A^h\boldsymbol{y}_*^h. \tag{3.13}$$

The approximate solution $\boldsymbol{x}_{T(\lambda)}^h$ is a solution of the following minimization problem,

$$\boldsymbol{x}_{T(\lambda)}^h = \arg\min_{\boldsymbol{x}^h\in\mathcal{X}^h} \Big\{\frac{\lambda}{2}||\boldsymbol{x}^h||_{L^2(\Omega)}^2 + \frac{1}{2}||\boldsymbol{y}_*^h - A^h\boldsymbol{x}^h||_{L^2(\Omega)}^2\Big\}.$$

When we set $\mathcal{C} = \mathcal{X}^h$, $f(\boldsymbol{x}^h) = ||\boldsymbol{x}^h||_{L^2(\Omega)}^2/2$, $A = A^h$, $\boldsymbol{y} = \boldsymbol{y}_*^h$, $\mathcal{Y} = \mathcal{X}^h$, and $||\cdot||_{\mathcal{Y}} = ||\cdot||_{L^2(\Omega)}$ in (3.11)*, we obtain the ℓ_2TR method.*

Considering the fact that the discrete Helmholtz operator A^h is SPD over $\mathcal{X}^h$, there is a SPD operator B such that $A^h = B^2$. The solution from the following

$$\boldsymbol{x}^h_{L(\lambda)} = \underset{\boldsymbol{x}^h \in \mathcal{X}^h}{\arg\min} \left\{ \frac{\lambda}{2} ||\boldsymbol{x}^h||^2_{L^2(\Omega)} + \frac{1}{2} ||B^{-1}\boldsymbol{y}^h_* - B\boldsymbol{x}^h||^2_{L^2(\Omega)} \right\} \tag{3.14}$$

also satisfies the equation

$$(B^2 + \lambda I)\boldsymbol{x}^h_{L(\lambda)} = B(B^{-1}\boldsymbol{y}^h_*) \Rightarrow (A^h + \lambda I)\boldsymbol{x}^h_{L(\lambda)} = \boldsymbol{y}^h_*.$$

This reasoning gives rise to the Tikhonov-Lavrentiev regularization (TLR) method (see [19, 33, 37, 41, 63]). The TLR method finds the FEM solution $\boldsymbol{x}^h_{L(\lambda)} \in \mathcal{X}^h$ such that for all $\boldsymbol{v}^h \in \mathcal{X}^h$, the following holds

$$\lambda\delta^2 \langle \nabla \boldsymbol{x}^h_{L(\lambda)}, \nabla \boldsymbol{v}^h \rangle + (1+\lambda)\langle \boldsymbol{x}^h_{L(\lambda)}, \boldsymbol{v}^h \rangle = \delta^2 \langle \nabla \boldsymbol{y}^h_*, \nabla \boldsymbol{v}^h \rangle + \langle \boldsymbol{y}^h_*, \boldsymbol{v}^h \rangle. \tag{3.15}$$

Remark 3.3.2. *Put in the operator form, the TLR method finds the FEM solution* $\boldsymbol{x}^h_{L(\lambda)}$ *satisfying the following equation*

$$(A^h + \lambda I)\boldsymbol{x}^h_{L(\lambda)} = \boldsymbol{y}^h_*. \tag{3.16}$$

Define a weighted L^2 *norm for* $\boldsymbol{x}^h \in \mathcal{X}^h$ *as follows:* $||\boldsymbol{x}^h||_{L^2_{B^{-1}}(\Omega)} = ||B^{-1}\boldsymbol{x}^h||_{L^2(\Omega)}$. *When we consider the following*

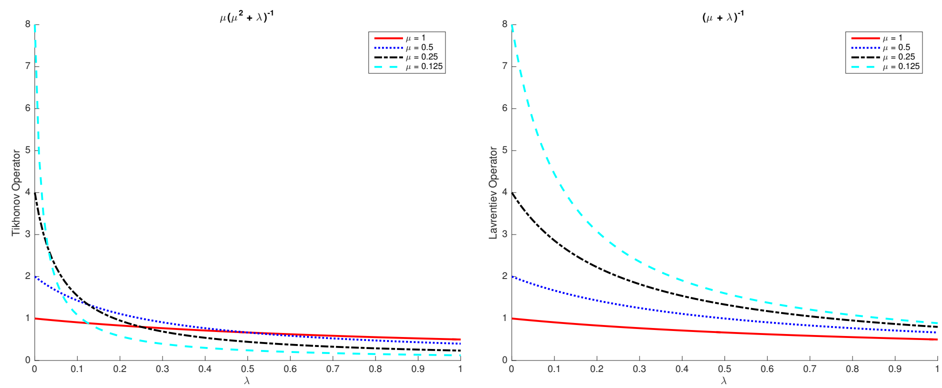
$$||B^{-1}\boldsymbol{y}^h_* - B\boldsymbol{x}^h||_{L^2(\Omega)} = ||B^{-1}(\boldsymbol{y}^h_* - B^2\boldsymbol{x}^h)||_{L^2(\Omega)} = ||\boldsymbol{y}^h_* - A^h\boldsymbol{x}^h||_{L^2_{B^{-1}}(\Omega)},$$

we obtain another formulation of (3.14)

$$\boldsymbol{x}^h_{L(\lambda)} = \underset{\boldsymbol{x}^h \in \mathcal{X}^h}{\arg\min} \left\{ \frac{\lambda}{2} ||\boldsymbol{x}^h||^2_{L^2(\Omega)} + \frac{1}{2} ||\boldsymbol{y}^h_* - A^h\boldsymbol{x}^h||^2_{L^2_{B^{-1}}(\Omega)} \right\}.$$

Such specific formulation can be obtained from (3.11) *when we set* $f(\boldsymbol{x}^h) = ||\boldsymbol{x}^h||_{L^2(\Omega)}/2$, $\mathcal{C} = \mathcal{X}^h$, $\mathcal{Y} = \mathcal{X}^h$, *and* $||\cdot||_{\mathcal{Y}} = ||\cdot||_{L^2_{B^{-1}}(\Omega)}$.

The TLR method is also known as the method of Lavrentiev regularization [35] or the method of Singular Perturbation [38]. We conclude this section by comparing the effects of the regularization parameter λ on the singular values of the two de-convolution operators: the Tikhonov de-convolution operator $((A^h)^2 + \lambda I)^{-1}A^h$ and the Lavrentiev de-convolution operator $(A^h + \lambda I)^{-1}$.



(a) Tikhonov (b) Lavrentiev

Figure 3.2: The Effect of λ

3.4 The HR Method for Recovery of $\boldsymbol{x}_*$ from Discrete Filered Output $\boldsymbol{y}_*^h$

We argue that since the regularization *parameter* λ in the TLR methods presents a regularization *scale*, there should be successive extraction of information from the previous residual at a *finer* scale. Recall that in chapter 2, we propose the HR method to tackle the ill-posedness from an under-determined matrix A. Along the same line of reasoning, we propose the HR method as a multi-scale generalization to the TLR method to counter the ill-posedness from the discrete linear operator A^h such that the discrete approximation error will be improved thanks to a ladder of decreasing hierarchical scales. Furthermore, we exploit the filter property of the discrete Helmholtz filter A^h, i.e., the filtering radius δ. To simplify the notations, we will use the operator form of the TLR method. Let

$$\boldsymbol{x}^h_\lambda = ((1-\lambda)A^h + \lambda I)^{-1}\boldsymbol{y}^h_*,$$

since there is information left in the residual, $\boldsymbol{r}^h_\lambda = \boldsymbol{y}^h_* - \boldsymbol{x}^h_\lambda$, i.e., $\boldsymbol{r}^h_\lambda \neq \boldsymbol{0}$ (or we are done), we can extract further information from $\boldsymbol{r}^h_\lambda$ at a *finer* scale, say $\frac{\lambda}{2}$,

$$\boldsymbol{x}^h_{\frac{\lambda}{2}} = ((1-\frac{\lambda}{2})A^h + \frac{\lambda}{2}I)^{-1}\boldsymbol{r}^h_\lambda \quad \text{with} \quad \boldsymbol{r}^h_{\frac{\lambda}{2}} := \boldsymbol{r}^h_\lambda - A\boldsymbol{x}^h_{\frac{\lambda}{2}}.$$

We obtain with a two scale decomposition of $\boldsymbol{y}^h_*$, i.e., $\boldsymbol{y}^h_* = A\boldsymbol{x}^h_\lambda + A\boldsymbol{x}^h_{\frac{\lambda}{2}} + \boldsymbol{r}^h_{\frac{\lambda}{2}}$. The previous extraction process can continue. To simplify the notations, we will use numbered subscripts from now on. We start from $\boldsymbol{x}^h_{(1)} = \boldsymbol{x}^h_\lambda$, $\boldsymbol{r}^h_0 = \boldsymbol{y}^h_*$, and choose hierarchical scales $\lambda_j = 2^{1-j}\lambda$. The HR method will solve the following,

$$\boldsymbol{x}^h_{(j)} = ((1-\lambda_j)A^h + \lambda_j I)^{-1}\boldsymbol{r}^h_{j-1}, \quad \text{for } 1 \leq j \leq J.$$

The discrete hierarchical residual $\boldsymbol{r}^h_j$ satisfies a recursive relation: $\boldsymbol{r}^h_j = \boldsymbol{r}^h_{j-1} - A\boldsymbol{x}^h_{(j)}$. We sum up the discrete hierarchical terms $\boldsymbol{x}^h_{(j)}$'s, and obtain an approximate solution in the form of a discrete hierarchical sum, i.e., $\boldsymbol{X}^h_J = \sum_{j=1}^J \boldsymbol{x}^h_{(j)}$. The discrete hierarchical observation $A\boldsymbol{X}^h_J$ will provide a multi-scale description of $\boldsymbol{y}^h_*$ as follows,

$$\boldsymbol{y}^h_* = A\boldsymbol{x}^h_{(1)} + A\boldsymbol{x}^h_{(2)} + \ldots + A\boldsymbol{x}^h_{(J)} + \boldsymbol{r}^h_J.$$

It also follows that the discrete hierarchical sum $\boldsymbol{X}^h_J$, as the sum of the discrete hierarchical terms $\boldsymbol{x}^h_{(j)}$, i.e., $\boldsymbol{X}^h_J = \sum_{j=1}^J \boldsymbol{x}^h_{(j)}$, would provide a multi-scale approximation of $\boldsymbol{x}_*$, i.e.,

$$\boldsymbol{X}^h_J = \boldsymbol{x}^h_{(1)} + \boldsymbol{x}^h_{(2)} + \ldots + \boldsymbol{x}^h_{(J)} \approx \boldsymbol{x}_*.$$

We note that different ladders of the hierarchical scales can be employed, e.g., $\lambda_j = \theta^{j-1}\lambda_1$ for $0 < \theta < 1$ and a carefully chosen λ_1. Such *finitely many* scales are important in optimizing the approximation error, $\boldsymbol{X}_J^h - \boldsymbol{x}_*$. We summarize the HR method with a general ladder of decreasing hierarchical scales $\{\lambda_j\}_{j=1}^J$ as follows. Given the initial hierarchical residual $\boldsymbol{r}_0 = \boldsymbol{y}_*^h$, the HR method chooses a suitable starting hierarchical scale $\lambda_1 > 0$ and finds the first discrete hierarchical term $\boldsymbol{x}_{(1)}^h \in \mathcal{X}^h$ as the unique FEM solution such that for all $\boldsymbol{v}^h \in \mathcal{X}^h$ the following holds

$$\lambda\delta^2\langle\nabla\boldsymbol{x}_{(1)}^h, \nabla\boldsymbol{v}^h\rangle + \langle\boldsymbol{x}_{(1)}^h, \boldsymbol{v}^h\rangle = \delta^2\langle\nabla\boldsymbol{r}_0^h, \nabla\boldsymbol{v}^h\rangle + \langle\boldsymbol{r}_0^h, \boldsymbol{v}^h\rangle.$$

The first hierarchical residual $\boldsymbol{r}_1^h \in \mathcal{X}^h$ is found at the unique FEM solution such that for all $\boldsymbol{v}^h \in \mathcal{X}^h$ the following holds

$$\delta^2\langle\nabla\boldsymbol{r}_1^h, \nabla\boldsymbol{v}^h\rangle + \langle\boldsymbol{r}_1^h, \boldsymbol{v}^h\rangle = \delta^2\langle\nabla\boldsymbol{r}_0^h, \nabla\boldsymbol{v}^h\rangle + \langle\boldsymbol{r}_0^h - \boldsymbol{x}_{(j)}^h, \boldsymbol{v}^h\rangle.$$

With a carefully chosen a ladder of decreasing hierarchical scales $\{\lambda_j\}_{j=1}^J$, the HR method solves for the unique FEM solution $\boldsymbol{x}_{(j)}^h \in \mathcal{X}^h$ such that for $2 \le j \le J$ and all $\boldsymbol{v}^h \in \mathcal{X}^h$, the following holds

$$\lambda\delta^2\langle\nabla\boldsymbol{x}_{(j)}^h, \nabla\boldsymbol{v}^h\rangle + \langle\boldsymbol{x}_{(j)}^h, \boldsymbol{v}^h\rangle = \delta^2\langle\nabla\boldsymbol{r}_{j-1}^h, \nabla\boldsymbol{v}^h\rangle + \langle\boldsymbol{r}_{j-1}^h, \boldsymbol{v}^h\rangle. \tag{3.17}$$

Here the discrete hierarchical residual $\boldsymbol{r}_j^h \in \mathcal{X}^h$ is found as the FEM solution such that for all $\boldsymbol{v}^h \in \mathcal{X}^h$, the following recursive relationship holds

$$\delta^2\langle\nabla\boldsymbol{r}_j^h, \nabla\boldsymbol{v}^h\rangle + \langle\boldsymbol{r}_j^h, \boldsymbol{v}^h\rangle = \delta^2\langle\nabla\boldsymbol{r}_{j-1}^h, \nabla\boldsymbol{v}^h\rangle + \langle\boldsymbol{r}_{j-1}^h - \boldsymbol{x}_{(j)}^h, \boldsymbol{v}^h\rangle. \tag{3.18}$$

Remark 3.4.1. *Using the operator notation, the HR method is expressed as follows: find the unique FEM solution $\boldsymbol{x}_{(j)}^h \in \mathcal{X}^h$ satisfying*

$$((1-\lambda_j)A^h + \lambda_j I)\boldsymbol{x}_{(j)}^h = \boldsymbol{r}_{j-1}^h, \quad 1 \le j \le J. \tag{3.19}$$

For the discrete hierarchical residual $\boldsymbol{r}_j^h$, it satisfies the following recursion,

$$\boldsymbol{r}_j^h = \boldsymbol{r}_{j-1} - A^h \boldsymbol{x}_{(j)}^h. \tag{3.20}$$

The discrete hierarchical sum is the sum of the discrete hierarchical terms, i.e., $\boldsymbol{X}_J^h = \sum_{j=1}^J \boldsymbol{x}_{(j)}^h$. Before we present the discrete approximation error, $\boldsymbol{x}_* - \boldsymbol{X}_J^h$, we have to discuss the continuous approximation error, $\boldsymbol{x}_* - \boldsymbol{X}_J$, where $\boldsymbol{X}_J$ is obtained by applying the HR method to the original problem: recovery of $\boldsymbol{x}_*$ via de-convolution from the continuous filtered output $\boldsymbol{y}_*$. The HR method for the recovery of $\boldsymbol{x}_*$ from $\boldsymbol{y}_*$ is set up as follows. With a ladder of decreasing hierarchical scales $\{\lambda_j\}_{j=1}^J$, the HR method initially solves for the first hierarchical term $\boldsymbol{x}_{(1)} \in \mathcal{X}$ from the following

$$((1-\lambda_1)A_\delta + \lambda_1 I)\boldsymbol{x}_{(1)} = \boldsymbol{r}_0 \quad \text{and} \quad \boldsymbol{r}_0 = \boldsymbol{y}_*.$$

Setting $\boldsymbol{r}_1 = \boldsymbol{r}_0 - A\boldsymbol{x}_{(1)}$, the HR method solves for the hierarchical term $\boldsymbol{x}_{(j)} \in \mathcal{X}$ from the following

$$((1-\lambda_j)A_\delta + \lambda_j I)\boldsymbol{x}_{(j)} = \boldsymbol{r}_{j-1}, \quad \text{for } 2 \le j \le J, \tag{3.21}$$

with hierarchical residual $\boldsymbol{r}_j$ is given in a recursion, $\boldsymbol{r}_j = \boldsymbol{r}_{j-1} - A_\delta \boldsymbol{x}_{(j)}$. We sum up the hierarchical terms to obtain the hierarchical sum, i.e., $\boldsymbol{X}_J = \sum_{j=1}^J \boldsymbol{x}_{(j)}$. Next, we present the following definitions on the hierarchical de-convolution operators D_{λ_j} and D_J for convenience of notation.

Definition 3.4.2. *For $\lambda_j > 0$, define the j^{th} step hierarchical de-convolution operator with respect to λ_j as D_{λ_j}, i.e., $\boldsymbol{x}_{(j)} = D_{\lambda_j}\boldsymbol{r}_{j-1}$, in operator form, as*

$$D_{\lambda_j} = ((1-\lambda_j)A_\delta + \lambda_j I)^{-1} \tag{3.22}$$

For $J \geq 1$, define the total hierarchical operator with respect to J as D_J, i.e.,

$$\boldsymbol{X}_J = D_J \boldsymbol{y}_*. \tag{3.23}$$

The next lemma is concerned with the bounds on the operators D_{λ_j}, $D_{\lambda_j} A_\delta$ and $I - D_{\lambda_j} A_\delta$.

Lemma 3.4.3. *For $\lambda_j \in (0, 1]$, the operators, D_{λ_j}, $D_{\lambda_j} A_\delta$ and $I - D_{\lambda_j} A_\delta$, are bounded. In particular, they satisfy*

$$||D_{\lambda_j}||_{L^2(\Omega)} \leq \frac{1}{\lambda_j}, \quad ||D_{\lambda_j} A_\delta||_{L^2(\Omega)} \leq 1, \quad \textit{and} \quad ||I - D_{\lambda_j} A_\delta||_{L^2(\Omega)} < 1. \tag{3.24}$$

Proof. For any $\lambda \in (0, 1]$, the range of the function $g : (0, 1] \to \mathbb{R}$, defined as $g(x) = \big((1-\lambda)x+\lambda\big)^{-1}$, is $(0, \lambda^{-1}]$. The range of the function $h : (0, 1] \to \mathbb{R}$, defined as $h(x) = x\big((1-\lambda)x+\lambda\big)^{-1}$, is $(0, 1]$. Recall that $||A_\delta||_{L^2(\Omega)} \leq 1$. Since $D_{\lambda_j} = ((1-\lambda_j)A_\delta+\lambda_j I)^{-1}$, $||D_{\lambda_j}||_{L^2(\Omega)} \leq \lambda_j^{-1}$. Again, since $D_{\lambda_j} A_\delta = ((1-\lambda_j)A_\delta + \lambda_j I)^{-1} A_\delta$, $||D_{\lambda_j} A_\delta||_{L^2(\Omega)} \leq 1$. It follows that $||I - D_{\lambda_j} A_\delta||_{L^2(\Omega)} < 1$. □

Theorem 3.4.4. *For the de-convolution from $\boldsymbol{y}_*$, the regularization error, $\boldsymbol{e}_J = \boldsymbol{x}_* - \boldsymbol{X}_J$, is given by*

$$\boldsymbol{e}_J = (-\delta^2)^J \Big(\prod_{j=1}^{J} (\lambda_j D_{\lambda_j} A_\delta) \Big) \Delta^J \boldsymbol{x}_*. \tag{3.25}$$

We also have

$$||\boldsymbol{e}_J||_{L^2(\Omega)} \leq \delta^{2J} (\prod_{j=1}^{J} \lambda_j) ||\Delta^J \boldsymbol{x}_*||_{L^2(\Omega)}. \tag{3.26}$$

Proof. First, let $1 \leq j \leq J$, define $\boldsymbol{X}_j = D_j \boldsymbol{y}_*$. Realize that for $j \geq 2$, $\boldsymbol{X}_j - \boldsymbol{X}_{j-1} = \boldsymbol{x}_{(j)}$, and $\boldsymbol{X}_1 = \boldsymbol{x}_{(1)}$. Meanwhile, the hierarchical residual,

$$\boldsymbol{r}_j = \boldsymbol{r}_{j-1} - A_\delta \boldsymbol{X}_j = \boldsymbol{y}_* - A_\delta \boldsymbol{X}_j$$

For $j = 1$, we have

$$[(1-\lambda_1)A_\delta + \lambda_1 I]\boldsymbol{x}_* = (1-\lambda_1)\boldsymbol{y}_* + \lambda_1 \boldsymbol{x}_* \quad \text{and}$$

$$[(1-\lambda_1)A_\delta + \lambda_1 I]\boldsymbol{X}_1 = \boldsymbol{y}_*.$$

Taking difference of the previous equations, we have

$$\boldsymbol{e}_1 = \boldsymbol{x}_* - \boldsymbol{X}_1 = \lambda_1 D_{\lambda_1}(I - A_\delta)\boldsymbol{x}_* = \lambda_1 D_{\lambda_1} A_\delta (A_\delta^{-1} - I)\boldsymbol{x}_*.$$

