ABSTRACT

Title of Dissertation: CONVEX GEOMETRY
AND ASYMPTOTICS
OF STABILITY THRESHOLDS
IN ALGEBRAIC GEOMETRY

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This thesis investigates stability thresholds in algebraic geometry, particularly the α- and δ-invariants, which play a central role in various areas, including the existence of Kähler-Einstein metrics on Fano manifolds.

The work is divided into four main parts. First, we provide a counterexample to Tian's stability conjecture for α_k-invariants, demonstrating that these invariants do not always stabilize or become monotone for large k. Second, we resolve the Cheltsov–Rubinstein problem for strongly asymptotically log del Pezzo surfaces by studying the last remaining pair (II.6A.n.m) and establishing necessary and sufficient conditions for the existence of Kähler-Einstein edge metrics. In the last two parts, we study the stabilization and asymptotics of toric α_k- and δ_k-invariants, showing that while α_k-invariants stabilize from $k = 1$ in the toric setting, δ_k-invariants rarely do, and we derive their asymptotic expansions. The results combine techniques from convex geometry, Ehrhart theory, and birational geometry, offering new insights into the interplay between convex geometry and algebraic geometry.

CONVEX GEOMETRY AND ASYMPTOTICS OF STABILITY THRESHOLDS IN ALGEBRAIC GEOMETRY

by

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Chapter 1

Overview

This thesis investigates stability thresholds, specifically the α- and δ-invariants, which play a crucial role in various areas, most notably in their original purpose of detecting the existence of (twisted) Kähler–Einstein metrics on Kähler manifolds. These invariants are related to singularities and stability conditions in algebraic geometry.

The α-invariant, introduced by Tian in 1987, provides a sufficient condition for the existence of Kähler–Einstein metrics on Fano manifolds, while the δ-invariant, developed by Fujita–Odaka in 2016, offers a necessary and sufficient criterion. This thesis explores the asymptotic behavior of these invariants, mostly in the toric case, where their computation can be reduced to combinatorial problems for lattice polytopes.

The main results of this thesis include: a counterexample to Tian's stability conjecture on α_k-invariants (Chapter 4), a resolution of the last case of Cheltsov–Rubinstein problem (Chapter 5), and the derivation of toric G-equivariant α_k-invariants (Chapter 6), as well as asymptotic expansions for quantized barycenters of lattice polytopes and toric δ_k-invariants (Chapter 7). Chapters 4 and 5 deal with surfaces that are not toric, while Chapters 6 and 7 deal with toric manifolds.

1.1 Motivation

1.1.1 The Kähler–Einstein problem

Let X be a compact Kähler manifold of dimension n. Recall that X is *Kähler–Eintein* if it admits a Kähler–Einstein metric, i.e., the Kähler form ω satisfies

$$\operatorname{Ric}\omega = \lambda\omega$$

for some $\lambda \in \mathbb{R}$.

Notice that there is a cohomological constraint for the existence of such a metric. Indeed, taking cohomology classes of both sides, we obtain $2\pi c_1(X) = \lambda[\omega]$. In particular, $c_1(X)$ is either negative, 0, or positive. For the first two cases, it was shown that a unique Kähler–Einstein metric always exists [5, 111, 112]. The last case, i.e., X is Fano, is more complicated. Unlike the former cases, Fano manifolds do not always admit Kähler–Einstein metrics. Indeed, a necessary condition for the existence of such a metric on X was found by Matsushima, namely the reductivity of $\operatorname{Aut}_0(X)$, the identity component of its automorphism group [78]. In particular, $\mathbb{P}^2$ blown up at 1 or 2 points does not admit Kähler–Einstein metric, as its automorphism group is not reductive.

On the other hand, Tian's α-invariant provided the first sufficient condition for the existence of Kähler–Einstein metric. We may assume $\lambda = 1$ by rescaling, and fix some reference metric ω in the class $2\pi c_1(X)$. Consider the space

$$\mathcal{H}_\omega := \left\{\varphi \,\middle|\, \omega_\varphi := \omega + \sqrt{-1}\partial\bar\partial\varphi > 0\right\} \subseteq C^\infty(X)$$

of Kähler potentials of metrics cohomologous to ω. Tian introduced the invariant

$$\alpha(X) := \sup\left\{c > 0 \,\middle|\, \sup_{\varphi\in\mathcal{H}_\omega} \int_X e^{-c(\varphi - \sup\varphi)}\omega^n < \infty\right\}$$

Note that α does not depend on the choice of ω.

Theorem 1.1.1. [101, Theorem 2.1] *Let X be a Kähler manifold with $c_1(X) > 0$. X admits a Kähler–Einstein metric if*

$$\alpha(X) > \frac{n}{n+1}.$$

There is also a discrete version of α-invariants,

$$\alpha_k(X) := \sup\left\{c > 0 \,\middle|\, \sup_{\varphi \in \mathcal{H}_{\omega,k}} \int_X e^{-c(\varphi - \sup \varphi)} \omega^n < \infty\right\},$$

where

$$\mathcal{H}_{\omega,k} := \left\{\varphi \,\middle|\, \omega_\varphi = \frac{1}{k}\sqrt{-1}\partial\bar{\partial} \log \sum_{i=1}^{d_k} |s_i|^2,\ \{s_i\}_{i=1}^{d_k} \text{ is a basis of } H^0(X, -kK_X)\right\}$$

and $-K_X$ is the anti-canonical bundle. This was introduced by Tian, as a result of the approximation of $\mathcal{H}_\omega$ by the quantized spaces $\mathcal{H}_{\omega,k}$ [102, 103]. The α_k-invariant is the quantized version of the α-invariant in the sense that

$$\alpha(X) = \lim_{k \to \infty} \alpha_k(X) = \inf_k \alpha_k(X).$$

In 1988 Tian conjectured that $\alpha_k(X) = \alpha(X)$ for sufficiently large k [103, Question 1]. This is known as Tian's stabilization problem. The stabilization of $\alpha_k(X)$ will provide a way to compute $\alpha(X)$, without the need to consider infinitely many k.

1.1.2 Complex singularity exponents and log canonical thresholds

On the algebraic side, it was later conjectured by Cheltsov and established by Demailly that α actually coincides with the global log canonical threshold from algebraic geometry [29].

The log canonical threshold is a fundamental invariant in algebraic geometry. It first appeared in a paper of Atiyah [4]. There he proved a conjecture of Gelfand proposed in the Amsterdam Congress of 1954 (see [60, §5.4]): for any $s \in \mathbb{C}$ with positive real part there is a distribution on $\mathbb{C}^n$ given by $\varphi \mapsto \int_{\mathbb{C}^n} |f(z)|^{2s} \varphi(z) dz d\bar{z}$ for any $\varphi \in C_c^\infty(\mathbb{C}^n)$. Atiyah proved that this can be extended to $\mathbb{C}$ as a meromorphic function, with the largest pole bounded above by $-\operatorname{lct}(f)$. Specifically, by taking a log resolution we may reduce to the case when $f(z) = z^I$ for some multi-index I. Then $\operatorname{lct}(f) = (\max_i I_i)^{-1}$.

Later it was Shokurov who first used the idea of log canonical threshold in the context of birational geometry [95, Definition 6.1]. The log canonical threshold is the largest multiple λD of a divisor D in X such that the pair $(X, \lambda D)$ is log canonical, that is, up to some log resolution, the coefficients of $\lambda D = \sum_i a_i D_i$ are bounded above by 1. This invariant plays a crucial role in the minimal model program, particularly in the study of singularities and termination of flips. Shokurov also proposed the ACC conjecture and proved it in dimension 2, which asserts that the set of all log canonical thresholds in a fixed dimension contains no strictly increasing sequences, i.e., any increasing sequence stabilizes.

The log canonical threshold is used to measure how bad a singularity is. Geometrically, it coincides with the notion of complex singularity exponent, which measures the integrability of $|f|^{-2c}$ near the singularity.

Proposition 1.1.2. [66, Proposition 8.2] *The log canonical threshold coincides with the complex singularity exponent, i.e.,*

$$\mathrm{lct}(f) = \sup\left\{c \geq 0 \,\middle|\, |f|^{-2c} \in L^1_{\mathrm{loc}}\right\}.$$

By Demailly–Kollár [37, Main Theorem 0.2], it is lower semi-continuous.

It was first shown by Demailly that α coincides with the global log canonical threshold, defined as the infimum of the log canonical thresholds of all effective $\mathbb{Q}$-divisors in the anti-canonical class. Another proof is also given by Shi.

Definition 1.1.3. Let X be a Fano manifold. For $k \in \mathbb{N}_+$, the *k-th global log canonical threshold*

$$\mathrm{glct}_k(X) := \inf_{D \in |-kK_X|} k\,\mathrm{lct}(D),$$

and the *global log canonical threshold*

$$\mathrm{glct}(X) := \inf_k \mathrm{glct}_k(X).$$

Theorem 1.1.4. [29, Theorem A.1],[94, Theorem 2.2] *Let X be a Fano manifold. Then*

$$\alpha(X) = \mathrm{glct}(X) = \lim_{k\to\infty} \mathrm{glct}_k(X).$$

Moreover, it was also shown that the discrete version, glct_k, coincides with α_k as well.

Proposition 1.1.5. [94, Remark 2.1] *Let X be a Fano manifold. Then for $k \in \mathbb{N}_+$,*

$$\alpha_k(X) = \mathrm{glct}_k(X).$$

This equivalence allows the study of Tian's α-invariants using tools from birational geometry.

1.1.3 K-stability, Ding stability, and δ-invariants

The notion of K-stability was first introduced by Ding–Tian [38] and Tian [106], and later refined by Donaldson [39], Berman [10], Fujita–Odaka [48] and Fujita [45]. It characterizes the existence of Kähler–Einstein metrics. While Tian's α-invariant gives only a suffcient condition, the δ-invariant, introduced by Fujita–Odaka, provides both necessary and sufficient conditions for the existence of Kähler–Einstein metrics [48]. Instead of taking into consideration all the divisors, it measures the singularity of a special class of divisors called basis divisors.

Definition 1.1.6. [48, Definitions 0.1, 0.2] Let X be a Fano manifold. For $k \in \mathbb{N}_+$, a divisor D is a *k-basis divisor* if there is a basis $\{s_i\}_{i=1}^{d_k}$ of $H^0(X, -kK_X)$ such that

$$D = \frac{1}{kd_k}\sum_{i=1}^{d_k}(s_i).$$

The *k-basis log canonical threshold*

$$\delta_k := \inf_{D\ k\text{-basis}} \mathrm{lct}(D).$$

The *basis log canonical threshold*

$$\delta := \lim_{k\to\infty} \delta_k.$$

The existence of the limit (i.e., the convergence of δ_k) is due to Blum–Jonsson [16, Theorem A]. They used valuations to reduce this problem to the approximation of the integral of a concave function by its restriction to lattice points.

Theorem 1.1.7. [48, Theorem 0.3],[16, Theorem B] *Let X be a Fano manifold. If X admits a Kähler–Einstein metric, then $\delta \geq 1$. If $\delta > 1$, then X admits a Kähler–Einstein metric.*

We also mention that orbifold Kähler–Einstein metrics can give rise to Sasaki–Einstein metrics on Seifert circle bundles [17].So the α- and δ-invariants have played a role in constructing such metrics.

The Kähler–Einstein problem can also be generalized to the twisted case. Let L be ample and take a semi-positive smooth form θ in the class $2\pi(c_1(X) - c_1(L))$. A *θ-twisted Kähler–Einstein metric* is a solution to the twisted Kähler–Einstein equation

$$\operatorname{Ric}\omega = \omega + \theta.$$

This corresponds to the notion of Ding stability, which can also be detected by δ-invariant, defined with respect to L instead of $-K_X$. The following generalizes Theorem 1.1.7.

Theorem 1.1.8. [11, Theorem A],[16, p. 2] *Let L be ample and take a semi-positive smooth form θ in the class $2\pi(c_1(X) - c_1(L))$. If X admits a θ-twisted Kähler–Einstein metric, then $\delta(L) \geq 1$. If $\delta > 1$, then X admits a θ-twisted Kähler–Einstein metric.*

1.1.4 Exceptional Fano manifolds

Tian's α-invariant can be generalized to the case when we are given a compact subgroup G of $\operatorname{Aut} X$. Indeed, we may define

$$\alpha_G(X) := \sup\left\{ c > 0 \,\middle|\, \sup_{\varphi \in \mathcal{H}_{\omega,G}} \int_X e^{-c(\varphi - \sup\varphi)}\omega^n < \infty \right\}$$

where $\mathcal{H}_{\omega,G}$ contains only the G-invariant metrics. Since $\alpha_G \geq \alpha$, α_G can give rise to more Kähler–Einstein manifolds.

When $\alpha_G > 1$, the first application in differential geometry was given by Rubinstein, namely that Kähler–Einstein metrics can be constructed via Ricci iteration.

Theorem 1.1.9. [87, Theorem 3.3] *Suppose $\alpha_G(X) > 1$. Then the Ricci iteration*

$$\operatorname{Ric}\omega_k = \omega_{k-1}$$

is solvable in the class $2\pi c_1(X)$, with $\omega_k \xrightarrow{C^\infty} \omega_{KE}$, where ω_{KE} is a G-invariant Kähler–Einstein metric.

Examples of smooth Fano manifolds satisfying $\alpha_G(X) > 1$ include the del Pezzo surface $S = dP_5$ (i.e., $\mathbb{P}^2$ blown up at four points with no three of them colinear) with $G = \operatorname{Aut}(S) \cong S_5$. In this case $\alpha_G(X) = 2$ [25, Example 1.11]. In 2009 Rubinstein asked whether such examples exist for $\mathbb{P}^n$.

Problem 1.1.10. Is there a finite subgroup $G \subseteq \operatorname{Aut}(\mathbb{P}^n)$ such that $\alpha_G(\mathbb{P}^n) > 1$?

Cheltsov–Shramov showed that such finite subgroup exists for $n \leq 4$ [30, Theorem 1.22], $n = 5$ [31, Theorem 1.4], and $n = 8$ [32, Theorem 6], but not for $n = 6$ [31, Theorem 1.5]. It turns out that this is almost equivalent to the existence of exceptional quotient singularities of dimension $n + 1$ [30, Theorem 1.17].

Recall that a singularity $O \in X$ is *exceptional* if for any log canonical pair (X, D) (i.e., $A_{(X,D)}(E) \geq 0$ for any E over X), there is at most one exceptional divisor E such that $A_{(X,D)}(E) = 0$ [96, Definition 1.5].

Theorem 1.1.11. [30, Theorem 3.17] *Let G be a finite subgroup of $GL_{n+1}(\mathbb{C})$ that does not contain reflections, and $\overline{G}$ its image under the projection $GL_{n+1}(\mathbb{C}) \to PGL_{n+1}(\mathbb{C})$. Then the singularity $\mathbb{C}^{n+1}$ is exceptional if and only if $\operatorname{lct}(D) > 1$ for any $\overline{G}$-invariant $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_{\mathbb{P}^n}$.*

Recall that

$$\alpha_{\overline{G}}(\mathbb{P}^n) = \inf_{\substack{D \sim_{\mathbb{Q}} -K_{\mathbb{P}^n} \\ \overline{G}\text{-invariant}}} \operatorname{lct}(D).$$

It follows that $\alpha_{\overline{G}}(\mathbb{P}^n) > 1$ implies that the singularity $\mathbb{C}^{n+1}$ is exceptional. Conversely, if the singularity $\mathbb{C}^{n+1}$ is exceptional, $\alpha_{\overline{G}}(\mathbb{P}^n) \geq 1$.

Using the notation of Theorem 1.1.11, when $n \leq 4$, the singularity $\mathbb{C}^{n+1}$ is exceptional if and only if $\alpha_{\overline{G}}(\mathbb{P}^n) \geq \frac{n+2}{n+1}$.

1.2 Main results

1.2.1 Tian's stability conjecture on α_k-invariants

In 1988 Tian proposed the following conjecture, that α_k stabilizes for sufficiently large k.

Conjecture 1.2.1. [103, Question 1] *For any Fano manifold X, there exists $k_0 \in \mathbb{N}_+$ such that for $k \geq k_0$, $\alpha_k(X) = \alpha(X)$.*

Tian generalized this conjecture to any polarization in 2012.

Conjecture 1.2.2. [107, Conjecture 5.3] *For any ample line bundle L over a projective manifold X, there exists $k_0 \in \mathbb{N}_+$ such that for $k \geq k_0$, $\alpha_k(L) = \alpha(L)$.*

There is also a weaker folklore conjecture, stating that the sequence of α_k-invariants is eventually monotone.

Conjecture 1.2.3. *For any ample line bundle L over a projective manifold X, there exists $k_0 \in \mathbb{N}_+$ such that for $k_0 \leq k_1 \leq k_2$, $\alpha_{k_1}(L) \geq \alpha_{k_2}(L)$.*

These conjectures were originally stated for Fano X and $L = -K_X$. That remains the most interesting case, due to the relation to Kähler–Eintein metrics. Some positive results are known for this case. The case of smooth del Pezzo surfaces can be deduced from [25]. In general, it was shown by Birkar that there exists some $\ell \in \mathbb{N}_+$ such that $\alpha_{k\ell}(-K_X) = \alpha(-K_X)$ for all $k \in \mathbb{N}_+$, provided that $\alpha(-K_X) \leq 1$ (even for $-K_X$ big and nef) [13]. Note that this result does not provide an answer to Conjecture 1.2.2 or 1.2.3. When the manifold is toric, this thesis confirms Conjecture 1.2.2 for any ample line bundle, and stabilization is disproved for other quantized invariants of the form $\alpha_{k,m}$ in the toric setting. See Chapter 6.

There is an even stronger version of Conjecture 1.2.2, predicting how large k_0 is.

Conjecture 1.2.4. [107, Conjecture 5.4] *For any ample line bundle L over a projective manifold X, if the section ring $R(X, L) := \oplus_{k=0}^{\infty} H^0(X, kL)$ is generated by $\oplus_{k=0}^{k_0} H^0(X, kL)$, then for $k \geq k_0$, $\alpha_k(L) = \alpha(L)$. In particular, if the section ring $R(X, L)$ is generated by $H^0(X, L)$, then for any $k \in \mathbb{N}_+$, $\alpha_k(L) = \alpha(L)$.*

A counterexample to Conjecture 1.2.4 was provided by Ahmadinezhad–Cheltsov–Schicho [1]. They found a smooth surface S with a very ample line bundle L such that the section ring $R(S,L)$ is generated by $H^0(S,L)$, but $\alpha_1(L) > \alpha(L)$.

The main result of Chapter 4 is a counterexample to Conjectures 1.2.2 and 1.2.3.

Theorem 1.2.5. *There is an ample line bundle L over a smooth surface S with $-K_S$ nef and big such that α_k does not stabilize, nor is it eventually monotone.*

As $L \neq -K_X$, Conjecture 1.2.1 is still open.

1.2.2 Cheltsov–Rubinstein Problem

The purpose of Chapter 5 is to complete the solution of the Cheltsov–Rubinstein problem [26, 88] on the classfication of strongly asymptotically log del Pezzo surfaces admitting a Kähler–Einstein edge metric. This has been a topic of intense research. See, e.g., [26, 27, 28, 47, 88, 89, 91]. As a consequence of these works, one (infinite) family of pairs remains to be understood: the pairs (II.6A.n.m) (see Definition 5.0.6). This is the only pair where the boundary consists of more than one component. In Chapter 5 we treat this family completely and hence solve the Cheltsov–Rubinstein problem for strongly asymptotically log del Pezzo surfaces. The Cheltsov–Rubinstein problem for general (i.e., not strongly) asymptotically log del Pezzo surfaces is very challenging and remains wide open.

Theorem 1.2.6. *Consider* (II.6A.n.m) *as in Definition 5.0.6. We have*

$$\delta\Big(S, (1-\beta_1)\widetilde{C}_1 + (1-\beta_2)\widetilde{C}_2\Big) \leq 1,$$

with equality if and only if

$$\beta_1^2 + \frac{1}{3}\widetilde{C}_1^2\beta_1^3 = \beta_2^2 + \frac{1}{3}\widetilde{C}_2^2\beta_2^3, \quad m \neq 1. \tag{1.1}$$

The pair is log K-polystable, i.e., it admits a Kähler–Einstein edge metric, if and only if (1.1) *holds.*

1.2.3 Stabilization of toric α_k-invariants

Chapter 6 starts with the by-now-classical relation between Tian's (holomorphic) invariants and the (algebraic) global log canonical thresholds. The relationship was first conjectured by Cheltsov and proved by Demailly and Shi [29, 94]. However, so far, this relationship has only been shown for the α- and α_k-invariants, or for the α_G-invariant (see [29, Theorem A.3], [94, Proposition 2.1], [29, (A.1)], respectively), and not for the more subtle invariants $\alpha_{k,G}$. Inspired by Demailly, we introduce (Definition 6.2.4) the k-th G-equivariant global log canonical threshold

$$\mathrm{glct}_{k,G}$$

as an algebraic counterpart of Tian's $\alpha_{k,G}$ (Definition 6.2.1). A natural question is:

Problem 1.2.7. Let X be Fano and $L = -K_X$, and let G be a compact subgroup $G \subseteq \mathrm{Aut}\, X$. Is $\mathrm{glct}_{k,G} = \alpha_{k,G}$?

In Chapter 6, we consider a toric manifold X and a compact subgroup of the normalizer

$$N((\mathbb{C}^*)^n)$$

of the complex torus $(\mathbb{C}^*)^n$ in $\mathrm{Aut}\, X$. Denote by

$$\mathrm{Aut}\, P \subseteq GL(M) \cong GL(n, \mathbb{Z})$$

the subgroup of the automorphism group of the lattice M that leaves the polytope P (3.18) invariant. It is necessarily a finite group. In fact, $\mathrm{Aut}\, P$ is the quotient of the normalizer $N((\mathbb{C}^*)^n)$ of the complex torus $(\mathbb{C}^*)^n$ in $\mathrm{Aut}\, X$ by $(\mathbb{C}^*)^n$, so that $N((\mathbb{C}^*)^n)$ consists of finitely many components each isomorphic to a complex torus [7, Proposition 3.1]. For $H \subseteq \mathrm{Aut}\, P$, let

$$G(H) := H \ltimes \left(S^1\right)^n \subseteq N((\mathbb{C}^*)^n) \subseteq \mathrm{Aut}\, X \tag{1.2}$$

denote the compact group generated by H and $(S^1)^n$ (the latter is the maximal compact subgroup of the complex torus $(\mathbb{C}^*)^n$).

Proposition 1.2.8. *Let X be toric Fano and $L = -K_X$. Let $P \subseteq M_{\mathbb{R}}$ (see (3.6), (3.18)) be the polytope associated to $(X, -K_X)$, let $H \subseteq \operatorname{Aut} P$, and let $G(H)$ be as in (6.5). Then $\operatorname{glct}_{k,G(H)} = \alpha_{k,G(H)}$.*

This solves Problem 1.2.7 in the toric setting.

Using this result, and several new estimates, we can resolve Tian's stabilization problem (Conjecture 1.2.2) in the toric setting in a surprisingly strong sense, showing that equality holds for *all* $k \in \mathbb{N}$. We also allow for all groups $G(H)$ (recall (1.2)) and not just the maximal toric one $G(\operatorname{Aut} P)$.

Theorem 1.2.9. *Let X be toric Fano associated to a fan Δ whose rays are generated by primitive elements v_i in the lattice N dual to M. Let $P \subseteq M_{\mathbb{R}}$ (see (3.6), (3.18)) be the polytope associated to $(X, -K_X)$, let $H \subseteq \operatorname{Aut} P$, and let $G(H)$ be as in (6.5). Then for any $k \in \mathbb{N}$,*

$$\begin{aligned}
\alpha_{k,G(H)} &= \sup\left\{ c \in (0,1) \,\middle|\, -\frac{c}{1-c} P^H \subseteq P \right\} \\
&= \min_{u \in \operatorname{Ver} P^H} \frac{1}{\max_i \langle u, v_i \rangle + 1} \\
&= \min_{u \in \pi_H(\operatorname{Ver} P)} \frac{1}{\max_i \langle u, v_i \rangle + 1},
\end{aligned}$$

where P^H and π_H are defined in (6.6)–(6.7) and $\operatorname{Ver}(\,\cdot\,)$ denotes the vertex set of a polytope. In particular, $\alpha_{k,G(H)}$ is independent of $k \in \mathbb{N}$ and is equal to $\alpha_{G(H)}$. Same is true for any $G(H)$-invariant line bundle.

1.2.4 Asymptotics of toric δ_k-invariants

Chapter 7 deals with the δ_k-invariants of Fujita–Odaka (also called k-th stability thresholds) on toric Fano manifolds. It turns out that for these invariants there is a quantitative dichotomy regarding stabilization, and when stabilization fails we derive their complete asymptotic expansion.

Chapter 7 also initiates the study of a combinatorial problem: *what is the asymptotic behavior of the quantized (or discrete) barycenters of lattice polytopes?*

To state this more precisely, let

$$M \cong \mathbb{Z}^n$$

denote a lattice and $P \subseteq M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ a convex lattice polytope, i.e., the vertex set satisfies $\operatorname{Ver} P \subseteq M$. Does the sequence $\{\mathrm{Bc}_k(P)\}_{k\in\mathbb{N}}$ encode other canonical quantities associated to P beside $\mathrm{Bc}(P)$? We pose the following:

Problem 1.2.10. Is there an asymptotic expansion

$$\mathrm{Bc}_k(P) = \sum_{i=0}^{\infty} a_i(P)k^{-i}?$$

If so, what are the coefficients $a_i(P)$?

Problem 1.2.10 is of purely combinatorial interest within Ehrhart theory. Thus, the bulk of Chapter 7 (Sections 7.1, 7.3, and 7.4) is purely enumerative combinatorics, and to some extent also Section 7.2 that is concerned with algebraic combinatorics. Sections 7.5–7.6 are concerned with the algebraic geometric applications and can be read independently by the reader interested only in those (or skipped by the combinatorics-oriented reader).

At the same time, Chapter 7 can be considered as a sequel to Chapter 6 where we initiate the study of asymptotics of algebro-geometric invariants arising from Kodaira embeddings using vector spaces of holomorphic sections of powers of a line bundle $H^0(X, L^k)$. The asymptotic parameter is the power k of the line bundle, where $1/k$ can be thought of as Planck's constant in geometric quantization. The over-arching geometric motivation for the series initiated in Chapter 6 is to obtain complete asymptotic expansions for these invariants. This is of course motivated by the foundational result of Catlin and Zelditch on the complete asymptotics of the Bergman kernel [24, 117], but the setting is quite different: the invariants we are interested in are Tian's α_k- and Fujita–Odaka's δ_k-invariants [103, 48] and they are not directly defined in terms of the Bergman kernel. Rather, they are more related to singularities of divisors in the linear series $|kL|$ on X. These invariants play a fundamental role in both the minimal model program in algebraic geometry and in the study of canonical Kähler metrics in differential geometry.

Problem 1.2.11. Is there an asymptotic expansion

$$\delta_k = \sum_{i=0}^{\infty} a_i(X, L)k^{-i}?$$

If so, what are the coefficients $a_i(X, L)$?

By a theorem of Blum–Jonsson $\lim_k \delta_k = \delta$ [16, Theorem 4.4], with δ the basis log canonical threshold of Fujita–Odaka [48]. Thus, if such an expansion exists then $a_0(X, L) = \delta$.

Another motivation is *Tian's stabilization problem* which asks whether such invariants "stabilize", i.e., become eventually constant in k. Tian asked this for his α_k-invariants in the smooth Fano setting [103, Question 1], and later for all polarizations on possibly singular varieties [107, Conjecture 5.3], and one can extend his problem to the δ_k-invariants.

Problem 1.2.12. Is $\delta_k = \delta$ for all sufficiently large k?

One can consider our Problem 7.0.2 as a refinement of Problem 7.0.3.

We completely solve Problem 1.2.12 in the (possibly singular) toric Fano setting. This shows there is a remarkable dichotomy between the α_k-invariants and the δ_k-invariants. Chapter 6 resolved Tian's stabilization problem for α_k-invariants by showing that in the toric Fano setting $\alpha_k = \alpha$ for all k. Here we show that δ_k-invariants behave very differently and *rarely stabilize,* but that if they do stabilize then they also satisfy $\delta_k = \delta$ for *all* $k \in \mathbb{N}$, and when Fano also $\delta_k = \delta = 1$. This precisely suggests that most likely there is a complete asymptotic expansion as conjectured in Problem 7.0.2, and indeed we prove that is the case.

The main result of Chapter 7 is this asymptotic expansion, along with computations of the coefficients in various special cases. The most general case is the following.

Theorem 1.2.13. *Let M be an n-dimensional lattice and $P \subseteq M_{\mathbb{R}}$ an n-dimensional convex lattice polytope. Then $\mathrm{Bc}_k(P) = \sum_{i=0}^{\infty} a_i(P) k^{-i}$ for some constants $a_i(P)$. In particular,*

$$a_0(P) = \mathrm{Bc}(P), \quad a_1(P) = \frac{\widetilde{\mathrm{Vol}}(\partial P)}{2\,\mathrm{Vol}(P)}\Big(\widetilde{\mathrm{Bc}}(\partial P) - \mathrm{Bc}(P)\Big),$$

where $\widetilde{\mathrm{Vol}}$ is the normalized measure on ∂P, and $\widetilde{\mathrm{Bc}}$ denotes barycenter with respect to this measure (see Definition 3.3.1).

Moreover, $\mathrm{Bc}_k(P)$ is the quotient of two polynomials of degree n. In particular, if $\mathrm{Bc}_k(P)$ is constant for $n+1$ values of k then it is constant for all $k \in \mathbb{N}$.

Theorem 1.2.14. *Let X be a (possibly singular) toric variety given by a fan Δ in a lattice N. Let $v_1, \ldots, v_d \in N$ denote the primitive generators of rays in Δ. Let L be an ample line bundle with the associated convex polytope*

$$P = \{x \in M_{\mathbb{R}} \mid \langle x, v_i \rangle \geq -b_i, \text{ for all } i = 1, \ldots, d\}$$

for some $b_i \in \mathbb{R}$. Let I be the set of indices i such that v_i and b_i compute δ. Then

$$\delta_k = \delta - \frac{\widetilde{\mathrm{Vol}}(\partial P)}{2\,\mathrm{Vol}(P)} \max_{i \in I} \Big\langle \widetilde{\mathrm{Bc}}(\partial P) - \mathrm{Bc}(P), v_i \Big\rangle \delta^2 k^{-1} + O\big(k^{-2}\big).$$

For sufficiently large k, it is a rational function of k with both numerator and denominator polynomial of degree n in k.

Chapter 2

Notions from algebraic geometry

This chapter introduces fundamental concepts from algebraic geometry that are essential for understanding stability thresholds.

2.1 Log canonical thresholds

In this section we explain the notion of log canonical threshold in the definition of the α- and δ-invariants.

Definition 2.1.1. Let D be an effective divisor on a complex manifold X given by local defining functions $\{(U_i, f_i)\}_{i=1}^N$. Its *log canonical threshold*

$$\mathrm{lct}(D) := \sup\Big\{c \geq 0 \,\Big|\, |f_i|^{-2c} \in L^1_{\mathrm{loc}}(U_i)\Big\}.$$

Proposition 2.1.2. *For any effective divisor D and $k \in \mathbb{N}_+$, $\mathrm{lct}(kD) = k^{-1}\,\mathrm{lct}(D)$.*

Proof. If f is a local defining function of D, then f^k is a local defining function of kD. The result follows from Definition 2.1.1. $\square$

Using this homogeneity we can extend the notion of log canonical threshold to effective $\mathbb{Q}$-divisors.

Definition 2.1.3. Let D be an effective $\mathbb{Q}$-divisor and suppose kD is a divisor for some $k \in \mathbb{N}_+$. Then

$$\operatorname{lct}(D) := k \operatorname{lct}(kD).$$

Definition 2.1.4. A $\mathbb{Q}$-divisor D on a complex manifold X is *normal crossing* if for any $p \in X$ there is a coordinate chart on some neighborhood U of p such that $\operatorname{Supp} D \cap U$ is a union of coordinate hyperplanes, i.e., it is the vanishing locus of $z_1 \cdots z_\ell$, for some $0 \leq \ell \leq n$.

Example 2.1.5. Consider $X = \mathbb{P}^n$. Let H_i denote the hyperplane $V(Z_i)$, $i = 0, \ldots, Z_n$. Then $D := \sum_{i=0}^n H_i$ is normal crossing.

Example 2.1.6. The nodal curve $V(Y^2Z - X^2(X+Z))$ in $\mathbb{P}^2$ is normal crossing. See Figure 2.1.

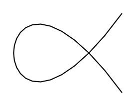

Figure 2.1: The nodal curve $V(Y^2Z - X^2(X+Z))$ in $\mathbb{P}^2$.

Lemma 2.1.7. *Consider the function* $f(z_1, \ldots, z_n) = \prod_{i=1}^n z_i^{a_i}$, *where* $a_i \in \mathbb{N}$. *Then for* $c \geq 0$, $|f|^{-2c}$ *is integrable around* 0 *if and only if*

$$c < \left(\max_{1 \leq i \leq n} a_i \right)^{-1}.$$

Proof. Let D denote the unit disc. We compute

$$\int_{D^n} |f(z)|^{-2c} dz = \prod_{i=1}^n \int_D |z_i|^{-2a_i c} dz_i.$$

Therefore $|f|^{-2c}$ is integrable around 0 if and only if for each i, $c < a_i^{-1}$. □

Proposition 2.1.8. *Suppose* $D = \sum_{i=1}^\ell a_i D_i$ *is a normal crossing* $\mathbb{Q}$*-divisor, where* $a_i > 0$. *Then* $\operatorname{lct}(D) = (\max_{1 \leq i \leq \ell} a_i)^{-1}$.

Proof. By homogeneity (Proposition 2.1.2) we may assume D is a divisor, i.e., $a_i \in \mathbb{N}_+$. Recall Definition 2.1.4. The result follows from Lemma 2.1.7. □

Example 2.1.9. Let H_i denote the hyperplane $V(Z_i)$ in $\mathbb{P}^n$. By Proposition 2.1.8, $\mathrm{lct}(\sum_{i=0}^n a_i H_i) = (\max_{1\le i\le \ell} a_i)^{-1}$.

Example 2.1.10. Let C be a nodal curve in $\mathbb{P}^2$. Since it is normal crossing, by Proposition 2.1.8, $\mathrm{lct}(aC) = a^{-1}$.

There is a more algebraic approach to define log canonical threshold, which we now explain. See, e.g., [66, §8].

Definition 2.1.11. Let X be $\mathbb{Q}$-Gorenstein, i.e., X is normal and some multiple of $-K_X$ is Cartier. We call E *a prime divisor over* X if there is a projective birational morphism $f : Y \to X$ with normal variety Y, and E is a prime divisor on Y. Its *log discrepancy* is

$$A(E) := 1 - \mathrm{ord}_E(f^*(K_X) - K_Y).$$

X is called *klt* if $A(E) > 0$ for any E over X.

For any $\mathbb{Q}$-divisor D on X,

$$\mathrm{ord}_E(D) := \mathrm{ord}_E(f^*(D)).$$

Example 2.1.12. Let E be on X and take $f = \mathrm{id}_X$. Then $A(E) = 1$.

Example 2.1.13. Let $Y = \mathrm{Bl}_p X \xrightarrow{f} X$ be the blow-up of a point $p \in X$, with E the exceptional divisor. Since $K_Y - f^*(K_X) = (\dim X - 1)E$, we have $A(E) = \dim X$. For any divisor D on X, $\mathrm{ord}_E(D) = \mathrm{ord}_E(f^*(D)) = \mathrm{mult}_p D$.

Theorem 2.1.14. *For any effective $\mathbb{Q}$-divisor D,*

$$\mathrm{lct}(D) = \inf_{E/X} \frac{A(E)}{\mathrm{ord}_E(D)}.$$

2.2 Nef and big classes and their volume

While α- and δ-invariants are useful in determining the existence of Kähler–Einstein metric, they are in general hard to compute. In this section, we recall some basic notions from algebraic geometry that will be needed in the study of these invariants, especially for surfaces.

Definition 2.2.1. Let X be a projective manifold of dimension n and D a divisor on X. D is called *numerically effective*, or *nef*, if $D \cdot C \geq 0$ for any (irreducible) curve C. Its *volume* is

$$\operatorname{Vol}(D) := \lim_{k \to \infty} \frac{n!}{k^n} \dim H^0(X, kD). \tag{2.1}$$

In general, if D is a $\mathbb{Q}$-divisor, assume its multiple mD is an integral divisor. Then its volume is defined by rescaling

$$\operatorname{Vol}(D) := \frac{1}{m^n} \operatorname{Vol}(mD).$$

We say D is *big* if $\operatorname{Vol}(D) > 0$.

Example 2.2.2. Consider $X = \mathbb{P}^n$. A hyperplane H is nef since it has positive intersection with any curve. For any $k \in \mathbb{N}$, $H^0(X, kH)$ is isomorphic to the space of homogeneous polynomial of degree k in $\mathbb{C}[Z_0, \ldots, Z_n]$. In particular, $\dim H^0(X, kH) = \binom{n+k}{k}$. By (2.1),

$$\operatorname{Vol}(H) = \lim_{k \to \infty} \frac{n!}{k^n} \binom{n+k}{k} = \lim_{k \to \infty} \prod_{i=1}^{n} \left(1 + \frac{i}{k}\right) = 1.$$

To compute the volume, the following standard result is useful.

Proposition 2.2.3. [68, Corollary 1.4.41] *Let X be a projective manifold of dimension n and D a nef divisor on X. Then*

$$\operatorname{Vol}(D) = D^n.$$

Remark 2.2.4. Suppose D is ample and let $\omega \in c_1(D)$ be a metric. Then $\operatorname{Vol}_\omega(X) = \int_X \omega^n = \int_X c_1(D)^n = D^n = \operatorname{Vol}(D)$, i.e., the notion of volume coincides with the volume in differential geometry.

Example 2.2.5. We can apply Proposition 2.2.3 to Example 2.2.2 and directly yield

$$\operatorname{Vol}(H) = H^n = 1.$$

If the divisor is not nef, the volume is harder to compute. In the next section we provide a way to compute the volume in dimension 2.

2.3 Zariski decomposition

When X is a surface, we can use Zariski decomposition to compute the volume [116, Theorems 7.7 and 8.1].

Definition 2.3.1. Let D be an effective $\mathbb{Q}$-divisor on a smooth projective surface S. There is a unique effective $\mathbb{Q}$-divisor

$$N = \sum_i a_i N_i$$

satisfying the following:

(i) $D - N$ is nef;

(ii) The intersection matrix $(N_i \cdot N_j)$ is negative definite;

(iii) For any i,

$$(D - N) \cdot N_i = 0. \tag{2.2}$$

The $\mathbb{Q}$-divisor N is the *negative part* of D, and the $\mathbb{Q}$-divisor

$$P := D - N$$

is the *positive part* of D.

Remark 2.3.2. We see from the definition that if two effective $\mathbb{Q}$-divisors are numerically equivalent, i.e., their intersection numbers with any given curve are equal, then their negative parts are the same.

Proposition 2.3.3. *Let D be an effective divisor on a smooth projective surface S with Zariski decomposition $D = P + N$. Then*

$$\mathrm{Vol}(D) = \mathrm{Vol}(P) = P^2.$$

Example 2.3.4. Consider $X = \mathbb{F}_n$ and $D = Z_{-n} + F$, where Z_{-n} is the $-n$-curve and F is a fiber. To obtain the Zariski decomposition $D = P + N$, assume the negative part $N = aZ_{-n}$ for some $a \in [0, 1]$. For $n \geq 1$, since Z_{-n} has negative self-intersection, (ii) is satisfied. To satisfy (iii), we need

$$(D - N) \cdot Z_{-n} = -n + 1 + na = 0, \tag{2.3}$$

i.e., $a = 1 - \frac{1}{n}$. We claim that $N = (1 - \frac{1}{n})Z_{-n}$ is the negative part of D by checking (i), i.e.,

$$P = D - N = \frac{1}{n} Z_{-n} + F$$

is nef. Indeed, by (2.3), $P \cdot Z_{-n} = 0$. Any curve other than Z_{-n} has non-negative self-intersection, and hence intersect non-negatively with any effective divisor. In particular, $P \cdot C \geq 0$ for any curve C, i.e., P is nef.

By Proposition 2.3.3,

$$\mathrm{Vol}(D) = \mathrm{Vol}(P) = P^2 = \frac{1}{n}.$$

We provide a simple proof of the existence and uniqueness of Zariski decomposition due to Bauer [8].

Definition 2.3.5. Let D, D' be effective $\mathbb{Q}$-divisors. $D \leq D'$ if $D' - D$ is effective.

Lemma 2.3.6. *Let D be an effective $\mathbb{Q}$-divisor on a smooth projective surface X. D admits a maximal nef subdivisor P. That is, for any nef divisor D' such that $D' \leq D$, $D' \leq P$. Moreover, any maximal nef subdivisor P is a positive part of D.*

Proof. Step 1: Existence of maximal nef subdivisor P. Suppose

$$D = \sum_{i=1}^{d} a_i C_i$$

where C_i is distinct and $a_i > 0$. We consider its subdivisor

$$D' := \sum_{i=1}^{d} x_i C_i$$

where

$$0 \le x_i \le a_i. \tag{2.4}$$

D' is nef if for all j,

$$\sum_{i=1}^{d} x_i C_i \cdot C_j \ge 0. \tag{2.5}$$

The constraints (2.4) and (2.5) cut out a compact convex set K in $[0, a_1] \times \cdots \times [0, a_d]$. Hence the function $f(x) := \sum_{i=1}^{d} x_i$ on K attains its maximum at some point $(b_1, \ldots, b_d) \in K$. Then

$$P := \sum_{i=1}^{d} b_i C_i$$

is a maximal nef subdivisor.

Step 2: P is a positive part of D. By construction, P is a nef subdivisor of D. Therefore

$$N := D - P = \sum_{i=1}^{d} (a_i - b_i) C_i$$

is effective. It remains to check (ii) and (iii). For simplicity we may assume the irreducible components of $\operatorname{Supp} N$ is $C_1, \ldots, C_e$ where $e \le d$, i.e., $a_i > b_i$ for $i \le e$, and $a_i = b_i$ otherwise.

To check (iii), suppose $P \cdot C_j > 0$ for some $j \le e$. Then we may take a sufficiently small $\varepsilon > 0$ such that

$$\varepsilon \le a_j - b_j \tag{2.6}$$

and

$$P \cdot C_j + \varepsilon C_j^2 \ge 0 \tag{2.7}$$

Now, consider the divisor

$$P' := P + \varepsilon C_j.$$

By (2.6),

$$P' = \sum_{i \neq j} b_i C_i + (b_j + \varepsilon) C_j \leq D.$$

By (2.7),

$$P' \cdot C_j = P \cdot C_j + \varepsilon C_j^2 \geq 0.$$

For any $i \neq j$, since P is nef,

$$P' \cdot C_i = P \cdot C_i + \varepsilon C_i \cdot C_j \geq 0.$$

Hence P' is also a nef subdivisor of D, contradicting the maximality of P.

To check (ii), it suffices to show that if the intersection matrix of N is not negative definite, then there is a non-zero effective divisor $E = \sum_{i=1}^{e} f_i C_i$ that is nef. Indeed, the existence of such divisor implies that $P + \varepsilon E$ is a nef subdivisor of D for sufficiently small ε, contradicting the maximality of P.

First we assume the intersection matrix is not negative semi-definite, i.e., there is a divisor

$$B = \sum_{i=1}^{e} c_i C_i \tag{2.8}$$

such that

$$B^2 > 0 \tag{2.9}$$

Note that B is not necessarily effective. Write $B = B_+ - B_-$, where

$$B_+ := \sum_{c_i > 0} c_i C_i, \quad B_- := -\sum_{c_i < 0} c_i C_i.$$

Then $0 < B^2 = B_+^2 + B_-^2 - 2B_+ \cdot B_- \leq B_+^2 + B_-^2$. Therefore at least one of B_+ and B_- has positive self-intersection. Hence by replacing B with B_+ or B_- we may assume B is effective while still satisfying (2.8) and (2.9). By Riemann–Roch, for sufficiently large k such that $h^0(X, K_X - kB) = 0$, we have

$h^0(X, kB) \geq \chi(kB) \sim \frac{1}{2}B^2k^2$. By (2.9), there is at least another divisor $B' \in |kB|$ different from kB. Write $B' = \sum_{i=1}^{e} c_i' C_i + B_0'$, where $\operatorname{Supp} B_0$ does not contain any curve C_i. Consider the effective divisor

$$E := \sum_{i=1}^{e} (kc_i - \min\{kc_i, c_i'\}) C_i.$$

Note that $E \neq 0$, i.e., there is some i such that $kc_i > c_i'$. Indeed, otherwise $kB < B'$, contradicting the fact that $B' \sim kB$. Moreover, E is nef. Indeed, for any C_j in its support, i.e., $kc_j > c_j'$, we have

$$E = kB - \sum_{i=1}^{e} \min\{kc_i, c_i'\} C_i \sim B' - \sum_{i=1}^{e} \min\{kc_i, c_i'\} C_i = B_0' + \sum_{i=1}^{e} (c_i' - \min\{kc_i, c_i'\}) C_i$$

and hence

$$E \cdot C_j = B_0' \cdot C_j + \sum_{i=1}^{e} (c_i' - \min\{kc_i, c_i'\}) C_i \cdot C_j \geq \left(c_j' - \min\left\{kc_j, c_j'\right\}\right) C_j^2 = 0.$$

We are left with the case when the intersection matrix is negative semi-definite, but not negative definite, i.e., its kernel is non-trivial. We prove by induction on e. When $e = 1$, the intersection matrix is C_1^2. Since it is singular, $C_1^2 = 0$. Hence $E = C_1$ is a non-zero effective nef divisor. For $e \geq 2$, since the intersection matrix has a non-trivial kernel, there is a non-zero divisor

$$B := \sum_{i=1}^{e} c_i C_i$$

such that

$$B \cdot C_i = 0 \tag{2.10}$$

for any $1 \leq i \leq e$. If B or $-B$ is effective, then it is nef and we can simply take it as the non-zero effective nef divisor E. Otherwise, write $B = B_+ - B_-$, where

$$B_+ := \sum_{c_i > 0} c_i C_i, \quad B_- := -\sum_{c_i < 0} c_i C_i.$$

By (2.10), $B^2 = 0$. On the other hand, since the intersection matrix is negative semi-definite, $B^2 = B_+^2 + B_-^2 - 2B_+ \cdot B_- \leq B_+^2 + B_-^2 \leq 0$. Hence equality holds and in particular $B_+^2 = 0$. Recall that B is assumed to be not effective. Hence $\operatorname{Supp} B_+$ has less than e irreducible components. In particular, its intersection matrix is singular and negative semi-definite with rank less than e, and the induction hypothesis guarantees the existence of such divisor. This completes the proof. □

We have shown that any maximal nef subdivisor is a positive part. The following lemma shows the converse.

Lemma 2.3.7. *Let P be a positive part of D. Then P is a maximal nef subdivisor of D.*

Proof. Suppose $D = P + N$ is a Zariski decomposition with $N = \sum_{i=1}^{e} g_i C_i$. If $P' \geq P$ is another nef subdivisor of D, since $P' - P \leq D - P = N$, we may write

$$P' - P = \sum_{i=1}^{e} g_i' C_i,$$

where $0 \leq g_i' \leq g_i$. Since P' is nef, by (iii),

$$(P' - P)^2 = \sum_{i=1}^{e} g_i' C_i \cdot (P' - P) = \sum_{i=1}^{e} g_i' C_i \cdot P' \geq 0.$$

But the intersection matrix of N is negative definite. It follows that $P' - P = 0$. □

Next, we show that maximal nef subdivisors are unique.

Lemma 2.3.8. *Any effective $\mathbb{Q}$-divisor admits a unique maximal nef subdivisor.*

Proof. It suffices to show that if $P = \sum_{i=1}^{d} b_i C_i$ and $P' = \sum_{i=1}^{d} b_i' C_i$ are two maximal nef subdivisors, then

$$P'' = \sum_{i=1}^{d} \max\{b_i, b_i'\} C_i$$

is also nef, i.e., for any j,

$$-\max\{b_j, b_j'\} C_j^2 \leq \sum_{i \neq j} \max\{b_i, b_i'\} C_i \cdot C_j.$$

To see this, we may assume $b_j \geq b'_j$. Then by nefness of P,

$$P \cdot C_j = \sum_{i=1}^{d} b_i C_i \cdot C_j \geq 0.$$

Therefore

$$-\max\{b_j, b'_j\} C_j^2 = -b_j C_j^2 \leq \sum_{i \neq j} b_i C_i \cdot C_j \leq \sum_{i \neq j} \max\{b_i, b'_i\} C_i \cdot C_j.$$

This completes the proof. □

The existence and uniqueness of Zariski decomposition follows from Lemmas 2.3.6, 2.3.7, and 2.3.8.

Theorem 2.3.9. *Let D be an effective $\mathbb{Q}$-divisor on a smooth projective surface X. Its Zariski decomposition exists and is unique. Indeed, the positive part P is the unique maximal nef subdivisor of D.*

Using Theorem 2.3.9 we can compute any Zariski decomposition via linear programming.

Example 2.3.10. We generalize Example 2.3.4 to any effective divisor $D = aZ_{-n} + bF$, where $a, b \geq 0$.

Since $D \cdot Z_{-n} = -na + b$, and $D \cdot F = a$, we know that D is nef if and only if $na \leq b$. In that case we have $P = D$ and $N = 0$.

Now let us assume that D is not nef, i.e., $na > b$. By Theorem 2.3.9, its positive part $P = a_p Z_{-n} + b_p F$ is the maximal nef subdivisor of D, i.e., it is the largest solution to the inequalities

$$\begin{cases} 0 \leq a_p \leq a, & (2.11) \\ 0 \leq b_p \leq b, & (2.12) \\ na_p \leq b_p. & (2.13) \end{cases}$$

By (2.12), $b_p \leq b$. Combining with (2.13), $a_p \leq \frac{b}{n}$. This automatically imply (2.11) since we have assumed $na > b$. Therefore the maximal nef subdivisor of D is $P = \frac{b}{n} Z_{-n} + bF$, and $N = D - P = (a - \frac{b}{n}) Z_{-n}$.

By Proposition 2.3.3, when D is nef, i.e., $na \leq b$, $\mathrm{Vol}(D) = D^2 = 2ab - na^2$. When D is not nef, i.e., $na > b$, $\mathrm{Vol}(D) = P^2 = \frac{b^2}{n}$.

2.4 α-invariants

The α- and δ-invariants, initially defined in the Fano case, can be generalized to arbitrary polarized projective manifolds. In this section and the next we study them in this generality.

Definition 2.4.1. Let X be a projective manifold, and L a big line bundle on X. Its *section ring* is

$$R := \bigoplus_{k=0}^{\infty} R_k := \bigoplus_{k=0}^{\infty} H^0(X, kL).$$

Let

$$d_k := \dim H^0(X, kL).$$

Definition 2.4.2. Let X be a projective manifold, and L a big line bundle on X. For $k \in \mathbb{N}_+$,

$$\alpha_k := \inf_{D \in |kL|} \operatorname{lct}\bigl(k^{-1} D\bigr). \tag{2.14}$$

Blum–Jonsson realized that a more algebraic approach can be taken to study these invariants, via valuations [16]. In particular, they provide an equivalent definition of these invariants, using notions from birational geometry. We now express α_k in terms of prime divisors over X using Theorem 2.1.14. First we introduce the notion of maximal vanishing order.

Definition 2.4.3. Fix a big line bundle $L \to X$. For a prime divisor E over X, let $f : Y \to X$ denote the map as in Definition 2.1.11. For each $k \in \mathbb{N}_+$, the *maximal vanishing order* of E is

$$\tau_k(E) := \frac{1}{k} \max\{x \in \mathbb{N} \,|\, f^*(kL) - xE \text{ is effective}\}.$$

By convention, if kL is not effective, $\tau_k(E) = -\infty$.

Remark 2.4.4. Equivalently, we have

$$\tau_k(E) = \frac{1}{k} \sup_{D \in |kL|} \operatorname{ord}_E(D). \tag{2.15}$$

Proposition 2.4.5. *For a big line bundle $L \to X$, if kL is effective,*

$$\alpha_k = \inf_{E/X} \frac{A(E)}{\tau_k(E)}.$$

Proof. By (2.14), Theorem 2.1.14, and (2.15),

$$\begin{aligned}
\alpha_k &= \inf_{D \in |kL|} \operatorname{lct}\big(k^{-1}D\big) && (2.16)\\
&= \inf_{D \in |kL|} \inf_{E/X} \frac{kA(E)}{\operatorname{ord}_E(D)} && (2.17)\\
&= \inf_{E/X} \frac{kA(E)}{\sup_{D \in |kL|} \operatorname{ord}_E(D)} && (2.18)\\
&= \inf_{E/X} \frac{A(E)}{\tau_k(E)}. && (2.19)
\end{aligned}$$

□

Notice that $k\tau_k$ satisfies the following super-additivity property.

Proposition 2.4.6. *For any E over X, $\{k\tau_k(E)\}_{k=1}^{\infty}$ is super-additive, i.e., For any $k, k' \in \mathbb{N}_+$,*

$$k\tau_k(E) + k'\tau_{k'}(E) \leq (k + k')\tau_{k+k'}(E).$$

Proof. If kL or $k'L$ is not effective, the inequlity holds. Otherwise, we can find divisors $D \in |kL|$ and $D' \in |k'L|$ such that

$$\operatorname{ord}_E(D) = k\tau_k(E), \quad \operatorname{ord}_E(D') = k'\tau_{k'}(E).$$

Hence we have $D + D' \in |(k + k')L|$ with

$$\operatorname{ord}_E(D + D') = k\tau_k(E) + k'\tau_{k'}(E).$$

In particular,

$$(k + k')\tau_{k+k'}(E) \geq \operatorname{ord}_E(D + D') = k\tau_k(E) + k'\tau_{k'}(E).$$

□

Recall Fekete's super-additive lemma.

