SOME SOLUTIONS TO OVERDETERMINED BOUNDARY VALUE PROBLEMS ON SUBSETS OF SPHERES

by

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ABSTRACT

Title of Dissertation: SOME SOLUTIONS TO OVERDETERMINED
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Dissertation directed by: Dr. Carlos Berenstein, Professor, Department of
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For Ω an open domain contained in a Riemannian manifold M, various researchers have considered the problem of finding functions $u : \Omega \to \mathbf{R}$ which satisfy overdetermined boundary value problems such as $\Delta u + \alpha u = 0$ in Ω and $u = 0$ and $\frac{\partial u}{\partial n} = $ constant on $\partial \Omega$. (Here Δ is the Laplace-Beltrami operator on M.) Their results demonstrate the relative difficulty of finding such solutions. It has been shown for various choices of M (e.g., $M = \mathbf{R}^n$ or S_n^+) that the only domains Ω with $\partial \Omega$ connected and sufficiently regular which admit solutions to problems such as the one above are metric balls (see, e.g., [Be1] or [Se]). The first result of this thesis is a set of domains contained in S^n which are not metric balls but which do admit solutions to various overdetermined boundary value problems. In the case of the problem stated above, solutions are found for infinitely many choices of α. It is observed that the solutions found are isoparametric functions. (A function g is isoparametric if Δg and the length of the gradient of g are both functions of g, see [Ca].) In some cases, it is shown that these functions are restrictions of spherical eigenfunctions. In some cases, they are not. Next, for these same domains, an original choice of variables is developed under which the Laplace operator can be separated. This separation of variables is used to find a complete set of Dirichlet eigenfunctions for the domains. Initial sequences of Dirichlet eigenvalues for some of the domains are computed numerically. Finally, some comments are made about the connection between solutions to overdetermined problems and isoparametric functions.

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Suppose M is a Riemannian manifold and $\Omega \subset M$ is an open, relatively compact, simply connected domain with $\partial\Omega$ connected and with some regularity on $\partial\Omega$ (e.g., $C^{2+\epsilon}$). By an overdetermined boundary value problem, we mean a problem such as the following. Do there exist functions $f : \Omega \to \mathbf{R}$ such that

$$(0.1) \qquad\qquad \Delta f + \lambda f = 0 \qquad \text{in } \Omega$$

with $f = 0$ and $\frac{\partial f}{\partial n} = $ constant on $\partial\Omega$, the boundary of Ω? Here Δ is the Laplace operator on M, $\frac{\partial f}{\partial n}$ is the derivative of f perpendicular to $\partial\Omega$, and λ is a fixed real number.

This problem is made overdetermined by the specification of two conditions on f on $\partial\Omega$. In general there exist infinitely many values of λ and corresponding functions f which satisfy equation (0.1) and the single condition $f = 0$ on $\partial\Omega$. This is the well-known Dirichlet eigenvalue problem for the domain Ω (see, e.g., [Cl], chapter 1). In contrast, various results known for specific spaces M indicate that it is much more difficult to find solutions to problems such as the one above with overdetermined boundary conditions. For example, if $M = \mathbf{R}^n$, it is known that unless Ω is a metric ball there are at most finitely many values of λ for which functions f can be found which satisfy the overdetermined problem given above ([Be1]). (In fact, no such functions f have actually been found unless Ω is a ball.) One of the first results in this area is that of Serrin ([Se]), who found that if $\Omega \subset \mathbf{R}^n$ admits solutions f to

$$(0.2) \qquad\qquad \Delta f = -1 \qquad \text{in } \Omega$$

such that $f = 0$ and $\frac{\partial f}{\partial n} = $ constant on $\partial\Omega$, then Ω is a ball. Amplification of this result and other results are given in section 1 of Chapter 1.

The first result of this thesis, presented in chapter 1, is to describe some domains contained in S^n which admit solutions to various overdetermined boundary value problems including those described above. In the case of the first problem given above, these domains admit solutions for infinitely many choices of λ. In some cases it is shown that the solutions are restrictions of eigenfunctions of the whole sphere. In other cases, they are not. The domains and functions given in Chapter 1 are based on an example found by Berenstein and Yang ([BY2]).

In chapter 2 of this thesis, the problem of finding all Dirichlet eigenvalues for the same domains of spheres given in Chapter 1 is attacked. The technique used is separation of variables. An original choice of variables is developed in this thesis which is closely related to the functions found in Chapter 1. The results of this chapter allow numerical computation of any initial sequence

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of the Dirichlet eigenvalues of any of the domains considered. For two such domains, the results of such computations are given. The computations were achieved by use of the program package Mathematica (see [Wo] for further information). The computations were done on a workstation of a Sun system 3/280.

It turns out that the functions found in Chapter 1 are isoparametric. (A function g is isoparametric if Δg and $\|\nabla g\|^2$ are functions of g.) In some sense, the success of the calculations in Chapter 1 is due to the functions being isoparametric. The connection between isoparametric functions and solutions of overdetermined boundary value problems is explored in Chapter 3.

1. Some Solutions to Overdetermined Boundary Value Problems On Spheres

1.1 Background.

Let Ω be an open relatively compact domain contained in M ($\Omega \subset\subset M$) for M a real analytic (or C^∞) Riemannian manifold, with $\partial\Omega$ connected and with some regularity on $\partial\Omega$ (e.g., $\partial\Omega$ Lipshitz or $C^{2+\epsilon}$). Various researchers have considered the existence of functions u which satisfy one of the following overdetermined boundary value problems.

$$(1.1) \qquad \begin{cases} \Delta u + \alpha u = 0 & \text{in } \Omega \\ u = 0, \frac{\partial u}{\partial n} = \text{constant} & \text{on } \partial\Omega \end{cases}$$

$$(1.2) \qquad \begin{cases} \Delta u + \alpha u = 0 & \text{in } \Omega \\ u = \text{constant}, \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

$$(1.3) \qquad \begin{cases} \Delta u = -1 & \text{in } \Omega \\ u = 0, \frac{\partial u}{\partial n} = \text{constant} & \text{on } \partial\Omega \end{cases}$$

$$(1.4) \qquad \begin{cases} \Delta u = -1 & \text{in } \Omega \\ u = \text{constant}, \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

The symbol Δ is used for the Laplace-Beltrami operator associated to the Riemannian structure on M. $\frac{\partial u}{\partial n}$ is the derivative of u in the direction of a unit vector perpendicular to $\partial\Omega$ (in the outward direction). We assume Ω is oriented.

We note that if a solution to one of (1-1) through (1-4) is found for a domain Ω (with the assumptions above), then in fact $\partial\Omega$ is C^∞ (or real analytic if M is real analytic). This result is used in some of the papers below.

In [Be1], [Be2], [BY1], and [BY2], C. Berenstein and P. Yang have shown by asymptotic techniques that for $M = \mathbf{R}^n$ or $\mathbf{H}^n$ (real hyperbolic n-space), if there exist solutions to (1.1) or to (1.2) for infinitely many values of α, then Ω must be a metric ball. It is unknown even in $\mathbf{R}^n$ if the existence of a single eigenfunction for either (1.1) or (1.2) implies Ω is a ball (except in two extreme cases, namely the existence of a solution to (1.1) for $\alpha = \lambda_1$, the first Dirichlet eigenvalue, or a solution to (1.2) for $\alpha = \lambda_2$, which both imply Ω is a ball; see[Be1]). An interesting result from [Be1] is that problem (1.2) is equivalent to the Pompeiu problem which asks: If $\int_{T(\Omega)} f \, dx = 0$ for all rigid motions T where Ω is a bounded smooth simply connected domain in $\mathbf{R}^2$ and $f : \mathbf{R}^2 \to \mathbf{R}$

is locally integrable, can one conclude $f = 0$? The answer is no exactly when Ω admits solutions to (1.2).

In [Se], J. Serrin showed that for $\Omega \subset \mathbf{R}^n$ with $\partial\Omega$ C^2, if there exists a solution to (1.3), then Ω is a ball. He also showed the same result for more general elliptic differential equations of the form

$$(1.5) \qquad a(u, \|\nabla u\|)\Delta u + h(u, \|\nabla u\|)\sum_{i,j} u_i u_j u_{ij} = f(u, \|\nabla u\|)$$

where $\nabla u = (u_1, u_2, \ldots, u_n)$ is the gradient vector of u, and for somewhat more general boundary conditions. His proof used symmetry arguments and a boundary maximum principle. In a note following Serrin's paper, H. Weinberger [We] derived the same basic result as Serrin (no generalizations) by use of maximum principles and an equality of Rellich type. Recently, R. Molzon [Mo] has extended Serrin's arguments to include domains $\Omega \subset \mathbf{H}^n$ and $\Omega \subset S_n^+$. And N. Garofalo and J. Lewis [GL] have extended Weinberger's techniques to more general partial differential equations for $\Omega \subset \mathbf{R}^n$.

In [Av], P. Aviles considered the following overdetermined problem for $\Omega \subset \mathbf{R}^n$

$$(1.6) \qquad \begin{cases} \Delta u + g(r, u) = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0, u = \text{constant} & \text{on } \partial\Omega \end{cases}$$

where $r = \|\overline{x}\|$ and $g \in C^1(\mathbf{R}^+ \times \mathbf{R})$. Aviles showed that if $\frac{\partial g}{\partial t}(r, t) < \lambda_2$, the second Dirichlet eigenvalue of Ω, for all r and t, then Ω is a ball and u is radially symmetric. He also showed that if $g(r, u)$ is of the form $g(r, u) = \nu u + \lambda$ and (a) $\nu \leq \nu_{n+3} = $ the $n + 3$rd Neumann eigenvalue of Ω with $\partial\Omega$ having nonnegative mean curvature or (b) $\nu \leq \nu_{2n+2}$ with Ω convex, then again Ω is a ball and u is radially symmetric.

Below we give a set of domains contained in S^n for which there exist solutions to equations (1.1)-(1.4). (These domains generalize an example of Berenstein and Yang [BY1].) Further, these domains are not metric balls in S^n, demonstrating that some of the results described above cannot be extended (at least in the simplest way) to S^n. This is interesting especially in light of Molzon's result on S_n^+.

Some of this material has been presented in talks given at the AMS meetings held in Manhattan, Kansas (March 1990) and in College Park, Maryland (April 1988).

1.2 First Examples.

Suppose k and l are integers greater then one. We will write a vector in $\mathbf{R}^{k+l} = \mathbf{R}^k \times \mathbf{R}^l$ as $(\overline{x}, \overline{y}) = (x_1, \ldots, x_k, y_1, \ldots, y_l)$.

We define $f_{k,l} : \mathbf{R}^{k+l} \to \mathbf{R}$ by $f_{k,l}(\overline{x}, \overline{y}) = l\|\overline{x}\|^2 - k\|\overline{y}\|^2$. Then f is a second degree polynomial in the variables $x_1, \ldots, y_l$. We note that $f_{k,l}$ is

harmonic. Then since $f_{k,l}$ is also homogeneous it is known that $f_{k,l}$ restricted to $S^{k+l-1} \subset \mathbf{R}^{k+l}$ is a solution to the eigenvalue problem

$$\Delta_S(u) + \lambda u = 0 \qquad \text{on } S^{k+l-1}$$

for $\lambda = 2(k+l)$ where Δ_S is the Laplace-Beltrami operator on S^{k+l-1} (Chavel, p.35).

We now define $\Omega_{k,l} = \{p \in S^{k+l-1} \,|\, f_{k,l}(p) > 0\}$. We note that, when restricted to S^{k+l-1}, the range of $f_{k,l}$ is $[-k, l]$ (because on S^{k+l-1}, both $\|\overline{x}\|^2$ and $\|\overline{y}\|^2$ are no greater than one). So we may write $\Omega_{k,l} = f^{-1}((0, l])$.

We have also

$$\begin{aligned}
\partial\Omega_{k,l} &= \{p \in S^{k+l-1} \,|\, f_{k,l}(p) = 0\} \\
&= \{(\overline{x}, \overline{y}) \in R^{k+l} \,|\, l\|\overline{x}\|^2 - k\|\overline{y}\|^2 = 0, \quad \|\overline{x}\|^2 + \|\overline{y}\|^2 = 1\} \\
&= \{(\overline{x}, \overline{y}) \in R^{k+l} \,|\, \|\overline{x}\|^2 = \frac{k}{k+l}, \quad \|\overline{y}\|^2 = \frac{l}{k+l}\}
\end{aligned}$$

from which we see that $\partial\Omega_{k,l}$ is homeomorphic to $S^{k-1} \times S^{l-1}$.

We now have

THEOREM 1. $f_{k,l}$ satisfies problem (1.1) with $\Omega = \Omega_{k,l}$ and $\alpha = 2(k+l)$.

PROOF: It remains only to show that $\dfrac{\partial f_{k,l}}{\partial n}$, the outward normal derivative of $f_{k,l}$ to $\partial\Omega$ is constant.

In the following we will suppress the subscripts k and l. We use the notations $\nabla_R f$ and $\nabla_S f$, respectively, for the gradient of f in $\mathbf{R}^{k+l}$ and the gradient of $f|_{S^{k+l-1}}$ with respect to the standard geometry of S^{k+l-1}. Then $\nabla_S f(p) = $ projection of $\nabla_R f(p)$ onto the tangent space of S^{k+l-1} at p for $p \in S^{k+l-1} \subset \mathbf{R}^{k+l}$. We note that $\nabla_R f = (2l\overline{x}, -2k\overline{y})$. (We continue to use the notation $(\overline{a}, \overline{b})$ for a vector in $\mathbf{R}^{k+l}$.) Then

$$\nabla_R f(\overline{x}, \overline{y}) \cdot (\overline{x}, \overline{y}) = 2l\|\overline{x}\|^2 - 2k\|\overline{y}\|^2 = 2f(\overline{x}, \overline{y}).$$

So on $\partial\Omega$ (where $f = 0$) $\nabla_R f$ is perpendicular to $(\overline{x}, \overline{y})$. So $\nabla_R f$ is tangent to S^{k+l-1} and $\nabla_R f = \nabla_S f$.

Now since $\partial\Omega$ is a level surface of f,

$$\frac{\partial f}{\partial n} = \nabla_S f \cdot \left(\frac{-\nabla_S f}{\|\nabla_S f\|} \right) = -\|\nabla_S f\|.$$

By above, on $\partial\Omega$ we have $\|\nabla_S f\| = \|\nabla_R f\|$. So

$$\begin{aligned}
(1.7) \qquad \|\nabla_R f\|^2 &= 4(l^2\|\overline{x}\|^2 + k^2\|\overline{y}\|^2) \\
&= 4[(l - k)(l\|\overline{x}\|^2 - k\|\overline{y}\|^2) + kl(\|\overline{x}\|^2 + \|\overline{y}\|^2)] \\
&= 4[(l - k)f + kl] \qquad \text{if } (\overline{x}, \overline{y}) \in S^{k+l-1}.
\end{aligned}$$

So on $\partial\Omega$

$$\frac{\partial f}{\partial n} = -2\sqrt{(l-k)\cdot 0 + kl} = -2\sqrt{kl}.$$

This completes the proof of Theorem 1.