For $j \geq 2$, we consider

$$[(1-\lambda_j)A_\delta + \lambda_j I](\boldsymbol{x}_* - \boldsymbol{x}_*) = \boldsymbol{y}_* - A_\delta \boldsymbol{x}_* \quad \text{and}$$

$$[(1-\lambda_j)A_\delta + \lambda_j I](\boldsymbol{X}_j - \boldsymbol{X}_{j-1}) = \boldsymbol{y}_* - A_\delta \boldsymbol{X}_{j-1}.$$

Taking difference of the previous equations, we obtain

$$\boldsymbol{e}_j = \lambda_j D_{\lambda_j}(I - A_\delta)\boldsymbol{e}_{j-1} = \lambda_j D_{\lambda_j} A_\delta (A_\delta^{-1} - I)\boldsymbol{e}_{j-1}.$$

Hence, by Math Induction together with $(A_\delta^{-1} - I)\boldsymbol{x}_* = -\delta^2 \Delta \boldsymbol{x}_*$, we arrive at the following

$$\boldsymbol{e}_J = (-\delta^2)^J \Big(\prod_{j=1}^{J} (\lambda_j D_{\lambda_j} A_\delta)(A_\delta^{-1} - I) \Big) \boldsymbol{x}_* = (-\delta^2)^J \Big(\prod_{j=1}^{J} (\lambda_j D_{\lambda_j} A_\delta) \Big) \Delta^J \boldsymbol{x}_*.$$

Regarding the bound on $\boldsymbol{e}_J$, we can use the bounds from lemma 3.4.3 to obtain

$$\begin{aligned} ||\boldsymbol{e}_J||_{L^2(\Omega)} &= ||(-\delta^2)^J \Big(\prod_{j=1}^{J} (\lambda_j D_{\lambda_j} A_\delta) \Big) \Delta^J \boldsymbol{x}_*||_{L^2(\Omega)} \\ &\leq \delta^{2J} \Big(\prod_{j=1}^{J} \lambda_j \Big) ||\Delta^J \boldsymbol{x}_*||_{L^2(\Omega)}. \end{aligned}$$

□

Remark 3.4.5. *The number of hierarchical iterations taken depends heavily on the regularity of the unknown $\boldsymbol{x}_*$, when the growth of $\Delta^J \boldsymbol{x}_*$ is controlled by the filtering radius δ, i.e., $||\delta^{2J}\Delta^J \boldsymbol{x}_*||_{L^2(\Omega)} < \infty$, we can take as many hierarchical iterations as possible. When the growth of $\Delta^J \boldsymbol{x}_*$ becomes impossible to manage by the filtering radius δ, we will pick the largest $J_{\max}$ such that $||\delta^{2J_{\max}}\Delta^{J_{\max}} \boldsymbol{x}_*||_{L^2(\Omega)} < \infty$.*

With the continuous approximation error bound established, we are now ready to show the discrete approximation error. We begin with the definition of two discrete deconvolution operators.

Definition 3.4.6. *For $\lambda_j > 0$, define the j^{th} step discrete hierarchical de-convolution operator with respect to λ_j as $D^h_{\lambda_j}$, i.e., $\boldsymbol{x}^h_{(j)} = D^h_{\lambda_j} \boldsymbol{r}^h_{j-1}$, in operator form,*

$$D^h_{\lambda_j} = ((1-\lambda_j)A^h + \lambda_j I)^{-1}.$$

For $J \geq 1$, define the total discrete hierarchical operator with respect to J as D^h_J, i.e.,

$$\boldsymbol{X}^h_J = D^h_J \boldsymbol{y}^h_*.$$

Theorem 3.4.7. *Let A^h be the discrete Helmholtz filter with filtering radius $\delta > 0$, and choose a ladder of decreasing scales $\{\lambda_j\}_{j=1}^J$. The discrete approximation error, $\boldsymbol{x}_* - \boldsymbol{X}^h_J$, where $\boldsymbol{X}^h_J = \sum_{j=1}^J \boldsymbol{x}^h_{(j)}$ with the discrete hierarchical term $\boldsymbol{x}^h_{(j)}$ in* (3.17)*, is bounded as follows,*

$$||\boldsymbol{x}_* - \boldsymbol{X}^h_J||_{L^2(\Omega)} \leq \delta^{2J}\Big(\prod_{j=1}^J \lambda_j\Big)||\Delta^J \boldsymbol{x}_*||_{L^2(\Omega)} + Ch^k \max_{1\leq j\leq J}(\sqrt{\lambda_j \delta^2} + h)||D_j A_\delta \boldsymbol{x}_*||_{H^{k+1}(\Omega)}.$$

Proof. Realizing $\boldsymbol{X}_J^h = D_J^h A^h \boldsymbol{x}_*$ and $\boldsymbol{X}_J = D_J A_\delta \boldsymbol{x}_*$, we break the discrete approximation error into two parts,

$$||\boldsymbol{x}_* - \boldsymbol{X}_J^h||_{L^2(\Omega)} \leq ||\boldsymbol{x}_* - \boldsymbol{X}_J||_{L^2(\Omega)} + ||\boldsymbol{X}_J - \boldsymbol{X}_J^h||_{L^2(\Omega)}. \tag{3.27}$$

The first half of (3.27) is bounded in theorem 3.4.4 as

$$||\boldsymbol{x}_* - \boldsymbol{X}_J||_{L^2(\Omega)} \leq \delta^{2J} \Big(\prod_{j=1}^{J} \lambda_j \Big) ||\Delta^J \boldsymbol{x}_*||_{L^2(\Omega)}.$$

For the second half of (3.27), we follow the ideas in [42] and choose $1 \leq j \leq J$, let $\boldsymbol{X}_j^h = D_j^h A^h \boldsymbol{x}_*$ and $\boldsymbol{X}_j = D_j A_\delta \boldsymbol{x}_*$. For all $\boldsymbol{v}^h \in \mathcal{X}^h$ and $2 \leq j \leq J$, we have

$$\lambda_j \delta^2 \langle \nabla(\boldsymbol{X}_j - \boldsymbol{X}_j^h), \nabla \boldsymbol{v}^h \rangle + \langle \boldsymbol{X}_j - \boldsymbol{X}_j^h, \boldsymbol{v}^h \rangle = \lambda_j \delta^2 \langle \nabla(\boldsymbol{X}_{j-1} - \boldsymbol{X}_{j-1}^h), \nabla \boldsymbol{v}^h \rangle.$$

The initial case when $j = 1$ follows similarly from [41]. We define

$$\boldsymbol{\eta}_j = \boldsymbol{X}_j - \boldsymbol{w}_j^h \quad \text{and} \quad \boldsymbol{\phi}_j^h = \boldsymbol{X}_j^h - \boldsymbol{w}_j^h$$

for some $\boldsymbol{w}_j^h \in \mathcal{X}^h$ to be chosen later for each $2 \leq j \leq J$. Using these definitions, for any $\boldsymbol{v}^h \in \mathcal{X}^h$, we have

$$\lambda_j \delta^2 \langle \nabla(\boldsymbol{\eta}_j - \boldsymbol{\phi}_j^h), \nabla \boldsymbol{v}^h \rangle + \langle \boldsymbol{\eta}_j - \boldsymbol{\phi}_j^h, \boldsymbol{v}^h \rangle = \lambda_j \delta^2 \langle \nabla(\boldsymbol{\eta}_{j-1} - \boldsymbol{\phi}_{j-1}^h), \nabla \boldsymbol{v}^h \rangle.$$

Set $\boldsymbol{v}^h = \boldsymbol{\phi}_j^h$ and let $\boldsymbol{d}_j^h = \boldsymbol{X}_j - \boldsymbol{X}_j^h = \boldsymbol{\eta}_j - \boldsymbol{\phi}_j^h$,

$$\begin{aligned}
\lambda_j \delta^2 ||\nabla \boldsymbol{\phi}_j^h||_{L^2(\Omega)}^2 + ||\boldsymbol{\phi}_j^h||_{L^2(\Omega)}^2 &= \lambda_j \delta^2 \langle \nabla \boldsymbol{\eta}_j, \nabla \boldsymbol{\phi}_j^h \rangle + \langle \boldsymbol{\eta}_j, \boldsymbol{\phi}_j^h \rangle \\
&\quad - \lambda_j \delta^2 \langle \nabla \boldsymbol{d}_{j-1}^h, \nabla \boldsymbol{\phi}_j^h \rangle \\
&\leq \lambda_j \delta^2 ||\nabla \boldsymbol{\eta}_j||_{L^2(\Omega)}^2 + \frac{\lambda_j \delta^2}{4} ||\nabla \boldsymbol{\phi}_j^h||_{L^2(\Omega)}^2 \\
&\quad + \frac{1}{2} ||\boldsymbol{\eta}_j||_{L^2(\Omega)}^2 + \frac{1}{2} ||\boldsymbol{\phi}_j^h||_{L^2(\Omega)}^2 \\
&\quad + \lambda_j \delta^2 ||\nabla \boldsymbol{d}_{j-1}^h||_{L^2(\Omega)}^2 + \frac{\lambda_j \delta^2}{4} ||\nabla \boldsymbol{\phi}_{j-1}^h||_{L^2(\Omega)}^2.
\end{aligned}$$

Keeping only terms with $||\nabla \boldsymbol{\phi}_j^h||^2_{L^2(\Omega)}$ and $||\boldsymbol{\phi}_j^h||^2_{L^2(\Omega)}$ on the left hand side, we obtain

$$\begin{aligned}\lambda_j\delta^2||\nabla \boldsymbol{\phi}_j^h||^2_{L^2(\Omega)} + ||\boldsymbol{\phi}_j^h||^2_{L^2(\Omega)} \leq 2\lambda_j\delta^2||\nabla \boldsymbol{\eta}_j||^2_{L^2(\Omega)} + ||\boldsymbol{\eta}_j||^2_{L^2(\Omega)} \\ + 2\lambda_j\delta^2||\nabla \boldsymbol{d}_{j-1}^h||^2_{L^2(\Omega)}.\end{aligned}$$

Using

$$||\nabla \boldsymbol{d}_j^h||_{L^2(\Omega)} \leq ||\nabla \boldsymbol{\eta}_j||_{L^2(\Omega)} + ||\nabla \boldsymbol{\phi}_j^h||_{L^2(\Omega)} \quad \text{and}$$

$$||\boldsymbol{d}_j^h||_{L^2(\Omega)} \leq ||\boldsymbol{\eta}_j||_{L^2(\Omega)} + ||\boldsymbol{\phi}_j^h||_{L^2(\Omega)},$$

we end up with the following recursion,

$$\begin{aligned}\lambda_j\delta^2||\nabla \boldsymbol{d}_j^h||^2_{L^2(\Omega)} + ||\boldsymbol{d}_j^h||^2_{L^2(\Omega)} \leq 3\lambda_j\delta^2||\nabla \boldsymbol{\eta}_j||^2_{L^2(\Omega)} + 2||\boldsymbol{\eta}_j||^2_{L^2(\Omega)} \\ + 2\lambda_j\delta^2||\nabla \boldsymbol{d}_{j-1}^h||^2_{L^2(\Omega)}.\end{aligned}$$

Therefore we have

$$||\boldsymbol{d}_J^h||_{L^2(\Omega)} \leq C \max_{1\leq j\leq J}(\sqrt{\lambda_j\delta^2}||\nabla \boldsymbol{\eta}_j||_{L^2(\Omega)} + ||\boldsymbol{\eta}_j||_{L^2(\Omega)}).$$

This inequality holds for any $\boldsymbol{w}_j^h \in \mathcal{X}^h$; Having taken the infimum over $\mathcal{X}^h$ and by (3.9), we arrive at the following,

$$||D_J A_\delta \boldsymbol{x}_* - D_J^h A^h \boldsymbol{x}_*||_{L^2(\Omega)} \leq Ch^k \max_{1\leq j\leq J}(\sqrt{\lambda_j\delta^2} + h)||D_j A_\delta \boldsymbol{x}_*||_{H^{k+1}(\Omega)}.$$

Combing the bounds for both parts, we prove our claim. □

Note that the number of hierarchical iterations taken, namely J, depends on the regularity of the unknown $\boldsymbol{x}_*$. When the term, $\sqrt{\lambda_j\delta^2}$, behaves roughly like h, we gain an extra degree of accuracy in the discrete approximation by using the HR method.

3.5 Recovery of $\boldsymbol{x}_*$ from Noisy Discrete Filtered Output $\boldsymbol{y}_*^{h,\varepsilon}$

When the residual stress tensor is approximated in some LES models, there is a modeling error $\boldsymbol{\epsilon}^h$ in the numerical output $\boldsymbol{y}_*^{h,\varepsilon}$, i.e., $\boldsymbol{y}_*^{h,\varepsilon} = A^h\boldsymbol{x}_* + \boldsymbol{\epsilon}^h$. Limited information is known about the modeling error $\boldsymbol{\epsilon}^h$, except that it is bounded, i.e., $||\boldsymbol{\epsilon}^h||_{L^2(\Omega)} \leq \varepsilon$. We analyze the approximation error when applying the HR method for finding an approximate solution to $\boldsymbol{x}_*$ via de-convolution from $\boldsymbol{y}_*^{h,\varepsilon}$. We start from a closed form expression for the discrete de-convolution operator D_J^h.

Proposition 3.5.1. *The discrete hierarchical sum after J steps of hierarchical iteration, namely $\boldsymbol{X}_J^h$, is given by*

$$\boldsymbol{X}_J^h = D_J^h \boldsymbol{y}_*^h = \Big(D_{\lambda_J}^h + \sum_{j=1}^{J-1}(\prod_{i=0}^{j-1}(I - D_{\lambda_{J-i}}^h A^h))D_{\lambda_{J-j}}^h\Big)\boldsymbol{y}_*^h, \quad \text{for } J \geq 2. \tag{3.28}$$

When $J = 1$, $\boldsymbol{X}_1^h = D_{\lambda_1}^h \boldsymbol{y}_*^h$.

Proof. The case for $J = 1$ is obvious. For any $J \geq 2$, we use the fact that $\boldsymbol{X}_J^h - \boldsymbol{X}_{J-1}^h = \boldsymbol{x}_{(J)}^h$, $\boldsymbol{r}_J^h = \boldsymbol{y}_*^h - \boldsymbol{X}_J^h$, and $\boldsymbol{X}_1^h = \boldsymbol{x}_{(1)}^h = D_{\lambda_1}^h \boldsymbol{y}_*^h$. Starting from the hierarchical sum after J steps (for $J > 1$),

$$\begin{aligned}
\boldsymbol{X}_J^h &= \boldsymbol{X}_{J-1}^h + D_{\lambda_J}^h(\boldsymbol{y}_*^h - A^h \boldsymbol{X}_{J-1}^h) = D_{\lambda_J}^h \boldsymbol{y}_*^h + (I - D_{\lambda_J}^h A^h)\boldsymbol{X}_{J-1}^h \\
&= D_{\lambda_J}^h \boldsymbol{y}_*^h + (I - D_{\lambda_J}^h A^h) D_{\lambda_{J-1}}^h \boldsymbol{y}_*^h + (I - D_{\lambda_J}^h A^h)(I - D_{\lambda_{J-1}}^h A^h)\boldsymbol{X}_{J-2}^h \\
&\vdots \quad \text{Inductively} \\
&= \Big(D_{\lambda_J}^h + \sum_{j=1}^{J-1}(\prod_{i=0}^{j-1}(I - D_{\lambda_{J-i}}^h A^h))D_{\lambda_{J-j}^h}\Big)\boldsymbol{y}_*^h.
\end{aligned}$$

□

The following lemmas provides the upper bounds on the operators: A^h, $D_{\lambda_j}^h$, and D_J^h.

Lemma 3.5.2. *The operators, $A^h : \mathcal{X} \to \mathcal{X}^h$, $D_{\lambda_j}^h : \mathcal{X}^h \to \mathcal{X}^h$, and $D_J^h : \mathcal{X}^h \to \mathcal{X}^h$ are* *bounded as follows,*

$$||A^h||_{L^2(\Omega)} \leq 1, \quad ||D^h_{\lambda_j}||_{L^2(\Omega)} \leq \frac{1}{\lambda_j}, \quad ||D^h_J||_{L^2(\Omega)} \leq \sum_{j=1}^{J} \frac{1}{\lambda_j}.,$$

Moreover, we also have

$$||I - D^h_{\lambda_j} A^h||_{L^2(\Omega)} \leq 1 \quad \textit{and} \quad ||D^h_{\lambda_j} A^h||_{L^2(\Omega)} \leq 1.$$

Proof. First, set $\boldsymbol{v}^h = \boldsymbol{y}^h_*$ in (3.6), then

$$||\boldsymbol{y}^h_*||_{L^2(\Omega)} \leq \delta^2 \langle \nabla \boldsymbol{y}^h_*, \nabla \boldsymbol{y}^h_* \rangle + \langle \boldsymbol{y}^h_*, \boldsymbol{y}^h_* \rangle = \langle \boldsymbol{x}_*, \boldsymbol{y}^h_* \rangle \quad \text{and}$$
$$\langle \boldsymbol{x}_*, \boldsymbol{y}^h_* \rangle \leq \frac{1}{2} ||\boldsymbol{y}^h_*||^2_{L^2(\Omega)} + \frac{1}{2} ||\boldsymbol{x}_*||^2_{L^2(\Omega)}.$$

We have $||\boldsymbol{y}^h_*||_{L^2(\Omega)} \leq ||\boldsymbol{x}_*||_{L^2(\Omega)}$, thus $||A^h||_{L^2(\Omega)} \leq 1$. Next, in the operator form, $D^h_{\lambda_j} = ((1-\lambda_j)A^h + \lambda_j I)^{-1}$, which is a convex combination of A^h and I, along the line of reasoning in lemma 3.4.3, we have $||D^h_{\lambda_j}||_{L^2(\Omega)} \leq \lambda_j^{-1}$. Similarly $D^h_{\lambda_j} A^h = ((1 - \lambda_j)A^h + \lambda_j I)^{-1} A^h$, we have $||D^h_{\lambda_j} A^h||_{L^2(\Omega)} \leq 1$. Therefore, $||I - D^h_{\lambda_{J-k}} A^h||_{L^2(\Omega)} < 1$. For the last inequality, using proposition 3.5.1, we have $D^h_J = D^h_{\lambda_J} + \sum_{j=1}^{J-1} (\prod_{i=0}^{j-1} (I - D^h_{\lambda_{J-i}} A^h)) D^h_{\lambda_{J-j}}$. For any $\boldsymbol{v}^h \in \mathcal{X}^h$, we derive the following,

$$||D^h_J \boldsymbol{v}^h||_{L^2(\Omega)} = ||\Big(D^h_{\lambda_J} + \sum_{j=1}^{J-1} (\prod_{i=0}^{j-1} (I - D^h_{\lambda_{J-i}} A^h)) D^h_{\lambda_{J-j}}\Big) \boldsymbol{v}^h||_{L^2(\Omega)}$$
$$\leq \sum_{j=1}^{J} \frac{1}{\lambda_j} ||\boldsymbol{v}^h||_{L^2(\Omega)},$$

hence $||D^h_J||_{L^2(\Omega)} \leq \sum_{j=1}^{J} \lambda_j^{-1}$. □

Theorem 3.5.3. *If the noise $\boldsymbol{\epsilon}^h \in \mathcal{X}^h$ is bounded with $||\boldsymbol{\epsilon}^h||_{L^2(\Omega)} \leq \varepsilon$ for some noise level $\varepsilon > 0$, the discrete approximation error, $\boldsymbol{x}_* - \boldsymbol{X}^{h,\varepsilon}_J$, where $\boldsymbol{X}^{h,\varepsilon}_J = D^h_J \boldsymbol{y}^{h,\varepsilon}_*$, is bounded as* *follows*

$$||\boldsymbol{x}_* - \boldsymbol{X}_J^{h,\varepsilon}||_{L^2(\Omega)} \leq \delta^{2J}\Big(\prod_{j=1}^{J}\lambda_j\Big)||\Delta^J \boldsymbol{x}_*||_{L^2(\Omega)}$$
$$+ Ch^k \max_{1\leq j\leq J}(\sqrt{\lambda_j\delta^2}+h)||D_j A_\delta \boldsymbol{x}_*||_{H^{k+1}(\Omega)} + \varepsilon\sum_{j=1}^{J}\frac{1}{\lambda_j}.$$

Proof. With the presence of noise, we have the discrete hierarchical sum expressed in terms of $\boldsymbol{y}_*^{h,\varepsilon}$, i.e., $\boldsymbol{X}_J^{h,\varepsilon} = D_J^h \boldsymbol{y}_*^{h,\varepsilon} = D_J^h(A^h\boldsymbol{x}_* + \boldsymbol{\epsilon}^h)$, therefore, using theorem 3.4.7, we have

$$\begin{aligned}||\boldsymbol{x}_* - \boldsymbol{X}_J^{h,\varepsilon}||_{L^2(\Omega)} &= ||\boldsymbol{x}_* - D_J^h A^h \boldsymbol{x}_* - D_J^h \boldsymbol{\epsilon}^h||_{L^2(\Omega)} \\ &\leq ||\boldsymbol{x}_* - D_J^h A^h \boldsymbol{x}_*||_{L^2(\Omega)} + ||D_J^h\boldsymbol{\epsilon}^h||_{L^2(\Omega)} \\ &\leq \delta^{2J}\Big(\prod_{j=1}^{J}\lambda_j\Big)||\Delta^J\boldsymbol{x}_*||_{L^2(\Omega)} + \varepsilon\sum_{j=1}^{J}\frac{1}{\lambda_j} \\ &\quad + Ch^k \max_{1\leq j\leq J}(\sqrt{\lambda_j\delta^2}+h)||D_j A_\delta\boldsymbol{x}_*||_{H^{k+1}(\Omega)}.\end{aligned}$$

□

Given the bound in theorem 3.5.3, we note that there is an amplification of noise when using the HR method. In order to balance the effect of noise amplification on the approximation error bound, we suggested an (near) optimal stopping criteria for the hierarchical iterations. We plan to investigate the condition for finding the (near) optimal stopping criteria for the HR method when $\boldsymbol{y}_*^{h,\varepsilon}$ is provided. We start from the following functional.

$$E_0(\boldsymbol{v}^h) = \frac{1}{2}\langle A^h\boldsymbol{v}^h, \boldsymbol{v}^h\rangle - \langle \boldsymbol{y}_*^{h,\varepsilon}, \boldsymbol{v}^h\rangle, \quad \text{for } \boldsymbol{v}^h \in \mathcal{X}^h. \tag{3.29}$$

Remark 3.5.4. *Consider the following minimization problem,*

$$\boldsymbol{w}^h = \underset{\boldsymbol{v}^h\in\mathcal{X}^h}{\arg\min}\Big\{E_0(\boldsymbol{v}^h)\Big\}. \tag{3.30}$$

The solution $\boldsymbol{w}^h$ *of* (3.30) *also satisfies the linear equation* $A^h \boldsymbol{w}^h = \boldsymbol{y}_*^{h,\varepsilon}$.

