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Generalized Riccati Equations

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GENERALIZED RICCATI EQUATIONS*

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I. Introduction

The matrix Riccati equation refers to the quadratic differential equation (RDE)

$$\dot{P}(t) = B_{21}(t) + B_{22}(t)P(t) - P(t)B_{11}(t) - P(t)B_{12}(t)P(t)$$

on the space $\mathbb{R}^{m\times n}$ of real $m \times n$ matrices. In this equation, $B_{21}(t), B_{22}(t), B_{11}(t)$ and $B_{12}(t)$ are time varying real matrices of dimensions $m \times n, m \times m, n \times n$ and $n \times m$ respectively. The Riccati equation is crucial to a diversity of applications including among others transmission line phenomena, optimal control and filtering, diffusion problems, and invariant imbedding [1]. Thus it is natural to consider the problem of understanding which class of differential equations should be regarded as a generalization of the Riccati equation on $\mathbb{R}^{m\times n}$. It is the purpose of this paper to illustrate a possible point of view in this direction.

It is well-known (see e.g. [5]) that the flow corresponding to the RDE can be considered as the description in local coordinates of the flow linearly induced by $B(t) = \begin{bmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{bmatrix}$ on the Grassmann manifold $G^n(\mathbb{R}^{n+m})$ of n-dimensional subspaces of $\mathbb{R}^{n+m}$. More precisely, define a mapping $\psi : \mathbb{R}^{m\times n} \to G^n(\mathbb{R}^{n+m})$ by $\psi(P) = sp\begin{bmatrix} I \\ P \end{bmatrix}$, the column space of the $(n+m) \times n$ matrix $\begin{bmatrix} I \\ P \end{bmatrix}$. The mapping ψ imbeds $\mathbb{R}^{m\times n}$ as an open and dense subset of $G^n(\mathbb{R}^{n+m})$. Consequently, we may view $G^n(\mathbb{R}^{n+m})$ as a compactification of the space of $m \times n$ matrices.

Let now $X_B(t, t_0)$ be the transition matrix associated with $B(t)$, and define a flow Φ_B on $G^n(\mathbb{R}^{n+m})$ by

$$\Phi_B(t, S_0, t_0) = X_B(t, t_0)(S_0),$$

$S_o \in G^n(\mathbb{R}^{n+m})$. The flow Φ_B is related to the flow $P(t, P_0, t_0)$ of the RDE on $\mathbb{R}^{m\times n}$ by

$$\psi(P(t, P_0, t_0)) = \Phi_B(t, \psi(P_0), t_0)$$

which is valid as long as the solution $P(t, P_0, t_0)$ continues to exist.

It is clear that the definition of the flow Φ_B linearly induced by B on $G^n(\mathbb{R}^{n+m})$ can be generalized to other homogeneous spaces of $GL(n, \mathbb{R})$, such as the so-called partial flag manifolds. In fact, such flows are of interest in some applications and have been considered in the literature (see e.g. [2], [3], [4], [6]). Our suggestion is to regard them as "completions" of generalized Riccati flows on the affine space $\mathbb{R}^N$, in the sense that a generalized Riccati flow on $\mathbb{R}^N$ should be the description in local coordinates of a flow linearly induced by B on a partial flag manifold.

We point out though that, even in the case under consideration, a clear distinction should be maintained between the flow $P(t, P_0, t_0)$ on $\mathbb{R}^{m\times n}$ and the flow $\Phi_B(t, S_0, t_0)$. The two flows, although closely related, present significant differences such as, for instance, finite escape time phenomena which generally occur for $P(t, P_0, t_0)$ but cannot appear for the complete flow $\Phi_B(t, S_0, t_0)$.