By a similar calculation to the above, we can find $\|\nabla_S f\|^2$ on all of S^{k+l-1}. We first project $\nabla_R f = (2l\overline{x}, -2k\overline{y})$ onto the tangent space of S^{k+l-1} at $(\overline{x}, \overline{y})$. This yields

$$\begin{aligned}
\nabla_S f &= (2l\overline{x}, -2k\overline{y}) - \frac{(2l\overline{x}, -2k\overline{y})\cdot(\overline{x}, \overline{y})}{(\overline{x}, \overline{y})\cdot(\overline{x}, \overline{y})}(\overline{x}, \overline{y}) \\
&= (2l\overline{x}, 2k\overline{y}) - (2l\|\overline{x}\|^2 - 2k\|\overline{y}\|^2)(\overline{x}, \overline{y}) \\
&= (2l\overline{x}, 2k\overline{y}) - 2f(\overline{x}, \overline{y}) \\
&= 2((l-f)\overline{x}, -(k+f)\overline{y})
\end{aligned}$$

Then

(1.8)
$$\begin{aligned}
\|\nabla_S f\|^2 &= 4\left[(l-f)^2\|\overline{x}\|^2 + (k+f)^2\|\overline{y}\|^2\right] \\
&= 4\left[l^2\|\overline{x}\|^2 + k^2\|\overline{y}\|^2 - 2f[l\|\overline{x}\|^2 - k\|\overline{y}\|^2] + f^2[\|\overline{x}\|^2 + \|\overline{y}\|^2]\right] \\
&= 4\left[(l-k)f + kl - 2f^2 + f^2\right] \\
&= 4\left[(l-k)f + kl - f^2\right]
\end{aligned}$$

We have already seen that $\Delta f = -2(k+l)f$. With (1.7) we now see that f is an *isoparametric* function as defined by E. Cartan (see, e.g., [Ca]), that is, a function for which $\|\nabla f\|^2$ and Δf are functions of f. (The fact that these functions are isoparametric was first pointed out to me by Dr. Karsten Grove.)

We may note now that Theorem 1 is also true if $f_{k,l}$ is replaced by $f_{k,l,\epsilon} = f_{k,l} - \epsilon$ and $\Omega_{k,l}$ is replaced by the smaller domain

$$\Omega_{k,l,\epsilon} = \{p \in S^{k+l-1} | f_{k,l}(p) > \epsilon\}.$$

Clearly, $f_{k,l,\epsilon}$ is zero on $\partial\Omega_{k,l,\epsilon}$. Since $f_{k,l}$ is isoparametric, so is $f_{k,l,\epsilon}$. Whence, $\|\nabla_S f_{k,l,\epsilon}\|$ is constant on its level curve $\partial\Omega_{k,l,\epsilon}$, and we have the result. Alternatively, one may replace $f_{k,l}$ by $f_{k,l} + \epsilon$ and $\Omega_{k,l}$ by the appropriate larger domain where the new function is positive.

The connection between functions which satisfy overdetermined boundary value problems and isoparametric functions will be investigated in Chapter 3 of this thesis.

1.3 Further Examples.

We will generate further solutions to (1.1) on $\Omega = \Omega_{k,l}$ and also solutions to (1.2), (1.3), and (1.4) by considering functions of the form $\phi(f)$ (where $f = f_{k,l}$) for $\phi : \mathbf{R} \to \mathbf{R}$ in $C^2(f(\Omega)) \cap C^1(f(\overline{\Omega}))$. We are led to calculate

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$\Delta_S(\phi(f))$. To do so we note ([Cl] p. 34) that for $G : \mathbf{R}^{k+l} \to \mathbf{R}$, with $g = G|_{S^n}$

(1.9) $$\Delta_S(g) = \Delta_R(G) - \partial_r(r^n\partial_r G)|_{r=1} \qquad (\Delta_R = \Delta_{\mathbf{R}^{n+1}})$$

(Functions and their restrictions will not be distinguished below.)

We now find $\Delta_R(\phi(f))$ and $\partial_r(r^n\partial_r(\phi(f)))|_{r=1}$. We calculate

(1.10)
$$
\begin{aligned}
\Delta_R(\phi(f)) &= \frac{\partial^2(\phi(f))}{\partial x_1^2} + \frac{\partial^2(\phi(f))}{\partial x_2^2} + \cdots + \frac{\partial^2(\phi(f))}{\partial y_l^2} \\
&= \frac{\partial}{\partial x_1}(\phi'(f) \cdot \frac{\partial f}{\partial x_1}) + \cdots + \frac{\partial}{\partial y_l}(\phi'(f) \cdot \frac{\partial f}{\partial y_l}) \\
&= \phi''(f)(\frac{\partial f}{\partial x_1})^2 + \phi'(f) \cdot \frac{\partial^2 f}{\partial x_1^2} + \cdots \\
&\qquad + \phi''(f)(\frac{\partial f}{\partial y_l})^2 + \phi'(f) \cdot \frac{\partial^2 f}{\partial y_l^2} \\
&= \phi''(f)\left[(\frac{\partial f}{\partial x_1})^2 + (\frac{\partial f}{\partial x_2})^2 + \cdots + (\frac{\partial f}{\partial y_l})^2\right] \\
&\qquad + \phi'(f)\left[\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial y_l^2}\right] \\
&= \phi''(f) \cdot \|\nabla_R f\|^2 + \phi'(f) \cdot \Delta_R f \\
&= \phi''(f) \cdot \|\nabla_R f\|^2 \qquad \text{(remember } \Delta_R f = 0) \\
&= \phi''(f) \cdot 4[(l-k)f + kl] \qquad \text{(from (1.7))}
\end{aligned}
$$

We also calculate

$$
\begin{aligned}
\partial_r(f) &= \nabla_R f \cdot \frac{(\overline{x}, \overline{y})}{r} = (2l\overline{x}, -2k\overline{y}) \cdot \frac{(\overline{x}, \overline{y})}{r} \\
&= \frac{2l\|\overline{x}\|^2 - 2k\|\overline{y}\|^2}{r} = \frac{2f}{r}.
\end{aligned}
$$

whence

$$
\begin{aligned}
\partial_{rr}(f) &= \partial_r(\frac{2f}{r}) = \frac{r2\partial_r(f) - 2f}{r^2} = \frac{r2(\frac{2f}{r}) - 2f}{r^2} \\
&= \frac{4f - 2f}{r^2} = \frac{2f}{r^2}.
\end{aligned}
$$

So

(1.11)
$$\partial_r(r^n\partial_r(\phi(f)))|_{r=1} = \partial_r(r^n\phi'(f)\partial_r f)|_{r=1}$$
$$= [r^n\partial_r(\phi'(f)\partial_r f) + nr^{n-1}\phi'(f)\partial_r(f)]|r=1$$
$$= [r^n\phi''(f)(\partial_r f)^2 + r^n\phi'(f)\partial_{rr}f]|_{r=1}$$
$$+ [nr^{n-1}\phi'(f)\partial_r(f)]|_{r=1}$$
$$= \left[r^n\phi''(f)(\frac{2f}{r})^2 + r^n\phi''(f)\frac{2f}{r^2} + nr^{n-1}\phi'(f)\frac{2f}{r}\right]|_{r=1}$$
$$= 4f^2\phi''(f) + 2(n+1)f\phi'(f).$$

We have $n + 1 = k + l$, then substitution of (1.10) and (1.11) into (1.9) yield

(1.12)
$$\Delta_S\phi(f) = 4[-f^2 + (l-k)f + kl]\phi''(f) - 2(k+l)f\phi'(f).$$

We now have

THEOREM 2. *For f and Ω as above, $\phi(f)$ restricted to Ω will be a solution to (1.1) if ϕ satisfies*

(1.13)
$$4[-z^2 + (l-k)z + kl]\phi''(z) - 2(k+l)z\phi'(z) + \alpha\phi(z) = 0$$

for all $z \in f(\Omega)$ and if $\phi(0) = 0$. Furthermore, if ϕ satisfies (1.13) and if $\phi'(0) = 0$ instead, then $\phi(f)$ is a solution to (1.2).

PROOF: It remains only to show that $\phi(f)$ satisfies the proper boundary conditions. By the definition of Ω, we have $f = 0$ on $\partial\Omega$. So $\phi(f) = \phi(0)$ on $\partial\Omega$. Whence $\phi(f)$ is constant on $\partial\Omega$, and if $\phi(0) = 0$ then $\phi(f) = 0$ on $\partial\Omega$. We also have $\frac{\partial}{\partial n}(\phi(f)) = \phi'(f)\frac{\partial f}{\partial n} = \phi'(0)\frac{\partial f}{\partial n}$ on $\partial\Omega$. So $\frac{\partial}{\partial n}(\phi(f)) = 0$ on $\partial\Omega$ if $\phi'(0) = 0$. In any case, $\frac{\partial}{\partial n}(\phi(f))$ is constant on $\partial\Omega$ because $\frac{\partial f}{\partial n}$ is constant on $\partial\Omega$.

In the same fashion we get

THEOREM 3. *For f and Ω as in Theorem 1, $\psi(f)$ restricted to Ω will be a solution to (1.3) (respectively (1.4)) if $\psi(0) = 0$ (repectively $\psi'(0) = 0$) and if ψ satisfies*

(1.14)
$$4[-z^2 + (l-k)z + kl]\psi''(z) - 2(k+l)z\psi'(z) = -1$$

for all $z \in f(\Omega)$.

As noted above, $f(S^{k+l-1})$, the range of f, is $[-k, l]$. We have then that $f(\Omega) = (0, l]$. In the two theorems above, we have differential equations over this range. In both, boundary conditions are to hold at $z = 0$. We note also

that both equations (1.13) and (1.14) have singular points at $z = l$. This raises the question of whether solutions exist to either equation over the full range $(0, l]$. In the case of equation (1.13), $r = l$ and $r = -k$ are regular singular points. So solutions to (1.13) exist over the range $(-k, l]$. Then one can find solutions ϕ with Φeither $\phi(0) = 0$ or $\phi'(0) = 0$ for infinitely many α. Indeed, there are polynomial solutions to (1.13) in both cases for infinitely many α. Such polynomial solutions are found by substituting polynomials with undetermined coefficients into equation (1.13), equating coefficients with zero, and solving the resulting simultaneous equations. When ϕ is a polynomial, it is of course defined over the entire range $[-k, l]$. Then, $\phi(f)$ is well defined over all of S^{k+l-1} and we observe that $\phi(f)$ is an eigenfunction for S^{k+l-1}. We recall that our first example of an overdetermined eigenfunction on Ω, namely f, was also the restriction to Ω of a global eigenfunction of the sphere. One is led to wonder if every overdetermined eigenfunction arises in this fashion. In fact, we have that if $k = l$ a solution to (1.1) which is in $C^2(\overline{\Omega}_{k,k})$ will be an eigenfunction of the sphere. See Theorem 4 below. However, In Chapter 2 below, we will see that if $k \neq l$, there exist solutions to (1.1) which are not restricted global eigenfunctions.

Although equation (1.14) is nonlinear, it is also possible to find solutions on the range $(0, l]$. Such solutions can be found by first replacing ψ' and ψ'' by y abd y'. The resulting first order equation can be solved by use of an appropriate integrating factor. One solution to (1.14) with $\phi(0) = 0$ is the function $\psi(z) = \frac{1}{4} \log(z + 1)$ where $k = 2$ and $l = 1$.

THEOREM 4. *If* $k = l$, *and if* $h \in C^2(\overline{\Omega}_{k,k})$ *satisfies* (1D), *then* h *satisfies* $\Delta_S h + \alpha h = 0$ *on* S^{2k-1}.

PROOF: In the case $k = l$, we have $\Omega_{k,k} = \Omega = \{(\overline{x}, \overline{y}) \in S^{2k-1} | \|\overline{x}\|^2 > \|\overline{y}\|^2\}$ and $\partial\Omega = \{(\overline{x}, \overline{y}) | \|\overline{x}\|^2 = \|\overline{y}\|^2\}$. Suppose $h \in C^2(\overline{\Omega})$ satisfies (1D). We define $\hat{h}$ on $S^{2k-1} \setminus \Omega$ by $\hat{h}(\overline{x}, \overline{y}) = -h(\overline{y}, \overline{x})$. Then for $(\overline{x}, \overline{y}) \in \partial\Omega$, we have $(\overline{y}, \overline{x}) \in \partial\Omega$ and $\hat{h}(\overline{x}, \overline{y}) = \hat{h}(\overline{y}, \overline{x}) = 0$. So h and $\hat{h}$ together define a continuous function on S^{2k-1}.

Now if $(\overline{x}, \overline{y}) \in S^n \setminus \Omega$, then $(\overline{y}, \overline{x}) \in \Omega$ and we have

$$\Delta_S \hat{h}(\overline{x}, \overline{y}) = -\Delta_S h(\overline{y}, \overline{x}) = \lambda h(\overline{y}, \overline{x}) = -\lambda \hat{h}(\overline{x}, \overline{y}).$$

For $(\overline{x}, \overline{y}) \in \partial\Omega$, let us calculate Δ_S as $\sum_{i=1}^{n} \frac{\partial_i^2}{\partial t_i^2}$ where $\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_{n-1}}$ are perpendicular unit vectors tangent to $\partial\Omega$ and $\frac{\partial}{\partial t_n} = \frac{\partial}{\partial n}$, the outward normal direction in S^n to $\partial\Omega$. Then, since $h = \hat{h} = 0$ on $\partial\Omega$, we have $\frac{\partial^2 h}{\partial t_i^2} = \frac{\partial^2 \hat{h}}{\partial t_i^2} = 0$ for $i < n$ on $\partial\Omega$. But $\Delta_S h = -\lambda h$ in Ω. So by continuity (since we have

assumed sufficient regularity), $\Delta_S h(\overline{x}, \overline{y}) = 0$ for $(\overline{x}, \overline{y}) \in \partial\Omega$. So $\dfrac{\partial^2 h}{\partial t_n^2}(\overline{x}, \overline{y}) = 0$ for any $(\overline{x}, \overline{y}) \in \partial\Omega$. Then

$$\frac{\partial^2 \hat{h}}{\partial t_n^2}(\overline{x}, \overline{y}) = -\frac{\partial^2 h}{\partial t_n^2}(\overline{y}, \overline{x}) = 0,$$

since if $(\overline{x}, \overline{y}) \in \partial\Omega$ so is $(\overline{y}, \overline{x})$.

In this chapter, we address the problem of finding all Dirichlet eigenvalues of the domains $\Omega_{k,l}$ given in Chapter 1. Our approach will be separation of variables. In the first section of this chapter we will define variables for $\Omega_{k,l}$.