Now we discuss the condition on the hierarchical scale λ_j's so that the sequence of discrete hierarchical partial sums, $\{\boldsymbol{X}_j^{h,\varepsilon}\}_{j=1}^{\infty}$, will form a decreasing sequence for E_0.

Proposition 3.5.5. *Let* A^h *be the discrete Helmholtz filter, when* $0 < \lambda_j \leq \frac{1}{2}$ *for any* $1 \leq j < \infty$*, the Hierarchical partial sums,* $\boldsymbol{X}_j^{h,\varepsilon} = \sum_{i=1}^{j} \boldsymbol{x}_{(i)}^{h,\varepsilon} = D_j^h \boldsymbol{y}_*^{h,\varepsilon}$*, after* j *steps of HR iterations are a decreasing sequence for* E_0*, in particular,*

$$\begin{aligned} E_0(\boldsymbol{X}_{j-1}^{h,\varepsilon}) - E_0(\boldsymbol{X}_j^{h,\varepsilon}) &= \langle \Big((\frac{1}{2} - \lambda_j) A^h + \lambda_j I \Big) (\boldsymbol{X}_j^{h,\varepsilon} - \boldsymbol{X}_{j-1}^{h,\varepsilon}), \boldsymbol{X}_j^{h,\varepsilon} - \boldsymbol{X}_{j-1}^{h,\varepsilon} \rangle \\ &\geq 0, \end{aligned} \tag{3.31}$$

with the equality is achieved if and only if $\boldsymbol{X}_j^{h,\varepsilon} = \boldsymbol{X}_{j-1}^{h,\varepsilon}$.

Proof. We will expand the energy functional E_0 using its definition and cancel terms to prove the identity along with the fact that A^h is SPD over $\mathcal{X}^h$,

$$
\begin{aligned}
E_0(\boldsymbol{X}_{j-1}^{h,\varepsilon}) - E_0(\boldsymbol{X}_j^{h,\varepsilon}) &= \frac{1}{2}\langle A^h \boldsymbol{X}_{j-1}^{h,\varepsilon}, \boldsymbol{X}_{j-1}^{h,\varepsilon}\rangle - \frac{1}{2}\langle A^h \boldsymbol{X}_j^{h,\varepsilon}, \boldsymbol{X}_j^{h,\varepsilon}\rangle \\
&\quad - \langle \boldsymbol{y}_*^{h,\varepsilon}, \boldsymbol{X}_{j-1}^{h,\varepsilon} - \boldsymbol{X}_j^{h,\varepsilon}\rangle \\
&= \frac{1}{2}\langle A^h \boldsymbol{X}_{j-1}^{h,\varepsilon}, \boldsymbol{X}_{j-1}^{h,\varepsilon} - \boldsymbol{X}_j^{h,\varepsilon}\rangle + \frac{1}{2}\langle A^h (\boldsymbol{X}_{j-1}^{h,\varepsilon} - \boldsymbol{X}_j^{h,\varepsilon}), \boldsymbol{X}_j^{h,\varepsilon}\rangle \\
&\quad - \langle \boldsymbol{y}_*^{h,\varepsilon}, \boldsymbol{X}_{j-1}^{h,\varepsilon} - \boldsymbol{X}_j^{h,\varepsilon}\rangle \\
&= \frac{1}{2}\langle A^h (\boldsymbol{X}_{j-1}^{h,\varepsilon} - \boldsymbol{X}_j^{h,\varepsilon}), \boldsymbol{X}_{j-1}^{h,\varepsilon} + \boldsymbol{X}_j^{h,\varepsilon}\rangle \\
&\quad - \langle \Big((1-\lambda_j)A^h + \lambda_j I\Big)(\boldsymbol{X}_j^{h,\varepsilon} - \boldsymbol{X}_{j-1}^{h,\varepsilon}), \boldsymbol{X}_{j-1}^{h,\varepsilon} - \boldsymbol{X}_j^{h,\varepsilon}\rangle \\
&\quad - \langle A^h \boldsymbol{X}_{j-1}^{h,\varepsilon}, \boldsymbol{X}_{j-1}^{h,\varepsilon} - \boldsymbol{X}_j^{h,\varepsilon}\rangle \\
&= \frac{1}{2}\langle A^h (\boldsymbol{X}_{j-1}^{h,\varepsilon} - \boldsymbol{X}_j^{h,\varepsilon}), \boldsymbol{X}_{j-1}^{h,\varepsilon} - \boldsymbol{X}_j^{h,\varepsilon}\rangle \\
&\quad - \lambda_j \langle (I - A^h)(\boldsymbol{X}_j^{h,\varepsilon} - \boldsymbol{X}_{j-1}^{h,\varepsilon}), \boldsymbol{X}_{j-1}^{h,\varepsilon} - \boldsymbol{X}_j^{h,\varepsilon}\rangle \\
&= \langle \Big(\frac{1}{2} - \lambda_j)A^h + \lambda_j I\Big)(\boldsymbol{X}_j^{h,\varepsilon} - \boldsymbol{X}_{j-1}^{h,\varepsilon}), \boldsymbol{X}_j^{h,\varepsilon} - \boldsymbol{X}_{j-1}^{h,\varepsilon}\rangle.
\end{aligned}
$$

The difference in (3.31) stays positive as long as $0 < \lambda_j \leq \frac{1}{2}$ for any $1 \leq j < \infty$, thus $E_0(\boldsymbol{X}_{j-1}^{h,\varepsilon}) > E_0(\boldsymbol{X}_j^{h,\varepsilon})$ unless $\boldsymbol{X}_j^{h,\varepsilon} = \boldsymbol{X}_{j-1}^{h,\varepsilon}$. □

From the definition of the HR method, we note that $\boldsymbol{X}_j^{h,\varepsilon} = \boldsymbol{X}_{j-1}^{h,\varepsilon}$ if and only if $\boldsymbol{x}_{(j)}^{h,\varepsilon} = \mathbf{0}$. It also follows that $A^h \boldsymbol{X}_j^{h,\varepsilon} = \boldsymbol{y}_*^{h,\varepsilon}$. However, such convergence is not desired. It is crucial for the HR method to stop after performing a certain number of hierarchical iterations since we seek an approximate solution $\boldsymbol{X}_j^{h,\varepsilon} \to \boldsymbol{x}_*$ as $j \to \infty$. Consider the following noisy functional,

$$
E_\varepsilon(\boldsymbol{v}^h) = \frac{1}{2}\langle A^h \boldsymbol{v}^h, \boldsymbol{v}^h\rangle - \langle \boldsymbol{y}_*^{h,\varepsilon} - \boldsymbol{\epsilon}^h, \boldsymbol{v}^h\rangle, \quad \text{for any } \boldsymbol{v}^h \in \mathcal{X}^h. \tag{3.32}
$$

Remark 3.5.6. *Consider the solution from the following minimization problem,*

$$\boldsymbol{w}^h = \underset{\boldsymbol{v}^h \in \mathcal{X}^h}{\arg\min} \Big\{ E_\varepsilon(\boldsymbol{v}^h) \Big\}. \tag{3.33}$$

The solution $\boldsymbol{w}^h$ *satisfies the linear equation* $A^h \boldsymbol{w}^h = \boldsymbol{y}_*^\varepsilon - \boldsymbol{\epsilon}^h = A^h \boldsymbol{x}_*$. *Moreover,* $E_\varepsilon(\boldsymbol{v}^h) = E_0(\boldsymbol{v}^h) - \langle \boldsymbol{\epsilon}^h, \boldsymbol{v}^h \rangle$. *Having expanded the difference,* $E_\varepsilon(\boldsymbol{X}_{j-1}^{h,\varepsilon}) - E_\varepsilon(\boldsymbol{X}_j^{h,\varepsilon})$, *we have*

$$E_\varepsilon(\boldsymbol{X}_{j-1}^{h,\varepsilon}) - E_\varepsilon(\boldsymbol{X}_j^{h,\varepsilon}) = E_0(\boldsymbol{X}_{j-1}^{h,\varepsilon}) - E_0(\boldsymbol{X}_j^{h,\varepsilon}) - \langle \boldsymbol{\epsilon}^h, \boldsymbol{X}_{j-1}^{h,\varepsilon} - \boldsymbol{X}_j^{h,\varepsilon} \rangle.$$

Next, we discuss the condition on the hierarchical scale λ_j's so that the hierarchical partial sums, $\{\boldsymbol{X}_j^{h,\varepsilon}\}_{j=1}^\infty$, in the functional E_ε will form a decreasing sequence.

Theorem 3.5.7. *Let* A^h *be the discrete Helmholtz filter and suppose the modeling error is bounded above as* $||\boldsymbol{\epsilon}||_{L^2(\Omega)} \leq \varepsilon$ *for some known noise level* $\varepsilon > 0$. *The discrete hierarchical partial sums from the HR method form a decreasing sequence for the energy functional* E_ε *as long as the following inequalities hold*

$$\frac{\varepsilon}{||\boldsymbol{x}_{(j)}^{h,\varepsilon}||_2} \leq \lambda_j \leq \frac{1}{2}, \quad \textit{for } j \geq 2. \tag{3.34}$$

Proof. We start from the Cauchy-Schwartz inequality, $\big|\langle \boldsymbol{\epsilon}^h, \boldsymbol{x}_{(j)}^{h,\varepsilon} \rangle\big| \leq \varepsilon ||\boldsymbol{x}_{(j)}^{h,\varepsilon}||_{L^2(\Omega)}$. When (3.34) holds, we have

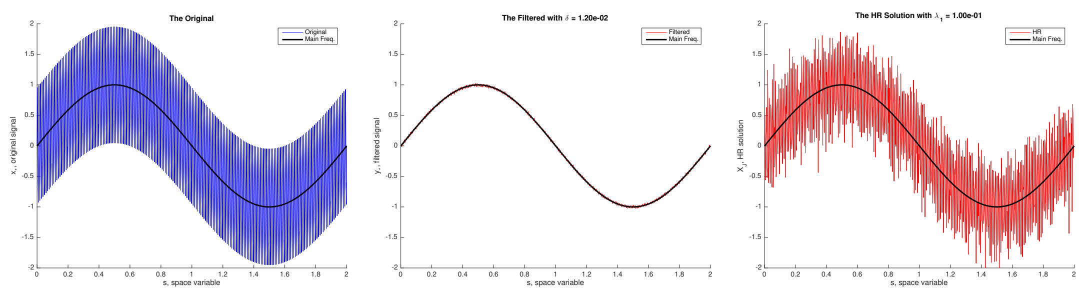
$$\begin{aligned}
0 &\leq \varepsilon ||\boldsymbol{x}_{(j)}^{h,\varepsilon}||_{L^2(\Omega)} - \big|\langle \boldsymbol{\epsilon}^h, \boldsymbol{x}_{(j)}^{h,\varepsilon} \rangle\big| \leq \lambda_j ||\boldsymbol{x}_{(j)}^{h,\varepsilon}||_{L^2(\Omega)}^2 - \big|\langle \boldsymbol{\epsilon}^h, \boldsymbol{x}_{(j)}^{h,\varepsilon} \rangle\big| \\
&\leq \langle \Big((\frac{1}{2} - \lambda_j) A^h + \lambda_j I \Big) (\boldsymbol{X}_j^{h,\varepsilon} - \boldsymbol{X}_{j-1}^{h,\varepsilon}), \boldsymbol{X}_j^{h,\varepsilon} - \boldsymbol{X}_{j-1}^{h,\varepsilon} \rangle - \big|\langle \boldsymbol{\epsilon}^h, \boldsymbol{X}_j^{h,\varepsilon} - \boldsymbol{X}_{j-1}^{h,\varepsilon} \rangle\big| \\
&\leq \langle \Big((\frac{1}{2} - \lambda_j) A^h + \lambda_j I \Big) (\boldsymbol{X}_j^{h,\varepsilon} - \boldsymbol{X}_{j-1}^{h,\varepsilon}), \boldsymbol{X}_j^{h,\varepsilon} - \boldsymbol{X}_{j-1}^{h,\varepsilon} \rangle + \langle \boldsymbol{\epsilon}^h, \boldsymbol{X}_j^{h,\varepsilon} - \boldsymbol{X}_{j-1}^{h,\varepsilon} \rangle \\
&= E_\varepsilon(\boldsymbol{X}_{j-1}^{h,\varepsilon}) - E_\varepsilon(\boldsymbol{X}_j^{h,\varepsilon}).
\end{aligned}$$

□

Theorem 3.5.7 indicates that when the size of the updates is larger than twice the noise, then the updates move the approximations closer to the desired unknown $\boldsymbol{x}_*$. As the updates become smaller and smaller, $\boldsymbol{X}_j^{h,\varepsilon}$ begins to accumulate more noise, unless the resolution scale λ_j is larger then the ratio between the noise level and the update, which prompts us to consider stopping the hierarchical update.

3.6 Numerical Experiements

First, we verify the optimal stopping criterion (Theorem 3.5.7) in MATLAB (version: $R2015b$) with the following details: first we choose a true solution to be $\boldsymbol{x}_* = \sin(\pi s) + \sin(200\pi s)$, plotted in Figure 3.3, over the interval $[0, 2]$.



(a) Original (b) Filtered (c) HR

Figure 3.3: Original Signal: $\boldsymbol{x}_* = \sin(\pi s) + \sin(200\pi s)$.

We discretize the interval with a step size of $h = \frac{2}{1000}$ (hence 1001 sample points) and choose the filtering radius for the Helmholtz filter to be $\delta = 6h$. To implement the discrete Helmholtz filter, we begin with approximating the Laplace operator with a center differencing scheme,

$$\Delta \boldsymbol{x}_* \approx \Delta^h \boldsymbol{x}_* = \frac{\boldsymbol{x}_*(s-h) - 2\boldsymbol{x}_*(s) + \boldsymbol{x}_*(s+h)}{h^2}.$$

Define the discrete operator $(A^h)^{-1}$ as

$$(A^h)^{-1}\boldsymbol{x}_* = -\delta^2 \Delta^h \boldsymbol{x}_* + \boldsymbol{x}_*.$$

Our simulated data was obtained by filtering the true solution and adding 1% (the noise to signal ratio) random noise to the filtered data, that is, $\boldsymbol{y}_*^{h,\varepsilon} = A^h \boldsymbol{x}_* + \boldsymbol{\epsilon}^h$ where $||\boldsymbol{\epsilon}^h||_{L^2(\Omega)} = 0.01 * ||A^h \boldsymbol{x}_*||_{L^2(\Omega)}$ ($\boldsymbol{\epsilon}^h$ is generated in MATLAB using the command "randn", and normalized to have a unit L^2 norm). For calculating the L^2 norm of a function g over $[a, b]$, we use either the composite Trapezoidal rule or the composite Simpson's rule. We select the initial hierarchical scale $\lambda_1 = 0.1$ and the hierarchical multiplier $\theta = 0.1$. To calculate λ_j, we use the formula $\lambda_j = \theta^{j-1}\lambda_1$ for $j \geq 2$. We use the following guideline for finding the optimal stopping J:

Step 1: At the j^{th} iterate, except the initial iterate, we calculate $||\boldsymbol{x}_{(j)}^{h,\varepsilon}||_{L^2(\Omega)}$.

Step 2: We then compare λ_j to $\varepsilon/||\boldsymbol{x}_{(j)}^{h,\varepsilon}||_{L^2(\Omega)}$.

Step 3: According to theorem 3.5.7 : if $\lambda_j \geq \varepsilon/||\boldsymbol{x}_{(j)}^{h,\varepsilon}||_{L^2(\Omega)}$, we proceed to next iterate; otherwise, we stop the iteration.

The actual simulation which we did for this demonstration, on the other hand, will not stop once we find the stopping J_{opt}; instead the (near) optimal J_{opt} will be recorded. Figure 3.4 shows the noisy functional, E_ε, calculated with the HR partial sum $\boldsymbol{X}_j^{h,\varepsilon}$'s. The calculated (near) optimal stopping point (via theorem 3.5.7) occurs after $J_{opt} = 2$ iteration steps and is shown as a solid green dot. The figure shows that the algorithm stops right where functional reaches its minimum and starts increasing again, hence avoiding the convergence to the noisy solution. However, because we do not have precise information of the noise (as it is always the case in real life applications), the algorithm will always try to stop before the functional reaches its minimum.

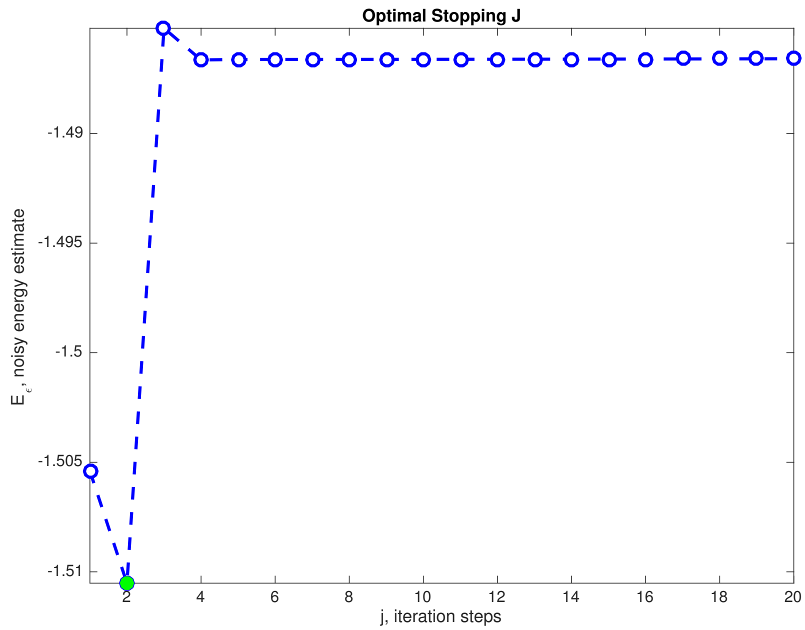
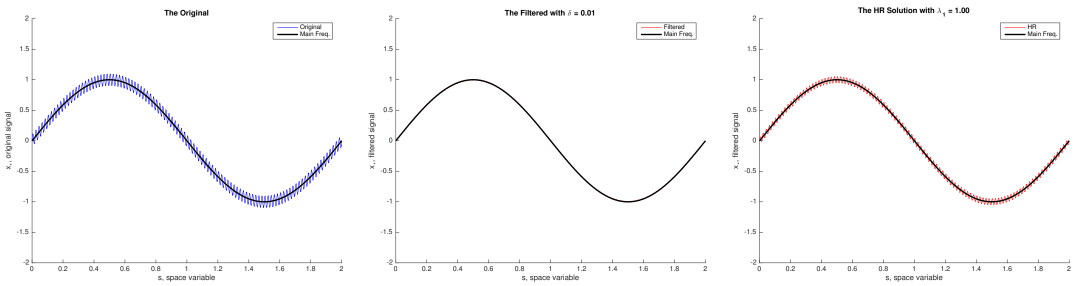


Figure 3.4: Noisy functional E_ε calculated for for $1 \leq j \leq 20$.