We now illustrate in more detail our point of view. Let $K = (k_1, \cdots, k_d)$ be a d-tuple of positive integers satisfying the condition $0 < k_1 < \cdots < k_d < k_{d+1} = n$. then the (partial) flag manifold $\mathrm{Flag}(K, \mathbb{R}^n)$ is defined by:

$$\mathrm{Flag}(K, \mathbb{R}^n) = \{(S_1, \cdots, S_d) \in \prod_{i=1}^{d} G^{k_i}(\mathbb{R}^n) \mid S_1 \subset \cdots \subset S_d\},$$

a non-singular subvariety of the indicated product of Grassmannians. In fact, $\mathrm{Flag}(K, \mathbb{R}^n)$ can be given the structure of a real analytic manifold of dimension $N(K) \triangleq \sum_{i=1}^{d} k_i(k_{i+1} - k_i)$. Clearly, in the special case $d = 1$, $\mathrm{Flag}(K, \mathbb{R}^n)$ is a Grassmann manifold.

If $B(t)$ and $X_B(t, t_0)$ are as before, then we can define a flow on $\mathrm{Flag}(K, \mathbb{R}^n)$ by

$$\Phi_B(t, (S_1, \cdots, S_d), t_0) = (X_B(t, t_0)(S_1), \cdots, X_B(t, t_0)(S_d)),$$

and refer to it as the flow linearly induced by B.

Let now $\psi : \mathbb{R}^{N(K)} \to \mathrm{Flag}(K, \mathbb{R}^n)$ be a regular imbedding (i.e. a one-to-one smooth mapping with an everywhere non-singular differential) onto an open and dense subset of $\mathrm{Flag}(K, \mathbb{R}^n)$. It is immediate to show that for each $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^{N(K)}$, there exist $\delta_{t_0,x_0} > 0$ and an open neighborhood U_{t_0,x_0} of x_0 in $\mathbb{R}^{N(K)}$ with the property that $\Phi_B(t, \psi(x), t_0) \subset \psi(\mathbb{R}^{N(K)})$, for all $(t, x) \in (t_0 - \delta_{t_0,x_0}, t_0 + \delta_{t_0,x_0}) \times U_{t_0,x_0}$. Thus, for each $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^{N(K)}$ we may define a map $R_{\psi,t_0,x_0} : (t_0 - \delta_{t_0,x_0}, t_0 + \delta_{t_0,x_0}) \times U_{t_0,x_0} \to \mathbb{R}^{N(K)}$ by

$$(t, x) \to \psi^{-1}\, \Phi_B(t, \psi(x), t_0).$$

It is easy to check that R_{ψ,t_0,x_0} is a local flow around x_0. Moreover, the family $\{R_{\psi,t_0,x_0} \mid x_0 \in \mathbb{R}^{N(K)}\}$ is compatible, in the sense that two flow lines that intersect, will coincide in their common domain. Therefore we obtain a time-varying vector field on $\mathbb{R}^{N(K)}$, which we denote by $Y_{B,\psi}(t)$ and refer to as a generalized Riccati field associated with B and ψ. We remark that if ψ and ϕ are two different imbeddings of the above type, there will be an open and dense subset of $\mathbb{R}^{N(K)}$ on which the integral curves of $Y_{B,\psi}(t)$ are carried diffeomorphically onto the integral curves of $Y_{B,\phi}(t)$.

In the next section we give an example of a Riccati field associated with a standard imbedding ψ, and we show that its integral curves are solutions of a polynomial differential equation of degree $d+1$. This equation coincides with the RDE in the special case $d = 1$.

II. GENERALIZED RICCATI EQUATIONS

Let $V_k(\mathbb{R}^m)$ denote the Stiefel manifold of full-rank $m \times k$ matrices, $k \leq m$, with entries in $\mathbb{R}$. Let $K = (k_1, \cdots, k_d)$ be as above, and let

$$V(K, \mathbb{R}^n) = \prod_{i=1}^{d-1} V_{k_{d-i}}(\mathbb{R}^{k_{d+1-i}})$$

$$G(K, \mathbb{R}) = \oplus_{i=0}^{d-1} GL(k_{d-i}, \mathbb{R}).$$

Define a right action of $G(K, \mathbb{R})$ on $V(K, \mathbb{R}^n)$ by

$$((g_d, \cdots, g_1), (X_d, \cdots, X_1)) \to (X_d g_d, d_d^{-1} X_{d-1} g_{d-1}, \cdots, g_2^{-1} X_1 g_1).$$