2.1 Choice of Variables.

Suppose k and l are integers greater than one. We will continue to write $\mathbf{R}^{k+l}$ as the set of vectors $(\overline{x}, \overline{y})$ where $\overline{x} \in \mathbf{R}^k$ and $\overline{y} \in \mathbf{R}^l$. In this notation, S^{k+l-1}, the unit sphere in $\mathbf{R}^{k+l}$, is the set $\{(\overline{x}, \overline{y}) | \|\overline{x}\|^2 + \|\overline{y}\|^2 = 1\}$.

We define polar coordinates on $\mathbf{R}^k$, $r_x : \mathbf{R}^k \to \mathbf{R}$ and $\xi : \mathbf{R}^k \setminus \{\overline{0}\} \to S^{k-1}$, in the usual fashion by $r_x(\overline{x}) = \|\overline{x}\|$ and $\xi(\overline{x}) = \frac{\overline{x}}{\|\overline{x}\|}$. Similarly, we let r_y and ζ be polar coordinates on $\mathbf{R}^l$. The functions r_x and ξ (or r_y and ζ) may be extended from $\mathbf{R}^k$ (or $\mathbf{R}^l$) to $\mathbf{R}^{k+l}$ in trivial fashion. For example, we will let $r_x(\overline{x}, \overline{y}) = r_x(\overline{x}) = \|\overline{x}\|$. These functions form a set of coordinates on $\mathbf{R}^{k+l}$ in the sense that the values $r_x(\overline{x}, \overline{y})$, $\xi(\overline{x}, \overline{y})$, $r_y(\overline{x}, \overline{y})$, and $\zeta(\overline{x}, \overline{y})$ uniquely determine the point $(\overline{x}, \overline{y}) = (r_x \xi, r_y \zeta)$. Further, any subset of these functions does not have this property.

We wish to restrict these coordinates to S^{k+l-1}. We note that for $(\overline{x}, \overline{y}) \in S^{k+l-1}$, $r_x^2(\overline{x}, \overline{y}) + r_y^2(\overline{x}, \overline{y}) = 1$. Thus, r_x and r_y are not independent functions on S^{k+l-1}. That is, given a point $(\overline{x}, \overline{y}) \in S^{k+l-1}$, if we know the value of $r_x(\overline{x}, \overline{y})$, we can derive the value of $r_y(\overline{x}, \overline{y})$. Consequently, we define a function $r : S^{k+l-1} \to \mathbf{R}$ by $r(\overline{x}, \overline{y}) = l\|\overline{x}\|^2 - k\|\overline{y}\|^2 = lr_x^2 - kr_y^2$. Then the functions r, ξ, and ζ form a set of coordinates on S^{k+l-1}. (We note that r is identical to $f_{k,l}$ as defined in Part I. The notation r will be more convenient below where we will use r as a pseudo-radial coordinate on S^{k+l-1}.) The image of r is the interval $[-k, l]$. We have $r^{-1}(-k) = \{(\overline{x}, \overline{y}) | \|\overline{x}\|^2 = 0, \|\overline{y}\|^2 = 1\} = \{(\overline{0}, \overline{y}) | \|\overline{y}\| = 1\} = S^{l-1} \subset \mathbf{R}^l \subset \mathbf{R}^{k+l}$. Similarly, $r^{-1}(l) = \{(\overline{x}, \overline{0}) | \|\overline{x}\| = 1\} = S^{k-1}$. For any t in the open interval $(-k, l)$, $r^{-1}(t) = \{(\overline{x}, \overline{y}) | \|\overline{x}\|^2 + \|\overline{y}\|^2 = 1, \quad l\|\overline{x}\|^2 - k\|\overline{y}\|^2 = t\} = \{(\overline{x}, \overline{y}) | \|\overline{x}\| = \sqrt{\frac{k+t}{k+l}}, \quad \|\overline{y}\| = \sqrt{\frac{l-t}{k+l}}\}$ which is homeomorphic to the torus $S^{k-1} \times S^{l-1}$. The set of tori $r^{-1}(t)$ for $t \in (-k, l)$ are the generalized Clifford tori ([EDM], vol. 2, p. 1033) contained in the sphere S^{k+l-1}. Together with the two spheres $r^{-1}(-k)$ and $r^{-1}(l)$, these tori completely fill out the sphere S^{k+l-1}.

The three functions r, ξ, and ζ assign three values to most points p on S^{k+l-1} (for some points only one of ξ and ζ is defined). $r(p)$ specifies which Clifford torus a point p is on. Then the values $\xi(p)$ and $\zeta(p)$ uniquely distinguish p from all other points on the torus. If $r(p)$ is $-k$ or l, then p lies on one of the spheres $r^{-1}(-k)$ or $r^{-1}(l)$ and the value of either $\xi(p)$ or $\zeta(p)$ (whichever is defined) distinguishes p.

The coordinates (r, ξ, ζ) on S^{k+l-1} may be fruitfully compared to polar coordinates (r, θ) on $\mathbf{R}^2$. In the case of polar coordinates, we define $r : \mathbf{R}^2 \to \mathbf{R}$

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(different r from above) by $r(\overline{x}) = \|\overline{x}\|$ and $\theta : \mathbf{R}^2 \setminus \{\overline{0}\} \to S^1$ by $\theta(\overline{x}) = \frac{\overline{x}}{\|\overline{x}\|}$. Every point in $\mathbf{R}^2$ is specified by its value for both r and θ. The origin is specified completely by $r = 0$.

We define $Q : S^{k+l-1} \setminus \{r^{-1}(-k) \cup r^{-1}(l)\} \to [-k, l] \times S^{k-1} \times S^{l-1}$ by $Q(p) = (r(p), \xi(p), \zeta(p))$. Inverse to Q is $P : [-k, l] \times S^{k-1} \times Sl - 1 \to S^{k+l-1}$ given by $P(r, \xi, \zeta) = (r_x(r)\xi, r_y(r)\zeta)$. ($S^{k-1}$ is the unit sphere in $\mathbf{R}^k$, so $r_x(r)\xi$ is a vector of length $r_x(r)$ in $\mathbf{R}^k$.)

Suppose $u : U \subset S^{k-1} \to \mathbf{R}^{k-1}$ and $v : V \subset S^{l-1} \to \mathbf{R}^{l-1}$ are charts on open subsets U and V of S^{k-1} and S^{l-1}. Then $W = r^{-1}(-k, l) \cap \xi^{-1}(U) \cap \zeta^{-1}(V)$ is an open subset of S^{k+l-1} and we define a chart $w : W \to \mathbf{R} \times \mathbf{R}^{k-1} \times \mathbf{R}^{l-1} = \mathbf{R}^{k+l-1}$ by $w(p) = (r(p), u(\xi(p)), v(\zeta(p))$ for $p \in W$. (See figure 2-1)
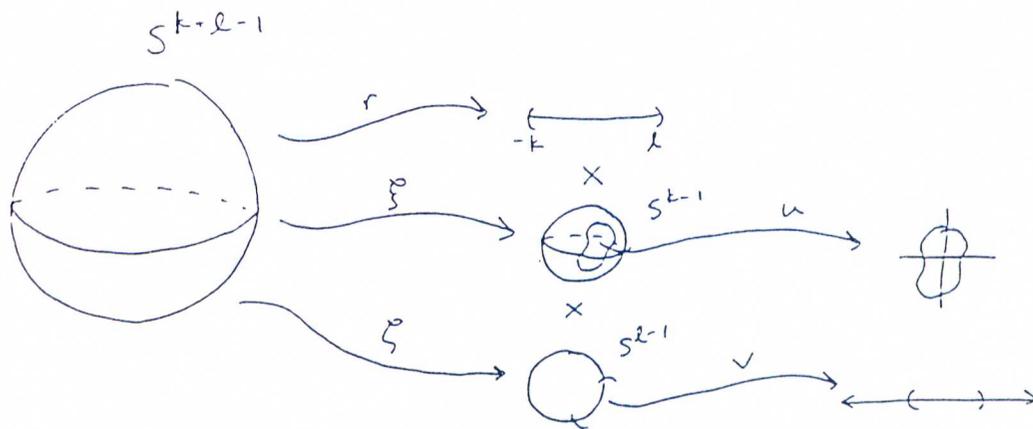

figure 2-1

For $x \in S^{k-1}$ we will write $u(x) = (u^1(x), u^2(x), \ldots, u^{k-1}(x))$. Similarly we will write $v = (v^1, \ldots, v^{l-1})$. For $p \in W$ we will write $u(\xi(p)) = (\hat{u}^1(p), \hat{u}^2(p), \ldots, \hat{u}^{k-1}(p))$ and $v(\zeta(p)) = (\hat{v}^1(p), \hat{v}^2(p), \ldots, \hat{v}^{k-1}(p))$. So we have

$$w(p) = (r(p), \hat{u}^1(p), \ldots, \hat{u}^{k-1}(p), \hat{v}^1(p), \ldots, \hat{v}^{l-1}(p)).$$

Corresponding to these local coordinate functions in W are coordinate vector fields $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial \hat{u}^i}$, and $\frac{\partial}{\partial \hat{v}^j}$ for $1 \leq i \leq k - 1$ and $1 \leq j \leq l - 1$. We also have coordinate vector fields $\frac{\partial}{\partial u^i}$ on U and $\frac{\partial}{\partial v^j}$ on V.

2.2 Calculation of $\triangle$.

We will now calculate $\triangle_{S^{k+l-1}} = \triangle$ in terms of the variables r, ξ, and ζ.

That is, we will calculate Δ in terms of the local coordinates $r, \hat{u}^1, \ldots, \hat{v}^{l-1}$. To do so we will use the standard formula for computing the Laplace operator in terms of local coordinates, equation (2.1) below ([Cl], p. 5). To perform this calculation we will first compute $G = [g_j k] = \langle \partial_j, \partial_k \rangle =$ the matrix of inner products between the coordinate vector fields. Then we will write $g = \det G$ and $G^{-1} = [g^{jk}]$. Then we have

$$(2.1) \qquad\qquad \Delta f = \frac{1}{\sqrt{g}} \sum_{j,k} \partial_j(g^{jk} \sqrt{g} \partial_k f)$$

The result of these calculations is equation (2.11) at the end of this section.

We begin by computing all inner products between the coordinate vector fields $\frac{\partial}{\partial r}, \frac{\partial}{\partial \hat{u}^i}$, and $\frac{\partial}{\partial \hat{v}^j}$. First, we will investigate the relationship between $\frac{\partial}{\partial u^i}$ and $\frac{\partial}{\partial \hat{u}^i}$.

For fixed $r \in (-k, l]$ and $\zeta \in S^{l-1}$, we define $\xi_{r,\zeta}^{-1} : S^{k-1} \to S^{k+l-1}$ by $\xi_{r,\zeta}^{-1}(\xi) = P(r, \xi, \zeta) = (r_x(r)\xi, r_y(r)\zeta)$. Then for $f : S^{k+l-1} \to \mathbf{R}$ and $g : S^{k-1} \to \mathbf{R}$, $\frac{\partial}{\partial u^i}$ and $\frac{\partial}{\partial \hat{u}^i}$ are defined by

$$\frac{\partial g}{\partial u^i}(\xi(p)) = \partial_i(g \circ u^{-1})(u(\xi(p)))$$

and

$$\frac{\partial f}{\partial \hat{u}^i}(p) = \partial_i(f \circ \xi_{r(p),\zeta(p)}^{-1} \circ u^{-1})(u(\xi(p)))$$

where $p \in W$ and ∂_i is the ith coordinate derivative in $\mathbf{R}^{k-1}$. It follows that for $f : S^{k+l-1} \to \mathbf{R}$ and $p \in W$,

$$\frac{\partial f}{\partial \hat{u}^i}(p) = \frac{\partial(f \circ \xi_{r(p),\zeta(p)}^{-1})}{\partial u^i}(\xi(p)).$$
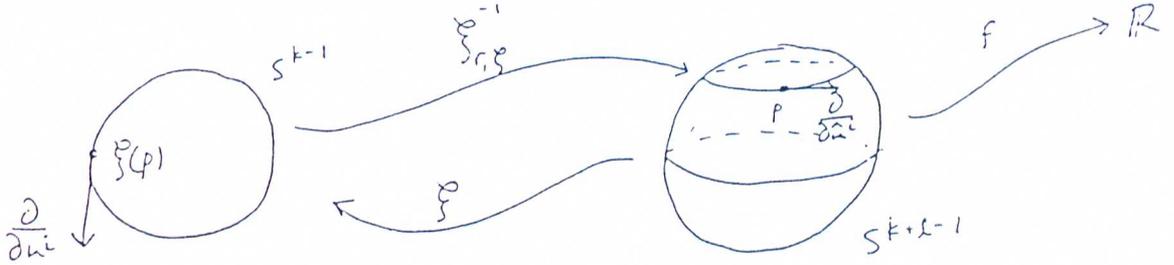
(See figure 2-2)

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figure 2-2

We note also that the map $\xi_{r,\zeta}^{-1}$ acts as a dilation; it carries S^{k-1} to a $(k-1)$-sphere of radius $r_x(r)$ in S^{k+l-1}. Since we are assuming that the unit spheres S^{k-1}, S^{l-1}, and S^{k+l-1} carry the usual Riemannian metrics induced by their embeddings in Euclidean space, we have

$$(2.2) \quad \left\langle \frac{\partial}{\partial \hat{u}^i}(p), \frac{\partial}{\partial \hat{u}^j}(p) \right\rangle = r_x^2 \left\langle \frac{\partial}{\partial u^i}(\xi(p)), \frac{\partial}{\partial u^j}(\xi(p)) \right\rangle \qquad 1 \leq i, j \leq k-1$$

so $\left\| \frac{\partial}{\partial \hat{u}^i}(p) \right\| = r_x \left\| \frac{\partial}{\partial u^i}(\xi(p)) \right\|$.