We check the efficiency of the HR method by comparing the relative error of a solution for a given initial λ_1 (θ is set at 0.1) to the relative errors found with ℓ_2TR method and TLR method with $\lambda = \lambda_1$. We implement the codes in MATLAB (version $R2015b$) with the following details: we start out with the original data as $\boldsymbol{x}_* = \sin(\pi s) + 0.1\sin(100\pi s)$, with 1001 sample points taken over the interval $[0, 2]$, see figure 3.5; hence the step size is $h = \frac{2}{1000}$. We set our filtering radius at $\delta = 0.01$. We create 101 sample points of the λ_1 from $[10^{-1.2}, 1]$ and calculate 2 steps the HR method (due to the possible regularity issue of the original $\boldsymbol{x}_*$). For an approximation $\tilde{\boldsymbol{x}}$ to a desired solution $\boldsymbol{x}_*$, the relative error, $\boldsymbol{e}_{rel}$, is defined as, $\boldsymbol{e}_{rel} = ||\boldsymbol{x}_* - \tilde{\boldsymbol{x}}||_{L^2(\Omega)}/||\boldsymbol{x}_*||_{L^2(\Omega)}$.



(a) Original (b) Filtered (c) HR

Figure 3.5: Original Signal: $\boldsymbol{x}_* = \sin(\pi s) + 0.1\sin(100\pi s)$.

The results are shown in Figures 3.6.

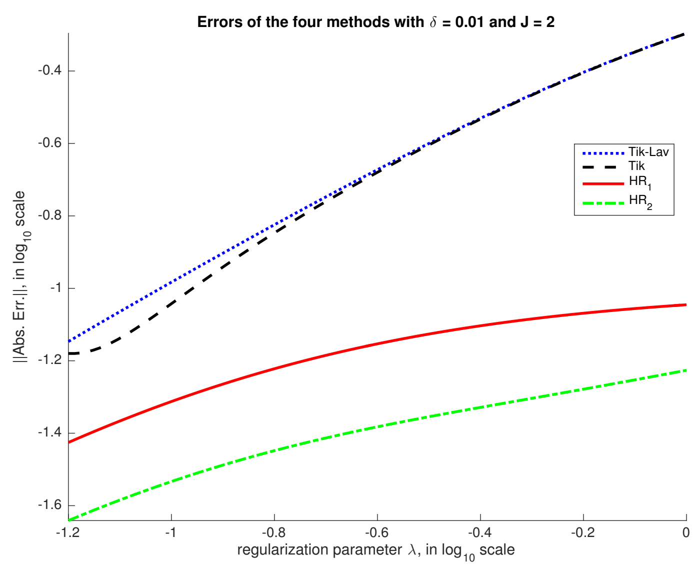


Figure 3.6: Comparison of the 4 regularization methods

As shown in the figure, the ℓ_2TR method and TLR method perform roughly the same. The HR method with only 1 hierarchical iteration (HR_1) taken performs superior than the two methods (from -1.2 to -1.4); the HR method with 2 hierarchical iterations (HR_1) improves from HR_1 (from -1.4 to -1.6). Next, we calculate the convergence rates of the HR method to verify the convergence rates predicted in theorem 3.4.7. We take a

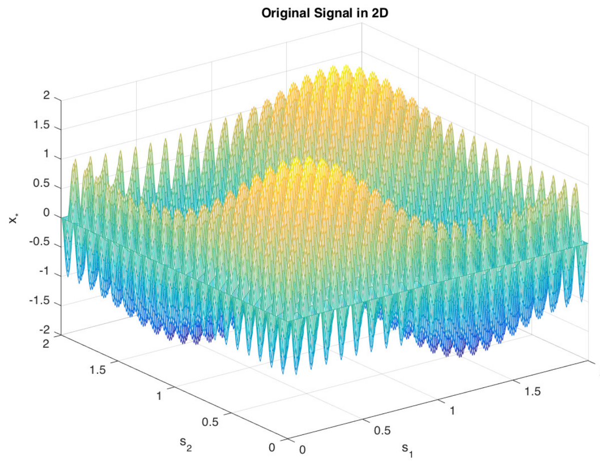


Figure 3.7: $\boldsymbol{x}_* = \sin(\pi s_1)\sin(\pi s_2) + \sin(20\pi s_1)\sin(20\pi s_2)$.

true solution over the domain $[0, 2] \times [0, 2]$ of

$$\boldsymbol{x}_* = \sin(\pi s_1)\sin(\pi s_2) + \sin(20\pi s_1)\sin(20\pi s_2).$$

see Figure 3.7. We discretize using the square command in FreeFEM++ [31] with n intervals in each of x and y coordinates and use piece-wise continuous linear polynomials. We use a filter radius of $\delta = 0.1(\frac{2\pi}{n})^{1/4} = \mathcal{O}(h^{0.25})$ and regularization parameter $\lambda_1 = 0.1(\frac{2\pi}{n})^{1/2} = \mathcal{O}(h^{0.5})$ with $\theta = 0.9$. And the results are presented in the following table.

Table 3.1: Convergence rates for the HR method.

n	L^2 error	rate	H^1 error	rate
60	$7.28251e-05$		27.7141	
120	$6.58783e-07$	110.5449	10.2005	2.7169
240	$4.52863e-07$	1.4547	2.65976	3.8351
480	$4.52255e-08$	10.0134	0.729082	3.6481
960	$3.52947e-09$	12.8137	0.19551	3.7291

The L^2 error is predicted to be of the order of h^3, since we used degree $k = 2$ for the basis function in FreeFEM++. In the case of the HR method, it shows super convergence since the convergence rate is reported at more than a factor of eight. For the H^1 error, the error is predicted to have a rate of h^2. The HR method is shown to have roughly a decaying factor of four.

3.7 Conclusion

We discussed the recovery of the unknown $\boldsymbol{x}_*$ via de-convolution on the Helmholtz filter from there different kinds of filtered quantity: the original filtered output $\boldsymbol{y}_*^h$, the discrete filtered output $\boldsymbol{y}_*^h$, and the noisy discrete filtered output $\boldsymbol{y}_*^{h,\varepsilon}$. We analyzed the application of the HR method on finding an approximate solution to $\boldsymbol{x}_*$ via de-convolution from three kinds of filtered output. We concluded that the HR method provided a much better approximation error on de-convolution from $\boldsymbol{y}_*$ and $\boldsymbol{y}_*^h$, when compared to the two Tikhonov regularization methods. When the noisy discrete filtered output $\boldsymbol{y}_*^{h,\varepsilon}$ is provided, we also supplied an (near) optimal stopping criteria for the HR method to stop the hierarchical iterations before the convergence sequence of approximation solutions from the HR method deviates from its convergent path to $\boldsymbol{x}_*$. We noted that our discrete approximation error analysis can be applied to other discretization scheme, such as the Finite Difference Method, as long as the bound on $D_j A_\delta \boldsymbol{x}_* - D_j^h A^h \boldsymbol{x}_*$ can be derived from such discretization scheme. Moreover, our analysis can be extended to other LES filters, especially for recovery of $\boldsymbol{x}_*$ via the de-convolution from $\boldsymbol{y}_*$, the analysis is done using only the upper bound of the Helmholtz filter A in L^2 norm, regardless of the specific definition of the Helmholtz filter A_δ.

Chapter 4: Hierarchical Reconstruction Method for Linear Regression

4.1 Introduction: Linear Regression

We study two inverse problems from Linear Regression. When given the data from a set of M observations: $\{A, \boldsymbol{y}_*^\varepsilon\}$ ($A \in \mathbb{R}^{M\times N}$ and $\boldsymbol{y}_*^\varepsilon \in \mathbb{R}^M$), where the i^{th} row of A and the i^{th} entry of $\boldsymbol{y}_*^\varepsilon$ are the regressors and response for the i^{th} observation respectively. With the regressor matrix A being over-determined ($M \gg N$) and $\boldsymbol{y}_*^\varepsilon$ not in the range of A, finding a linear model $\boldsymbol{x}_* \in \mathbb{R}^N$ satisfying the linear equation $A\boldsymbol{x} = \boldsymbol{y}_*^\varepsilon$ exactly is impossible. Instead of finding an exact solution to the equation $A\boldsymbol{x} = \boldsymbol{y}_*^\varepsilon$, we seek a most suitable linear model $\boldsymbol{x}_*$ such that $A\boldsymbol{x}_* \approx \boldsymbol{y}_*^\varepsilon$, where the modeling error, $\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}_*$, is well-controlled, such as being smallest in ℓ_2 norm. The Least Square (LS) method provides a linear model from the following

$$\boldsymbol{x}_{LS} = \underset{\boldsymbol{x}\in\mathbb{R}^N}{\arg\min} \left\{ ||\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}||_2 \right\}.$$

The LS linear model comes with a modeling error, $\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}_{LS}$, being smallest in the ℓ_2 norm. However, the LS linear model, $\boldsymbol{x}_{LS}$, is never sparse. Since sparse linear models have few non-zero entries making them easier to interprete, these sparse linear models are sought after in this chapter. In order to provide a sparse linear model through a continuous selection process, the Least Absolute Shrinkage and Selection Operator (LASSO) method

[56] produces a sparse linear model, $\boldsymbol{x}_{LA(\lambda)}$, via the usage of a ℓ_1 constraint or a ℓ_1 penalty function. The constrained LASSO method finds a linear model from a convex feasible set $\mathcal{B}_\lambda = \{\boldsymbol{x} \in \mathbb{R}^N \Big| ||\boldsymbol{x}||_1 \leq \lambda\}$ such that $\boldsymbol{x}_{LA(\lambda)}$ satisfies the following

$$\boldsymbol{x}_{LA(\lambda)} = \underset{\boldsymbol{x} \in \mathcal{B}_\lambda}{\arg\min} \Big\{ ||\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}||_2 \Big\}.$$

The penalized LASSO method finds a linear model satisfying

$$\boldsymbol{x}_{LA(\lambda)} = \underset{\boldsymbol{x} \in \mathbb{R}^N}{\arg\min} \Big\{ \lambda||\boldsymbol{x}||_1 + \frac{1}{2}||\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}||_2^2 \Big\}.$$

However, choosing a suitable λ is significant in obtaining a useful linear model $\boldsymbol{x}_{LA(\lambda)}$. Considering that the regularization *parameter* λ controls the distance between $\boldsymbol{x}_{LA(\lambda)}$ and $\boldsymbol{x}_{LS}$, we propose the Hierarchical Reconstruction (HR) method as an multi-scale generalization to the LASSO method. Recall that in previous chapters, the HR method was used to provide approximate solutions to linear equation either with an under-determined matrix or a linear operator with an eigenvalue of 0 as the ill-posed operator. In this chapter, the HR method is used to tackle the ill-posedness from an over-determined matrix. The constrained HR method solves for the hierarchical term $\boldsymbol{x}_{(j)}$ satisfying

$$\boldsymbol{x}_{(j)} = \underset{\boldsymbol{x} \in \mathcal{B}_{\lambda_j}}{\arg\min} \Big\{ ||\boldsymbol{r}_{j-1} - A\boldsymbol{x}||_2 \Big\}, \quad \text{for } 1 \leq j \leq J.$$

The penalized HR method solves for the hierarchical term $\boldsymbol{x}_{(j)}$ satisfying

$$\boldsymbol{x}_{(j)} = \underset{\boldsymbol{x} \in \mathbb{R}^N}{\arg\min} \Big\{ \lambda_j||\boldsymbol{x}||_1 + \frac{1}{2}||\boldsymbol{r}_{j-1} - A\boldsymbol{x}||_2^2 \Big\}, \quad \text{for } 1 \leq j \leq J.$$

In both HR methods, the hierarchical residual is defined in a recursive manner $\boldsymbol{r}_j = \boldsymbol{r}_{j-1} - A\boldsymbol{x}_{(j)}$ and the initial hierarchical residual is set as $\boldsymbol{r}_0 = \boldsymbol{y}_*^\varepsilon$. The hierarchical scales $\{\lambda_j\}_{j=1}^J$ are a sequence of decreasing scales. Utilizing a ladder of hierarchical residuals with their corresponding hierarchical scales, the HR method *successively* extracts information from the previous hierarchical residual to the current hierarchical term at a *finer* hierarchical scale. The hierarchical sum, as the sum of all hierarchical terms $\boldsymbol{X}_J = \sum_{j=1}^{J} \boldsymbol{x}_{(j)}$, becomes the linear model produced by the HR method, and its distance to $\boldsymbol{x}_{LS}$ is controlled by the number of hierarchical iterations taken. Since the main focus of this thesis is on the recovery of the unknown, we also investigate the de-noising problem in Linear Regression, where the responses $\boldsymbol{y}_*^{\varepsilon}$ are the noisy linear combination of the unknown linear model $\boldsymbol{x}_*$, i.e., $\boldsymbol{y}_*^{\varepsilon} = A\boldsymbol{x}_* + \boldsymbol{\epsilon}$ with a given noise level $\varepsilon > 0$ ($||\boldsymbol{\epsilon}||_2 \leq \varepsilon$). We compare the linear models from the LS method and the LASSO method, and discuss their de-noising capability. We also consider the linear model from the HR method, and show that the de-nosing capability from using the hierarchical extraction is superior than the other two methods.

The remaining sections of the chapter are structured as follows. In section 4.2, we discuss the difference between the LS linear model and the LASSO linear models. In section 4.2.1, we propose the HR method and discuss its ability to control the distance between the HR linear model and the LS linear model. We move on to the de-noising problem in section 4.3 and discuss the difference in de-noising capability between the LS linear model and the LASSO linear models. In section 4.3.1, we show that the HR linear model has superior de-noising capability than the LASSO linear models due to its multi-scale hierarchical extraction approach. In section 4.4, we report numerical experiments on comparing the three linear models. We conclude this chapter 4.5 with discussion on extending the HR method to linear regression with non-linear basis function.

4.2 The Original Problem: Finding the Most Suitable Linear Model

The original problem in Linear Regression is concerned with finding the most suitable linear model $\boldsymbol{x}_*$ so that the modeling error, $\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}_*$, is well-controlled, e.g., being smallest in some ℓ_p norm. The relationship between responses $\boldsymbol{y}_*^\varepsilon$ and the regressor matrix A is unknown before hand, the original problem is investigating the performance of using a linear model to approximate such relationship. Considering that the regressor matrix A is over-determined and the response $\boldsymbol{y}_*^\varepsilon$ is not in the range of A, we employ the general Tikhonov regularization method, which finds a linear model from a convex feasible set $\mathcal{C} \subset \mathbb{R}^N$ such that the linear model satisfies the following

$$\boldsymbol{x}_{T(\lambda)} = \operatorname*{arg\,min}_{\boldsymbol{x}\in\mathcal{C}} \left\{\lambda f(\boldsymbol{x}) + \frac{1}{2}||\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}||_2^2\right\}. \tag{4.1}$$

Now we discuss the different implementation of the stabilizing function f and the regularization parameter λ in order to counter the ill-posedness from the regressor matrix A. The LS method finds a linear model $\boldsymbol{x}_{LS}$ which satisfies the following

$$\boldsymbol{x}_{LS} = \operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{R}^N} \left\{||\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}||_2\right\}. \tag{P_{LS}}$$

Remark 4.2.1. *When we set $\mathcal{C} = \mathbb{R}^N$ and $\lambda = 0$ in* (4.1) *(eliminating the need to specify a stabilizing function f), the general Tikhonov regularization method becomes the LS method. Here, we used the fact that the minimizer of $||\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}||_2^2/2$ is the same as that of $||\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}||_2$.*

Such linear model $\boldsymbol{x}_{LS}$ has the its modeling error being the smallest in the ℓ_2 norm, which is one of the desired features for the original problem. Hence, we will take

$\boldsymbol{x}_* = \boldsymbol{x}_{LS}$ as the reference linear model for the original problem throughout this chapter. However the LS linear model is rarely sparse. To attain sparsity (having few non-zero entries), several approaches had been introduced. The Best Subset Selection [56] (BSS) method provides a k-sparse[1] linear model by keeping the k largest (in magnitude) entries of the LS linear model $\boldsymbol{x}_{LS}$. However, such selection process is discrete, slight changes in value of one of the entries of $\boldsymbol{x}_{LS}$ will produce a totally different BSS linear model. In order to provide a continuous selection process, the Ridge Regression [56] (RR) method finds a linear model $\boldsymbol{x}_\lambda$ which satisfies the following

$$\boldsymbol{x}_\lambda = \underset{\boldsymbol{x}\in\mathbb{R}^N}{\arg\min}\, \{\frac{\lambda}{2}||\boldsymbol{x}||_2^2 + \frac{1}{2}||\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}||_2^2\}.$$

Remark 4.2.2. *When we set $\mathcal{C} = \mathbb{R}^N$ and $f(\boldsymbol{x}) = ||\boldsymbol{x}||_2^2/2$ in* (4.1)*, the general Tikhonov regularization method becomes the RR method.*

While the RR method is robust against small changes in $\boldsymbol{y}_*^\varepsilon$, the RR linear model is rarely sparse. The LASSO method was introduced to provide a sparse linear model through a continuous selection process. There are two known definitions for the LASSO method. The original definition uses a ℓ_1 constraint, known as the constrained regression [50, 56]. The constrained LASSO method finds a linear model $\boldsymbol{x}_{LA(\lambda)}$ from a convex feasible set $\mathcal{B}_\lambda = \{\boldsymbol{x} \in \mathbb{R}^N \Big| ||\boldsymbol{x}||_1 \leq \lambda\} \subset \mathbb{R}^N$ with a regularization parameter $\lambda > 0$ satisfying the following,

$$\boldsymbol{x}_{LA(\lambda)} = \underset{\boldsymbol{x}\in\mathcal{B}_\lambda}{\arg\min}\, \Big\{||\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}||_2\Big\}. \qquad (P_{CL})$$

The constrained LASSO linear model $\boldsymbol{x}_{LA(\lambda)}$ has a closed form expression when the re-

[1] A linear model $\boldsymbol{x} \in \mathbb{R}^N$ is k-sparse if and only if it has at most k non-zero entries.

gressor matrix A is an isometry ($A^\top A = I$), i.e.,

$$(\boldsymbol{x}_{LA(\lambda)})_i = \text{sgn}((\boldsymbol{x}_{LS})_i)(\big|(\boldsymbol{x}_{LS})_i\big| - \gamma)^+, \quad 1 \leq i \leq N. \tag{4.2}$$

The parameter $\gamma > 0$ is chosen such that $||\boldsymbol{x}_{LA(\lambda)}||_1 = \lambda$. This close form expression is exactly the same as the soft shrinkage operator in [15, 17]. We will use the constrained LASSO method for isometry regressor matrix A.

Remark 4.2.3. *When we set $\mathcal{C} = \mathcal{B}_\lambda$ and $f(\boldsymbol{x}) = 0$ (moving the regularization parameter λ to the constraint) in* (4.1)*, the general Tikhonov regularization method becomes the constrained LASSO method.*

The other definition uses a ℓ_1 penalty function, known as the penalized regression [50]. The penalized LASSO method finds a linear model $\boldsymbol{x}_{LA(\lambda)}$ from the following,

$$\boldsymbol{x}_{LA(\lambda)} = \underset{\boldsymbol{x}\in\mathbb{R}^N}{\arg\min} \Big\{\lambda||\boldsymbol{x}||_1 + \frac{1}{2}||\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}||_2^2\Big\}. \tag{P_{PL}}$$

Remark 4.2.4. *When we set $\mathcal{C} = \mathbb{R}^N$ and $f(\boldsymbol{x}) = ||\boldsymbol{x}||_1$ in* (4.1)*, the general Tikhonov regularization method becomes the penalized LASSO method.*

The connection between the constrained LASSO method and penalized LASSO method was explained in [16, 50]. We will use the penalized LASSO method for any other regressor matrix A. Thanks to using the ℓ_1 constraint (or the ℓ_1 penalty function), the LASSO method is able to produce a sparse linear model. Since the LASSO method is also minimizing the modeling error, $\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}_{LA(\lambda)}$, the LASSO linear model is robust against small changes in $\boldsymbol{y}_*^\varepsilon$. The following lemma discusses the difference between the LS linear model and the LASSO linear models.