It is easy to check that this action is free and proper. Consequently, the orbit space $V(K, \mathbb{R}^n)/G(K, \mathbb{R})$ is a manifold. The orbit manifold $V(K, \mathbb{R}^n)/G(k, \mathbb{R})$ may be identified with Flag$(K, \mathbb{R}^n)$ via the natural bijection η given by:

$$\eta(< X_d, \cdots, X_1 >) = (spX_d \cdot \cdots \cdot X_1, spX_d \cdot \cdots \cdot X_2, \cdots, spX_d X_{d-1}, spX_d),$$

where $< X_d, \cdots, X_1 >$ denotes the $G(K, \mathbb{R})$-orbit of the point $(X_d, \cdots, X_1)) \in V(K, \mathbb{R}^n)$. (For more details, see [7].)

Let now $M_{ij}(\mathbb{R})$ denote the space of $i \times j$ matrices with entries in $\mathbb{R}$. We will make the following natural identification: $\prod_{i=0}^{d-1} M_{(k_{d+1-i} - k_{d-i})}, k_{d-i}(\mathbb{R}) \cong \mathbb{R}^{N(K)}$, and we define a mapping $\psi : \mathbb{R}^{N(K)} \to \mathrm{Flag}(K, \mathbb{R}^n)$ by

$$(P_d, \cdots, P_1) \to < \begin{bmatrix} I \\ P_d \end{bmatrix}, \begin{bmatrix} I \\ P_{d-1} \end{bmatrix}, \cdots, \begin{bmatrix} I \\ P_1 \end{bmatrix} > .$$

It is easy to see that ψ is a regular imbedding of $\mathbb{R}^{N(K)}$ onto an open and dense subset of $\mathrm{Flag}(K, \mathbb{R}^n)$. Thus, if $X_{d+1}(t, t_0) \triangleq X_B(t, t_0)$ is the transition matrix associated with $B(t) \in M_{n,n}(\mathbb{R})$, we may consider the Riccati field $Y_{B,\psi}(t)$.

If Φ_B denotes as usual the flow linearly induced by $B(t)$ on $\mathrm{Flag}(K, \mathbb{R}^n)$, then

$$\Phi_B(t, \psi(P_d^0, \cdots, P_1^0), t_0) = < X_{d+1}(t, t_0) \begin{bmatrix} I \\ P_d^0 \end{bmatrix}, \begin{bmatrix} I \\ P_{d-1}^0 \end{bmatrix}, \cdots, \begin{bmatrix} I \\ P_1^0 \end{bmatrix} > .$$

Next, if we put

$$X_{d+1}(t, t_0) \begin{bmatrix} I \\ P_d^0 \end{bmatrix} \triangleq \begin{bmatrix} X_d(t, t_0) \\ Z_d(t, t_0) \end{bmatrix}, P_d(t) \triangleq Z_d(t, t_0) X_d(t, t_0)^{-1},$$

then we see at once that $P_d(t)$ satisfies the RDE

$$\dot{P}_d(t) = B_{21}(t) + B_{22}(t) P_d(t) - P_d(t) B_{11}(t) - P_d(t) B_{12}(t) P_d(t)$$

and the initial condition $P_d(t_0) = P_d^0$, which expresses the dependency on t_0 that has been omitted in the definition of $P_d(t)$. Here $B(t) = \begin{bmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{bmatrix}$ is a block-partition of $B(t)$ corresponding to the partition $(k_d, n - k_d)$ of n. Thus $B_{11}(t), B_{12}(t), B_{21}(t)$ and $B_{22}(t)$ are $k_d \times k_d, k_d \times (n - k_d), (n - k_d) \times k_d$ and $(n - k_d) \times (n - k_d)$ respectively. Moreover, we observe that $X_d(t_0, t_0) = I$ and

$$\begin{aligned} \dot{X}_d(t, t_0) &= B_{11}(t) X_d(t, t_0) + B_{12}(t) Z_d(t, t_0) \\ &= (B_{11}(t) + B_{12}(t) P_d(t)) X_d(t, t_0). \end{aligned}$$