Lemma 2.4.7. [43] *Suppose* $\{a_k\}_{k=1}^{\infty}$ *is super-additive. Then*

$$\lim_{k\to\infty} \frac{a_k}{k} = \sup_k \frac{a_k}{k}.$$

Proof. Fix any $\ell \in \mathbb{N}_+$. For any $k \in \mathbb{N}_+$, consider the Euclidean division

$$k = q(k)\ell + r(k),$$

where $0 \le r(k) < \ell$. By super-additivity,

$$a_k \ge q(k)a_\ell + a_{r(k)}.$$

Therefore

$$\liminf_{k\to\infty} \frac{a_k}{k} \ge \lim_{k\to\infty} \frac{q(k)}{k} a_\ell + \liminf_{k\to\infty} \frac{a_{r(k)}}{k} = \frac{a_\ell}{\ell}.$$

Since ℓ is arbitrary,

$$\liminf_{k\to\infty} \frac{a_k}{k} \ge \sup_\ell \frac{a_\ell}{\ell} \ge \limsup_{\ell\to\infty} \frac{a_\ell}{\ell} \ge \liminf_{\ell\to\infty} \frac{a_\ell}{\ell}.$$

This completes the proof. □

Since $\{k\tau_k(E)\}_{k=1}^{\infty}$ is super-additive, it follows that $\tau_k(E)$ converges to its supremum.

Definition 2.4.8. Fix a big line bundle $L \to X$. For a prime divisor E over X, let $f : Y \to X$ denote the map as in Definition 2.1.11. The *pseudo-effective threshold* of E is

$$\tau(E) := \sup\{x \in \mathbb{Q}_{>0} \mid f^*(L) - xE \text{ is effective}\}, \tag{2.20}$$

and

$$\alpha := \inf_{E/X} \frac{A(E)}{\tau(E)}. \tag{2.21}$$

Proposition 2.4.9. *For any E over X,*

$$\lim_{k\to\infty} \tau_k(E) = \sup_k \tau_k(E) = \tau(E).$$

Proof. This follows from Lemma 2.4.7 and (2.20). □

Theorem 2.4.10. [29, Theorem A.3], [94, Theorem 2.2] *For a big line bundle $L \to X$,*

$$\alpha = \lim_{k\to\infty} \alpha_k = \inf_k \alpha_k.$$

Proof. Recall (2.21). By Propositions 2.4.9 and 2.4.5, if kL is effective,

$$\alpha = \inf_{E/X} \frac{A(E)}{\tau(E)} \le \inf_{E/X} \frac{A(E)}{\tau_k(E)} = \alpha_k.$$

If kL is not effective, we have $\alpha \le +\infty = \alpha_k$. Therefore

$$\alpha \le \inf_k \alpha_k. \tag{2.22}$$

For any E_0 over X, by Propositions 2.4.5 and 2.4.9,

$$\limsup_{k\to\infty} \alpha_k = \limsup_{k\to\infty} \inf_{E/X} \frac{A(E)}{\tau_k(E)} \le \lim_{k\to\infty} \frac{A(E_0)}{\tau_k(E_0)} = \frac{A(E_0)}{\tau(E_0)}.$$

Since E_0 is arbitrary, by (2.21),

$$\limsup_{k\to\infty} \alpha_k \le \inf_{E/X} \frac{A(E)}{\tau(E)} = \alpha. \tag{2.23}$$

Combining (2.22) and (2.23), we get

$$\alpha \le \inf_k \alpha_k \le \liminf_{k\to\infty} \alpha_k \le \limsup_{k\to\infty} \alpha_k \le \alpha.$$

This completes the proof. $\square$

Remark 2.4.11. Shi's proof only works on ample line bundles as metrics on the line bundle are assumed to be smooth. Demailly's proof works for any big line bundle, as singular metrics are included, but it was not explicitly stated that $\alpha = \lim_{k\to\infty} \alpha_k$.

Remark 2.4.12. Note that

$$\alpha = \inf_k \alpha_k = \inf_{kD\in|kL|} \operatorname{lct}(D) = \inf_{D\sim_{\mathbb{Q}} L} \operatorname{lct}(D). \tag{2.24}$$

2.5 δ-invariants

We continue in this section to study δ-invariants. To define them we use the notion of basis divisors.

Definition 2.5.1. [48, Definition 0.1] Let X be a projective manifold, and L a big line bundle on X. A $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} L$ is a (k-)basis divisor if there is a basis $\{s_i\}_{i=1}^{d_k}$ of $H^0(X, kL)$ such that

$$D = \frac{1}{kd_k} \sum_{i=1}^{d_k} (s_i).$$

Definition 2.5.2. [48, Definition 0.2] Let X be a projective manifold, and L a big line bundle on X. For $k \in \mathbb{N}_+$,

$$\delta_k := \inf_{\substack{D\sim_{\mathbb{Q}} L \\ k\text{-basis divisor}}} \operatorname{lct}(D),$$

and

$$\delta := \limsup_{k\to\infty} \delta_k.$$

Theorem 2.5.3. [16, Theorem A] *For a big line bundle $L \to X$,*

$$\delta = \lim_{k\to\infty} \delta_k.$$

Similar to α-invariants, we can also define δ-invariants in terms of valuations due to Blum–Jonsson.

Definition 2.5.4. Fix a big line bundle $L \to X$. For a prime divisor E over X, let $f: Y \to X$ denote the map as in Definition 2.1.11. The *expected vanishing order* of E is

$$S(E) := \frac{1}{\mathrm{Vol}(L)} \int_0^{\tau_L(E)} \mathrm{Vol}(f^*(L) - xE)dx.$$

Theorem 2.5.5. [16, Theorem C] *For a big line bundle $L \to X$,*

$$\delta = \inf_{E/X} \frac{A(E)}{S(E)}.$$

Finally, we note that the above definitions can be generalized to log pairs as follows. Recall the following definitions.

Definition 2.5.6. [11, Theorem 3.3],[67, Definition 2.24] Let $\Delta = \sum_i a_i D_i$ with $a_i \in (0,1)$. Assume the canonical divisor of the pair (X, Δ),

$$K_{X,\Delta} := K_X + \Delta$$

is $\mathbb{Q}$-Cartier.

Recall that E is a *prime divisor* over X if there is a projective birational morphism $f: Y \to X$ with normal variety Y, and E is a prime divisor on Y. Its *log discrepancy*

$$A(E) := 1 - \mathrm{ord}_E(f^*(K_{X,\Delta}) - K_Y). \tag{2.25}$$

(X, Δ) is called *klt* if $A(E) > 0$ for any E over X.

It can be verified that (2.21) and Theorem 2.5.5 generalize to pairs as well.

Theorem 2.5.7. [14, Theorem E] *For a pair (X, Δ) and a big line bundle $L \to X$,*

$$\alpha = \inf_{E/X} \frac{A(E)}{\tau(E)}, \quad \delta = \inf_{E/X} \frac{A(E)}{S(E)}.$$

Chapter 3

Toric Geometry

This chapter introduces the basics in toric geometry, as well as T-varieties, which will be crucial tools to be used in the subsequent chapters.

3.1 Notions from convexity

Consider an n-dimensional real vector space V and let V^* denote its dual with the pairing denoted by $\langle \cdot, \cdot \rangle$. Given a set $A \subseteq V$, denote by

$$A^\circ = \{y \in V^* \mid \langle x, y \rangle \leq 1, \text{ for all } x \in A\}$$

the polar of A [86, p. 125], and by

$$\operatorname{co} A := \left\{ \sum_{i=1}^{\ell} \lambda_i p_i \,\middle|\, \sum_{i=1}^{\ell} \lambda_i = 1,\ \lambda \in [0,1]^\ell, p_i \in A, \ell \in \mathbb{N}_+ \right\} \tag{3.1}$$

the convex hull of A [86, p. 12].

Set

$$-K := \{-x \mid x \in K\}.$$

Note $(-K)^\circ = -K^\circ$. Also, $K^\circ = (\operatorname{co} K)^\circ \subseteq V^*$ whenever $K \subseteq V$. The polar can also be described via the support function $h_K : V^* \to \mathbb{R}$,

$$h_K(y) := \sup_{x \in K} \langle x, y \rangle, \tag{3.2}$$

by $K^\circ = \{h_K \leq 1\}$.

A dual notion to the support function is the near-norm function

$$\|x\|_K := \inf\{t \geq 0 \,|\, x \in tK\}, \tag{3.3}$$

associated to any compact convex set K with $0 \in \operatorname{int} K$. Note that [86, Corollary 14.5]

$$\|\cdot\|_K = h_{K^\circ}. \tag{3.4}$$

When K is centrally symmetric (i.e., $K = -K$), $\|\cdot\|_K$ is a norm. Otherwise it is only a near-norm in the sense that it satisfies all the properties of a norm but is only $\mathbb{R}_+$-homogeneous: $\|\lambda x\|_K = \lambda \|x\|_K$ for $\lambda \in \mathbb{R}_+$ (and not fully $\mathbb{R}$-homogeneous). The infimum in (3.3) is achieved: for a minimizing sequence $\{t_i\}$, $\frac{x}{t_i} \in K$ for any i, so

$$x \in \|x\|_K K \tag{3.5}$$

since K is closed.

Example 3.1.1. Let $V = \mathbb{R}^2$ and $A = \{(1,1), (1,-1), (-1,1), (-1,-1)\}$. Then $K := \operatorname{co} A$ is the unit ∞-ball $\{\|x\|_\infty \leq 1\}$. The polar body

$$A^\circ = \left\{y \in \mathbb{R}^2 \,\middle|\, -1 \leq y_1 + y_2 \leq 1, -1 \leq y_1 - y_2 \leq 1\right\} = \{\|y\|_1 \leq 1\}$$

is the unit 1-ball. See Figure 3.1.

By (3.3),

$$\|x\|_K = \inf\{t \geq 0 \,|\, \|x\|_\infty \leq t\} = \|x\|_\infty.$$

Similarly,

$$\|\cdot\|_{A^\circ} = \|\cdot\|_1.$$

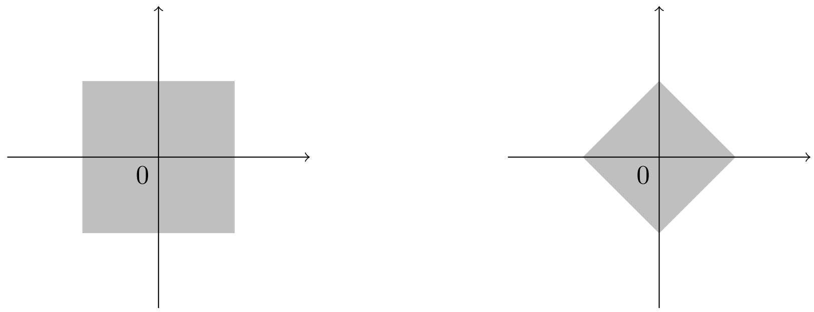

Figure 3.1: The unit ∞-ball and its polar body.

Lemma 3.1.2. *Let V be an $\mathbb{R}$-vector space. Consider the polytope*

$$A = \bigcap_{j=1}^{d} \{x \in V \mid \langle x, v_j \rangle \leq 1\},$$

where $v_j \in V^*$. Then,

$$\|x\|_A = \max_{1 \leq j \leq d} \langle x, v_j \rangle.$$

Proof. Notice that $x \in tA$ if and only if for any $1 \leq j \leq d$, $\langle x, v_j \rangle \leq t$. Thus,

$$\|x\|_A = \inf \left\{ t \geq 0 \,\middle|\, \max_{1 \leq j \leq d} \langle x, v_j \rangle \leq t \right\} = \max_{1 \leq j \leq d} \langle x, v_j \rangle.$$

Alternatively, observe that $A = \{v_1, \ldots, v_d\}^\circ$ and $A^\circ = (\{v_1, \ldots, v_d\}^\circ)^\circ = \mathrm{co}\{v_1, \ldots, v_d\}$ [86, Theorem 14.5] and then use (6.34) and (3.4). $\square$

3.2 Toric algebra

We start from the standard n-dimensional torus $(\mathbb{C}^*)^n$. The space of 1-parameter subgroups $\lambda : \mathbb{C}^* \to (\mathbb{C}^*)^n$ forms a free abelian group of rank n. Indeed, it is always in the form of $\lambda_v : t \mapsto (t^{v_1}, \ldots, t^{v_n})$, for $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ [49, §2.3]. Let N denote this group. Its dual $M := \mathrm{Hom}(N, \mathbb{Z})$ is also a free abelian group of rank n, isomorphic to the space of characters $\chi : (\mathbb{C}^*)^n \to \mathbb{C}^*$. Explicitly, it is in the form of $\chi^u : (t_1, \ldots, t_n) \mapsto t_1^{u_1} \cdots t_n^{u_n}$, for $u = (u_1, \ldots, u_n) \in \mathbb{Z}^n$. Let $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ denote the corresponding

$\mathbb{R}$-vector space and its dual

$$N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}, \quad M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} = N_{\mathbb{R}}^*. \tag{3.6}$$

A rational convex polyhedral cone in $N_{\mathbb{R}}$ takes the form

$$\sigma = \sigma(v_1, \ldots, v_d) := \left\{ \sum_{i=1}^{d} a_i v_i \,\middle|\, a_i \geq 0, v_i \in N \right\}. \tag{3.7}$$

The rays $\mathbb{R}_+ v_i$, $i = 1, \ldots, d$ are called the generators of the cone [49, p. 9]. They are (1-dimensional) cones themselves, of course. Our convention will be that

$$v_1, \ldots, v_d \text{ are primitive elements of the lattice } N, \tag{3.8}$$

which means there is no $m \in \mathbb{N}_+ \setminus \{1\}$ such that $v_i \in mN$. A cone is called *strongly convex* if $\sigma \cap -\sigma = \{0\}$ [49, p. 14]. A face of σ is any intersection of σ with a supporting hyperplane.

For any rational convex polyhedral cone in $N_{\mathbb{R}}$ as in (3.7), its dual cone in $M_{\mathbb{R}}$ is given by

$$\begin{aligned} \sigma^\vee &:= \{u \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq 0, \text{ for all } v \in N_{\mathbb{R}}\} \\ &= \bigcap_{i=1}^{d} \{u \in M_{\mathbb{R}} \mid \langle u, v_i \rangle \geq 0\}. \end{aligned}$$

Example 3.2.1. Consider $\mathbb{C}^n$. It has a natural toric structure. Recall that each lattice point $u \in M$ gives rise to a character χ^u, which is a function on $\mathbb{C}^*$ given by $\chi^u(t) = t^u$. Such function can be extended to an entire function on $\mathbb{C}^n$ if and only if $u_i \geq 0$ for all i, i.e., u lies in the first orthant. We associate this cone (and its dual cone in $N_{\mathbb{R}}$) to this toric variety. See Figure 3.2.

Example 3.2.2. Notice that $\mathbb{C}^* \times \mathbb{C}^{n-1} \subseteq \mathbb{C}^n$ also has a toric structure inherited from $\mathbb{C}^n$. A character χ^u extends to an entire function on $\mathbb{C}^* \times \mathbb{C}^{n-1}$ if and only if $u_i \geq 0$ for $i \geq 2$, i.e., u lies in the cone

$$\sigma := \bigcap_{i=2}^{n} \{u \in M_{\mathbb{R}} \mid \langle u, e_i \rangle \geq 0\},$$

where $e_1, \ldots, e_n$ is the standard basis of N. See Figure 3.3. The dual cone $\sigma^\vee$ is generated by $e_2, \ldots, e_n$.

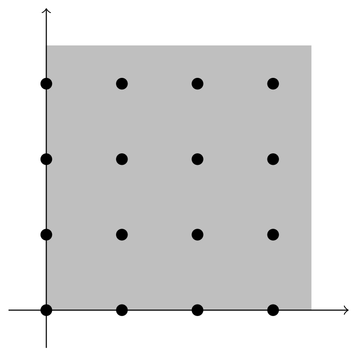

Figure 3.2: The cone σ in $M_{\mathbb{R}}$ associated to $\mathbb{C}^n$. The lattice points in σ are the characters that extend to an entire function on $\mathbb{C}^n$.

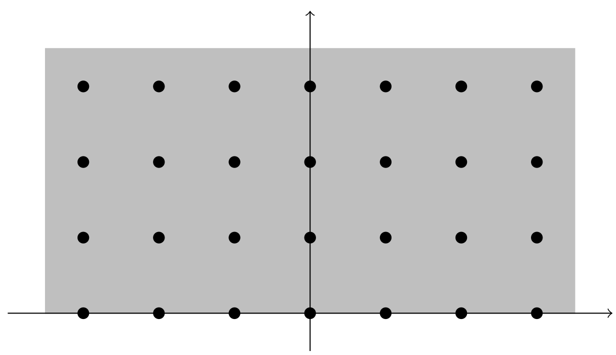

Figure 3.3: The cone σ in $M_{\mathbb{R}}$ associated to $\mathbb{C}^* \times \mathbb{C}^{n-1}$. The lattice points in σ are the characters that extend to an entire function on $\mathbb{C}^* \times \mathbb{C}^{n-1}$.

We associate this cone to $\mathbb{C}^* \times \mathbb{C}^{n-1}$. See Figure 3.4.

In general, each face of the first orthant is associated to a toric subvariety, which is the product of copies of $\mathbb{C}$ and $\mathbb{C}^*$. Specifically, the i-th factor is $\mathbb{C}$ if and only if e_i is a generator of the face.

Definition 3.2.3. A *fan* $\Delta = \{\sigma_i\}_{i=1}^{\delta}$ in N is a finite set of rational strongly convex polyhedral cones σ_i in $N_{\mathbb{R}}$ such that:

(i) each face of a cone in Δ is also (a cone) in Δ;

(ii) the intersection of two cones in Δ is a face of each.

Such a fan gives rise to a toric variety $X(\Delta)$: each cone σ_i in Δ gives rise to an affine toric variety [49, §1.3], that serves as a (Zariski open) chart in $X(\Delta)$ with the transition between the charts constructed by (i) and (ii) above [49, p. 21]. For instance, the zero cone corresponds to the open dense orbit $(\mathbb{C}^*)^n$ [23, p. 64], and more generally there is a bijection between the cones $\{\sigma_i\}_{i=1}^{\delta}$ and the orbits of the complex torus $(\mathbb{C}^*)^n$ in $X(\Delta)$ [23, Proposition 5.6.2], which is further in bijection with irreducible closed

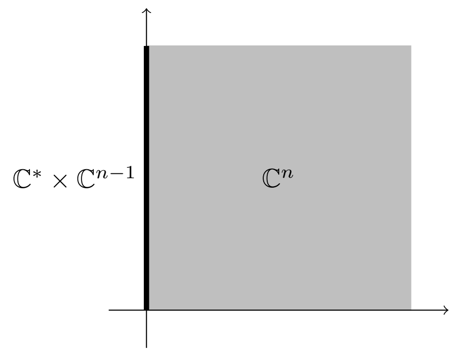

Figure 3.4: The cone in $N_{\mathbb{R}}$ associated to $\mathbb{C}^* \times \mathbb{C}^{n-1}$ is generated by $e_2, \ldots, e_n$. It is a face of the first orthant, which is the cone associated to $\mathbb{C}^n$.

toric subvarieties (via taking the closure of the orbit, and the dense orbit of the subvariety), with the non-zero cones corresponding precisely to all the irreducible closed toric subvarieties of $X(\Delta)$ of positive codimension. Indeed, the dimension of the cone coincides with the codimension of the orbit and the subvariety. In particular, the 1-dimensional cones correspond to the torus invariant prime (Weil) divisors.

The following examples explain intuitively the correspondence between toric manifolds and fans.

Example 3.2.4. Consider $\mathbb{P}^n$. It has a natural toric structure. We can cover $\mathbb{P}^n$ using the standard coordinate charts $U_i = \{Z_i \neq 0\}$, $i = 0, \ldots, n$.

Let the torus $(\mathbb{C}^*)^n \overset{\iota}{\hookrightarrow} \mathbb{P}^n$ be given by $\iota(t_1, \ldots, t_n) = [1 : t_1 : \cdots, t_n]$. On the chart U_0 with coordinates $z_{0,1}, \ldots, z_{0,n}$,

$$\chi^u(t) = t^u = z_{0,1}^{u_1} \cdots z_{0,n}^{u_n},$$

which extends to an entire function on U_0 when u lies in the first orthant, which is the dual of the cone

$$\sigma_0 := \sum_{i=1}^{n} \mathbb{R}_+ e_i \subseteq N_{\mathbb{R}}.$$

On the chart U_i, $i = 1, \ldots, n$, the coordinate $(z_{i,0}, \ldots, \widehat{z_{i,i}}, \ldots, z_{i,n})$ of a point $t = [1 : t_1 : \cdots : t_n]$ on $(\mathbb{C}^*)^n$ is given by

$$\begin{cases} z_{i,0} = \frac{1}{t_i}, \\ z_{i,j} = \frac{t_j}{t_i}, \quad j \neq i, \end{cases}$$

i.e.,

$$\begin{cases} t_i = \frac{1}{z_{i,0}}, \\ t_j = \frac{z_{i,j}}{z_{i,0}}, \quad j \neq i. \end{cases}$$

Therefore

$$\chi^u(t) = t_i^{u_i} \cdot \prod_{j \neq i} t_j^{u_j} = {z_{i,0}}^{-u_1 - \cdots - u_n} \prod_{j \neq i} {z_{i,j}}^{u_j}.$$

This function extends to an entire function on $U_i \cong \mathbb{C}^n$ when $-u_1 - \cdots - u_n \geq 0$ and $u_j \geq 0$ for all $j \neq i$, i.e., $\langle u, v_j \rangle \geq 0$ for all $j = 0, \ldots, \hat{i}, \ldots, n$, where $v_0 = -e_1 - \cdots - e_n$, and $v_i = e_i$ for $i = 1, \ldots, n$. Equivalently, u lies in the dual of the cone

$$\sigma_i := \sum_{j \neq i} \mathbb{R}_+ v_j.$$

These cones are precisely the n-dimensional cones in the fan associated to $\mathbb{P}^n$. See Figure 3.5. Under the correspondence between rays and prime toric divisors, the ray generated by v_i corresponds to the toric divisor $V(Z_i)$.

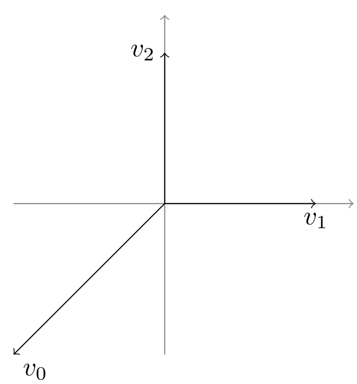

Figure 3.5: The fan corresponding to $\mathbb{P}^2$. The ray generated by v_i corresponds to the toric divisor $V(Z_i)$.

Example 3.2.5. Consider $\mathbb{F}_n$. It is also equipped with a toric structure. Indeed, we can obtain $\mathbb{F}_n$ by gluing two copies of $\mathbb{P}^1 \times \mathbb{C}$ as follows. Let each of the two copies of $\mathbb{P}^1 \times \mathbb{C}$ be covered by two charts $U_{i0} = \{(z_i, w_{i0})\}$ and $U_{i1} = \{(z_1, w_{i1})\}$, $i = 0, 1$, with transition functions

$$w_{i0} w_{i1} = 1. \tag{3.9}$$

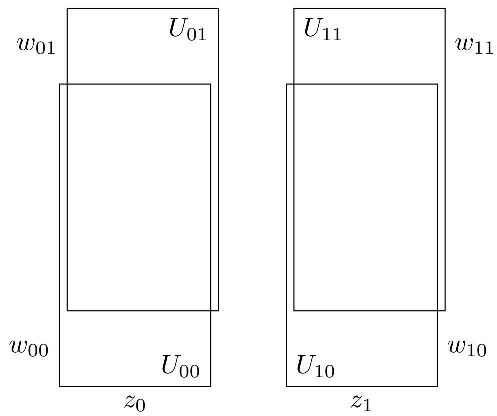

Figure 3.6: $\mathbb{F}_n$ is constructed by gluing two copies of $\mathbb{P}^1 \times \mathbb{C}$. In total there are four charts.

Glue them along $\{z_i \neq 0\}$ by

$$z_0 z_1 = 1, \tag{3.10}$$

and

$$w_{00} = z_1^n \cdot w_{10}.$$

See Figure 3.6.

On the chart U_{00}, let the torus $(\mathbb{C}^*)^2 \overset{\iota}{\hookrightarrow} \mathbb{F}_n$ be given by

$$(z_0, w_{00}) = \iota(t_1, t_2) = (t_1, t_2).$$

Then

$$\chi^u(t) = t^u = z_0^{u_1} w_{00}^{u_2}, \tag{3.11}$$

which extends to an entire function on U_0 when $u_1, u_2 \geq 0$, i.e., u lies in the dual of the cone generated by $(1,0)$ and $(0,1)$.

On the chart U_{01}, rewriting (3.11) using the transition function (3.9), we get

$$\chi^u(t) = z_0^{u_1} w_{00}^{u_2} = z_0^{u_1} w_{01}^{-u_2},$$

which extends to an entire function on U_{01} when $u_1, -u_2 \geq 0$, i.e., u lies in the dual of the cone generated by $(1,0)$ and $(0,-1)$.

On the chart U_{10}, rewriting (3.11) using the transition functions (3.10) and (3.9), we get

$$\chi^u(t) = z_0^{u_1} w_{00}^{u_2} = z_1^{-u_1}(z_1^n w_{10})^{u_2} = z_1^{-u_1+nu_2} w_{10}^{u_2}, \tag{3.12}$$

which extends to an entire function on U_{10} when $-u_1 + nu_2, u_2 \geq 0$, i.e., u lies in the dual of the cone generated by $(-1,n)$ and $(0,1)$.

On the chart U_{11}, rewriting (3.12) using the transition functions (3.9), we get

$$\chi^u(t) = z_1^{-u_1+nu_2} w_{10}^{u_2} = z_1^{-u_1+nu_2} w_{11}^{-u_2},$$

which extends to an entire function on U_{11} when $-u_1 + nu_2, -u_2 \geq 0$, i.e., u lies in the dual of the cone generated by $(-1,n)$ and $(0,-1)$.

These cones are precisely the n-dimensional cones in the fan associated to $\mathbb{F}_n$. See Figure 3.7. Under the correspondence between rays and prime toric divisors, the ray generated by v_1, v_2, v_3, and v_4 corresponds to the $-n$-curve, the toric n-curve, and the two toric fibers of self-intersection 0, respectively.

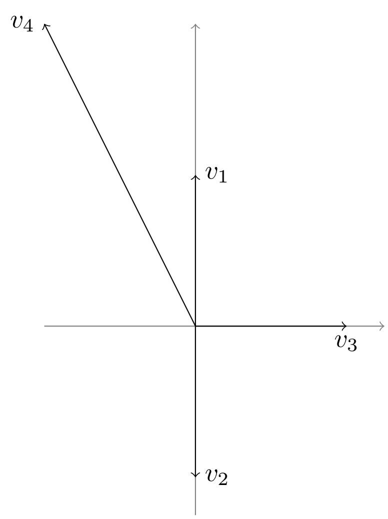

Figure 3.7: The fan corresponding to $\mathbb{F}_n$. The rays generated by $v_1, \ldots, v_4$ correspond to the $-n$-curve, the toric n-curve, and the two toric fibers, respectively.

We can obtain new toric manifold by blowing up torus-invariant submanifolds. In particular, we explain the blowing up of points.

Proposition 3.2.6. [33, Proposition 3.3.15] *Let $U \cong \mathbb{C}^n$ be a chart corresponding to a cone spanned by a $\mathbb{Z}$-basis $v_1, \ldots, v_n$ of N. Let $v_0 = \sum_{i=1}^n v_i$. Then the fan of $\mathrm{Bl}_0 U$ consists of all cones that are generated by any subset of $\{v_0, \ldots, v_n\}$ that does not contain $\{v_1, \ldots, v_n\}$.*

Remark 3.2.7. In particular, to blow up a point in a toric surface that corresponds to a cone generated by v_1 and v_2, we replace this cone by a new 1-dimensional cone generated by $v_0 := v_1 + v_2$ and two 2-dimensional cones generated by v_0, v_1, and v_0, v_2, respectively.

Example 3.2.8. Recall the fan of $\mathbb{P}^2$ (See Figure 3.8). Blowing up a point that corresponds to, say, the cone generated by v_1 and v_2, we obtain the fan for dP_8. Similarly, blowing up the two points that correspond to the other two 2-dimensional cones gives the fan for dP_7, and blowing up the three points that correspond to all three 2-dimensional cones gives the fan for dP_6.

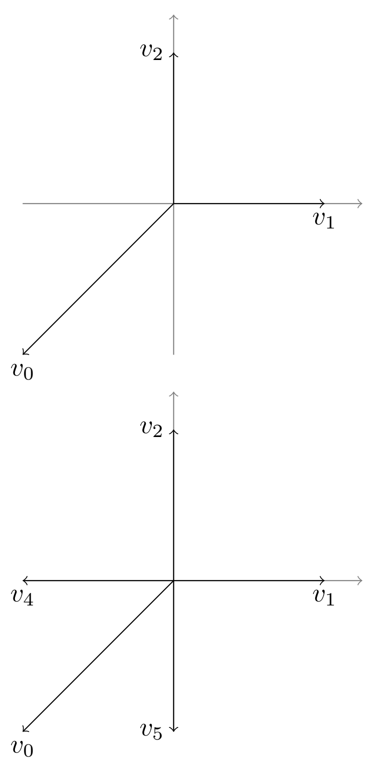

Figure 3.8: The fans corresponding to $\mathbb{P}^2$ and its blow up at up to 3 points.

When $X(\Delta)$ is a smooth toric Fano variety, the fan Δ must arise from an integral polytope as follows. Let Δ be a fan such that $X(\Delta)$ is smooth Fano and let $\sigma_1, \ldots, \sigma_d$ be its 1-dimensional cones (i.e., rays) generated by primitive generators

$$\Delta_1 := \{v_1, \ldots, v_d\} \subseteq N, \tag{3.13}$$

so $\sigma_i = \mathbb{R}_+ v_i$, and set (recall (3.1))

$$Q := \operatorname{co} \Delta_1 = \operatorname{co}\{v_1, \ldots, v_d\} = \operatorname{co} \Delta_1 \subseteq N_{\mathbb{R}}. \tag{3.14}$$

Then Δ is equal to the collection of cones over each face of Q plus the zero cone [49, p. 25], in other words if $F \subseteq Q$ is a face, then

$$\sigma_F := \{rx \in N_{\mathbb{R}} \,|\, r \geq 0, x \in F\} \tag{3.15}$$

is the union of all rays through F and the origin, and

$$\Delta = \{\sigma_F\}_{F \subseteq Q} \cup \{0\},$$

where 0 denotes the zero cone. Denote by

$$\operatorname{Ver} A \tag{3.16}$$

the vertices of a polytope A. Note that $\operatorname{Ver} F \subseteq \operatorname{Ver} Q = \Delta_1$, and by (3.7)–(3.8),

$$\sigma_F = \sigma(\operatorname{Ver} F). \tag{3.17}$$

Oftentimes we will work with

$$P := -Q^\circ = \{-v_1, \ldots, -v_d\}^\circ = -\{v_1, \ldots, v_d\}^\circ \subseteq M_{\mathbb{R}}, \tag{3.18}$$

as it has the nice geometric property of faces of (real) dimension k corresponding to toric subvarieties of dimension k, and since the metric properties (e.g., volume) of P correspond to those of X. (Moreover, P can also be realized as the Delzant (moment) polytope associated to any $(S^1)^n$-invariant Kähler metric representing the anticanonical class.) In particular,

$$P = \bigcap_{i=1}^{d} \{u \in M_{\mathbb{R}} \,|\, \langle u, -v_i \rangle \le 1\} = \{h_{-\Delta_1} \le 1\} = \left\{ u \in M_{\mathbb{R}} \,\middle|\, \min_j \langle u, v_j \rangle \ge -1 \right\}, \tag{3.19}$$

and the prime toric divisors correspond to facets of P

$$\{u \in P \,|\, \langle u, v_i \rangle = -1\}. \tag{3.20}$$

Note that P contains the origin in its interior. Also note that (3.18) is the standard convention since then lattice points of P correspond to monomials via (6.11). Batyrev–Selivanova use $-P$ instead.

Example 3.2.9. Recall that $X = \mathbb{P}^n$ has a natural toric structure. The corresponding polytope P is the simplex $(n+1)\triangle - (1, \dots, 1)$, where $\triangle$ is the standard simplex. See Figure 3.9.

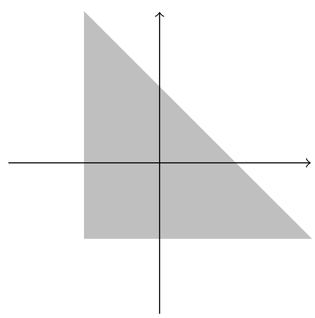

Figure 3.9: The polytope corresponding to $\mathbb{P}^2$.

Example 3.2.10. Combining Examples 3.2.4, 3.2.5, and 3.2.8, we obtain the polytopes for all toric del Pezzo surfaces. See Figure.

Smoothness of X means that the generators of σ_F form a $\mathbb{Z}$-basis for N [49, p. 29]. By (3.17) this means

$$\text{the vertices of } F \text{ form a } \mathbb{Z}\text{-basis for } N \text{ (for any facet } F \subseteq Q). \tag{3.21}$$

Thus each facet F of Q is an $(n-1)$-simplex whose vertices form a $\mathbb{Z}$-basis of N. This means the vertices of the polar polytope $P := -Q^\circ$ belong to the dual lattice M.

Lemma 3.2.11. *Let $P \subseteq M_{\mathbb{R}}$ be the polytope associated to a smooth toric variety X. Then P is an integral*

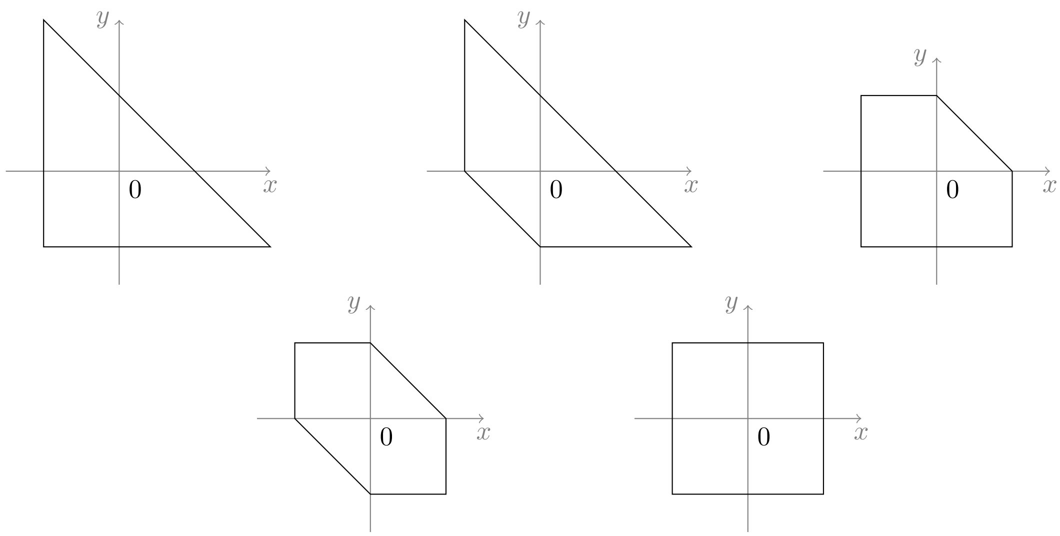

Figure 3.10: The polytope P for del Pezzo surfaces, namely $\mathbb{P}^2$ blown up at no more than 3 generically positioned points, and $\mathbb{P}^1 \times \mathbb{P}^1$.

lattice polytope, i.e., $\operatorname{Ver} P \subseteq M$.

Proof. By duality, if $u \in M_{\mathbb{R}}$ is a vertex of P, then

$$F = \{v \in Q \mid \langle u, v\rangle = -1\} \subseteq N_{\mathbb{R}} \tag{3.22}$$

is a facet of $Q = -P^\circ \subset N_{\mathbb{R}}$. Since X is smooth, the vertices of F form a $\mathbb{Z}$-basis for N by (3.21). Choose coordinates on N associated to this $\mathbb{Z}$-basis, i.e., the vertices of F are the standard basis vectors $e_1, \ldots, e_n$. Then the facet $F = \operatorname{co}\{e_1, \ldots, e_n\}$ is the standard $(n-1)$-simplex in $\mathbb{R}^n$ cut out by the equation $\langle u, v\rangle = -1$ where $u = (-1, \ldots, -1) \in M$, as desired.

Alternatively, if one does not wish to choose coordinates but rather work invariantly, denote by $\operatorname{Ver} F = \{f_1, \ldots, f_n\} \subseteq N$, and note any $v \in N$ can be written uniquely as $v = \sum_{i=1}^n a_i f_i$ with $a_i \in \mathbb{Z}$. Thus $u \in M_{\mathbb{R}} = \operatorname{Hom}(N, \mathbb{R})$ can be identified with the map

$$N \ni v \mapsto -\sum_{i=1}^n a_i,$$

As $\sum_{i=1}^n a_i \in \mathbb{Z}$, this map actually belongs to $\operatorname{Hom}(N, \mathbb{Z}) = M$. $\square$

In fact, the associated polytope P is a reflexive Delzant lattice polytope [22, 36].

Definition 3.2.12. A convex lattice polytope P in $\mathbb{R}^n$ is *Delzant* if

(i) there are exactly n edges meeting at each vertex p (i.e., P is *simple*);

(ii) the edges meeting at a vertex p satisfy that each edge is of the form $p+tu_i$ with $u_i \in \mathbb{Z}^n$, $i = 1, \ldots, n$;

(iii) the $u_1, \ldots, u_n$ in (ii) can be chosen to be a basis of $\mathbb{Z}^n$.

It is *reflexive* if P° is a lattice polytope.

Such polytopes are an important sub-class of lattice polytopes since there is a one-to-one correspondence between Delzant lattice polytopes and polarized toric manifolds. We collect in one statement that correspondence and in fact a more general one including the singular and reflexive cases.

Theorem 3.2.13. [22, Theorem 28.2], [36], [49, §1.5], [6, Proposition 2.2.23] *There is a one-to-one correspondence between lattice polytopes and polarized toric varieties. Specifically, let*

$$P = \bigcap_{i=1}^{d} \{u \in M_{\mathbb{R}} \mid \langle u, v_i \rangle \geq -b_i\} \tag{3.23}$$

be a lattice polytope, where each $v_i \in N := \mathrm{Hom}(M, \mathbb{Z})$ is chosen to be primitive, i.e., not a multiple of another lattice point, and $b_i \in \mathbb{Z}$. It determines a fan where $\{v_i\}_{i=1}^d$ is the set of primitive generators of the one-dimensional cones. This fan then determines a toric variety X, on which the prime toric divisors $\{D_i\}_{i=1}^d$ are in one-to-one correspondence with $\{v_i\}_{i=1}^d$. The divisor

$$D := \sum_{i=1}^{d} b_i D_i$$

is ample and Cartier. Let L be the corresponding line bundle. Then the polarized variety (X, L) is the one in correspondence with P.

The variety X is smooth if and only if P is Delzant (Definition 3.2.12).

The variety X is Gorenstein Fano (i.e., $-K_X$ is Cartier and ample) with $L = -K_X$ if and only if P is reflexive (Definition 3.2.12).

When $L = -K_X$, there is an $\mathrm{Aut}\, X$ action on $H^0(X, -kK_X)$ for every $k \in \mathbb{N}$. To get an induced linear action on $M_{\mathbb{Q}}$ we must restrict to the normalizer $N((\mathbb{C}^*)^n)$ of the complex torus $(\mathbb{C}^*)^n$ in $\mathrm{Aut}\, X$.

The representation of $(\mathbb{C}^*)^n$ on $H^0(X, -kK_X)$ splits into 1-dimensional spaces, whose generators are called the monomial basis. There is a one-to-one correspondence between the monomial basis of $H^0(X, -kK_X)$ and points in $kP \cap M$, and the quotient $N((\mathbb{C}^*)^n)/(\mathbb{C}^*)^n$ is a linear group, that can be identified with $\operatorname{Aut} P \subseteq GL(M) \cong GL(n, \mathbb{Z})$ (6.4). Since P is defined as the convex hull of vertices in M it follows that $\operatorname{Aut} P$ is finite. Alternatively, this can be seen by observing that $N_{\mathbb{R}}$ is canonically isomorphic to the quotient of $(\mathbb{C}^*)^n$ by its maximal compact subgroup $(S^1)^n$ [7, p. 229] and the induced action on $M_{\mathbb{R}}$ is then defined by transposing via the pairing. Conversely, all compact subgroups of $N((\mathbb{C}^*)^n)$ that contain $(S^1)^n$ are generated by $(S^1)^n$ and a finite subgroup H of $\operatorname{Aut} P$ [7, Proposition 3.1], and we denote such a group by $G(H) \subseteq \operatorname{Aut} X$ as in (6.5). We describe in the proof of Lemma 6.1.1 concretely how the action of $\operatorname{Aut} P$ is expressed in coordinates. For a finite group H or finite set $\mathcal{A}$ we denote by

$$|H|, \text{ respectively } |\mathcal{A}|,$$

its order or cardinality.

Another useful fact is a sort of maximum principle for convex polytopes, saying essentially that a convex function on a polytope achieves its maximum at some vertex (regardless of continuity).

Lemma 3.2.14. *Let A be a convex polytope and $f : A \to \mathbb{R} \cup \{\infty\}$ a convex function. Then:*

(i) $\sup_A f = \sup_{\operatorname{Ver} A} f$*;*

(ii) if f is bounded on $\operatorname{Ver} A$*, it is bounded on A and its maximum is achieved in a vertex;*

(iii) if f attains its finite maximum on $\operatorname{int} A$ *it is constant;*

(iv) if f attains its finite maximum on the relative interior of a face $F \subseteq A$ it is constant on F.

Proof. Write $\operatorname{Ver} A = \{p_1, \ldots, p_\ell\}$ and assume $f(p_1) \leq \cdots \leq f(p_\ell)$. By convexity, $A = \operatorname{co} \operatorname{Ver} A$, and (3.1) implies that any $x \in A$ can be expressed as

$$x = \sum_{i=1}^{\ell} \lambda_i p_i, \quad \sum_{i=1}^{\ell} \lambda_i = 1, \quad \lambda \in [0,1]^\ell. \tag{3.24}$$

Then

$$f(x) \leq \sum_{i=1}^{\ell} \lambda_i f(p_i) \leq \sum_{i=1}^{\ell} \lambda_i f(p_\ell) = f(p_\ell),$$

proving (i) and (ii).

To see (iii), again express any $x \in A$ using (3.24). Suppose that $x_{\text{int}} \in \operatorname{int} A$ achieves the (finite) maximum of f. Since $x_{\text{int}} \in \operatorname{int} A$, there is $\delta = \delta(x) \in (0,1)$ such that

$$x' := \frac{1}{1-\delta}(x_{\text{int}} - \delta x) \in A.$$

Notice that $\delta x + (1-\delta)x' = x_{\text{int}}$. Thus,

$$\max_A f = f(x_{\text{int}}) \leq \delta f(x) + (1-\delta) f(x') \leq \max_A f,$$

forcing equality, i.e., $f(x) = \max_A f$ (since $\delta > 0$ and $\max_A f < \infty$), proving (iii). The proof of (iv) is identical by working on the polytope F. $\square$

3.3 Ehrhart theory

In this section we set up some basic terminology from Ehrhart theory.

A real Euclidean space V of dimension n and an $\mathbb{R}$-basis $\{v_1, \ldots, v_n\}$ of V define a lattice

$$\mathbb{Z}^n \cong M \equiv M(v_1, \ldots, v_n) := \left\{ \sum_{i=1}^{n} a_i v_i \,\middle|\, a = (a_1, \ldots, a_n) \in \mathbb{Z}^n \right\} \subseteq V \cong \mathbb{R}^n,$$

and $\{v_1, \ldots, v_n\}$ is called a basis of M, and $M \otimes_{\mathbb{Z}} \mathbb{R} = V \cong \mathbb{R}^n$. Any other basis of M differs by an element of $GL(M)$, i.e., an $n \times n$ matrix with integer entries and determinant ± 1 [57, Theorem 21.1]. The determinant of M is the Euclidean volume of the unit (or fundamental) parallelotope $\{\sum_{i=1}^n a_i v_i \,|\, a \in [0,1)^n\} \subseteq M \otimes_{\mathbb{Z}} \mathbb{R}$, i.e.,

$$\det M = |\det[v_1 \ldots v_n]|,$$

independent of the choice of the basis.

A convex lattice polytope $P \subseteq M_{\mathbb{R}}$ is the convex hull of a finite collection of lattice points, assumed to be n-dimensional.

The *k-th quantized barycenter* (or *k-th discrete barycenter*) of P is defined as the average of the lattice points in kP divided by k:

$$\mathrm{Bc}_k(P) := \frac{1}{k|kP \cap M|} \sum_{u \in kP \cap M} u, \quad k \in \mathbb{N}.$$

The barycenter of P is the first moment of the uniform probability measure on P, considered as a convex body in $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$, or

$$\mathrm{Bc}(P) := \frac{1}{|P|} \int_P x dx \in M_{\mathbb{R}}.$$

The empirical measures $\frac{1}{|kP \cap M|} \sum_{u \in P \cap k^{-1}M} \delta_u$ (where δ_u is the Dirac measure at u) converge weakly to $\mathbb{1}_P dx/|P|$ so

$$\mathrm{Bc}(P) = \lim_{k \to \infty} \mathrm{Bc}_k(P) \in M_{\mathbb{R}}.$$

Definition 3.3.1. Let L be a n-dimensional lattice and L_H an affine sublattice obtained by intersecting L with a hyperplane H.

Let Vol denote the standard Lebesgue $(n-1)$-dimensional measure on H. Define the *lattice-normalized measure on H* as the measure with respect to the affine sublattice L_H, i.e.,

$$\widetilde{\mathrm{Vol}} := \frac{1}{\mathrm{Vol}(C)} \mathrm{Vol},$$

where C is a fundamental parallelotope of the affine sublattice L_H. That is, C is the parallelotope spaned by a basis of L_H. Note that while fundamental parallelotopes are not unique, their volumes are the same [57, p. 358].

Let P be a lattice polytope. The *lattice-normalized measure on ∂P* is defined to be the measure whose restriction to any facet of P is the normalized volume on that facet. Let $\widetilde{\mathrm{Bc}}(\partial P)$ denote the barycenter of

∂P with respect to this measure, i.e.,

$$\widetilde{\mathrm{Bc}}(\partial P) = \frac{1}{\widetilde{\mathrm{Vol}}(\partial P)} \int_{\partial P} u \cdot d\,\widetilde{\mathrm{Vol}}$$

Example 3.3.2. Let P be the associated polytope of $\mathbb{P}^2$. Specifically, the three edges of ∂P are $F_1 := \{(-1, y) \in \mathbb{R}^2 \,|\, -1 \leq y \leq 2\}$, $F_2 := \{(x, -1) \in \mathbb{R}^2 \,|\, -1 \leq x \leq 2\}$, and $F_3 := \{(x, y) \in \mathbb{R}^2 \,|\, x + y - 1 = 0,\ -1 \leq x \leq 2\}$. Let C_1, C_2, and C_3 denote the standard cubes of the induced affine sublattices on F_1, F_2, and F_3, respectively. Precisely, we may choose $C_1 := \{(-1, y) \in \mathbb{R}^2 \,|\, -1 \leq y \leq 0\}$, $C_2 := \{(x, -1) \in \mathbb{R}^2 \,|\, -1 \leq x \leq 0\}$ and $C_3 := \{(x, y) \in \mathbb{R}^2 \,|\, x + y - 1 = 0,\ -1 \leq x \leq 0\}$. Then $\mathrm{Vol}(C_1) = \mathrm{Vol}(C_2) = 1$ and $\mathrm{Vol}(C_3) = \sqrt{2}$. Thus one has

$$\begin{aligned}
\widetilde{\mathrm{Vol}}(\partial P) &= \widetilde{\mathrm{Vol}}(F_1) + \widetilde{\mathrm{Vol}}(F_2) + \widetilde{\mathrm{Vol}}(F_3) \\
&= \frac{\mathrm{Vol}(F_1)}{\mathrm{Vol}(C_1)} + \frac{\mathrm{Vol}(F_2)}{\mathrm{Vol}(C_2)} + \frac{\mathrm{Vol}(F_3)}{\mathrm{Vol}(C_3)} \\
&= 3 + 3 + \frac{3\sqrt{2}}{\sqrt{2}} \\
&= 9, \\
\widetilde{\mathrm{Bc}}(\partial P) &= \frac{1}{\widetilde{\mathrm{Vol}}(\partial P)} \left(\int_{F_1} (x, y) d\,\widetilde{\mathrm{Vol}} + \int_{F_2} (x, y) d\,\widetilde{\mathrm{Vol}} + \int_{F_3} (x, y) d\,\widetilde{\mathrm{Vol}} \right) \\
&= \frac{1}{9} \left(\left(-3, \frac{3}{2} \right) + \left(\frac{3}{2}, -3 \right) + \left(\frac{3}{2}, \frac{3}{2} \right) \right) \\
&= 0.
\end{aligned}$$

Example 3.3.3. Let P' be the associated polytope of blow-up of $\mathbb{P}^2$ at one point. Specifically, the four edges of $\partial P'$ by $F_1' := \{(-1, y) \in \mathbb{R}^2 \,|\, 0 \leq y \leq 2\}$, $F_2' := \{(x, -1) \in \mathbb{R}^2 \,|\, 0 \leq x \leq 2\}$, $F_3' := \{(x, y) \in \mathbb{R}^2 \,|\, x + y = 1,\ -1 \leq x \leq 2\}$, and $F_4' := \{(x, y) \in \mathbb{R}^2 \,|\, x + y = -1,\ -1 \leq x \leq 0\}$. Let C_1', C_2', C_3', and C_4' denote the standard cube of the induced lattice on F_1', F_2', F_3', and F_4', respectively. Then $\mathrm{Vol}(C_1') = \mathrm{Vol}(C_2') = 1$ and $\mathrm{Vol}(C_3) = \mathrm{Vol}(C_4') = \sqrt{2}$. Thus one has

$$\widetilde{\mathrm{Vol}}(\partial P') = \widetilde{\mathrm{Vol}}(F_1') + \widetilde{\mathrm{Vol}}(F_2') + \widetilde{\mathrm{Vol}}(F_3') + \widetilde{\mathrm{Vol}}(F_4')$$

$$
\begin{aligned}
&= \frac{\mathrm{Vol}(F_1')}{\mathrm{Vol}(C_1')} + \frac{\mathrm{Vol}(F_2')}{\mathrm{Vol}(C_2')} + \frac{\mathrm{Vol}(F_3')}{\mathrm{Vol}(C_3')} + \frac{\mathrm{Vol}(F_4')}{\mathrm{Vol}(C_4')} \\
&= 2 + 2 + \frac{3\sqrt{2}}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{2}} \\
&= 8,
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\mathrm{Bc}}(\partial P') &= \frac{1}{\widetilde{\mathrm{Vol}}(\partial P')}\left(\int_{F_1'}(x,y)d\widetilde{\mathrm{Vol}} + \int_{F_2'}(x,y)d\widetilde{\mathrm{Vol}} + \int_{F_3'}(x,y)d\widetilde{\mathrm{Vol}} + \int_{F_4'}(x,y)d\widetilde{\mathrm{Vol}}\right) \\
&= \frac{1}{8}\left((-2,2) + (2,-2) + \left(\frac{3}{2},\frac{3}{2}\right) + \left(-\frac{1}{2},-\frac{1}{2}\right)\right) \\
&= \left(\frac{1}{8},\frac{1}{8}\right).
\end{aligned}
$$

This measure is important in Ehrhart theory, as it governs the first non-trivial coefficient of the *Ehrhart polynomial*, a classical object recalled in the next result:

Theorem 3.3.4. [41, 42],[57, Theorem 19.1],[79, Theorem 12.2, Exercise 12.5],[9, Corollary 5.5] *Let M be a lattice of dimension n. For any lattice polytope P, there is a polynomial*

$$E_P(k) = \sum_{i=0}^{n} a_i k^i$$

such that

$$E_P(k) = \#(kP \cap M)$$

for any $k \in \mathbb{N}$. In particular,

$$a_n = \mathrm{Vol}(P), \quad a_{n-1} = \frac{1}{2}\widetilde{\mathrm{Vol}}(\partial P), \quad a_0 = 1.$$

Next, embed M as a sublattice $M \times \{0\}$ of $M \times \mathbb{Z}$. For $c = (c_1, \ldots, c_n) \in M$ such that

$$P + c \subseteq \mathbb{R}^n_+ \times \{0\},$$

and for each $i \in \{1, \dots, n\}$, define the convex lattice polytope of one dimension higher ("rooftop polytope"),

$$P(c; i) := \{(u, h) \mid u = (u_1, \dots, u_n) \in P + c, h \in [0, u_i]\} \subseteq M_{\mathbb{R}} \times \mathbb{R}.$$

3.4 Stability thresholds for toric varieties

When X is toric, there are explicit formulas for stability thresholds of big line bundles, in terms of the associated polytope given by (3.23).