Lemma 4.2.5. *Given the data, $\{A, \boldsymbol{y}_*^\varepsilon\}$, from a set of M observations, we assume that the regressor matrix A has linearly independent columns. The difference between the LS linear model in (P_{LS}) and the constrained LASSO linear model in (P_{CL}) is bounded as follows,*

$$||\boldsymbol{x}_{LS} - \boldsymbol{x}_{LA(\lambda)}||_1 = ||\boldsymbol{x}_{LS}||_1 - \lambda.$$

The difference between the LS linear model in (P_{LS}) and the penalized LASSO linear model in (P_{PL}) is given as follows,

$$\boldsymbol{x}_{LS} - \boldsymbol{x}_{LA(\lambda)} = \lambda(A^\top A)^{-1}\boldsymbol{sgn}(\boldsymbol{x}_{LA(\lambda)}).$$

Proof. Since the regressor matrix used in (P_{CL}) is an isometry, by lemma A.5.2 and (4.2), we have

$$||\boldsymbol{x}_{LS} - \boldsymbol{x}_{LA(\lambda)}||_1 = ||\boldsymbol{x}_{LS}||_1 - ||\boldsymbol{x}_{LA(\lambda)}||_1 = ||\boldsymbol{x}_{LS}||_1 - \lambda.$$

For the LASSO linear model in (P_{PL}), we combine (A.10) with $A^\top A\boldsymbol{x}_{LS} = A^\top \boldsymbol{y}_*^\varepsilon$,

$$\begin{aligned} &\lambda\mathbf{sgn}(\boldsymbol{x}_{LA(\lambda)}) + A^\top(A\boldsymbol{x}_{LA(\lambda)} - A\boldsymbol{x}_{LS}) = \mathbf{0} \\ \Rightarrow\quad &\boldsymbol{x}_{LS} - \boldsymbol{x}_{LA(\lambda)} = \lambda(A^\top A)^{-1}\mathbf{sgn}(\boldsymbol{x}_{LA(\lambda)}). \end{aligned}$$

□

We end this section by the following remark.

Remark 4.2.6. *For the constrained LASSO method, when $\lambda = ||\boldsymbol{x}_{LS}||_1$, $\boldsymbol{x}_{LA(\lambda)} = \boldsymbol{x}_{LS}$; whereas $\lambda = 0$ would lead to $\boldsymbol{x}_{LA(\lambda)} = \mathbf{0}$. In fact, when $\lambda = ||\boldsymbol{x}_{LS}||_1/2$, the size of support of $\boldsymbol{x}_{LA(\lambda)}$ is roughly half of the size of the support of $\boldsymbol{x}_{LS}$. For the penalized LASSO method, when $\lambda = 0$, $\boldsymbol{x}_{LA(\lambda)} = \boldsymbol{x}_{LS}$; when $\lambda = ||A^\top \boldsymbol{y}_*^\varepsilon||_\infty$, $\boldsymbol{x}_{LA(\lambda)} = \mathbf{0}$.*

As we learn from the previous remark that choosing a suitable regularization *parameter* λ is crucial for finding a useful LASSO linear model. When a LASSO linear model is found not desired, a re-trial with a different λ is deemed necessary. The single scale approach would re-start from scratch, leaving the residual term, $\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}_{LA(\lambda)}$, totally un-used.

4.2.1 The HR Method for the Original Problem

From lemma 4.2.5, we learn that the regularization *parameter* λ controls the distance between the LS linear model $\boldsymbol{x}_{LS}$ and the LASSO linear model $\boldsymbol{x}_{LA(\lambda)}$. The regularization *parameter* λ can be viewed as a regularization *scale*. Based on this understanding, we propose a multi-scale generalization to the LASSO method, the HR method. Recall that in chapter 2, the HR method was used to provide approximate solutions to the ill-posed $A\boldsymbol{x} = \boldsymbol{y}_*^\varepsilon$ where the operator A is an under-determined matrix; in chapter 3, the HR methods used to provide approximate solutions to the ill-posed $A^h\boldsymbol{x} = \boldsymbol{y}_*^{h,\varepsilon}$ where the linear operator A^h (the discrete Helmholtz filter) has an eigenvalue of 0. In this chapter, the HR method is used for an over-determined matrix A. From previous remarks, we learn that the two different LASSO methods are specific implementations of the general Tikhonov regularization method. We begin with the general form for the LASSO method. Let

$$\boldsymbol{x}_\lambda = \boldsymbol{x}_{LA(\lambda)} = \arg\min_{\boldsymbol{x}\in\mathcal{C}} \Big\{\lambda f(\boldsymbol{x}) + \frac{1}{2}||\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}||_2^2\Big\},$$

since there is information left in the residual, $\boldsymbol{r}_\lambda = \boldsymbol{y}_*^\varepsilon - \boldsymbol{x}_\lambda$, i.e., $\boldsymbol{r}_\lambda \neq \boldsymbol{0}$ (or we are done), we can extract further information from $\boldsymbol{r}_\lambda$ at a finer scale, say $\frac{\lambda}{2}$,

$$\boldsymbol{x}_{\frac{\lambda}{2}} := \arg\min_{\boldsymbol{x}\in\mathcal{C}} \left\{\frac{\lambda}{2} f(\boldsymbol{x}) + \frac{1}{2}||\boldsymbol{r}_\lambda - A\boldsymbol{x}||_2^2\right\} \quad \text{with} \quad \boldsymbol{r}_{\frac{\lambda}{2}} := \boldsymbol{r}_\lambda - A\boldsymbol{x}_{\frac{\lambda}{2}}.$$

We obtain with a two scale decomposition of $\boldsymbol{y}_*^\varepsilon$, i.e., $\boldsymbol{y}_*^\varepsilon = A\boldsymbol{x}_\lambda + A\boldsymbol{x}_{\frac{\lambda}{2}} + \boldsymbol{r}_{\frac{\lambda}{2}}$. The previous extraction process can continue. To simplify notations, we will use numbered subscripts from now on. We start from setting $\boldsymbol{x}_{(1)} = \boldsymbol{x}_\lambda$, $\boldsymbol{r}_0 = \boldsymbol{y}_*^\varepsilon$, and choose hierarchical scales $\lambda_j = 2^{1-j}\lambda$. The HR method will solve the following,

$$\boldsymbol{x}_{(j)} = \arg\min_{\boldsymbol{x}\in\mathcal{C}} \left\{\lambda_j f(\boldsymbol{x}) + \frac{1}{2}||\boldsymbol{r}_{j-1} - A\boldsymbol{x}||_2^2\right\}, \quad \text{for } 1 \leq j \leq J.$$

The hierarchical residual $\boldsymbol{r}_j$ satisfies a recursive relation: $\boldsymbol{r}_j = \boldsymbol{r}_{j-1} - A\boldsymbol{x}_{(j)}$. We sum up the hierarchical terms, $\boldsymbol{x}_{(j)}$, and obtain a linear model in the form of a hierarchical sum, i.e., $\boldsymbol{X}_J = \sum_{j=1}^J \boldsymbol{x}_{(j)}$. The hierarchical observation $A\boldsymbol{X}_J$ will provide a multi-scale description of $\boldsymbol{y}_*^\varepsilon$ as follows,

$$\boldsymbol{y}_*^\varepsilon = A\boldsymbol{x}_{(1)} + A\boldsymbol{x}_{(2)} + \ldots + A\boldsymbol{x}_{(J)} + \boldsymbol{r}_J.$$

It follows that the hierarchical sum $\boldsymbol{X}_J$ presents a multi-scale approximate description of $\boldsymbol{x}_{LS}$. To be exact, the distance between $\boldsymbol{X}_J$ and $\boldsymbol{x}_{LS}$ is controlled by the number of hierarchical steps taken. We will present two HR methods based on the two different LASSO methods. For the constrained HR method, the hierarchical term $\boldsymbol{x}_{(j)}$ is found from a shrinking convex feasibility set $\mathcal{B}_{\lambda_j}$ and it satisfies the following,

$$\boldsymbol{x}_{(j)} = \arg\min_{\boldsymbol{x}\in\mathcal{B}_{\lambda_j}} \left\{||\boldsymbol{r}_{j-1} - A\boldsymbol{x}||_2\right\}. \tag{P_{CH}}$$

For the penalized HR method, the hierarchical term $\boldsymbol{x}_{(j)}$ is found satisfying the following,

$$\boldsymbol{x}_{(j)} = \underset{\boldsymbol{x}\in\mathbb{R}^N}{\arg\min} \left\{\lambda_j||\boldsymbol{x}||_1 + \frac{1}{2}||\boldsymbol{r}_{j-1} - A\boldsymbol{x}||_2^2\right\}. \tag{P_{PH}}$$

The following remark discusses properties of the hierarchical least square term $\boldsymbol{u}_j = A^\top \boldsymbol{r}_j$ from the constrained HR method.

Remark 4.2.7. *Let $\boldsymbol{u}_j \in \mathbb{R}^N$ be such that $\boldsymbol{u}_j = A^\top \boldsymbol{r}_j$, then the following holds,*

- $\boldsymbol{u}_0 = A^\top \boldsymbol{r}_0 = A^\top \boldsymbol{y}_*^\varepsilon = \boldsymbol{x}_{LS}$.
- *For $i \in$ supp$(\boldsymbol{u}_{j-1})$, $(\boldsymbol{u}_{j-1})_i = 0 \Rightarrow (\boldsymbol{u}_j)_i = 0$, hence supp$(\boldsymbol{u}_j) \subset$ supp$(\boldsymbol{u}_{j-1})$.*
- *For $i \in$ supp$(\boldsymbol{u}_j)$, $(\boldsymbol{u}_j)_i$ and $(\boldsymbol{u}_{j-1})_i$ have the same sign and $\left|(\boldsymbol{u}_j)_i\right| < \left|\boldsymbol{u}_{j-1})_i\right|$.*
- $\boldsymbol{u}_j = A^\top \boldsymbol{r}_j = A^\top(\boldsymbol{r}_{j-1} - A\boldsymbol{x}_{(j)}) = \boldsymbol{u}_{j-1} - \boldsymbol{x}_{(j)}$, *and* $\boldsymbol{u}_J = \boldsymbol{x}_{LS} - \boldsymbol{X}_J$.

The proof of the statement made in remark 4.2.7 is easily obtained by following the remark A.5.3. Given the additional properties in remark 4.2.7, we can show the following lemma.

Lemma 4.2.8. *The hierarchical term $\boldsymbol{x}_{(j)}$ defined in (P_{CH}) satisfies the following,*

- $||\boldsymbol{x}_{(j)}||_1 = \lambda_j$.
- *supp$(\boldsymbol{x}_{(j)}) \subset$ supp$(\boldsymbol{x}_{(j-1)})$.*
- *For $i \in$ supp$(\boldsymbol{x}_{(j)})$, $(\boldsymbol{x}_{(j)})_i$ and $(\boldsymbol{x}_{(j-1)})_i$ have the same sign and $\left|(\boldsymbol{x}_{(j)})_i\right| < \left|(\boldsymbol{x}_{(j-1)})_i\right|$.*

The hierarchical sum, $\boldsymbol{X}_J = \sum_{j=1}^J \boldsymbol{x}_{(j)}$, satisfies the following,

- $||\boldsymbol{X}_J||_1 = \sum_{j=1}^{J} \lambda_j$.

- *For the LS linear model $\boldsymbol{x}_{LS}$ defined in* (P_{LS}), $||\boldsymbol{x}_{LS} - \boldsymbol{X}_J||_1 = ||\boldsymbol{x}_{LS}||_1 - \sum_{j=1}^{J} \lambda_j$.

Proof. Following remark 4.2.7, the hierarchical term $\boldsymbol{x}_{(j)}$ satisfies the following (for $j \geq 2$),

- $||\boldsymbol{x}_{(j)}||_1 = \lambda_j$.

- $\text{supp}(\boldsymbol{x}_{(j)}) \subset \text{supp}(\boldsymbol{x}_{(j-1)})$.

- For $i \in \text{supp}(\boldsymbol{x}_{(j)})$, $(\boldsymbol{x}_{(j)})_i \cdot (\boldsymbol{x}_{(j-1)})_i > 0$ and $\left|(\boldsymbol{x}_{(j)})_i\right| < \left|(\boldsymbol{x}_{(j-1)})_i\right|$.

The initial hierarchical term, $\boldsymbol{X}_1 = \boldsymbol{x}_{LA(\lambda_1)}$, and therefore it satisfies remark A.5.3 with $\lambda = \lambda_1$ and $\boldsymbol{x}_{LS} = A^\top \boldsymbol{y}_*^\varepsilon$. It follows that,

$$||\boldsymbol{X}_J||_1 = ||\sum_{j=1}^{J} \boldsymbol{x}_{(j)}||_1 = \sum_{j=1}^{J} ||\boldsymbol{x}_{(j)}||_1 = \sum_{j=1}^{J} \lambda_j.$$

Since $\text{supp}(\boldsymbol{x}_{(1)}) \subset \text{supp}(\boldsymbol{x}_{LS})$, using remark A.5.3, we have

$$||\boldsymbol{x}_{LS} - \boldsymbol{X}_J||_1 = ||\boldsymbol{x}_{LS}||_1 - ||\boldsymbol{X}_J||_1 = ||\boldsymbol{x}_{LS}||_1 - \sum_{j=1}^{J} \lambda_j.$$

□

Remark 4.2.9. *As shown in lemma 4.2.8, we learn that the distance between $\boldsymbol{X}_J$ and $\boldsymbol{x}_{LS}$ is controlled by the number of hierarchical steps taken from a pre-determined hierarchy of scales $\{\lambda_j\}_{j=1}^{J}$.*

Next, we are concerned with the distance between the HR linear model $\boldsymbol{X}_J$ and $\boldsymbol{x}_{LS}$, when the hierarchical terms $\boldsymbol{x}_{(j)}$'s are given in (P_{PH}).

Lemma 4.2.10. *The distance between the HR linear model,* $\boldsymbol{X}_J = \sum_{j=1}^{J} \boldsymbol{x}_{(j)}$ *with the hierarchical term* $\boldsymbol{x}_{(j)}$ *in* (P_{PH})*, and the LS linear model* $\boldsymbol{x}_{LS}$ *is given as follows,*

$$\boldsymbol{x}_{LS} - \boldsymbol{X}_J = \lambda_J (A^\top A)^{-1} \boldsymbol{sgn}(\boldsymbol{x}_{(J)}).$$

Proof. For the hierarchical term $\boldsymbol{x}_{(j)}$ defined in (P_{PH}), it satisfies the following signum equation,

$$\lambda_j \mathbf{sgn}(\boldsymbol{x}_{(j)}) + A^\top (A\boldsymbol{x}_{(j)} - \boldsymbol{r}_{j-1}) = \mathbf{0}, \quad \text{for } \lambda_j \leq ||A^\top \boldsymbol{r}_{j-1}||_\infty.$$

Using the fact that $A^\top A \boldsymbol{x}_{LS} = A^\top \boldsymbol{y}_*^\varepsilon$ and $\boldsymbol{r}_J = \boldsymbol{y}_*^\varepsilon - A\boldsymbol{X}_J$, we arrive at the following,

$$\lambda_J \mathbf{sgn}(\boldsymbol{x}_{(J)}) + A^\top (AA\boldsymbol{x}_{(J)} - \boldsymbol{r}_{J-1}) = \lambda_J \mathbf{sgn}(\boldsymbol{x}_{(J)}) + A^\top (A\boldsymbol{X}_J - \boldsymbol{y}_*^\varepsilon) = \mathbf{0}.$$

It follows that,

$$\boldsymbol{x}_{LS} - \boldsymbol{X}_J = \lambda_J (A^\top A)^{-1} \mathbf{sgn}(\boldsymbol{x}_{(J)}).$$

□

Remark 4.2.11. *The distance between* $\boldsymbol{X}_J$ *and* $\boldsymbol{x}_{LS}$ *is only determined by the final hierarchical scale* λ_J*.*

4.3 The De-noising Problem in Linear Regression

Given the noisy response $\boldsymbol{y}_*^\varepsilon = A\boldsymbol{x}_* + \boldsymbol{\epsilon}$ as the unknown linear model $\boldsymbol{x}_*$ with a bounded observation noise $\boldsymbol{\epsilon}$, i.e., $||\boldsymbol{\epsilon}||_2 \leq \varepsilon$, the de-noising problem is inquiring about the possibility of recovering $\boldsymbol{x}_*$ from $\boldsymbol{y}_*^\varepsilon$. However since exact recovery is not possible, the de-noising problem is interested in the performance of the available linear models. The following lemma discusses the de-noising performance of the LS linear model $\boldsymbol{x}_{LS}$.

Lemma 4.3.1. *Given the noisy responses $\boldsymbol{y}_*^\varepsilon$, we assume that A has linearly independent columns. The LS linear model, $\boldsymbol{x}_{LS}$ in (P_{LS}), approximates $\boldsymbol{x}_*$ with the following error,*

$$\boldsymbol{x}_{LS} - \boldsymbol{x}_* = (A^\top A)^{-1} A^\top \boldsymbol{\epsilon}.$$

Proof. Since the LS linear model $\boldsymbol{x}_{LS}$ solves the normal equation, $A^\top A \boldsymbol{x}_{LS} = A^\top \boldsymbol{y}_*^\varepsilon$, it follows that,

$$A^\top A \boldsymbol{x}_{LS} = A^\top (A\boldsymbol{x}_* + \boldsymbol{\epsilon}) = A^\top A \boldsymbol{x}_* + A^\top \boldsymbol{\epsilon}. \tag{4.3}$$

Multiplying the inverse of $A^\top A$ to the right of both sides of (4.3), we obtain the error formula in lemma 4.3.1. □

Remark 4.3.2. *When $\varepsilon = 0$, the LS linear model recovers exactly $\boldsymbol{x}_*$. When $\varepsilon \neq 0$, the observation noise is amplified by $(A^\top A)^{-1} A^\top$. Note that when $A^\top A = I$, $\boldsymbol{x}_{LS} - \boldsymbol{x}_* = A^\top \boldsymbol{\epsilon}$.*

The following lemma is concerned with the de-noising performance of the LASSO linear models.

Lemma 4.3.3. *Given the noise responses $\boldsymbol{y}_*^\varepsilon$, we assume that A has linearly independent columns. The LASSO linear model $\boldsymbol{x}_{LA(\lambda)}$ in (P_{CL}) approximates $\boldsymbol{x}_*$ with the error bounded above as follows,*

$$||\boldsymbol{x}_* - \boldsymbol{x}_{LA(\lambda)}||_1 \leq ||\boldsymbol{x}_{LS}||_1 + \varepsilon ||A^\top||_2 - \lambda.$$

For the LASSO linear model $\boldsymbol{x}_{LA(\lambda)}$ in (P_{PL}), it approximates $\boldsymbol{x}_*$ with the following error,

$$\boldsymbol{x}_* - \boldsymbol{x}_{LA(\lambda)} = (A^\top A)^{-1} (\lambda \boldsymbol{sgn}(\boldsymbol{x}_{LA(\lambda)}) - A^\top \boldsymbol{\epsilon}).$$

Proof. Using lemma A.5.2 and $\boldsymbol{x}_{LS} - \boldsymbol{x}_* = A^\top \boldsymbol{\epsilon}$, we have

$$||\boldsymbol{x}_* - \boldsymbol{x}_{LA(\lambda)}||_1 \leq ||\boldsymbol{x}_{LS} - \boldsymbol{x}_{LA(\lambda)}||_1 + ||A^\top \boldsymbol{\epsilon}||_1 \leq ||\boldsymbol{x}_{LS}||_1 - ||\boldsymbol{x}_{LA(\lambda)}||_1 + ||\boldsymbol{\epsilon}||_2 ||A^\top||_2$$
$$= ||\boldsymbol{x}_{LS}||_1 - \lambda + \varepsilon ||A^\top||_2.$$

For the second definition of the LASSO linear model, we start with the signum equation in (A.10),

$$\mathbf{0} = \lambda \mathbf{sgn}(\boldsymbol{x}_{LA(\lambda)}) + A^\top (A\boldsymbol{x}_{LA(\lambda)} - \boldsymbol{y}_*^\varepsilon) = \lambda \mathbf{sgn}(\boldsymbol{x}_{LA(\lambda)}) + A^\top (A\boldsymbol{x}_{LA(\lambda)} - A\boldsymbol{x}_* - \boldsymbol{\epsilon})$$
$$= A^\top A(\boldsymbol{x}_{LA(\lambda)} - \boldsymbol{x}_*) + \lambda \mathbf{sgn}(\boldsymbol{x}_{LA(\lambda)}) - A^\top \boldsymbol{\epsilon}.$$

Multiplying the inverse of $A^\top A$ to both sides of the previous equation, we move the difference term $\boldsymbol{x}_{LA(\lambda)} - \boldsymbol{x}_*$ to the right to obtain the following,

$$\boldsymbol{x}_* - \boldsymbol{x}_{LA(\lambda)} = (A^\top A)^{-1} (\lambda \mathbf{sgn}(\boldsymbol{x}_{LA(\lambda)}) - A^\top \boldsymbol{\epsilon}).$$

□

Remark 4.3.4. *First, we consider constrained LASSO method. When $\varepsilon = 0$, the LASSO linear model recovers $\boldsymbol{x}_*$ when $\lambda_1 = ||\boldsymbol{x}_{LS}||_1$; we also have $\boldsymbol{x}_{LA(\lambda_1)} = \boldsymbol{x}_{LS}$. When $\varepsilon \neq 0$, we can set $\lambda_2 = \min\{||\boldsymbol{x}_{LS}||_1, \varepsilon ||A^\top||_2\}$; however, due to the restriction on λ ($0 \leq \lambda \leq ||\boldsymbol{x}_{LS}||_1$), using the LASSO method for de-noising is at best performing the same as using the LS method. Second, we consider the penalized LASSO method. When $\varepsilon = 0$, the LASSO linear model recovers $\boldsymbol{x}_*$ when $\lambda_3 = 0$; we also have $\boldsymbol{x}_{LA(\lambda_3)} = \boldsymbol{x}_{LS}$. For $\varepsilon \neq 0$, the noise amplification is reduced due to the presence of the term $\lambda \mathbf{sgn}(\boldsymbol{x}_{LA(\lambda)})$. It follows that the penalized LASSO method offers a better de-noising capability than the constrained LASSO method.*

4.3.1 The HR Method for the De-noising Problem

We argue that the HR method demonstrates superior de-noising capability than the LASSO method, thanks to its usage of a ladder of hierarchical residuals with their corresponding hierarchical scales. We begin the de-noising analysis with the following lemma.