Therefore, $X_d(t, t_0)$ is the transition matrix for

$$B^{d-1}(t) \triangleq (B_{11}(t) + B_{12}(t) P_d(t)) \in M_{k_d, k_d}(\mathbb{R}).$$

Finally, keeping in mind the group action of $G(K,\mathbb{R})$ on $V(K,\mathbb{R}^n)$, we have:

$$\begin{aligned}\Phi_B(t,\psi(P_d^0,\cdots,P_1^0)t_0) &=< X_{d+1}(t,t_0)\begin{bmatrix} I \\ P_d^0\end{bmatrix},\begin{bmatrix} I \\ P_{d-1}^0\end{bmatrix},\cdots,\begin{bmatrix} I \\ P_1^0\end{bmatrix}> \\ &=< \begin{bmatrix} X_d(t,t_0) \\ Z_d(t,t_0)\end{bmatrix} X_d(t,t_0)^{-1}, X_d(t,t_0)\begin{bmatrix} I \\ P_{d-1}^0\end{bmatrix},\cdots,\begin{bmatrix} I \\ P_1^0\end{bmatrix}> \\ &=< \begin{bmatrix} I \\ P_d(t)\end{bmatrix}, X_d(t,t_0)\begin{bmatrix} I \\ P_{d-1}^0\end{bmatrix},\cdots,\begin{bmatrix} I \\ P_1^0\end{bmatrix}>.\end{aligned}$$

Thus, we can proceed inductively as before, to get matrices $X_{d-i}(t,t_0), Z_{d-i}(t,t_0), B^{d-i}(t)$, $P^{d-i}(t), i=0,\cdots,d-1$ defined as follows:

$$\begin{aligned}\begin{bmatrix} X_{d-i}(t,t_0) \\ Z_{d-i}(t,t_0)\end{bmatrix} &\triangleq X_{d-i+1}(t,t_0)\begin{bmatrix} I \\ P_{d-i}^0\end{bmatrix} \\ P_{d-i}(t) &\triangleq Z_{d-i}(t,t_0)X_{d-i}(t,t_0)^{-1} \\ B^{d-i-1}(t) &\triangleq (B_{11}^{d-1}(t) + B_{12}^{d-i}(t)P_{d-i}(t)) \\ B^d(t) &\triangleq B(t),\end{aligned}$$

where $(B_{\mu\nu}^{d-i}(t)), \mu,\nu = 1,2$ is a block-partition of $B^{d-i}(t)$ corresponding to the partition $(k_{d-i}, k_{d-i+1} - k_{d-i})$ of k_{d-i+1}. Hence, $X_{d-i}(t,t_0)$ is $k_{d-i}\times k_{d-i}, Z_{d-i}(t,t_0)$ is $(k_{d-i+1} - k_{d-i})\times k_{d-i}, P_{d-i}(t)$ is $(k_{d-i+1}-k_{d-i})\times k_{d-i}$ and $B^{d-i}(t)$ is $k_{d-i+1}\times k_{d-i}$. The above matrices are such that

i) $X_{d-i}(t,t_0)$ is the transition matrix associated with $B^{d-i-1}(t)$.

ii) $P_{d-i}(t)$ satisfies the differential equation:

(2) $\dot{P}_{d-i}(t) = B_{21}^{d-i}(t) + B_{22}^{d-i}(t)P_{d-i}(t) - P_{d-i}(t)B_{11}^{d-i}(t) - P_{d-i}(t)B_{12}^{d-i}(t)P_{d-i}(t)$

and the initial condition $P_{d-i}(t_0) = P_{d-i}^0$.

iii) $\Phi_B(t,\psi(P_d^0,\cdots,P_1^0),t_0) =< \begin{bmatrix} I \\ P_d(t)\end{bmatrix},\begin{bmatrix} I \\ P_{d-1}(t)\end{bmatrix},\cdots,\begin{bmatrix} I \\ P_1(t)\end{bmatrix}>$
$= \psi(P_d(t),\cdots,P_1(t))$.