Theorem 3.4.1. [16, Corollary 7.16],[90, Corollary 7.1],[63, §A] *Let*

$$P = \bigcap_{i=1}^{d} \{u \in M_{\mathbb{R}} \mid \langle u, v_i \rangle \geq -b_i\}$$

be the polytope associated to the big line bundle $L = \sum_{i=1}^{d} b_i D_i$, where D_i is the prime divisor corresponding to v_i. Then

$$\alpha_k = \left(\max_{u \in \mathrm{Ver}\, P} \max_{1 \leq i \leq d} \langle u, v_i \rangle + b_i \right)^{-1}, \qquad \alpha = \left(\max_{u \in \mathrm{Ver}\, P} \max_{1 \leq i \leq d} \langle u, v_i \rangle + b_i \right)^{-1},$$
$$\delta_k = \left(\max_{1 \leq i \leq d} \langle \mathrm{Bc}_k(P), v_i \rangle + b_i \right)^{-1}, \qquad \delta = \left(\max_{1 \leq i \leq d} \langle \mathrm{Bc}(P), v_i \rangle + b_i \right)^{-1}.$$

Example 3.4.2. Recall Example 3.2.10. We apply Theorem 3.4.1 to all toric del Pezzo surfaces. The result is collected in Table 3.1.

S	$\alpha(S) = \alpha_k(S)$	$\mathrm{Bc}_k(P)$	$\mathrm{Bc}(P)$	$\delta_k(S)$	$\delta(S)$
$\mathbb{P}^2$	$1/3$	0	0	1	1
$\mathbb{P}^1 \times \mathbb{P}^1 \cong \mathbb{F}_0$	$1/2$	0	0	1	1
$\mathbb{F}_1$	$1/3$	$(\frac{k+1}{6(2k+1)}, \frac{k+1}{6(2k+1)})$	$(1/12, 1/12)$	$\frac{3(2k+1)}{7k+4}$	$6/7$
dP_7	$1/3$	$(-\frac{(k+1)(2k+1)}{21k(k+1)+6}, -\frac{(k+1)(2k+1)}{21k(k+1)+6})$	$(-2/21, -2/21)$	$\frac{21k^2+21k+6}{25k^2+27k+8}$	$21/25$
dP_6	$1/2$	0	0	1	1

Table 3.1: Stability thresholds for toric del Pezzo surfaces.

3.5 T-varieties

Toric varieties are a special case of a more general varieties called T-varieties.

Definition 3.5.1. A *T-variety of complexity* k is a normal variety X admitting an effective torus action of codimension k.

Example 3.5.2. Any toric variety of dimension n can be thought of as a T-variety of complexity $k \leq n$ by restricting the torus action to a sub-torus of codimension k. For example, while $\mathbb{F}_n$ is a toric variety, we can restrict ourselves to the sub-torus action that preserves fibers. This equips $\mathbb{F}_n$ with the structure of T-variety of complexity 1. See Figure 3.11.

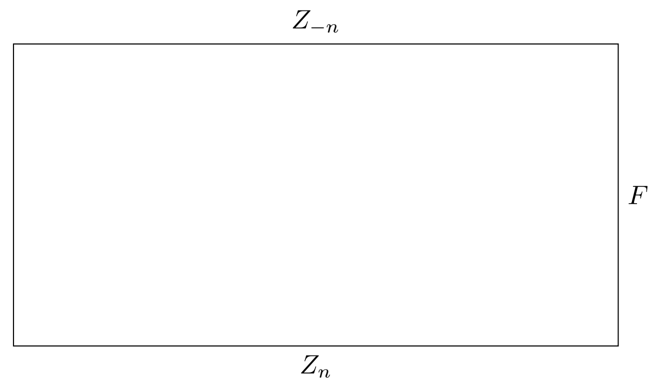

Figure 3.11: The Hirzebruch surface $\mathbb{F}_n$ as a T-variety. The torus-invariant curves are Z_n, Z_{-n}, and all the fibers F.

Example 3.5.3. Given a surface $\mathbb{F}_n$ with the T-variety structure as in Example 3.5.2, let X be the surface obtained by blowing up some points that are invariant under the torus action. Then the torus action can be lifted to X, equipping it with the structure of T-variety of complexity 1.

To compute α_k-invariants on T-varieties it suffices to look at torus-invariant divisors.

Theorem 3.5.4. [98, Proposition 2.6] *Let X be a T-variety. For any effective divisor D, there is a torus-invariant divisor $D' \sim D$ with* $\operatorname{lct}(D') \leq \operatorname{lct}(D)$.

As a result, the computation of α_k-invariants can be simplified for T-varieties. If we can classify the torus-invariant divisors, we can compute their log canonical thresholds and α_k-invariants.

Chapter 4

Tian's stability conjecture on α_k-invariants

In 1988 Tian proposed the following conjecture (refined in 2012), that α_k stabilizes for sufficiently large k.

Conjecture 4.0.1. [103, Question 1],[107, Conjecture 5.3] *For any ample line bundle L over a projective manifold X, there exists $k_0 \in \mathbb{N}_+$ such that for $k \geq k_0$, $\alpha_k(L) = \alpha(L)$.*

There is also a folklore conjecture, weaker than Conjecture 4.0.1, stating that the sequence of α_k-invariants is eventually monotone (see Chapter 6 for a discussion and references).

Conjecture 4.0.2. *For any ample line bundle L over a projective manifold X, there exists $k_0 \in \mathbb{N}_+$ such that for $k_0 \leq k_1 \leq k_2$, $\alpha_{k_1}(L) \geq \alpha_{k_2}(L)$.*

Conjectures 4.0.1 and 4.0.2 were originally stated for Fano X and $L = -K_X$. That remains the most interesting case, due to the relation to Kähler–Eintein metrics. Some positive results are known for this case. The case of smooth del Pezzo surfaces can be deduced from [25]. In general, it was shown by Birkar that $\alpha_k(-K_X) = \alpha(-K_X)$ if k is sufficiently large and *divisible*, provided that $\alpha(-K_X) \leq 1$ (even for $-K_X$ big and nef) [13]. Note that this result does not provide an answer to Conjecture 4.0.1. When the manifold is toric, Conjecture 4.0.1 is confirmed for any ample line bundle, and stabilization is disproved for other quantized invariants of the form $\alpha_{k,m}$ in the toric setting. See Chapter 6.

There is an even stronger version of Conjecture 4.0.1, predicting how large k_0 is.

Conjecture 4.0.3. [107, Conjecture 5.4] *For any ample line bundle L over a projective manifold X, if the section ring $R(X, L) := \oplus_{k=0}^{\infty} H^0(X, kL)$ is generated by $\oplus_{k=0}^{k_0} H^0(X, kL)$, then for $k \geq k_0$, $\alpha_k(L) = \alpha(L)$. In particular, if the section ring $R(X, L)$ is generated by $H^0(X, L)$, then for any $k \in \mathbb{N}_+$, $\alpha_k(L) = \alpha(L)$.*

A counterexample to Conjecture 4.0.3 was provided by Ahmadinezhad–Cheltsov–Schicho [1]. They found a smooth surface S with a very ample line bundle L such that the section ring $R(S, L)$ is generated by $H^0(S, L)$, but $\alpha_1(L) > \alpha(L)$.

In this chapter, we prove Theorem 4.0.4 by explicitly providing a pair (S, L), disproving Conjectures 4.0.1 and 4.0.2.

Theorem 4.0.4. *There is an ample line bundle L over a smooth surface S such that α_k does not stabilize, nor is it eventually monotone.*

Note that in this counterexample $-K_S$ is big and nef, although we are not taking $L = -K_S$. It still remains open whether for Fano X and $L = -K_X$ such an example exists. Further counterexamples can be constructed if the positivity of $-K_X$ is relaxed (e.g., curves of genus $g \geq 2$, or ruled surfaces over them, as observed already by Zhuang [122, Example A.8]. These are counterexamples to Conjecture 4.0.1, but not necessarily to Conjecture 4.0.2). However, as noted earlier, the main impetus of Tian's original conjecture in 1988 was the positive regime. In the big and nef setting of Birkar's theorem, Theorem 4.0.4 shows that stabilization along a divisible subsequence, i.e., Birkar's theorem, is optimal.

This chapter can be considered as part of the program to tackle problems on stabilization of quantized algebraic invariants arising in K-stability, and in the absence of such stabilization to consider the refined problem of determining the large k asymptotics of such invariants. In this direction, Theorem 4.4.1 below also shows that the asymptotics $\alpha_k(L) = \alpha(L) + O(k^{-1})$ [16, Corollary 5.2] [64, Theorem 1.5] is optimal in the setting that either L is ample or $\alpha(L) = \alpha_\ell(L)$ for some ℓ. As far as we are aware, this is also the first time the sequence $\{\alpha_k\}$ is explicitly computed in any example where the sequence is non-constant.

4.1 The construction of the pair

In this section we explain the construction of the pair (S, L) and fix some notations. See Figure 4.1 for an illustration.

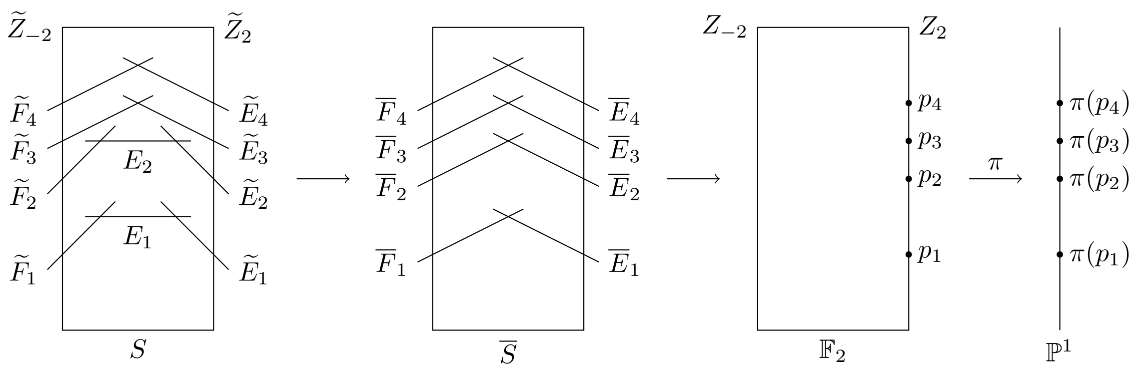

Figure 4.1: The construction of the surface S.

Recall that the Hirzebruch surface $\mathbb{F}_2$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Let $\pi : \mathbb{F}_2 \to \mathbb{P}^1$ denote this projection map. We also fix two sections of this bundle, namely the -2-curve Z_{-2}, and a smooth 2-curve Z_2.

Pick four points $p_1, \dots, p_4 \in Z_2$. Let F_i denote the fiber $\pi^{-1}(\pi(p_i))$, and F a general fiber on $\mathbb{F}_2$, i.e., $F = \pi^{-1}(p)$ for some $p \in U$, where

$$U := \mathbb{P}^1 \setminus \{\pi(p_1), \dots, \pi(p_4)\}.$$

Let $\overline{S}$ denote the blow-up of $\mathbb{F}_2$ at the four points $p_1, \dots, p_4$. Let $\overline{E}_i$ denote the corresponding exceptional divisor, and $\overline{F}_i$ the proper transform of F_i.

Finally, let S denote the blow-up of $\overline{S}$ at the two points $\overline{E}_1 \cap \overline{F}_1$ and $\overline{E}_2 \cap \overline{F}_2$. Let E_1 and E_2 denote the corresponding exceptional divisor, and $\widetilde{E}_i$ (resp. $\widetilde{F}_i$) the proper transform of $\overline{E}_i$ (resp. $\overline{F}_i$).

Let $f : S \to \mathbb{F}_2$ denote the composition of blow-up maps, $\widetilde{Z}_2$ (resp. $\widetilde{Z}_{-2}$, $\widetilde{F}$) the proper transform of Z_2 (resp. Z_{-2}, F). We consider the line bundle

$$L := \mathcal{O}_S\Big(2\widetilde{Z}_2 + 2\widetilde{Z}_{-2} + 3\widetilde{F} + \widetilde{E}_1 + \widetilde{F}_1 + E_1 + \widetilde{E}_2 + \widetilde{F}_2 + E_2\Big). \tag{4.1}$$

Note that $-K_X = \widetilde{Z}_2 + \widetilde{Z}_{-2} + \widetilde{E}_1 + \widetilde{F}_1 + E_1 + \widetilde{E}_2 + \widetilde{F}_2 + E_2$.

4.2 Picard group and intersection numbers

We compute the intersection numbers on $\mathrm{Pic}(S)$ in this section. As an application, we show that L is ample (Theorem 4.2.2).

Recall that $\mathrm{Pic}(\mathbb{F}_2)$ is a free abelian group of rank 2, generated by $[Z_2]$ and $[F]$, with 2-by-2 intersection matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix},$$

and

$$Z_{-2} \sim Z_2 - 2F.$$

After the six blow-ups, $\mathrm{Pic}(S)$ is a free abelian group of rank 8, generated by $[\widetilde{Z}_2]$, $[\widetilde{F}]$, and the exceptional divisors $[\widetilde{E}_1], \ldots, [\widetilde{E}_4], [E_1], [E_2]$, with 8-by-8 intersection matrix (off-diagonal zero entries are omitted)

$$\begin{pmatrix} -2 & 1 & 1 & 1 & 1 & 1 & & \\ 1 & 0 & & & & & & \\ 1 & & -2 & & & & 1 & \\ 1 & & & -2 & & & & 1 \\ 1 & & & & -1 & & & \\ 1 & & & & & -1 & & \\ & & 1 & & & & -1 & \\ & & & 1 & & & & -1 \end{pmatrix}, \tag{4.2}$$

and

$$\begin{gathered} \widetilde{F}_1 \sim \widetilde{F} - \widetilde{E}_1 - 2E_1, \qquad \widetilde{F}_2 \sim \widetilde{F} - \widetilde{E}_2 - 2E_2, \qquad \widetilde{F}_3 \sim \widetilde{F} - \widetilde{E}_3, \qquad \widetilde{F}_4 \sim \widetilde{F} - \widetilde{E}_4, \\ \widetilde{Z}_{-2} \sim \widetilde{Z}_2 - 2\widetilde{F} + \widetilde{E}_1 + \widetilde{E}_2 + \widetilde{E}_3 + \widetilde{E}_4 + E_1 + E_2. \end{gathered} \tag{4.3}$$

See also Figure 4.2.

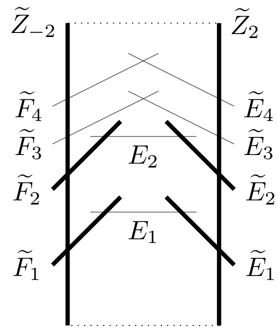

Figure 4.2: Some curves with negative self-intersection on S. The thick curves have self-intersection -2, and the thin curves have self-intersection -1.

Recall (4.1). We compute the following intersection numbers.

$$\begin{gathered} L\cdot\widetilde{Z}_2=1,\qquad L\cdot\widetilde{Z}_{-2}=1,\qquad L\cdot\widetilde{F}=4,\\ L\cdot\widetilde{E}_1=1,\qquad L\cdot\widetilde{F}_1=1,\qquad L\cdot E_1=1,\\ L\cdot\widetilde{E}_2=1,\qquad L\cdot\widetilde{F}_2=1,\qquad L\cdot E_2=1,\\ L\cdot\widetilde{E}_3=2,\qquad L\cdot\widetilde{F}_3=2,\qquad L\cdot\widetilde{E}_4=2,\qquad L\cdot\widetilde{F}_4=2. \end{gathered} \tag{4.4}$$

To prove ampleness we use Nakai–Moishezon criterion (see, e.g., [59, Ch. V, Theorem 1.10]).

Theorem 4.2.1. *A divisor D on the surface X is ample if and only if $D^2>0$ and $D\cdot C>0$ for all irreducible curves C in X.*

Theorem 4.2.2. *The line bundle L defined by* (4.1) *is ample.*

Proof. Since L is effective, it suffices to prove $L\cdot C>0$ for all irreducible curves C.

Let

$$D:=2\widetilde{Z}_2+2\widetilde{Z}_{-2}+3\widetilde{F}+\widetilde{E}_1+\widetilde{F}_1+E_1+\widetilde{E}_2+\widetilde{F}_2+E_2\sim L.$$

If C lies in $\operatorname{Supp} D$, it has been shown in (4.4) that $L\cdot C>0$. If C does not lie in $\operatorname{Supp} D$, it suffices to show that C intersects $\operatorname{Supp} D$. Recall the blow-down map $f:S\to\mathbb{F}_2$ and the projection $\pi:\mathbb{F}_2\to\mathbb{P}^1$. The image $(\pi\circ f)(C)$ is either 0-dimensional (i.e., a point $p\in U\subseteq\mathbb{P}^1$) or 1-dimensional (i.e., the whole $\mathbb{P}^1$). In the former case, $C=(\pi\circ f)^{-1}(p)$, so C intersects $\widetilde{Z}_2$ and $\widetilde{Z}_{-2}$. In the latter case, C intersects $\widetilde{F}$. This completes the proof. $\square$

Note that $-K_X$ is nef but not ample. It is big with $\mathrm{Vol}(-K_X) = 2$.

4.3 Torus-invariant divisors

In this section we discuss the T-variety structure of S, and reduce the family of divisors to be considered in computing α_k-invariants.

Recall that $\mathbb{F}_2$ admits a toric variety structure, with Z_2 fixed by the torus action. We regard it as a T-variety of complexity 1 by restricting the torus action to the one dimensional subgroup that leaves each fiber invariant. More specifically, recall that $\mathbb{F}_2 \setminus Z_{-2}$ is the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(2)$. The $\mathbb{C}^*$-action is the scalar multiplication on each fiber of this line bundle.

This action can then be lifted to S, equipping S with the structure of T-variety of complexity 1.

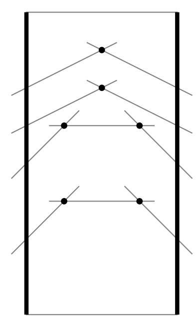

Figure 4.3: The fixed points under the torus action.

Lemma 4.3.1. *The torus-invariant curves on S are $\widetilde{Z}_2, \widetilde{Z}_{-2}, \widetilde{E}_1, \ldots, \widetilde{E}_4, \widetilde{F}_1, \ldots, \widetilde{F}_4, E_1, E_2,$ and $(\pi \circ f)^{-1}(p)$ for $p \in U$ (recall §4.1).*

Proof. Notice that the set of fixed points under the torus action consists of $\widetilde{Z}_2 \cup \widetilde{Z}_{-2}$ and six points $E_1 \cap \widetilde{E}_1, E_1 \cap \widetilde{F}_1, E_2 \cap \widetilde{E}_2, E_2 \cap \widetilde{F}_2, \widetilde{E}_3 \cap \widetilde{F}_3, \widetilde{E}_4 \cap \widetilde{F}_4$ (see Figure 4.3). Let C be a torus-invariant curve. If C is fixed by the torus action, then $C = \widetilde{Z}_2$ or $\widetilde{Z}_{-2}$. If C is not fixed by the torus action, then C contains a one-dimensional orbit. In that case, $C = \widetilde{E}_1, \ldots, \widetilde{E}_4, \widetilde{F}_1, \ldots, \widetilde{F}_4, E_1, E_2$, or $(\pi \circ f)^{-1}(p)$ for $p \in U$. $\square$

Lemma 4.3.2. *For $k \in \mathbb{N}_+$,*

$$\alpha_k(L) = \inf_D k \operatorname{lct}(D),$$

where the infimum is taken over divisors of the form

$$D = a_1\widetilde{Z}_2 + a_2\widetilde{Z}_{-2} + a_3\widetilde{E}_1 + \cdots + a_6\widetilde{E}_4 + a_7\widetilde{F}_1 + \cdots + a_{10}\widetilde{F}_4 + a_{11}E_1 + a_{12}E_2 \sim kL.$$

Proof. By Theorem 3.5.4 and Lemma 4.3.1, it suffices to look at divisors of the form

$$D = a_1\widetilde{Z}_2 + a_2\widetilde{Z}_{-2} + a_3\widetilde{E}_1 + \cdots + a_6\widetilde{E}_4 + a_7\widetilde{F}_1 + \cdots + a_{10}\widetilde{F}_4 + a_{11}E_1 + a_{12}E_2 + \sum_{p\in U} a_p(\pi\circ f)^{-1}(p).$$

We may drop the last term. Indeed, by (4.3),

$$\begin{aligned} D \sim D' := a_1\widetilde{Z}_2 + a_2\widetilde{Z}_{-2} + a_3\widetilde{E}_1 + a_4\widetilde{E}_2 + a_5\widetilde{E}_3 + \left(a_6 + \sum_{p\in U} a_p\right)\widetilde{E}_4 \\ + a_7\widetilde{F}_1 + a_8\widetilde{F}_2 + a_9\widetilde{F}_3 + \left(a_{10} + \sum_{p\in U} a_p\right)\widetilde{F}_4 + a_{11}E_1 + a_{12}E_2. \end{aligned}$$

By Proposition 2.1.8, $\mathrm{lct}(D') \le \mathrm{lct}(D)$. This completes the proof. $\square$

4.4 The computation of α_k-invariants

In this section we finish the computation of α_k-invariants, providing an explicit counterexample to Tian's stabilization conjecture.

Theorem 4.4.1. *Let (S, L) be as in §4.1. Then*

$$\alpha_k(L) = \begin{cases} \dfrac{1}{8}, & k \text{ is even;} \\ \dfrac{k}{8k-1}, & k \text{ is odd,} \end{cases}$$

and

$$\alpha(L) = \frac{1}{8}.$$

In particular, $\alpha_k(L) \ge \alpha(L)$ with equality if and only if k is even, and $\alpha_k(L)$ is not eventually monotone.

Proof. Let

$$D = a_1\widetilde{Z}_2 + a_2\widetilde{Z}_{-2} + a_3\widetilde{E}_1 + \cdots + a_6\widetilde{E}_4 + a_7\widetilde{F}_1 + \cdots + a_{10}\widetilde{F}_4 + a_{11}E_1 + a_{12}E_2 \sim kL,$$

where $a_1, \ldots, a_{12} \in \mathbb{N}_+$. By (4.2) and (4.3) (see also Figure 4.2),

$$\begin{aligned}
&D \cdot \widetilde{F} = a_1 + a_2,\\
&D \cdot \left(-\widetilde{E}_2 + \widetilde{E}_3\right) = 2a_4 - a_5 + a_9 - a_{12},\\
&D \cdot \left(-\widetilde{E}_1 + \widetilde{E}_3\right) = 2a_3 - a_5 + a_9 - a_{11},\\
&D \cdot \left(\widetilde{Z}_2 + \widetilde{E}_3 + \widetilde{E}_4\right) = a_3 + a_4 + a_9 + a_{10},\\
&D \cdot \left(\widetilde{Z}_{-2} + \widetilde{F}_3 + \widetilde{E}_4\right) = a_5 + a_6 + a_7 + a_8,\\
&D \cdot \left(2\widetilde{Z}_2 + \widetilde{E}_1 + \widetilde{E}_2 + 2\widetilde{E}_3\right) = 2a_6 + 2a_9 + a_{11} + a_{12},\\
&D \cdot \left(2\widetilde{Z}_2 + \widetilde{E}_1 + \widetilde{E}_2 + 2\widetilde{E}_4\right) = 2a_5 + 2a_{10} + a_{11} + a_{12}.
\end{aligned}$$

Since $D \sim kL$, combining with (4.4) we get

$$a_1 + a_2 = 4k, \tag{4.5}$$

$$2a_4 - a_5 + a_9 - a_{12} = k, \tag{4.6}$$

$$2a_3 - a_5 + a_9 - a_{11} = k, \tag{4.7}$$

$$a_3 + a_4 + a_9 + a_{10} = 5k, \tag{4.8}$$

$$a_5 + a_6 + a_7 + a_8 = 5k, \tag{4.9}$$

$$2a_6 + 2a_9 + a_{11} + a_{12} = 8k, \tag{4.10}$$

$$2a_5 + 2a_{10} + a_{11} + a_{12} = 8k. \tag{4.11}$$

By (4.5),

$$a_1, a_2 \leq 4k. \tag{4.12}$$

By (4.8) and (4.9),

$$a_3, \ldots, a_{10} \leq 5k. \tag{4.13}$$

By (4.10) or (4.11),

$$a_{11}, a_{12} \leq 8k. \tag{4.14}$$

Combining (4.12), (4.13), and (4.14), by Proposition 2.1.8,

$$\operatorname{lct}(D) \geq (8k)^{-1}.$$

Since D is arbitrary, by Lemma 4.3.2,

$$\alpha_k(L) \geq \frac{1}{8}. \tag{4.15}$$

If k is even, consider the divisor

$$D' := 2k\widetilde{Z}_2 + 2k\widetilde{Z}_{-2} + \frac{9k}{2}\widetilde{E}_1 + \frac{9k}{2}\widetilde{F}_1 + 8kE_1 + \frac{k}{2}\widetilde{E}_2 + \frac{k}{2}\widetilde{F}_2.$$

Using (4.3) we can check that $D' \sim kL$. By Proposition 2.1.8,

$$\operatorname{lct}(D') = (8k)^{-1}.$$

By Lemma 4.3.2 and (4.15),

$$\alpha_k(L) = \frac{1}{8}.$$

Finally, let k be odd. We claim that $a_{11}, a_{12} \neq 8k$. Indeed, assume $a_{11} = 8k$. By (4.10) and (4.11),

$$a_5 = a_6 = a_9 = a_{10} = a_{12} = 0.$$

Hence by (4.6),

$$2a_4 = k,$$

contradicting the fact that k is odd. Similarly, assume $a_{12} = 8k$. By (4.10) and (4.11),

$$a_5 = a_6 = a_9 = a_{10} = a_{11} = 0.$$

Hence by (4.7),

$$2a_3 = k,$$

contradicting the fact that k is odd. Therefore, (4.14) can be strengthened to

$$a_{11}, a_{12} \le 8k - 1.$$

Combining with (4.12) and (4.13), by Proposition 2.1.8,

$$\mathrm{lct}(D) \ge (8k-1)^{-1}.$$

Since D is arbitrary, by Lemma 4.3.2,

$$\alpha_k(L) \ge \frac{k}{8k-1}. \tag{4.16}$$

On the other hand, consider the divisor

$$D'' := 2k\widetilde{Z}_2 + 2k\widetilde{Z}_{-2} + \frac{9k-1}{2}\widetilde{E}_1 + \frac{9k-1}{2}\widetilde{F}_1 + (8k-1)E_1 + \frac{k+1}{2}\widetilde{E}_2 + \frac{k+1}{2}\widetilde{F}_2 + E_2.$$

Using (4.3) we can check that $D'' \sim kL$. By Proposition 2.1.8,

$$\mathrm{lct}(D'') = (8k-1)^{-1}.$$

By Lemma 4.3.2 and (4.16),

$$\alpha_k(L) = \frac{k}{8k-1}.$$

This completes the proof. □

Chapter 5

Cheltsov–Rubinstein Problem

The purpose of this chapter is to complete the solution of the Cheltsov–Rubinstein problem [26, 88] on the classfication of strongly asymptotically log del Pezzo surfaces admitting a Kähler–Einstein edge metric. This has been a topic of intense research. See, e.g., [26, 27, 28, 47, 88, 89, 91]. As a consequence of these works, one (infinite) family of pairs remains to be understood: the pairs (II.6A.n.m) (see Definition 5.0.6). In this chapter we treat this family completely and hence solve the Cheltsov–Rubinstein problem for strongly asymptotically log del Pezzo surfaces. The Cheltsov–Rubinstein problem for general asymptotically log del Pezzo surfaces is very challenging and remains wide open.

To explain the problem and our work let us introduce some notation. Let X be a Kähler manifold. Fix a divisor $D = D_1 + \cdots + D_r$ with simple normal crossings on X, and angle vector $\beta = (\beta_1, \ldots, \beta_r) \in (0,1]^r$. A Kähler–Einstein edge metric is a smooth Kähler–Einstein metric on $X \setminus \bigcup_i D_i$ that has a conical singularity of angle $2\pi\beta_i$ along D_i [61, 88, 105]. The existence of a Kähler–Einstein edge metric with positive Ricci curvature imposes the cohomological condition that

$$-K_X - \sum_{i=1}^{r} (1-\beta_i) D_i$$

is ample. When $\beta_i = 1$, the metric reduces to the ordinary (smooth) Kähler–Einstein metric. On the other hand, the case when the β_i's are *small* is also of much interest; this corresponds to the notion of asymptotically log Fano varieties, introduced by Cheltsov–Rubinstein.

Definition 5.0.1. [26] $(X, D_1 + \cdots + D_r)$ is *(strongly) asymptotically log Fano* if $-K_X - \sum_{i=1}^{r}(1-\beta_i)D_i$ is ample for (all) sufficiently small $\beta \in (0,1]^r$. In dimension 2, these are also called *(strongly) asymptotically log del Pezzo surfaces.*

Strongly asymptotically log del Pezzo surfaces have been classified by Cheltsov–Rubinstein [26, 89]. A natural question to ask is the existence of Kähler–Einstein edge metrics on such surfaces, namely the following uniformization problem posed by Cheltsov–Rubinstein.

Problem 5.0.2. [88, Problem 9.1] Determine which strongly asymptotically log del Pezzo surfaces admit Kähler–Einstein edge metrics for sufficiently small β.

Assume the boundary is smooth. By Cheltsov–Rubinstein classification, we are reduced to the following 10 families.

Theorem 5.0.3. [26, Theorems 2.1 and 3.1] *Let S be a smooth surface, $C_1, \ldots, C_r$ be disjoint smooth irreducible curves on S. Let Z_n denote the curve of self-intersection $-n$ on the Hirzebruch surface $\mathbb{F}_n$, and F a fiber on it. Then $(S, C_1 + \cdots + C_r)$ is strongly asymptotically log del Pezzo if and only if it is one of the following pairs.*

(I.3A) *$S \cong \mathbb{F}_1$, and $C_1 \in |2(Z_1 + F)|$;*

(I.4A) *$S \cong \mathbb{P}^1 \times \mathbb{P}^1$, and C_1 is a smooth elliptic curve of bi-degree $(2,2)$;*

(I.5.m) *S is a blow-up of $\mathbb{P}^2$ at $0 \leq m \leq 8$ distinct points on a smooth cubic elliptic curve such that $-K_S$ is ample, and C_1 is the proper transform of the elliptic curve;*

(I.6B.m) *S is a blow-up of $\mathbb{P}^2$ at $m \geq 0$ distinct points on a smooth conic, and C_1 is the proper transform of the conic;*

(I.6C.m) *S is a blow-up of $\mathbb{P}^2$ at $m \geq 0$ distinct points on a line, and C_1 is the proper transform of the line;*

(I.7.n.m) *S is a blow-up of $\mathbb{F}_n$ at $m \geq 0$ distinct points on Z_n, and C_1 is the proper transform of Z_n, for any $n \geq 0$;*

(I.8B.m) *S is a blow-up of $\mathbb{F}_1$ at $m \geq 0$ distinct points on a smooth rational curve in the linear system $|Z_1 + F|$, and C_1 is the proper transform of the rational curve;*

(I.9B.m) *S is a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at $m \geq 0$ distinct points on a smooth rational curve of bi-degree $(2,1)$ with no two of them on a single curve of bi-degree $(0,1)$, and C_1 is the proper transform of the rational curve;*

(I.9C.m) *S is a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at $m \geq 0$ distinct points on a smooth rational curve of bi-degree $(1,1)$, and C_1 is the proper transform of the rational curve;*

(II.6A.n.m) *S is a blow-up of $\mathbb{F}_n$ at $m \geq 0$ distinct points on Z_n and a smooth rational curve in the linear system $|Z_n + nF|$ with no two of them on a single curve in the linear system $|F|$, C_1 is the proper transform of Z_n, and C_2 is the proper transform of the rational curve, for any $n \geq 0$.*

As a step towards Problem 5.0.2, Cheltsov–Rubinstein proposed the following conjecture.

Conjecture 5.0.4. [26, Conjecture 1.6] *Let (S, C) be a strongly asymptotically log del Pezzo surface with $r = 1$. Then (S, C) admits a Kähler–Einstein edge metric for sufficiently small β if and only if $(K_S + C)^2 = 0$.*

The necessary part and all but 6 sub-families of the sufficient part were verified. Namely, the pairs (I.6B.m), (I.6C.m), (I.7.n.m), (I.8B.m), (I.9C.m) do not admit Kähler–Einstein edge metrics with small angles [27, Theorem 1.6], while the pairs (I.3A), (I.4A), (I.5.m), and (I.9B.m) for $m = 0$ or $m \geq 7$ admit Kähler–Einstein edge metrics with small angles [26, Propositions 7.4 and 7.5], [28, Theorem 1.3], [61, Corollary 1].

Let us now discuss the remaining cases, namely, the sub-families (I.9B.m) with $1 \leq m \leq 6$ and (II.6A.n.m).

For the sub-families (I.9B.m) with $1 \leq m \leq 6$, each sub-family is parameterized by the position of the m blown-up points on the smooth $(2,1)$-curve. Let p_0 and p_∞ denote the points where the $(2,1)$-curve on $\mathbb{P}^1 \times \mathbb{P}^1$ is tangent to a $(0,1)$-curve. Fujita–Liu–Süß–Zhang–Zhuang showed that the pairs (I.9B.m) contradict the conjecture if $m = 1, 2$ and exactly 1 of the points p_0 and p_∞ is blown-up, by computing an upper bound for the δ-invariant [47, Corollary 2.9]. Fujita verified that these 2 pairs are the only counterexamples in the 6 sub-families [45, Theorem 1.3]. Thus, Conjecture 5.0.4 of Cheltsov–Rubinstein holds for the vast majority of the cases, and the remaining cases are semi-stable degenerations that are now completely understood by the aforementioned works.

In this chapter we restrict to the remaining tenth and final family, (II.6A.n.m), where $\operatorname{Supp} C$ has multiple components, and hence is not within the scope of Conjecture 5.0.4.

Problem 5.0.5. Determine whether (II.6A.n.m) admit Kähler–Einstein edge metrics for sufficiently small β.

First we introduce some notation to describe the curves appearing in the study of (II.6A.n.m).

Definition 5.0.6. Let $\mathbb{F}_n$ denote the Hirzebruch surface containing a curve C_1 whose self intersection number is $-n$, and pick a smooth curve C_2 in $|C_1 + nF|$ where F denotes a fiber. Let S be the blow-up of $\mathbb{F}_n$ at m points on $C_1 \cup C_2$ such that no two of them lie on the same fiber, and $C = \widetilde{C}_1 + \widetilde{C}_2$, where $\widetilde{C}_i \subseteq S$ is the proper transform of C_i. This family is named (II.6A.n.m). See Figure 5.1.

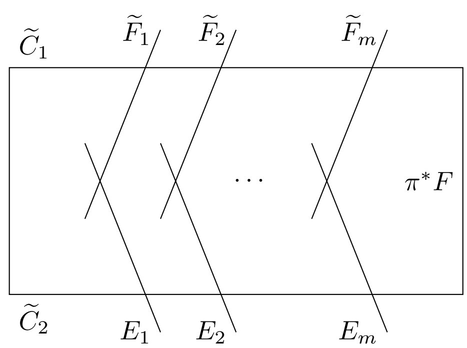

Figure 5.1: The pair (II.6A.n.m). The surface is given by blowing up m points $p_1, \dots, p_m$ on the curve $C_2 \subseteq \mathbb{F}_n$, where $\mathbb{F}_n$ is the Hirzebruch surface containing a curve C_1 whose self intersection number is $-n$ and a generic fiber F, and C_2 is a smooth element of $|C_1 + nF|$. The fiber through the point p_i is denoted by F_i, and the exceptional curve over p_i is denoted E_i. The curves $\widetilde{C}_1, \widetilde{C}_2, \widetilde{F}_1, \dots, \widetilde{F}_m$ are the proper transforms of $C_1, C_2, F_1, \dots, F_m$, respectively.

This family is the generalization of the football to complex surfaces. In particular, the pair (II.6A.0.0) can precisely be equipped with the standard conical metric on the football times $\mathbb{P}^1$, if $\beta_1 = \beta_2$. Generalizing the product football metric, using the $\mathbb{P}^1$-bundle structure of Hirzebruch surfaces, Rubinstein–Zhang constructed a family of Kähler–Einstein edge metrics using the Calabi ansatz [91, Theorem 1.2], and settled the case $m = 0$ for all n by finding an explicit relation between β_1 and β_2.

Theorem 5.0.7. [91, Theorem 1.2]. *Consider* (II.6A.n.0). *Namely, S is the Hirzebruch surface $\mathbb{F}_n$ containing a curve C_1 whose self intersection number is $-n$, and C_2 is a smooth curve in $|C_1 + nF|$ where F denotes a fiber. Let β_1 and β_2 be the cone angles along the boundary curves C_1 and C_2. Then Kähler–Einstein edge metrics exist if and only if*

$$\beta_1^2 - \frac{n}{3}\beta_1^3 = \beta_2^2 + \frac{n}{3}\beta_2^3. \tag{5.1}$$

It is difficult to generalize such a construction to the cases $m \geq 1$, as these pairs no longer have a $\mathbb{P}^1$-bundle structure. In this chapter, we study Problem 5.0.5 for $m \geq 1$ using algebraic methods, i.e., verifying log K-stability.

The notion of K-stability introduced by Tian [106] and refined by Donaldson [39], Berman [10], Fujita–Odaka [48] and Fujita [45] turned out to characterize the existence of Kähler–Einstein metrics. It was also extended to log setting [40], namely Kähler–Einstein edge metrics exist if and only if the pair is log K-polystable [11]. Moreover, Fujita–Odaka [48] introduced the δ-invariant (Definition 2.5.6) that characterizes K-stability. In fact, it has been shown by Blum–Jonsson [16] that $\delta > 1$ is equivalent to (uniform) K-stability, which is stronger than K-polystability, and $\delta \geq 1$ is equivalent to K-semistability, which is weaker than K-polystability. This was also extended to log setting [14].

The δ-invariant is the main technical tool used in this chapter. Our main result generalizes (5.1), imposing a relation on the cone angles.

Theorem 5.0.8. *Consider* (II.6A.n.m) *as in Definition 5.0.6. We have*

$$\delta\Big(S, (1-\beta_1)\widetilde{C}_1 + (1-\beta_2)\widetilde{C}_2\Big) \leq 1,$$

with equality if and only if

$$\beta_1^2 + \frac{1}{3}\widetilde{C}_1^2\beta_1^3 = \beta_2^2 + \frac{1}{3}\widetilde{C}_2^2\beta_2^3, \quad m \neq 1. \tag{5.2}$$

The pair is log K-polystable, i.e., it admits a Kähler–Einstein edge metric, if and only if (5.2) *holds.*

The equivalence between stability and existence of Kähler–Einstein edge metrics used in the statement is due to the resolution of the logarithmic Yau–Tian–Donaldson conjecture [73, Theorem 1.6].

Theorem 5.0.8 generalizes Rubinstein–Zhang's Theorem 5.0.7 and completely solves Problem 5.0.5.

5.1 The pairs (II.6A.n.m)

In this section, we start by explaining the definition of the family (II.6A.n.m). As a preparation, we also compute the intersection numbers of curves as well as the log anti-canonical divisor of the surface S.

For $n \geq 0$, let $\mathbb{F}_n$ denote the Hirzebruch surface containing a curve C_1 whose self intersection number is $-n$. A fiber F is a smooth curve on $\mathbb{F}_n$ whose self intersection number is 0. Pick a smooth element of $|C_1 + nF|$ and call it C_2. Let S be the blow-up of $\mathbb{F}_n$ at m points $p_1, \ldots, p_m$ on $C_1 \cup C_2$ such that no two of them lie on the same fiber, and $\widetilde{C}_i \subseteq S$ the proper transform of C_i. Consider the pair $(S, \widetilde{C}_1 + \widetilde{C}_2)$. Let $\pi : S \to \mathbb{F}_n$ denote the blow-up morphism, E_i the exceptional divisor $\pi^{-1}(p_i)$, F a generic fiber, i.e., a fiber not through any p_i, and F_i the fiber through p_i. For an irreducible curve $C \subseteq \mathbb{F}_n$, let $\widetilde{C}$ denote its proper transform. For $i = 1, 2$, define

$$I_i := \{j \,|\, p_j \in C_i\}, \quad m_i := |I_i|.$$

Remark 5.1.1. Recall that $\mathbb{F}_{n+1}$ can be obtained by blowing up a point on the $-n$-curve of $\mathbb{F}_n$ and then blow down the proper transform of the fiber through that point. In other words, if S is obtained from $\mathbb{F}_n$ by blowing up m_1 points on $C_1 \subseteq \mathbb{F}_n$ and m_2 points on $C_2 \subseteq \mathbb{F}_n$ such that no two of them lie on the same fiber, then it can also be obtained from $\mathbb{F}_{n+m_1}$ by blowing up $m_1 + m_2$ points on $C_2 \subseteq \mathbb{F}_{n+m_1}$. Because of this, we may assume

$$m_1 = 0, \quad m_2 = m.$$

The Picard group of S is generated by the linear equivalent classes of $\pi^* C_2$, $\pi^* F$ and E_i, giving intersection matrix

$$\begin{pmatrix} n & 1 & & & \\ 1 & 0 & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{pmatrix}.$$

Note that

$$\widetilde{C}_1 \sim \pi^*(C_2 - nF),$$

$$\widetilde{C}_2 \sim \pi^* C_2 - \sum_i E_i,$$

$$-K_S = \widetilde{C}_1 + \widetilde{C}_2 + 2\pi^* F \sim \pi^*(2C_2 + (2-n)F) - \sum_i E_i.$$

Let

$$\Delta_\beta := (1-\beta_1)\widetilde{C}_1 + (1-\beta_2)\widetilde{C}_2 \sim_{\mathbb{Q}} \pi^*((2-\beta_1-\beta_2)C_2 - (n - n\beta_1)F) - (1-\beta_2)\sum_i E_i.$$

Then

$$-K_{S,\Delta_\beta} := -K_S - \Delta_\beta = \pi^*((\beta_1+\beta_2)C_2 + (2-n\beta_1)F) - \beta_2 \sum_i E_i. \tag{5.3}$$

5.2 Nefness criterion

For the remainder of the chapter we let (S, C) be the pair (II.6A.n.m). See Section 5.1.

To compute Zariski decomposition, we need to check nefness of various divisors. To simplify this process, in this section we first compute the nef cone. The result is the following.

Lemma 5.2.1. *A* $\mathbb{Q}$*-divisor*

$$D \sim_{\mathbb{Q}} \pi^*(aC_2 + bF) - \sum_i c_i E_i$$

is nef if and only if

$$
\begin{aligned}
&0 \leq D \cdot E_i = c_i, \\
&0 \leq D \cdot \widetilde{F}_i = a - c_i, \\
&0 \leq D \cdot \widetilde{C}_1 = b, \\
&0 \leq D \cdot \widetilde{C}_2 = na + b - \sum_i c_i.
\end{aligned}
\tag{5.4}
$$

Proof. Recall Definition 2.2.1. If D is nef, then (5.4) is necessary.

On the other hand, if (5.4) is satisfied, then the $\mathbb{Q}$-divisor

$$D' := a\widetilde{C}_2 + \sum_i (a - c_i)E_i + b\pi^* F \sim_{\mathbb{Q}} D$$

is effective. To prove that D is nef, it suffices to show that $D \cdot C \geq 0$ for each irreducible component C of D'. Since (5.4) is satisfied,

$$
\begin{aligned}
&D \cdot \widetilde{C}_2 = na + b - \sum_i c_i \geq 0, \\
&D \cdot E_i = c_i \geq 0, \\
&D \cdot \pi^* F = a \geq 0.
\end{aligned}
$$

This shows D is nef. □

Corollary 5.2.2. *The log anti-canonical divisor $-K_{S,\Delta_\beta}$ is ample if and only if*

$$\beta_1 \in \left(0, \frac{2}{n}\right), \quad \beta_2 \in \begin{cases} (0, +\infty), & n \geq m \\ \left(0, \frac{2}{m-n}\right), & n < m. \end{cases} \tag{5.5}$$

Proof. Since the ample cone is the interior of the nef cone [68, Theorem 1.4.23], $-K_{S,\Delta_\beta}$ is ample if and only if the inequalities of (5.4) are strict. By (5.3),

$$\beta_2 > 0,$$

$$\beta_1 > 0,$$

$$2 - n\beta_1 > 0,$$

$$2 + n\beta_2 > m\beta_2.$$

This is (5.5). $\square$

5.3 Intersection matrices

Since the Zariski decomposition of a divisor also involves curves with negative definite intersection matrices, in this section we give some collections of curves that have negative definite intersection matrices. They will be used in the computation of Zariski decompositions.

Claim 5.3.1. *The following families of divisors have negative definite intersection matrices.*

(1) $\widetilde{F}_1, \ldots, \widetilde{F}_m$*;*

(2) $E_1, \ldots, E_m$*;*

(3) $\widetilde{C}_1, \widetilde{F}_i$*, where* $n > 1$ *and* $1 \leq i \leq m$*;*

(4) $\widetilde{C}_1, \widetilde{F}_1, \ldots, \widetilde{F}_m$*, where* $n > m$*;*

(5) $\widetilde{C}_2, \widetilde{F}_i$*, where* $n < m$ *and* $1 \leq i \leq m$*;*

(6) $\widetilde{C}_1, \widetilde{C}_2, \widetilde{F}_i$*, where* $1 < n < m$ *and* $1 \leq i \leq m$*;*

(7) $\widetilde{C}_2, \widetilde{F}_i, E_1, \ldots, \widehat{E}_i, \ldots, E_m$*, where* $n = 0$ *and* $1 \leq i \leq m$*.*

Proof. (1) The intersection matrix is $-I$, which is negative definite.

(2) The intersection matrix is $-I$, which is negative definite.

(3) The intersection matrix is

$$\begin{pmatrix} -n & 1 \\ 1 & -1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} -n & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix} \begin{pmatrix} -(n-1) & \\ & -1 \end{pmatrix} \begin{pmatrix} 1 & \\ -1 & 1 \end{pmatrix},$$

it is negative definite.

(4) The intersection matrix is

$$\begin{pmatrix} -n & 1 & \cdots & 1 \\ 1 & -1 & & \\ \vdots & & \ddots & \\ 1 & & & -1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} -n & 1 & \cdots & 1 \\ 1 & -1 & & \\ \vdots & & \ddots & \\ 1 & & & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & \cdots & -1 \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} -(n-m) & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ \vdots & & \ddots & \\ -1 & & & 1 \end{pmatrix},$$

it is negative definite.

(5) The intersection matrix is

$$\begin{pmatrix} -(m-n) & \\ & -1 \end{pmatrix},$$

which is negative definite.

(6) The intersection matrix is

$$\begin{pmatrix} -n & & 1 \\ & -(m-n) & \\ 1 & & -1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} -n & & 1 \\ & -(m-n) & \\ 1 & & -1 \end{pmatrix} = \begin{pmatrix} 1 & & -1 \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} -(n-1) & & \\ & -(m-n) & \\ & & -1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{pmatrix},$$

it is negative definite.

(7) We may assume $i = 1$. The intersection matrix is

$$\begin{pmatrix} -m & 0 & 1 & \cdots & 1 \\ 0 & -1 & & & \\ 1 & & -1 & & \\ \vdots & & & \ddots & \\ 1 & & & & -1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} -m & 0 & 1 & \cdots & 1 \\ 0 & -1 & & & \\ 1 & & -1 & & \\ \vdots & & & \ddots & \\ 1 & & & & -1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & -1 & \cdots & -1 \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{pmatrix} \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ -1 & & 1 & & \\ \vdots & & & \ddots & \\ -1 & & & & 1 \end{pmatrix},$$

it is negative definite. $\square$

5.4 Upper bounds for the delta-invariant

In this section, we give some upper bounds for the δ-invariant by computing the ratio

$$\frac{A_{S,\Delta_\beta}(E)}{S_L(E)}$$

for various divisors E. In particular, by letting $E = \widetilde{C}_1, \widetilde{C}_2$ we prove Theorem 5.4.5, and by letting $E = E_i$ we prove Theorem 5.4.11. They combine to give the necessary part of the main theorem.

Lemma 5.4.1. *The pseudo-effective threshold*

$$\tau_{-K_{S,\Delta_\beta}}\left(\widetilde{C}_1\right) = \beta_1 + \beta_2,$$

and

$$\mathrm{Vol}\Big(-K_{S,\Delta_\beta} - x\widetilde{C}_1\Big) = \begin{cases} \mathrm{Vol}\big(-K_{S,\Delta_\beta}\big) - 2(2-n\beta_1)x - nx^2, & x \in [0,\beta_1], \\ \\ (\beta_1+\beta_2-x)(4-(m-n)(x-\beta_1+\beta_2)), & x \in (\beta_1, \beta_1+\beta_2]. \end{cases}$$

Proof. By (5.4) and (5.5),

$$-K_{S,\Delta_\beta} - x\widetilde{C}_1 = \pi^*((\beta_1+\beta_2-x)C_2 + (2-n\beta_1+nx)F) - \beta_2\sum_i E_i$$

is nef when $x \in [0,\beta_1]$. Therefore by Proposition 2.2.3,

$$\mathrm{Vol}\Big(-K_{S,\Delta_\beta} - x\widetilde{C}_1\Big) = \mathrm{Vol}\big(-K_{S,\Delta_\beta}\big) - 2(2-n\beta_1)x - nx^2.$$

When x reaches β_1, the intersection number $(-K_{S,\Delta_\beta} - x\widetilde{C}_1)\cdot\widetilde{F}_i$ decreases to 0. When $x > \beta_1$, assume there is an effective $\mathbb{R}$-divisor N supported on $\bigcup_i \widetilde{F}_i$ that satisfies (2.2). We can solve

$$N = (x-\beta_1)\sum_i \widetilde{F}_i.$$

By (5.4) and (5.5),

$$-K_{S,\Delta_\beta} - x\widetilde{C}_1 - N = \pi^*((\beta_1+\beta_2-x)C_2 + (2+(m-n)(\beta_1-x))F) - (\beta_1+\beta_2-x)\sum_i E_i$$

is nef when $x \in (\beta_1, \beta_1+\beta_2]$. Also note Claim 5.3.1(1). By Definition 2.3.1, N is indeed the negative part of $-K_{S,\Delta_\beta} - x\widetilde{C}_1$ for $x \in (\beta_1, \beta_1+\beta_2]$. Therefore by Proposition 2.3.3,

$$\operatorname{Vol}\Big(-K_{S,\Delta_\beta} - x\widetilde{C}_1\Big) = (\beta_1+\beta_2-x)(4-(m-n)(x-\beta_1+\beta_2)).$$

When x reaches $\beta_1+\beta_2$, the volume $\operatorname{Vol}(-K_{S,\Delta_\beta} - x\widetilde{C}_1)$ decreases to 0. This implies

$$\tau_{-K_{S,\Delta_\beta}}\Big(\widetilde{C}_1\Big) = \beta_1+\beta_2.$$

□

Lemma 5.4.2. *The expected vanishing order*

$$S_{-K_{S,\Delta_\beta}}\Big(\widetilde{C}_1\Big) = \beta_1 - \frac{2}{\operatorname{Vol}\big(-K_{S,\Delta_\beta}\big)}\left(\beta_1^2 - \beta_2^2 + \frac{1}{3}\big(-n\beta_1^3 + (m-n)\beta_2^3\big)\right).$$

Proof. By Lemma 5.4.1,

$$\int_0^{\beta_1} \operatorname{Vol}\Big(-K_{S,\Delta_\beta} - x\widetilde{C}_1\Big)dx = \beta_1 \operatorname{Vol}\big(-K_{S,\Delta_\beta}\big) - 2\beta_1^2 + \frac{2}{3}n\beta_1^3,$$
$$\int_{\beta_1}^{\beta_1+\beta_2} \operatorname{Vol}\Big(-K_{S,\Delta_\beta} - x\widetilde{C}_1\Big)dx = 2\beta_2^2 - \frac{2}{3}(m-n)\beta_2^3.$$

Therefore

$$\begin{aligned} S_{-K_{S,\Delta_\beta}}\Big(\widetilde{C}_1\Big) &= \frac{1}{\operatorname{Vol}\big(-K_{S,\Delta_\beta}\big)} \int_0^{\beta_1+\beta_2} \operatorname{Vol}\Big(-K_{S,\Delta_\beta} - x\widetilde{C}_1\Big)dx \\ &= \beta_1 - \frac{2}{\operatorname{Vol}\big(-K_{S,\Delta_\beta}\big)}\left(\beta_1^2 - \beta_2^2 + \frac{1}{3}\big(-n\beta_1^3 + (m-n)\beta_2^3\big)\right). \end{aligned}$$

□

Lemma 5.4.3. *The pseudo-effective threshold*

$$\tau_{-K_{S,\Delta_\beta}}\left(\widetilde{C}_2\right) = \beta_1 + \beta_2,$$

and

$$\operatorname{Vol}\left(-K_{S,\Delta_\beta} - x\widetilde{C}_2\right) = \begin{cases} \operatorname{Vol}\left(-K_{S,\Delta_\beta}\right) - 2(2-(m-n)\beta_2)x - (m-n)x^2, & x \in [0,\beta_2], \\ (\beta_1+\beta_2-x)(4-n(x+\beta_1-\beta_2)), & x \in (\beta_2, \beta_1+\beta_2]. \end{cases}$$

Proof. By (5.4) and (5.5),

$$-K_{S,\Delta_\beta} - x\widetilde{C}_2 = \pi^*((\beta_1+\beta_2-x)C_2 + (2-n\beta_1)F) - (\beta_2 - x)\sum_i E_i$$

is nef when $x \in [0,\beta_2]$. Therefore by Proposition 2.2.3,

$$\operatorname{Vol}\left(-K_{S,\Delta_\beta} - x\widetilde{C}_2\right) = \operatorname{Vol}\left(-K_{S,\Delta_\beta}\right) - 2(2-(m-n)\beta_2)x - (m-n)x^2.$$

When x reaches β_2, the intersection number $(-K_{S,\Delta_\beta} - x\widetilde{C}_2)\cdot E_i$ decreases to 0. When $x > \beta_2$, assume there is an effective $\mathbb{R}$-divisor N supported on $\bigcup_i E_i$ that satisfies (2.2). We can solve

$$N = (x-\beta_2)\sum_i E_i.$$

By (5.4) and (5.5),

$$-K_{S,\Delta_\beta} - x\widetilde{C}_2 - N = \pi^*((\beta_1+\beta_2-x)C_2 + (2-n\beta_1)F)$$

is nef when $x \in (\beta_2, \beta_1+\beta_2]$. Also note Claim 5.3.1(2). By Definition 2.3.1, N is indeed the negative part of $-K_{S,\Delta_\beta} - x\widetilde{C}_2$ for $x \in (\beta_2, \beta_1+\beta_2]$. Therefore by Proposition 2.3.3,

$$\operatorname{Vol}\left(-K_{S,\Delta_\beta} - x\widetilde{C}_2\right) = (\beta_1+\beta_2-x)(4-n(x+\beta_1-\beta_2)).$$

When x reaches $\beta_1+\beta_2$, the volume $\mathrm{Vol}(-K_{S,\Delta_\beta}-x\widetilde{C}_2)$ decreases to 0. This implies

$$\tau_{-K_{S,\Delta_\beta}}\left(\widetilde{C}_2\right)=\beta_1+\beta_2.$$

□

Lemma 5.4.4. *The expected vanishing order*

$$S_{-K_{S,\Delta_\beta}}\left(\widetilde{C}_2\right)=\beta_2+\frac{2}{\mathrm{Vol}\big(-K_{S,\Delta_\beta}\big)}\left(\beta_1^2-\beta_2^2+\frac{1}{3}\big(-n\beta_1^3+(m-n)\beta_2^3\big)\right).$$

Proof. By Lemma 5.4.3,

$$\begin{aligned}\int_0^{\beta_2}\mathrm{Vol}\Big(-K_{S,\Delta_\beta}-x\widetilde{C}_2\Big)dx&=\beta_2\,\mathrm{Vol}\big(-K_{S,\Delta_\beta}\big)-2\beta_2^2+\frac{2}{3}(m-n)\beta_2^3,\\ \int_{\beta_2}^{\beta_1+\beta_2}\mathrm{Vol}\Big(-K_{S,\Delta_\beta}-x\widetilde{C}_2\Big)dx&=2\beta_1^2-\frac{2}{3}n\beta_1^3.\end{aligned}$$

Therefore

$$\begin{aligned}S_{-K_{S,\Delta_\beta}}\left(\widetilde{C}_1\right)&=\frac{1}{\mathrm{Vol}\big(-K_{S,\Delta_\beta}\big)}\int_0^{\beta_1+\beta_2}\mathrm{Vol}\Big(-K_{S,\Delta_\beta}-x\widetilde{C}_2\Big)dx\\&=\beta_2+\frac{2}{\mathrm{Vol}\big(-K_{S,\Delta_\beta}\big)}\left(\beta_1^2-\beta_2^2+\frac{1}{3}\big(-n\beta_1^3+(m-n)\beta_2^3\big)\right).\end{aligned}$$

□

Theorem 5.4.5. *If*

$$\beta_1^2-\beta_2^2+\frac{1}{3}\big(-n\beta_1^3+(m-n)\beta_2^3\big)=0,$$

then $\delta(S,\Delta_\beta)\leq 1$. *Otherwise* $\delta(S,\Delta_\beta)<1$.