Lemma 4.3.5. *Given the noisy responses $\boldsymbol{y}_*^{\varepsilon}$, we assume that A has linearly independent columns. The hierarchical sum, $\boldsymbol{X}_J = \sum_{j=1}^{J} \boldsymbol{x}_{(j)}$ with the hierarchical term $\boldsymbol{x}_{(j)}$ in (P_{CH}), approximates $\boldsymbol{x}_*$ with the following error bound,*

$$||\boldsymbol{x}_* - \boldsymbol{X}_J||_1 \leq ||\boldsymbol{x}_{LS}||_1 + \varepsilon||A^\top||_2 - \sum_{j=1}^{J} \lambda_j.$$

For the hierarchical term $\boldsymbol{x}_{(j)}$ in (P_{PH}), the hierarchical sum, $\boldsymbol{X}_J = \sum_{j=1}^{J} \boldsymbol{x}_{(j)}$, approximates $\boldsymbol{x}_*$ with the following error,

$$\boldsymbol{x}_* - \boldsymbol{X}_J = (A^\top A)^{-1}(\lambda_J \boldsymbol{sgn}(\boldsymbol{x}_{(J)}) - A^\top \boldsymbol{\epsilon}).$$

Proof. For the constrained HR method, we use lemma 4.2.8 together with $\boldsymbol{x}_* = \boldsymbol{x}_{LS} - A^\top \boldsymbol{\epsilon}$ to derive the following,

$$\begin{aligned}||\boldsymbol{x}_* - \boldsymbol{X}_J||_1 \leq ||\boldsymbol{x}_{LS} - \boldsymbol{X}_J||_1 + ||\boldsymbol{\epsilon}||_2||A^\top||_2 &\leq ||\boldsymbol{x}_{LS}||_1 - ||\boldsymbol{X}_J||_1 + \varepsilon||A^\top||_2 \\ &= ||\boldsymbol{x}_{LS}||_1 + \varepsilon||A^\top||_2 - \sum_{j=1}^{J} \lambda_j.\end{aligned}$$

For the penalized method, we use remark 4.2.7 together with $\boldsymbol{r}_J = \boldsymbol{y}_*^{\varepsilon} - A\boldsymbol{X}_J$ to obtain the following,

$$\begin{aligned}\mathbf{0} &= \lambda_J \mathbf{sgn}(\boldsymbol{x}_{(J)}) + A^\top(A\boldsymbol{x}_{(J)} - \boldsymbol{r}_{J-1}) = \lambda_J \mathbf{sgn}(\boldsymbol{x}_{(J)}) + A^\top(A\boldsymbol{X}_J - \boldsymbol{y}_*^{\varepsilon}) \\ &= \lambda_J \mathbf{sgn}(\boldsymbol{x}_{(J)}) + A^\top(A\boldsymbol{X}_J - A\boldsymbol{x}_* - \boldsymbol{\epsilon}) = A^\top A(\boldsymbol{X}_J - \boldsymbol{x}_*) + \lambda_J \mathbf{sgn}(\boldsymbol{x}_{(J)}) - A^\top \boldsymbol{\epsilon}.\end{aligned}$$

Multiplying the inverse of $A^\top A$ to both sides of the previous equation, we move the different term, $\boldsymbol{X}_J - \boldsymbol{x}_*$, to the right to obtain,

$$\boldsymbol{x}_* - \boldsymbol{X}_J = (A^\top A)^{-1}(\lambda_J \mathbf{sgn}(\boldsymbol{x}_{(J)}) - A^\top \boldsymbol{\epsilon}).$$

□

Remark 4.3.6. *For the constrained HR method, we first consider $\varepsilon = 0$. When we have $\sum_{j=1}^{J} \lambda_j = ||\boldsymbol{x}_{LS}||_1$, we have $\boldsymbol{X}_J = \boldsymbol{x}_*$ as well as $\boldsymbol{X}_J = \boldsymbol{x}_{LS}$. Next, we consider $\varepsilon \neq 0$. Due to the usage of a ladder of hierarchical scales, we can decrease the term, $||\boldsymbol{x}_{LS}||_1 + \varepsilon||A^\top||_2$, by a sum of hierarchical scales $\sum_{j=1}^{J} \lambda_j$; however when $\sum_{j=1}^{J} \lambda_j \geq ||\boldsymbol{x}_{LS}||_1$, the HR linear model will become the LS linear model. For the penalized HR method, we first consider $\varepsilon = 0$. When the final hierarchical scale $\lambda_J = 0$, we have $\boldsymbol{X}_J = \boldsymbol{x}_*$ as well as $\boldsymbol{X}_J = \boldsymbol{x}_{LS}$. Next, we consider $\varepsilon \neq 0$. Since the noise term is reduced by the extra $\lambda_J \boldsymbol{sgn}(\boldsymbol{x}_{(J)})$, we can design the ladder of hierarchical scales, so that the difference, $\lambda_J \boldsymbol{sgn}(\boldsymbol{x}_{(J)}) - A^\top \boldsymbol{\epsilon}$, can be at its minimum.*

4.4 Numerical Experiments

We run the numerical simulation by comparing the HR method to the LS method and LASSO method on the de-noising problem for Linear Regression. The unknown linear model $\boldsymbol{x}_* \in \mathbb{R}^N$ is generated with $k = 32$ non-zero entries, and the values of those entries are randomly picked as ± 1, ± 0.5 and ± 0.25 to demonstrate the HR method's multi-scale capability, and the location of the non-zero entries is also randomly picked. We set the number of observations M to 512, and the dimension of the unknown linear model is set at $N = 120$. The entries of the regressor matrix $A \in \mathbb{R}^{M \times N}$ are samples of identically distributed standard normal variables, i.e., $A = \text{randn}(M, N)$. We orthonormalize the columns of A by doing $[A, \;] = \text{qr}(A, 0)$. The noise level is set at $\varepsilon = 1.00$. The noise is generated as $\boldsymbol{\epsilon} = \text{randn}(M, 1)$ and normalized to have unit ℓ_2 norm; then $\boldsymbol{\epsilon} = \varepsilon * \boldsymbol{\epsilon}$ and $\boldsymbol{y}_*^\varepsilon = A\boldsymbol{x}_* + \boldsymbol{\epsilon}$. The LS linear model is obtained by doing $\boldsymbol{x}_{LS} = A^\top \boldsymbol{y}_*^\varepsilon$. The regularization parameter λ for the constrained LASSO method is set as $\lambda = 0.4 * \text{norm}(\boldsymbol{x}_{LS}, 1)$. The HR method uses the initial hierarchical scale $\lambda_1 = \lambda$ and performs 2 hierarchical iterations. The linear models are plotted in the same plot shown in figure 4.1.

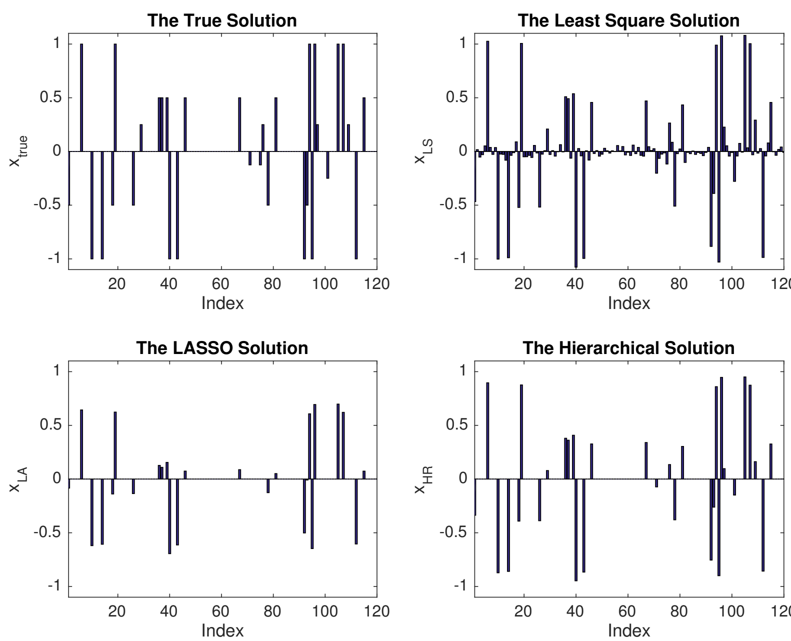


Figure 4.1: Comparison of 3 Linear Models on De-noising

As shown in the figure 4.1, the LS linear model $\boldsymbol{x}_{LS}$ recovers most of the features of $\boldsymbol{x}_*$ long with the noisy part embedded in $\boldsymbol{y}_*^\varepsilon$. For the LASSO linear model, when $A^\top A = I$, the LASSO method works similar to the soft shrinkage operator and starts from the maximum entries (in magnitude) of $\boldsymbol{x}_{LS}$ and move down until it accumulates enough energy, i.e., $||\boldsymbol{x}_{LA(\lambda)}||_1 = \lambda$. The HR method, on the other hand, due to its multi-scale approach, works similar to an iterative soft shrinkage, starting from where the previous hierarchical residual is and continuing picking up energy from $\boldsymbol{x}_{LS}$, until it finishes J hierarchical iterations, thus ending up recovering more features of $\boldsymbol{x}_*$ and not allowing the noisy information enter the approximate solution.

4.5 Conclusion

We discussed two inverse problems in Linear Regression. We compared the LS method and the LASSO method on addressing these inverse problems. We proposed the HR method based on the LASSO method and showed that the HR method controlled the distance between its linear model and the LS linear model. We also discussed the de-noising capability of the HR method. We note that normally each column of the regressor matrix A contains samples of a certain independent variable. When a nonlinear basis function is used, i.e., $y_i^\varepsilon = f_{1,i}(a_{1,i})x_1 + f_{2,i}(a_{2,i})x_2 + \ldots + f_{N,i}(a_{N,i})x_N + \epsilon_i$, we can assemble a different regressor matrix $\bar{A}$, such that each entry of $\bar{A}$ corresponds to $f_{i,j}(a_{i,j})$. We end up with a different linear system $\bar{A}\boldsymbol{x} \approx \boldsymbol{y}_*^\varepsilon$; however the analysis and convergence results will follow through.

Chapter 5: Conclusion

5.1 Conclusion

We presented the analysis of the applicaiton of the Hierarchical Reconstruction (HR) method for solving three different inverse problems. These inverse problems which we studied were concerned with the recovery of an unknown quantity $\boldsymbol{x}_*$ in a Hilbert space $\mathcal{X}$ from its observation $\boldsymbol{y}_*$ in another Hilbert space $\mathcal{Y}$ equipped with norm $||\cdot||_{\mathcal{Y}}$. The observation $\boldsymbol{y}_*$ is obtained by applying an observation operator $A : \mathcal{X} \to \mathcal{Y}$ to $\boldsymbol{x}_*$. When the observation operator A is ill-posed, recovery of $\boldsymbol{x}_*$ is done via the general Tikhonov regularization method. The general Tikhonov regularization method finds an approximation solution to $\boldsymbol{x}_*$ with an extra regularization parameter $\lambda > 0$ from the following,

$$\boldsymbol{x}_{T(\lambda)} = \underset{\boldsymbol{x}\in\mathcal{C}}{\arg\min} \left\{ \lambda f(\boldsymbol{x}) + \frac{1}{2}||\boldsymbol{y}_* - A\boldsymbol{x}||_{\mathcal{Y}}^2 \right\}. \tag{5.1}$$

Here $\mathcal{C}$ is a convex feasibility subset in $\mathcal{X}$ and the non-negative auxiliary function $f : \mathcal{X} \to \mathbb{R}^+$ is a stabilizing function which targets the specific ill-posedness of A. The choice of the stabilizing function f depends on the desired features of the unknown $\boldsymbol{x}_*$, which the approximate solution $\boldsymbol{x}_{T(\lambda)}$ would also attain.

For the recovery of a general unknown $\boldsymbol{x}_* \in \mathbb{R}^N$ from its noisy observation $\boldsymbol{y}_*^\varepsilon =$

$A\boldsymbol{x}_* + \boldsymbol{\epsilon} \in \mathbb{R}^M$ ($||\boldsymbol{\epsilon}||_2 \leq \varepsilon$), we suggested using the unconstrained ℓ_1 method. The unconstrained ℓ_1 method finds a solution from the following,

$$\boldsymbol{x}_\lambda = \underset{\boldsymbol{x}\in\mathbb{R}^N}{\arg\min} \left\{\lambda||\boldsymbol{x}||_1 + \frac{1}{2}||\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}||_2^2\right\}.$$

Having understood the relationship between $\boldsymbol{x}_\lambda$ and $\boldsymbol{x}_*$, we proposed the HR method to decrease the number of total iterations. The HR method finds the approximate solution as a sum of hierarchical terms, $\boldsymbol{x}_{(j)}$'s, with each of them satisfying the following,

$$\boldsymbol{x}_{(j)} = \underset{\boldsymbol{x}\in\mathbb{R}^N}{\arg\min} \left\{\lambda_j||\boldsymbol{x}||_1 + \frac{1}{2}||\boldsymbol{r}_{j-1} - A\boldsymbol{x}||_2^2\right\}, \quad \text{for } 1 \leq j \leq J,$$

with the hierarchical residual residual defined recursively $\boldsymbol{r}_j = \boldsymbol{r}_{j-1} - A\boldsymbol{x}_{(j)}$ and the initial hierarchical residual is set as $\boldsymbol{r}_0 = \boldsymbol{y}_*^\varepsilon$. The ladder of finitely many hierarchical scales is set as: $\lambda_j = \theta^{j-1}\lambda_1$ where $0 < \theta < 1$ and the initial hierarchical scale λ_1 depends on the problem at hand. With a ladder of finitely many and gradually decreasing hierarchical scales, the HR method was shown to decrease the total number of iterations, when compared to the unconstrained ℓ_1 method on recovery of a general $\boldsymbol{x}_*$ from its noisy observation $\boldsymbol{y}_*^\varepsilon$ with $\lambda = \lambda_J$.

For the recovery of the unknown $\boldsymbol{x}_* \in \mathcal{X}$ via de-convolution from its discrete filtered output $\boldsymbol{y}_*^h \in \mathcal{X}^h$ ($\mathcal{X}^h$ is a finite dimensional subspace of $\mathcal{X}$) where $\boldsymbol{y}_*^h$ is obtained by application of the discrete Helmholtz filter A^h, i.e., $\boldsymbol{y}_*^h = A^h\boldsymbol{x}_*$, the recovery cannot be done by direct solution of the linear equation $A^h\boldsymbol{x} = \boldsymbol{y}_*^h$ due to the ill-posedness of A^h. Therefore, we suggested the Tikhonov-Lavrentiev regularization method to find an approximate solution, which finds a FEM solution $\boldsymbol{x}_{L(\lambda)}^h$ with an extra regularization parameter $\lambda > 0$ such that for all $\boldsymbol{v}^h \in \mathcal{X}^h$, the following holds

$$\lambda\delta^2\langle\nabla\boldsymbol{x}_{L(\lambda)}^h, \nabla\boldsymbol{v}^h\rangle + (1+\lambda)\langle\boldsymbol{x}_{L(\lambda)}^h, \boldsymbol{v}^h\rangle = \delta^2\langle\nabla\boldsymbol{y}_*^{h,\varepsilon}, \nabla\boldsymbol{v}^h\rangle + \langle\boldsymbol{y}_*^{h,\varepsilon}, \boldsymbol{v}^h\rangle.$$

Having understood that the λ contributes to the accuracy of the approximate solution $\boldsymbol{x}^h_{L(\lambda)}$ alongside with the spatial resolution scale h, which is used by a certain Finite Element Method, we proposed the HR method in order to improve the approximation error. With a carefully designed ladder of hierarchical scales $\{\lambda_j\}_{j=1}^J$, the HR approximate solution, namely the discrete hierarchical sum as the same of all the discrete hierarchical terms, was shown to be a better approximate solution to $\boldsymbol{x}_*$ than the TLR approximate solution $\boldsymbol{x}^h_{L(\lambda)}$. When the noisy discrete filtered output $\boldsymbol{y}_*^{h,\varepsilon} = A^h\boldsymbol{x}_* + \boldsymbol{\epsilon}^h$ ($||\boldsymbol{\epsilon}^h||_{L^2(\Omega)} \leq \varepsilon$), we provided a (near) optimal stopping criteria for stopping the hierarchical iterations.

For the recovery of the linear model $\boldsymbol{x}_* \in \mathbb{R}^N$ from a set of data given by M observations, $\{A, \boldsymbol{y}_*^\varepsilon\}$, the problem is ill-posed since the regressor matrix $A \in \mathbb{R}^{M\times N}$ is over-determined and $\boldsymbol{y}_*^\varepsilon$ is not in the range of A. We suggested the Least Absolute Shrinkage and Selection Operator (LASSO) method. The LASSO method has two different versions. The constrained LASSO method finds a linear model from the following,

$$\boldsymbol{x}_{LA(\lambda)} = \operatorname*{arg\,min}_{\boldsymbol{x}\in\mathcal{B}_\lambda} \Big\{||\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}||_2\Big\}.$$

Here the feasible set $\mathcal{B}_\lambda = \{\boldsymbol{x} \in \mathbb{R}^N \Big| ||\boldsymbol{x}||_1 \leq \lambda\}$ is convex. The penalized LASSO method finds a linear model from the following,

$$\boldsymbol{x}_{LA(\lambda)} = \operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{R}^N} \Big\{\lambda||\boldsymbol{x}||_1 + \frac{1}{2}||\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}||_2^2\Big\}.$$

The LASSO linear model $\boldsymbol{x}_{LA(\lambda)}$ retains the sparsity feature thanks to its usage of ℓ_1 constraint (or the ℓ_1 penalty function). In order to offer better control on the sparsity level, we proposed the HR method based on the two different version of LASSO methods. With a careful designed ladder of decreasing hierarchical scales, the HR method was shown to control the approximation error, i.e., $\boldsymbol{X}_J - \boldsymbol{x}_*$, by the number of hierarchical iterations taken.

5.2 Future Work

We are now ready to discuss the approximation error from using the HR method to recovery of the unknown $\boldsymbol{x}_*$ from its noisy observation $\boldsymbol{y}_*^\varepsilon = A\boldsymbol{x}_* + \boldsymbol{\epsilon}$ ($||\boldsymbol{\epsilon}||_2 \leq \varepsilon$) where the ill-posed observation operator A is linear with no further information given. The HR method finds an approximate solution to $\boldsymbol{x}_*$, namely $\boldsymbol{X}_J$, as the sum of hierarchical terms $\boldsymbol{x}_{(j)}$ which is obtained from

$$\boldsymbol{x}_{(j)} = \arg\min_{\boldsymbol{x}\in\mathcal{X}} \left\{\lambda f(\boldsymbol{x}) + \frac{1}{2}||\boldsymbol{r}_{j-1} - A\boldsymbol{x}||_{\mathcal{Y}}^2\right\}, \tag{5.2}$$

with the hierarchical residual defined in a recursive equation $\boldsymbol{r}_j = \boldsymbol{r}_{j-1} - A\boldsymbol{x}_{(j)}$ and the initial hierarchical residual set as $\boldsymbol{r}_0 = \boldsymbol{y}_*^\varepsilon$. Before we present the error analysis, we will need the following assumptions:

A. 1: $f(\boldsymbol{u}+\boldsymbol{v}) \leq f(\boldsymbol{u}) + f(\boldsymbol{v})$, for any $\boldsymbol{u}$, $\boldsymbol{v} \in \mathcal{X}$;

A. 2: If $f(\boldsymbol{u}-\boldsymbol{v}) = 0$, then $\boldsymbol{u} = \boldsymbol{v}$, for any $\boldsymbol{u}$, $\boldsymbol{v} \in \mathcal{X}$;

A. 3: $f(\boldsymbol{u}-\boldsymbol{v}) \leq \alpha(f(\boldsymbol{u}) - f(\boldsymbol{v})) + \beta G(\boldsymbol{u},\boldsymbol{v}) + \gamma||A(\boldsymbol{u}-\boldsymbol{v})||_{\mathcal{Y}}$, for any $\boldsymbol{u}$, $\boldsymbol{v} \in \mathcal{X}$; α, β and $\gamma > 0$ are constants and the function G depends on the desired features which we want to preserve from the unknown;

A. 4: Let

$$\boldsymbol{x}_{T(\lambda)} = \arg\min_{\boldsymbol{x}\in\mathcal{X}} \left\{\lambda f(\boldsymbol{x}) + \frac{1}{2}||\boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}||_{\mathcal{Y}}^2\right\},$$

and define $\boldsymbol{r} = \boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}_{T(\lambda)}$, then $\zeta_1\lambda \leq ||\boldsymbol{r}||_{\mathcal{Y}} \leq \zeta_2\lambda$ for some $0 < \zeta_1 < \zeta_2$.