In analogous fashion, we define $\zeta_{r,\xi}^{-1} : S^{l-1} \to S^{k+l-1}$ by

$$\zeta_{r,\xi}^{-1}(\zeta) = P(r, \xi, \zeta) = (r_x(r)\xi, r_y(r)\zeta).$$

Then it follows that

$$\frac{\partial f}{\partial \hat{v}^j}(p) = \frac{\partial(f \circ \zeta_{r(p),\xi(p)}^{-1})}{\partial v^j}(\zeta(p))$$

for $p \in W$ and $f : S^{k+l-1} \to \mathbf{R}$. And also

$$(2.3) \quad \left\langle \frac{\partial}{\partial \hat{v}^i}(p), \frac{\partial}{\partial \hat{v}^j}(p) \right\rangle = r_y^2 \left\langle \frac{\partial}{\partial v^i}(\zeta(p)), \frac{\partial}{\partial v^j}(\zeta(p)) \right\rangle \qquad 1 \leq i, j \leq l-1$$

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To calculate inner products involving $\frac{\partial}{\partial r}$, we will write $\frac{\partial}{\partial r}$ in terms of the rectangular vector fields $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial y^j}$, $1 \le i \le k$, $1 \le j \le l$ defined in $\mathbf{R}^k$ and $\mathbf{R}^l$. Recall that $\frac{\partial}{\partial r}$ is a vector field on $S^{k+l-1} \subset \mathbf{R}^k \times \mathbf{R}^l$. We let $p \in W$ be a point with $\xi(p) = \xi$ and $\zeta(p) = \zeta$. Suppose $\xi \in S^{k-1} \subset \mathbf{R}^k$ has coordinates $(\xi_1, \ldots, \xi_k)$ and $\zeta \in S^{l-1} \subset \mathbf{R}^l$ has coordinates $(\zeta_1, \ldots, \zeta_l)$. From above we have

$$
\begin{aligned}
P(r, \xi, \zeta) &= (r_x(r)\xi, r_y(r)\zeta) \\
&= (r_x(r)\xi_1, r_x(r)\xi_2, \ldots, r_x(r)\xi_k, r_y(r)\zeta_1, \ldots, r_y(r)\zeta_l).
\end{aligned}
$$

So

$$
\begin{aligned}
\frac{\partial}{\partial r}(p) &= r'_x(r)\xi_1 \frac{\partial}{\partial x_1}(p) + r'_x(r)\xi_2 \frac{\partial}{\partial x_2}(p) + \cdots + r'_y(r)\zeta_1 \frac{\partial}{\partial y_1}(p) \\
&\quad + \cdots + r'_y(r)\zeta_l \frac{\partial}{\partial y_l}(p) \\
&= r'_x(r) \cdot [\text{deriv. in } \xi\text{-direction}](p) + r'_y(r) \cdot [\text{deriv. in } \zeta\text{-direction}](p)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(2.4) \qquad \left\| \frac{\partial}{\partial r}(p) \right\|^2 &= [r'_x(r)]^2 \|\xi\|^2 + [r'_y(r)]^2 \|\zeta\|^2 \\
&= [r'_x(r)]^2 + [r'_y(r)]^2.
\end{aligned}
$$

We also note that the direction represented by the vector ξ is perpendicular to the sphere $\xi_{r,\zeta}^{-1}(S^{k-1})$. (See figure 2-3)
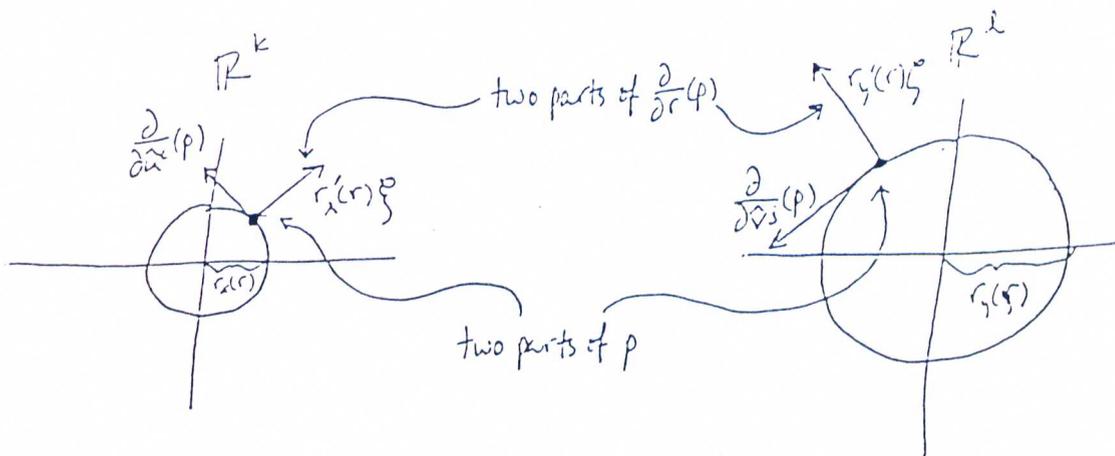
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figure 2-3

Hence

$$\left\langle \frac{\partial}{\partial \hat{u}^i}(p), [\text{derivative in } \xi\text{-direction}](p) \right\rangle = 0.$$

Also,

$$\left\langle \frac{\partial}{\partial \hat{u}^i}(p), [\text{derivative in } \zeta\text{-direction}](p) \right\rangle = 0$$

since $\frac{\partial}{\partial \hat{u}^i}(p)$ is in the span of $\{\frac{\partial}{\partial x^1}(p), \dots, \frac{\partial}{\partial x^k}(p)\}$ and the derivative in the ζ-direction is in the span of $\{\frac{\partial}{\partial y^1}(p), \dots, \frac{\partial}{\partial y^l}(p)\}$. Thus we have that

$$(2.5) \qquad \left\langle \frac{\partial}{\partial \hat{u}^i}(p), \frac{\partial}{\partial r}(p) \right\rangle = 0 \qquad 1 \le i \le k-1.$$

Similarly,

$$(2.6) \qquad \left\langle \frac{\partial}{\partial \hat{v}^j}(p), \frac{\partial}{\partial r}(p) \right\rangle = 0 \qquad 1 \le j \le l-1.$$

And also

$$(2.7) \qquad \left\langle \frac{\partial}{\partial \hat{u}^i}(p), \frac{\partial}{\partial \hat{v}^j}(p) \right\rangle = 0 \qquad \text{for all } 1 \le i \le k-1$$
$$1 \le j \le l-1.$$

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In equations (2.2)-(2.7) we have computed expressions for the inner products between all of the coordinate vector fields. Following usual notation (also noted above), we define the matrix $G = [g_{j,k}] = \langle \partial_j, \partial_k \rangle$. Then we have at any $p \in W$

$$G = \begin{bmatrix} (r'_x)^2 + (r'_y)^2 & 0 & 0 \\ 0 & r_x^2 G_x & 0 \\ 0 & 0 & r_y^2 G_y \end{bmatrix}$$

where

$$G_x = [\langle \frac{\partial}{\partial u^i}(\xi(p)), \frac{\partial}{\partial u^j}(\xi(p)) \rangle] = [g_{ij}^x]$$

and

$$G_y = [\langle \frac{\partial}{\partial v^i}(\zeta(p)), \frac{\partial}{\partial v^j}(\zeta(p)) \rangle] = [g_{ij}^y].$$

We then calculate that

$$\det G = ((r'_x)^2 + (r'_y)^2) r_x^{2(k-1)} r_y^{2(l-1)} \det G_x \det G_y.$$

Given a C^2 function $f : S^{k+l-1} \to \mathbf{R}$ we will compute the Laplacian of f, $\Delta_{S^{k+l-1}} f = \Delta f$, in terms of the local coordinates we have introduced. We recall the standard formula for the Laplacian in local coordinates given above,

(2.1)
$$\Delta f = \frac{1}{\sqrt{g}} \sum_{j,k} \partial_j (g^{jk} \sqrt{g} \partial_k f)$$

where again $g = \det G$, $[g^{jk}] = G^{-1}$, and $\partial_k =$ the kth coordinate vector. Since we have expressed G in terms of G_x and G_y, our expression for Δf will involve $\Delta_{S^{k-1}}(f \circ \xi_{r,\zeta}^{-1})$ and $\Delta_{S^{l-1}}(f \circ \zeta_{r,\xi}^{-1})$.

Below we will write $R = (r'_x)^2 + (r'_y)^2$, $g_x = \det G_x$, $g_y = \det G_y$, $G_x^{-1} = [g_x^{ij}]$, and $G_y^{-1} = [g_y^{ij}]$. In these terms, $g = R r_x^{2(k-1)} r_y^{2(l-1)} g_x g_y$. So $\sqrt{g} = \sqrt{R g_x g_y} r_x^{k-1} r_y^{l-1}$ and $\frac{1}{\sqrt{g}} = \frac{1}{\sqrt{R g_x g_y}} \cdot \frac{1}{r_x^{k-1} r_y^{l-1}}$.

We compute

$$G^{-1} = \begin{bmatrix} \frac{1}{R} & 0 & 0 \\ 0 & \frac{1}{r_x^2} G_x^{-1} & 0 \\ 0 & 0 & \frac{1}{r_y^2} G_y^{-1} \end{bmatrix}$$

Then we expand (2.1) as

(2.8)
$$\Delta f = \frac{1}{\sqrt{R}} \frac{1}{\sqrt{g_x g_y}} \frac{1}{r_x^{k-1} r_y^{l-1}} \frac{\partial}{\partial r} \left(\frac{1}{R} \sqrt{R} \sqrt{g_x g_y} r_x^{k-1} r_y^{l-1} \frac{\partial f}{\partial r} \right)$$

$$+ \frac{1}{\sqrt{R g_y}} \frac{1}{r_x^{k-1} r_y^{l-1}} \frac{1}{\sqrt{g_x}} \left[\sum_{i,j=1}^{k-1} \frac{\partial}{\partial \hat{u}^i} \left(\frac{1}{r_x^2} g_x^{ij} \sqrt{R g_y} r_x^{k-1} r_y^{l-1} \sqrt{g_x} \frac{\partial f}{\partial \hat{u}^j} \right) \right]$$

$$+ \frac{1}{\sqrt{R g_x}} \frac{1}{r_x^{k-1} r_y^{l-1}} \frac{1}{\sqrt{g_y}} \left[\sum_{i,j=1}^{l-1} \frac{\partial}{\partial \hat{v}^i} \left(\frac{1}{r_y^2} g_y^{ij} \sqrt{R g_x} r_x^{k-1} r_y^{l-1} \sqrt{g_y} \frac{\partial f}{\partial \hat{v}^j} \right) \right].$$

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R, r_x, and r_y are functions of r only, so they factor through $\frac{\partial}{\partial \hat{u}^i}$ and $\frac{\partial}{\partial \hat{v}^j}$ for all i and j. Also, the elements of the matrix G_x, $g_{ij}^x = \langle \frac{\partial}{\partial u^i}(\xi(p)), \frac{\partial}{\partial u^i}(\xi(p)) \rangle$, depend only on the value of $\xi(p)$, so $\frac{\partial}{\partial r} g_{ij}^x = 0$ and $\frac{\partial}{\partial \hat{v}^k} g_{ij}^x = 0$ for all i. Similarly, we find that g_y and g_y^{ij} factor through $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \hat{u}^i}$ for all i.

After factoring through and cancelling we get

(2.9)
$$\Delta f = \frac{1}{\sqrt{R}} \frac{1}{r_x^{k-1} r_y^{l-1}} \frac{\partial}{\partial r} \left(\frac{1}{\sqrt{R}} r_x^{k-1} r_y^{l-1} \frac{\partial f}{\partial r} \right)$$
$$+ \frac{1}{r_x^2} \frac{1}{\sqrt{g_x}} \left[\sum_{i,j=1}^{k-1} \frac{\partial}{\partial \hat{u}^i} \left(g_x^{ij} \sqrt{g_x} \frac{\partial f}{\partial \hat{u}^j} \right) \right]$$
$$+ \frac{1}{r_y^2} \frac{1}{\sqrt{g_y}} \left[\sum_{i,j=1}^{l-1} \frac{\partial}{\partial \hat{v}^i} \left(g_y^{ij} \sqrt{g_y} \frac{\partial f}{\partial \hat{v}^j} \right) \right]$$

From above we have $\frac{\partial f}{\partial \hat{u}^i}(p) = \frac{\partial (f \circ \xi_{r(p)\varsigma(p)}^{-1})}{\partial u^i}(\xi(p))$. Also, the functions g_x^{ij} and $\sqrt{g_x}$ are defined at p as the evalutions of appropriate functions at $\xi(p)$. In consequence we have that

$$\frac{1}{\sqrt{g_x}} \sum_{i,j=1}^{k-1} \frac{\partial}{\partial \hat{u}^i} \left(g_x^{ij} \sqrt{g_x} \frac{\partial f}{\partial \hat{u}^j} \right) = \frac{1}{\sqrt{g_x}} \sum_{i,j=1}^{k-1} \frac{\partial}{\partial \hat{u}^i} \left(g_x^{ij} \sqrt{g_x} \frac{\partial}{\partial u^j} \left(f \circ \xi_{r(p),\varsigma(p)}^{-1} \right) \right)$$

(now g_x^{ij} and g_x are defined directly on S^{k-1})

$$= \Delta_{S^{k-1}}(f \circ \xi_{r(p),\varsigma(p)}^{-1}) \qquad \text{by (2.1).}$$

Similarly we have

$$\frac{1}{\sqrt{g_y}} \sum_{i,j=1}^{l-1} \frac{\partial}{\partial \hat{v}^i} \left(g_y^{ij} \sqrt{g_y} \frac{\partial f}{\partial \hat{v}^j} \right) = \Delta_{S^{l-1}}(f \circ \varsigma_{r(p),\xi(p)}^{-1}).$$

So (2.9) becomes

(2.10)
$$\Delta f = \frac{1}{\sqrt{R}} \frac{1}{r_x^{k-1} r_y^{l-1}} \frac{\partial}{\partial r} \left(\frac{1}{\sqrt{R}} r_x^{k-1} r_y^{l-1} \frac{\partial f}{\partial r} \right)$$
$$+ \frac{1}{r_x^2} \Delta_{S^{k-1}}(f \circ \xi_{r(p),\varsigma(p)}^{-1})$$
$$+ \Delta_{S^{l-1}}(f \circ \varsigma_{r(p),\xi(p)}^{-1}).$$

We expand the first term in (2.10) by the product rule to get

$$\frac{1}{R}\frac{\partial^2 f}{\partial r^2} + \frac{1}{\sqrt{R}}r_x^{k-1}r_y^{l-1}\frac{\partial}{\partial r}\left(\frac{r_x^{k-1}r_y^{l-1}}{\sqrt{R}}\right)\frac{\partial f}{\partial r}.$$

We recall that $r_x = \sqrt{\frac{k+r}{k+l}}$, $r_y = \sqrt{\frac{l-r}{k+l}}$, and $R = (r_x')^2 + (r_y')^2$, and calculate that $R = \dfrac{1}{4(k+l)(l-r)}$ and

$$\frac{1}{\sqrt{R}}r_x^{k-1}r_y^{l-1}\frac{\partial}{\partial r}\left(\frac{r_x^{k-1}r_y^{l-1}}{\sqrt{R}}\right) = -2(k+l)r.$$

So (2.10) becomes

(2.11)

$$\Delta f = 4(k+r)(l-r)\frac{\partial^2 f}{\partial r^2} - 2(k+l)r\frac{\partial f}{\partial r} + \frac{k+l}{k+r}\Delta_{S^{k-1}}\left(f \circ \xi^{-1}_{r(p),\zeta(p)}\right)$$
$$+ \frac{k+l}{l-r}\Delta_{S^{l-1}}\left(f \circ \zeta^{-1}_{r(p),\xi(p)}\right)$$

(2.11) holds for all $p \in W$.