Proof. By (2.25), $A_{S,\Delta_\beta}(\widetilde{C}_i)=1-\mathrm{ord}_{\widetilde{C}_i}\Delta_\beta=\beta_i$, for $i=1,2$. Recall Lemmas 5.4.2 and 5.4.4,

$$S_{-K_{S,\Delta_\beta}}\left(\widetilde{C}_i\right)=\begin{cases}\beta_1-\frac{2}{\mathrm{Vol}\left(-K_{S,\Delta_\beta}\right)}\big(\beta_1^2-\beta_2^2+\frac{1}{3}\big(-n\beta_1^3+(m-n)\beta_2^3\big)\big), & i=1,\\ \beta_2+\frac{2}{\mathrm{Vol}\left(-K_{S,\Delta_\beta}\right)}\big(\beta_1^2-\beta_2^2+\frac{1}{3}\big(-n\beta_1^3+(m-n)\beta_2^3\big)\big), & i=2.\end{cases}$$

By Definition 2.5.6,

$$\delta(S,\Delta_\beta) \le \min_{i=1,2} \frac{A_{S,\Delta_\beta}\left(\widetilde{C}_i\right)}{S_{-K_{S,\Delta_\beta}}\left(\widetilde{C}_i\right)}. \tag{5.6}$$

When

$$\beta_1^2 - \beta_2^2 + \frac{1}{3}\big(-n\beta_1^3 + (m-n)\beta_2^3\big) = 0,$$

(5.6) becomes $\delta(S,\Delta_\beta) \le 1$. Otherwise if

$$\beta_1^2 - \beta_2^2 + \frac{1}{3}\big(-n\beta_1^3 + (m-n)\beta_2^3\big) < 0,$$

then (5.6) gives $\delta(S,\Delta_\beta) \le \frac{A_{S,\Delta_\beta}(\widetilde{C}_1)}{S_{-K_{S,\Delta_\beta}}(\widetilde{C}_1)} < 1$; if

$$\beta_1^2 - \beta_2^2 + \frac{1}{3}\big(-n\beta_1^3 + (m-n)\beta_2^3\big) > 0,$$

then (5.6) gives $\delta(S,\Delta_\beta) \le \frac{A_{S,\Delta_\beta}(\widetilde{C}_2)}{S_{-K_{S,\Delta_\beta}}(\widetilde{C}_2)} < 1$. □

Recall that we assume $m \ge 1$, i.e., S contains some exceptional curve E_i.

Lemma 5.4.6. *Assume $n \ge m$. The pseudo-effective threshold*

$$\tau_{-K_{S,\Delta_\beta}}(E_i) = 2 + (n-m)\beta_2,$$

and

$$\mathrm{Vol}\big(-K_{S,\Delta_\beta} - xE_i\big) =$$
$$\begin{cases} 4(\beta_1+\beta_2) - n\beta_1^2 + (n-m)\beta_2^2 - 2\beta_2 x - x^2, & x \in [0,\beta_1], \\ 4(\beta_1+\beta_2) - (n-1)\beta_1^2 + (n-m)\beta_2^2 - 2(\beta_1+\beta_2)x, & x \in (\beta_1, 2-(n-1)\beta_1], \\ (n-1)\Big(\beta_2 + \frac{2-x}{n-1}\Big)^2 - (m-1)\beta_2^2, & x \in (2-(n-1)\beta_1, 2], \\ \frac{(2+(n-m)\beta_2 - x)^2}{n-m}, & x \in (2, 2+(n-m)\beta_2]. \end{cases}$$

Proof. By (5.4) and (5.5),

$$-K_{S,\Delta_\beta} - xE_i = \pi^*((\beta_1+\beta_2)C_2 + (2-n\beta_1)F) - (\beta_2+x)E_i - \sum_{j\neq i}\beta_2 E_j$$

is nef when $x \in [0,\beta_1]$. Therefore by Proposition 2.2.3,

$$\mathrm{Vol}\big(-K_{S,\Delta_\beta} - xE_i\big) = 4(\beta_1+\beta_2) - n\beta_1^2 + (n-m)\beta_2^2 - 2\beta_2 x - x^2.$$

When x reaches β_1, the intersection number $(-K_{S,\Delta_\beta} - xE_i)\cdot\widetilde{F}_i$ decreases to 0. When $x > \beta_1$, assume there is an effective $\mathbb{R}$-divisor N supported on $\widetilde{F}_i$ that satisfies (2.2). We can solve

$$N = (x-\beta_1)\widetilde{F}_i.$$

By (5.4) and (5.5),

$$-K_{S,\Delta_\beta} - xE_i - N = \pi^*((\beta_1+\beta_2)C_2 + (2-(n-1)\beta_1 - x)F) - (\beta_1+\beta_2)E_i - \sum_{j\neq i}\beta_2 E_j$$

is nef when $x \in (\beta_1, 2-(n-1)\beta_1]$. Also note Claim 5.3.1(1). By Definition 2.3.1, N is indeed the negative part of $-K_{S,\Delta_\beta} - xE_i$ for $x \in (\beta_1, 2-(n-1)\beta_1]$. Therefore by Proposition 2.3.3,

$$\mathrm{Vol}\big(-K_{S,\Delta_\beta} - xE_i\big) = 4(\beta_1+\beta_2) - (n-1)\beta_1^2 + (n-m)\beta_2^2 - 2(\beta_1+\beta_2)x.$$

When x reaches $2-(n-1)\beta_1$, the intersection number $(-K_{S,\Delta_\beta} - xE_i)\cdot\widetilde{C}_1$ decreases to 0. If $n>1$, when $x > 2-(n-1)\beta_1$, assume there is an effective $\mathbb{R}$-divisor N supported on $\widetilde{F}_i \cup \widetilde{C}_1$ that satisfies (2.2). We can solve

$$N = \left(\beta_1 - \frac{2-x}{n-1}\right)\widetilde{C}_1 + \frac{nx-2}{n-1}\widetilde{F}_i.$$

By (5.4) and (5.5),

$$-K_{S,\Delta_\beta} - xE_i - N = \pi^*\left(\left(\beta_2 + \frac{2-x}{n-1}\right)C_2\right) - \left(\beta_2 + \frac{2-x}{n-1}\right)E_i - \sum_{j\neq i}\beta_2 E_j$$

is nef when $x \in (2-(n-1)\beta_1, 2]$. Also note Claim 5.3.1(3). By Definition 2.3.1, N is indeed the negative part of $-K_{S,\Delta_\beta} - xE_i$ for $x \in (2-(n-1)\beta_1, 2]$. Therefore by Proposition 2.3.3,

$$\mathrm{Vol}\big(-K_{S,\Delta_\beta} - xE_i\big) = (n-1)\left(\beta_2 + \frac{2-x}{n-1}\right)^2 - (m-1)\beta_2^2.$$

When x reaches 2, the intersection number $(-K_{S,\Delta_\beta} - xE_i)\cdot \widetilde{F}_j$ decreases to 0 for $j \neq i$. If $n > m$, when $x > 2$, assume there is an effective $\mathbb{R}$-divisor N supported on $\widetilde{C}_1 \cup \bigcup_j \widetilde{F}_j$ that satisfies (2.2). We can solve

$$N = \left(\beta_1 + \frac{x-2}{n-m}\right)\widetilde{C}_1 + x\widetilde{F}_i + \sum_j \frac{x-2}{n-m}\widetilde{F}_j.$$

By (5.4) and (5.5),

$$-K_{S,\Delta_\beta} - xE_i - N = \pi^*\left(\left(\beta_2 - \frac{x-2}{n-m}\right)C_2\right) - \sum_j\left(\beta_2 - \frac{x-2}{n-m}\right)E_i$$

is nef when $x \in (2, 2+(n-m)\beta_2]$. Also note Claim 5.3.1(4). By Definition 2.3.1, N is indeed the negative part of $-K_{S,\Delta_\beta} - xE_i$ for $x \in (2, 2+(n-m)\beta_2]$. Therefore by Proposition 2.3.3,

$$\mathrm{Vol}\big(-K_{S,\Delta_\beta} - xE_i\big) = \frac{\left(2+(n-m)\beta_2 - x\right)^2}{n-m}.$$

When x reaches $2+(n-m)\beta_2$, the volume $\mathrm{Vol}(-K_{S,\Delta_\beta} - xE_i)$ decreases to 0. This implies

$$\tau_{-K_{S,\Delta_\beta}}(E_i) = 2+(n-m)\beta_2.$$

$\square$

Lemma 5.4.7. *Assume $n < m$ and $(n-1)\beta_1 \geq (m-n)\beta_2$. The pseudo-effective threshold*

$$\tau_{-K_{S,\Delta_\beta}}(E_i) = 2,$$

and

$$\mathrm{Vol}\big(-K_{S,\Delta_\beta} - xE_i\big) = \begin{cases} 4(\beta_1+\beta_2) - n\beta_1^2 + (n-m)\beta_2^2 - 2\beta_2 x - x^2, & x \in [0,\beta_1], \\ 4(\beta_1+\beta_2) - (n-1)\beta_1^2 + (n-m)\beta_2^2 - 2(\beta_1+\beta_2)x, & x \in (\beta_1, 2-(n-1)\beta_1], \\ (n-1)\Big(\beta_2 + \frac{2-x}{n-1}\Big)^2 - (m-1)\beta_2^2, & x \in (2-(n-1)\beta_1, 2-(m-n)\beta_2], \\ \frac{(m-1)}{(m-n)(n-1)}(x-2)^2, & x \in (2-(m-n)\beta_2, 2]. \end{cases}$$

Proof. By (5.4) and (5.5),

$$-K_{S,\Delta_\beta} - xE_i = \pi^*((\beta_1+\beta_2)C_2 + (2-n\beta_1)F) - (\beta_2+x)E_i - \sum_{j\neq i}\beta_2 E_j$$

is nef when $x \in [0,\beta_1]$. Therefore by Proposition 2.2.3,

$$\mathrm{Vol}\big(-K_{S,\Delta_\beta} - xE_i\big) = 4(\beta_1+\beta_2) - n\beta_1^2 + (n-m)\beta_2^2 - 2\beta_2 x - x^2.$$

When x reaches β_1, the intersection number $(-K_{S,\Delta_\beta} - xE_i)\cdot \widetilde{F}_i$ decreases to 0. When $x > \beta_1$, assume there is an effective $\mathbb{R}$-divisor N supported on $\widetilde{F}_i$ that satisfies (2.2). We can solve

$$N = (x-\beta_1)\widetilde{F}_i.$$

By (5.4) and (5.5),

$$-K_{S,\Delta_\beta} - xE_i - N = \pi^*((\beta_1+\beta_2)C_2 + (2-(n-1)\beta_1 - x)F) - (\beta_1+\beta_2)E_i - \sum_{j\neq i}\beta_2 E_j$$

is nef when $x \in (\beta_1, 2-(n-1)\beta_1]$. Also note Claim 5.3.1(1). By Definition 2.3.1, N is indeed the negative part of $-K_{S,\Delta_\beta} - xE_i$ for $x \in (\beta_1, 2-(n-1)\beta_1]$. Therefore by Proposition 2.3.3,

$$\mathrm{Vol}\big(-K_{S,\Delta_\beta} - xE_i\big) = 4(\beta_1+\beta_2) - (n-1)\beta_1^2 + (n-m)\beta_2^2 - 2(\beta_1+\beta_2)x.$$

When x reaches $2-(n-1)\beta_1$, the intersection number $(-K_{S,\Delta_\beta} - xE_i)\cdot \widetilde{C}_1$ decreases to 0. When $x > 2-(n-1)\beta_1$, assume there is an effective $\mathbb{R}$-divisor N supported on $\widetilde{F}_i \cup \widetilde{C}_1$ that satisfies (2.2). We can solve

$$N = \left(\beta_1 - \frac{2-x}{n-1}\right)\widetilde{C}_1 + \frac{nx-2}{n-1}\widetilde{F}_i.$$

By (5.4) and (5.5),

$$-K_{S,\Delta_\beta} - xE_i - N = \pi^*\left(\left(\beta_2 + \frac{2-x}{n-1}\right)C_2\right) - \left(\beta_2 + \frac{2-x}{n-1}\right)E_i - \sum_{j\neq i}\beta_2 E_j$$

is nef when $x \in (2-(n-1)\beta_1, 2-(m-n)\beta_2]$. Also note Claim 5.3.1(3). By Definition 2.3.1, N is indeed the negative part of $-K_{S,\Delta_\beta} - xE_i$ for $x \in (2-(n-1)\beta_1, 2-(m-n)\beta_2]$. Therefore by Proposition 2.3.3,

$$\mathrm{Vol}\big(-K_{S,\Delta_\beta} - xE_i\big) = (n-1)\left(\beta_2 + \frac{2-x}{n-1}\right)^2 - (m-1)\beta_2^2.$$

When x reaches $2-(m-n)\beta_2$, the intersection number $(-K_{S,\Delta_\beta} - xE_i)\cdot \widetilde{C}_2$ decreases to 0. When $x > 2-(m-n)\beta_2$, assume there is an effective $\mathbb{R}$-divisor N supported on $\widetilde{F}_i \cup \widetilde{C}_1 \cup \widetilde{C}_2$ that satisfies (2.2). We can solve

$$N = \left(\beta_1 - \frac{2-x}{n-1}\right)\widetilde{C}_1 + \frac{nx-2}{n-1}\widetilde{F}_i + \left(\beta_2 + \frac{x-2}{m-n}\right)\widetilde{C}_2.$$

By (5.4) and (5.5),

$$-K_{S,\Delta_\beta} - xE_i - N = \pi^*\left(\frac{m-1}{(n-1)(m-n)}(2-x)C_2\right) - \frac{m-1}{(n-1)(m-n)}(2-x)E_i - \sum_{j\neq i}\frac{2-x}{m-n}E_j$$

is nef when $x \in (2-(m-n)\beta_2, 2]$. Also note Claim 5.3.1(6). By Definition 2.3.1, N is indeed the negative part of $-K_{S,\Delta_\beta} - xE_i$ for $x \in (2-(m-n)\beta_2, 2]$. Therefore by Proposition 2.3.3,

$$\mathrm{Vol}\big(-K_{S,\Delta_\beta} - xE_i\big) = \frac{(m-1)}{(m-n)(n-1)}(x-2)^2.$$

When x reaches 2, the volume $\mathrm{Vol}(-K_{S,\Delta_\beta} - xE_i)$ decreases to 0. This implies

$$\tau_{-K_{S,\Delta_\beta}}(E_i) = 2.$$

□

Lemma 5.4.8. *Assume $(n-1)\beta_1 < (m-n)\beta_2$ and $n \geq 1$. The pseudo-effective threshold*

$$\tau_{-K_{S,\Delta_\beta}}(E_i) = 2,$$

and

$$\mathrm{Vol}\big(-K_{S,\Delta_\beta} - xE_i\big) = \begin{cases} 4(\beta_1+\beta_2) - n\beta_1^2 + (n-m)\beta_2^2 - 2\beta_2 x - x^2, & x \in [0, \beta_1], \\ 4(\beta_1+\beta_2) - (n-1)\beta_1^2 + (n-m)\beta_2^2 - 2(\beta_1+\beta_2)x, & x \in (\beta_1, 2-(m-n)\beta_2], \\ \frac{4+(m-n)(4-(n-1)\beta_1)\beta_1 - 2(2+(m-n)\beta_1)x + x^2}{m-n}, & x \in (2-(m-n)\beta_2, 2-(n-1)\beta_1], \\ \frac{(m-1)}{(m-n)(n-1)}(x-2)^2, & x \in (2-(n-1)\beta_1, 2]. \end{cases}$$

Proof. By (5.4) and (5.5),

$$-K_{S,\Delta_\beta} - xE_i = \pi^*((\beta_1+\beta_2)C_2 + (2-n\beta_1)F) - (\beta_2+x)E_i - \sum_{j\neq i}\beta_2 E_j$$

is nef when $x \in [0, \beta_1]$. Therefore by Proposition 2.2.3,

$$\mathrm{Vol}\big(-K_{S,\Delta_\beta} - xE_i\big) = 4(\beta_1+\beta_2) - n\beta_1^2 + (n-m)\beta_2^2 - 2\beta_2 x - x^2.$$

When x reaches β_1, the intersection number $(-K_{S,\Delta_\beta} - xE_i) \cdot \widetilde{F}_i$ decreases to 0. When $x > \beta_1$, assume there is an effective $\mathbb{R}$-divisor N supported on $\widetilde{F}_i$ that satisfies (2.2). We can solve

$$N = (x - \beta_1)\widetilde{F}_i.$$

By (5.4) and (5.5),

$$-K_{S,\Delta_\beta} - xE_i - N = \pi^*((\beta_1 + \beta_2)C_2 + (2 - (n-1)\beta_1 - x)F) - (\beta_1 + \beta_2)E_i - \sum_{j \neq i} \beta_2 E_j$$

is nef when $x \in (\beta_1, 2-(m-n)\beta_2]$. Also note Claim 5.3.1(1). By Definition 2.3.1, N is indeed the negative part of $-K_{S,\Delta_\beta} - xE_i$ for $x \in (\beta_1, 2-(m-n)\beta_2]$. Therefore by Proposition 2.3.3,

$$\mathrm{Vol}\big(-K_{S,\Delta_\beta} - xE_i\big) = 4(\beta_1 + \beta_2) - (n-1)\beta_1^2 + (n-m)\beta_2^2 - 2(\beta_1 + \beta_2)x.$$

When x reaches $2 - (m-n)\beta_2$, the intersection number $(-K_{S,\Delta_\beta} - xE_i) \cdot \widetilde{C}_2$ decreases to 0. When $x > 2 - (m-n)\beta_2$, assume there is an effective $\mathbb{R}$-divisor N supported on $\widetilde{F}_i \cup \widetilde{C}_2$ that satisfies (2.2). We can solve

$$N = \left(\beta_2 - \frac{2-x}{m-n}\right)\widetilde{C}_2 + (x - \beta_1)\widetilde{F}_i.$$

By (5.4) and (5.5),

$$\begin{aligned} -K_{S,\Delta_\beta} - xE_i - N = \pi^*\left(\left(\beta_1 + \frac{2-x}{m-n}\right)C_2 + (2-(n-1)\beta_1 - x)F\right) - \left(\beta_1 + \frac{2-x}{m-n}\right)E_i \\ - \sum_{j \neq i} \frac{2-x}{m-n} E_j \end{aligned}$$

is nef when $x \in (2-(m-n)\beta_2, 2-(n-1)\beta_1]$. Also note Claim 5.3.1(5). By Definition 2.3.1, N is indeed the negative part of $-K_{S,\Delta_\beta} - xE_i$ for $x \in (2-(m-n)\beta_2, 2-(n-1)\beta_1]$. Therefore by Proposition 2.3.3,

$$\mathrm{Vol}\big(-K_{S,\Delta_\beta} - xE_i\big) = \frac{4 + (m-n)(4-(n-1)\beta_1)\beta_1 - 2(2+(m-n)\beta_1)x + x^2}{m-n}.$$

When x reaches $2-(n-1)\beta_1$, the intersection number $(-K_{S,\Delta_\beta}-xE_i)\cdot\widetilde{C}_1$ decreases to 0. If $n>1$, when $x>2-(n-1)\beta_1$, assume there is an effective $\mathbb{R}$-divisor N supported on $\widetilde{F}_i\cup\widetilde{C}_1\cup\widetilde{C}_2$ that satisfies (2.2). We can solve

$$N=\left(\beta_1-\frac{2-x}{n-1}\right)\widetilde{C}_1+\left(\beta_2-\frac{2-x}{m-n}\right)\widetilde{C}_2+\frac{nx-2}{n-1}\widetilde{F}_i.$$

By (5.4) and (5.5),

$$-K_{S,\Delta_\beta}-xE_i-N=\pi^*\left(\frac{m-1}{(n-1)(m-n)}(2-x)C_2\right)-\frac{m-1}{(n-1)(m-n)}(2-x)E_i-\sum_{j\neq i}\frac{2-x}{m-n}E_j$$

is nef when $x\in(2-(n-1)\beta_1,2]$. Also note Claim 5.3.1(6). By Definition 2.3.1, N is indeed the negative part of $-K_{S,\Delta_\beta}-xE_i$ for $x\in(2-(n-1)\beta_1,2]$. Therefore by Proposition 2.3.3,

$$\operatorname{Vol}\big(-K_{S,\Delta_\beta}-xE_i\big)=\frac{(m-1)}{(m-n)(n-1)}(x-2)^2.$$

When x reaches 2, the volume $\operatorname{Vol}(-K_{S,\Delta_\beta}-xE_i)$ decreases to 0. This implies

$$\tau_{-K_{S,\Delta_\beta}}(E_i)=2.$$

$\square$

Lemma 5.4.9. *Assume $n=0$. The pseudo-effective threshold*

$$\tau_{-K_{S,\Delta_\beta}}(E_i)=2+\beta_1,$$

and

$$\operatorname{Vol}\big(-K_{S,\Delta_\beta}-xE_i\big)=\begin{cases}4(\beta_1+\beta_2)-m\beta_2^2-2\beta_2x-x^2, & x\in[0,\beta_1],\\ 4(\beta_1+\beta_2)+\beta_1^2-m\beta_2^2-2(\beta_1+\beta_2)x, & x\in(\beta_1,2-m\beta_2],\\ \frac{4+m(4+\beta_1)\beta_1-2(2+m\beta_1)x+x^2}{m}, & x\in(2-m\beta_2,2],\\ (2+\beta_1-x)^2, & x\in(2,2+\beta_1].\end{cases}$$

Proof. By (5.4) and (5.5),

$$-K_{S,\Delta_\beta} - xE_i = \pi^*((\beta_1+\beta_2)C_2 + 2F) - (\beta_2 + x)E_i - \sum_{j\neq i} \beta_2 E_j$$

is nef when $x \in [0, \beta_1]$. Therefore by Proposition 2.2.3,

$$\mathrm{Vol}\big(-K_{S,\Delta_\beta} - xE_i\big) = 4(\beta_1+\beta_2) - m\beta_2^2 - 2\beta_2 x - x^2.$$

When x reaches β_1, the intersection number $(-K_{S,\Delta_\beta} - xE_i) \cdot \widetilde{F}_i$ decreases to 0. When $x > \beta_1$, assume there is an effective $\mathbb{R}$-divisor N supported on $\widetilde{F}_i$ that satisfies (2.2). We can solve

$$N = (x - \beta_1)\widetilde{F}_i.$$

By (5.4) and (5.5),

$$-K_{S,\Delta_\beta} - xE_i - N = \pi^*((\beta_1+\beta_2)C_2 + (2+\beta_1 - x)F) - (\beta_1+\beta_2)E_i - \sum_{j\neq i} \beta_2 E_j$$

is nef when $x \in (\beta_1, 2 - m\beta_2]$. Also note Claim 5.3.1(1). By Definition 2.3.1, N is indeed the negative part of $-K_{S,\Delta_\beta} - xE_i$ for $x \in (\beta_1, 2 - m\beta_2]$. Therefore by Proposition 2.3.3,

$$\mathrm{Vol}\big(-K_{S,\Delta_\beta} - xE_i\big) = 4(\beta_1+\beta_2) + \beta_1^2 - m\beta_2^2 - 2(\beta_1+\beta_2)x.$$

When x reaches $2 - m\beta_2$, the intersection number $(-K_{S,\Delta_\beta} - xE_i) \cdot \widetilde{C}_2$ decreases to 0. When $x > 2 - m\beta_2$, assume there is an effective $\mathbb{R}$-divisor N supported on $\widetilde{F}_i \cup \widetilde{C}_2$ that satisfies (2.2). We can solve

$$N = \left(\beta_2 - \frac{2-x}{m}\right)\widetilde{C}_2 + (x - \beta_1)\widetilde{F}_i.$$

By (5.4) and (5.5),

$$-K_{S,\Delta_\beta} - xE_i - N = \pi^*\Big(\Big(\beta_1 + \frac{2-x}{m}\Big)C_2 + (2+\beta_1 - x)F\Big) - \Big(\beta_1 + \frac{2-x}{m}\Big)E_i - \sum_{j\neq i}\frac{2-x}{m}E_j$$

is nef when $x \in (2 - m\beta_2, 2]$. Also note Claim 5.3.1(5). By Definition 2.3.1, N is indeed the negative part of $-K_{S,\Delta_\beta} - xE_i$ for $x \in (2-m\beta_2, 2]$. Therefore by Proposition 2.3.3,

$$\mathrm{Vol}\big(-K_{S,\Delta_\beta} - xE_i\big) = \frac{4 + m(4+\beta_1)\beta_1 - 2(2+m\beta_1)x + x^2}{m}.$$

When x reaches 2, the intersection number $(-K_{S,\Delta_\beta} - xE_i)\cdot E_j$ decreases to 0 for $j \neq i$. When $x > 2$, assume there is an effective $\mathbb{R}$-divisor N supported on $\widetilde{F}_i \cup \widetilde{C}_2 \cup \bigcup_{j\neq i} E_j$ that satisfies (2.2). We can solve

$$N = (\beta_2 + x - 2)\widetilde{C}_2 + (x-\beta_1)\widetilde{F}_i + \sum_{j\neq i}(x-2)E_j.$$

By (5.4) and (5.5),

$$-K_{S,\Delta_\beta} - xE_i - N = \pi^*((2+\beta_1-x)C_2 + (2+\beta_1-x)F) - (2+\beta_1-x)E_i$$

is nef when $x \in (2, 2+\beta_1]$. Also note Claim 5.3.1(7). By Definition 2.3.1, N is indeed the negative part of $-K_{S,\Delta_\beta} - xE_i$ for $x \in (2, 2+\beta_1]$. Therefore by Proposition 2.3.3,

$$\mathrm{Vol}\big(-K_{S,\Delta_\beta} - xE_i\big) = (2+\beta_1-x)^2.$$

When x reaches $2+\beta_1$, the volume $\mathrm{Vol}(-K_{S,\Delta_\beta} - xE_i)$ decreases to 0. This implies

$$\tau_{-K_{S,\Delta_\beta}}(E_i) = 2 + \beta_1.$$

□

Lemma 5.4.10. *The expected vanishing order*

$$S_{-K_{S,\Delta_\beta}}(E_i) = 1 + \frac{1}{\mathrm{Vol}\big(-K_{S,\Delta_\beta}\big)}\left(-(n-2)\beta_1^2 + (n-m)\beta_2^2 + \frac{n(n-2)\beta_1^3 + (n-m)^2\beta_2^3}{3}\right).$$

Proof. By Lemmas 5.4.6, 5.4.7, 5.4.8 and 5.4.9,

$$\begin{aligned} S_{-K_{S,\Delta_\beta}}(E_i) &= \frac{1}{\mathrm{Vol}\big(-K_{S,\Delta_\beta}\big)}\int_0^{\tau_{-K_{S,\Delta_\beta}}(E_i)} \mathrm{Vol}\big(-K_{S,\Delta_\beta} - xE_i\big)dx \\ &= 1 + \frac{1}{\mathrm{Vol}\big(-K_{S,\Delta_\beta}\big)}\left(-(n-2)\beta_1^2 + (n-m)\beta_2^2 + \frac{n(n-2)\beta_1^3 + (n-m)^2\beta_2^3}{3}\right). \end{aligned}$$

□

Theorem 5.4.11. *Assume* $\beta_1^2 - \beta_2^2 - \frac{1}{3}\big(n\beta_1^3 + (n-m)\beta_2^3\big) = 0$*. If* $m = 1$*, then* $\delta(S, \Delta_\beta) < 1$*.*

Proof. By (2.25), $A_{S,\Delta_\beta}(E_i) = 1 - \mathrm{ord}_{E_i}\Delta_\beta = 1$. Recall Lemma 5.4.10,

$$S_{-K_{S,\Delta_\beta}}(E_i) = 1 + \frac{1}{\mathrm{Vol}\big(-K_{S,\Delta_\beta}\big)}(2-m)\beta_1^2\Big(1 - \frac{n}{3}\beta_1\Big).$$

By definition 2.5.6, when $m = 1$,

$$\delta(S, \Delta_\beta) \leq \frac{A_{S,\Delta_\beta}(E_i)}{S_{-K_{S,\Delta_\beta}}(E_i)} < 1.$$

□

Theorems 5.4.5 and 5.4.11 combined imply the necessary part of the main theorem.

Theorem 5.4.12. *Consider* (II.6A.n.m) *as in Definition 5.0.6. We have*

$$\delta\Big(S, (1-\beta_1)\widetilde{C}_1 + (1-\beta_2)\widetilde{C}_2\Big) \leq 1,$$

with equality only if

$$\beta_1^2 + \frac{1}{3}\widetilde{C}_1^2\beta_1^3 = \beta_2^2 + \frac{1}{3}\widetilde{C}_2^2\beta_2^3$$

and

$$m \neq 1.$$

Proof. By Theorem 5.4.5,

$$\delta(S, \Delta_\beta) \leq 1.$$

If equality holds, by Theorem 5.4.5,

$$\beta_1^2 + \frac{1}{3}\widetilde{C}_1^2\beta_1^3 = \beta_2^2 + \frac{1}{3}\widetilde{C}_2^2\beta_1^3;$$

by Theorem 5.4.11,

$$m \neq 1.$$

□

5.5 Rational T-varieties of complexity 1

In this section we collect some definitions and properties of rational T-varieties of complexity 1 that will be used in the proof of the main theorem.

5.5.1 Construction of rational T-varieties of complexity 1

We first recall some definitions for T-varieties [2, Sections 2.2 and 2.3], in particular, rational T-varieties of complexity 1 [118, Section 1].

Definition 5.5.1. Let N be a lattice and M its dual lattice. We say $\Delta \subseteq N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ is a *polyhedron* if it is a finite intersection of half-spaces. Its *tailcone* is defined by

$$\mathrm{tail}(\Delta) := \{a \in N_{\mathbb{Q}} \mid a + \Delta \subseteq \Delta\}.$$

A cone is always assumed to be *pointed*, i.e., it does not contain non-trivial linear subspaces. Fix a cone σ, a formal sum

$$\mathcal{D} = \sum_{p \in \mathbb{P}^1} \mathcal{D}_p \otimes p$$

is called a *polyhedral divisor* on $\mathbb{P}^1$ with $\mathrm{tail}(\mathcal{D}) := \sigma$ if each coefficient $\mathcal{D}_p$ is either a polyhedron with

$\operatorname{tail}(\mathcal{D}_p) = \sigma$, or the empty set, and all but finitely many coefficients are σ. Define its *locus*

$$\operatorname{Loc}\mathcal{D} := \mathbb{P}^1 \setminus \bigcup_{\mathcal{D}_p = \varnothing} p$$

and *degree*

$$\deg \mathcal{D} := \sum_{p \in \mathbb{P}^1} \mathcal{D}_p.$$

Here the sum is in the sense of Minkowski sum, i.e.,

$$\sum_{i=1}^{N} \Delta_i = \left\{ \sum_{i=1}^{N} a_i \,\middle|\, a_i \in \Delta_i \right\}.$$

Note that if the tailcone of a polyhedron Δ is σ, then $\Delta \subseteq \Delta + \sigma \subseteq \Delta$, i.e., $\Delta + \sigma = \Delta$. Hence the definition of $\deg \mathcal{D}$ makes sense. Indeed,

$$\sum_{p \in \mathbb{P}^1} \mathcal{D}_p = \sum_{\mathcal{D}_p \neq \sigma} \mathcal{D}_p$$

is a finite sum, as any polyhedron $\mathcal{D}_p$ equaling σ can be ignored. By convention, if $\mathcal{D}_p = \sigma$ for all p, then $\deg \mathcal{D} = \sigma$. We assume $\mathcal{D}$ is a *p-divisor*, that is, $\deg \mathcal{D} \subsetneq \sigma$. A ray ρ of σ is called *extremal* if $\rho \cap \deg \mathcal{D} = \varnothing$. For $u \in \sigma^\vee \cap M$, the evaluation

$$\mathcal{D}(u) := \sum_{p \in \operatorname{Loc}\mathcal{D}} \min_{v \in \mathcal{D}_p} \langle u, v \rangle \cdot p$$

is a $\mathbb{Q}$-divisor on $\operatorname{Loc}\mathcal{D}$. Consider the graded $\mathcal{O}_{\operatorname{Loc}\mathcal{D}}$-algebra

$$\mathcal{A}(\mathcal{D}) := \bigoplus_{u \in \sigma^\vee \cap M} \mathcal{O}_{\operatorname{Loc}\mathcal{D}}(\mathcal{D}(u))\chi^u.$$

We can define two T-varieties

$$\operatorname{TV}(\mathcal{D}) := \mathbf{Spec}\, \mathcal{A}(\mathcal{D}), \quad \operatorname{tv}(\mathcal{D}) := \operatorname{Spec} H^0(\operatorname{Loc}\mathcal{D}, \mathcal{A}(\mathcal{D})),$$

equipped with an action of $T := \operatorname{Spec}\mathbb{C}[M]$ induced by the action of $\mathbb{C}[M]$ on $\mathcal{A}(\mathcal{D})$.

Theorem 5.5.2. [83, Proposition 3.13], [118, Section 1]. *There are two types of T-invariant prime divisors on* $\mathrm{TV}(D)$*: vertical and horizontal divisors. Vertical divisors are classified by pairs (p, v), where v is a vertex of $\mathcal{D}_p$. Such divisor is denoted $D_{p,v}$. Horizontal divisors are classified by rays ρ of σ. Such divisor is denoted D_ρ. The natural morphism $r : \mathrm{TV}(D) \to \mathrm{tv}(D)$ is a proper birational T-equivariant contraction morphism which contracts any horizontal divisor D_ρ where ρ is not extremal. The other divisors survive and are precisely the T-invariant prime divisors on* $\mathrm{tv}(D)$.

Similar to the toric case, general T-varieties are obtained by gluing affine T-varieties.

Definition 5.5.3. [2, Definition 13], [118, Definition 1.4]. For two p-divisors $\mathcal{D} = \sum_{p \in \mathbb{P}^1} \mathcal{D}_p \otimes p$ and $\mathcal{D}' = \sum_{p \in \mathbb{P}^1} \mathcal{D}'_p \otimes p$, define their intersection

$$\mathcal{D} \cap \mathcal{D}' := \sum_{p \in \mathbb{P}^1} \left(\mathcal{D}_p \cap \mathcal{D}'_p\right) \otimes p.$$

If $\mathcal{D}' \subseteq \mathcal{D}$, then there is an induced map $\mathrm{tv}(\mathcal{D}') \to \mathrm{tv}(\mathcal{D})$. If this map is an open embedding, then $\mathcal{D}'$ is called a *face* of $\mathcal{D}$. A finite set $\mathcal{S} = \{\mathcal{D}^i\}$ of p-divisors on $\mathbb{P}^1$ is called a *divisorial fan* if the intersection of two p-divisors from $\mathcal{S}$ is a common face of both and also belongs to $\mathcal{S}$. For $p \in \mathbb{P}^1$, define a *slice* $\mathcal{S}_p := \{\mathcal{D}^i_p\}$. $\mathcal{S}$ is called *complete* if each slice is a complete polyhedral subdivision of $N_\mathbb{Q}$.

Example 5.5.4. Consider the toric variety $\mathbb{F}_n$. We restrict its torus action to that of a subtorus, consisting of the actions that fix the fibers. This induces the structure of a rational T-variety of complexity 1. For such T-varieties, we can use toric downgrade to obtain its divisorial fan [83, Remark 2.8] (see also [118, Section 4.1]). See Figure 5.2.

Start from the fan corresponding to the toric variety $\mathbb{F}_n$. As explained in Figure 5.2, let $pr_k : N_\mathbb{Q} \to N_\mathbb{Q}$ be the projection map onto the k-th variable, $k = 1, 2$. The fan corresponding to $\mathbb{F}_n$ is mapped via pr_2 to the fan corresponding to $\mathbb{P}^1$, which has two rays corresponding to the two torus-invariant divisors $D_1 = 0$ and $D_{-1} = \infty$, with primitive generators 1 and -1, respectively. For each 2-dimensional cone σ_i, define a polyhedral divisor $\mathcal{D}_i$, whose coefficient over D_j is the polyhedron $pr_1(\sigma_i \cap pr_2^{-1}(D_j))$, $j = \pm 1$.

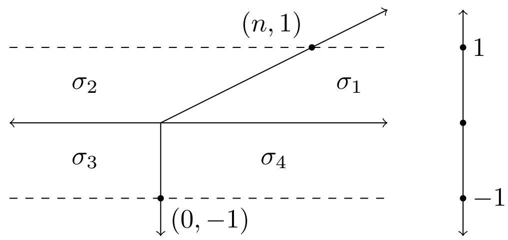

Figure 5.2: The downgrading of the torus action on $\mathbb{F}_n$. The projection to the second variable maps the fan corresponding to $\mathbb{F}_n$ to the fan corresponding to the base $\mathbb{P}^1$. Let $0, \infty \in \mathbb{P}^1$ denote the torus-invariant divisors corresponding to the rays with primitive generators 1 and -1, respectively. Then each 2-dimensional cone σ_i gives rise to a polyhedral divisor $\mathcal{D}_i$ whose coefficient over a torus-invariant divisor is the intersection of that cone and its preimage under the projection map.

That is,

$$\mathcal{D}_1 = [n, +\infty) \otimes 0 + \varnothing \otimes \infty,$$

$$\mathcal{D}_2 = (-\infty, n] \otimes 0 + \varnothing \otimes \infty,$$

$$\mathcal{D}_3 = \varnothing \otimes 0 + (-\infty, 0] \otimes \infty,$$

$$\mathcal{D}_4 = \varnothing \otimes 0 + [0, +\infty) \otimes \infty.$$

Note that $\deg \mathcal{D}_i = \varnothing$ for any i. These polyhedral divisors now give rise to a divisorial fan corresponding to $\mathbb{F}_n$.

Example 5.5.5. Recall how we blow up a torus-invariant point on a smooth toric variety. We can also interpret it under toric downgrade. See Figure 5.3. The cone $\sigma = \sigma_1 \cup \sigma_2$ corresponds to $\mathbb{C}^2$, while the cones σ_1 and σ_2 give rise to a fan that corresponds to $Bl_0\mathbb{C}^2$. After the blow-up, we replace the polyhedral divisor corresponding to σ by two polyhedral divisors corresponding to σ_1 and σ_2, whose coefficients over $0 \in \mathbb{C}$ are $[0, 1]$ and $[1, +\infty)$, respectively. Note that since blow-ups are local, similar construction applies if we replace the base to some open neighborhood $U \subseteq \mathbb{C}$ of 0.

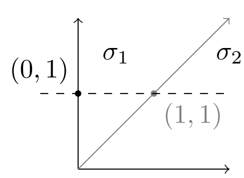

Figure 5.3: The blow-up of $\mathbb{C}^2$ at 0. Since the cone $\sigma = \sigma_1 \cup \sigma_2$ is replaced by the cones σ_1 and σ_2, we replace the polyhedral divisor corresponding to σ with two polyhedral divisors, whose coefficients over 0 are $[0, 1]$ and $[1, +\infty)$, respectively.

Remark 5.5.6. [2, Section 4.4]. In general, the slices of a divisorial fan $\mathcal{S}$ do not capture the entire information of $\mathcal{S}$, since the information of which polyhedra from different slices belong to the same p-divisor is lost. However, in complexity 1, with the additional information of the degree, this problem can be overcome, which leads to the definition of f-divisors.

Definition 5.5.7. [2, Definition 19]. Consider a pair $(\sum_{p\in\mathbb{P}^1} \mathcal{S}_p \otimes p, \mathfrak{deg})$ where each $\mathcal{S}_p$ is a polyhedral subdivision with common tailfan Σ and $\mathfrak{deg} \subseteq |\Sigma|$. For any full-dimensional $\sigma \in \operatorname{tail}(\mathcal{S})$ that intersects $\mathfrak{deg}$, let Δ_p^σ be the unique polyhedron in $\mathcal{S}_p$ with tailcone σ, and $\mathcal{D}^\sigma := \sum_{p\in\mathbb{P}^1} \Delta_p^\sigma$. The pair $(\sum_{p\in\mathbb{P}^1} \mathcal{S}_p \otimes p, \mathfrak{deg})$ is called an *f-divisor* if each $\mathcal{D}^\sigma$ is a p-divisor with $\deg \mathcal{D}^\sigma = \mathfrak{deg} \cap \sigma$.

Theorem 5.5.8. [2, Proposition 20]. *For any f-divisor* $(\sum_{p\in\mathbb{P}^1} \mathcal{S}_p \otimes p, \mathfrak{deg})$*, there is a divisorial fan* $\mathcal{S}$ *with slices* $\mathcal{S}_p$ *and degree* $\mathfrak{deg}$*. Moreover, the T-variety constructed from* $\mathcal{S}$ *is independent of the choice of* $\mathcal{S}$*.*

Let X be a rational T-variety of complexity 1, characterized by a complete divisorial fan $\mathcal{S}$. By [83, Section 3.1], a T-invariant Cartier divisor is uniquely characterized by its divisorial support function $h = \sum_{p\in\mathbb{P}^1} h_p \otimes p$, where each h_p is defined on the slice $\mathcal{S}_p$ and is affine on each polyhedron. Moreover, the linear part h_p on a polyhedron depends only on its tailcone but not on p. The corresponding Weil divisor is given by the following theorem.

Theorem 5.5.9. [83, Corollary 3.19]. *The Weil divisor corresponding to the divisorial support function* h *is given by*

$$-\sum_{\rho} h_t(n_\rho) D_\rho - \sum_{(p,v)} \mu(v) h_p(v) D_{p,v},$$

where h_t *is the linear part of* h*,* n_ρ *is the primitive generator of the extremal ray* ρ*, and* $\mu(v)$ *is the smallest natural number such that* $\mu(v)v$ *is a lattice point.*

Theorem 5.5.10. [83, Theorem 3.21]. *Fix an anti-canonical divisor* $-K_{\mathbb{P}^1} = \sum_{p\in\mathbb{P}^1} a_p \cdot p$*. Then the anti-canonical divisor*

$$-K_X = \sum_{\rho} D_\rho + \sum_{(p,v)} (\mu(v)a_p + 1 - \mu(v)) D_{p,v}.$$

Definition 5.5.11. [83, Definition 3.22]. For a divisorial support function h with linear part h_t, define

$$\square_h := \bigcap_{\rho} \{u \in M_{\mathbb{R}} \mid \langle u, n_\rho \rangle \geq h_t(n_\rho)\},$$

where ρ runs through all rays in $\mathrm{tail}(\mathcal{S})$. The *dual* function of h is defined to be

$$\Psi := \sum_{p \in \mathbb{P}^1} \Psi_p \otimes p,$$

where

$$\Psi_p(u) := h_p^*(u) = \min_v(\langle u, v\rangle - h_p(v)\}$$

is the Legendre transform of h_p on $\square_h$. Note that it suffices to let v run through all vertices in $\mathcal{S}_p$. We also define its degree

$$\deg \Psi := \sum_{p \in \mathbb{P}^1} \Psi_p.$$

5.5.2 δ-invariant and K-stability

In this section, we provide results relevant to the δ-invariant and K-stability.

Theorem 5.5.12. [118, Section 3], [99, Theorem 3.15]. *Let X be a rational Fano T-variety of complexity 1 with log terminal singularities, characterized by a complete divisorial fan $\mathcal{S}$. Let h be the divisorial support function corresponding to $-K_X$ specified by Theorem 5.5.10, and Ψ its dual function. Define*

$$\begin{aligned} \mathrm{Vol}(\Psi) &:= \int_{\square_h} \deg \Psi(u) du, \\ \mathrm{Bc}(\Psi) &:= \frac{1}{\mathrm{Vol}(\Psi)} \int_{\square_h} u \cdot \deg \Psi(u) du, \\ \mathrm{Bc}_p &:= \frac{1}{\mathrm{Vol}(\Psi)} \int_{\square_h} \left(\frac{1}{2}\deg \Psi(u) - \Psi_p(u)\right) \deg \Psi(u) du. \end{aligned} \tag{5.7}$$

Then

$$\delta(X) = \min\left\{\frac{-h_t(n_\rho)}{\langle \mathrm{Bc}(\Psi), n_\rho\rangle - h_t(n_\rho)}\right\}_\rho \cup \left\{\frac{1}{\mu(v)(\mathrm{Bc}_p + \langle \mathrm{Bc}(\Psi), v\rangle - h_p(v))}\right\}_{(p,v)},$$

where each term corresponds to the valuation given by D_ρ and $D_{p,v}$.

The Futaki character vanishes precisely when $\mathrm{Bc}(\Psi) = 0$.

Theorem 5.5.13. [3, Theorem 1.31], [72, Theorem 3.2]. *Let (X, Δ) be log Fano with Kawamata log terminal singularities. Let T be the maximal torus in* $\mathrm{Aut}(X, \Delta)$. *Suppose* $\dim T = \dim X - 1$. *Then*

(X,Δ) is log K-polystable if and only if the Futaki character vanishes, and for any T-invariant prime divisor E on X,

$$\frac{A_{X,\Delta}(E)}{S_{-K_{X,\Delta}}(E)} \geq 1,$$

with equality if and only if E is horizontal.

If (X,Δ) is toric, recall the following theorem.

Theorem 5.5.14. [14, 16] *Let (X,Δ) be log Fano and toric. Then it is log K-polystable if and only if for any torus-invariant prime divisor E on X,*

$$\frac{A_{X,\Delta}(E)}{S_{-K_{X,\Delta}}(E)} = 1.$$

5.6 Proof of main theorem

In this section we make use of the T-variety structure of the pairs to compute the δ-invariant, independent of the computation from Section 5.4, and prove Theorem 5.0.8.

Start from the toric variety $\mathbb{F}_n$. The $\mathbb{C}^*$-action along the fibers can be lifted to S, equipping the surface with the structure of a T-variety of complexity 1. We choose the f-divisor $(\sum_{i=0}^{m} \mathcal{S}_i \otimes p_i, \mathfrak{deg})$ on $\mathbb{P}^1$, where

$$\mathcal{S}_i = \begin{cases} \{(-\infty, n], [n, +\infty)\}, & i = 0, \\ \{(-\infty, -1], [-1, 0], [0, +\infty)\}, & i \geq 1, \end{cases}$$

(intersections omitted), and $\mathfrak{deg} = \varnothing$, as explained in Figure 5.4. By Example 5.5.4, when $m = 0$, this f-divisor corresponds to $\mathbb{F}_n$. By Example 5.5.5, for general m, this f-divisor corresponds to S.

Recall Theorem 5.5.10. Consider

$$-K_S = \widetilde{C}_1 + \widetilde{C}_2 + 2\pi^* F,$$

$$-K_{S,\Delta_\beta} = \beta_1 \widetilde{C}_1 + \beta_2 \widetilde{C}_2 + 2\pi^* F.$$

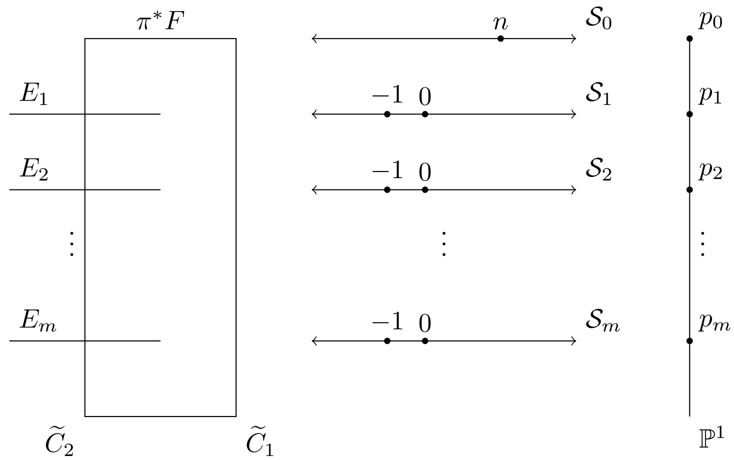

Figure 5.4: An f-divisor corresponding to S. The rays $[0,+\infty)$ and $(-\infty,0]$ correspond to the horizontal divisors $\widetilde{C}_1$ and $\widetilde{C}_2$, respectively. The vertices correspond to vertical divisors. The vertex n on $\mathcal{S}_0$ correspond to the fiber over p_0. The vertices -1 and 0 on $\mathcal{S}_i$ correspond to E_i and $\widetilde{F}_i$, respectively. The vertex 0 on any trivial slice over $p \in \mathbb{P}^1$ correspond to the fiber over p.

By Theorem 5.5.9, $-K_{S,\Delta_\beta}$ corresponds to the divisorial support function h with linear part

$$h_t(v) = \begin{cases} \beta_2 v, & v < 0, \\ -\beta_1 v, & v \geq 0, \end{cases}$$

and satisfies

$$h_{p_0}(n) = -2, \quad h_{p_1}(-1) = h_{p_1}(0) = \cdots = h_{p_m}(-1) = h_{p_m}(0) = 0,$$

as shown in Figure 5.5.

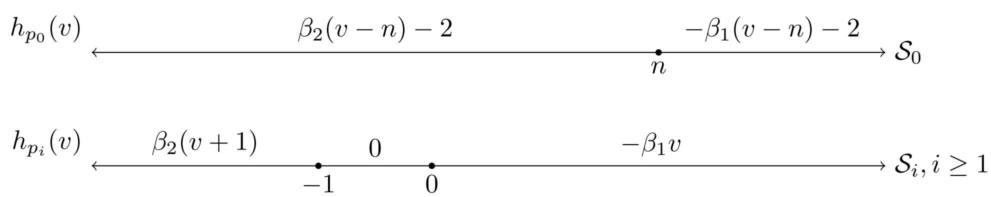

Figure 5.5: The divisorial support function h corresponding to the log anti-canonical divisor $-K_{S,\Delta_\beta}$.