The following theorem is concerned with the approximation error from using the HR method.

Theorem 5.2.1. *For an unknown $\boldsymbol{x}_* \in \mathcal{X}$, a solution $\boldsymbol{X}_J$, as the sum of the hierarchical terms $\boldsymbol{x}_{(j)}$ in* (5.2) *with the initial hierarchical residual set as $\boldsymbol{r}_0 = \boldsymbol{y}_*^{\varepsilon}$ where $\boldsymbol{y}_*^{\varepsilon} = A\boldsymbol{x}_* + \boldsymbol{\epsilon}$ ($||\boldsymbol{\epsilon}||_{\mathcal{Y}} \leq \varepsilon$), will approximate $\boldsymbol{x}_*$ with the following error,*

$$f(\boldsymbol{X}_J - \boldsymbol{x}_*) \leq \alpha\Big(\frac{\varepsilon}{2\lambda_1} - \frac{\zeta_1^2\lambda_1}{2} + \frac{\lambda_1}{2}(\zeta_2^2\theta^{-2} - \zeta_1^2)\frac{\theta - \theta^J}{1-\theta}\Big) + \beta G(\boldsymbol{X}_J, \boldsymbol{x}_*) + \gamma(\zeta_2\lambda_J + \varepsilon).$$

Here the ladder of hierarchical scales $\{\lambda_j\}_{j=1}^J$ is designed as $\lambda_j = \theta^{j-1}\lambda_1$ with $0 < \theta < 1$ and λ_1 depending on the particular kind of recovery problem at hand.

Proof. We begin with $f(\boldsymbol{X}_J) \leq \sum_{j=1}^J f(\boldsymbol{x}_{(j)})$. By the optimality condition of $\boldsymbol{x}_{(1)}$, we have

$$\lambda_1 f(\boldsymbol{x}_{(1)}) + \frac{1}{2}||\boldsymbol{r}_0 - A\boldsymbol{x}_{(1)}||_{\mathcal{Y}}^2 \leq \lambda_1 f(\boldsymbol{x}_*) + \frac{1}{2}||\boldsymbol{r}_0 - A\boldsymbol{x}_*||_{\mathcal{Y}}^2 = \lambda_1 f(\boldsymbol{x}_*) + \frac{\varepsilon^2}{2}.$$

It follows that

$$f(\boldsymbol{x}_{(1)}) - f(\boldsymbol{x}_*) \leq \frac{\varepsilon^2}{2\lambda_1} - \frac{\zeta_1^2\lambda_1^2}{2}.$$

For other hierarchical terms $\boldsymbol{x}_{(j)}$ (for $2 \leq j \leq J$), we use an energy estimate,

$$\lambda_1 f(\boldsymbol{x}_{(j)}) + \frac{1}{2}||\boldsymbol{r}_{j-1} - A\boldsymbol{x}_{(j)}||_{\mathcal{Y}}^2 \leq \frac{1}{2}||\boldsymbol{r}_{j-1}||_{\mathcal{Y}}^2.$$

Hence, we obtain

$$f(\boldsymbol{x}_{(j)}) \leq \frac{1}{2\lambda_j}(\zeta_2^2\lambda_{j-1}^2 - \zeta_1^2\lambda_j^2) \leq \frac{(\zeta_2^2\theta^{-2} - \zeta_1^2)\lambda_j}{2}$$

Therefore, the difference, $f(\boldsymbol{X}_J) - f(\boldsymbol{x}_*)$, is bounded as

$$\begin{aligned} f(\boldsymbol{X}_J) - f(\boldsymbol{x}_*) &\leq (\sum_{j=1}^J f(\boldsymbol{x}_{(j)})) - f(\boldsymbol{x}_*) \leq \frac{\varepsilon^2}{2\lambda_1} - \frac{\zeta_1^2\lambda_1}{2} + (\zeta_2^2\theta^{-2} - \zeta_1^2)\sum_{j=2}^J \frac{\lambda_j}{2} \\ &\leq \frac{\varepsilon^2}{2\lambda_1} - \frac{\zeta_1^2\lambda_1}{2} + \frac{(\zeta_2^2\theta^{-2} - \zeta_1^2)\lambda_1}{2}\frac{\theta - \theta^J}{1-\theta}. \end{aligned}$$

Combing all the previous bounds together, we obtain

$$
\begin{aligned}
f(\boldsymbol{X}_J - \boldsymbol{x}_*) &\leq \alpha(f(\boldsymbol{X}_J) - f(\boldsymbol{x}_*)) + \beta G(\boldsymbol{X}_J, \boldsymbol{x}_*) + \gamma||A(\boldsymbol{X}_J - \boldsymbol{x}_*)||_{\mathcal{Y}} \\
&\leq \alpha\Big(\frac{\varepsilon^2}{2\lambda_1} - \frac{\zeta_1^2\lambda_1}{2} + \frac{(\zeta_2^2\theta^{-2} - \zeta_1^2)\lambda_1}{2}\frac{\theta - \theta^J}{1-\theta}\Big) + \beta G(\boldsymbol{X}_J, \boldsymbol{x}_*) \\
&\quad + \gamma(||A\boldsymbol{X}_J - \boldsymbol{y}_*^{\varepsilon}||_{\mathcal{Y}} + ||\boldsymbol{y}_*^{\varepsilon} - A\boldsymbol{x}_*||_{\mathcal{Y}}) \\
&\leq \alpha\Big(\frac{\varepsilon^2}{2\lambda_1} - \frac{\zeta_1^2\lambda_1}{2} + \frac{(\zeta_2^2\theta^{-2} - \zeta_1^2)\lambda_1}{2}\frac{\theta - \theta^J}{1-\theta}\Big) + \beta G(\boldsymbol{X}_J, \boldsymbol{x}_*) \\
&\quad + \gamma(\zeta_2\lambda_J + \varepsilon).
\end{aligned}
$$

□

The next step in my research would be the analysis of applying the HR method for recovery of $\boldsymbol{x}_*$ from $\boldsymbol{y}_*^{\varepsilon}$ when $\boldsymbol{y}_*^{\varepsilon} = A(\boldsymbol{x}_*) + \boldsymbol{\epsilon}$ and the observation operator A is non-linear. A natural starting point for this future research would be to study the general Tikhonov regularization method for providing approximate solutions to the non-linear equation $A(\boldsymbol{x}) = \boldsymbol{y}_*^{\varepsilon}$.

Appendix A: More on Tikhonov Regularization

We devote this chapter to the discussion of individual implementations of the general Tikhonov regularization method. Given $\boldsymbol{y}$ in a Hilbert space $\mathcal{Y}$ equipped with a norm $||\cdot||_{\mathcal{Y}}$, it is generally difficult to find an unknown $\boldsymbol{x}$ from another Hilbert space $\mathcal{X}$ satisfying the linear equation $A\boldsymbol{x} = \boldsymbol{y}$ exactly when the linear operator A is ill-posed. The general Tikhonov regularization method finds an approximate solution to $A\boldsymbol{x} = \boldsymbol{y}$ from a convex feasible set $\mathcal{C} \subset \mathcal{X}$ with the help of an extra regularization parameter $\lambda > 0$, and the approximate solution satisfies the following,

$$\boldsymbol{x}_{T(\lambda)} = \arg\min_{\boldsymbol{x}\in\mathcal{C}} \left\{ \lambda f(\boldsymbol{x}) + \frac{1}{2}||\boldsymbol{y} - A\boldsymbol{x}||_{\mathcal{Y}}^2 \right\}. \tag{A.1}$$

Here the non-negative auxiliary function $f : \mathcal{X} \to \mathbb{R}^+$ is a stabilizing function. The appropriate choice of the stabilizing function f and the feasible set $\mathcal{C}$ will be discussed in details in each individual section.

A.1 The Constrained ℓ_2 Method

The constrained ℓ_2 method is used to tackle the ill-posed linear system $A\boldsymbol{x} = \boldsymbol{y}$ where the matrix $A \in \mathbb{R}^{M\times N}$ is under-determined ($M \ll N$). It finds an approximate solution within a convex feasible set $\mathcal{B} = \{\boldsymbol{x} \in \mathbb{R}^N \big| A\boldsymbol{x} = \boldsymbol{y}\}$, and it satisfies the following

$$\boldsymbol{x}_2 \coloneqq \operatorname*{arg\,min}_{\boldsymbol{x}\in\mathcal{B}} \Big\{||\boldsymbol{x}||_2\Big\}. \tag{A.2}$$

Remark A.1.1. *We set* $\mathcal{X} = \mathbb{R}^N$, $\mathcal{Y} = \mathbb{R}^M$, $||\cdot||_{\mathcal{Y}} = ||\cdot||_2$, $\mathcal{C} = \mathcal{B}$, *and* $f(\boldsymbol{x}) = ||\boldsymbol{x}||_2$ *in* (A.1). *When we let* $\lambda \to \infty$, *we obtain the constrained* ℓ_2 *method.*

The minimizer to the constrained ℓ_2 method has a closed form expression.

Lemma A.1.2. *Assume that the matrix* A *has linearly independent rows. The minimizer* $\boldsymbol{x}_2$ *to* (A.2) *has a closed form expression, i.e.,*

$$\boldsymbol{x}_2 \coloneqq A^\top (AA^\top)^{-1}\boldsymbol{y}. \tag{A.3}$$

Proof. First, we want to show the solution defined in (A.3) satisfies the feasibility condition, i.e., $A\boldsymbol{x} = \boldsymbol{y}$. Let $\tilde{\boldsymbol{x}} = A^\top(AA^\top)^{-1}\boldsymbol{y}$, we have

$$A\tilde{\boldsymbol{x}} = AA^\top(AA^\top)^{-1}\boldsymbol{y} = \boldsymbol{y}.$$

Moreover, for any other $\boldsymbol{x} \in \mathbb{R}^N$ such that $A\boldsymbol{x} = \boldsymbol{y}$, we have,

$$\langle \boldsymbol{x} - \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{x}}\rangle = \langle \boldsymbol{x} - \tilde{\boldsymbol{x}}, A^\top(AA^\top)^{-1}\boldsymbol{y}\rangle = \langle A(\boldsymbol{x} - \tilde{\boldsymbol{x}}), (AA^\top)^{-1}\boldsymbol{y}\rangle = 0.$$

Hence the difference, $\boldsymbol{x} - \tilde{\boldsymbol{x}}$, is orthogonal to $\tilde{\boldsymbol{x}}$. Therefore,

$$||\boldsymbol{x}||_2^2 = ||\boldsymbol{x} - \tilde{\boldsymbol{x}}||_2^2 + ||\tilde{\boldsymbol{x}}||_2^2 \geq ||\tilde{\boldsymbol{x}}||_2^2$$

with the equality achieved only when $||\boldsymbol{x} - \tilde{\boldsymbol{x}}||_2 = 0$. □

Remark A.1.3. *The pair,* $(\boldsymbol{x}_2, A^\top\boldsymbol{r})$ *with* $\boldsymbol{r} = \boldsymbol{y} - A\boldsymbol{x}_2$, *is an extremal pair, because*

$$\langle \boldsymbol{x}_2, A^\top\boldsymbol{r}\rangle = 0 = ||\boldsymbol{x}_2||_2||A^\top\boldsymbol{r}||.$$

Here, we used the fact that $\boldsymbol{y} - A\boldsymbol{x}_2 = \boldsymbol{0}$.

A.2 The $\ell_2^2 - \ell_2^2$ Tikhonov Regularization Method

The $\ell_2^2 - \ell_2^2$ Tikhonov regularization method finds an approximate solution to the linear system $A\boldsymbol{x} = \boldsymbol{y}$ where $A \in \mathbb{R}^{M\times N}$ is ill-posed. The approximate solution satisfies the following

$$\boldsymbol{x}_{T(\lambda)} = \arg\min_{\boldsymbol{x}\in\mathbb{R}^N} \left\{\frac{\lambda}{2}||\boldsymbol{x}||_2^2 + \frac{1}{2}||\boldsymbol{y} - A\boldsymbol{x}||_2^2\right\}. \tag{A.4}$$

Remark A.2.1. *We set* $\mathcal{X} = \mathbb{R}^N$, $\mathcal{Y} = \mathbb{R}^M$, $||\cdot||_{\mathcal{Y}} = ||\cdot||_2$, $\mathcal{C} = \mathbb{R}^N$, *and* $f(\boldsymbol{x}) = ||\boldsymbol{x}||_2^2/2$ *in* (A.1)*, then we obtain the* $\ell_2^2 - \ell_2^2$ *Tikhonov regularization method.*

Lemma A.2.2. *The minimizer* $\boldsymbol{x}_{T(\lambda)}$ *of* (A.4) *has a closed form expression,*

$$\boldsymbol{x}_{T(\lambda)} = (A^\top A + \lambda I)^{-1} A^\top \boldsymbol{y}.$$

Proof. Define the energy functional $J : \mathbb{R}^N \to \mathbb{R}^+$ as $J(\boldsymbol{x}) = \frac{\lambda}{2}||\boldsymbol{x}||_2^2 + \frac{1}{2}||\boldsymbol{y} - A\boldsymbol{x}||_2^2$. The derivative of J is $D(\boldsymbol{x}) = \lambda\boldsymbol{x} - A^\top\boldsymbol{y} + A^\top A\boldsymbol{x}$. When the shifted normal equation has an solution, i.e., $(A^\top A + \lambda I)\boldsymbol{x}_c = A^\top\boldsymbol{y}$, the energy functional J has a critical point. The Hessian of J is $H(J) = A^\top A + \lambda I$. H is Symmetric Positive Definite. Hence the critical point is a minima. □

Remark A.2.3. *The pair* $(\boldsymbol{x}_{T(\lambda)}, A^\top\boldsymbol{r})$ *with* $\boldsymbol{r} = \boldsymbol{y} - A\boldsymbol{x}_{T(\lambda)}$ *is an extremal pair, since*

$$\langle\boldsymbol{x}_{T(\lambda)}, A^\top\boldsymbol{r}\rangle = \langle\boldsymbol{x}_{T(\lambda)}, \lambda\boldsymbol{x}_{T(\lambda)}\rangle = \lambda||\boldsymbol{x}_{T(\lambda)}||_2^2.$$

From (A.4)*, we have,*

$$A^\top\boldsymbol{r} = A^\top(\boldsymbol{y} - A\boldsymbol{x}_{T(\lambda)}) = \lambda\boldsymbol{x}_{T(\lambda)}.$$

Thus, $||A^\top\boldsymbol{r}||_2 = \lambda||\boldsymbol{x}_{T(\lambda)}||_2$.

A.3 The Unconstrained ℓ_1 Method

The unconstrained ℓ_1 method finds an approximate solution to the linear system $A\boldsymbol{x} = \boldsymbol{y}$ where $A \in \mathbb{R}^{M\times n}$ is ill-posed. The approximate solution satisfies the following,

$$\boldsymbol{x}_\lambda = \underset{\boldsymbol{x}\in\mathbb{R}^N}{\arg\min} \left\{\lambda||\boldsymbol{x}||_1 + \frac{1}{2}||\boldsymbol{y} - A\boldsymbol{x}||_2^2\right\}. \tag{A.5}$$

Remark A.3.1. *We set* $\mathcal{X} = \mathbb{R}^N$*,* $\mathcal{Y} = \mathbb{R}^M$*,* $||\cdot||_{\mathcal{Y}} = ||\cdot||_2$*,* $\mathcal{C} = \mathbb{R}^N$*, and* $f(\boldsymbol{x}) = ||\boldsymbol{x}||_1$ *in* (A.1)*, then we obtain the unconstrained* ℓ_1 *method.*

We present the following lemma on the Euler-Lagrange equation which the solution from (A.5) would satisfy.

Lemma A.3.2. *The solution,* $\boldsymbol{x}_\lambda$ *of* (A.5)*, also satisfies the following equation,*

$$\lambda\boldsymbol{sgn}(\boldsymbol{x}_\lambda) + A^\top(A\boldsymbol{x}_\lambda - \boldsymbol{y}) = \mathbf{0}.$$

Proof. We start from the following energy functional $J : \mathbb{R}^N \to \mathbb{R}$,

$$J(\boldsymbol{x}) = \lambda||\boldsymbol{x}||_1 + \frac{1}{2}||\boldsymbol{y} - A\boldsymbol{x}||_2^2.$$

We define a sub-gradient to the ℓ_1 norm,

$$(\mathbf{sgn}(\boldsymbol{x}))_i = \mathrm{sgn}((\boldsymbol{x})_i) = \begin{cases} 1, & (\boldsymbol{x})_i > 0 \\ 0, & (\boldsymbol{x})_i = 0 \\ -1, & (\boldsymbol{x})_i < 0 \end{cases}.$$

Let $\boldsymbol{x}_\lambda$ be a solution of (A.5). For any $\boldsymbol{x}_\lambda + \epsilon\boldsymbol{u}$, $J(\boldsymbol{x}_\lambda) \le J(\boldsymbol{x}_\lambda + \epsilon\boldsymbol{u})$. We expand the difference term, $J(\boldsymbol{x}_\lambda + \epsilon\boldsymbol{u}) - J(\boldsymbol{x}_\lambda)$, and obtain

$$J(\boldsymbol{x}_\lambda + \epsilon\boldsymbol{u}) - J(\boldsymbol{x}_\lambda) = \epsilon\langle\lambda\mathbf{sgn}(\boldsymbol{x}_\lambda) + A^\top(A\boldsymbol{x}_\lambda - \boldsymbol{y}), \boldsymbol{u}\rangle + \frac{\epsilon^2}{2}||A\boldsymbol{u}||_2^2$$

Since $J(\boldsymbol{x}_\lambda + \epsilon \boldsymbol{u}) - J(\boldsymbol{x}_\lambda) \geq 0$ for any small ϵ, we let $\epsilon \downarrow 0$ and divide both sides by ϵ to obtain

$$\epsilon\langle \lambda \mathbf{sgn}(\boldsymbol{x}_\lambda) + A^\top(A\boldsymbol{x}_\lambda - \boldsymbol{y}), \boldsymbol{u}\rangle \geq 0;$$

Next we let $\epsilon \uparrow 0$ and divide both sides by ϵ again to obtain

$$\epsilon\langle \lambda \mathbf{sgn}(\boldsymbol{x}_\lambda) + A^\top(A\boldsymbol{x}_\lambda - \boldsymbol{y}), \boldsymbol{u}\rangle \leq 0.$$

It follows that $\epsilon\langle \lambda \mathbf{sgn}(\boldsymbol{x}_\lambda) + \lambda A^\top(A\boldsymbol{x}_\lambda - \boldsymbol{y}), \boldsymbol{u}\rangle = 0$. Since $\boldsymbol{u}$ is picked arbitrarily, we have the minimizer, $\boldsymbol{x}_\lambda$, satisfying the following Euler-Lagrange equation,

$$\lambda \mathbf{sgn}(\boldsymbol{x}_\lambda) + A^\top(A\boldsymbol{x}_\lambda - \boldsymbol{y}) = \mathbf{0}.$$

□

The following lemma is concerned with the extremal pair relationship.