We remark that $B^{d-1}(t) = (B_{11}(t) + B_{12}(t)P_d(t))$ is of degree one in $P_d(t), B^{d-2}(t) = (B_{11}^{d-1}(t) + B_{12}^{d-1} + B_{12}^{d-1}(t)P_{d-1}(t))$ is of degree two in $P_d(t), P_{d-1}(t)$ and, in general, $B^{d-i}(t)$

is of degree $\leq i+2$ in $P_d(t), \cdots, P_{d-i+1}(t)$. Thus the differential equation (2) is of degree $\leq i+2$ in $P_d(t), \cdots, P_{d-i}(t)$. In particular, for $i = d-1$, we see that the overall degree of the system is $\leq d+1$. Also, from iii) we see that the flow line for $Y_{B,\psi}(t)$ passing through $(P_d^0, \cdots, P_1^0)$ at time t_0 is precisely $(P_d(t), \cdots, P_1(t))$.

We summarize the above discussion in the following.

Theorem The flow line $(P_d(t), \cdots, P_1(t))$ of the Riccati field $Y_{B,\psi}(t)$ passing through $(P_d^0, \cdots, P_1^0)$ at time t_0 is the solution of a polynomial differential equation of degree $d+1$ which may be written as a nested system of Riccati equations of the type

$$\dot{P}_{d-i}(t) = B_{21}^{d-i}(t) + B_{22}^{d-i}(t)P_{d-i}(t) - P_{d-i}(t)B_{11}^{d-i}(t) - P_{d-i}(t)B_{12}^{d-i}(t)P_{d-i}(t)$$

where $B^{d-i-1}(t) = (B_{11}^{d-i}(t) + B_{12}^{d-i}(t)P_{d-i}(t)), i = 0, \cdots, d-1$, and $B^d(t) = B(t)$. In particular, if $d = 1, P_1(t)$ is the solution of the single Riccati equation associated with $B(t)$ passing through P_1^0 at time t_0.

We conclude with an example. Let $n = 3, K = (1,2)$, so that $N(K) = 3$. Thus $\mathbb{R}^3$ will be identified with the space $\{[A, B] \mid A \in M_{12}(\mathbb{R}), B \in M_{11}(\mathbb{R})\}$. We want to describe the differential equation that the flow line $(P_2(t), P_1(t)) \triangleq ([p_2^{(1)}(t), p_2^{(2)}(t)], [p_1(t)])$ has to satisfy. For simplicity of notation, we will omit the explicit time dependencies.

$$B^2 = \left[\begin{array}{cc|c} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ \hline b_{31} & b_{32} & b_{33} \end{array}\right]$$

$$B^1 = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} b_{13} \\ b_{23} \end{bmatrix} [p_2^{(1)}, p_2^{(2)}]$$

$$= \begin{bmatrix} b_{11} + b_{13}p_2^{(1)} & b_{12} + b_{13}p_2^{(2)} \\ b_{21} + b_{23}p_2^{(1)} & b_{22} + b_{23}p_2^{(2)} \end{bmatrix}.$$

Thus we obtain

$$\left[\dot{p}_2^{(1)}, \dot{p}_2^{(2)}\right] = [b_{31}, b_{32}] + b_{33}\left[p_2^{(1)}, p_2^{(2)}\right] - \left[p_2^{(1)}, p_2^{(2)}\right]\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$- \left[p_2^{(1)}, p_2^{(2)}\right]\begin{bmatrix} b_{13} \\ b_{23} \end{bmatrix}\left[p_2^{(1)}, p_2^{(2)}\right]$$

$$\dot{p}_1 = (b_{21} + b_{23}p_2^{(1)}) + (b_{22} + b_{23}p_2^{(2)})p_1 - p_1(b_{11} + b_{13}p_2^{(1)})$$
$$- p_1(b_{12} + b_{13}p_2^{(2)})p_1.$$

Hence, if $b_{13} = b_{13}(t)$ is not identically zero, this is a cubic differential equation.

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