By Definition 5.5.11, the dual function Ψ of h is given by

$$\Psi_{p_i}(u) = \begin{cases} nu+2, & i=0, \\ \min\{0,-u\}, & i\geq 1, \end{cases}$$

where the domain $\square_h = [-\beta_1, \beta_2]$, and its degree

$$\deg \Psi(u) = \begin{cases} nu+2, & u<0, \\ (n-m)u+2, & u\geq 0. \end{cases}$$

By (5.7), we compute its volume

$$\mathrm{Vol}(\Psi) = 2(\beta_1+\beta_2) - \frac{n}{2}\beta_1^2 + \frac{n-m}{2}\beta_2^2,$$

and barycenter

$$\mathrm{Bc}(\Psi) = -\frac{2}{\mathrm{Vol}\big(-K_{S,\Delta_\beta}\big)}\Big(\beta_1^2 - \beta_2^2 - \frac{1}{3}\big(n\beta_1^3 + (n-m)\beta_2^3\big)\Big),$$

$$\mathrm{Bc}_P(\Psi) = \begin{cases} -\frac{2}{\mathrm{Vol}\left(-K_{S,\Delta_\beta}\right)}\Big((2-n\beta_1+n\beta_2)(\beta_1+\beta_2) + \frac{1}{6}\big(n^2\beta_1^3 + \big(n^2-m^2\big)\beta_2^3\big)\Big), & i=0, \\ \frac{2}{\mathrm{Vol}\left(-K_{S,\Delta_\beta}\right)}\Big(2(\beta_1+\beta_2) - n\beta_1^2 + (n-m+1)\beta_2^2 + \frac{n^2\beta_1^3+(n-m+2)(n-m)\beta_2^3}{6}\Big), & i\geq 1. \end{cases}$$

Recall that by Theorem 5.5.12, the Futaki character vanishes precisely when $\mathrm{Bc}(\Psi)=0$, i.e.,

$$\beta_1^2 - \beta_2^2 - \frac{1}{3}\big(n\beta_1^3 + (n-m)\beta_2^3\big) = 0. \tag{5.8}$$

We are now ready to compute the δ-invariant.

$$\begin{aligned}
\frac{A_{S,\Delta_\beta}\left(\widetilde{C}_1\right)}{S_{-K_{S,\Delta_\beta}}\left(\widetilde{C}_1\right)} &= \frac{\beta_1}{\beta_1 - \frac{2}{\mathrm{Vol}\left(-K_{S,\Delta_\beta}\right)}\left(\beta_1^2 - \beta_2^2 - \frac{1}{3}(n\beta_1^3 + (n-m)\beta_2^3)\right)},\\
\frac{A_{S,\Delta_\beta}\left(\widetilde{C}_2\right)}{S_{-K_{S,\Delta_\beta}}\left(\widetilde{C}_2\right)} &= \frac{\beta_2}{\beta_2 + \frac{2}{\mathrm{Vol}\left(-K_{S,\Delta_\beta}\right)}\left(\beta_1^2 - \beta_2^2 - \frac{1}{3}(n\beta_1^3 + (n-m)\beta_2^3)\right)},\\
\frac{A_{S,\Delta_\beta}(E_i)}{S_{-K_{S,\Delta_\beta}}(E_i)} &= \frac{1}{1 + \frac{1}{\mathrm{Vol}\left(-K_{S,\Delta_\beta}\right)}\left(-(n-2)\beta_1^2 + (n-m)\beta_2^2 + \frac{n(n-2)\beta_1^3 + (n-m)^2\beta_2^3}{3}\right)},\\
\frac{A_{S,\Delta_\beta}(F_i)}{S_{-K_{S,\Delta_\beta}}(F_i)} &= \frac{1}{1 + \frac{1}{\mathrm{Vol}\left(-K_{S,\Delta_\beta}\right)}\left(-n\beta_1^2 + (n-m+2)\beta_2^2 + \frac{n^2\beta_1^3 + (n-m+2)(n-m)\beta_2^3}{3}\right)}.
\end{aligned} \tag{5.9}$$

This reproduces the computation in Section 5.4 and hence proves Theorem 5.4.12, the necessary part of Theorem 5.0.8. We now prove the sufficient part of Theorem 5.0.8.

Proof of Theorem 5.0.8. Assume

$$\beta_1^2 - \beta_2^2 - \frac{1}{3}\left(n\beta_1^3 + (n-m)\beta_2^3\right) = 0$$

and $m \geq 2$. By (5.9),

$$\begin{aligned}
\frac{A_{S,\Delta_\beta}\left(\widetilde{C}_1\right)}{S_{-K_{S,\Delta_\beta}}\left(\widetilde{C}_1\right)} &= \frac{A_{S,\Delta_\beta}\left(\widetilde{C}_2\right)}{S_{-K_{S,\Delta_\beta}}\left(\widetilde{C}_2\right)} = 1,\\
\frac{A_{S,\Delta_\beta}(E_i)}{S_{-K_{S,\Delta_\beta}}(E_i)} &= \frac{A_{S,\Delta_\beta}(F_i)}{S_{-K_{S,\Delta_\beta}}(F_i)} = \frac{1}{1 + \frac{1}{\mathrm{Vol}\left(-K_{S,\Delta_\beta}\right)}(2-m)\beta_1^2\left(1 - \frac{n}{3}\beta_1\right)}.
\end{aligned}$$

When $m \geq 3$,

$$\frac{A_{S,\Delta_\beta}(E_i)}{S_{-K_{S,\Delta_\beta}}(E_i)} = \frac{A_{S,\Delta_\beta}(F_i)}{S_{-K_{S,\Delta_\beta}}(F_i)} > 1.$$

Moreover, since $\pi^*F - F_i \geq 0$,

$$\frac{A_{S,\Delta_\beta}(\pi^*F)}{S_{-K_{S,\Delta_\beta}}(\pi^*F)} \geq \frac{A_{S,\Delta_\beta}(F_i)}{S_{-K_{S,\Delta_\beta}}(F_i)} > 1.$$

By (5.8), the Futaki character vanishes. Therefore by Theorem 5.5.13, the pair (S, Δ_β) is K-polystable.

When $m = 2$, for $i = 1, 2$,

$$\frac{A_{S,\Delta_\beta}\left(\widetilde{C}_i\right)}{S_{-K_{S,\Delta_\beta}}\left(\widetilde{C}_i\right)} = \frac{A_{S,\Delta_\beta}(E_i)}{S_{-K_{S,\Delta_\beta}}(E_i)} = \frac{A_{S,\Delta_\beta}(F_i)}{S_{-K_{S,\Delta_\beta}}(F_i)} = 1.$$

Therefore by Theorem 5.5.14, the pair (S, Δ_β) is K-polystable. $\square$

Remark 5.6.1. By [46, Theorem 5.1], the vanishing of the Futaki invariant also follows directly from the fact that $A_{S,\Delta_\beta}(\widetilde{C}_i) = S_{-K_{S,\Delta_\beta}}(\widetilde{C}_i)$.

Chapter 6

Stabilization of toric α_k-invariants

In this chapter we focus on Tian's $\alpha_{k,G}$- and $\alpha_{k,m,G}$-invariants that were defined in 1988 and 1991, and to a large extent are still rather mysterious. The presence of the compact symmetry group G is a major source of difficulty and new ideas are needed here as these invariants have not been previously systematically studied or computed. In particular, we provide a formula for such invariants, valid for all toric Fano manifolds, leading to a resolution of Tian's stabilization conjecture in this setting, which is also the first general result on Tian's conjecture.

Let (X, L, ω) be a polarized Kähler manifold of dimension n with L a very ample line bundle over X, and ω a Kähler form representing $c_1(L)$. The space

$$\mathcal{H}_L := \left\{\varphi \,\middle|\, \omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\right\} \subseteq C^\infty(X) \tag{6.1}$$

of Kähler potentials of metrics cohomologous to ω was introduced by Calabi in a short visionary talk in the Joint AMS–MAA Annual Meetings held at Johns Hopkins University in December, 1953 [19]. In a groundbreaking article some 35 years later, Tian proved (motivated by a question of Yau [113, p. 139]) that $\mathcal{H}_L$ is approximated (or "quantized") in the C^2 sense by the finite-dimensional spaces $\mathcal{H}_k$ consisting of pull-backs of Fubini–Study metrics on $\mathbb{P}(H^0(X, L^k)^*)$ under all possible Kodaira embeddings induced by $H^0(X, L^k)$ [103]. A decade later this was improved to a complete asymptotic expansion [24, 117] and so the $\mathcal{H}_k$ can be considered as the Taylor (or Fourier, depending on the point of view) expansion of $\mathcal{H}_L$. The theme that holomorphic and other invariants associated to X and $\mathcal{H}$ may be quantized using the spaces $\mathcal{H}_k$ has dominated Kähler geometry for the last 35 years.

In the 1980's, Futaki's invariant was a new obstruction for the existence of Kähler–Einstein metrics, but there were no invariants that guaranteed existence. At best, there were constructions that utilized symmetry to reduce the Kähler–Einstein equation to a simpler equation that could be solved and lead to specific examples (another theme pioneered by Calabi, which makes a surprise appearance to close this chapter, see Example 6.6.12). Given a maximal compact subgroup G of the automorphism group $\operatorname{Aut} X$, Tian introduced the invariant

$$\alpha_G := \sup\left\{ c > 0 \,\middle|\, \sup_{\varphi\in\mathcal{H}^G} \int_X e^{-c(\varphi-\sup\varphi)}\omega^n < \infty \right\} \tag{6.2}$$

(where $\mathcal{H}^G$ denotes the G-invariant elements of $\mathcal{H}_L$), and its quantized version (Definition 6.2.1)

$$\alpha_{k,G},$$

computed over the G-invariant elements of $\mathcal{H}_k$, and obtained the sufficient condition

$$\alpha_G > \frac{n}{n+1} \tag{6.3}$$

for the existence of a Kähler–Einstein metric when $L = -K_X$ (which we will henceforth assume unless otherwise stated) [101, Theorem 4.1]. Initially, the main interest in the invariants α_G was as the first systematic tool for constructing Kähler–Einstein metrics on Fano manifolds, but later it was also conjectured by Cheltsov and established by Demailly that α_G actually coincides with the G-equivariant global log canonical threshold from algebraic geometry [29]. Since $\mathcal{H}_k \subseteq \mathcal{H}_L$ it follows that $\alpha_G \leq \inf_k \alpha_{k,G}$ [103, p. 128], yet this does not help obtain (6.3). Instead, Tian posed the following difficult question that would reduce the computation of the invariant α_G from the infinite-dimensional space $\mathcal{H}_L$ to a finite-dimensional one $\mathcal{H}_k$, and establish a highly non-trivial relation between the different $\mathcal{H}_k$'s.

Problem 6.0.1. [103, Question 1],[102] Let X be Fano and $L = -K_X$, and let G be a maximal compact subgroup $G \subseteq \operatorname{Aut} X$. Is $\alpha_{k,G} = \alpha_G$ for all sufficiently large $k \in \mathbb{N}$?

It is interesting to note that a slight variation of Tian's α- and α_k-invariants turned out to lead about three decades later [119, 90] to the very closely related δ- and δ_k-invariants of Fujita–Odaka [48] that are in turn a slight (and ingenius) variation on global log canonical thresholds, and turn out to essentially characterize the existence of Kähler–Einstein metrics. We return to these invariants in Chapter 7.

A Demailly type identity in the presence of symmetry

Another (easier) problem motivating this chapter concerns the by-now-classical relation between Tian's (holomorphic) invariants and the (algebraic) global log canonical thresholds. The relationship was first conjectured by Cheltsov and proved by Demailly and Shi [29, 94]. However, so far, this relationship has only been shown for the α- and α_k-invariants, or for the α_G-invariant (see [29, Theorem A.3], [94, Proposition 2.1], [29, (A.1)], respectively), and not for the more subtle invariants $\alpha_{k,G}$. Inspired by Demailly, we introduce (Definition 6.2.4) the k-th G-equivariant global log canonical threshold

$$\operatorname{glct}_{k,G}$$

as an algebraic counterpart of Tian's $\alpha_{k,G}$ (Definition 6.2.1). A natural question is:

Problem 6.0.2. Let X be Fano and $L = -K_X$, and let G be a compact subgroup $G \subseteq \operatorname{Aut} X$. Is $\operatorname{glct}_{k,G} = \alpha_{k,G}$?

Note that the proof of Demailly in the equivariant case [29, (A.2)] works for the limit as $k \to \infty$ but not on a fixed level k.

Stabilization of toric α_k-invariants

In this chapter, we resolve both Problems 6.0.1 and 6.0.2 in the toric setting. It is perhaps not well-known, but Calabi was interested in toric geometry and computed certain geodesics in $\mathcal{H}_L$ in the toric setting, although he never published the result [21].

As standard, we allow the slightly more flexible situation of any (and not just a maximal) compact subgroup of the normalizer

$$N((\mathbb{C}^*)^n)$$

of the complex torus $(\mathbb{C}^*)^n$ in $\operatorname{Aut} X$. Denote by

$$\operatorname{Aut} P \subseteq GL(M) \cong GL(n, \mathbb{Z}) \tag{6.4}$$

the subgroup of the automorphism group of the lattice M that leaves the polytope P (3.18) invariant. It is necessarily a finite group. In fact, $\operatorname{Aut} P$ is the quotient of the normalizer $N((\mathbb{C}^*)^n)$ of the complex torus $(\mathbb{C}^*)^n$ in $\operatorname{Aut} X$ by $(\mathbb{C}^*)^n$, so that $N((\mathbb{C}^*)^n)$ consists of finitely many components each isomorphic to a complex torus [7, Proposition 3.1]. For $H \subseteq \operatorname{Aut} P$, let

$$G(H) := H \ltimes \left(S^1\right)^n \subseteq N((\mathbb{C}^*)^n) \subseteq \operatorname{Aut} X \tag{6.5}$$

denote the compact group generated by H and $(S^1)^n$ (the latter is the maximal compact subgroup of the complex torus $(\mathbb{C}^*)^n$).

Our first result resolves Problem 6.0.2 in this generality.

Proposition 6.0.3. *Let X be toric Fano and $L = -K_X$. Let $P \subseteq M_{\mathbb{R}}$ (see (3.6), (3.18)) be the polytope associated to $(X, -K_X)$, let $H \subseteq \operatorname{Aut} P$, and let $G(H)$ be as in (6.5). Then $\operatorname{glct}_{k,G(H)} = \alpha_{k,G(H)}$.*

The proof of this Proposition 6.0.3 relies on the structure of ring of holomorphic sections on a toric line bundle established in §6.2.3. For general non-toric cases, such a proposition will hold once a similar result on the space of sections can be established.

Using this result, and several new estimates, we can resolve Tian's Problem 6.0.1 in the toric setting in a surprisingly strong sense, showing that equality holds for *all* $k \in \mathbb{N}$. We also allow for all groups $G(H)$ (and not just the maximal toric one $G(\operatorname{Aut} P)$).

To state the precise result we introduce some more notation. For $H \subseteq \operatorname{Aut} P$, denote by

$$P^H := \{y \in P \mid h.y = y, \quad \forall h \in H\} \subseteq P \subseteq M_{\mathbb{R}} \tag{6.6}$$

the fixed-point set of H in P, and let

$$\pi_H := \frac{1}{|H|} \sum_{\eta \in H} \eta \in \operatorname{End}(M_{\mathbb{Q}}) \tag{6.7}$$

be the map that takes a point in $M_{\mathbb{R}}$ to the average of its H-orbit. Note that π_H is a projection map (see §6.4.3 for details).

Theorem 6.0.4. *Let X be toric Fano associated to a fan Δ whose rays are generated by primitive elements v_i in the lattice N dual to M. Let $P \subseteq M_{\mathbb{R}}$ (see* (3.6), (3.18)*) be the polytope associated to $(X, -K_X)$, let $H \subseteq \operatorname{Aut} P$, and let $G(H)$ be as in* (6.5)*. Then for any $k \in \mathbb{N}$,*

$$\begin{aligned} \alpha_{k,G(H)} &= \sup\left\{ c \in (0,1) \,\middle|\, -\frac{c}{1-c} P^H \subseteq P \right\} \\ &= \min_{u \in \operatorname{Ver} P^H} \frac{1}{\max_i \langle u, v_i \rangle + 1} \\ &= \min_{u \in \pi_H(\operatorname{Ver} P)} \frac{1}{\max_i \langle u, v_i \rangle + 1}, \end{aligned} \tag{6.8}$$

where P^H and π_H are defined in (6.6)–(6.7) *and* $\operatorname{Ver}(\,\cdot\,)$ *denotes the vertex set of a polytope. In particular, $\alpha_{k,G(H)}$ is independent of $k \in \mathbb{N}$ and is equal to $\alpha_{G(H)}$.*

This is the first time $\alpha_{k,G}$-invariant is computed. There are a few new ingredients in the proof of Theorem 6.0.4. The first is a useful formula for the spaces $\mathcal{H}_k^{G(H)}$ in terms of the H-orbits of the finite group action (Lemma 6.2.5). This together with a trick that amounts to estimating the singularities associated to a basis of sections in terms of the finite group action orbit of a section yields a useful formula for $\alpha_{k,G(H)}$ (Proposition 6.2.6) as well as the equality $\alpha_{k,G(H)} = \operatorname{glct}_{k,G(H)}$, i.e., a solution to Problem 6.0.2 (Proposition 6.0.3). These then yield a corresponding useful formula for $\alpha_{G(H)}$ (Corollary 6.2.7). One may prove using the results of §6.1–§6.2 that $\alpha_{k,G(H)} \geq \alpha_{k\ell,G(H)}$ for any fixed k and all $\ell \in \mathbb{N}$ (Proposition

6.3.1). In Proposition 6.3.3 it is shown that there is a special k_0 for which $\alpha_{k_0\ell,G(H)} = \alpha_{G(H)}$ for all $\ell \in \mathbb{N}$, as well as observed that this does not seem to imply Tian's conjecture (Remark 6.3.6). Finally, key new estimates occur in §6.4. First, we show that rather general complex singularity exponents associated to collections of toric monomials are independent of k (Proposition 6.4.2). The proof of this uses a new observation about the relation between the support functions of collections of lattice points associated to the toric monomials and complex singularity exponents. We then apply this to our $G(H)$-invariant setting, using the aforementioned expression of $\mathcal{H}_k^{G(H)}$ and a reduction lemma to the H-invariant subspace (Lemma 6.4.10), to conclude the proof of Theorem 6.0.4.

It is perhaps of some interest to include here a rather immediate application of this circle of ideas to a slightly more technical set of invariants, also introduced by Tian, that we call Tian's Grassmannian α-invariants. These invariants are defined a little differently, algebraically, and are denoted $\alpha_{k,m}$ or $\alpha_{k,m,G}$ (Definition 6.5.1). At least in the non-equivariant setting (as well as in the torus-equivariant setting, see Remark 6.5.2) these can be considered as generalizations of the α_k as $\alpha_k = \alpha_{k,1}$ [94, 29]. The invariants $\alpha_{k,2}$ were used by Tian implicitly in his proof of Calabi's conjecture for del Pezzo surfaces [103, Appendix A] (cf. [104, Theorem 6.1]) to overcome the most difficult case (of a cubic surface with an Eckardt point) where equality holds in (6.3), and this was improved by Shi to $\alpha_{k,2} > 2/3 = \alpha_{k,1}$ in that case [94, Theorem 1.3],[25].

Tian also posed a stabilization conjecture for these invariants in 2012 [107, Conjecture 5.3]:

Conjecture 6.0.5. *Let X be Fano. Fix $m \in \mathbb{N}$. For sufficiently large k, $\alpha_{k,m,G}$ is constant.*

We completely resolve Conjecture 6.0.5 in the toric setting. Theorem 6.0.4 resolved Problem 6.0.1 in the affirmative (corresponding to the case $m = 1$ of Conjecture 6.0.5). For $m \geq 2$, Conjecture 6.0.5 turn out to be only partially true as determined by a novel convex geometric obstruction we introduce:

$$\|\cdot\|_{-P}\Big|_{P\setminus\operatorname{Ver} P} < \max_P \|\cdot\|_{-P}. \tag{$*_P$}$$

Note that this condition depends only on P (and not on m, k). The condition $(*_P)$ means that the function

$\|\cdot\|_{-P}$ on P achieves its maximum only at the vertices of P, i.e.,

$$\operatorname{argmax}_P \|\cdot\|_{-P} \subseteq \operatorname{Ver} P. \tag{$*_P$}$$

When $(*_P)$ fails the maximum is achieved also at some point that is not a vertex of P.

Theorem 6.0.6. *Let X be toric Fano with associated polytope P (3.19) and let $\alpha := \alpha_{(S^1)^n}$ be as in (6.2) with $L = -K_X$. Conjecture 6.0.5 holds if and only if $(*_P)$ fails. More precisely, if $(*_P)$ holds,*

$$\alpha_{k,m,(S^1)^n} > \alpha, \quad \textit{for } k \in \mathbb{N} \textit{ and } m \in \mathbb{N} \setminus \{1\}, \tag{6.9}$$

otherwise

$$\alpha_{k,m,(S^1)^n} = \alpha, \quad \textit{for } m \in \mathbb{N} \textit{ and for sufficiently large } k \in \mathbb{N}. \tag{6.10}$$

Theorem 6.0.6 is proven in §6.5.1 where we also explain the intuition behind it (see also Examples 6.6.6 and 6.6.9). For now, let us elucidate the condition $(*_P)$ a bit. The level set $\{\|\cdot\|_{-P} = \lambda\}$ is the dilation $\lambda\partial(-P)$, and $\{\|\cdot\|_{-P} = \max_P \|\cdot\|_{-P}\}$ is the largest dilation that intersects P (by Lemma 6.5.4). Thus, condition $(*_P)$ states that P intersects $\max_P \|\cdot\|_{-P}\partial(-P)$ only at vertices. When $(*_P)$ fails, convexity arguments show the intersection will contain a positive-dimensional face of P. See Figure 6.1 for two examples.

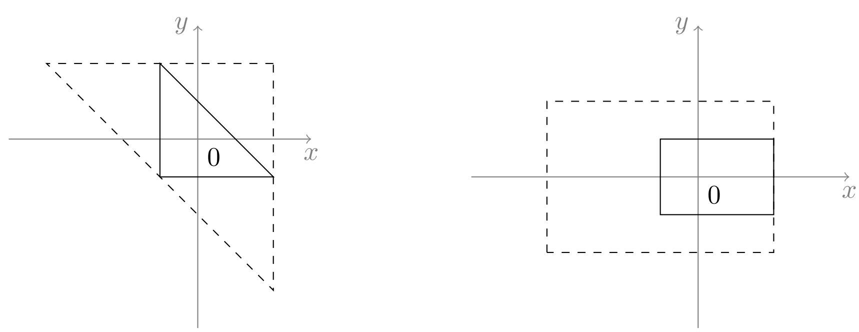

Figure 6.1: The polytope P (solid line) and the level set $\{\|\cdot\|_{-P} = \max_P \|\cdot\|_{-P}\}$ (dashed line). For $P = \mathrm{co}\{(-1,-1),(2,-1),(-1,2)\}$, the maximum is only attained at the vertices of P. In particular, $(*_P)$ holds. For $P = [-1,2] \times [-1,1]$, the maximum is attained on the line segment $\{2\} \times [-1,1]$. In particular, $(*_P)$ does not hold.

Relation to earlier works. Theorem 6.0.4 strengthens and clarifies work of Song [97] and Li–Zhu [71].

Song obtained a formula for α_G but not for $\alpha_{k,G}$. In particular, there seems to be a gap in the proof of [97, Theorem 1.2] that claims that $\alpha_G = \alpha_{k,G}$ for all sufficiently large k. This claim relies on proving that for some $k_0 \in \mathbb{N}$ and all $\ell \in \mathbb{N}$, $\alpha_G = \alpha_{k_0\ell,G}$ and then invoking that $\alpha_{k,G}$ is eventually monotone in k, and hence must be independent of k for sufficiently large k. Unfortunately, the proof of monotonicity is omitted from [97, p. 1257, line 7], and it appears to be difficult to reproduce. It seems that such monotonicity is not currently known (cf. Remark 6.3.6). Indeed, there is no obvious relationship between the various $\mathcal{H}_k^G$ coming from different Kodaira embeddings. As noted above, Song showed (for $H = \mathrm{Aut}\, P$) that $\alpha_{G(H)} = \alpha_{k_0\ell,G(H)}$ for some $k_0 \in \mathbb{N}$ and all $\ell \in \mathbb{N}$. Li–Zhu showed the same identity (essentially for $H = \{\mathrm{id}\}$) for a group compactification of a reductive complex Lie group and also claimed, similarly to Song, that this implies eventual constancy in k in that setting [71, Theorem 1.3, p. 233]. Unfortunately, also they do not provide a proof of the needed eventual monotonicity or constancy. In the non-equivariant setting, and for rather general Fano varieties for which $\alpha \leq 1$, Birkar showed the deep result that $\alpha = \alpha_{k_0\ell}$ for some $k_0 \in \mathbb{N}$ and all $\ell \in \mathbb{N}$ [13, Theorem 1.7]. However, also this result does not imply Tian stabilization due to the aforementioned unknown monotonicity. Thus, Theorem 6.0.4 seems to be the first general result on Tian's stabilization Problem 6.0.1.

Similarly, Theorem 6.0.6 seems to be the first general result on Conjecture 6.0.5. Indeed, Li–Zhu showed the same type of result Song obtained in the $m = 1$ setting, i.e., that $\alpha_{k_0\ell,m,(S^1)^n} = \alpha_{(S^1)^n}$ under a condition depending on k_0 and m, and hence different from our $(*_P)$ (with the minor caveat that their statement as written [71, Theorem 1.4] is incorrect, though can be easily fixed by replacing "facet" by "face", see §6.5.1). However, again, due to the lack of monotonicity, they do not obtain a resolution of Conjecture 6.0.5 though they do obtain the first counterexamples to it when $m \geq 2$.

Combining Theorem 6.0.4 and Demailly's theorem [29, (A.1)] also recovers Song's formula for the $\alpha_{G(\mathrm{Aut}\, P)}$ (that itself generalized Batyrev–Selivanova's formula that $\alpha_{G(\mathrm{Aut}\, P)} = 1$ whenever $P^{\mathrm{Aut}\, P} = \{0\}$ (recall (6.6)) [7, Theorem 1.1, p. 233]). Cheltsov–Shramov claimed a more general formula for α_H (i.e., without the real torus symmetry included in $G(H)$, recall (6.5)) however (as kindly pointed out to us by I. Cheltsov) there is an error in the proof of [29, Lemma 5.1] as the toric degeneration used there need not respect the H-invariance. Further generalizations of Song's formula for α_G to general polarizations and group compactifications are due to Delcroix [34, 35] and Li–Shi–Yao [70] and our methods should generalize to those settings as well as to the setting of log toric Fano pairs and edge singularities [26, §6–7]. Although this is not the topic of our chapter, for completeness we mention in passing that the stabilization of the $\alpha_k(L)$-invariants for arbitrary toric varieties (i.e., not necessarily Fano) but for the *non-equivariant* setting (i.e., $H = \{id\}$) is true, i.e., $\alpha_{k,(S^1)^n}(L) = \alpha(L)$ for all $k \in \mathbb{N}$, yet is much-simpler; the crux of our work is to address the equivariant setting that presents new challenges.

Theorem 6.0.7. *Let X be a toric variety and L ample, with associated lattice polytope*

$$P = \bigcap_{i=1}^{d} \{y \in M_{\mathbb{R}} \mid \langle y, v_i \rangle \geq -b_i\}$$

with $v_i \in N$, $b_i \in \mathbb{Z}$. For $k \in \mathbb{N}$,

$$\alpha_{k,(S^1)^n}(L) = \min_{u \in P \cap M/k} \frac{1}{\max_i \langle u, v_i \rangle + b_i} = \min_{u \in \operatorname{Ver} P} \frac{1}{\max_i \langle u, v_i \rangle + b_i} = \alpha(L).$$

The same proof for Theorem 6.0.4 works in this case, with the additional assumption that $H = \{id\}$, except that the coefficient 1 is replaced by b_i. Proposition 6.2.6, for instance, gives the left-hand side. Blum–Jonsson's formula [16, (7.2)] gives the right-hand side. And the equality again relies on $\operatorname{Ver} P \subseteq M$ and the fact that a convex function on a convex polytope attains its maximum on a vertex.

Finally, it is also worth mentioning that Tian also posed more general conjectures [107, Conjecture 5.4] for general polarizations (i.e., L not being $-K_X$) for which there are already some counterexamples [1].

6.1 A natural equivariant Hermitian metric and volume form

In light of Lemma 6.2.2 below it makes sense to choose a convenient pair (μ, h) of a volume form and a Hermitian metric. In fact, we are free to choose such a pair for each k. The special feature of working with $L = -K_X$ is that in fact a volume form essentially doubles as a Hermitian metric, which is sometimes a bit confusing to keep track of in terms of notation, but is quite convenient for computations. This section serves to explain this choice ($\mu_k = h_k^{1/k}, h_k$), see (6.17) and (6.19), originally due to Song [97, Lemma 4.3]. We emphasize that the Hermitian metric must additionally be chosen G-invariant in Definition 6.2.1, and this is confirmed for h_k in Lemma 6.1.1.

Let X be a toric Fano manifold and P its associated polytope. There is a natural basis of the space of holomorphic sections $H^0(X, -kK_X)$ defined by the monomials z^{ku} where $u \in P \cap \frac{1}{k}M$. That is, there exists an invariant frame e over the open orbit such that

$$s_{k,u}(z) = z^{ku}e. \tag{6.11}$$

What does e actually look like? This is most naturally expressed in terms of the *monomial basis*. When $k = 1$ and $u \in P \cap M$ [49, §4.3],

$$s_{1,u} = z^u \prod_{i=1}^{n} z_i \cdot \partial_{z_1} \wedge \cdots \wedge \partial_{z_n}.$$

In general, for any $k \in \mathbb{N}$ and $u \in P \cap \frac{1}{k}M$,

$$s_{k,u} = z^{ku} \left(\prod_{i=1}^{n} z_i \right)^k (\partial_{z_1} \wedge \cdots \wedge \partial_{z_n})^{\otimes k}. \tag{6.12}$$

In other words,

$$e = \left(\prod_{i=1}^{n} z_i \right)^k (\partial_{z_1} \wedge \cdots \wedge \partial_{z_n})^{\otimes k}. \tag{6.13}$$

Next, let us construct a canonical Hermitian metric h_k on $-kK_X$. Since $-kK_X$ is very ample [49, p. 70], it is natural to pull-back the Fubini–Study Hermitian metric via the Kodaira embedding. It turns out that choosing the Kodaira embedding given by the monomial basis

$$\iota_k : X \ni z \mapsto [s_{k,u}(z)/e(z)]_{u \in P \cap \frac{1}{k}M} = [z^{ku}]_{u \in P \cap \frac{1}{k}M} \in \mathbb{P}^{E_P(k)-1}, \tag{6.14}$$

will yield the desired h_k; importantly, the resulting h_k will be *torus-invariant*, smooth, and essentially transform computations on X to P. To wit, the Fubini–Study metric on $\mathcal{O}(1) \to \mathbb{P}^{E_P(k)-1}$ is (where $E_P(k)$

is the number of lattice points in kP)

$$h_{FS}(Z_i, Z_j) := \frac{Z_i \bar{Z}_j}{\sum_\ell |Z_\ell|^2},$$

and we define

$$h_k := \iota_k^* h_{FS}. \tag{6.15}$$

Note that each homogeneous coordinate $Z_i \in H^0(\mathbb{P}^{E_P(k)-1}, \mathcal{O}(1))$ pulls-back via ι_k to one of the monomial sections $s_{k,u}$ (which one depends on the ordering for the elements of $P \cap \frac{1}{k}M$ chosen in (6.14)). Thus, to express h_k it suffices to compute it on the monomial basis of $H^0(X, -kK_X)$:

$$h_k(s_{k,u_1}, s_{k,u_2})(z) = h_{FS}\big(s_{k,u_1}, s_{k,u_2}\big)(\iota_k(z)) = \frac{z^{ku_1}\overline{z^{ku_2}}}{\sum\limits_{u \in P \cap \frac{1}{k}M} |z^{ku}|^2}. \tag{6.16}$$

Comparing (6.12) and (6.16) means that h_k can be written as

$$h_k = \frac{\left(dz^1 \wedge \overline{dz^1} \wedge \cdots \wedge dz^n \wedge \overline{dz^n}\right)^{\otimes k}}{\left(\prod\limits_{i=1}^n |z_i|^2\right)^k \sum\limits_{u \in P \cap \frac{1}{k}M} |z^{ku}|^2}. \tag{6.17}$$

In conclusion, $h_k^{1/k}$ is a smooth metric on $-K_X$ (h_k being obtained as a pull-back of a smooth metric under the Kodaira *embedding*), hence it is a smooth volume form on X. To express this volume form, on the open orbit $(\mathbb{C}^*)^n = \mathbb{R}^n \times (S^1)^n$ consider the holomorphic coordinates

$$w_i := x_i/2 + \sqrt{-1}\theta_i = \log z_i \in \mathbb{C}^n. \tag{6.18}$$

In these coordinates then, this volume form, on the open orbit, is

$$\mu_k := h_k^{\frac{1}{k}} = \frac{dx_1 \wedge \cdots \wedge dx_n \wedge d\theta_1 \wedge \cdots \wedge d\theta_n}{\left(\sum\limits_{u \in P \cap M/k} e^{\langle ku, x \rangle}\right)^{\frac{1}{k}}}. \tag{6.19}$$

Lemma 6.1.1. *Let $G(H) \subseteq \operatorname{Aut} X$ (6.5) be a subgroup generated by $(S^1)^n$ and a subgroup H of* $\operatorname{Aut} P$.

Then h_k (6.17) *is* $G(H)$*-invariant.*

Proof. From (6.19) it is evident that h_k is independent of $(\theta_1, \ldots, \theta_n)$, i.e., it is $(S^1)^n$-invariant. An automorphism $\sigma \in \operatorname{Aut} P \subseteq GL(M)$ can be represented (via choosing a basis for the lattice M) by a matrix in $GL(n, \mathbb{Z}) \cong GL(M)$. Since σ preserves the polytope P, then $\det \sigma \in \{\pm 1\}$ (σ could be orientation-reversing, e.g., in the case of a reflection). The induced action of σ on the dual space $N_{\mathbb{R}}$ is naturally represented (via the pairing between M and N) by the transpose matrix, that we denote by σ^T, and this action is actually coming from the $\mathbb{C}$-linear action of σ^T on $N_{\mathbb{C}} \cong \mathbb{C}^n$ (6.18). Thus,

$$\sigma.(dx_1 \wedge \cdots \wedge dx_n) = d(x_1 \circ \sigma) \wedge \cdots \wedge d(x_n \circ \sigma) = \det(\sigma^T) dx_1 \wedge \cdots \wedge dx_n$$

(here $\sigma.$ denotes the action of σ on forms, i.e., by pull-back), and $\sigma.(d\theta_1 \wedge \cdots \wedge d\theta_n) = \det(\sigma^T) d\theta_1 \wedge \cdots \wedge d\theta_n$. Since $(\det \sigma^T)^2 = 1$ it remains to consider the denominator of (6.19):

$$\begin{aligned} \sigma. \sum_{u \in P \cap M/k} e^{k\langle u, x\rangle} &= \sum_{u \in P \cap M/k} e^{k\langle u, \sigma^T . x\rangle} \\ &= \sum_{u \in P \cap M/k} e^{k\langle \sigma . u, x\rangle} \\ &= \sum_{u \in \sigma(P \cap M/k)} e^{k\langle u, x\rangle} \\ &= \sum_{u \in P \cap M/k} e^{k\langle u, x\rangle}, \end{aligned}$$

since σ preserves both P and M/k. In particular, h_k is invariant under σ, concluding the proof. □

Remark 6.1.2. An alternative, more invariant, proof of Lemma 6.1.1 is as follows. By (6.15), $\sigma . h_k = (\iota_k \circ \sigma)^* h_{\mathrm{FS}}$. Now $\iota_k \circ \sigma$ induces the exact same Kodaira embedding if $\sigma \in (S^1)^n < G(H)$. So it suffices to consider $\sigma \in H$; then, since H preserves $P \cap M/k$, one obtains the same Kodaira embedding up to permutation of the coordinates in $\mathbb{P}^{E_P(k)-1}$. Either way, one obtains the same pull-back of the Fubini–Study metric as can be from the definition of the Fubini–Study metric or directly from (6.17), i.e.,

$\sigma.h_k = h_k$.

6.2 An algebraic $\alpha_{k,G}$-invariant and a Demailly type result

6.2.1 Analytic definition

Analogous to the classical α_G-invariant, Tian [103, p. 128] defined the $\alpha_{k,G}$-invariant. For this one restricts to a G-invariant subset of $\mathcal{H}_k$. To write the subset explicitly in terms of global Kähler potentials it is necessary to choose a continuous G-invariant Hermitian metric h on $-kK_X$:

$$\mathcal{H}_k^G(h) := \left\{ \varphi = \frac{1}{k} \log \sum_i |s_i|_h^2 \,\middle|\, \varphi \text{ is } G\text{-invariant}, \{s_i\} \text{ is a basis of } H^0(X, -kK_X) \right\} \subseteq C^\infty(X). \quad (6.20)$$

Definition 6.2.1. Let $G \subseteq \operatorname{Aut} X$ be a compact subgroup. Let h be a fixed continuous G-invariant Hermitian metric on $-kK_X$, and μ a fixed continuous volume form on X. Then

$$\alpha_{k,G}(h, \mu) := \sup \left\{ c > 0 \,\middle|\, \sup_{\varphi \in \mathcal{H}_k^G} \int_X e^{-c(\varphi - \sup \varphi)} d\mu < \infty \right\}.$$

Lemma 6.2.2. *Definition 6.2.1 does not depend on the choice of h or μ.*

For this reason we will simply denote the invariants by $\alpha_{k,G}$ from now on.

Proof. Since X is compact, any two continuous volume forms are uniformly bounded and hence define the same L^1 spaces. Next, given two continuous G-invariant Hermitian metrics h and $\tilde{h}$ on $-kK_X$ there is an isomorphism from $\mathcal{H}_k^G(h)$ to $\mathcal{H}_k^G(\tilde{h})$ given by $\varphi \mapsto \varphi + \frac{1}{k} \log \frac{\tilde{h}}{h}$. Observe that $\frac{1}{k} \log \frac{\tilde{h}}{h}$ is (again by compactness of X) a uniformly bounded function on X. Hence, for a fixed $c > 0$,

$$\sup_{\varphi \in \mathcal{H}_k^G(h)} \int_X e^{-c(\varphi - \sup \varphi)} d\mu < \infty \quad \Leftrightarrow \sup_{\varphi \in \mathcal{H}_k^G(\tilde{h})} \int_X e^{-c(\varphi - \sup \varphi)} d\mu < \infty,$$

as desired. □

6.2.2 Algebraic definition

Consider a (complex) non-zero vector subspace V of $H^0(X, kL)$. Associated to it is the (not necessarily complete) linear system $|V| := \mathbb{P}V \subseteq |kL| := \mathbb{P}H^0(X, kL)$ [56, p. 137].

Lemma 6.2.3. *Let V be vector subspace of $H^0(X, kL)$ of dimension $p > 0$. For any basis $\nu_1, \ldots, \nu_p \in H^0(X, kL)$ of V, the number*

$$\sup\left\{ c > 0 \,\middle|\, \left(\sum_{j=1}^{p} |\nu_j(z)|^2\right)^{-c} \text{ is locally integrable on } X \right\}$$

is the same.

Proof. Let $\{\nu_1^{(\ell)}, \ldots, \nu_p^{(i)}\}$, $\ell \in \{1, 2\}$, be two bases for $V \in H^0(X, kL)$. Let $A \in GL(p, \mathbb{C})$ be the change-of-basis matrix, i.e., $\nu_j^{(2)} = A_j^i \nu_i^{(1)}$. Observe that $A^H A$ is a positive Hermitian matrix-valued on X, and denote its eigenvalues $0 < \lambda_1 \leq \cdots \leq \lambda_p$. Denote $\nu^{(\ell)}(z) := (\nu_1^{(\ell)}(z), \ldots, \nu_p^{(\ell)}(z)) \in \mathbb{C}^p$ and $|\nu^{(\ell)}(z)|^2 = \sum_{i=1}^p |\nu_i^{(\ell)}(z)|^2$. Then,

$$\frac{|\nu^{(2)}(z)|^2}{|\nu^{(1)}(z)|^2} = \frac{|A\nu^{(1)}(z)|^2}{|\nu^{(1)}(z)|^2} \in [\lambda_1, \lambda_p],$$

Thus, $|\nu^{(2)}|^2$ is locally integrable if and only if $|\nu^{(1)}|^2$ is. □

Thus, define the log canonical threshold of the linear system $|V|$ by

$$\operatorname{lct}|V| := \sup\left\{ c > 0 \,\middle|\, \left(\sum_{j} |\nu_j(z)|^2\right)^{-c} \text{ is locally integrable on } X \right\}. \tag{6.21}$$

When $L = -K_X$, there is an $\operatorname{Aut} X$ action on $H^0(X, -kK_X)$ for every $k \in \mathbb{N}$. Demailly [29, Theorem A.3, (A.1)] noted that then α_G-invariants (for compact subgroup $G \subseteq \operatorname{Aut} X$) can be algebraically computed as

$$\alpha_G = \inf_{k \in \mathbb{N}} k \inf_{\substack{|V| \subseteq |-kK_X| \\ V^G = V \neq 0}} \operatorname{lct}|V|, \tag{6.22}$$

where

$$V^G := \{v \in V \mid g.v \in V \quad \forall g \in G\}.$$

Note that in (6.22) $V \neq 0$ ranges over all G-invariant vector subspaces of $H^0(X, -kK_X)$ (i.e., of any positive dimension). For example, $\operatorname{lct} |-kK_X| = \infty$. From the definition, if $V_1 \subseteq V_2$ are two such subspaces it suffices to compute $\operatorname{lct} |V_1|$ since

$$\operatorname{lct} |V_1| \leq \operatorname{lct} |V_2|. \tag{6.23}$$

It is thus natural to define the algebraic counterpart of the $\alpha_{k,G}$-invariant as follows.

Definition 6.2.4. Let X be a Fano manifold and $G \subseteq \operatorname{Aut} X$ be a compact subgroup of the automorphism group. Define

$$\operatorname{glct}_{k,G} := k \inf_{\substack{|V| \subseteq |-kK_X| \\ V^G = V}} \operatorname{lct} |V|. \tag{6.24}$$

6.2.3 Characterization of the equivariant Bergman spaces

First, we show that in the toric setting the space $\mathcal{H}_k^{G(H)}$ consists of Kähler potentials induced by Kodaira embeddings of multiples of monomials sections, with the norming constants constant along orbits of H.

Denote by

$$O_1^{(k)}, \ldots, O_N^{(k)}, \tag{6.25}$$

the orbits of H in $k^{-1}M \cap P$ (see Figure 6.2 for an example).

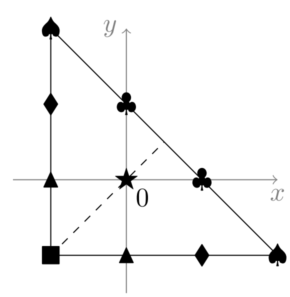

Figure 6.2: The six orbits $O_1^{(1)}, \ldots, O_6^{(1)}$ of the action of the group generated by the reflection about $y = x$ on the polytope corresponding to $\mathbb{P}^2$ with $k = 1$.

Lemma 6.2.5. *Let $\{s_{k,u}\}_{u \in k^{-1}M \cap P}$ be the monomial basis* (6.11) *of* $H^0(X, -kL)$. *Let* $G(H) \subseteq \operatorname{Aut} X$

(6.5) *be a subgroup generated by* $(S^1)^n$ *and a subgroup* H *of* $\operatorname{Aut} P$*. Then (recall* (6.20) *and* (6.25)*),*

$$\mathcal{H}_k^{G(H)} := \mathcal{H}_k^{G(H)}(h_k) = \left\{ k^{-1} \log \sum_{i=1}^{N} \lambda_i \sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k} \, : \, \lambda_i > 0 \right\}.$$

Proof. Recall the natural basis $\{s_{k,u}\}_{u \in P \cap M/k}$ defined by the lattice monomials (6.11). Given any other basis $\{s_i\}$ for $H^0(X, -kK_X)$, let $A \in GL(E_P(k), \mathbb{C})$ be the change-of-basis matrix, so

$$s_i = A_i^u s_{k,u}, \quad i \in \{1, \dots, E_P(k)\}$$

(we use the Einstein summation convention).

Let $\varphi \in \mathcal{H}_k^{G(H)}$ and let $\{s_i\}$ be the associated basis (6.20). Expanding φ in terms of the monomials,

$$\begin{aligned} e^{k\varphi} = \sum_i |s_i|^2_{h_k} &= \sum_{u,u' \in P \cap M/k} \sum_{i,j=1}^{E_P(k)} A_i^u \overline{A_j^{u'}} \langle s_{k,u}, s_{k,u'} \rangle_{h_k} \\ &=: |s_{k,0}|^2_{h_k} \sum_{u,u' \in P \cap M/k} c_{u,u'} z^{ku} \bar{z}^{ku'} \end{aligned}$$

for some coefficients $\{c_{u,u'}\}_{u,u' \in P \cap M/k}$. We claim that $\{z^u \bar{z}^{u'}\}_{u,u' \in M}$ are linearly independent. To see that, suppose

$$f(z) = \sum_{u,u' \in M} c_{u,u'} z^u \bar{z}^{u'} = 0$$

with all but finitely many coefficients being 0. We may assume for any $c_{u,u'} \neq 0$, u, u' lie in the positive orthant, by multiplying by $\prod_i |z_i|^2$ to some sufficiently large power. Then

$$c_{u,u'} = \frac{1}{u!u'!} \frac{\partial^{|u|+|u'|}}{\partial^u z \partial^{u'} \bar{z}} \bigg|_{z=0} f(z) = 0,$$

proving the claim. Now, the subgroup $(S^1)^n < G(H)$ acts on the open orbit $(\mathbb{C}^*)^n$ by

$$(\beta_1, \dots, \beta_n).(z_1, \dots, z_n) = (e^{\sqrt{-1}\beta_1} z_1, \dots, e^{\sqrt{-1}\beta_n} z_n). \tag{6.26}$$

So by $(S^1)^n$-invariance and Lemma 6.1.1,

$$\beta.e^{k\varphi} = |s_{k,0}|^2_{h_k} \sum_{u,u' \in P \cap M/k} c_{u,u'} e^{\sqrt{-1}\langle \beta, k(u-u') \rangle} z^{ku} \bar{z}^{ku'} = |s_{k,0}|^2_{h_k} \sum_{u,u' \in P \cap M/k} c_{u,u'} z^{ku} \bar{z}^{ku'} = e^{k\varphi},$$

for any $\beta \in (S^1)^n = (\mathbb{R}/2\pi\mathbb{Z})^n$. By comparing the coefficients and using the claim we just demonstrated, it follows that for $e^{k\varphi}$ to be $(S^1)^n$-invariant we must have $c_{u,u'} = 0$ whenever $u \neq u'$ (the converse is also true, of course).

Moreover, we also require $e^{k\varphi}$ to be invariant under the action of H, i.e., $c_{u,u} = c_{\sigma u, \sigma u}$ for any $\sigma \in H$. In conclusion, let $O_1^{(k)}, \ldots, O_N^{(k)}$ be the orbits of the action of H on $P \cap M/k$. Then

$$e^{k\varphi} = \sum_{i=1}^{N} \lambda_i \sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k},$$

for some coefficients $\{\lambda_i\}_{i=1}^N$. Thus (6.20) simplifies to

$$\mathcal{H}_k^{G(H)}(h) = \left\{ \varphi = \frac{1}{k} \log \sum_{i=1}^{N} \lambda_i \sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k} \;\middle|\; \lambda_i > 0 \right\},$$

as claimed. □

6.2.4 Replacing a basis of sections by an orbit of a section

The next result generalizes [94, Proposition 2.1] to the equivariant setting using Lemma 6.2.5.

Proposition 6.2.6. *For a subgroup $G(H) \subseteq \operatorname{Aut} X$ (6.5) generated by $(S^1)^n$ and a subgroup H of $\operatorname{Aut} P$ (recall (6.19) and Definition 6.2.1),*

$$\alpha_{k,G(H)} = \sup \left\{ c > 0 \;\middle|\; \int_X \Big(\sum_{\sigma \in H} |s_{k,\sigma u}|^2_{h_k} \Big)^{-\frac{c}{k}} d\mu_k < \infty, \ \forall\, u \in P \cap M/k \right\}. \tag{6.27}$$

Proof. Step 1: estimate a basis-type element using the worst orbit present in its expansion. Comparing (6.15) and (6.16) gives a useful expression for h_k (which is not strictly needed for the computation below, but makes it slightly easier to follow),

$$h_k = \Big(\sum_{u \in P \cap M/k} |s_{k,u}|^2 \Big)^{-1}. \tag{6.28}$$

Now, assuming $\lambda_N = \max_{i=1,\dots,N} \lambda_i$, following the notation of Lemma 6.2.5,

$$\begin{aligned}
\sup \varphi &\leq \sup \left(\frac{1}{k} \log \sum_{i=1}^{N} \lambda_N \sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k} \right) \\
&= \sup \left(\frac{1}{k} \log \lambda_N \sum_{i=1}^{N} \sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k} \right) \\
&= \sup \left(\frac{1}{k} \log \lambda_N \sum_{u \in P \cap M/k} |s_{k,u}|^2_{h_k} \right) \\
&= \frac{1}{k} \log \lambda_N,
\end{aligned}$$

since by (6.28),

$$|s|^2_{h_k} = \frac{|s|^2}{\sum\limits_{u \in P \cap M/k} |s_{k,u}|^2}, \qquad \text{for any } s \in H^0(X, -kK_X).$$

If $c > 0$ is such that for each $i \in \{1, \dots, N\}$,

$$\int_X \left(\sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k} \right)^{-\frac{c}{k}} d\mu_k \leq C_c,$$

then for $\varphi \in \mathcal{H}_k^{G(H)}$ (using Lemma 6.2.5),

$$\begin{aligned}
\int_X e^{-c(\varphi - \sup \varphi)} d\mu_k &= e^{c \sup \varphi} \int_X \left(\sum_{i=1}^{N} \lambda_i \sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k} \right)^{-\frac{c}{k}} d\mu_k \\
&\leq e^{\frac{c}{k} \log \lambda_N} \int_X \left(\lambda_N \sum_{u \in O_N^{(k)}} |s_{k,u}|^2_{h_k} \right)^{-\frac{c}{k}} d\mu_k \\
&= \lambda_N^{\frac{c}{k}} \cdot \lambda_N^{-\frac{c}{k}} \int_X \left(\sum_{u \in O_N^{(k)}} |s_{k,u}|^2_{h_k} \right)^{-\frac{c}{k}} d\mu_k \leq C_c.
\end{aligned}$$

Step 2: estimate an orbit-type element using an approximation by degenerating basis-type elements.

Conversely, assume $c > 0$ is such that for any $\varphi \in \mathcal{H}_k^{G(H)}$,

$$\int_X e^{-c(\varphi - \sup \varphi)} d\mu_k \leq C_c.$$

Since, for any $\ell \in \{1, \dots, N\}$,

$$\left(\sum_{u \in O_\ell^{(k)}} |s_{k,u}|^2_{h_k} \right)^{-\frac{c}{k}} = \lim_{\lambda_i \to 0, i \neq \ell} \left(\sum_{i=1}^N \lambda_i \sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k} \right)^{-\frac{c}{k}},$$

where $\lambda_\ell = 1$, by Fatou's Lemma [44, Lemma 2.18],

$$\begin{aligned}
\int_X \left(\sum_{u \in O_\ell^{(k)}} |s_{k,u}|^2_{h^k} \right)^{-\frac{c}{k}} d\mu_k &\leq \liminf_{\lambda_i \to 0, i \neq \ell} e^{-c \sup \varphi_\lambda} e^{c \sup \varphi_\lambda} \int_X \left(\sum_{i=1}^N \lambda_i \sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k} \right)^{-\frac{c}{k}} d\mu_k \\
&= \left(\sup \sum_{u \in O_\ell^{(k)}} |s_{k,u}|^2_{h_k} \right)^{-\frac{c}{k}} \liminf_{\lambda_i \to 0, i \neq \ell} \int_X e^{-c(\varphi_\lambda - \sup \varphi_\lambda)} d\mu_k \\
&\leq \left(\sup \sum_{u \in O_\ell^{(k)}} |s_{k,u}|^2_{h_k} \right)^{-\frac{c}{k}} C_c,
\end{aligned}$$

where

$$\varphi_\lambda := \frac{1}{k} \log \sum_{i=1}^N \lambda_i \sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k}.$$

Thus, we have shown that

$$\begin{aligned}
\alpha_{k,G(H)} &= \sup \left\{ c > 0 \,\middle|\, \int_X \left(\sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k} \right)^{-\frac{c}{k}} d\mu_k < \infty, \ \forall\ i \in \{1, \dots, N\} \right\} \qquad (6.29) \\
&= \sup \left\{ c > 0 \,\middle|\, \int_X \left(\sum_{\sigma \in H} |s_{k,\sigma u}|^2_{h_k} \right)^{-\frac{c}{k}} d\mu_k < \infty, \ \forall\ u \in P \cap M/k \right\},
\end{aligned}$$

proving (6.27). $\square$

From (6.29):

Corollary 6.2.7. *Fix $k \in \mathbb{N}$ and let $O_1^{(k)}, \ldots, O_N^{(k)}$ be the orbits of the action of H on $k^{-1}M \cap P$. Then*

$$\alpha_{k,G(H)} = \min_{1 \leq i \leq N} \sup \left\{ c > 0 \,\middle|\, \int_X \left(\sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k} \right)^{-\frac{c}{k}} d\mu_k < \infty \right\}.$$

We can now answer affirmatively Problem 6.0.2 in our setting.