Lemma A.3.3. *The pair, $(\boldsymbol{x}_\lambda, A^\top \boldsymbol{r})$ with $\boldsymbol{r} = \boldsymbol{y} - A\boldsymbol{x}_\lambda$, is called an extremal pair, because the pair satisfies the following equality,*

$$\langle \boldsymbol{x}_\lambda, A^\top \boldsymbol{r}\rangle 2 = ||\boldsymbol{x}_\lambda||_1 ||A^\top \boldsymbol{r}||_\infty.$$

Proof. From signum equation in lemma A.3.2, we have

$$\langle \boldsymbol{x}_\lambda, A^\top \boldsymbol{r}\rangle 2 = \langle \boldsymbol{x}_\lambda, A^\top(\boldsymbol{y} - A\boldsymbol{x}_\lambda)\rangle = \langle \boldsymbol{x}_\lambda, \lambda \mathbf{sgn}(\boldsymbol{x}_\lambda)\rangle = \lambda ||\boldsymbol{x}_\lambda||_1$$

Furthermore, we have

$$||A^\top \boldsymbol{r}||_\infty = ||A^\top(\boldsymbol{y} - A\boldsymbol{x}_\lambda)||_\infty = ||\lambda \mathbf{sgn}(\boldsymbol{x}_\lambda)||_\infty = \lambda.$$

Putting the two together, we have

$$\langle \boldsymbol{x}_\lambda, A^\top \boldsymbol{r}\rangle = ||\boldsymbol{x}_\lambda||_1 ||A^\top \boldsymbol{r}||_\infty.$$

□

Remark A.3.4. *When we use the usual Hölder's Inequality on any pair* $(\boldsymbol{x}, A^\top(\boldsymbol{y}-A\boldsymbol{x}))$*,*

$$\langle \boldsymbol{x}, A^\top(\boldsymbol{y} - A\boldsymbol{x})\rangle 2 \leq ||\boldsymbol{x}||_1 ||A^\top(\boldsymbol{y} - A\boldsymbol{x})||_\infty.$$

The equality is realized when $\boldsymbol{x} = \boldsymbol{x}_\lambda$*, which is another optimal condition for the minimizer of* (A.5) *to hold.*

Using lemma A.3.2, we can show an upper bound on λ to avoid trivial solution.

Lemma A.3.5. *The minimizer,* $\boldsymbol{x}_\lambda$*, of* (A.5) *is non-trivial if and only if* $\lambda \leq ||A^\top \boldsymbol{y}_*^\varepsilon||_\infty$*.*

Proof. We consider the extremal pair, $(\boldsymbol{x}_\lambda, A^\top \boldsymbol{r})$, where $\boldsymbol{r} = \boldsymbol{y}_*^\varepsilon - A\boldsymbol{x}_\lambda$. When $\boldsymbol{x}_\lambda$ is non-trivial, it follows that,

$$\begin{aligned}\lambda||\boldsymbol{x}_\lambda||_1 &\leq \lambda||\boldsymbol{x}_\lambda||_1 + ||A\boldsymbol{x}_\lambda||_2^2 = \langle \boldsymbol{x}_\lambda, A^\top \boldsymbol{r}\rangle + ||A\boldsymbol{x}_\lambda||_2^2 = \langle \boldsymbol{x}_\lambda, A^\top(\boldsymbol{r} + A\boldsymbol{x}_\lambda)\rangle \\ &\leq \langle \boldsymbol{x}_\lambda, A^\top \boldsymbol{y}\rangle \leq ||\boldsymbol{x}_\lambda||_1 ||A^\top \boldsymbol{y}||_\infty.\end{aligned}$$

Therefore $\lambda \leq ||A^\top \boldsymbol{y}||_\infty$. We use a similar idea from [49] to show that when $\lambda > ||A^\top \boldsymbol{y}||_\infty$, $\boldsymbol{x}_\lambda = \boldsymbol{0}$. □

Now, we are ready to show the bounds on the residual term, $\boldsymbol{r} = \boldsymbol{y} - A\boldsymbol{x}_\lambda$.

Lemma A.3.6. *Assume that the matrix* A *has linealry independent rows. The residual term,* $\boldsymbol{r} = \boldsymbol{y} - A\boldsymbol{x}_\lambda$*, satisfies the following bounds,*

$$\lambda||A^\top||_p^{-1} \leq ||\boldsymbol{r}||_p \leq \lambda\sqrt[p]{N}||(AA^\top)^{-1}A||_p, \quad \textit{for } 1 \leq p \leq \infty.$$

Proof. We begin from the equation in lemma A.3.2, we have $\boldsymbol{r} = \lambda(AA^\top)^{-1}A\mathbf{sgn}(\boldsymbol{x}_\lambda)$. It follows that,

$$||\boldsymbol{r}||_p = \lambda||(AA^\top)^{-1}A\mathbf{sgn}(\boldsymbol{x}_\lambda)||_p \leq \lambda||(AA^\top)^{-1}A||_p||\mathbf{sgn}(\boldsymbol{x}_\lambda)||_p$$
$$\leq \lambda\sqrt[p]{N}||(AA^\top)^{-1}A||_p.$$

For the lower bound, we take the ℓ_p norm of the signum equation in lemma A.3.2 to obtain

$$||\lambda\mathbf{sgn}(\boldsymbol{x}_\lambda)||_p = ||A^\top\boldsymbol{r}||_p \leq ||A^\top||_p||\boldsymbol{r}||_p.$$

It follows that

$$||\boldsymbol{r}||_p \geq \lambda||\mathbf{sgn}(\boldsymbol{x}_\lambda)||_p||A^\top||_p^{-1} \geq \lambda||A^\top||_p^{-1}.$$

□

A.4 The Least Square Method

The Least Square (LS) method finds a linear model as an approximate solution to the linear equation $A\boldsymbol{x} = \boldsymbol{y}$ where $A \in \mathbb{R}^{M\times N}$ is ill-posed, and the linear model satisfies the following,

$$\boldsymbol{x}_{LS} = \underset{\boldsymbol{x}\in\mathbb{R}^N}{\arg\min}\left\{||\boldsymbol{y} - A\boldsymbol{x}||_2\right\}. \tag{A.6}$$

Remark A.4.1. *We set $\mathcal{X} = \mathbb{R}^N$, $\mathcal{Y} = \mathbb{R}^M$, $||\cdot||_{\mathcal{Y}} = ||\cdot||_2$, $\mathcal{C} = \mathbb{R}^N$, and $f(\boldsymbol{x}) = 0$ in* (A.1)*, then we obtain the LS method. Here we used the fact that minimizer of $||\boldsymbol{y} - A\boldsymbol{x}||_2$ is the same as $||\boldsymbol{y} - A\boldsymbol{x}||_2^2/2$.*

Lemma A.4.2. *Assume that A has linearly independent columns. The solution of* (A.6) *has a closed form expression, $\boldsymbol{x}_{LS} = (A^\top A)^{-1}A^\top\boldsymbol{y}$.*

Proof. We define an energy functional $J : \mathbb{R}^N \to \mathbb{R}$ as $J(\boldsymbol{x}) = ||\boldsymbol{y} - A\boldsymbol{x}||_2^2 = ||\boldsymbol{y}||_2^2 - 2\langle A^\top \boldsymbol{y}, \boldsymbol{x}\rangle + ||A\boldsymbol{x}||_2^2$. The derivative of J is $D(\boldsymbol{x}) = -2A^\top \boldsymbol{y} + 2A^\top A\boldsymbol{x}$. The only critical point of J is $\boldsymbol{x}_{cp} = (A^\top A)^{-1} A^\top \boldsymbol{y}$. The Hessian of J is $H(\boldsymbol{x}) = 2A^\top A$, which is SPD. Thus $\boldsymbol{x}_{cp}$ is the unique minimizer. $\boldsymbol{x}_{LS} = \boldsymbol{x}_{cp}$ as claimed. □

Remark A.4.3. *Let $\boldsymbol{r} = \boldsymbol{y} - A\boldsymbol{x}_{LS}$, and we consider the pair $(\boldsymbol{x}_{LS}, A^\top \boldsymbol{r})$. Since $A^\top \boldsymbol{r} = A^\top(\boldsymbol{y} - A\boldsymbol{x}_{LS}) = \boldsymbol{0}$, we have*

$$\langle \boldsymbol{x}_{LS}, A^\top \boldsymbol{r}\rangle = 0 = 0||\boldsymbol{x}_{LS}||_2.$$

Therefore, we call the pair $(\boldsymbol{x}_{LS}, A^\top \boldsymbol{r})$ an extremal pair.

A.5 The Least Absolute Shrinkage and Selection Operator Method

When given a data from M observations, $\{A, \boldsymbol{y}\}$ where $A \in \mathbb{R}^{M\times N}$ and $\boldsymbol{y} \in \mathbb{R}^M$, the constrained Least Absolute Shrinkage and Selection Operator (LASSO) method finds a linear model from a convex feasible set $\mathcal{B}_\lambda = \{\boldsymbol{x} \in \mathbb{R}^N \Big| ||\boldsymbol{x}||_1 \leq \lambda\}$ such that the linear model satisfies the following

$$\boldsymbol{x}_{LA(\lambda)} = \underset{\boldsymbol{x}\in\mathcal{B}_\lambda}{\arg\min} \Big\{||\boldsymbol{y} - A\boldsymbol{x}||_2\Big\}. \tag{A.7}$$

We will use the following notation $(\cdot)^+$, which is defined as

$$(x)^+ = \begin{cases} x & x > 0 \\ 0 & x \geq 0 \end{cases}, \quad \text{for } x \in \mathbb{R}.$$

Remark A.5.1. *We set $\mathcal{X} = \mathbb{R}^N$, $\mathcal{Y} = \mathbb{R}^M$, $||\cdot||_{\mathcal{Y}} = ||\cdot||_2$, $\mathcal{C} = \mathcal{B}_\lambda$, and $f(\boldsymbol{x}) = 0$ in* (A.1), *then we obtain the constrained LASSO method. Here we used the fact that minimizer of $||\boldsymbol{y} - A\boldsymbol{x}||_2$ is the same as $||\boldsymbol{y} - A\boldsymbol{x}||_2^2/2$.*

Lemma A.5.2. *The constrained LASSO linear model,* $\boldsymbol{x}_{LA(\lambda)}$ *of* (A.7) *has the following closed form expression,*

$$(\boldsymbol{x}_{LA(\lambda)})_i = sgn((\boldsymbol{x}_{LS})_i)(\big|(\boldsymbol{x}_{LS})_i\big| - \gamma)^+, \quad \text{for } 1 \leq i \leq N.$$

The parameter $\gamma > 0$ *is chosen such that* $||\boldsymbol{x}_{LA(\lambda)}||_1 = \lambda$.

Proof. We start from the definition of the constrained LASSO method, minimizing the objective function $||\boldsymbol{y} - A\boldsymbol{x}||_2$ is also equivalent to minimizing $||\boldsymbol{y} - A\boldsymbol{x}||_2^2$. It follows that,

$$\begin{aligned} ||\boldsymbol{y} - A\boldsymbol{x}||_2^2 &= ||\boldsymbol{y}||_2^2 - 2\langle \boldsymbol{y}, A\boldsymbol{x}\rangle + ||A\boldsymbol{x}||_2^2 \\ &= ||\boldsymbol{x}_{LS} - \boldsymbol{x}||_2^2 + ||\boldsymbol{y}||_2^2 - ||\boldsymbol{x}_{LS}||_2^2. \end{aligned}$$

We used the fact that $A^\top A = I$. Since $\boldsymbol{y}$ and $\boldsymbol{x}_{LS}$ are considered fixed, the constrained LASSO method finds a linear model from the following,

$$\boldsymbol{x}_{LA(\lambda)} = \underset{\boldsymbol{x} \in \mathcal{B}_\lambda}{\arg\min} \Big\{ ||\boldsymbol{x}_{LS} - \boldsymbol{x}||_2^2 \Big\}.$$

Using the Lagrange multiplier method, there exists a $\gamma > 0$ such that

$$\boldsymbol{x}_{LA(\lambda)} = \underset{\boldsymbol{x} \in \mathbb{R}^N}{\arg\min} \Big\{ ||\boldsymbol{x}_{LS} - \boldsymbol{x}||_2^2 + \gamma||\boldsymbol{x}||_1 \Big\}.$$

The objective function is completely de-coupled, i.e.,

$$||\boldsymbol{x}_{LS} - \boldsymbol{x}||_2^2 + \gamma||\boldsymbol{x}||_1 = \sum_{i=1}^{N} \Big((\boldsymbol{x})_i^2 - 2(\boldsymbol{x})_i(\boldsymbol{x}_{LS})_i + (\boldsymbol{x}_{LS})_i^2 + \gamma\big|(\boldsymbol{x})_i\big| \Big) = \sum_{i=1}^{N} L_i((\boldsymbol{x})_i).$$

Here each function $L_i(x) = x^2 - 2x(\boldsymbol{x}_{LS})_i + (\boldsymbol{x}_{LS})_i^2 + \gamma\big|x\big|$ for $x \in \mathbb{R}$. The minima of the function L_i is $x_{\min} = \text{sgn}((\boldsymbol{x}_{LS})_i)(\big|(\boldsymbol{x}_{LS})_i\big| - \gamma/2)^+$. Since each L_i depends only on $(\boldsymbol{x})_i$, we have

$$(\boldsymbol{x}_{LA(\lambda)})_i = \text{sgn}((\boldsymbol{x}_{LS})_i)(\big|(\boldsymbol{x}_{LS})_i\big| - \gamma/2)^+, \quad \text{for } 1 \leq i \leq N.$$

□

Remark A.5.3. *The following list shows additional properties which the LASSO minimier would satisfy when it is found using the formula in lemma A.5.2.*

- $||\boldsymbol{x}_{LA(\lambda)}||_1 = \lambda$.
- *When* $(\boldsymbol{x}_{LS})_i = 0$ *for some* $1 \leq i \leq N$, $(\boldsymbol{x}_{LA(\lambda)})_i = 0$. *Thus, supp*$(\boldsymbol{x}_{LA(\lambda)}) \subset$ *supp*$(\boldsymbol{x}_{LS})$.
- *When* $\big|(\boldsymbol{x}_{LS})_i\big| \leq \gamma$, $(\boldsymbol{x}_{LA(\lambda)})_i = 0$.
- *When* $\gamma = 0 \Rightarrow \lambda = ||\boldsymbol{x}_{LS}||_1$, $\boldsymbol{x}_{LA(\lambda)} = \boldsymbol{x}_{LS}$.
- *When* $\gamma > ||\boldsymbol{x}_{LS}||_\infty \Rightarrow \lambda = 0$, $\boldsymbol{x}_{LA(\lambda)} = \mathbf{0}$.
- *For* $i \in$ *supp*$(\boldsymbol{x}_{LA(\lambda)})$, $(\boldsymbol{x}_{LA(\lambda)})_i$ *and* $(\boldsymbol{x}_{LS})_i$ *have the same sign, and* $\big|(\boldsymbol{x}_{LA(\lambda)})_i\big| < \big|(\boldsymbol{x}_{LS})_i\big|$.

Lemma A.5.4. *The pair,* $(\boldsymbol{x}_{LA(\lambda)}, A^\top \boldsymbol{r})$, *where* $\boldsymbol{r} = \boldsymbol{y}_* - A\boldsymbol{x}_{LA(\lambda)}$, *is called an extremal pair, because it satisfies the following equality,*

$$\langle \boldsymbol{x}_{LA(\lambda)}, A^\top \boldsymbol{r} \rangle = ||\boldsymbol{x}_{LA(\lambda)}||_1 ||A^\top \boldsymbol{r}||_\infty. \tag{A.8}$$

Moreover, we have $||A^\top \boldsymbol{r}||_\infty = \gamma$ *and* $||\boldsymbol{x}_{LA(\lambda)}||_1 = \lambda$.

Proof. First, let $\otimes$ denote the component wise multiplication for vectors of the same size, $\mathbf{1} \in \mathbb{R}^N$ with all of its entries being 1's, $\big|\cdot\big|$ and $(\cdot)^+$ are defined component wise for vectors. Starting from the inner product, we have,

$$\begin{aligned}\langle \boldsymbol{x}_{LA(\lambda)}, A^\top \boldsymbol{r}\rangle &= \langle \boldsymbol{x}_{LA(\lambda)}, A^\top(\boldsymbol{y}_* - A\boldsymbol{x}_{LA(\lambda)})\rangle = \langle \boldsymbol{x}_{LA(\lambda)}, \boldsymbol{x}_{LS} - \boldsymbol{x}_{LA(\lambda)}\rangle \\ &= \langle \mathbf{sgn}(\boldsymbol{x}_{LS}) \otimes (\lvert\boldsymbol{x}_{LS}\rvert - \gamma\mathbf{1})^+, \boldsymbol{x}_{LS}\rangle - ||\mathbf{sgn}(\boldsymbol{x}_{LS}) \otimes (\lvert\boldsymbol{x}_{LS}\rvert - \gamma\mathbf{1})^+||_2^2 \\ &= \langle (\lvert\boldsymbol{x}_{LS}\rvert - \gamma\mathbf{1})^+, \lvert\boldsymbol{x}_{LS}\rvert\rangle - ||(\lvert\boldsymbol{x}_{LS}\rvert - \gamma\mathbf{1})^+||_2^2 \\ &= \langle \lvert\boldsymbol{x}_{LS}\rvert - \gamma\mathbf{1})^+, \gamma\mathbf{1}\rangle = \gamma||\boldsymbol{x}_{LA(\lambda)}||_1.\end{aligned}$$

Meanwhile,

$$\begin{aligned}||A^\top \boldsymbol{r}||_\infty &= ||A^\top(\boldsymbol{y}_* - A\boldsymbol{x}_{LA(\lambda)}||_\infty = ||\boldsymbol{x}_{LS} - \boldsymbol{x}_{LA(\lambda)}||_\infty \\ &= ||\lvert\boldsymbol{x}_{LS}\rvert - (\lvert\boldsymbol{x}_{LS}\rvert - \gamma\mathbf{1})^+||_\infty = \gamma.\end{aligned}$$

Therefore, $\langle \boldsymbol{x}_{LA(\lambda)}, A^\top \boldsymbol{r}\rangle = ||\boldsymbol{x}_{LA(\lambda)}||_1 ||A^\top \boldsymbol{r}||_\infty$ as claimed. Moreover, $||\boldsymbol{x}_{LA(\lambda)}||_1 = \lambda$ is shown in remark A.5.3. □

Given the set of data from M observations, $\{A, \boldsymbol{y}\}$, the penalized LASSO method finds a linear model from the following,

$$\boldsymbol{x}_{LA(\lambda)} = \underset{\boldsymbol{x}\in\mathbb{R}^N}{\arg\min} \left\{\lambda||\boldsymbol{x}||_1 + \frac{1}{2}||\boldsymbol{y} - A\boldsymbol{x}||_2^2\right\}. \tag{A.9}$$

Remark A.5.5. *We set* $\mathcal{X} = \mathbb{R}^N$, $\mathcal{Y} = \mathbb{R}^M$, $||\cdot||_\mathcal{Y} = ||\cdot||_2$, $\mathcal{C} = \mathbb{R}^N$, *and* $f(\boldsymbol{x}) = ||\boldsymbol{x}||_1$ *in* (A.1)*, then we obtain the penalized LASSO method.*

Following the idea shown in lemma A.3.2, we can show that $\boldsymbol{x}_{LA(\lambda)}$ also satisfies the following,

$$\lambda\mathbf{sgn}(\boldsymbol{x}_{LA(\lambda)}) + A^\top(A\boldsymbol{x}_{LA(\tau)} - \boldsymbol{y}_*) = \mathbf{0}. \tag{A.10}$$

With this signum equation established for the penalized LASSO linear model, we show the following lemma on the extremal pair relationship.

Lemma A.5.6. *The pair* $(\boldsymbol{x}_{LA(\lambda)}, A^\top \boldsymbol{r})$ *with* $\boldsymbol{x}_{LA(\lambda)}$ *in* (A.9) *and* $\boldsymbol{r} = \boldsymbol{y}_* - A\boldsymbol{x}_{LA(\lambda)}$, *is an extremal pair. It satisfies the following,*

$$\langle \boldsymbol{x}_{LA(\lambda)}, A^\top \boldsymbol{r} \rangle = ||\boldsymbol{x}_{LA(\lambda)}||_1 ||A^\top \boldsymbol{r}||_\infty.$$

Proof. Since $\boldsymbol{x}_{LA(\lambda)}$ satisfies (A.10), we have

$$\begin{aligned}\langle \boldsymbol{x}_{LA(\lambda)}, A^\top \boldsymbol{r} \rangle &= \langle \boldsymbol{x}_{LA(\lambda)}, A^\top (\boldsymbol{y}_* - A\boldsymbol{x}_{LA(\lambda)}) \rangle = \langle \boldsymbol{x}_{LA(\lambda)}, \lambda \textbf{sgn}(\boldsymbol{x}_{LA(\lambda)}) \rangle \\ &= \lambda ||\boldsymbol{x}_{LA(\lambda)}||_1.\end{aligned}$$

Meanwhile,

$$\begin{aligned}||A^\top \boldsymbol{r}||_\infty &= ||A^\top (\boldsymbol{y}_* - A\boldsymbol{x}_{LA(\lambda)})||_\infty = ||\lambda \textbf{sgn}(\boldsymbol{x}_{LA(\lambda)})||_\infty \\ &= \lambda ||\textbf{sgn}(\boldsymbol{x}_{LA(\lambda)})||_\infty = \lambda.\end{aligned}$$

Therefore $\langle \boldsymbol{x}_{LA(\lambda)}, A^\top \boldsymbol{r} \rangle = ||\boldsymbol{x}_{LA(\lambda)}||_1 ||A^\top \boldsymbol{r}||_\infty$. □

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