We recall that $W = r^{-1}(-k,l) \cap \xi^{-1}(U) \cap \zeta^{-1}(V)$ where U and V are open sets in S^{k-1} and S^{l-1} respectively with corresponding coordinate charts u and v. Given $p \in S^{k+l-1}$, if $r(p) \in (-k,l)$ then an open set W as above with $p \in W$ can be chosen by choosing open sets U and V containing $\xi(p)$ and $\zeta(p)$. Consequently, (2.11) holds for all $p \in S^{k+l-1}$ such that $r(p) \in (-k,l)$. That is, (2.11) holds except on the spheres $r^{-1}(-k)$ and $r^{-1}(l)$.

3. Dirichlet Eigenvalues of $\Omega_{k,l}$.

We continue to assume that k and l are positive integers. We recall that the open set $\Omega_{k,l} \subset S^{k+l-1}$ was defined by

$$\Omega_{k,l} \subset S^{k+l-1} = \{(\bar{x},\bar{y}) \in S^{k+l-1} \mid r(\bar{x},\bar{y}) > 0\}$$
$$= \{(\bar{x},\bar{y}) \in \mathbf{R}^{k+l} \mid \|\bar{x}\|^2 + \|\bar{y}\|^2 = 1 \text{ and } l\|\bar{x}\|^2 - k\|\bar{y}\|^2 > 0\}.$$

Then the boundary of $\Omega_{k,l}$ is $r^{-1}(0)$ which we have seen above is a Clifford torus homeomorphic to $S^{k-1} \times S^{l-1}$. (In fact, $r^{-1}(0) = \{(\bar{x},\bar{y}) \in \mathbf{R}^{k+l} \mid \|\bar{x}\| = \sqrt{\frac{k}{k+l}}, \|\bar{y}\| = \sqrt{\frac{l}{k+l}}\}$. So $r^{-1}(0)$ is isometric to $S^{k-1}(\sqrt{\frac{k}{k+l}}) \times S^{l-1}(\sqrt{\frac{l}{k+l}})$ where $S^{k-1}(t) = (k-1)$-sphere of radius t in $\mathbf{R}^k$.)

We will attack the Dirichlet eigenvalue problem on $\Omega_{k,l}$. That is, we will try to find real numbers λ for which there exist corresponding functions $f : \Omega_{k,l} \to \mathbf{R}$ with $f \in C^2(\Omega_{k,l}) \cap C^0$(closure of $\Omega_{k,l}$) which solve

(2.12a,b)
$$\begin{cases} \Delta f + \lambda f = 0 & \text{in } \Omega_{k,l} \\ f = 0 & \text{on } \partial\Omega_{k,l} \end{cases}$$

Using (2.11) we write (2.12a & b) in terms of the variables r, ξ, and ζ as

(2.13a)

$$4(k+l)(l-r)\frac{\partial^2 f}{\partial r^2}(p) - 2(k+l)r\frac{\partial f}{\partial r}(p) + \frac{k+l}{k+r}\Delta_{S^{k-1}}(f \circ \xi_{r(p),\zeta(p)}^{-1})(\xi(p))$$
$$+ \frac{k+l}{l-r}\Delta_{S^{l-1}}(f \circ \zeta_{r(p),\xi(p)}^{-1})(\zeta(p)) + \lambda f(p) = 0$$

for p with $r(p) \in (0,l)$ and

(2.13b) $\qquad\qquad f(p) = 0 \qquad\qquad$ for p with $r(p) = 0$.

(2.13a) is not defined for p such that $r(p) = l$. In fact, $\zeta(p)$ is undefined for such p. However, we note that the closure of $\Omega_{k,l} \setminus r^{-1}(l)$ is $\Omega_{k,l}$. Thus if a function f is found which satisfies (2.13a) for some λ on $\Omega_{k,l} \setminus r^{-1}(l)$ and which is C^2 on $\Omega_{k,l}$, then in fact f satisfies (2.12a). In consequense we will look for functions that satisfy (2.13) and are C^2 on $\Omega_{k,l}$.

We will use separation of variables to solve (2.13). So we will look for solutions to (2.13) of the form $f(p) = u(r(p))w^x(\xi(p))w^y(\zeta(p))$ where we have $u : [0,l] \to \mathbf{R}$, $w^x : S^{k-1} \to \mathbf{R}$, and $w^y : S^{l-1} \to \mathbf{R}$. We assume that w^x and w^y are C^2 and that $u \in C^2((0,l)) \cap C^0([0,l])$.(We will eventually show that the solutions we create via this separation of variables are C^2 on all of $\Omega_{k,l}$. Whence they are solutions to (2.12) and therefore C^∞ functions.) After substitution (2.13) becomes

(2.14a)

$$4(k+r)(l-r)u_{rr}w^x w^y - 2(k+l)ru_r w^x w^y + \frac{k+l}{k+r}(\Delta_{S^{k-1}}w^x)uw^y$$
$$+ \frac{k+l}{l-r}(\Delta_{S^{l-1}}w^y)uw^x + \lambda uw^x w^y = 0$$

and

(2.14b) $\qquad\qquad u(0) = 0$

We divide (2.14a) by $f = uw^x w^y$ to get
(2.15)
$$4(k+r)(l-r)\frac{u_r r}{u} - 2(k+l)r\frac{u_r}{u} + \frac{k+l}{k+r}\frac{\Delta_{S^{k-1}}w^x}{w^x} + \frac{k+l}{l-r}\frac{\Delta_{S^{l-1}}w^y}{w^y} + \lambda = 0$$

In (2.15), only the term $\frac{\Delta_{S^{k-1}}w^x}{w^x}$ depends upon ξ. That is, if p is varied so that $r(p)$ and $\zeta(p)$ remain constant then all other terms in (15) will remain constant. Whence, we have $\frac{\Delta_{S^{k-1}}w^x}{w^x} = \hat{c} = $ constant. So w^x is an eigenfunction on S^{k-1}. Then we have from standard theory (e.g., [Cl], p. 35) that

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$\hat{c} = -m_x(m_x + k - 2)$ where m_x is a nonnegative integer. Similarly, we find that $\dfrac{\Delta_{S^{l-1}} w^y}{w^y} = -m_y(m_y + l - 2)$ with $m_y = 0, 1, 2, \ldots$. By substituting these results into (2.15) and multiplying by u we get the following ODE for u.

(2.16)
$$4(k+r)(l-r)u_{rr} - 2(k+l)ru_r$$
$$- (k+l)\left[\frac{m_x(m_x + k - 2)}{k+r} + \frac{m_y(m_y + l - 2)}{l-r}\right]u + \lambda u = 0$$

We require C^2 solutions u to (2.16) for all $r \in (0, l)$ with boundary condition $u(0) = 0$. (2.16) is simplified by the substitution $u(r) = (k+r)^{\frac{m_x}{2}}(l-r)^{\frac{m_y}{2}}v(r)$. Direct substitution into (2.16) yields

(2.17)
$$4(k+r)^{\frac{m_x}{2}+1}(l-r)^{\frac{m_y}{2}+1}v_{rr} + Av_r + Bv = 0$$

where

$$A = 4m_x(k+r)^{\frac{m_x}{2}}(l-r)^{\frac{m_y}{2}+1} - 4m_y(k+r)^{\frac{m_x}{2}+1}(l-r)^{\frac{m_y}{2}}$$
$$- 2(k+l)r(k+r)^{\frac{m_x}{2}}(l-r)^{\frac{m_y}{2}}.$$

and

$$B = 2m_x(\frac{m_x}{2} - 1)(k+r)^{\frac{m_x}{2}-1}(l-r)^{\frac{m_y}{2}+1}$$
$$- 2m_x m_y(k+r)^{\frac{m_x}{2}}(l-r)^{\frac{m_y}{2}}$$
$$+ 2m_y(\frac{m_y}{2} - 1)(k+r)^{\frac{m_x}{2}+1}(l-r)^{\frac{m_y}{2}-1}$$
$$- (k+l)rm_x(k+r)^{\frac{m_x}{2}-1}(l-r)^{\frac{m_y}{2}}$$
$$+ (k+l)rm_y(k+r)^{\frac{m_x}{2}}(l-r)^{\frac{m_y}{2}-1}$$
$$- (k+l)m_x(m_x + k - 2)(k+r)^{\frac{m_x}{2}-1}(l-r)^{\frac{m_y}{2}}$$
$$- (k+l)m_y(m_y + l - 2)(k+r)^{\frac{m_x}{2}}(l-r)^{\frac{m_y}{2}-1}$$
$$+ \lambda(k+r)^{\frac{m_x}{2}}(l-r)^{\frac{m_y}{2}}$$

B can be simplified by factoring out $(k+r)^{\frac{m_x}{2}-1}(l-r)^{\frac{m_y}{2}-1}$ and regrouping the remaining terms. Then

$$B = (k+r)^{\frac{m_x}{2}-1}(l-r)^{\frac{m_y}{2}-1} \times$$
$$\{(l-r)[m_x(m_x - 2)(l-r) - (k+l)rm_x - (k+l)m_x(m_x + k - 2)]$$
$$+ (k+r)[m_y(m_y - 2)(k+r) + (k+l)rm_y - (k+l)m_y(m_y + l - 2)]$$
$$+ (k+r)(l-r)[\lambda - 2m_x m_y]\}$$

$$= (k+r)^{\frac{m_x}{2}-1}(l-r)^{\frac{m_y}{2}-1} \times$$
$$\{m_x(l-r)[m_x l - m_x r - 2l + 2r - kr$$
$$- lr - km_x - k^2 + 2k - lm_x - lk + 2l]$$
$$+ m_y(k+r)[m_y k + m_y r - 2k - 2r + kr$$
$$+ lr - km_y - kl + 2k - lm_y - l^2 + 2l]$$
$$+ (k+r)(l-r)[\lambda - 2m_x m_y]\}$$

$$= (k+r)^{\frac{m_x}{2}-1}(l-r)^{\frac{m_y}{2}-1}\{m_x(l-r)[(k+r)(2-k-m_x-l)]$$
$$+ m_y(k+r)[(l-r)(2-l-m_y-k)]$$
$$+ (k+r)(l-r)[\lambda - 2m_x m_y]\}$$

$$= (k+r)^{\frac{m_x}{2}}(l-r)^{\frac{m_y}{2}}\{m_x(2-k-m_x-l)+m_y(2-l-m_y-k)+\lambda-2m_x m_y\}$$

Now $(k+r)^{\frac{m_x}{2}}(l-r)^{\frac{m_y}{2}}$ can be factored completely out of equation (2.17), yielding,

(2.18)
$$4(k+r)(l-r)v_{rr} + \{4m_x(l-r) - 4m_y(k+r) - 2(k+l)r\}v_r$$
$$+ \{m_x(2-k-m_x-l) + m_y(2-l-m_y-k) + \lambda - 2m_x m_y\}v = 0$$

We divide (2.18) by 4 and rewrite slightly to get

(2.19) $(k+r)(l-r)v_{rr}+[(m_x l-m_y k)-(m_x+m_y+\frac{k+l}{2})r]v_r+\frac{1}{4}[\lambda-K]v = 0$

where $K = (m_x + m_y)(m_x + m_y + k + l - 2)$. This differential equation has singular points at $r = -k$ and $r = l$. As noted above we want f to be C^2 on $\Omega_{k,l} = r^{-1}([0,l])$. This will place conditions on the regularity of v at $r = l$. Also, since $u(r) = (k+r)^{\frac{m_x}{2}}(l-r)^{\frac{m_y}{2}}v(r)$ the boundary condition $u(0) = 0$ becomes $v(0) = 0$.

We make the change of variable $t = \frac{1}{k+l}(l-r)$ in (2.19) to get

(2.20) $\quad t(1-t)v_{tt} + [(m_y + \frac{l}{2}) - (m_x + m_y + \frac{k+l}{2})t]v_t + \frac{1}{4}[\lambda - K]v = 0$

Corresponding to $0 < r \le l$ we have $0 \le t < \frac{l}{k+l}$ for $\Omega_{k,l}$. Our boundary condition at $r = 0$ becomes $v(t = \frac{l}{k+l}) = 0$. Also, certain regularity conditions on v will hold at $t = 0$.

We recognize (2.20) as the hypergeometric differential equation which is usually written

(2.21) $\qquad\qquad t(1-t)v'' + [c-(a+b+1)t]v' - abv = 0$

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In these terms we have from (2.20)

$$\begin{cases} c = m_y + \dfrac{l}{2} \\[2mm] a + b + 1 = m_x + m_y + \dfrac{k+l}{2} \\[2mm] ab = \dfrac{1}{4}[K - \lambda] \end{cases}$$

Which are solved by

$$a = \frac{m_x}{2} + \frac{m_y}{2} + L$$

$$b = \frac{m_x}{2} + \frac{m_y}{2} + \frac{k+l}{2} - (1+L)$$

$$c = m_y + \frac{l}{2}$$

for $L = \dfrac{(\frac{k+l}{2} - 1) + \sqrt{(\frac{k+l}{2} - 1)^2 + \lambda}}{2}$.

The solution of (2.21) which is regular (continuous) at $t = 0$ is the hypergeometric function $f(t) = {}_2F_1(a, b, c; t)$. So the solution of (2.20) which is regular at $t = 0$ is

$$v(t) = {}_2F_1(\frac{m_x}{2} + \frac{m_y}{2} + L, \frac{m_x}{2} + \frac{m_y}{2} + \frac{k+l}{2} - (1+L), m_y + \frac{l}{2}; t).$$

Equation (2.21) also has a solution (independent from ${}_2F_1(a, b, c; t)$) which in general does not converge at $t = 0$. Let us call this solution $\hat{v}(t)$. In the variable r, $\hat{v}(r)$ will have some kind of pole at $r = l$. However, a priori, it might be possible that $u(r) = (k + r)^{\frac{m_x}{2}}(l - r)^{\frac{m_y}{2}}\hat{v}(r)$ is well-defined at $r = l$. In fact, this does not happen. Our argument is at the end of this section. The argument is somewhat involved as the form of $\hat{v}(t)$ varies depending on the values of the parameters a, b, and c. We must allow a wide range of possibilities for a and b in particular because as yet we have no specific choices for λ.

We continue to assume that k and l are fixed integers greater than one, specifying a particular $\Omega_{k,l}$. We now also suppose that m_x and m_y are fixed nonnegative integers. Now we determine λ by the boundary condition $v(\dfrac{l}{k+l}) = 0$. That is, we want the zeroes of the function

$$g_{m_x, m_y}(\lambda) = v(\frac{l}{k+l})$$

$$= {}_2F_1(\frac{m_x}{2} + \frac{m_y}{2} + L(\lambda), \frac{m_x}{2} + \frac{m_y}{2} + \frac{k+l}{2} - (1 + L(\lambda)), m_y + \frac{l}{2}; \frac{l}{k+l}).$$

(figures 2.4–2.7 at the end of chapter 2 below show the graphs of $g_{m_x,m_y}(\lambda)$ for some specific choices of m_x, m_y, k, and l. These graphs were produced by the software package Mathematica on a Sun System 3/280. For information on Mathematica, see [Wo].)