Proof of Proposition 6.0.3. Let $\left|V_{O_i^{(k)}}\right|$ denote the linear system generated by $\{(s_{k,u}) \mid u \in O_i^{(k)}\}$ (recall (6.25)). By (6.21) and Corollary 6.2.7,

$$\alpha_{k,G(H)} = k \min_{i \in \{1,\ldots,N\}} \operatorname{lct} \left|V_{O_i^{(k)}}\right|.$$

It remains to show

$$\operatorname{glct}_{k,G(H)} = k \min_{i \in \{1,\ldots,N\}} \operatorname{lct} \left|V_{O_i^{(k)}}\right|.$$

By (6.23), it suffices in (6.24) to restrict to irreducible $G(H)$-invariant linear systems (cf. [70, p. 146]). Now, any $(S^1)^n$-invariant linear system $|V|$ is spanned by monomials (see Lemma 6.2.8 below). By irreducibility, this means $|V|$ is spanned by the monomials coming from some H-orbit O_i. i.e., $|V| = \left|V_{O_i^{(k)}}\right|$. □

Lemma 6.2.8. *Let $V \subseteq H^0(X, -kK_X)$ be a complex vector subspace invariant under $(S^1)^n$. Then there exists a unique subset $\mathcal{F} \subseteq P \cap M/k$ such that*

$$V = \operatorname{Span}\{s_{k,u}\}_{u \in \mathcal{F}}.$$

Proof. It suffices to prove that any section in V is generated by monomials in V. That is, given any section

$$s = \sum_{u \in P \cap M/k} a_u s_{k,u} \in V,$$

if $a_u \neq 0$, then $s_{k,u} \in V$.

By $(S^1)^n$-invariance, for any $\beta \in (S^1)^n$ and $s \in V$ also $\beta.s \in V$ (recall (6.26)), i.e.,

$$\beta.s = \sum_{u \in P \cap M/k} a_u \beta.s_{k,u} = \sum_{u \in P \cap M/k} a_u e^{\sqrt{-1}\langle \beta, ku\rangle} s_{k,u} \in V.$$

Thus for any $u_0 \in P \cap M/k$,

$$\begin{aligned}
\int_{(S^1)^n} e^{-\sqrt{-1}\langle \beta, ku_0\rangle} \beta.s \, d\beta &= \sum_{u \in P \cap M/k} a_u s_{k,u} \int_{(S^1)^n} e^{\sqrt{-1}\langle \beta, ku - ku_0\rangle} d\beta \\
&= \sum_{u \in P \cap M/k} a_u s_{k,u} \cdot (2\pi)^n \delta_{u-u_0} \\
&= (2\pi)^n a_{u_0} s_{k,u_0} \in V.
\end{aligned}$$

If $a_{u_0} \neq 0$, then $s_{k,u_0} \in V$. This completes the proof. $\square$

Corollary 6.2.9.

$$\alpha_{G(H)} = \inf_k \alpha_{k,G(H)} = \sup\left\{ c > 0 \,\middle|\, \int_X \left(\sum_{\sigma \in H} |s_{k,\sigma u}|^2_{h_k} \right)^{-\frac{c}{k}} d\mu_k < \infty, \ \forall \ u \in P \cap M/k, \ k \in \mathbb{N} \right\}. \quad (6.30)$$

Proof. By Demailly's theorem (6.22) [29, (A.1)], (6.24), and Proposition 6.0.3,

$$\alpha_{G(H)} = \inf_{k \in \mathbb{N}} \mathrm{glct}_{k,G(H)} = \inf_{k \in \mathbb{N}} \alpha_{k,G(H)}.$$

By (6.27),

$$\begin{aligned}
\alpha_{G(H)} &= \inf_{k \in \mathbb{N}} \sup\left\{ c > 0 \,\middle|\, \int_X \left(\sum_{\sigma \in H} |s_{k,\sigma u}|^2_{h_k} \right)^{-\frac{c}{k}} d\mu_k < \infty, \ \forall \ u \in P \cap M/k \right\} \\
&= \sup\left\{ c > 0 \,\middle|\, \int_X \left(\sum_{\sigma \in H} |s_{k,\sigma u}|^2_{h_k} \right)^{-\frac{c}{k}} d\mu_k < \infty, \ \forall \ u \in P \cap M/k, \ k \in \mathbb{N} \right\}.
\end{aligned}$$

$\square$

Remark 6.2.10. Note that here it would not have been enough to invoke Tian's theorem [103, Proposition 6.1], and it was necessary to invoke Demailly's theorem [29, (A.1)]. The reason is that the $\alpha_{k,G}$-invariants are defined by requiring certain integrals to be merely finite, i.e., bounded, but with a constant possibly depending on k. Tian's theorem says that $\mathcal{H}^G$ is approximated by $\mathcal{H}_k^G$, but in approximating $\varphi \in \mathcal{H}^G$ by a sequence $\varphi_k \in \mathcal{H}_k^G$ it might happen that the integrals $\int_X e^{-c(\varphi_k - \sup \varphi_k)}\omega^n$ blow up as k tends to infinity. Demailly precludes that from happening by using the Demailly–Kollár lower semi-continuity of complex singularity exponents.

6.3 The case of divisible k

We ultimately improve on the results of this section, but we include them since they serve to emphasize the difficulties that still need to be dealt with (see Remark 6.3.6).

Proposition 6.3.1. *Let $G(H) \subseteq \operatorname{Aut} X$ (6.5) be a subgroup generated by $(S^1)^n$ and a subgroup H of $\operatorname{Aut} P$. For $k, \ell \in \mathbb{N}$, $\alpha_{k,G(H)} \geq \alpha_{k\ell,G(H)}$.*

Proof. Since $h_{k\ell}$ and h_k^ℓ (6.16) are both smooth metrics on $-k\ell K_X$, by compactness of X, they are equivalent. We also use the following lemma.

Lemma 6.3.2. *Let $a_1, \ldots, a_N \geq 0$. Then for $\ell \in \mathbb{N}$,*

$$\sum_{i=1}^N a_i^\ell \leq \left(\sum_{i=1}^N a_i\right)^\ell \leq N^{\ell-1} \sum_{i=1}^N a_i^\ell.$$

Proof. The first inequality follows by expanding $(\sum_{i=1}^N a_i)^\ell$. For the second inequality, consider the function $f(x) := x^\ell$ on $[0, +\infty)$. Since f is convex, by Jensen's inequality,

$$f\left(\frac{1}{N}\sum_{i=1}^N a_i\right) \leq \frac{1}{N}\sum_{i=1}^N f(a_i),$$

i.e., $\left(\sum_{i=1}^N a_i\right)^\ell \leq N^{\ell-1} \sum_{i=1}^N a_i^\ell.$ □

By Proposition 6.2.6, and since $M/k\ell \subseteq M/k$,

$$\alpha_{k\ell,G(H)} = \sup\left\{ c > 0 \;\middle|\; \int_X \Big(\sum_{\sigma \in H} |s_{k\ell,\sigma u}|^2_{h_{k\ell}} \Big)^{-\frac{c}{k\ell}} d\mu_k < \infty, \ \forall\, u \in P \cap M/k\ell \right\}$$

$$
\begin{aligned}
&\leq \sup\left\{ c>0 \,\middle|\, \int_X \Big(\sum_{\sigma\in H} |s_{k\ell,\sigma u}|^2_{h_{k\ell}} \Big)^{-\frac{c}{k\ell}} d\mu_k < \infty,\ \forall\, u \in P\cap M/k \right\} \\
&= \sup\left\{ c>0 \,\middle|\, \int_X \Big(\sum_{\sigma\in H} \Big|s^{\otimes\ell}_{k,\sigma u}\Big|^2_{h^\ell_k} \Big)^{-\frac{c}{k\ell}} d\mu_k < \infty,\ \forall\, u \in P\cap M/k \right\} \\
&= \sup\left\{ c>0 \,\middle|\, \int_X \Big(\sum_{\sigma\in H} |s_{k,\sigma u}|^{2\ell}_{h_k} \Big)^{-\frac{c}{k\ell}} d\mu_k < \infty,\ \forall\, u \in P\cap M/k \right\} \\
&= \sup\left\{ c>0 \,\middle|\, \int_X \Big(\sum_{\sigma\in H} |s_{k,\sigma u}|^2_{h_k} \Big)^{-\frac{c}{k}} d\mu_k < \infty,\ \forall\, u \in P\cap M/k \right\} \\
&= \alpha_{k,G(H)},
\end{aligned}
$$

where Lemma 6.3.2 was invoked in the penultimate equality. □

Proposition 6.3.3. *Let $G(H) \subseteq \operatorname{Aut} X$ (6.5) be a subgroup generated by $(S^1)^n$ and a subgroup H of $\operatorname{Aut} P$. There exists $k_0 \in \mathbb{N}$ such that $\alpha_{K,G(H)} = \alpha_{G(H)}$ for all K divisible by k_0.*

Remark 6.3.4. In fact, the proof will show that k_0 is determined by the fan as follows: let $u_0 \in P^H$ attain $\sup_{u\in P^H} \max_i \langle u, v_i\rangle$. Since P^H is a convex polytope u_0 will be a vertex of P^H, i.e., cut out by P^H and the supporting hyperplanes of P containing u_0. The equations defining P^H are determined by $H \subseteq GL(M)$ hence are linear equations with integer coefficients. So are the equations cutting out ∂P. It follows that u_0 is a rational point, i.e., $u_0 \in M/k_0$ for some k_0. Let k_0 be the smallest such positive integer. The fact that u_0 is a rational point is originally due to Song [97, p. 1257] who proved that $\alpha_{k_0,G(\operatorname{Aut} P)} = \alpha_{G(\operatorname{Aut} P)}$.

Proof. We can further simplify (6.30). Recall (6.7). By the geometric-arithmetic mean inequality,

$$
\begin{aligned}
\left(\frac{1}{|H|} \sum_{\sigma\in H} |s_{k,\sigma u}|^2_{h_k} \right)^{-\frac{\alpha}{k}} \leq \prod_{\sigma\in H} |s_{k,\sigma u}|^{-\frac{2\alpha}{|H|k}}_{h_k} &= \big|s_{|H|k,\pi_H(u)}\big|^{-\frac{2\alpha}{|H|k}}_{h_{|H|k}} \\
&= \left(\frac{1}{|H|} \sum_{\sigma\in H} \big|s_{|H|k,\sigma\pi_H(u)}\big|^2_{h_{|H|k}} \right)^{-\frac{\alpha}{|H|k}},
\end{aligned}
$$

with the last equality since $\pi_H(u)$ is fixed by H (recall (6.6)–(6.7)) so each term in the sum is identical. Therefore the supremum in (6.30) is unchanged if restricted to those u fixed by H. Thus, using Lemma

6.3.5 (proven below), (6.30) simplifies to (recall (6.6))

$$\begin{aligned}
\alpha_{G(H)} &= \sup\left\{c>0 \;\middle|\; \int_X |s_{k,u}|_{h_k}^{-\frac{2c}{k}}\, d\mu_k < \infty,\ \forall\, u\in P^H\cap \frac{1}{k}M,\ k\in\mathbb{N}\right\}\\
&= \inf\left\{k\cdot \operatorname{lct}(s_{k,u}) \;\middle|\; u\in P^H\cap \frac{1}{k}M,\ k\in\mathbb{N}\right\}\\
&= \inf_{k\in\mathbb{N}} \inf_{u\in P^H\cap\frac{1}{k}M} \frac{1}{\max_i\langle u,v_i\rangle+1}\\
&= \inf_{u\in P^H} \frac{1}{\max_i\langle u,v_i\rangle+1}\\
&= \frac{1}{\max_i\langle u_0,v_i\rangle+1},
\end{aligned}$$

where we used the notation of Remark 6.3.4. By that same Remark, $u_0\in P^H\cap\frac{1}{K}M$ whenever $K\in\mathbb{N}$ is divisible by k_0. In particular, by (6.27), and using Lemma 6.3.5 again,

$$\begin{aligned}
\alpha_{K,G(H)} &\le \sup\left\{c>0 \;\middle|\; \int_X |s_{K,u_0}|_{h_K}^{-\frac{2c}{K}}\, d\mu_k < \infty\right\}\\
&= K\cdot\operatorname{lct}(s_{K,u_0})\\
&= \min_i \frac{1}{\langle u_0,v_i\rangle+1}\\
&= \alpha_{G(H)}.
\end{aligned}$$

Since by definition $\alpha_{G(H)}\le\alpha_{K,G(H)}$, equality is achieved and $\alpha_{K,G(H)}=\alpha_{G(H)}$ for all K divisible by k_0. ☐

Lemma 6.3.5. *For $u\in P\cap k^{-1}M$, $\operatorname{lct}(s_{k,u})=\dfrac{1}{k}\dfrac{1}{1+\max_i\langle u,v_i\rangle}$ (recall* (3.13) *and* (6.11)*).*

Proof. This is well-known (see, e.g., [16, Corollary 7.4], [70, Theorem 4.1]). It is also a consequence of Proposition 6.4.8 proven below (put $\mathcal{F}=\{u\}$). ☐

Remark 6.3.6. According to Proposition 6.3.3, the sequence $\{\alpha_{k,G(H)}\}_{k\in\mathbb{N}}$ is constant (equal to $\alpha_{G(H)}$) along the subsequence $k_0, 2k_0, \ldots$. Proposition 6.3.3 does not yield information for all $k\in\mathbb{N}$ unless $k_0=1$, and examples show (see §6.6) that oftentimes $k_0>1$. Moreover, even though the sequence also satisfies

$\alpha_{k,G(H)} \geq \alpha_{k\ell,G(H)} \geq \alpha_{G(H)}$ for any $k, \ell \in \mathbb{N}$ (Proposition 6.3.1), these facts combined are still not enough to conclude Tian's conjecture without further work. For instance, the sequence

$$a_k := \begin{cases} \alpha_{G(H)}, & k \text{ even}, \\ \alpha_{G(H)} + k^{-1}, & k \text{ odd}, \end{cases}$$

satisfies these requirements for $k_0 = 2$ (and also satisfy $\lim_k a_k = \alpha_{G(H)}$). Perhaps a more natural sequence that satisfies all the requirements and even for the k_0 of Remark 6.3.4 is

$$a_k := \inf\left\{ k \cdot \operatorname{lct}(s_{k,u}) \,\middle|\, u \in P^H \cap \frac{1}{k}M \right\},$$

with the proof of Proposition 6.3.3 showing that $a_{k_0\ell} = a_{k_0} = \alpha_{G(H)}$ for $\ell \in \mathbb{N}$ but possibly $a_k > \alpha_{G(H)}$ for k not divisible by k_0. Thus, we are led to develop more refined estimates that are the topic of the next section.

6.4 Estimating singularities associated to orbits

6.4.1 Real singularity exponents and support functions

In this subsection we develop a key new technical estimate that expresses real singularity exponents associated to collections of toric monomials in terms of support functions.

Definition 6.4.1. For non-empty finite sets $\mathcal{F}, \mathcal{U} \subseteq \mathbb{R}^n$ and $k \in \mathbb{N}$,

$$c_k(\mathcal{F}, \mathcal{U}) := \sup\left\{ c \in (0,1) \,\middle|\, \int_{\mathbb{R}^n} \frac{\left(\sum_{u \in \mathcal{F}} e^{\langle ku, x\rangle}\right)^{-\frac{c}{k}}}{\left(\sum_{u \in \mathcal{U}} e^{\langle ku, x\rangle}\right)^{\frac{1-c}{k}}} dx < \infty \right\}.$$

Proposition 6.4.2. *For $\mathcal{F}, \mathcal{U} \subseteq \mathbb{R}^n$ non-empty finite sets with $0 \in \operatorname{int} \operatorname{co} \mathcal{U}$ (recall (3.1)),*

$$\begin{aligned} c_k(\mathcal{F}, \mathcal{U}) &= \sup\{ c \in (0,1) \,|\, 0 \in (1-c)\operatorname{co}\mathcal{U} + c \operatorname{co}\mathcal{F} \} \\ &= \sup\left\{ c \in (0,1) \,\middle|\, \left(-\frac{c}{1-c} \operatorname{co}\mathcal{F} \right) \cap \operatorname{co}\mathcal{U} \neq \emptyset \right\} > 0. \end{aligned}$$

In particular, $c_k(\mathcal{F},\mathcal{U})$ is independent of k.

Proof. Using polar coordinates $x = r\nu$ and $dx = r^{n-1}dr \wedge dS(\nu)$ with $r \in \mathbb{R}_+$, $\nu \in S^{n-1}(1) := \{x \in \mathbb{R}^n \,|\, |x| = 1\}$,

$$\int_{\mathbb{R}^n} \frac{\left(\sum_{u\in\mathcal{F}} e^{\langle ku,x\rangle}\right)^{-\frac{c}{k}}}{\left(\sum_{u\in\mathcal{U}} e^{\langle ku,x\rangle}\right)^{\frac{1-c}{k}}} dx = \int_{S^{n-1}(1)} f_c(\nu) dS(\nu), \tag{6.31}$$

where $f_c : S^{n-1}(1) \to \mathbb{R}_+$ is defined by

$$f_c(\nu) := \int_0^\infty \frac{\left(\sum_{u\in\mathcal{F}} e^{kr\langle u,\nu\rangle}\right)^{-\frac{c}{k}}}{\left(\sum_{u\in\mathcal{U}} e^{kr\langle u,\nu\rangle}\right)^{\frac{1-c}{k}}} r^{n-1} dr.$$

Notice that for any finite set $\mathcal{A} \subseteq \mathbb{R}^n$ and $r \in \mathbb{R}_+$ (recall (3.2)),

$$e^{krh_{\mathcal{A}}(\nu)} \le \sum_{u\in\mathcal{A}} e^{kr\langle u,\nu\rangle} \le |\mathcal{A}| e^{krh_{\mathcal{A}}(\nu)}.$$

Hence for $c \in (0,1)$,

$$\frac{|\mathcal{F}|^{-\frac{c}{k}}}{|\mathcal{U}|^{\frac{1-c}{k}}} e^{-krg_c(\nu)} \le \frac{\left(\sum_{u\in\mathcal{F}} e^{kr\langle u,\nu\rangle}\right)^{-\frac{c}{k}}}{\left(\sum_{u\in\mathcal{U}} e^{kr\langle u,\nu\rangle}\right)^{\frac{1-c}{k}}} \le e^{-krg_c(\nu)}, \tag{6.32}$$

where

$$\begin{aligned} g_c(x) &:= c \max_{u\in\mathcal{F}} \langle u,x\rangle + (1-c) \max_{u\in\mathcal{U}} \langle u,x\rangle \\ &= h_{c\mathcal{F}+(1-c)\mathcal{U}}(x) \\ &= c \max_{u\in\operatorname{co}\mathcal{F}} \langle u,x\rangle + (1-c) \max_{u\in\operatorname{co}\mathcal{U}} \langle u,x\rangle \\ &= c h_{\operatorname{co}\mathcal{F}}(x) + (1-c) h_{\operatorname{co}\mathcal{U}}(x) = h_{c\operatorname{co}\mathcal{F}+(1-c)\operatorname{co}\mathcal{U}}(x), \end{aligned} \tag{6.33}$$

where $A + B := \{x + y \,|\, x \in A, y \in B\}$ is the Minkowski sum.

We need the following property of support functions.

Claim 6.4.3. *Let $\mathcal{A} \subseteq \mathbb{R}^n$ be a nonempty finite set. Then $h_{\mathcal{A}}|_{S^{n-1}(1)} > 0$ if and only if $0 \in \operatorname{int}\operatorname{co}\mathcal{A}$.*

Proof. Note first that

$$h_{\mathcal{A}} = h_{\operatorname{co}\mathcal{A}} : \tag{6.34}$$

$h_{\mathcal{A}} \le h_{\mathrm{co}\,\mathcal{A}}$ since $\mathcal{A} \subseteq \mathrm{co}\,\mathcal{A}$, while if $h_{\mathrm{co}\,\mathcal{A}}(y) = \langle a(y), y\rangle$ for some $a(y) \in \mathrm{co}\,\mathcal{A}$, and writing $a(y) = \sum_{i=1}^{|\mathcal{A}|} \lambda_i a_i$ with $\sum_{i=1}^{|\mathcal{A}|} \lambda_i = 1$ and $\lambda_i \ge 0$, where $\mathcal{A} = \{a_i\}$ yields $h_{\mathrm{co}\,\mathcal{A}}(y) = \sum_{i=1}^{|\mathcal{A}|} \lambda_i \langle a_i, y\rangle \le \sum_{i=1}^{|\mathcal{A}|} \lambda_i h_{\mathcal{A}}(y) = h_{\mathcal{A}}(y)$.

Next, if $0 \in \mathrm{int}\,\mathrm{co}\,\mathcal{A}$ then $\varepsilon B_2^n \subseteq \mathrm{co}\,\mathcal{A}$ where $B_2^n := \{x \in \mathbb{R}^n \,|\, |x| \le 1\}$ and $h_{\mathrm{co}\,\mathcal{A}} \ge h_{\varepsilon B_2^n} = \varepsilon h_{B_2^n} = \varepsilon$ when restricted to $S^{n-1}(1)$.

Conversely, if $0 \notin \mathrm{int}\,\mathrm{co}\,\mathcal{A}$ then by convexity of $\mathrm{co}\,\mathcal{A}$ there exists a hyperplane H passing through a boundary point of $\mathrm{co}\,\mathcal{A}$ so that $\mathrm{co}\,\mathcal{A}$ lies on one side of it and 0 lies on the other side of it. If $\nu \in S^{n-1}(1)$ is normal to H and points toward the side not containing $\mathrm{co}\,\mathcal{A}$ then $h_{\mathrm{co}\,\mathcal{A}}(\nu) \le 0$. □

Claim 6.4.4. *Let $\mathcal{A} \subseteq \mathbb{R}^n$ be a nonempty finite set. Then $h_{\mathcal{A}}|_{S^{n-1}(1)}$ is somewhere negative if and only if $0 \in \mathrm{int}(\mathbb{R}^n \setminus \mathrm{co}\,\mathcal{A}) = \mathbb{R}^n \setminus \mathrm{co}\,\mathcal{A}$.*

Proof. Suppose $0 \notin \mathrm{int}(\mathbb{R}^n \setminus \mathrm{co}\,\mathcal{A}) = \mathbb{R}^n \setminus \mathrm{co}\,\mathcal{A}$. Equivalently $0 \in \mathrm{co}\,\mathcal{A}$. Then $h_{\mathcal{A}} \ge h_{\{0\}} = 0$ so $h_{\mathcal{A}}$ is nowhere negative.

Conversely, if $0 \notin \mathrm{co}\,\mathcal{A}$ then by convexity of $\mathrm{co}\,\mathcal{A}$ there exists a hyperplane H passing through the origin so that $\mathrm{co}\,\mathcal{A}$ lies *strictly* on one side. Let $\nu \in S^{n-1}(1)$ be a unit normal to H that points toward the side not containing $\mathrm{co}\,\mathcal{A}$. By the strictness mentioned above and compactness of $\mathrm{co}\,\mathcal{A}$, for some small $\varepsilon > 0$, $\mathrm{co}\,\mathcal{A} + \varepsilon\nu$ still lies on the same side of H as $\mathrm{co}\,\mathcal{A}$. Hence, as in the proof of Claim 6.4.3, $h_{\mathrm{co}\,\mathcal{A}+\varepsilon\nu}(\nu) \le 0$. In other words,

$$h_{\mathrm{co}\,\mathcal{A}}(\nu) = h_{\mathrm{co}\,\mathcal{A}+\varepsilon\nu}(\nu) - \varepsilon|\nu|^2 \le -\varepsilon,$$

as claimed. □

Corollary 6.4.5. *For $\mathcal{F}, \mathcal{U} \subseteq \mathbb{R}^n$ non-empty finite sets with $0 \in \mathrm{int}\,\mathrm{co}\mathcal{U}$ (recall (3.1)),*

$$\sup\left\{c \in (0,1) \,\middle|\, \left(-\frac{c}{1-c}\,\mathrm{co}\,\mathcal{F}\right) \cap \mathrm{co}\mathcal{U} \ne \emptyset\right\} = \begin{cases} 1, & 0 \in \mathrm{co}\,\mathcal{F}, \\ \left[1 - \min\limits_{S^{n-1}(1)} \dfrac{h_{\mathcal{F}}}{h_{\mathcal{U}}}\right]^{-1}, & 0 \notin \mathrm{co}\,\mathcal{F}. \end{cases} \tag{6.35}$$

Proof. By assumption $0 \in \mathrm{int}\,\mathrm{co}\mathcal{U}$, so by Claim 6.4.3, $h_{\mathcal{U}}|_{S^{n-1}(1)} > 0$. By (6.33),

$$g_c = h_{\mathrm{co}\mathcal{U}} \cdot \left(1 - c\left(1 - \frac{h_{\mathrm{co}\,\mathcal{F}}}{h_{\mathrm{co}\mathcal{U}}}\right)\right). \tag{6.36}$$

Case 1: $0 \in \operatorname{co}\mathcal{F}$. If $0 \in \operatorname{co}\mathcal{F}$, i.e., $h_{\operatorname{co}\mathcal{F}} \geq 0$, then $g_c \geq h_{\operatorname{co}\mathcal{U}} \cdot (1-c) > 0$ for any $c \in (0,1)$. By (6.36) and Claim 6.4.3 then $0 \in \operatorname{int}(c \operatorname{co}\mathcal{F} + (1-c)\operatorname{co}\mathcal{U}) \subseteq c\operatorname{co}\mathcal{F} + (1-c)\operatorname{co}\mathcal{U}$ and so (6.35) holds in this case.

Case 2: $0 \notin \operatorname{co}\mathcal{F}$. By assumption and Claim 6.4.4, $h_{\operatorname{co}\mathcal{F}} < 0$ somewhere, by continuity/compactness let ν_0 be a minimizer of the function

$$\frac{h_{\operatorname{co}\mathcal{F}}}{h_{\operatorname{co}\mathcal{U}}}$$

on $S^{n-1}(1)$. In particular,

$$\frac{h_{\operatorname{co}\mathcal{F}}(\nu_0)}{h_{\operatorname{co}\mathcal{U}}(\nu_0)} = \min_{S^{n-1}(1)} \frac{h_{\mathcal{F}}}{h_{\mathcal{U}}} < 0.$$

Set

$$c(\mathcal{F},\mathcal{U}) := \left[1 - \frac{h_{\operatorname{co}\mathcal{F}}(\nu_0)}{h_{\operatorname{co}\mathcal{U}}(\nu_0)}\right]^{-1} = \left[1 - \min_{S^{n-1}(1)} \frac{h_{\mathcal{F}}}{h_{\mathcal{U}}}\right]^{-1}.$$

First, if $c \in (0, c(\mathcal{F},\mathcal{U}))$, for all $\nu \in S^{n-1}(1)$,

$$\begin{aligned} g_c(\nu) &= h_{\operatorname{co}\mathcal{U}}(\nu)\left(1 - c\left(1 - \frac{h_{\operatorname{co}\mathcal{F}}(\nu)}{h_{\operatorname{co}\mathcal{U}}(\nu)}\right)\right) \\ &\geq h_{\operatorname{co}\mathcal{U}}(\nu)\left(1 - c\left(1 - \frac{h_{\operatorname{co}\mathcal{F}}(\nu_0)}{h_{\operatorname{co}\mathcal{U}}(\nu_0)}\right)\right) \\ &= h_{\operatorname{co}\mathcal{U}}(\nu)\left(1 - \frac{c}{c(\mathcal{F},\mathcal{U})}\right) > 0. \end{aligned}$$

Thus, by (6.33) and Claim 6.4.3, $0 \in \operatorname{int}(c\operatorname{co}\mathcal{F} + (1-c)\operatorname{co}\mathcal{U}) \subseteq c\operatorname{co}\mathcal{F} + (1-c)\operatorname{co}\mathcal{U}$; in particular,

$$\left(-\frac{c}{1-c}\operatorname{co}\mathcal{F}\right) \cap \operatorname{co}\mathcal{U} \neq \emptyset.$$

So,

$$\sup\left\{c \in (0,1) \,\middle|\, \left(-\frac{c}{1-c}\operatorname{co}\mathcal{F}\right) \cap \operatorname{co}\mathcal{U} \neq \emptyset\right\} \geq c(\mathcal{F},\mathcal{U}).$$

Second, if $c \in (c(\mathcal{F},\mathcal{U}), 1)$,

$$\begin{aligned} g_c(\nu_0) &= h_{\operatorname{co}\mathcal{U}}(\nu_0)\left(1 - c\left(1 - \frac{h_{\operatorname{co}\mathcal{F}}(\nu_0)}{h_{\operatorname{co}\mathcal{U}}(\nu_0)}\right)\right) \\ &= h_{\operatorname{co}\mathcal{U}}(\nu_0)\left(1 - \frac{c}{c(\mathcal{F},\mathcal{U})}\right) < 0. \end{aligned}$$

Thus, by Claim 6.4.4, $0 \in \mathbb{R}^n \setminus (c\,\mathrm{co}\,\mathcal{F} + (1-c)\,\mathrm{co}\,\mathcal{U})$, equivalently,

$$\Big(-\frac{c}{1-c}\,\mathrm{co}\,\mathcal{F}\Big) \cap \mathrm{co}\,\mathcal{U} = \emptyset,$$

and

$$\sup\Big\{c \in (0,1) \,\Big|\, \Big(-\frac{c}{1-c}\,\mathrm{co}\,\mathcal{F}\Big) \cap \mathrm{co}\,\mathcal{U} \neq \emptyset\Big\} \leq c(\mathcal{F},\mathcal{U}).$$

Thus, also in this case (6.35) holds. $\square$

The following corollary follows from the proof of Corollary 6.4.5.

Corollary 6.4.6. *For $\mathcal{F},\mathcal{U} \subseteq \mathbb{R}^n$ non-empty finite sets with $0 \in \mathrm{int}\,\mathrm{co}\,\mathcal{U}$ (recall (3.1)), let*

$$\begin{aligned} c(\mathcal{F},\mathcal{U}) :&= \sup\Big\{c \in (0,1) \,\Big|\, \Big(-\frac{c}{1-c}\,\mathrm{co}\,\mathcal{F}\Big) \cap \mathrm{co}\,\mathcal{U} \neq \emptyset\Big\} \\ &= \sup\{c \in (0,1) \,|\, 0 \in (1-c)\,\mathrm{co}\,\mathcal{U} + c\,\mathrm{co}\,\mathcal{F}\}. \end{aligned}$$

Recall (6.33). If $c < c(\mathcal{F},\mathcal{U})$, then $g_c > 0$; if $c > c(\mathcal{F},\mathcal{U})$, then $g_c < 0$ somewhere.

We can now complete the proof of Proposition 6.4.2. Set

$$c(\mathcal{F},\mathcal{U}) := \sup\Big\{c \in (0,1) \,\Big|\, \Big(-\frac{c}{1-c}\,\mathrm{co}\,\mathcal{F}\Big) \cap \mathrm{co}\,\mathcal{U} \neq \emptyset\Big\}.$$

By assumption $0 \in \mathrm{int}\,\mathrm{co}\,\mathcal{U}$, thus $c(\mathcal{F},\mathcal{U}) > 0$. We consider two cases.

Case 1: $c \in (0, c(\mathcal{F},\mathcal{U}))$. By the proof of Corollary 6.4.5, $g_c|_{S^{n-1}(1)} > 0$. By continuity/compactness it attains its minimum $g_c(\hat{\nu}) > 0$. Therefore,

$$f_c(\nu) \leq \int_0^\infty e^{-krg_c(\nu)} r^{n-1} dr \leq \int_0^\infty e^{-krg_c(\hat{\nu})} r^{n-1} dr < \infty, \tag{6.37}$$

so by (6.31), $c_k(\mathcal{F},\mathcal{U}) \geq c$, i.e., $c_k(\mathcal{F},\mathcal{U}) \geq c(\mathcal{F},\mathcal{U})$.

Case 2: $c \in (c(\mathcal{F},\mathcal{U}), 1)$. By the proof of Corollary 6.4.5, $g_c(\nu_0) < 0$ for some open neighborhood $U \subseteq$

$S^{n-1}(1)$. Thus, by (6.32), for some constant $C = C(c, k, \mathcal{F}, \mathcal{U}) > 0$,

$$f_c(\nu) \geq C \int_0^\infty e^{-krg_c(\nu)} r^{n-1} dr = \infty, \quad \text{for } \nu \in U,$$

so $c_k(\mathcal{F}, \mathcal{U}) \leq c$, i.e., $c_k(\mathcal{F}, \mathcal{U}) \leq c(\mathcal{F}, \mathcal{U})$.

In conclusion, $c_k(\mathcal{F}, \mathcal{U}) = c(\mathcal{F}, \mathcal{U})$ and Proposition 6.4.2 is proved. □

6.4.2 Complex singularity exponents

Now, we go back to our setting of estimating complex singularity exponents, where P was a reflexive polytope coming from a fan. For any $k \in \mathbb{N}$ and a non-empty subset $\mathcal{F} \subseteq P \cap M/k$ set

$$c_k(\mathcal{F}) := \sup \left\{ c \in (0,1) \;\middle|\; \int_X \left(\sum_{u \in \mathcal{F}} |s_{k,u}|^2_{h_k} \right)^{-\frac{c}{k}} d\mu_k < \infty \right\}. \tag{6.38}$$

Remark 6.4.7. Consider

$$\widetilde{c_k}(\mathcal{F}) := \sup \left\{ c \in (0,\infty) \;\middle|\; \int_X \left(\sum_{u \in \mathcal{F}} |s_{k,u}|^2_{h_k} \right)^{-\frac{c}{k}} d\mu_k < \infty \right\}.$$

Then, by definition (6.21), $\widetilde{c_k}(\mathcal{F}) = k \operatorname{lct} \left| \operatorname{Span}\{s_{k,u}\}_{u \in \mathcal{F}} \right|$. However, Proposition 6.4.2 would not be applicable then. The point is that for the proof of Tian's conjecture we need to compute equivariant *global* log canonical thresholds and there will always be 'admissible' invariant linear series for which k times the log canonical threshold is at most 1 (i.e., the "worst" ones will not have $\widetilde{c_k} > 1$), i.e., for which $c_k = \widetilde{c_k}$. This is essentially because our H are linear groups so $\{0\}$ is always an H-orbit and $\operatorname{lct}(s_{k,0}) = 1/k = c_k(\{0\}) = \widetilde{c_k}(\{0\})$ (by Lemma 6.3.5).

Proposition 6.4.8. *For any $k \in \mathbb{N}$ and a non-empty subset $\mathcal{F} \subseteq P \cap M/k$,*

$$c_k(\mathcal{F}) = \sup \left\{ c \in (0,1) \;\middle|\; P \cap \left(-\frac{c}{1-c} \operatorname{co} \mathcal{F} \right) \neq \emptyset \right\} > 0.$$

In particular, it is independent of k.

Proof. By (6.19),

$$\int_X \left(\sum_{u\in\mathcal{F}} |s_{k,u}|^2_{h_k}\right)^{-\frac{c}{k}} d\mu_k = (2\pi)^n \int_{\mathbb{R}^n} \frac{\left(\sum_{u\in\mathcal{F}} e^{\langle ku,x\rangle}\right)^{-\frac{c}{k}}}{\left(\sum_{u\in P\cap M/k} e^{\langle ku,x\rangle}\right)^{\frac{1-c}{k}}} dx.$$

By Proposition 6.4.2 with $\mathcal{U} = P\cap M/k$,

$$c_k(\mathcal{F}) = \sup\left\{c\in(0,1)\,\middle|\, P\cap\Big(-\frac{c}{1-c}\operatorname{co}\mathcal{F}\Big)\neq\emptyset\right\} > 0,$$

since $\operatorname{co}\mathcal{U} = P$ (as the vertices of P belong to M/k for all $k\in\mathbb{N}$). □

Remark 6.4.9. Although not needed for our analysis, it might be of independent interest to understand whether the supremum in (6.38) (and in the definition of $\alpha_{k,G(H)}$ (6.27)) is attained. In other words, is (recall the notation of Corollary 6.4.6)

$$\int_{S^{n-1}(1)} f_c(\nu)dS(\nu) \tag{6.39}$$

finite for $c = c(\mathcal{F},\mathcal{U})$? By (6.31)–(6.32) it is equivalent to consider this question for the integral

$$\int_{S^{n-1}(1)}\int_0^\infty e^{-krg_c(\nu)}r^{n-1}drdS(\nu) = \int_{\mathbb{R}^n} e^{-kh_{(1-c)\operatorname{co}\mathcal{U}+c\operatorname{co}\mathcal{F}}(x)}dx. \tag{6.40}$$

A classical formula in convex geometry says that whenever $K\subseteq\mathbb{R}^n$ is a compact convex set with the origin in its interior, then $n!|K^\circ| = \int_{\mathbb{R}^n} e^{-h_K(y)}dy$ (for a detailled proof see [12, (4.2)]). In fact, it is possible to show that when $0\in\partial K$ the formula still holds in the sense that both sides are equal to ∞ (this is related to, but stronger than, the classical fact that $0\in\operatorname{int}K$ if and only if K° is bounded [86, Corollary 14.5.1]). To summarize, both (6.39) and (6.40) are infinite when $c = c(\mathcal{F},\mathcal{U})$. We remark that one can also use this point of view to give an alternative, but equivalent, proof of Corollary 6.4.6.

6.4.3 Group orbits and singularities

Consider the orbit-averaging map π_H (6.7), taking a point to the average of its image under H. In particular, points on the same orbit have the same image. Let $M^H_{\mathbb{R}}$ be the subspace of fixed points of H

in $M_{\mathbb{R}}$. Then $\pi_H|_{M_{\mathbb{R}}^H} = id_{M_{\mathbb{R}}^H}$, and $M_{\mathbb{R}}^H$ is the image of π_H. Note that unless H is trivial, this is a proper subspace of $M_{\mathbb{R}}$, so π_H is a projection from $M_{\mathbb{R}}$ to the invariant subspace $M_{\mathbb{R}}^H$, justifying the notation.

Recall (6.6),

$$P^H := P \cap M_{\mathbb{R}}^H,$$

Since P is convex, $\pi_H(P) \subseteq P$. Hence $\pi_H(P) \subseteq P^H$. On the other hand, $P^H = \pi_H(P^H) \subseteq \pi_H(P)$. This implies

$$P^H = \pi_H(P). \tag{6.41}$$

Then (recall (3.16))

$$P^H = \pi_H(P) = \pi_H(\mathrm{co}(\mathrm{Ver}\, P)) = \mathrm{co}(\pi_H(\mathrm{Ver}\, P)).$$

In particular,

$$\mathrm{Ver}\, P^H \subseteq \pi_H(\mathrm{Ver}\, P). \tag{6.42}$$

Note that equality does not hold in general (for instance, consider $P = [-1, 1]^2$ and H the reflection group about a diagonal).

Lemma 6.4.10. *Fix $u_0 \in M_{\mathbb{R}}^H$, and let $\mathcal{F}$ be a non-empty H-invariant convex subset of some fiber $\pi_H^{-1}(u_0)$ of π_H. Then $\mathcal{F} \cap P \neq \emptyset$ if and only if $u_0 \in P$.*

Proof. If $\mathcal{F} \cap P \neq \emptyset$, then $\pi_H(\mathcal{F} \cap P) \neq \emptyset$. By (6.41), $\pi_H(\mathcal{F} \cap P) \subseteq \pi_H(P) = P^H \subseteq P$, and since $\emptyset \neq \mathcal{F} \subseteq \pi_H^{-1}(u_0)$, then $\pi_H(\mathcal{F} \cap P) \subseteq \pi_H(\mathcal{F}) = \{u_0\}$. Thus, $\pi_H(\mathcal{F} \cap P) \subseteq \{u_0\} \cap P$ and for this to be nonempty it follows that $u_0 \in P$.

For the converse, since $\mathcal{F}$ is H-invariant and convex, $\pi_H(\mathcal{F}) \subseteq \mathcal{F}$. Additionally, as before $\emptyset \neq \mathcal{F} \subseteq \pi_H^{-1}(u_0)$ implies $\pi_H(\mathcal{F}) = \{u_0\}$. Thus $u_0 \in \mathcal{F}$. Therefore, if $u_0 \in P$ then actually $u_0 \in \mathcal{F} \cap P$. $\square$

6.4.4 Proof of Tian's conjecture

We can now prove our main result.

Proof of Theorem 6.0.4. Fix $k \in \mathbb{N}$ and let $O_1^{(k)}, \ldots, O_N^{(k)}$ be the orbits of the action of H on $P \cap M/k$.

Recall that π_H maps an orbit to a singleton,

$$\big\{o_i^{(k)}\big\} := \pi_H\big(O_i^{(k)}\big).$$

Note that $O_i^{(k)} \subseteq \pi_H^{-1}\big(o_i^{(k)}\big)$ and hence (as the action of H and hence also π_H is linear) also $\operatorname{co} O_i^{(k)} \subseteq \pi_H^{-1}\big(o_i^{(k)}\big)$. Moreover, $\operatorname{co} O_i^{(k)}$ is convex and H-invariant. By Corollary 6.2.7, Proposition 6.4.8, and Lemma 6.4.10,

$$\begin{aligned}
\alpha_{k,G(H)} &= \min_{1\le i\le N} c\big(O_i^{(k)}\big) \\
&= \min_{1\le i\le N} \sup\left\{c \in (0,1) \,\middle|\, \Big(-\frac{c}{1-c}\operatorname{co} O_i^{(k)}\Big) \cap P \neq \emptyset\right\} \\
&= \min_{1\le i\le N} \sup\left\{c \in (0,1) \,\middle|\, -\frac{c}{1-c} o_i^{(k)} \in P\right\} \\
&= \sup\left\{c \in (0,1) \,\middle|\, -\frac{c}{1-c}\big\{o_1^{(k)}, \ldots, o_N^{(k)}\big\} \subseteq P\right\}.
\end{aligned}$$

Since

$$\big\{o_1^{(k)}, \ldots, o_N^{(k)}\big\} = \pi_H\left(\bigcup_{i=1}^{N} O_i^{(k)}\right) = \pi_H\big(k^{-1}M \cap P\big),$$

then

$$\begin{aligned}
\alpha_{k,G(H)} &= \sup\left\{c \in (0,1) \,\middle|\, -\frac{c}{1-c}\pi_H(k^{-1}M \cap P) \subseteq P\right\} \\
&= \sup\left\{c \in (0,1) \,\middle|\, -\frac{c}{1-c}\operatorname{co}(\pi_H(k^{-1}M \cap P)) \subseteq P\right\} \\
&= \sup\left\{c \in (0,1) \,\middle|\, -\frac{c}{1-c}\pi_H(\operatorname{co}(k^{-1}M \cap P)) \subseteq P\right\} \\
&= \sup\left\{c \in (0,1) \,\middle|\, -\frac{c}{1-c}\pi_H(P) \subseteq P\right\} \\
&= \sup\left\{c \in (0,1) \,\middle|\, -\frac{c}{1-c}P^H \subseteq P\right\} \\
&= \sup\left\{c \in (0,1) \,\middle|\, -\frac{c}{1-c}\operatorname{Ver} P^H \subseteq P\right\} \\
&= \min_{u \in \operatorname{Ver} P^H} \sup\left\{c \in (0,1) \,\middle|\, -\frac{c}{1-c}u \in P\right\}.
\end{aligned} \tag{6.43}$$

Also, by (6.42),

$$\begin{aligned}
\alpha_{k,G(H)} &= \sup\left\{ c \in (0,1) \,\middle|\, -\frac{c}{1-c} \operatorname{Ver} P^H \subseteq P \right\} \\
&\geq \sup\left\{ c \in (0,1) \,\middle|\, -\frac{c}{1-c} \pi_H(\operatorname{Ver} P) \subseteq P \right\} \\
&\geq \sup\left\{ c \in (0,1) \,\middle|\, -\frac{c}{1-c} \pi_H(P) \subseteq P \right\} \\
&= \alpha_{k,G(H)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\alpha_{k,G(H)} &= \sup\left\{ c \in (0,1) \,\middle|\, -\frac{c}{1-c} \pi_H(\operatorname{Ver} P) \subseteq P \right\} \\
&= \min_{u \in \pi_H(\operatorname{Ver} P)} \sup\left\{ c \in (0,1) \,\middle|\, -\frac{c}{1-c} u \in P \right\}.
\end{aligned} \tag{6.44}$$

Recall (3.19),

$$P = \bigcap_{j=1}^{d} \{ x \in M_{\mathbb{R}} \mid \langle x, -v_j \rangle \leq 1 \}.$$

For any point u and $c \in (0,1)$,

$$-\frac{c}{1-c} u \in P \quad \text{if and only if} \quad c \leq \min_j \frac{1}{\langle u, v_j \rangle + 1} \quad \text{for all } j \in \{1, \ldots, d\}.$$

In sum, combining (6.43) and (6.44) implies (6.8). In particular, $\alpha_{k,G(H)}$ is independent of $k \in \mathbb{N}$ and, by Corollary 6.2.9, is equal to $\alpha_{G(H)}$. This completes the proof of Theorem 6.0.4. □

6.5 Grassmannian Tian invariants

The following definition is due to Tian [104, (6.1)].

Definition 6.5.1. Let X be a Fano manifold and $G \subseteq \operatorname{Aut} X$ a compact subgroup. For $k, m \in \mathbb{N}$, define (recall (6.21))

$$\alpha_{k,m,G} := k \inf_{\substack{|V|\subseteq|-kK_X| \\ \dim V=m \\ V^G=V}} \operatorname{lct}|V|,$$

with the convention $\inf \emptyset = \infty$.

Equivalently, these invariants are the minimum of the function $V \mapsto k \operatorname{lct}|V|$ over the Grassmannian restricted to the subset of G-invariant m-dimensional subspaces of $H^0(X, -kK_X)$. This subset can be, in some situations, even a finite set (see Remark 6.5.9). By Remark 6.5.2 below, these invariants generalize the invariants $\alpha_{k,G}$. Note also that Definition 6.5.1 is an *algebraic* one. Also, $\alpha_{k,E_P(k),G} = \infty$ so it is *not* equal to $\alpha_{k,G}$ from Definition 6.2.1.

In the toric setting it is natural to consider Conjecture 6.0.5 for the invariants $\alpha_{k,m,(S^1)^n}$. For these invariants we show that Conjecture 6.0.5 does not quite hold, and we give a precise breakdown of when it does in terms of a natural convex geometric criterion. In contrast with previous sections, we do not work with the additional symmetry corresponding to the groups $G(H)$ (6.5). The reason for that is that even in simple situations $\alpha_{k,m,G(H)}$ will not be well-behaved: see Example 6.6.6.

Remark 6.5.2. Putting $H = \{\mathrm{id}\}$ in Proposition 6.2.6, notice that $\alpha_{k,(S^1)^n} = \alpha_{k,1,(S^1)^n}$. In this sense, $\alpha_{k,m,(S^1)^n}$ is a generalization of the $\alpha_{k,(S^1)^n}$-invariants.

First, we show, as a rather immediate consequence of our work, an explicit formula for the invariants of Definition 6.5.1.

Corollary 6.5.3. *Let X be toric Fano with associated polytope P* (3.19)*. For $k, m \in \mathbb{N}$ (recall* (6.38)*),*

$$\alpha_{k,m,(S^1)^n} = \min_{\substack{\mathcal{F}\subseteq P\cap M/k \\ |\mathcal{F}|=m}} c_k(\mathcal{F}).$$

Proof. Fix $k, m \in \mathbb{N}$. Let $V \subseteq H^0(X, -kK_X) = \operatorname{Span}\{s_u\}_{u\in P\cap M/k}$ be a subspace invariant under $(S^1)^n$. By Lemma 6.2.8, there exists a unique $\mathcal{F} \subseteq P \cap M/k$ such that

$$V = \operatorname{Span}\{s_u\}_{u\in\mathcal{F}}.$$

Note that $|\mathcal{F}| = \dim V$. By (6.21) and (6.38),

$$\alpha_{k,m,(S^1)^n} = k \min_{\substack{\mathcal{F}\subseteq P\cap M/k \\ |\mathcal{F}|=m}} \operatorname{lct}\left|\operatorname{Span}\{s_u\}_{u\in\mathcal{F}}\right| = \min_{\substack{\mathcal{F}\subseteq P\cap M/k \\ |\mathcal{F}|=m}} c_k(\mathcal{F}).$$

$\square$

Lemma 6.5.4. *Let $K \subseteq \mathbb{R}^n$ be a compact convex set with $0 \in \operatorname{int} K$. For any set $S \subseteq \mathbb{R}^n$,*

$$\inf_S \|\cdot\|_K = \inf\{\lambda \geq 0 \,|\, S \cap \lambda K \neq \emptyset\}.$$

Proof. If $S \cap \lambda K \neq \emptyset$ for some $\lambda \geq 0$, pick $x \in S \cap \lambda K$. Since $x \in \lambda K$, by (3.3), $\|x\|_K \leq \lambda$. Therefore $\inf_{x\in S} \|x\|_K \leq \inf\{\lambda \geq 0 \,|\, S \cap \lambda K \neq \emptyset\}$.

On the other hand, for any $x \in S$, $S \cap \|x\|_K K \supseteq \{x\} \neq \emptyset$ by (3.5). So $\inf_{x\in S} \|x\|_K \geq \inf\{\lambda \geq 0 \,|\, S \cap \lambda K \neq \emptyset\}$. $\square$

Proposition 6.4.8 can be reformulated in terms of near-norms:

Corollary 6.5.5. *For $k \in \mathbb{N}$ and a non-empty set $\mathcal{F} \subseteq P \cap M/k$ (recall (6.38)),*

$$c_k(\mathcal{F}) = \frac{1}{1 + \min_{\operatorname{co}\mathcal{F}} \|\cdot\|_{-P}}.$$

Proof. By Proposition 6.4.8 and Lemma 6.5.4,

$$\begin{aligned}
c_k(\mathcal{F}) &= \sup\left\{c \in (0,1) \,\middle|\, P \cap \left(-\frac{c}{1-c}\operatorname{co}\mathcal{F}\right) \neq \emptyset\right\} \\
&= \sup\left\{c \in (0,1) \,\middle|\, \operatorname{co}\mathcal{F} \cap \frac{1-c}{c}(-P) \neq \emptyset\right\} \\
&= \sup\left\{\frac{1}{1+a} \,\middle|\, a \in (0,\infty),\ \operatorname{co}\mathcal{F} \cap (-aP) \neq \emptyset\right\} \\
&= [1 + \inf\{a \in (0,\infty) \,|\, \operatorname{co}\mathcal{F} \cap (-aP) \neq \emptyset\}]^{-1} \\
&= \left[1 + \inf_{\operatorname{co}\mathcal{F}} \|\cdot\|_{-P}\right]^{-1} = \left[1 + \min_{\operatorname{co}\mathcal{F}} \|\cdot\|_{-P}\right]^{-1},
\end{aligned}$$

using compactness of $\operatorname{co}\mathcal{F}$. $\square$

Combining Corollaries 6.5.3 and 6.5.5 yields an explicit formula for $\alpha_{k,m,(S^1)^n}$.

Corollary 6.5.6. *Let X be toric Fano with associated polytope P (3.19). For $k, m \in \mathbb{N}$,*

$$\alpha_{k,m,(S^1)^n} = \left[1 + \max_{\substack{\mathcal{F} \subseteq P \cap M/k \\ |\mathcal{F}|=m}} \min_{\operatorname{co} \mathcal{F}} \| \cdot \|_{-P}\right]^{-1}.$$

Remark 6.5.7. For $m = 1$, Corollary 6.5.6 reads

$$\alpha_{k,(S^1)^n} = \frac{1}{1 + \max\limits_{u \in P \cap M/k} \|u\|_{-P}}.$$

Being a near-norm, the function $\| \cdot \|_{-P}$ satisfies the triangle inequality and is positively 1-homogeneous, hence it is convex. By Lemma 3.2.11, the vertex set of P is in the integral lattice M, hence also contained in M/k for all $k \in \mathbb{N}$. This combined with Lemma 3.2.14 gives,

$$\alpha_{k,(S^1)^n} = \frac{1}{1 + \max\limits_{\operatorname{Ver} P} \| \cdot \|_{-P}} = \frac{1}{1 + \max\limits_{P} \| \cdot \|_{-P}}. \tag{6.45}$$

This is, of course, a special case of Theorem 6.0.4 (proved in §6.4), derived here in slightly different notation (using the near-norm instead of the support function). Thus,

$$\alpha_{k,(S^1)^n} = \alpha_{(S^1)^n} = \alpha, \tag{6.46}$$

where the first equality follows from (6.45) and Demailly's result [29, (A.1)], and the last equality follows by comparing Theorem 6.0.4 with Blum–Jonsson's formula [16, (7.2)].

Proposition 6.5.8. *Let X be toric Fano with associated polytope P (3.19). For $k \in \mathbb{N}$ and $m' > m$,*

$$\alpha_{k,m',(S^1)^n} \geq \alpha_{k,m,(S^1)^n}. \tag{6.47}$$

In particular,

$$\alpha_{k,m,(S^1)^n} \geq \alpha_{k,1,(S^1)^n} = \alpha_{k,(S^1)^n} = \alpha.$$

Equality holds if and only if there is $\mathcal{F} \subseteq P \cap M/k$ with $|\mathcal{F}| = m$ satisfying

$$\|\cdot\|_{-P}\Big|_{\operatorname{co}\mathcal{F}} = \max_P \|\cdot\|_{-P}. \tag{6.48}$$

Remark 6.5.9. For a general Fano X, it does not follow from Definition 6.5.1 that $\alpha_{k,m',G} \geq \alpha_{k,m,G}$, i.e., there may be no monotonicity in m. For instance, there might be no m-dimensinal G-invariant subspaces. Thanks to Lemma 6.2.8 the situation for $G = (S^1)^n$ is particularly simple: there are finitely-many m-dimensional $(S^1)^n$-invariant subspaces (in fact, the Grassmannian of m-dimensional subspaces has exactly $\binom{E_P(k)}{m}$ fixed points by the $(S^1)^n$-action), and moreover every $(S^1)^n$-invariant subspace consists of 1-dimensional blocks.

Proof. Pick $\mathcal{F}' \subseteq P \cap M/k$ that computes $\alpha_{k,m',(S^1)^n}$ (such a subset exists since $P \cap M/k$ is a finite set). That is, $|\mathcal{F}'| = m'$ and $\alpha_{k,m',(S^1)^n} = \big[1 + \min_{\operatorname{co}\mathcal{F}'} \|\cdot\|_{-P}\big]^{-1}$ (recall Corollary 6.5.6). For $\mathcal{F}$ be a subset of $\mathcal{F}'$ with $|\mathcal{F}| = m$, Corollary 6.5.6 implies

$$\alpha_{k,m,(S^1)^n} = \frac{1}{1 + \max\limits_{\substack{\mathcal{A} \subseteq P \cap M/k \\ |\mathcal{A}| = m}} \min\limits_{\operatorname{co}\mathcal{A}} \|\cdot\|_{-P}} \leq \frac{1}{1 + \min\limits_{\operatorname{co}\mathcal{F}} \|\cdot\|_{-P}} \leq \frac{1}{1 + \min\limits_{\operatorname{co}\mathcal{F}'} \|\cdot\|_{-P}} = \alpha_{k,m',(S^1)^n},$$

proving (6.47).

Next, suppose $\alpha_{k,m,(S^1)^n} = \alpha$, i.e. (recall (6.45)–(6.46)),

$$\max_{\substack{\mathcal{F} \subseteq P \cap M/k \\ |\mathcal{F}| = m}} \min_{\operatorname{co}\mathcal{F}} \|\cdot\|_{-P} = \max_P \|\cdot\|_{-P}. \tag{6.49}$$

For any $\mathcal{F} \subseteq P$, convexity of P (recall (3.19)) implies $\operatorname{co}\mathcal{F} \subseteq P$. Hence,

$$\min_{\operatorname{co}\mathcal{F}} \|\cdot\|_{-P} \leq \max_P \|\cdot\|_{-P}.$$

The equality (6.49) holds if and only if there is $\mathcal{F} \subseteq P \cap M/k$ such that $|\mathcal{F}| = m$ and

$$\min_{\operatorname{co}\mathcal{F}} \|\cdot\|_{-P} = \max_P \|\cdot\|_{-P},$$

which forces $\|\cdot\|_{-P}$ to be the constant $\max_P \|\cdot\|_{-P}$ on $\mathrm{co}\,\mathcal{F}$. $\square$

6.5.1 Proof of Theorem 6.0.6 and the intuition behind it

Theorem 6.0.4 resolved Tian's classical stabilization Problem 6.0.1 in the affirmative. This corresponds to the case $m = 1$ of Conjecture 6.0.5. Next we show that Conjecture 6.0.5 is only partially true for $m \geq 2$.

Proof of Theorem 6.0.6. Let $\mathcal{F} \subseteq P \cap M/k$ with $|\mathcal{F}| = m$. By convexity of P and as $m \geq 2$, $\mathrm{co}\,\mathcal{F}$ contains an interval in P. In particular, $\mathcal{F}$ contains a point of $P \setminus \mathrm{Ver}\,P$. Thus, if $(*_P)$ holds then (6.48) does not hold (for any such $\mathcal{F}$), and Proposition 6.5.8 implies (6.9).

Next, suppose $(*_P)$ fails (and now we relax to $m \geq 1$). Let $\hat{x} \in P \setminus \mathrm{Ver}\,P$ achieve the maximum of $\|\cdot\|_{-P}$ on P. Express $\hat{x}$ as a convex combination of a *subset* of vertices of P, $\{p_1, \ldots, p_\ell\} \subseteq \mathrm{Ver}\,P$, namely,

$$\hat{x} = \sum_{i=1}^{\ell} \hat{\lambda}_i p_i, \quad \sum_{i=1}^{\ell} \hat{\lambda}_i = 1, \quad \hat{\lambda} \in (0,1)^\ell. \tag{6.50}$$

Note that necessarily

$$\ell > 1. \tag{6.51}$$

We claim that

$$\|x\|_{-P} = \max_{x \in P} \|\cdot\|_{-P}, \text{ for any } x \in \mathrm{co}\{p_i\}_{i=1}^{\ell}.$$

Indeed, letting

$$P' := \mathrm{co}\{p_i\}_{i=1}^{\ell} \subseteq P,$$

the claim follows from $\hat{x} \in \mathrm{int}\,P' \subseteq P$ and Lemma 3.2.14 (iii).

Now, the lattice polytope $P' := \mathrm{co}\{p_i\}_{i=1}^{\ell}$ has positive dimension by (6.51). Equivalently, the Ehrhart polynomial of P' has degree at least 1. Hence, by Proposition 3.3.4, $|P' \cap M/k| \geq m$ for all sufficiently large k. Pick $\mathcal{F} \subseteq P' \cap M/k \subseteq P \cap M/k$ such that $|\mathcal{F}| = m$. Then $\mathrm{co}\,\mathcal{F} \subseteq P'$. By Proposition 6.5.8, (6.10) follows. The case $m = 1$ was already treated in Theorem 6.0.4 but also from this proof we see that any $k \in \mathbb{N}$ works for $m = 1$. $\square$

The proof of Theorem 6.0.6 and condition $(*_P)$ have a pleasing geometric interpretation. Geometrically, by Proposition 6.5.8, $\alpha_{k,m,(S^1)^n} = \alpha$ if and only if there is a subset $\mathcal{F} \subseteq P \cap M/k$ such that $|\mathcal{F}| = m$ and $\operatorname{co}\mathcal{F}$ lies entirely in the level set

$$\left\{\| \cdot \|_{-P} = \max_P \| \cdot \|_{-P}\right\} \subseteq \partial P \tag{6.52}$$

Recall that any convex subset of the boundary of a polytope must lie in a single face. Hence $\mathcal{F}$ must be a subset of a face F of P. We may assume F is the minimal face that contains $\mathcal{F}$, i.e., $\mathcal{F}$ intersects the relative interior of F. In particular, $\| \cdot \|_{-P}|_F$ attains its maximum in the interior of F and so F has to lie entirely in the level set (6.52) by Lemma 3.2.14. Therefore $\alpha_{k,m,(S^1)^n} = \alpha$ if and only if there is a subset $\mathcal{F} \subseteq P \cap M/k$ such that $|\mathcal{F}| = m$ and $\mathcal{F}$ is a subset of a face that lies in the level set (6.52), or equivalently, there is a face F that lies in the level set (6.52), and $|F \cap M/k| \geq m$.

Notice that the level set $\{\| \cdot \|_{-P} = \lambda\}$ is the dilation $\lambda\partial(-P)$, and (6.52) is the largest dilation that intersects P (by Lemma 6.5.4); see Figure 6.1. Combining all the above then, $(*_P)$ states that

$$P \text{ intersects } \max_P \| \cdot \|_{-P}\partial(-P) \text{ only at vertices.}$$

If $(*_P)$ holds, then any such face F consists of a single lattice point. In particular, $|F \cap M/k| = 1$ for any k, and $\alpha_{k,m,(S^1)^n}$ stabilizes if and only if $m = 1$. On the other hand, if $(*_P)$ fails, then there is a face F of positive dimension that lies in the level set (6.52). Since $|F \cap M/k|$ increases to infinity, for any m, we can find k_0 such that $|F \cap M/k| \geq m$ for $k \geq k_0$, i.e., $\alpha_{k,m,(S^1)^n}$ stabilizes in k.

6.5.2 A Demailly result for Grassmannian invariants

The next result was obtained independently by Li–Zhu [71, Proposition 4.1].

Proposition 6.5.10. *Let X be toric Fano with associated polytope P (3.19). For $m \in \mathbb{N}$,*

$$\lim_{k\to\infty} \alpha_{k,m,(S^1)^n} = \alpha.$$

Proof. By continuity, given $\varepsilon > 0$ there exists an open set $U_\varepsilon \subseteq P$ such that

$$\|\cdot\|_{-P}\Big|_{U_\varepsilon} > \max_P \|\cdot\|_{-P} - \varepsilon. \tag{6.53}$$

Fix $\delta > 0$ sufficiently small so that there is a closed cube $C_\delta \subseteq U_\varepsilon$ with edge length δ. Now, for $k \geq K$, $|C_\delta \cap M/k| \geq \lfloor K\delta \rfloor^n$. Given $m \in \mathbb{N}$, choose K so $\lfloor K\delta \rfloor^n \geq m$. Then for each $k \geq K$ there exists $\mathcal{F} \subseteq C_\delta \cap M/k \subseteq P \cap M/k$ such that $|\mathcal{F}| = m$. Since $\mathcal{F} \subseteq C_\delta$ and C_δ is convex, $\operatorname{co}\mathcal{F} \subseteq C_\delta \subseteq U_\varepsilon$. By (6.53),

$$\min_{\operatorname{co}\mathcal{F}} \|\cdot\|_{-P} \geq \min_{C_\delta} \|\cdot\|_{-P} > \max_P \| \cdot \|_{-P} - \varepsilon.$$

So by Corollary 6.5.6 and (6.45)–(6.46),

$$\alpha_{k,m,(S^1)^n} \leq \frac{1}{1 + \min_{\operatorname{co}\mathcal{F}} \|\cdot\|_{-P}} < \frac{1}{1 + \max_P \| \cdot \|_{-P} - \varepsilon} = \frac{\alpha}{1 - \varepsilon\alpha},$$

for any k such that $\lfloor k\delta \rfloor \geq \sqrt[n]{m}$. Hence $\limsup_{k\to\infty} \alpha_{k,m,(S^1)^n} \leq \frac{\alpha}{1-\varepsilon\alpha}$. On the other hand, by Proposition 6.5.8, $\liminf_{k\to\infty} \alpha_{k,m,(S^1)^n} \geq \alpha$. Since $\varepsilon > 0$ is arbitrary the proof is complete. □

6.6 Examples

In §6.6.1 we carefully illustrate Theorems 6.0.4 and 6.0.6 in dimensions 2 and for all toric del Pezzo surfaces. In §6.6.2 we discuss central symmetry and its characterization in dimension 2 in terms of the α-invariant. In §6.6.3 we draw the connection between $(*_P)$ and work in the convex geometry literature on the asymmetry constant. We illustrate this with computations related to a remarkable toric 3-fold that uniquely minimizes α among all toric (and possibly even non-toric) 3-folds.

6.6.1 del Pezzo surfaces

Example 6.6.1. Let $v_1 = (1, 0), v_2 = (0, 1), v_3 = (-1, -1)$. Then $\mathbb{P}^2 = X(\Delta)$ for

$$\Delta = \{\{0\}, \mathbb{R}_+v_1, \mathbb{R}_+v_2, \mathbb{R}_+v_3, \mathbb{R}_+v_1 + \mathbb{R}_+v_2, \mathbb{R}_+v_1 + \mathbb{R}_+v_3, \mathbb{R}_+v_2 + \mathbb{R}_+v_3\}$$

and $P = \{-v_1, -v_2, -v_3\}^\circ = \{y \in M_\mathbb{R} \,|\, \langle y, -v_i \rangle \leq 1, i \in \{1,2,3\}\} \subseteq M_\mathbb{R}$ is depicted in Figure 6.3. Then $\mathrm{Aut}\, P \cong D_6 = S_3$, the dihedral/cyclic group of order 6 consisting of the permutations of the 3 vertices, or the 3 homogeneous coordinates $\mathbb{P}^2$, as in Figure 6.2. For $H = \mathrm{Aut}\, P$, P^H is the origin, so by Theorem 6.0.4, $\alpha_{k,G(H)} = \alpha_{G(H)} = 1$. When H is the trivial group then $P^H = P$ and Theorem 6.0.4 gives $\alpha_{k,G(H)} = \alpha_{G(H)} = \frac{1}{3}$, with minimum achieved at any vertex of P. When H is the subgroup $\mathbb{Z}_2$ given by reflection about, say, $y = x$ (i.e., generated by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ associated to switching two of the homogeneous coordinates of $\mathbb{P}^2$), P^H is the line segment $\mathrm{co}\{(-1,-1), (\frac{1}{2}, \frac{1}{2})\}$ and $\alpha_{k,G(H)} = \alpha_{G(H)} = \frac{1}{3}$, with minimum achieved at $(-1,-1)$. When H is the subgroup of order 3, $P^H = \{0\}$, and $\alpha_{k,G(H)} = \alpha_{G(H)} = 1$,

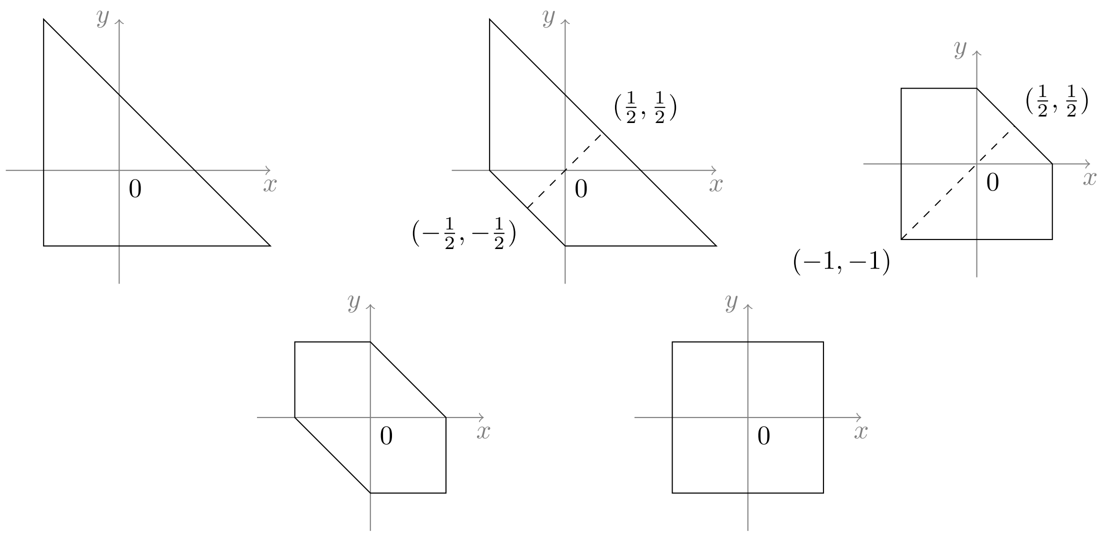

Figure 6.3: The polytope P for del Pezzo surfaces, namely $\mathbb{P}^2$ blown up at no more than 3 generically positioned points, and $\mathbb{P}^1 \times \mathbb{P}^1$. For the two K-unstable examples, the automorphism group $H = \mathrm{Aut}\, P$ is generated by the reflection about $y = x$, and P^H is the intersection of P with this reflection axis. For the other three examples, the automorphism group $H = \mathrm{Aut}\, P$ is the dihedral group associated to the polygon P, and $P^H = \{0\}$.

Example 6.6.2. Let P be the polytope for $\mathbb{P}^2$ blown up at one point, i.e., v_1, v_2, v_3 as above and $v_4 = (1,1)$. Then $\mathrm{Aut}\, P \cong \mathbb{Z}_2$, as shown in Figure 6.3. There are therefore two choices for H. When $H = \mathrm{Aut}\, P$, P^H is the line segment $\mathrm{co}\{(-\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\}$. By Theorem 6.0.4, $\alpha_{k,G(H)} = \alpha_{G(H)} = \frac{1}{2}$, with minimum achieved at $\pm(\frac{1}{2}, \frac{1}{2})$. When H is the trivial group then $P^H = P$ and Theorem 6.0.4 gives $\alpha_{k,G(H)} = \alpha_{G(H)} = \frac{1}{3}$, with minimum achieved at either of the vertices $(-1, 2)$ or $(2, -1)$.

Example 6.6.3. Let P be the polytope for $\mathbb{P}^2$ blown up at two points, i.e., v_1, v_2, v_3 as above and

$v_4 = (-1, 0), v_5 = (0, -1)$. Again $\operatorname{Aut} P \cong \mathbb{Z}_2$, as shown in Figure 6.3. For $H = \operatorname{Aut} P$, P^H is the line segment $\operatorname{co}\{(-1,-1), (\frac{1}{2}, \frac{1}{2})\}$. By Theorem 6.0.4, $\alpha_{k,G} = \alpha_G = \frac{1}{3}$, with minimum achieved at $(-1,-1)$. When H is the trivial group then $P^H = P$ and Theorem 6.0.4 gives $\alpha_{k,G(H)} = \alpha_{G(H)} = \frac{1}{3}$, with minimum achieved still at $(-1,-1)$.

Example 6.6.4. Let P be the polytope for $\mathbb{P}^2$ blown up at three non-colinear points, i.e., $v_1, \ldots, v_5$ as in the previous example and $v_6 = (1,1)$. Now $\operatorname{Aut} P \cong D_{12}$, the dihedral group of order 12 consisting of the cyclic permutations of the 6 vertices and the 6 reflections, as in Figure 6.3. For $H = \operatorname{Aut} P$, P^H is the origin, so by Theorem 6.0.4, $\alpha_{k,G(H)} = \alpha_{G(H)} = 1$. When H is the trivial group then $P^H = P$ and Theorem 6.0.4 gives $\alpha_{k,G(H)} = \alpha_{G(H)} = \frac{1}{2}$, with minimum achieved at any vertex of P. When H is the subgroup $\mathbb{Z}_2$ given by reflection about $y = x$, P^H is the line segment $\operatorname{co}\{(-\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\}$ and $\alpha_{k,G(H)} = \alpha_{G(H)} = \frac{1}{2}$, with minimum achieved at $\pm(\frac{1}{2}, \frac{1}{2})$. When H is the subgroup $\mathbb{Z}_2$ given by reflection about $y = -x$, $P^H = \operatorname{co}\{(-1,1), (1,-1)\}$, and $\alpha_{k,G(H)} = \alpha_{G(H)} = \frac{1}{2}$, with minimum achieved at $(\pm 1, \mp 1)$. Same for the other reflections. When H contains a cyclic permutation, or more than one reflection, P^H is the origin and $\alpha_{k,G(H)} = \alpha_{G(H)} = 1$.

Example 6.6.5. Let P be the polytope for $\mathbb{P}^1 \times \mathbb{P}^1$, i.e., with v_1, v_2 as above and $v_3 = (-1, 0)$ and $v_4 = (0, -1)$, so $P = [-1, 1]^2$ and $\operatorname{Aut} P \cong D_8$, the dihedral group of order 8 generated by 4 reflections shown in Figure 6.3. Similar to previous examples, we obtain $\alpha_{k,G(H)} = \alpha_{G(H)} = 1$ whenever H contains a cyclic permutation or more than one reflection. Otherwise it equals $1/2$.

Example 6.6.6. Let P be the polytope for $X = \mathbb{P}^2$ from Example 6.6.1. Recall that $\operatorname{Aut} P = S_3$. Let $H = A_3$, the alternating group of order 3, generated by a cyclic permutation of the three vertices. The group is represented by $\{I, A, A^2\}$ where

$$A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

All orbits of H on $M \cong \mathbb{Z}^2$ have cardinality 3 except the orbit of the origin, that has cardinality 1. Any $G(H)$-invariant subspace of $H^0(X, -kK_X)$ is spanned by a collection of monomials indexed by a union

$\cup_{i=1}^{\ell} O_i^{(k)}$ of H-orbits on $kP \cap M$. This forces the subspace to have dimension $m = 3\ell$ or $m = 3(\ell-1)+1$. Thus, $m \equiv_3 0, 1$ (see Figure 6.4 for the case $k=1$). As a result, if $m \equiv_3 2$ then $\alpha_{k,m,G(H)} = \infty$. For this reason we only consider in §6.5 the case $H = \{\mathrm{id}\}$.

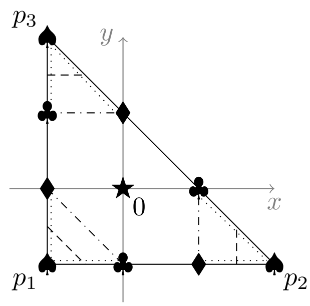

Figure 6.4: The four orbits $O_1^{(1)}, \ldots, O_4^{(1)}$ of the action of the cyclic group of order 3 (generated by cyclic permutation of the homogeneous coordinates of $\mathbb{P}^2$) on the polytope corresponding to $\mathbb{P}^2$ with $k=1$. The dashed-dotted lines depict the sets on which h_{Δ_1} is constant equal to 1 and that compute $\alpha_{1,2,(S^1)^2} = 1/2$. These sets are the three components of $-\partial P \cap P$. The dotted lines depict the sets on which h_{Δ_1} is not constant but whose minimum is 1 and that also compute $\alpha_{1,2,(S^1)^2} = 1/2$. The dashed lines depict the sets on which h_{Δ_1} is constant equal to 3/2 and that compute $\alpha_{2,2,(S^1)^2} = 2/5$. These sets are the three components of $-\frac{3}{2}\partial P \cap P$.

Example 6.6.7. Let P be the polytope for $X = \mathbb{P}^2$ from Example 6.6.1. We compute the obstruction $(*_P)$, i.e., compute $\mathrm{argmax}_P \|\cdot\|_{-P} = \mathrm{argmax}_P h_Q = \mathrm{argmax}_P h_{\Delta_1}$. Recall $\Delta_1 = \{v_1, v_2, v_3\} = \{(1,0),(0,1),(-1,-1)\}$. Let $y = (y_1, y_2)$ be coordinates on P. Then,

$$h_{v_1}(y) = y_1, \quad h_{v_2}(y) = y_2, \quad h_{v_3}(y) = -y_1 - y_2,$$

so

$$h_{\Delta_1}(y) = \max\{y_1, y_2, -y_1 - y_2\}.$$

By Lemma 3.2.14 $\mathrm{Ver}\, P \cap \mathrm{argmax}_P h_{\Delta_1} \neq \emptyset$. Note,

$$\mathrm{Ver}\, P = \{p_1, p_2, p_3\} = \{(-1,-1),(2,-1),(-1,2)\}.$$

So

$$\max_P h_Q = \max\{h_Q(p_1), h_Q(p_2), h_Q(p_3)\} = \max\{2,2,2\} = 2,$$

and $\operatorname{Ver} P \subseteq \operatorname{argmax}_P h_{\Delta_1}$. Suppose that $h_Q(y) = 2$. Then a computation shows that $y \in \operatorname{Ver} P$. Thus $\operatorname{Ver} P = \operatorname{argmax}_P h_{\Delta_1}$. By Theorem 6.0.6, $\alpha_{k,m,(S^1)^n} > \alpha = \frac{1}{3}$ for all $k \in \mathbb{N}$ and $m \geq 2$.

Let us compute $\alpha_{1,2,(S^1)^n}$. Let $\mathcal{F} = \{f_1, f_2\} \subseteq P \cap M$. By Corollary 6.5.6 it suffices to consider 'minimal' $\mathcal{F}$ in the sense that for no other $\mathcal{F}'$, $\operatorname{co}\mathcal{F}' \subseteq \operatorname{co}\mathcal{F}$. So for instance, we do not need to consider $\{p_1, p_2\}$ but instead

$$\{p_1, (0,-1)\}, \{(0,-1), (1,-1)\}, \{(1,-1), p_2\}.$$

In fact, $\min_{\operatorname{co}\{p_1,(0,-1)\}} h_Q = h_Q(0,-1) = 1$ while $\min_{\{p_1,p_2\}} h_Q = h_Q(1/2,-1) = 1/2$.

Also, again by Corollary 6.5.6, we do not need to consider any $\mathcal{F}$ such that $0 \in \operatorname{co}\mathcal{F}$ since on such an interval the support function attains its absolute minimum, i.e., vanishes.

Next, observe that $h_{\Delta_1} = 1$ on the vertices of the hexagon

$$P_1 := \operatorname{co}\{\pm(1,0), \pm(0,1), \pm(-1,1)\}.$$

By Lemma 3.2.14,

$$h_{\Delta_1}|_{P_1} \leq h_{\Delta_1}|_{\operatorname{Ver} P_1} = 1.$$

Thus (by Corollary 6.5.6) we do not need to consider any $\mathcal{F}$ such that $\operatorname{co}\mathcal{F}$ intersects P_1, as such an $\mathcal{F}$ will satisfy $\min_{\operatorname{co}\mathcal{F}} h_Q \leq 1$. Since every $\mathcal{F}$ intersects P_1 it follows that $\alpha_{1,2,(S^1)^2} = 1/2$ by Corollary 6.5.6. See Figure 6.4.

By the same reasoning we can compute $\alpha_{k,2,(S^1)^2}$ using the hexagons

$$\begin{aligned}
P_k := &\operatorname{co}\Big\{\Big(2-\frac{1}{k}, \frac{1}{k}-1\Big), \Big(2-\frac{1}{k}, -1\Big), \Big(-1+\frac{1}{k}, -1\Big), \Big(-1, -1+\frac{1}{k}\Big), \\
&\qquad \Big(-1, 2-\frac{1}{k}\Big), \Big(-1+\frac{1}{k}, 2-\frac{1}{k}\Big)\Big\} \\
=&\frac{1}{k}\operatorname{co}\Big\{\Big(2k-1, 1-k\Big), \Big(2k-1, -k\Big), \Big(-k+1, -k\Big), \Big(-k, -k+1\Big), \\
&\qquad \Big(-k, 2k-1\Big), \Big(-k+1, 2k-1\Big)\Big\} \subseteq P \cap M/k.
\end{aligned}$$

By Lemma 3.2.14,

$$h_{\Delta_1}|_{P_k} \leq h_{\Delta_1}|_{\mathrm{Ver}\, P_k} = 2 - 1/k.$$

Since every $\mathcal{F}$ intersects P_k it follows that

$$\alpha_{k,2,(S^1)^2} = \Big[1 + 2 - \frac{1}{k}\Big]^{-1} = \frac{k}{3k-1}$$

by Corollary 6.5.6. Note that $\lim_k \frac{k}{3k-1} = 1/3 = \alpha$ (recall Example 6.6.1 and (6.46)) in accordance with Proposition 6.5.10 and that this is a monotone sequence in accordance with Proposition 6.5.8.

6.6.2 Central symmetry

The next result should be well-known but we could not find a reference.

Lemma 6.6.8. *Let X be toric Fano. Let $P \subseteq M_{\mathbb{R}}$ (see* (3.6), (3.18)*) be the polytope associated to $(X, -K_X)$. Then $\alpha \leq 1/2$, and P is centrally symmetric (i.e., $P = -P$) if and only if $\alpha = 1/2$. If P is not centrally symmetric then $\alpha \leq 1/3$.*

Proof. If $P \neq -P$, then there is $u_0 \in P \setminus -P$. By (3.19), $-P = \{y \in M_{\mathbb{R}} \,|\, \max_j \langle v_j, y\rangle \leq 1\}$. Thus $\langle u_0, v_{i_0}\rangle > 1$ for some $v_{i_0} \in \Delta_1 \subseteq N$ (recall (3.13)). Recalling Theorem 6.0.4 and (6.46), or [16, Corollary 7.16],

$$\alpha = \min_{u\in P} \min_i \frac{1}{1 + \langle u, v_i\rangle} \leq \frac{1}{1 + \langle u_0, v_{i_0}\rangle} < \frac{1}{2}.$$

In fact, by convexity one may take $u_0 \in \mathrm{Ver}\, P \backslash -P$, so $u_0 \in M$ by Lemma 3.2.11 and hence $\mathbb{Z} \ni \langle u_0, v_{i_0}\rangle \geq 2$, i.e., $\alpha \leq 1/3$.

If $P = -P$, by (6.45) and (6.46),

$$\alpha = \frac{1}{1 + \max\limits_P \|\cdot\|_{-P}} = \frac{1}{1 + \max\limits_P \|\cdot\|_P} = \frac{1}{1+1} = \frac{1}{2}$$

by (3.3) and (3.5). □

It turns out that any centrally symmetric toric Fano manifold can be written as the product of

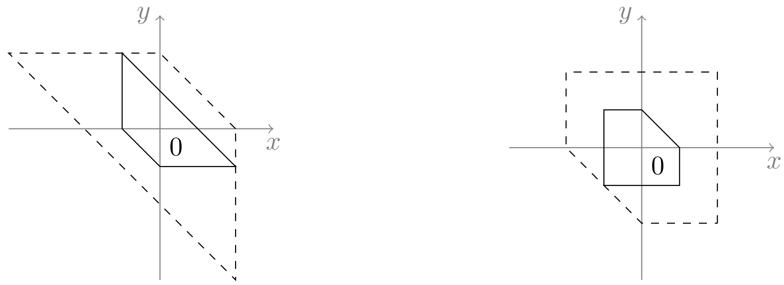

Figure 6.5: The polytope P for $\mathbb{P}^2$ blown up 1 or 2 points, and the level set $\{\|\cdot\|_{-P} = 2\}$.

factors that are either $\mathbb{P}^1$ or the so-called toric del Pezzo variety V^{2k}, namely the unique indecomposable centrally symmetric toric Fano manifold of even dimension $2k$ [109, Theorem 6]. For example, V^2 is $\mathbb{P}^2$ blown-up at 3 non-colinear points, and the only centrally symmetric toric Fano 3-folds are $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times V^2$.

Example 6.6.9. The purpose of this paragraph is to show that when $n = 2$, condition $(*_P)$ holds for a Delzant polytope P if and only if P is not centrally symmetric, i.e., $P \neq -P$. Alternatively, if $n = 2$, Conjecture 6.0.5 holds if and only if $\alpha = 1/2$, by Lemma 6.6.8. However, in dimension $n \geq 3$ this is no longer the case: for instance $\mathbb{P}^2 \times \mathbb{P}^1$ has a non-centrally symmetric polytope P but condition $(*_P)$ fails. For a counterexample that does not admit a centrally symmetric factor, consider the Fano 3-fold 4-11 [3, p. 90], namely the blow-up of $\mathbb{F}_1 \times \mathbb{P}^1$ along $E \times \{0\}$, where E is the -1-curve on $\mathbb{F}_1$. Condition $(*_P)$ fails along 3 edges of the polytope.

When $n = 2$, the centrally symmetric Delzant polytopes are associated to $\mathbb{P}^2$ blown-up at the three non-colinear points, and $\mathbb{P}^1 \times \mathbb{P}^1$ (Examples 6.6.4–6.6.5). Recall the description of $(*_P)$ given in §6.5.1. Since $P = -P$, the intersection of P and $\max_P \|\cdot\|_{-P}\partial(-P)$ is precisely all of ∂P, i.e., $(*_P)$ fails.

Thus, it remains to check $\mathbb{P}^2$ blown-up at up to two points. We have already showed that $(*_P)$ holds for (the non-centrally symmetric) P coming from $\mathbb{P}^2$ (Example 6.6.7). It thus remains to check the remaining two cases. The polytope for the 1-point blow-up is a subset of the one for $\mathbb{P}^2$ by chopping a corner. Thus the intersection of P and $\max_P \|\cdot\|_{-P}\partial(-P) = -2\partial P$ is still just the two vertices $(-1, 2)$ and $(2, -1)$. For the 2-point blow-up the intersection of P and $\max_P \|\cdot\|_{-P}\partial(-P) = -2\partial P$ is just the vertex $(-1, -1)$, concluding the proof. See Figure 6.5.

6.6.3 The asymmetry constant and a remarkable Fano 3-fold

The asymmetry constant measures how far a body is from being centrally symmetric.

Definition 6.6.10. [54, 58] For a convex polytope $P \ni 0$, define

$$\operatorname{asym}(P) := \min_{a\in P} \max_{P} \|\cdot\|_{-(P-a)} = \min_{a\in P} \min\{\lambda \geq 0 \,|\, P \subseteq -\lambda(P-a)\} \geq 1.$$

In other words, $\operatorname{asym}(P)(-P)$ is the smallest dilation of $-P$ that admits a translation containing P.

A corollary of (6.45)–(6.46) is

$$\alpha \leq \big[1 + \operatorname{asym}(P)\big]^{-1}. \tag{6.54}$$

As we will see in Example 6.6.11, this bound is often not sharp. In order to see that let us first describe how to compute $\operatorname{asym}(P)$ in general.

In practice, $\operatorname{asym}(P)$ is the smallest $\lambda \geq 0$ such that

$$-\lambda P + \tau \supseteq P$$

for some $\tau \in \mathbb{R}^n$. In other words, it is the smallest λ among all the pairs $(\tau, \lambda) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ satisfying

$$-\lambda P + \tau \supseteq P.$$

Hence it is the solution to the following linear programming problem: minimize

$$f(\tau, \lambda) := \lambda$$

on the region

$$\{(\tau, \lambda) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0} \,|\, -\lambda P + \tau \supseteq P\}.$$

More precisely, write $\operatorname{Ver} P = \{u_i\}_{i=1}^{p}$ and recall (3.20) the d facets of P are given by $\langle\,\cdot\,, v_j\rangle = 1$, $j \in$

$\{1, \ldots, d\}$. Then by (3.19), $\mathrm{asym}(P)$ is the solution to the following linear programming problem: minimize

$$f(\tau, \lambda) := \lambda$$

under the constraints that $(\tau, \lambda) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ satisfies

$$\langle \tau, v_j \rangle - \lambda \leq \langle u_i, v_j \rangle, \quad \text{for all } j \in \{1, \ldots, d\} \text{ and } i \in \{1, \ldots, p\}. \tag{6.55}$$

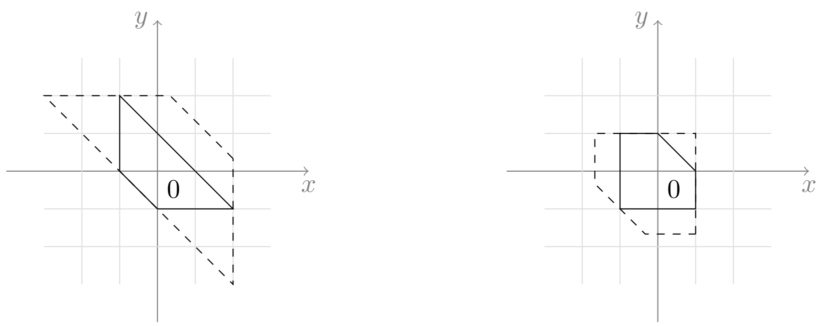

Figure 6.6: The polytope P for $\mathbb{P}^2$ blown up 1 or 2 points, and the translation of the smallest dilation of $-P$ that contains P. We can read from the figure that $\mathrm{asym}(P) = 5/3, 4/3$, respectively.

Example 6.6.11. For the polytopes of Figure 6.5 one has $\mathrm{asym}(P) = 5/3$ for the 1-point blow-up and $\mathrm{asym}(P) = 4/3$ for the 2-point blow-up, confirming the intution that the latter is "more centrally symmetric" than the former. This also serves to show that in many instances

$$\max_{x \in P} \| \cdot \|_{-P} > \mathrm{asym}(P),$$

i.e., (6.54) is rarely sharp for Delzant polytopes. The only examples we know it to be sharp are $\mathbb{P}^n$ and centrally symmetric P.

By John's theorem [65], $\mathrm{asym}(P) \leq n$, with equality attained for the simplex associated to $\mathbb{P}^n$ [54]. However, the next example shows that for Delzant polytopes arising from toric Fano manifolds one may have $\max_{x \in P} \| \cdot \|_{-P} > n$, even though this seems rare (at least in small dimensions). This first happens in $n = 3$ and just for a unique Fano 3-fold.

Example 6.6.12. Consider the Fano 3-fold 2-36 [3, p. 90], namely the Calabi–Hirzebruch manifold $\mathbb{F}_{3,2} := \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2})$ [20]. It is a toric manifold with [110, (4), p. 42]

$$\Delta_1 = \{v_1, \ldots, v_5\} = \{(0,0,1), (0,0,-1), (1,0,0), (0,1,0), (-1,-1,-2)\}.$$

Let P be the corresponding polytope, i.e., (recall (3.19))

$$P = \bigcap_{i=1}^{5} \{\langle\,\cdot\,, v_i\rangle \geq -1\}.$$

To compute its vertices, first notice that

$$\bigcap_{i=1}^{4} \{\langle\,\cdot\,, v_i\rangle \geq -1\} = [-1, +\infty) \times [-1, +\infty) \times [-1, 1],$$

which is an infinite cuboid with vertices $(-1,-1,\pm 1)$. See Figure 6.7. The polytope P is obtained by

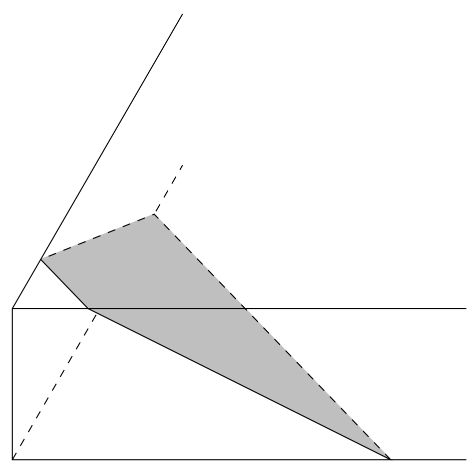

Figure 6.7: The polytope P, obtained by cutting the infinite cuboid with vertices $(-1,-1,\pm 1)$ with the hyperplane $\langle\cdot, v_5\rangle = -1$.

cutting it with the hyperplane $\langle\cdot, v_5\rangle = -1$, introducing the other 4 vertices $(-1,0,1)$, $(0,-1,1)$, $(-1,4,-1)$, $(4,-1,-1)$, namely the vertices of the isosceles trapezoid which is the facet cut out by the last equation $\langle\cdot, v_5\rangle = -1$. Therefore

$$\operatorname{Ver} P = \{(-1,-1,1), (-1,0,1), (0,-1,1), (-1,-1,-1), (-1,4,-1), (4,-1,-1)\}.$$

For α we use (6.45). We compute

$$\max_{x \in P} \| \cdot \|_{-P} = \max_{x \in \operatorname{Ver} P} \| \cdot \|_{-P} = 4,$$

with maximum obtained at the vertices $(-1,-1,-1), (-1,4,-1), (4,-1,-1)$, i.e.,

$$(-4\partial P) \cap P = \{(-1,-1,-1), (-1,4,-1), (4,-1,-1)\}.$$

See Figure 6.8.

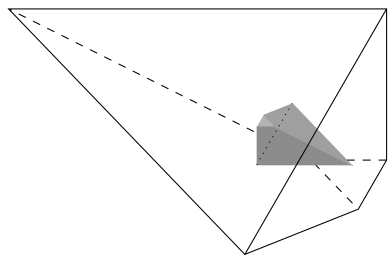

Figure 6.8: The frustum P (the solid inside) with upper base $\operatorname{co}\{(-1,-1,1), (-1,0,1), (0,-1,1)\}$ and lower base $\operatorname{co}\{(-1,-1,-1), (-1,4,-1), (4,-1,-1)\}$, and the boundary of $-4P$ (edges in solid and dashed lines). Their intersection consists of the 3 vertices of the lower base of P.

Hence $\alpha = \alpha_{k,1,(S^1)^n} = 1/5 < \alpha_{k,m,(S^1)^n}$, for $m \geq 2$. Solving the linear programming problem (6.55) yields $\operatorname{asym}(P) = 14/5 < 3$.

Chapter 7

Asymptotics of toric δ_k-invariants

This chapter deals with the δ_k-invariants of Fujita–Odaka (also called k-th stability thresholds) on toric Fano manifolds. It turns out that for these invariants there is a quantitative dichotomy regarding stabilization, and when stabilization fails we derive their complete asymptotic expansion.

This chapter also initiates the study of a combinatorial problem: *what is the asymptotic behavior of the quantized (or discrete) barycenters of lattice polytopes?*

To state this more precisely, let

$$M \cong \mathbb{Z}^n$$

denote a lattice and $P \subseteq M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ denote a convex lattice polytope, i.e., the vertex set satisfies $\operatorname{Ver} P \subseteq M$. Does the sequence $\{\mathrm{Bc}_k(P)\}_{k\in\mathbb{N}}$ encode other canonical quantities associated to P beside $\mathrm{Bc}(P)$? We pose the following:

Problem 7.0.1. Is there an asymptotic expansion

$$\mathrm{Bc}_k(P) = \sum_{i=0}^{\infty} a_i(P) k^{-i}?$$

If so, what are the coefficients $a_i(P)$?

Problem 7.0.1 is of purely combinatorial interest within Ehrhart theory. Thus, the bulk of this chapter (Sections 7.1, 7.3, and 7.4) can be read as a purely enumerative combinatorics paper, and to some extent also Section 7.2 that is concerned with algebraic combinatorics. Sections 7.5–7.6 are concerned with the algebraic geometric applications and can be read independently by the reader interested only in those (or skipped by the combinatorics-oriented reader).

At the same time, this chapter can be considered as a sequel to Chapter 6 where we initiate the study of asymptotics of algebro-geometric invariants arising from Kodaira embeddings using vector spaces of holomorphic sections of powers of a line bundle $H^0(X, L^k)$. The asymptotic parameter is the power k of the line bundle, where $1/k$ can be thought of as Planck's constant in geometric quantization. The over-arching geometric motivation for the series initiated in Chapter 6 is to obtain complete asymptotic expansions for these invariants. This is of course motivated by the foundational result of Catlin and Zelditch on the complete asymptotics of the Bergman kernel[24, 117], but the setting is quite different: the invariants we are interested in are Tian's α_k- and Fujita–Odaka's δ_k-invariants [103, 48] and they are not directly defined in terms of the Bergman kernel. Rather, they are more related to singularities of divisors in the linear series $|kL|$ on X. These invariants play a fundamental role in both the minimal model program in algebraic geometry and in the study of canonical Kähler metrics in differential geometry.

Problem 7.0.2. Is there an asymptotic expansion

$$\delta_k = \sum_{i=0}^{\infty} a_i(X, L) k^{-i}?$$

If so, what are the coefficients $a_i(X, L)$?

By a theorem of Blum–Jonsson $\lim_k \delta_k = \delta$ [16, Theorem 4.4], with δ the basis log canonical threshold of Fujita–Odaka [48]. Thus, if such an expansion exists then $a_0(X, L) = \delta$.

Another motivation is *Tian's stabilization problem* which asks whether such invariants "stabilize", i.e., become eventually constant in k. Tian asked this for his α_k-invariants in the smooth Fano setting [103, Question 1], and later for all polarizations on possibly singular varieties [107, Conjecture 5.3], and one can extend his problem to the δ_k-invariants.

Problem 7.0.3. Is $\delta_k = \delta$ for all sufficiently large k?

One can consider our Problem 7.0.2 as a refinement of Problem 7.0.3.

We completely solve Problem 7.0.3 in the (possibly singular) toric Fano setting (Corollary 7.0.14). This shows there is a remarkable dichotomy between the α_k-invariants and the δ_k-invariants. The previous chapter resolved Tian's stabilization problem for α_k-invariants by showing that in the toric Fano setting $\alpha_k = \alpha$ for all k. Here we show that δ_k-invariants behave very differently and *rarely stabilize,* but that if they do stabilize then they also satisfy $\delta_k = \delta$ for *all* $k \in \mathbb{N}$, and when Fano also $\delta_k = \delta = 1$. This precisely suggests that most likely there is a complete asymptotic expansion as conjectured in Problem 7.0.2, and indeed we prove that is the case.

Problems 7.0.2 and 7.0.3 are intimately related to Problem 7.0.1 as we explain next.

The relation between stability thresholds and barycenters has a rich and long history. Let $\Omega \in H^2(M, \mathbb{R})$ be a Kähler class. In 1983 Futaki introduced the eponymous invariant $F_\Omega(v)$ that associates to each holomorphic vector field v on X a number; he proved that F_Ω must vanish on all holomorphic vector fields in order for a constant scalar curvature Kähler metric to exist in Ω [50]. In the case X is toric it suffices to consider the n vector fields $t_1, \ldots, t_n$ generated by the complex torus $(\mathbb{C}^\star)^n$ acting on X. A beautiful theorem of Mabuchi states that (when suitably normalized) $(F(t_1), \ldots, F(t_n)) \in \mathbb{R}^n$ is the barycenter $\mathrm{Bc}(P_{X,\Omega})$ of the Delzant polytope $P_{X,\Omega}$ associated to (X, Ω) [76, Theorem 5.3]. Thus, Futaki's obstruction can be rephrased geometrically as $\mathrm{Bc}(P_{X,\Omega}) = 0$. Some thirty years later, Blum–Jonsson showed that Mabuchi's theorem can be rephrased in terms of the Fujita–Odaka δ invariant [16, Corollary 7.16]:

$$\delta^{-1} = \max_{i=1,\ldots,d} \left[\langle \mathrm{Bc}(P_{X,\Omega}), v_i \rangle + b_i \right],$$

where $\{v_1, \ldots, v_d\}$ are the primitive generators of the one-dimensional cones of the fan determining X and $(b_1, \ldots, b_d) \in \mathbb{R}^d$ determine the class Ω, so that $P_{X,\Omega} = \bigcap_{i=1}^d \{u \in M_\mathbb{R} \,|\, \langle u, v_i \rangle + b_i \geq 0\}$. In particular, in the Fano case $\Omega = -K_X$, $b_1 = \ldots = b_d = 1$, so $\delta \leq 1$ with equality if and only if $\mathrm{Bc}(P) = 0$. It was then realized by Rubinstein–Tian–Zhang that Blum–Jonsson's formula in the Fano case extends also to quantized barycenters [90, Corollary 7.1],

$$\delta_k^{-1} = \max_{i \in \{1,\ldots,d\}} \left[\langle \mathrm{Bc}_k(P_{X,-K_X}), v_i \rangle + 1 \right], \tag{7.1}$$

and that similarly $\mathrm{Bc}_k(P_{X,-K_X})$ is essentially a quantized Futaki character that obstructs the existence of k-th balanced metrics [90, Theorem 2.3]. The formula (7.1) is the link, at the heart of the present chapter, between quantized barycenters and stability thresholds. It is shown that (7.1) extends to all polarizations [63, Corollary A.8].

Thus, the study of asymptotics of $\mathrm{Bc}_k(P)$ for lattice polytopes is essentially equivalent to the study of the asymptotics of δ_k for (possibly singular) toric varieties. We employ this link in both directions in this chapter. For general P we derive a formula for $\mathrm{Bc}_k(P)$ which answers the first part of Problem 7.0.1 as well as determines $a_1(P)$; we then use this to obtain corresponding formula and asymptotics for δ_k for general (possibly very singular) toric varieties. Restricting to toric manifolds we then use (as well as add some entries to) the well-developed dictionary between Riemann–Roch theory and Ehrhart theory to answer Problem 7.0.1 completely for *Delzant lattice polytopes* P, determining all $a_i(P)$ in terms of mixed volumes of virtual polytopes. Thirdly, restricting to Gorenstein Fano varieties amounts to working with the sub-class of reflexive polytopes for which we refine the general expansion and show that $\mathrm{Bc}_k(P)$ are colinear, to first order, with $\mathrm{Bc}(P)$. This is not true for general P. Finally, for toric del Pezzos we show how the combinatorial and algebraic approach give two completely different proofs of the same formula for $\mathrm{Bc}_k(P)$ or δ_k. In this dimension $\mathrm{Bc}_k(P)$ are actually colinear with $\mathrm{Bc}(P)$. The beauty of this special case is that either convexity arguments or Ehrhart theory allows to replace the otherwise mysterious Ehrhart polynomial expressions by completely explicit formulas involving the lattice-normalized measure or intersection numbers.

Our overarching goal is to make this chapter accessible to both combinatorists and algebraic geometers. As part of that, we carefully separate the sections dealing with each. Sections 7.1–7.4 detail our combinatorial results on Problem 7.0.1. Section 7.5 describes our results on Problems 7.0.2 and 7.0.3. Section 7.6 revisits Section 7.2 in terms of test configurations and is relevant to both audiences.

Quantized barycenters of general lattice polytopes

Our first main result answers the first part of Problem 7.0.1, as well as determines the first non-trivial coefficient.

Denote

$$\mathrm{Bc}_k(P) = (\mathrm{Bc}_{k,1}(P), \ldots, \mathrm{Bc}_{k,n}(P)) \in M_{\mathbb{R}} \cong \mathbb{R}^n. \tag{7.2}$$

Theorem 7.0.4. *Let $M \cong \mathbb{Z}^n$ be a lattice and $P \subseteq M_{\mathbb{R}}$ be an n-dimensional convex lattice polytope. For $c \in M$ such that $P + c \subseteq \mathbb{R}^n_+$,*

$$\mathrm{Bc}_{k,i} = \frac{E_{P(c;i)}(k) - E_P(k)}{kE_P(k)} - c = \frac{E_{P(c;i)}(k) - E_{P\times[0,c_i]}(k)}{kE_P(k)}. \tag{7.3}$$

In particular, $\mathrm{Bc}_k(P) = \sum_{i=0}^{\infty} a_i(P)k^{-i}$ *for some a_i that depend only on P. Moreover,*

$$a_0(P) = \mathrm{Bc}(P), \quad a_1(P) = \frac{\widetilde{\mathrm{Vol}}(\partial P)}{2\,\mathrm{Vol}(P)}(\widetilde{\mathrm{Bc}}(\partial P) - \mathrm{Bc}(P)),$$

where $\widetilde{\mathrm{Vol}}(\partial P)$ is the mass of the lattice-normalized measure of ∂P, and $\widetilde{\mathrm{Bc}}(\partial P)$ is its barycenter (see Definition 3.3.1). Finally, if $\mathrm{Bc}_k(P)$ *is constant for $n+1$ values of k then $a_i(P) = 0$ for all $i \geq 1$ and* $\mathrm{Bc}_k(P) = \mathrm{Bc}(P)$ *for all $k \in \mathbb{N}$.*

Theorem 7.0.4 implies that if $\mathrm{Bc}(P) \neq \mathrm{Bc}_k(P)$ for *some* $k \in \mathbb{N}$ then the same is true for *all sufficiently large* k. In this context, it raises the following interesting problem:

Problem 7.0.5. Assume that P is a lattice polytope satisfying $\mathrm{Bc}(P) \neq \mathrm{Bc}_k(P)$ for some $k \in \mathbb{N}$. Does there exist such a P for which $\{\mathrm{Bc}_i(P)\}_{i\in I}$ is a singleton for some subset $I \subseteq \mathbb{N}$ with $|I| \geq 2$?

Theorem 7.0.4 says that if such a P exists then $|I| \leq n$. In Remark 7.4.3 we resolve Problem 7.0.5 in dimension $n = 2$.

Quantized barycenters of Delzant lattice polytopes

In this part, we assume that P belongs to the sub-class of Delzant polytopes.

To generalize this correspondence for divisors that are not necessarily ample, we use the notion of virtual polytopes (see [84, Corollary 2]).

Definition 7.0.6. • Let $\mathcal{P}$ denote the monoid of (possibly degenerate, i.e., of lower dimension) polytopes in $M_{\mathbb{R}}$. Recall that a monoid is a set equipped with an associative multiplication with identity. Let $\mathcal{P}^*$

denote the Grothendieck group of $\mathcal{P}$, i.e., the group of equivalent classes of formal differences $P_1 - P_2$ where $P_1 - P_2 \sim P_1' - P_2'$ if and only if $P_1 + P_2' = P_1' + P_2$. The elements in $\mathcal{P}^*$ are called *virtual polytopes.*

• The notion of *mixed volumes* (including volumes) can be extended by multi-linearity to virtual polytopes. For $P_1, \ldots, P_k \in \mathcal{P}^*$ and $m_1, \ldots, m_k \in \mathbb{N}$ with $m_1 + \ldots + m_k = n$, set

$$V(P_1, m_1; \ldots; P_k, m_k) := V(P_1, \ldots, P_1, \ldots, P_k, \ldots, P_k),$$

where each P_i appears m_i times.

• Let X be a smooth toric variety (toric manifold). Let

$$\{v_i\}_{i=1}^d$$

denote the primitive generators of the rays in the associated fan. Let

$$D_i$$

denote the toric divisor corresponding to v_i [49, §3.3]. For any ample toric divisor

$$D = \sum_{i=1}^d b_i D_i$$

where $b_i \in \mathbb{Z}$, define the *polytope associated to* (X, D),

$$P_D := \bigcap_{i=1}^d \{u \in M_\mathbb{R} \mid \langle u, v_i \rangle \geq -b_i\}.$$

In general, since any divisor can be written as the difference of two ample divisors [68, Example 1.2.10], for any divisor D there is an associated virtual polytope $P_D \in \mathcal{P}^*$.

There are efficient numerical algorithms to compute mixed volumes, see, e.g., [108, 55].