For the values of λ that are zeroes of g_{m_x,m_y} the corresponding eigenfunctions of λ are of the form

$$f(p) = (k + r(p))^{\frac{m_x}{2}}(l - r(p))^{\frac{m_y}{2}} v(r(p))w^x(\xi(p))w^y(\zeta(p))$$

where $v(r) = {}_2F_1(t = \frac{l-r}{k+l})$.

For w^x we have used one of the eigenfunctions of S^{k-1} which corresponded to the eigenvalue $m_x(m_x + k - 2)$. There are $\binom{m_x + k - 1}{m_x - 1} - \binom{m_x + k - 2}{m_x - 2}$ independent choices for w^x ([Cl], p. 35). Similarly, for w^y we have chosen one of the $\binom{m_y + l - 1}{m_y - 1} - \binom{m_y + l - 2}{m_y - 2}$ eigenfunctions of S^{l-1} which correspond to the eigenvalue $m_y(m_y + l - 2)$.

The multiplicity of an eigenvalue λ will be at least

$$N = [\binom{m_x + k - 1}{m_x - 1} - \binom{m_x + k - 2}{m_x - 2}] \cdot [\binom{m_y + l - 1}{m_y - 1} - \binom{m_y + l - 2}{m_y - 2}]$$

for the appropriate values of m_x and m_y. The multiplicity of λ will be greater than N if λ is a root of g_{m_x,m_y} for more than one choice of m_x and m_y.

The zeroes of $g_{m_x,m_y}(\lambda)$ may be found numerically (using the package Mathematica, for example, see [Wo]). The results of some computations are given in section 2.5.

The eigenfunctions above can be written more neatly as follows. We recall first that a complete set of eigenfunctions of S^{k-1} corresponding to the eigenvalue $m(m + k - 2)$ can be formed by restricting to S^{k-1} harmonic homogeneous polynomials on $\mathbf{R}^k$ of degree m. So we may suppose that the spherical eigenfunction w^x is chosen so that $r_x^{m_x}w^x$ is a harmonic homogeneous polynomial $p(\overline{x})$ in rectangular coordinates $\overline{x}$ on $\mathbf{R}^k$. As above, $r_x = \|\overline{x}\|^2 =$ radial distance in $\mathbf{R}^k$, Similarly, we assume that $r_y^{m_y}w^y = q(\overline{y})$. Now we recall from Part I above that $r_x = \|\overline{x}\| = \sqrt{\frac{k+r}{k+l}}$ and $r_y = \sqrt{\frac{l-r}{k+l}}$. So $(k+r)^{\frac{m_x}{2}} = (k+l)^{\frac{m_x}{2}}r_x^{m_x}$ and $(l-r)^{\frac{m_y}{2}} = (k+l)^{\frac{m_y}{2}}r_y^{m_y}$. Then we can rewrite our eigenfunction as follows

$$f = (k + r)^{\frac{m_x}{2}}(l - r)^{\frac{m_y}{2}} v(r)w^x(\xi)w^y(\zeta)$$
$$= (k + l)^{\frac{m_x}{2}}(k + l)^{\frac{m_y}{2}} v(r)r_x^{m_x}w^x(\xi)r_y^{m_y}w^y(\zeta)$$
$$= (k + l)^{\frac{m_x + m_y}{2}} v(r)p(\overline{x})q(\overline{y}).$$

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From the Bateman Manuscript ([BM] vol. 1, page 57), we have that $_2F_1(t)$ is defined by a power series which converges absolutely for $\|t\| < 1$. So $_2F_1(t)$ is C^∞ for $\|t\| < 1$. We are interested in $0 < r \leq l$ which corresponds to $0 \leq t < \dfrac{l}{k+l}$, so $v(r)$ is C^∞ in our range of interest. Since we can write $v(r)$ as $v(l\|\overline{x}\|^2 - k\|\overline{y}\|^2)$ it is now clear that the functions we have defined are C^2 (in fact C^∞) on all of $\Omega_{k,l}$. Thus they are indeed eigenfunctions for $\Omega_{k,l}$.

$_2F_1(t)$ is a polynomial in t if the parameter b is a negative integer. So $v(r)$ is a polynomial if $\frac{m_x}{2} + \frac{m_y}{2} + \frac{k+l}{2} - (1+L)$ is a negative integer. This sometimes happens for our eigenvalues but not generally. See section 2.5 below. Otherwise $_2F_1(t)$ converges at $t = 1$ only if $a+b-c < 0$. So $v(r)$ converges at $r = -k$ only if $m_x + \frac{k}{2} - 1 < 0$. But m_x is a nonnegative integer and k is an integer greater than one, so except for the occasional polynomial cases, our eigenfunctions for $\Omega_{k,l}$ do not extend to eigenfunctions of the sphere S^{k+l-1}. This is reflected by the fact that many of our numerically computed eigenvalues are not integers. See section 2.5 below.

We now argue that the second solution to (2.21), $\hat{v}(t)$, never leads to a solution $u(r)$ to (2.16) which is sufficiently regular at $r = l$. The results below are taken from [Ra] chapters 7 and 8, and from the Bateman Manuscript Project ([BM], chapter 2).

We recall equation (2.21), the hypergeometric differential equation, is

$$(2.21) \qquad t(1-t)v'' + [c - (a+b+1)t]v' - abv = 0.$$

We want solutions v to (2.21) on the range $0 \leq t < \frac{l}{k+l}$ from which we will construct a function $u(t) = (1-t)^{\frac{m_x}{2}} t^{\frac{m_y}{2}} v(t)$ (here we have rewritten $u(r)$ as afunction of t, recalling that $t = \frac{1}{k+l}(l-r)$). In the case $c \neq 0, -1, -2, \ldots$ (which is true for us), one solution of (2.21) is $_2F_1(a, b, c, t)$ which is defined for $0 \leq t < 1$. In general, there will be a second solution, $\hat{v}(t)$, the form of which varies depending on the parameters a, b, and c.

In the case that $c \neq$ an integer, $\hat{v}(t) = t^{1-c}\,_2F_1(a+1-c, b+1-c, 2-c, t)$. Then

$$t^{\frac{m_y}{2}} v(t) = t^{1-c+\frac{m_y}{2}}\,_2F_1(a+1-c, b+1-c, 2-c, t)$$
$$= t^{1-\frac{m_y}{2}-\frac{l}{2}}\,_2F_1(a+1-c, b+1-c, 2-c, t).$$

The exponent $1 - \frac{m_y}{2} - \frac{l}{2}$ will be negative unless $m_y = 0$ and $l = 2$. In that case, we have $c = 1$ which is a contradiction of the case at hand. Thus, when c is not an integer, the solution $\hat{v}(t)$ does not lead to a viable choice for $u(t)$. If we recall that $c = m_y + \frac{1}{2}$, we see that this case covers one half of our interest, namely $l = $ an odd integer.

In the case that $c = 1$ (so $m_y = 0$ and $l = 2$) and neither a nor b is zero or

a negative integer, the second solution to (2.21) is

$$\hat{v}(t) = {}_2F_1(a, b, 1, t) \ln t$$

$$+ \sum_{n=1}^{\infty} \frac{(a)_n (b)_n t^n}{(n!)^2} \{ H(a, n) + H(b, n) - 2H(1, n) \}$$

where $(a)_n = a(a+1)(a+2) \ldots (a+n-1)$ and $H(a, n) = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1}$. The sum converges at zero, so $\hat{v}$ behaves like $\ln t$ near zero. Then $t^{-\frac{m_y}{2}} \hat{v}(t) = t^0 \hat{v}(t)$ is bad at zero. So again no viable choice of $u(t)$ is created.

In the case that $c = 1$ and a or b is zero or a negative integer, the form of $\hat{v}(t)$ varies, depending on the particular values of a and b. However, in every instance $\hat{v}$ will either include $\ln t$ or t^{-p} for p a positive integer. Since $c = 1$ implies $m_y = 0$, $\hat{v}(t)$ undefined at zero will mean that the corresponding $u(t)$ is also undefined at zero.

In the case that $c = p$ for p a positive integer and neither a nor b is an integer less than c, the second solution to (2.21) is

$$\hat{v}(t) = {}_2F_1(a, b, c, t) \ln t$$

$$- \sum_{n=0}^{c-2} \frac{n!(1-c)_{n+1}}{(1-a)_{n+1}(1-b)_{n+1} t^{n+1}}$$

$$+ \sum_{n=0}^{\infty} \frac{(a)_n (b)_n t^n}{(c)_n n!} \{ H(a, n) + H(b, n) - H(c, n) - H(1, n) \}.$$

Now if $c > 2$, $\hat{v}$ includes a term $\frac{1}{t^{c-1}}$. Then $t^{-\frac{m_y}{2}} \frac{1}{t^{c-1}}$ is $t^{-\frac{m_y}{2}+1-c}$. We have seen above that this exponent is negative unless $c = 1$ which is not true here, so again $u(t)$ is undefined at zero. If $c = 2$, then we must have $m_y = 1$ and $l = 2$. Now the only bad term in $\hat{v}(t)$ is $\ln t$. Although $t^{\frac{1}{2}} \ln t$ is defined at zero, its derivative diverges there. Now $u(t)$ is no good because it is insufficiently differentiable at zero.

Finally, in the case that c is an integer greater than one and one of b or c is an integer less than c, we find that the form of $\hat{v}(t)$ varies depending on the particular values of a and b. However, in each instance $\hat{v}$ will either include a term $\frac{1}{t^{c-1}}$ which will lead $u(t)$ to be undefined at zero, or $\hat{v}$ will include a term $\ln t$. In the latter case the corresponding $u(t)$ will only be differentiable to a finite degree. But an eigenfunction will always be C^∞, so these u's will not lead to eigenfunctions.

We have now completed the argument that only the regular solutions of (2.21) were necessary.

2.4 Completeness.

We argue now that the collection of Dirichlet eigenfunctions for $\Omega_{k,l}$ found above is complete. The point of the argument will be to show that an arbitrary

Dirichlet eigenfunction can be found as a linear combination of the eigenfunctions given in section 2.3. It then follows that the collection is complete.

Let us suppose $f : S^{k+l-1} \to \mathbf{R}$ is a Dirichlet eigenfunction of $\Omega_{k,l}$. That is, we suppose f satisfies $\Delta f + \lambda f = 0$ in $\Omega_{k,l}$ for some number λ with $f = 0$ on $\partial \Omega_{k,l}$. Then in terms of the variables r, ξ, and ζ we have that $f(r, \xi, \zeta)$ satisfies (11). (Recall (11) holds except where $r = -k$ or l.) We let $\{\phi_n\}_{n=1}^{\infty}$ be a complete, orthonormal set of eigenfunctions for S^{k-1} with corresponding eigenvalues $\{\lambda_n\}$. (We will suppose that the functions w^x above were chosen from this set.) If r and ζ are fixed, with $r \neq -k$ or l, then we have a function $g_{r,\zeta} : S^{k-1} \to \mathbf{R}$ given by $g_{r,\zeta}(\xi) = f(r, \zeta, \xi)$. This function is C^2 and can be expanded as a sum of the eigenfunctions ϕ_n as

$$(2.22) \qquad g_{r,\zeta}(\xi) = \sum_{n=1}^{\infty} g_n(r, \zeta)\phi_n(\xi).$$

Furthermore, all derivatives of $g_{r,\zeta}(\xi)$ can be expanded by the same sum where $\phi_n(\xi)$ is replaced by its derivative of appropriate order.

We now consider $g_{r,\zeta}$ as a function of r and ζ. We wish to argue that derivatives with respect to r and ζ may also be taken through the sum in (2.22). We make the argument by looking at the details of the standard proof that (2.22) is uniform in ξ (see, e.g., [CH], vol. 1, pp. 426-427). We suppose that p and q are positive integers with $q > p$. Then from the reference we have

$$\left(\sum_{n=p}^{q} g_n(r, \zeta)\phi_n(\xi) \right)^2 \leq \left(\int_{S^{k-1}} \|\nabla g_{r,\zeta}\|^2 \, dV \right) \times h(p, q)$$

where $h(p, q) \to 0$ as $p \to \infty$ and $q \to \infty$. Since $g_{r,\zeta}(\xi)$ is C^{∞} on the compact set $\overline{\Omega_{k,l}}$, its first derivatives are each uniformly bounded on $\Omega_{k,l}$. Whence the integral in the preceding equation is uniformly bounded, and we see that $\sum_{n=1}^{\infty} g_n(r, \zeta)\phi_n(\xi)$ converges uniformly in all three variables. Since indeed all derivatives of $g_{r,\zeta}(\xi)$ are uniformly bounded on $\Omega_{k,l}$, the argument above may be repeated indefinitely and we have the desired result.

Now if we substitute (2.22) into (2.11) we get

$$\sum_{n=1}^{\infty} [4(k + r)(l - r)\frac{\partial^2 g_n}{\partial r^2} - 2(k + l)r\frac{\partial g_n}{\partial r} - \frac{k + l}{k + r}g_n\lambda_n$$

$$+ \frac{k + l}{l - r}\Delta_{S^{l-1}}g_n + \lambda g_n] = 0$$

If $\sum a_n\phi_n = 0$, we have $a_n = 0$. So we have

$$(2.23) \qquad 4(k + r)(l - r)\frac{\partial^2 g_n}{\partial r^2} - 2(k + l)r\frac{\partial g_n}{\partial r} - \frac{k + l}{k + r}g_n\lambda_n$$

$$+ \frac{k + l}{l - r}\Delta_{S^{l-1}}g_n + \lambda g_n = 0$$

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We now write $g_n(r,\zeta) = \sum_{m=1}^{\infty} h_{n,m}(r)\gamma_m(\zeta)$ where $\{\gamma_m\}_{m=1}^{\infty}$ is a complete, orthonormal set of eigenfunctions on S^{l-1} with eigenvalues $\{\mu_m\}_{m=1}^{\infty}$. Similar arguments to the above and substitution into (2.23) yield

(2.24)

$$4(k+r)(l-r)\frac{\partial^2 h_{n,m}}{\partial r^2} - 2(k+l)r\frac{\partial h_{n,m}}{\partial r} - \frac{k+l}{k+r}h_{n,m}\lambda_n$$
$$- \frac{k+l}{l-r}h_{n,m}\mu_n + \lambda h_{n,m} = 0$$

where for the Dirichlet boundary condition we need $h_{n,m} = 0$ for all n and m. Thus we have expanded an arbitrary solution f to the Dirichlet eigenvalue problem on $\Omega_{k,l}$ as the sum

(25)
$$f(r,\xi,\zeta) = \sum_{n,m}^{\infty} h_{n,m}\phi_n(\xi)\gamma_m(\zeta)$$

where ϕ_n and γ_m are eigenfunctions of S^{k-1} and S^{l-1} respectively and $h_{n,m}$ satisfies (2.24) with $h_{n,m}(0) = 0$. But we recall that $\lambda_n = m_x(m_x + k - 2)$ for some nonnegative integer m_x, and $\mu_n = m_y(m_y + l - 2)$ for some nonnegative integer m_y. Hence we see that (2.24) is identical to (2.16). If in fact we have found all solutions to (2.16) (up to constant multiples) in the work above , then the sum (2.25) is a linear combination of the eigenfunctions we found in section 2.3 above. In which case that set of eigenfunctions is complete.

2.5 Numerics.

For k and l fixed integers greater than one we have defined an open domain $\Omega_{k,l} \subset S^{k+l-1}$. In section 2.3 above we have identified the Dirichlet eigenvalues of this domain as the zeroes of a class of functions $g_{m_x,m_y}(\lambda)$ for m_x and m_y nonnegative integers . In section 2.4 above we argued that these zeroes form a complete Dirichlet spectrum for $\Omega_{k,l}$. We would like now to numerically estimate some of these zeroes for particular choices of k and l. We recall that the function $g_{m_x,m_y}(\lambda)$ is defined by

$$g_{m_x,m_y}(\lambda) = v(\frac{l}{k+l})$$
$$= {}_2F_1(\frac{m_x}{2} + \frac{m_y}{2} + L(\lambda), \frac{m_x}{2} + \frac{m_y}{2} + \frac{k+l}{2} - (1 + L(\lambda)), m_y + \frac{l}{2}; \frac{l}{k+l}).$$

The function ${}_2F_1$ is well-known and can be evaluated numerically by its power series. In fact, the function is available in various numerics packages. To generate the zeroes given below we have used the package Mathematica on a Sun System 3/280 (see [Wo] for information on Mathematica).

In table 2-1 below, we have fixed $k = 2$ and $l = 2$. Then for each pair of values for m_x and m_y where $0 \leq m_x, m_y \leq 5$, we list the first five zeroes of

the function $g_{m_x,m_y}(\lambda)$. Each list of five zeroes is headed by a term $\{m_x, m_y\}$. Thus, the five numbers in table 2-1 underneath $\{1,2\}$ are the first five zeroes of $g_{1,2}$ computed numerically.

Table 2-2 gives the same information for $k = 2$ and $l = 3$.

Every number in tables 2-1 and 2-2 is an approximation of a Dirichlet eigenvalue of the corresponding $\Omega_{k,l}$ ($\Omega_{2,2}$ or $\Omega_{2,3}$). Each eigenvalue appears only once in a table unless it is a zero of more than on g_{m_x,m_y}. However, The true multiplicities of these eigenvalues is larger. (See section 2-3 above.)

In tables 2-3 and 2-4, we have sorted the numbers from tables 2-1 and 2-2 repectively into increasing order. These two tables give Dirichlet eigenvalues of the domains $\Omega_{2,2}$ and $\Omega_{2,2}$ in increasing order. We note however that these two tables do not provide the first 180 eigenvalues of each of the domains $\Omega_{2,2}$ and $\Omega_{2,3}$. Many functions g_{m_x,m_y} not evaluated in forming these tables (i.e., those with m_x or m_y greater than 5) have smaller first and second (and even higher) zeroes than some of the numbers found. For example, with $k = 2$ and $l = 2$, the first zero of $g_{6,0}$ is approximately 49.1393. If this function had been included in forming table 2-3, this number would be the 16th entry. Analysis of the pattern of zeroes suggests that table 2-3 is a complete ordered list of eigenvalues for $\Omega_{2,2}$ discounting multiplicity through the 15th entry. (However, Table 2-3 includes some duplicates. For example, the value 48 appears twice in table 3-3 because 48 is a zero of both $g_{0,0}$ and $g_{2,2}$. Discounting this extra appearance, we have the first 14 distinct eigenvalues of $\Omega_{2,2}$.) Table 2-4 is complete through the 15th entry as well. In general, we observe that for $m_x + m_y =$ constant, the smallest zeroes come from g_{m_x,m_y} with $m_y = 0$. The next smallest when $m_y = 1$, etc. Thus, the most efficient way to find the first hundred eigenvalues of one of these domains would be to work with a rectangular (as opposed to square) array of choices for m_x and m_y. For example, one might let m_x range from 0 to 10 while m_y ranges from 0 to 4.

We observe that some of the eigenvalues in table 2-1 appear to be integers. In fact when $k = l = 2$ and $m_x = m_y$, apparently every zero of the function g_{m_x,m_y} is an integer. Furthermore, the particular integer values–8, 24, 48, 80, etc.–would suggest that the corresponding eigenfunctions are eigenfunctions of the whole sphere S^3. Indeed, one can find a complete set of polynomial solutions to equation (2.19) in this case. With $k = l$ and $m_x = m_y$ (2.19) becomes

$$(2.26) \qquad [k^2 - r^2]v_rr - (2m_x + k)rv_r + \frac{1}{4}[\lambda - K]v = 0$$

where $K = 4m_x(m_x + k - 1)$.

Now if we substitute $v = a_1r + a_2r^2 + \cdots + a_dr^d$ into (2.26) we get a degree d polynomial on the left-hand side. Equating its coefficients with zero yields a series of conditions on the a_i's and λ.

$$\text{constant coeff.:} \qquad k^2 2a_2 = 0$$

coeff. of r: $\qquad 6k^2 a_3 - (2m_x + k)a_1 + \frac{1}{4}[\lambda - K]a_1 = 0$

coeff. of r^d: $\qquad -d(d-1)a_d - (2m_x + k)da_d + \frac{1}{4}[\lambda - K]a_d = 0$

Under the assumption that $a_d \neq 0$, the last condition gives $\lambda = 4d(d-1) + 4(2m_x + k)d + K$. The first condition gives $a_2 = 0$. The third condition (not shown) gives a_4 in terms of a_2, and we find $a_4 = 0$. We may set $a_1 = 1$ (a multiple of a solution to (2.26) is also a solution). From the second condition, we will get a_3 in terms of a_1. We continue recursively and create an odd polynomial solution to (2.26) for any odd degree d. These odd polynomials form a complete set of solutions to (2.26) as any set of odd polynomials of every odd degree is complete on $[0, k = l]$ under the assumption $v(0) = 0$.

We note also that any polynomial solution to (2.26) will satisfy (2.26) not just on the range $r \in [0, k]$ (where $\Omega_{k,k}$ is defined) but on the entire interval $[-k, k]$ corresponding to the circle S^{2k-1}. Whence the solution to the original eigenvalue problem created using this polynomial solution $v(r)$ is an eigenfunction for the whole sphere, not merely for $\Omega_{k,k}$.

Some particular polynomial solutions to (2.26) when $k = l = 2$ are $v = r$ for $m_x = m_y = 0$ with $\lambda = 8$ and $v = r - \frac{7}{3}r^3$ for $m_x = m_y = 1$ with $\lambda = 80$.

We can attempt to find other polynomial solutions to (2.19) by the same technique as above even when k and l or m_x and m_y are different. We may substitute $v(r) = a_1 r + a_2 r^2 + \cdots + a_d r^d$ into (2.19) and try to find appropriate choices for the a_i's. In general, this problem is overdetermined and has no solution. However, some isolated polynomial solutions do occur in these "non-symmetric" cases. One such solution is $v(r) = 2r + r^3$ where $k = l = 2$ and $m_x = 1$ and $m_y = 4$ with $\lambda = 99$. We note that λ approximately 99 appears in Table 2-1 under the group labelled $\{1, 4\}$.

We have appended the Mathematica procedure used to create Tables 2-1 and 2-2 following Table 2-4.
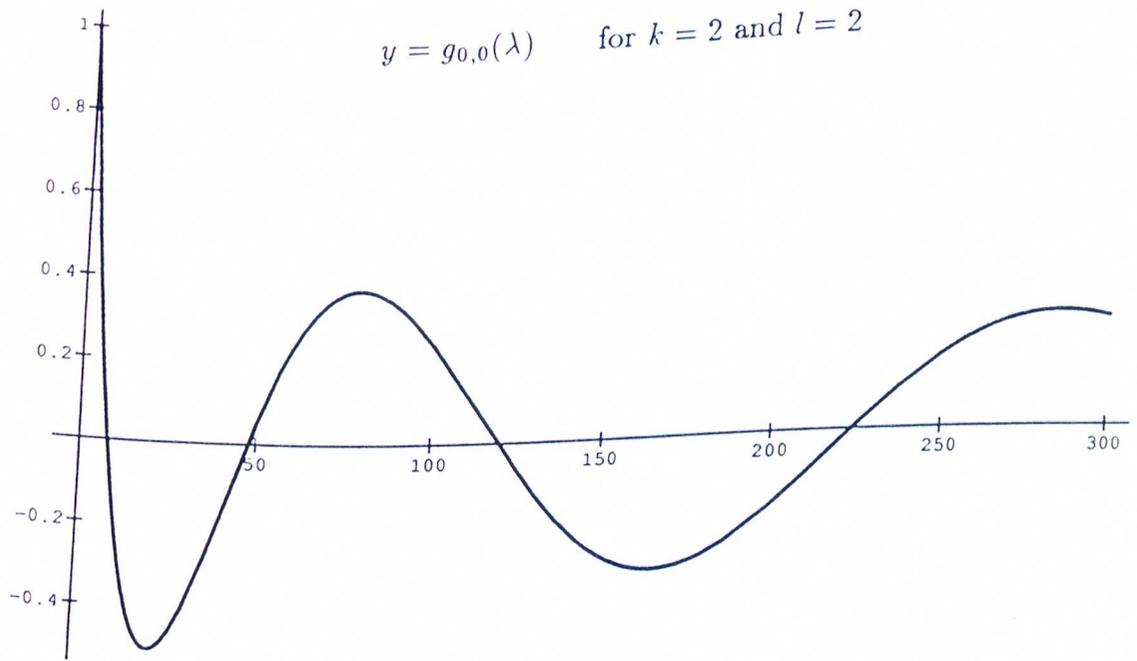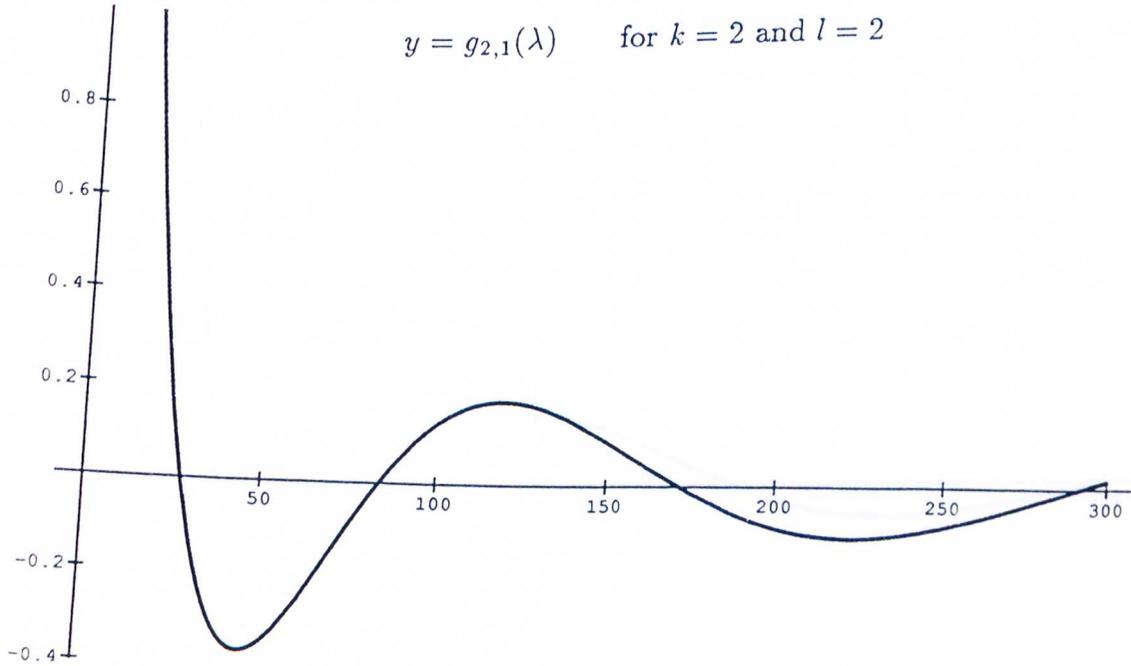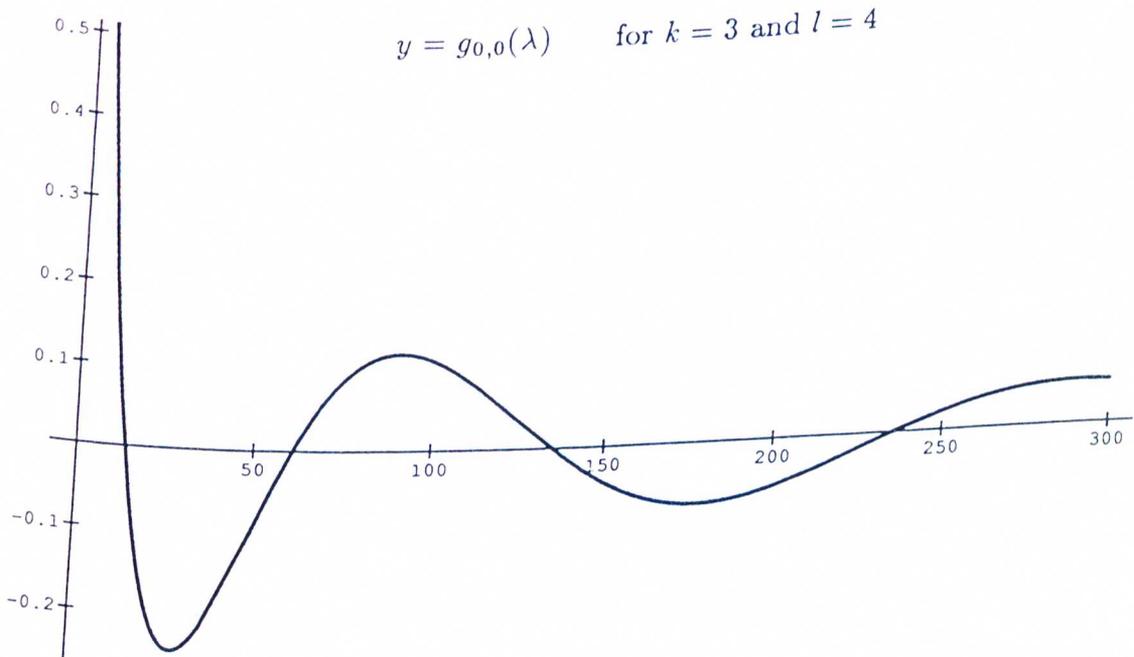
$y = g_{0,0}(\lambda)$ for $k = 2$ and $l = 2$

figure 2-4

$y = g_{2,1}(\lambda)$ for $k = 2$ and $l = 2$

figure 2-5

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$$y = g_{0,0}(\lambda) \qquad \text{for } k = 3 \text{ and } l = 4$$

figure 2-6

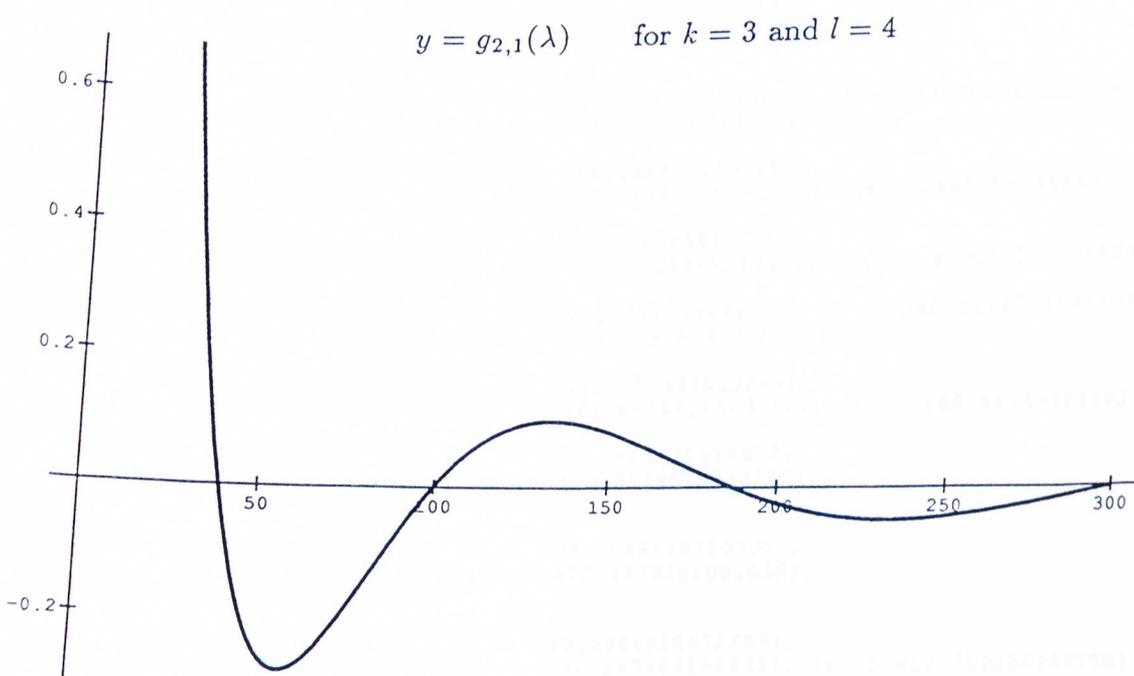$$y = g_{2,1}(\lambda) \qquad \text{for } k = 3 \text{ and } l = 4$$

figure 2-7

Table 2-1 (first page)

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Table 2-1 (second page)

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Table 2-2 (first page)

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Table 2-2 (second page)

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{5, 4}
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{5, 5}
{{t -> 160.1111295935442}, {t -> 259.8281220388808}, {t -> 385.181186380463},
  {t -> 535.8418144117165}, {t -> 711.7182740803474}}
```

Table 2-3

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 t -> 26.42009178606956, t -> 27.74608590223241, t -> 33.97232187904021,
 t -> 36.68237361420346, t -> 42.65346403681546, t -> 42.74436853156356,
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Table 2-4

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Mathematica procedure for tables 2-1 and 2-2

```
hz/: hz[k_, l_, m_, p1_, p2_, q1_, q2_] :=
    Do[gz[k, l, i, j, m]; PutAppend[{i, j}, "data"];
      PutAppend[Union[li], "data"]; lj=Append[lj, Union[li]],
      {i, p1, p2, 1},
      {j, q1, q2, 1}]

gz/: gz[k_, l_, mx_, my_, m_] :=
    Block[{x1, x2, n}, n = 1; x1 = 2; x2 = 12; li = {};
      While[n <= m, If[N[g[k, l, mx, my, x1]*g[k, l, mx, my, x2]] > 0,
        x2 = x2 + 10, AppendTo[li,
          FindRoot[g[k, l, mx, my, t], {t, {x1, x2}}]]; n = n + 1; x1 = x2+1;
        x2 = x2 + 11]]]

g/: g[k_, l_, mx_, my_, lambda_] :=
    Hypergeometric2F1[mx/2 + my/2 + h[k, l, lambda],
      (mx/2 + my/2 + (k + l)/2) - (1 + h[k, l, lambda]), my + 1/2, 1/(k + l)]

h/: h[k_, l_, lambda_] := ((k + l)/2 - 1 + Sqrt[((k + l)/2 - 1)^2 + lambda])/2
```

In this chapter we recall some of the basic definitions made in chapters 1 and 2 above, and we investigate the connections among them in a more abstract setting.

Suppose M is a Riemannian manifold. A function $f : M \to \mathbf{R}$ is an *isoparametric* function if Δf and $\|\nabla f\|^2$ are both functions of f. As above, Δ is the Laplace-Beltrami operator on M, and ∇f is the gradient of f.

Suppose Ω is an open domain in M. A function $g : \Omega \to \mathbf{R}$ is an *overdetermined (Dirichlet) eigenfunction* of Ω if g satisfies $\Delta g + \lambda g = 0$ in Ω for some number λ and satisfies both boundary conditions $g = 0$ and $\frac{\partial g}{\partial n} = \text{constant}$ on $\partial \Omega$. Not every domain in a manifold will admit overdetermined eigenfunctions (see the references at the beginning of Part I).

There are interesting overlaps between isoparametric functions and overdetermined eigenfunctions. Any eigenfunction g satisfies the isoparametric condition that Δg is a function of g, because in fact $\Delta g = -\lambda g$. Every isoparametric function f will satisfy overdetermined boundary conditions on a domain Ω if it satisfies only the Dirichlet boundary condition $f = 0$ on $\partial \Omega$. For suppose $\|\nabla f\|^2 = h(f)$, then on $\partial \Omega$, $\|\nabla f\|^2 = h(0) = \text{constant}$. Now because $\partial \Omega$ is a level surface of f, a normal vector to it will be $\frac{\nabla f}{\|\nabla f\|}$. So

$$\frac{\partial f}{\partial n} = \frac{\nabla f}{\|\nabla f\|} \cdot \nabla f = \|\nabla f\| = \sqrt{h(0)}.$$

Indeed, an isoparametric function will have constant normal derivatives on all of its level surfaces.

It follows that if an isoparametric function is also a Dirichlet eigenfunction for a domain Ω then in fact it is an overdetermined eigenfunction for the domain. The overdetermined eigenfunctions shown in Part I above for certain subdomains of spheres are all isoparametric functions. In $\mathbf{R}^n$, S^{n+} (hemisphere), and $\mathbf{H}^n$ (real hyperbolic space) the only examples known of overdetermined eigenfunctions are also isoparametric functions. As far as we know, it is an open question whether every overdetermined eigenfunction must be an isoparametric function.

It is clearly not true that an isoparametric function will necessarily be an overdetermined eigenfunction of some domain. (The most natural domain would be the set of points where the function is positive.) An isoparametric function need not satisfy the equation $\Delta f + \lambda f = 0$, only some condition of the type $\Delta f + h(f) = 0$. However, as was done in Part I above, given an isoparametric function f with $\Omega = \{p \in M | f(p) > 0\}$, one may search for overdetermined eigenfunctions on Ω of the form $\phi(f)$ where $\phi : \mathbf{R} \to \mathbf{R}$. Then we have $\Delta(\phi(f)) = \Delta(f)\phi'(f) + \|\nabla f\|^2 \phi''(f)$. Precisely because Δf and

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$\|\nabla f\|^2$ are functions of f, the equation $\Delta(\phi(f)) + \lambda f = 0$ will become an ODE for ϕ where the single boundary condition $\phi(0) = 0$ will guarantee that $\phi(f)$ satisfies overdetermined boundary conditions on $\partial\Omega$.

Cartan ([Ca]) has shown that if f is an isoparametric function, the level surfaces of f will be parallel and each level surface will have constant principal curvatures. Following his techniques, we are able to show that if a function f is an overdetermined eigenfunction on a domain Ω where $\partial\Omega$ has constant curvatures, then f will be isoparametric locally to $\partial\Omega$. See Theorem 1 below. We conjecture that a proof of the fact that an overdetermined eigenfunction must be isoparametric might go as follows.

1. The existence of one overdetermined (Dirichlet) eigenfunction on a domain $\Omega \Longrightarrow$ the first (Dirichlet) eigenfunction of Ω is overdetermined.

2. The first eigenfunction of Ω overdetermined $\Longrightarrow \partial\Omega$ has constant curvatures (by Rayleigh methods).

3. The first eigenfunction of Ω overdetermined and constant curvatures on the boundary of $\Omega \Longrightarrow$ the first eigenfunction (and any other overdetermined eigenfunction) is an isoparametric function (Theorem 1 plus a global geometric argument covering level surfaces having cusps, etc.).

THEOREM 1. *Let M be an n-dimensional Riemannian manifold. Suppose Ω is an open domain of M with the surface $\partial\Omega$ having constant principal curvatures $K_1, K_2, \ldots, K_{n-1}$. Suppose $f : \Omega \to \mathbf{R}$ satisfies $\Delta f + \lambda f = 0$ in Ω for some number λ and $f = 0$ and $\frac{\partial f}{\partial n} = k$ on $\partial\Omega$ for some constant k. Then $\|\nabla f\|$ is constant on level surfaces of f near $\partial\Omega$.*

PROOF: We will suppose $n = 3$. We also suppose that $\partial\Omega$ and level surfaces of f sufficiently near to it are smooth surfaces of dimension 2. In this region we define an orthonormal frame of coordinate vectors e_1, e_2, and e_3 in the following manner. e_1 and e_2 are unit vectors that lie in the directions of principal curvature of each level surface of f (including $\partial\Omega$). e_3 is always perpendicular to the level surfaces (say inward from $\partial\Omega$).

We will calculate Δf in terms of this moving frame. We let $f_i = e_i(f)$. Then $\nabla f = f_3 e_3$. We will write $D_u v$ for the Levi-Civita connection of the vector field v in the direction u. Then we calculate

$$D_{e_1}(\nabla f) = D_{e_1}(f_3 e_3) = f_{31} e_3 + f_3 D_{e_1}(e_3)$$
$$= f_{31} e_3 + f_3 K_1 e_1$$

$$D_{e_2}(\nabla f) = D_{e_2}(f_3 e_3) = f_{32} e_3 + f_3 K_2 e_2$$

$$D_{e_3}(\nabla f) = D_{e_3}(f_3 e_3) = f_{33} e_3 + f_3 D_{e_3}(e_3)$$

Then

$$\Delta f = e_1 \cdot (D_{e_1}(\nabla f)) + e_2 \cdot (D_{e_2}(\nabla f)) + e_3 \cdot (D_{e_3}(\nabla f)) = f_3 K_1 + f_3 K_2 + f_{33}.$$

Here we have used the fact that

$$e_3 \cdot (D_{e_3}(e_3)) = \frac{1}{2}e_3(e_3 \cdot e_3) = \frac{1}{2}e_3(1) = 0$$

Now we have assumed that f is an eigenfunction, so there exists a number λ such that

(3.1)
$$\Delta f = \lambda f = f_3 K_1 + f_3 K_2 + f_{33}$$

Let us consider this equation on the boundary surface $\partial\Omega$. Here we have $f_3 = \|\nabla f\| = $ constant. Also K_1 and K_2 are constants and $f = 0$. Whence we see that f_{33} is constant. But $f_{33} = e_3(f_3) = e_3(\|\nabla f\|)$. So we may rewrite (3.1) as

(3.2)
$$\lambda f = \|\nabla f\|(K_1(f) + K_2(f)) + e_3(\|\nabla f\|)$$

We recall that on $\partial\Omega$, $\|\nabla f\|$ is a constant. Now if we imagine solving for $\|\nabla f\|$ by integrating equation (3.2) in the e_3 direction from the boundary, we see that the value of $\|\nabla f\|$ depends only on the value of f. This result holds as far as our frame holds.

REFERENCES

[Av] P. Aviles, *Symmetry theorems related to Pompeiu's problem*, Amer. J. Math. **108** (1986), 1023–1036.

[Be1] C. Berenstein, *An inverse spectral theorem and its relation to the Pompeiu problem*, J. Anal. Math. **37** (1980), 128–144

[Be2] C. Berenstein, *On an overdetermined Neumann problem*, Seminari di Geometria, Univ. di Bologna (1986), 21–26

[BM] A. Erdelyi, ed., "Higher Transcendental Functions," vol. 1, McGraw-Hill, New York (1953)

[BY1] C. Berenstein and P. Yang, *An overdetermined Neumann problem in the unit disk*, Advances in Math. **44** (1982) 1–17

[BY2] C. Berenstein and P. Yang, *An inverse Neumann problem*, J. Reine Angew. Math. **382** (1987) 1–21

[Ca] E. Cartan, *Families de surfaces isoparametrique dans le space a courbure constante*, Ann. of Math. **17** (1938) 177–191

[CH] R. Courant and D. Hilbert, "Methods of Mathematical Physics, Vol. I," Interscience Publishers, Inc., New York (1953)

[Cl] I. Chavel, "Eigenvalues in Riemannian Geometry," Academic Press, Inc., New York (1984)

[EDM] "Encyclopediac Dictionary of Mathematics," Kiyosi Ito, ed., Mathematical Society of Japan, English translation published by MIT Press, Cambridge, Mass. (1987)

[GL] N. Garofalo and J. Lewis, *A symmetry result related to some overdetermined boundary value problems*, to appear in Amer. Jour. Math.

[Mo] R. Molzon, *Symmetry and a boundary value problem for the Laplacian on the sphere and hyperbolic space*, Institute Fourier Grenoble **101** (1988)

[Ra] E. Rainville, "Intermediate Differential Equations," The Macmillan Company, New York (1964)

[Se] J. Serrin, *A symmetry problem in potential theory*, Arch. Rat. Mech. Anal. **43** (1971) 304–318

[We] H. F. Weinberger, *Remark on the preceding paper of Serrin*, Arch. Rat. Mech. Anal. **43** (1971) 319–320

[Wo] S. Wolfram, "Mathematica, A System for Doing Mathematics by Computer," Addison-Wesley Publishing Company, Inc., New York (1988). The software package Mathematica is sold by Wolfram Research, Inc., Champaign, IL