Let P be a Delzant lattice polytope, and let (X, L) be the associated polarized toric manifold given by Theorem 3.2.13, where

$$L = \sum_{i=1}^{d} b_i D_i.$$

Fix $v \in N$. Pick

$$q \in \mathbb{Z} \text{ such that } \langle \cdot, v\rangle + q > 0 \text{ on } P \text{ (possible as } P \text{ is compact).} \tag{7.4}$$

Consider the lattice polytope of one dimension higher,

$$P_{v,q} := \{(u,h) \in \mathbb{R}^n \times \mathbb{R} \,|\, u \in P,\ 0 \le h \le \langle u, v\rangle + q\} \tag{7.5}$$

$$= \{(u,h) \in \mathbb{R}^n \times \mathbb{R} \,|\, \langle u, v_i\rangle \ge -b_i, i \in \{1, \dots, d\},\ 0 \le h \le \langle u, v\rangle + q\}. \tag{7.6}$$

It corresponds to a polarized toric manifold of dimension $n+1$

$$(\overline{\mathcal{X}}, \overline{\mathcal{L}}) \tag{7.7}$$

(see Figures 7.1 and 7.2). This polytope has $d+2$ facets. Under the correspondence of Theorem 3.2.13, the associated primitive generators of the rays in the fan are

$$(v_1, 0), \dots, (v_d, 0), (0, 1), (v, -1) \in N \times \mathbb{Z},$$

and we denote

$$\mathcal{D}_1, \dots, \mathcal{D}_d, \mathcal{D}_{d+1}, \mathcal{D}_{d+2} \tag{7.8}$$

the corresponding torus-invariant divisors. Rewrite (7.5),

$$P_{v,q} = \{(u,h) \in \mathbb{R}^n \times \mathbb{R} \,|\, \langle (u,h), (v_i, 0)\rangle \ge -b_i,\ \langle (u,h), (0,1)\rangle \ge 0,\ \langle (u,h), (v,-1)\rangle \ge -q\},$$

By Definition 7.0.6,

$$P_{v,q} = P_{\mathcal{D}},$$

where

$$\mathcal{D} := \sum_{i=1}^{d} b_i \mathcal{D}_i + q\mathcal{D}_{d+2},$$

and

$$\overline{\mathcal{L}} \sim \mathcal{D}.$$

Define a virtual polytope

$$P'_v := P_{\sum_{i=1}^{d} b_i \mathcal{D}_i} = P_{v,q} - qP_{\mathcal{D}_{d+2}}.$$

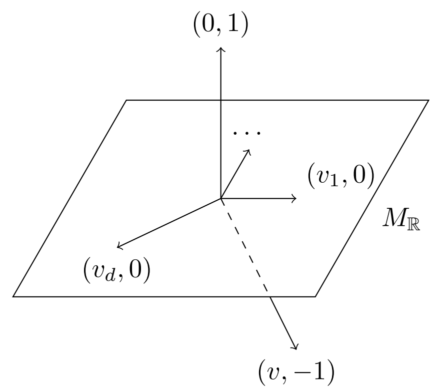

Figure 7.1: The fan of the new toric manifold $\overline{\mathcal{X}}$. Note that X corresponds to the subfan consisting of the cones in $M_\mathbb{R} \subseteq M_\mathbb{R} \times \mathbb{R}$. The cones above $M_\mathbb{R}$ are spanned by the cones in $M_\mathbb{R}$ and the ray generated by $(0, 1)$, while the cones below $M_\mathbb{R}$ are spanned by the cones in $M_\mathbb{R}$ and the ray generated by $(v, -1)$.

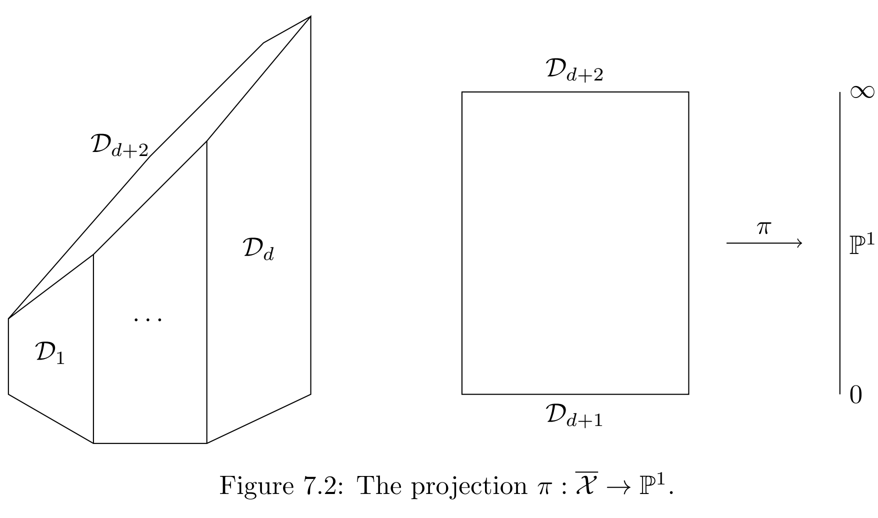

Figure 7.2: The projection $\pi : \overline{\mathcal{X}} \to \mathbb{P}^1$.

Our second main result is the complete resolution of Problem 7.0.1 when P is Delzant:

Theorem 7.0.7. *Let $P \subseteq M \otimes \mathbb{R}$ be a Delzant lattice polytope. Then for any $v \in N$,*

$$\langle \mathrm{Bc}_k(P), v\rangle = \frac{\sum_{j=0}^{n} c'_{j+1} k^j}{\sum_{j=0}^{n} a_j k^j} = \sum_{j=0}^{\infty} k^{-j} \sum_{i=0}^{j} c'_{n+1-j+i} \left(\frac{\delta_{i0}}{a_n} + \sum_{\ell=1}^{i} \frac{(-1)^\ell}{a_n^{\ell+1}} \sum_{\sum_{m=1}^{\ell} i_m = i-\ell} \prod_{m=1}^{\ell} a_{n-i_m-1} \right),$$

where

$$a_j = \sum_{\sum_{i=1}^{d} \ell_i = n-j} \frac{n! B_{\ell_1} \cdots B_{\ell_d}}{j! \ell_1! \cdots \ell_d!} V(P, j; P_{D_1}, \ell_1; \ldots; P_{D_d}, \ell_d),$$

$$c'_j = \sum_{\sum_{i=1}^{d} \ell_i = n+1-j} \frac{(n+1)! B_{\ell_1} \cdots B_{\ell_d}}{j! \ell_1! \cdots \ell_d!} V(P'_v, j; P_{\mathcal{D}_1}, \ell_i; \ldots; P_{\mathcal{D}_d}, \ell_d).$$

In particular,

$$a_n = \mathrm{Vol}(P),$$

$$a_{n-1} = \frac{n}{2} V(P, n-1; P_{-K_X}, 1),$$

$$a_{n-2} = \frac{n(n-1)}{12} \left(V(P, n-2; P_{-K_X}, 2) + \sum_{i<j} V(P, n-2; D_i, 1; D_j, 1) \right),$$

$$c'_{n+1} = \mathrm{Vol}(P'_v),$$

$$c'_n = \frac{n+1}{2} V\left(P'_v, n; P_{-K_{\overline{\mathcal{X}}/\mathbb{P}^1}}, 1\right),$$

$$c'_{n-1} = \frac{(n+1)n}{12} \left(V\left(P'_v, n-1; P_{-K_{\overline{\mathcal{X}}/\mathbb{P}^1}}, 2\right) + \sum_{1 \le i < j \le d} V\left(P'_v, n-1; P_{\mathcal{D}_i}, 1; P_{\mathcal{D}_j}, 1\right) \right),$$

where

$$P_{-K_X} = P_{\sum_{i=1}^{d} D_i}, \qquad P_{-K_{\overline{\mathcal{X}}/\mathbb{P}^1}} = P_{\sum_{i=1}^{d} \mathcal{D}_i},$$

and

$$\langle \mathrm{Bc}_k(P), v\rangle = \frac{c'_{n+1}}{a_n} + \left(\frac{c'_n}{a_n} - \frac{c'_{n+1} a_{n-1}}{a_n^2} \right) \frac{1}{k}$$

$$+\left(\frac{c'_{n-1}}{a_n}-\frac{c'_n a_{n-1}+c'_{n+1}a_{n-2}}{a_n^2}+\frac{c'_{n+1}a_{n-1}^2}{a_n^3}\right)\frac{1}{k^2}+O\left(\frac{1}{k^3}\right).$$

Quantized barycenters of reflexive polytopes

Another subclass of lattice polytopes is that of reflexive polytopes.

Definition 7.0.8. [33, Definition 2.3.12] A lattice polytope is called *reflexive* if it contains the origin, and each of its facet is given by $\langle u, v\rangle = 1$ where $v \in N$ is some primitive vector.

Remark 7.0.9. We are not aware of a relationship between being reflexive and being Delzant. For instance, the square with vertices $(\pm 1, 0)$ and $(0, \pm 1)$ is reflexive but not Delzant, and the square with vertices $(\pm 2, \pm 2)$ and $(\pm 2, \mp 2)$ is Delzant but not reflexive.

Our third main result is the specialization of Theorem 7.0.4 to the reflexive setting. It is quite remarkable that in this setting the *quantized barycenters are, to first order, colinear*. Furthermore, in the planar case, they are in fact *colinear*, and have a completely explicit expression.

Theorem 7.0.10. *Let P be a reflexive polytope. Then $a_1(P) = \mathrm{Bc}(P)/2$ (recall Problem 7.0.1), i.e.,*

$$\mathrm{Bc}_k(P) = \mathrm{Bc}(P)\left(1+\frac{1}{2k}\right)+O(k^{-2}).$$

Moreover, if $\dim P = 2$*, then*

$$\mathrm{Bc}_k(P) = \frac{(k+1)(2k+1)\,\widetilde{\mathrm{Vol}}(\partial P)}{4+2k(k+1)\,\widetilde{\mathrm{Vol}}(\partial P)}\,\mathrm{Bc}(P).$$

We give two completely different proofs of the two-dimensional part of Theorem 7.0.10.

Applications to asymptotics of δ_k-invariants

Theorem 7.0.4 together with (7.1) answer the first part of Problem 7.0.2 as well as determine the first non-trivial coefficient for a general polarized (possibly singular) toric variety.

Corollary 7.0.11. *Let X be a (possibly singular) toric variety given by a fan Δ in a lattice $N \simeq \mathbb{Z}^n$.*

Let $v_1, \dots, v_d \in N$ *denote the primitive generators of rays in* Δ*. Let* L *be an ample line bundle with the associated convex polytope (recall Theorem 3.2.13)*

$$P = \{x \in M_{\mathbb{R}} \,|\, \langle x, v_i \rangle \geq -b_i, \ \textit{for all } i = 1, \dots, d\}$$

for some $b_i \in \mathbb{R}$*, where* $M_{\mathbb{R}} := \mathrm{Hom}(N, \mathbb{Z}) \otimes \mathbb{R}$*. Let* I *be the set of indices* i *such that* v_i *and* b_i *compute* δ*. Then there is a complete asymptotic expansion for* δ_k *as in Problem 7.0.2, with*

$$a_0(X, L) = \delta = \frac{1}{\langle \mathrm{Bc}(P), v_i \rangle + b_i}, \ i \in I, \quad a_1(X, L) = -\frac{\widetilde{\mathrm{Vol}}(\partial P)}{2\,\mathrm{Vol}(P)} \max_{i \in I} \langle \widetilde{\mathrm{Bc}}(\partial P) - \mathrm{Bc}(P), v_i \rangle \delta^2.$$

In particular,

$$\delta_k = \delta - \frac{\widetilde{\mathrm{Vol}}(\partial P)}{2\,\mathrm{Vol}(P)} \max_{i \in I} \langle \widetilde{\mathrm{Bc}}(\partial P) - \mathrm{Bc}(P), v_i \rangle \delta^2 k^{-1} + O(k^{-2}), \tag{7.9}$$

where $\widetilde{\mathrm{Vol}}$ *is the normalized volume of the boundary* ∂P *and* $\widetilde{\mathrm{Bc}}$ *is the barycenter of* ∂P *with respect to* $\widetilde{\mathrm{Vol}}$ (*see* Definition 3.3.1).

In fact, for sufficiently large k*,* (7.9) *is a complete asymptotic expansion of a rational function of* k *with both numerator and denominator polynomial of degree* n *in* k*.*

In the case of most interest, namely when X is Fano polarized by $-K_X$, Theorem 7.0.10 and (7.1) give a more explicit result.

Corollary 7.0.12. *For a toric Fano (possibly singular) variety* X *and all* $k \in \mathbb{N}$*,*

$$\delta_k(-K_X) = \delta(-K_X) - \frac{1}{2k}\delta(-K_X)(1 - \delta(-K_X)) + O(k^{-2}).$$

Moreover, if $\dim X = 2$*,*

$$\delta_k(-K_X) = \left(1 + \frac{(k+1)(2k+1)K_X^2}{4 + 2k(k+1)K_X^2}\big((\delta(-K_X))^{-1} - 1\big)\right)^{-1}.$$

Test configurations, higher Donaldson–Futaki invariants, and Tian's stabilization problem

Finally, we make contact with the much-studied notion of test configurations [106, 39]. We show that in the case P is Delzant, the coefficients $a_i(P)$ conjectured by Problem 7.0.1 and established by Theorem 7.0.7 are closely related to the higher Donaldson–Futaki invariants of certain test configurations.

Theorem 7.0.13. *Let $P \subseteq M \otimes_{\mathbb{Z}} \mathbb{R}$ be a lattice polytope. Let (X, L) be the associated (possibly singular) toric variety given by Theorem 3.2.13. Let $v \in N$ be arbitrary. Then*

$$\langle a_i(P), v\rangle = DF_i\big(\mathcal{X}, \mathcal{L}, \sigma_{q(v)}\big), \quad i \in \mathbb{N} \cup \{0\},$$

where $\big(\mathcal{X}, \mathcal{L}, \sigma_{q(v)}\big)$ denotes a product test configuration (7.30) *constructed in §7.6.1.*

In particular, if (X, L) is K-semistable then DF_1 vanishes for all product test configurations [15, Remark 2.13(2)], thus $a_1(P) = 0$ by Theorem 7.0.13, i.e., $\widetilde{\mathrm{Bc}}(\partial P) = \mathrm{Bc}(P)$ by Theorem 7.0.4, recovering a well-known result of Donaldson [39, Proposition 4.2.1]. In the reflexive case the additional relation $\widetilde{\mathrm{Bc}}(\partial P) = (1 + \frac{1}{n})\,\mathrm{Bc}(P)$ (by Theorems 7.0.4 and 7.3.5) forces $\widetilde{\mathrm{Bc}}(\partial P) = \mathrm{Bc}(P) = 0$, generalizing one direction of a result of Ono [81, Corollary 1.7] to the non-smooth setting (notice Ono assumes P is reflexive *and Delzant*), that is, asymptotic Chow semistability implies $\mathrm{Bc}_k(P) = 0$ for all k. By (7.1), $\mathrm{Bc}_k(P) = 0$ if and only if $\delta_k = 1$. Conversely, if δ_k stabilizes then by Corollary 7.0.12 $\delta = 1$, and hence $\delta_k = 1$ for large k, that by the previous sentence means $\mathrm{Bc}_k = 0$ for large k. In this case we also know by Theorem 7.0.4 that $\mathrm{Bc}_k = 0$ for all k, that in turn by the other direction of Ono's result (only valid if P is also Delzant) implies asymptotic Chow semistability. In sum we have the following solution of Problem 7.0.3 for toric Fanos:

Corollary 7.0.14. *For a toric Fano manifold X, δ_k stabilizes in k if and only if $(X, -K_X)$ is asymptotically Chow semistable. In this case $\delta_k = \delta = 1$ for all $k \in \mathbb{N}$.*

For (possibly singular) toric Fano variety X, δ_k stabilizes in k if and only if $\delta_k = \delta = 1$ for all $k \in \mathbb{N}$.

Corollary 7.0.14 is in interesting contrast to the situation for the $\alpha_{k,G}$-invariant, that was shown to stabilize always (see Chapter 6). Thus $\mathbb{P}^2$ blown-up at one or two points are examples for which $\delta_k(-K_X)$ does not stabilize, while for the three remaining smooth toric del Pezzos stabilization occurs. This might give the impression that the existence of the Kähler–Einstein metric dictates stabilization; this is true precisely when $n \leq 6$ (Example 7.7.4).

Remark 7.0.15. The first part of Corollary 7.0.14 generalizes to non-toric smooth Fano X using difficult results from algebraic and differential geometry, e.g., the Yau–Tian–Donaldson correspondence, and finite generation. Indeed, when $\delta > 1$ this follows from [120, Corollary 1.11]. When $\delta < 1$, here is a sketch: there exists a "dreamy" divisor E over X that computes δ [73, Theorem 1.2], and corresponds to a test configuration $(\mathcal{X}, \mathcal{L})$ with $S_k(E) = \sum_i DF_i(\mathcal{X}, \mathcal{L})k^{-i}$ and $2\,\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = A(E) - S(E) < 0$. In particular, $DF_1(\mathcal{X}, \mathcal{L}) = -\,\mathrm{Fut}(\mathcal{X}, \mathcal{L}) > 0$. Therefore, for sufficiently large k, $S_k(E) > S(E)$. Hence $\delta_k < \delta$. We emphasize that these results do not seem to give any information concerning the more refined Problem 7.0.2 nor an alternative approach to our Corollary 7.0.11.

Remark 7.0.16. Let (X, L) be a polarized toric manifold with associated Delzant lattice polytope P. A result of Ono can be interpreted as (he does not implicitly consider the quantized barycenters) $\mathrm{Bc}_k(P) = \mathrm{Bc}(P)$ (for a fixed k) is a necessary condition for (X, kL) to be Chow semistable [81, Theorem 1.2] (for Fano, and not necessarily toric manifolds see [90, Theorem 2.3] and [90, Corollary 7.1]). In particular, if (X, L) is asymptotically Chow semistable, i.e., (X, kL) is Chow semistable for sufficiently large k, then Ono's result can be interpreted as implying $\mathrm{Bc}_k(P) = \mathrm{Bc}(P)$ for all k [81, Theorem 1.4] (by using the existence of the generalized Ehrhart polynomial for homogeneous polynomials (cf. Remark 7.1.2)). These results are of course special cases of Theorems 7.0.4 and 7.0.13, together with the fact that $DF_1\big(\mathcal{X}, \mathcal{L}, \sigma_{q(v)}\big) = 0$ for all product test configurations whenever (X, L) is K-semistable [39, Definition 2.1.2], a condition that is implied by asymptotic Chow semistability [100, §5]. Resolving a conjecture of Ono, Futaki [52] showed Ono's result can be interpreted in terms of integral invariants related to Hilbert series as in the work of Futaki [51] and Futaki–Ono–Sano [53]. See also [114, 115, 93] for more on relations between Chow stability, combinatorics and δ_k-invariants.

Problem 7.0.17. Does the stabilization of $\mathrm{Bc}_k(P)$ imply asymptotic Chow polystability of X_P, assuming it is smooth (i.e., P is Delzant by Theorem 3.2.13)?

The answer is likely affirmative: by a result of Lee–Yotsutani the stabilization of $\mathrm{Bc}_k(P)$ (they phrase this in an equivalent manner) combined with K-semistability for toric degenerations implies asymptotic Chow polystability [69, Corollary 1.5]. In general, the stabilization of $\mathrm{Bc}_k(P)$ does not imply asymptotic Chow stability. See [69, Corollary C.3] for a reflexive, non-Delzant example.

7.1 First order expansion of quantized barycenters of lattice polytopes

In this section we study the first order expansion of $\mathrm{Bc}_k(P)$ for any lattice polytope P.

Proposition 7.1.1. *Let P be a lattice polytope. Then*

$$\mathrm{Bc}_k(P) = \mathrm{Bc}(P) + \frac{\widetilde{\mathrm{Vol}}(\partial P)}{2\,\mathrm{Vol}(P)}(\widetilde{\mathrm{Bc}}(\partial P) - \mathrm{Bc}(P))k^{-1} + O(k^{-2}).$$

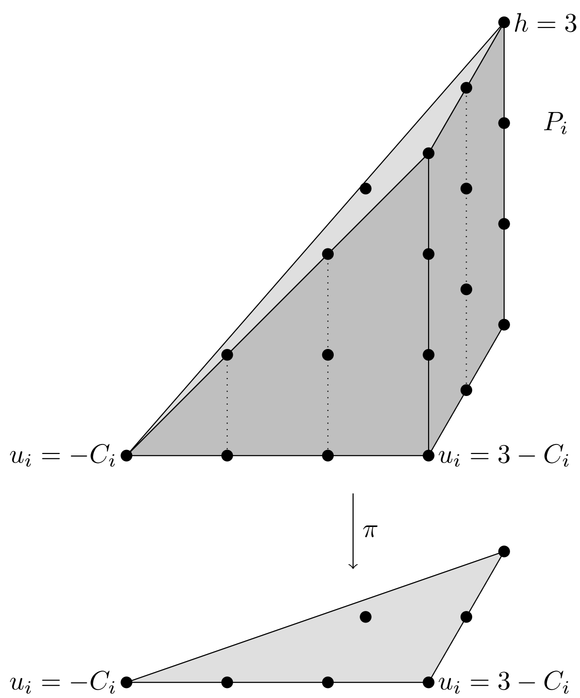

Figure 7.3: The lattice polytope P_i: $u_i + C_i$ lattice points above each lattice point in P with i-th coordinate equal to u_i. The dotted lines are fibers of $\pi : M \times \mathbb{R} \ni (u, h) \mapsto u \in M$.

Proof. Fix $1 \le i \le n$. Pick an integer $C_i \ge -\min\{u_i : (u_1, \dots, u_n) \in P\}$. Then for any $u \in P$, the function $h_i(u) := u_i + C_i$ is non-negative. Consider the lattice polytope (see Figure 7.3)

$$P_i := \{(u, h) \in \mathbb{R}^n \times \mathbb{R} \,|\, u \in P,\ 0 \le h \le h_i(u)\}. \tag{7.10}$$

Notice that

$$\begin{aligned} E_{P_i}(k) &= \sum_{u \in P \cap \frac{1}{k} L} \#\left\{(u, 0), \left(u, \frac{1}{k}\right), \cdots, (u, h_i(u))\right\} \\ &= \sum_{u \in P \cap \frac{1}{k} L} (k(u_i + C_i) + 1) \\ &= (kC_i + 1) E_P(k) + k \sum_{u \in P \cap \frac{1}{k} L} u_i. \end{aligned}$$

Therefore,

$$\frac{1}{E_P(k)} \sum_{u \in P \cap \frac{1}{k} L} u_i = \frac{E_{P_i}(k)}{k E_P(k)} - C_i - \frac{1}{k}. \tag{7.11}$$

Now,

$$\operatorname{Vol}(P_i) = \int_P h_i(u) d \operatorname{Vol} = \int_P u_i d \operatorname{Vol} + C_i \operatorname{Vol}(P). \tag{7.12}$$

Notice that the projection to $P \subseteq \mathbb{R}^n \times \{0\}$ from the upper boundary of P_i, denoted by

$$H_i := \{(u, h) \in P_i \,|\, h = h_i(u)\},$$

preserves normalized volumes, since this projection restricted to the lattice points on the hyperplanes is an isomorphism of lattices. In particular, $\operatorname{Vol}(P) = \widetilde{\operatorname{Vol}}(H_i)$. It follows that

$$\widetilde{\operatorname{Vol}}(\partial P_i) = \int_{\partial P} h_i(u) d \widetilde{\operatorname{Vol}} + 2 \operatorname{Vol}(P) = \int_{\partial P} u_i d \widetilde{\operatorname{Vol}} + C_i \widetilde{\operatorname{Vol}}(\partial P) + 2 \operatorname{Vol}(P). \tag{7.13}$$

Combining (7.11), (7.12), (7.13), and Theorem 3.3.4,

$$
\begin{aligned}
\frac{1}{E_P(k)} \sum_{u \in P \cap \frac{1}{k} L} u_i &= \frac{\operatorname{Vol}(P_i) k^{n+1} + \frac{1}{2} \widetilde{\operatorname{Vol}}(\partial P_i) k^n + O(k^{n-1})}{\operatorname{Vol}(P) k^{n+1} + \frac{1}{2} \widetilde{\operatorname{Vol}}(\partial P) k^n + O(k^{n-1})} - C_i - \frac{1}{k} \\
&= \frac{\operatorname{Vol}(P_i)}{\operatorname{Vol}(P)} - C_i + \left(\frac{\operatorname{Vol}(P) \widetilde{\operatorname{Vol}}(\partial P_i) - \widetilde{\operatorname{Vol}}(\partial P) \operatorname{Vol}(P_i)}{2 \operatorname{Vol}(P)^2} - 1 \right) k^{-1} + O(k^{-2}) \\
&= \frac{\int_P u_i d \operatorname{Vol}}{\operatorname{Vol}(P)} + \frac{\widetilde{\operatorname{Vol}}(\partial P)}{2 \operatorname{Vol}(P)} \left(\frac{\int_{\partial P} u_i d \widetilde{\operatorname{Vol}}}{\widetilde{\operatorname{Vol}}(\partial P)} - \frac{\int_P u_i d \operatorname{Vol}}{\operatorname{Vol}(P)} \right) k^{-1} + O(k^{-2}),
\end{aligned}
$$

concluding the proof of Proposition 7.1.1. □

Remark 7.1.2. An alternative proof of Proposition 7.1.1 can be obtained from the first-order expansion of the *sum* of the function $\phi(u) = u$ over the integral lattice points (as opposed to Bc_k that is its *average*), see, e.g., [115, (5)], [74, (7)]. Such a formula follows directly from the existence of the generalized Ehrhart polynomial for homogeneous polynomials and from Ehrhart–MacDonald reciprocity for such polynomials that generalizes Proposition 7.3.2 [18, Proposition 4.1]. While such a proof would be perhaps more standard for experts in Ehrhart theory, the proof we give is certainly more elementary and allows us to make the connection to the *rooftop polytope* (that, in the toric geometry literature, goes back to Donaldson [39, §4.2]) that then leads to the connection to Donaldson–Futaki invariants in Theorem 7.0.13.

Corollary 7.1.3. *Each coordinate of the quantized barycenter* $\mathrm{Bc}_k(P)$ *is a rational function of* k. *More precisely, it is the quotient of two polynomials of degree at most* n. *In particular, if* $\mathrm{Bc}_k(P)$ *is constant for* $n+1$ *values of* k, *then* $\mathrm{Bc}_k(P) = \mathrm{Bc}(P)$ *for all* k. *Consequently, if* $\mathrm{Bc}_k(P)$ *is constant for sufficiently large* k, *then* $\mathrm{Bc}_k(P) = \mathrm{Bc}(P)$ *for all* k.

Proof. Let $\mathrm{Bc}_{k,i}$ denote the i-th coordinate of the quantized barycenter $\mathrm{Bc}_k(P) \in \mathbb{R}^n$. Using the same notation as in the proof of Proposition 7.1.1, by (7.11), we have

$$
\mathrm{Bc}_{k,i} = \frac{1}{E_P(k)} \sum_{u \in P \cap \frac{1}{k} L} u_i = \frac{E_{P_i}(k) - (C_i k + 1) E_P(k)}{k E_P(k)}. \tag{7.14}
$$

Notice that the numerator

$$Q_i(k) := E_{P_i}(k) - (C_i k + 1)E_P(k) = E_{P_i}(k) - E_{P\times[0,C_i]}(k),$$

is a polynomial with respect to k of degree at most $n+1$ and with constant term 0 by Theorem 3.3.4. Hence $Q_i(k) = k\widetilde{Q}_i(k)$ for some polynomial $\widetilde{Q}_i$ of degree at most n. Now,

$$\mathrm{Bc}_{k,i} = \frac{\widetilde{Q}_i(k)}{E_P(k)}.$$

Since there are $n+1$ values of k such that $\mathrm{Bc}_{k,i}$ is equal to a constant, say, λ_i, the polynomial

$$\widetilde{Q}_i(k) - \lambda_i E_P(k)$$

has degree at most n and has at least $n+1$ zeros. It follows that this polynomial vanishes everywhere, i.e., $\mathrm{Bc}_{k,i}$ is constant for all k. □

Proof of Theorem 7.0.4. The second half of the theorem is proved by Proposition 7.1.1 and Corollary 7.1.3. It remains to show (7.3). Notice that for any i,

$$P(c;i) - c = \{(u,h)\,|\, u \in P + c,\ h \in [0,u_i]\} - c = \{(u,h)\,|\, u \in P,\ h \in [0, u_i + c_i]\} = P_i$$

(Recall (7.10)). In particular, their Ehrhart polynomials coincide. This reduces (7.3) to (7.14). □

7.2 Asymptotics of quantized barycenters of Delzant lattice polytopes

In this section, we express the asympototic expansion of the quantized barycenters in terms of mixed volumes of virtual polytopes using toric geometry. Here we restrict ourselves to Delzant lattice polytopes, namely the lattice polytopes that determines a smooth fan, i.e., the primitive generators of any

n-dimensional cone form a $\mathbb{Z}$-basis. Recall Definition 7.0.6.

Lemma 7.2.1. *For divisors $D_1, \ldots, D_n$ with associated virtual polytopes $P_1, \ldots, P_n$,*

$$V(P_1, \ldots, P_n) = \frac{1}{n!} D_1 \cdots D_n.$$

Proof. Since any divisor can be written as the difference of two ample divisors, by multi-linearity we may assume $D_1, \ldots, D_n$ are ample. In that case the corresponding virtual polytopes $P_1, \ldots, P_n$ are polytopes. For $k_1, \ldots, k_n > 0$, the divisor $k_1 D_1 + \cdots + k_n D_n$ is ample with associated polytope $k_1 P_1 + \cdots + k_n P_n$. Therefore

$$\mathrm{Vol}(k_1 P_1 + \cdots + k_n P_n) = \frac{1}{n!}(k_1 D_1 + \cdots + k_n D_n)^n.$$

Both sides are polynomials in $k_1, \ldots, k_n$. In particular, the coefficients of $k_1 \cdots k_n$ in both sides of the equation should be equal, i.e., $n! V(P_1, \ldots, P_n) = D_1 \cdots D_n$. $\square$

Definition 7.2.2. Define the Todd class

$$\mathrm{Td}(X) = \prod_{i=1}^{n} \frac{\xi_i}{1 - e^{-\xi_i}},$$

where the ξ_i's are the Chern roots of the tangent bundle T_X, i.e., they are symbolic objects such that

$$c(T_X) = \prod_{i=1}^{n} (1 + \xi_i),$$

or more explicitly,

$$c_i(T_X) = \sigma_i(\xi_1, \ldots, \xi_n),$$

where σ_i is the i-th elementary symmetric polynomial.

$\mathrm{Td}(X)$ can be expressed in terms of the cohomology classes of the torus invariant divisors.

Definition 7.2.3. Define

$$\mathrm{Td}(x) := \frac{x}{1-e^{-x}} = \sum_{j=0}^{\infty} \frac{B_j}{j!} x^j = 1 + \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \cdots ,$$

where B_j is called the j-th Bernoulli number.

Lemma 7.2.4. [33, Theorem 13.1.6] *Let D_i denote the divisor corresponding to v_i. Then*

$$\mathrm{Td}(X) = \prod_{i=1}^{d} \mathrm{Td}([D_i]).$$

Proposition 7.2.5. *Suppose L is ample. Then*

$$h^0(X, kL) = \sum_{j=0}^{n} a_j k^j,$$

where

$$a_j = \frac{L^j}{j!} \cdot \sum_{\sum_{i=1}^{d} \ell_i = n-j} \prod_{i=1}^{d} \frac{B_{\ell_i}}{\ell_i!} D_i^{\ell_i}.$$

In particular,

$$a_n = \frac{L^n}{n!}, \qquad a_{n-1} = \frac{L^{n-1} \cdot (-K_X)}{2(n-1)!}, \qquad a_{n-2} = \frac{L^{n-2} \cdot \left((-K_X)^2 + \sum_{i<j} D_i \cdot D_j \right)}{12(n-2)!}.$$

Proof. By the Kodaira vanishing theorem, $h^0(X, kL) = \chi(X, kL)$ for sufficiently large k. Since both are polynomials, they are equal for all k. Therefore

$$\begin{aligned}
h^0(X, kL) &= \chi(X, kL) \\
&= \int_X \mathrm{ch}(kL) \cdot \mathrm{Td}(X) \\
&= \int_X e^{k[L]} \cdot \prod_{i=1}^{d} \mathrm{Td}([D_i]) \\
&= \int_X \sum_{j=0}^{n} \frac{k^j}{j!} [L]^j \cdot \prod_{i=1}^{d} \sum_{\ell=0}^{\infty} \frac{B_\ell}{\ell!} [D_i]^\ell
\end{aligned}$$

$$= k^j \sum_{j=0}^{n} \frac{L^j}{j!} \cdot \sum_{\sum_{i=1}^{d} \ell_i = n-j} \prod_{i=1}^{d} \frac{B_{\ell_i}}{\ell_i!} D_i^{\ell_i}.$$

$\square$

Remark 7.2.6. Recall the average scalar curvature

$$\overline{S} := \frac{\int_X S_\omega \cdot \omega^n}{\int_X \omega^n} = \frac{\int_X n \operatorname{Ric} \omega \wedge \omega^{n-1}}{\int_X \omega^n} = \frac{n(-K_X) \cdot L^{n-1}}{L^n} = \frac{2a_{n-1}}{a_n}.$$

By Theorem 3.3.4,

$$\overline{S} = \frac{\widetilde{\mathrm{Vol}}(\partial P)}{\mathrm{Vol}(P)},$$

as observed by Donaldson [39, p. 309].

Next, using the construction of Proposition 7.1.1, we relate $\mathrm{Bc}_k(P)$ to this expansion.

Corollary 7.2.7. *Let $P \subseteq M \otimes \mathbb{R}$ be a Delzant lattice polytope. Fix $v \in N$. Let (X, L) denote the corresponding polarized toric manifold as in Theorem 3.2.13, where*

$$L = \sum_{i=1}^{d} b_i D_i.$$

Also recall (7.8)*. Then*

$$\langle \mathrm{Bc}_k(P), v \rangle = \frac{\sum_{j=0}^{n} c'_{j+1} k^j}{\sum_{j=0}^{n} a_j k^j} = \sum_{j=0}^{\infty} k^{-j} \sum_{i=0}^{j} c'_{n+1-j+i} \left(\frac{\delta_{i0}}{a_n} + \sum_{\ell=1}^{i} \frac{(-1)^\ell}{a_n^{\ell+1}} \sum_{\sum_{m=1}^{\ell} i_m = i-\ell} \prod_{m=1}^{\ell} a_{n-i_m-1} \right),$$

where

$$a_j = \frac{L^j}{j!} \cdot \sum_{\sum_{i=1}^{d} \ell_i = n-j} \prod_{i=1}^{d} \frac{B_{\ell_i}}{\ell_i!} D_i^{\ell_i}, \qquad c'_j = \frac{\overline{\mathcal{L}}'^j}{j!} \cdot \sum_{\sum_{i=1}^{d} \ell_i = n+1-j} \prod_{i=1}^{d} \frac{B_{\ell_i}}{\ell_i!} \mathcal{D}_i^{\ell_i},$$

and

$$\overline{\mathcal{L}}' = \sum_{i=1}^{d} b_i \mathcal{D}_i.$$

In particular,

$$\begin{aligned}
a_n &= \frac{L^n}{n!}, \\
a_{n-1} &= \frac{L^{n-1} \cdot (-K_X)}{2(n-1)!}, \\
a_{n-2} &= \frac{L^{n-2} \cdot \left((-K_X)^2 + \sum_{i<j} D_i \cdot D_j \right)}{12(n-2)!}, \\
c'_{n+1} &= \frac{\overline{\mathcal{L}}'^{n+1}}{(n+1)!}, \\
c'_n &= \frac{\overline{\mathcal{L}}'^{n} \cdot \left(-K_{\overline{\mathcal{X}}/\mathbb{P}^1} \right)}{2n!}, \\
c'_{n-1} &= \frac{\overline{\mathcal{L}}'^{n-1} \cdot \left(\left(-K_{\overline{\mathcal{X}}/\mathbb{P}^1} \right)^2 + \sum_{1 \le i<j \le d} \mathcal{D}_i \cdot \mathcal{D}_j \right)}{12(n-1)!},
\end{aligned}$$

where

$$-K_{\overline{\mathcal{X}}/\mathbb{P}^1} := -K_{\overline{\mathcal{X}}} + \pi^* K_{\mathbb{P}^1} = -K_{\overline{\mathcal{X}}} - \mathcal{D}_{d+1} - \mathcal{D}_{d+2} = \sum_{i=1}^{d} \mathcal{D}_i,$$

and

$$\begin{aligned}
\langle \mathrm{Bc}_k(P), v \rangle = \frac{c'_{n+1}}{a_n} &+ \left(\frac{c'_n}{a_n} - \frac{c'_{n+1} a_{n-1}}{a_n^2} \right) \frac{1}{k} \\
&+ \left(\frac{c'_{n-1}}{a_n} - \frac{c'_n a_{n-1} + c'_{n+1} a_{n-2}}{a_n^2} + \frac{c'_{n+1} a_{n-1}^2}{a_n^3} \right) \frac{1}{k^2} + O\left(\frac{1}{k^3} \right).
\end{aligned}$$

Proof. We use the setup in Section 7. By (7.11),

$$\langle \mathrm{Bc}_k(P), v \rangle = \frac{E_{P_{v,q}}(k)}{k E_P(k)} - q - \frac{1}{k}.$$

Write

$$E_P(k) = \sum_{j=0}^{n} a_j k^j,$$
$$E_{P_{v,q}}(k) = \sum_{j=0}^{n+1} c_j k^j.$$

We have

$$a_j = \frac{L^j}{j!} \cdot \sum_{\sum_{i=1}^{d} \ell_i = n-j} \prod_{i=1}^{d} \frac{B_{\ell_i}}{\ell_i!} D_i^{\ell_i},$$
$$c_j = \frac{\overline{\mathcal{L}}^j}{j!} \cdot \sum_{\sum_{i=1}^{d+2} \ell_i = n+1-j} \prod_{i=1}^{d+2} \frac{B_{\ell_i}}{\ell_i!} \mathcal{D}_i^{\ell_i} = q a_{j-1} + a_j + \frac{\overline{\mathcal{L}}'^j}{j!} \cdot \sum_{\sum_{i=1}^{d} \ell_i = n+1-j} \prod_{i=1}^{d} \frac{B_{\ell_i}}{\ell_i!} \mathcal{D}_i^{\ell_i}.$$

For simplicity, we define

$$a_j := 0$$

for $j < 0$ or $j > n$, and

$$c_j := 0$$

for $j < 0$ or $j > n+1$. We also define

$$c'_j := c_j - q a_{j-1} - a_j = \frac{\overline{\mathcal{L}}'^j}{j!} \cdot \sum_{\sum_{i=1}^{d} \ell_i = n+1-j} \prod_{i=1}^{d} \frac{B_{\ell_i}}{\ell_i!} \mathcal{D}_i^{\ell_i}.$$

Note c_j depends on the choice of q while c'_j is canonical. Then

$$\begin{aligned}
\langle \mathrm{Bc}_k(P), v \rangle &= \frac{E_{P_{v,q}}(k)}{k E_P(k)} - q - \frac{1}{k} \\
&= \frac{E_{P_{v,q}}(k) - q k E_P(k) - E_P(k)}{k E_P(k)} \\
&= \frac{\sum_{j=0}^{n+1} (c_j - q a_{j-1} - a_j) k^j}{\sum_{j=0}^{n} a_j k^{j+1}}
\end{aligned}$$

$$= \frac{\sum_{j=0}^{n+1} c'_j k^j}{\sum_{j=0}^{n} a_j k^{j+1}}$$

$$= \sum_{j=0}^{\infty} k^{-j} \sum_{i=0}^{j} c'_{n+1-j+i} \left(\frac{\delta_{i0}}{a_n} + \sum_{\ell=1}^{i} \frac{(-1)^\ell}{a_n^{\ell+1}} \sum_{\sum_{m=1}^{\ell} i_m = i-\ell} \prod_{m=1}^{\ell} a_{n-i_m-1} \right).$$

Finally, recall that $a_0 = c_0 = 1$. Therefore $c'_0 = c_0 - a_0 = 0$, and

$$\langle \mathrm{Bc}_k(P), v \rangle = \frac{\sum_{j=1}^{n+1} c'_j k^j}{\sum_{j=0}^{n} a_j k^{j+1}} = \frac{\sum_{j=0}^{n} c'_{j+1} k^j}{\sum_{j=0}^{n} a_j k^j}.$$

This concludes the proof. □

Theorem 7.0.7 is a reformulation of Corollary 7.2.7 in terms of mixed volumes of virtual polytopes.

Proof of Theorem 7.0.7. This follows from Corollary 7.2.7 and Lemma 7.2.1. □

7.3 First order expansion of quantized barycenters of reflexive polytopes

In this section, we study the special case that the lattice polytope is reflexive.

Definition 7.3.1. [33, Definition 2.3.12] A lattice polytope in $M \otimes \mathbb{R}$ is called *reflexive* if it contains the origin, and each of its facet is given by $\langle u, v \rangle = 1$ where v is some primitive vector of N.

Proposition 7.3.2. [77], [57, Theorem 19.1] *For $k \in \mathbb{N}$,*

$$E_P(-k) = (-1)^n \#((\operatorname{int} kP) \cap M).$$

Corollary 7.3.3. *Let $P \subseteq M_{\mathbb{R}}$ be an n-dimensional reflexive polytope. Then*

$$E_P(-k) = (-1)^n E_P(k-1).$$

Proof. By Definition 7.0.8, there exist $v_1, \dots, v_d \in N$ so that P can be written

$$P = \bigcap_{i=1}^{d} \{u \in M_{\mathbb{R}} \mid \langle u, v_i \rangle \geq -1\}. \tag{7.15}$$

Then for $k \in \mathbb{N}$,

$$\begin{aligned}
(\operatorname{int} kP) \cap M &= \bigcap_i \{u \in M \mid \langle u, v_i \rangle > -k\} \\
&= \bigcap_i \{u \in M \mid \langle u, v_i \rangle \geq -k+1\} \\
&= (k-1)P \cap M.
\end{aligned}$$

By Theorem 7.3.2,

$$E_P(-k) = (-1)^n \#((\operatorname{int} kP) \cap M) = (-1)^n \#((k-1)P \cap M) = (-1)^n E_P(k-1),$$

as claimed. $\square$

Proposition 7.3.4. *Let P be an n-dimensional reflexive polytope. Then $\widetilde{\operatorname{Vol}}(\partial P) = n \operatorname{Vol}(P)$. In other words,*

$$E_P(k) = \operatorname{Vol}(P)k^n + \frac{n}{2} \operatorname{Vol}(P)k^{n-1} + \cdots + 1.$$

In particular,

$$E_P(k) = \begin{cases} \operatorname{Vol}(P)k^2 + \operatorname{Vol}(P)k + 1, & n = 2; \\ \\ \operatorname{Vol}(P)k^3 + \frac{3}{2} \operatorname{Vol}(P)k^2 + \left(\frac{1}{2} \operatorname{Vol}(P) + 2\right)k + 1, & n = 3. \end{cases}$$

This proposition is well-known [41, 57], except the case $n = 3$ that seems new.

Proof. By Corollary 7.3.3, since

$$E_P(-k) = (-1)^n \operatorname{Vol}(P)k^n + (-1)^{n-1} \widetilde{\operatorname{Vol}}(\partial P)k^{n-1} + \cdots + 1,$$

and

$$(-1)^n E_P(k-1) = (-1)^n \operatorname{Vol}(P)(k-1)^n + (-1)^n \widetilde{\operatorname{Vol}}(\partial P)(k-1)^{n-1} + \cdots + (-1)^n,$$

comparing the coefficients we get

$$(-1)^{n-1} \widetilde{\operatorname{Vol}}(\partial P) = (-1)^{n-1} n \operatorname{Vol}(P) + (-1)^n \widetilde{\operatorname{Vol}}(\partial P),$$

i.e.,

$$\widetilde{\operatorname{Vol}}(\partial P) = \frac{n}{2} \operatorname{Vol}(P).$$

It follows that when $n = 2$,

$$E_P(k) = \operatorname{Vol}(P)k^2 + \operatorname{Vol}(P)k + 1.$$

When $n = 3$,

$$E_P(k) = \operatorname{Vol}(P)k^3 + \frac{3}{2} \operatorname{Vol}(P)k^2 + a_1 k + 1 \tag{7.16}$$

for some coefficient a_1. To determine a_1, we may simply plug in special values for k. By Corollary 7.3.3,

$$E_P(-1) = -E_P(0) = -1.$$

Letting $k = -1$ in (7.16) gives

$$-1 = \frac{1}{2} \operatorname{Vol}(P) - a_1 + 1,$$

i.e.,

$$a_1 = \frac{1}{2} \operatorname{Vol}(P) + 2,$$

as claimed. □

Theorem 7.3.5. *Let P be reflexive. Then*

$$\operatorname{Bc}_k(P) = \operatorname{Bc}(P)\left(1 + \frac{1}{2k}\right) + O(k^{-2}).$$

Proof. Recall (7.15). Fix $1 \leq i \leq d$. Consider the reflexive polytope

$$P_i := \{(u,h) \in \mathbb{R}^n \times \mathbb{R} \,|\, u \in P,\ -1 \leq h \leq \langle u, v_i \rangle + 1\}.$$

Notice that

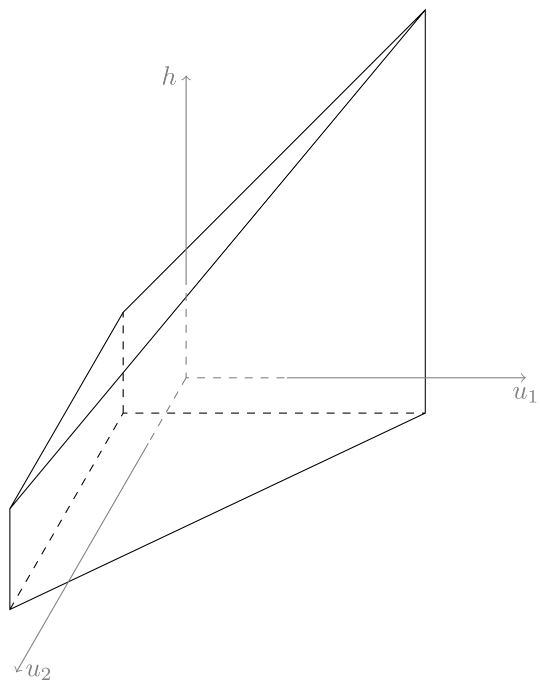

Figure 7.4: The polytope P_1 corresponding to $\mathbb{P}^2$ with $v_1 = (1,0)$, $v_2 = (0,1)$, $v_3 = (-1,-1)$.

$$\begin{aligned}
E_{P_i}(k) &= \sum_{u \in P \cap \frac{1}{k}L} \# \left\{ (u,-1), \left(u, -1 + \frac{1}{k}\right), \cdots, (u, \langle u, v_i \rangle + 1) \right\} \\
&= \sum_{u \in P \cap \frac{1}{k}L} (k\langle u, v_i \rangle + 2k + 1) \\
&= (2k+1)E_P(k) + k \sum_{u \in P \cap \frac{1}{k}L} \langle u, v_i \rangle \\
&= (2k+1)E_P(k) + kE_P(k)\langle \mathrm{Bc}_k(P), v_i \rangle.
\end{aligned}$$

Therefore,

$$\langle \mathrm{Bc}_k(P), v_i \rangle = \frac{E_{P_i}(k)}{kE_P(k)} - 2 - \frac{1}{k}. \tag{7.17}$$

Now,

$$\operatorname{Vol}(P_i) = \int_P (\langle u, v_i\rangle + 2) d\operatorname{Vol} = (\langle \operatorname{Bc}(P), v_i\rangle + 2)\operatorname{Vol}(P). \tag{7.18}$$

By Theorem 7.3.4,

$$\begin{aligned}
\langle \operatorname{Bc}_k(P), v_i\rangle &= \frac{\operatorname{Vol}(P_i)k^{n+1} + \frac{n+1}{2}\operatorname{Vol}(P_i)k^n + O(k^{n-1})}{\operatorname{Vol}(P)k^{n+1} + \frac{n}{2}\operatorname{Vol}(P)k^n + O(k^{n-1})} - 2 - \frac{1}{k} \\
&= \frac{\operatorname{Vol}(P_i)}{\operatorname{Vol}(P)}\left(1 + \frac{1}{2k}\right) - 2 - \frac{1}{k} + O(k^{-2}) \\
&= \langle \operatorname{Bc}(P), v_i\rangle \left(1 + \frac{1}{2k}\right) + O(k^{-2}).
\end{aligned}$$

Since P is bounded, $\{v_i\}_{i=1}^d$ spans $N \otimes \mathbb{R}$. This completes the proof. $\square$

7.4 Quantized barycenters of reflexive polygons

In this section, we give an explicit formula for the quantized barycenters of reflexive polygons without using Ehrhart theory.

Theorem 7.4.1. *Let P be a reflexive polygon, i.e., $P \subseteq \mathbb{R}^2$ is a lattice polytope given by*

$$P = \bigcap_{j=1}^{d} \{x \in M_{\mathbb{R}} \mid \langle x, v_j\rangle \geq -1\} \tag{7.19}$$

for some primitive vectors $v_j \in N$. Then

$$\operatorname{Bc}_k(P) = \frac{(k+1)(2k+1)\widetilde{\operatorname{Vol}}(\partial P)}{4 + 2k(k+1)\widetilde{\operatorname{Vol}}(\partial P)}\operatorname{Bc}(P). \tag{7.20}$$

Proof. Since $\dim P = 2$, each facet of P is a line segment on the line $\langle x, v_j\rangle = -1$. Let

$$F_j^{(1)}$$

denote this facet. We may index the lattice points $F_j^{(1)} \cap M$ on the polygon ∂P counterclockwise as $u_{j,0}^{(1)}, \ldots, u_{j,N_j}^{(1)}$. Note that $u_{j,0}^{(1)}$ and $u_{j,N_j}^{(1)}$ are vertices of P and will be indexed twice as they belong to two facets. Define $w_j := u_{j,1}^{(1)} - u_{j,0}^{(1)}$, i.e., w_j characterizes the distance between two neighboring lattice points on a line perpendicular to v_j. For any $0 \le l \le N_j$,

$$u_{j,l}^{(1)} = u_{j,0}^{(1)} + l w_j. \tag{7.21}$$

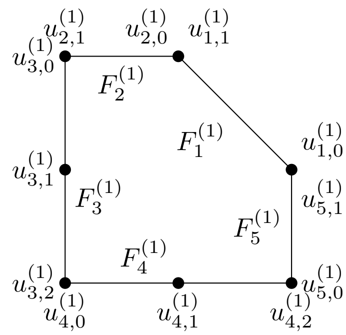

Figure 7.5: The indexing of the edges and the lattice points of the polygon corresponding to the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$.

We repeat this construction for any $k \in \mathbb{N}$. Each facet of kP is a line segment on the line $\langle x, v_j \rangle = -k$ and are denoted by

$$F_j^{(k)}.$$

Since kP is a dilation of P,

$$F_j^{(k)} = k \cdot F_j^{(1)},$$

and in particular, the end points are $k \cdot u_{j,0}^{(1)}$ and $k \cdot u_{j,N_j}^{(1)} = k \cdot u_{j,0}^{(1)} + kN_j \cdot w_j$ (recall (7.21)). We may also index the lattice points $F_j^{(k)} \cap M$ counterclockwise as $u_{j,0}^{(k)}, \dots, u_{j,kN_j}^{(k)}$, where for any $0 \le l \le kN_j$,

$$u_{j,l}^{(k)} = u_{j,0}^{(k)} + l w_j = k \cdot u_{j,0}^{(1)} + l w_j. \tag{7.22}$$

Notice that the lattice points of kP are

$$kP \cap M = \{0\} \cup \bigcup_{i=1}^{k} (\partial(iP) \cap M) \tag{7.23}$$

$$= \{0\} \cup \bigcup_{i=1}^{k} \bigcup_{j=1}^{d} \{u_{j,0}^{(i)}, \dots, u_{j,iN_j}^{(i)}\}. \tag{7.24}$$

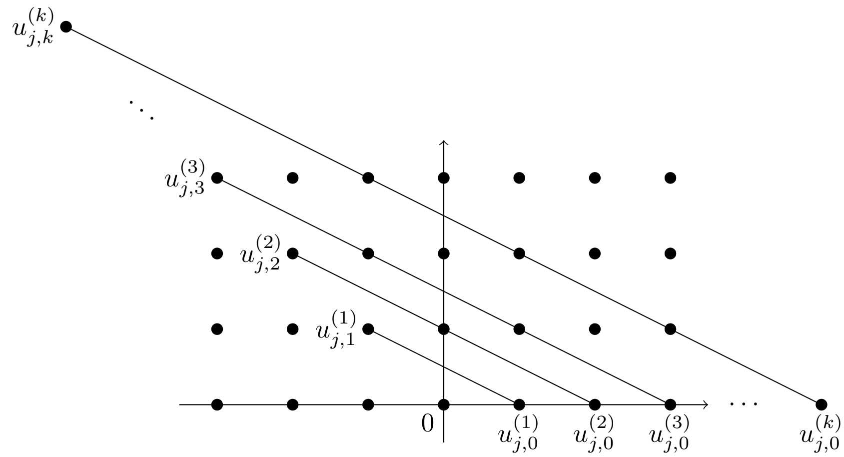

Figure 7.6: The relationship between $F_j^{(1)}$ and a general $F_j^{(k)}$.

Indeed, $0 \in kP \cap M$. For any $1 \leq i \leq k$, $\partial(iP) \subseteq kP$. Hence

$$kP \cap M \supseteq \{0\} \cup \bigcup_{i=1}^{k} (\partial(iP) \cap M).$$

On the other hand, suppose there exists $x \in kP \cap M$ other than the origin that does not belong to any $\partial(iP)$. Then there is a unique integer $0 \leq k_x < k$ such that $x \notin k_x P$ and $x \in \operatorname{int}(k_x + 1)P$. Recall (7.19). Since $x \notin k_x P$, there is some j such that $\langle x, v_j \rangle < -k_x$. On the other hand, since $x \in \operatorname{int}(k_x + 1)P$, we know $\langle x, v_j \rangle > -k_x - 1$. However, $\langle x, v_j \rangle \in \mathbb{Z}$, which is a contradiction.

We are now ready to compute the barycenters using (7.24). Recall that in the expression (7.24) every vertex of iP shows up twice.

Let $m_j^{(k)}$ denote the midpoint of $F_j^{(k)}$, i.e.,

$$m_j^{(k)} := u_{j,0}^{(k)} + \frac{kN_j}{2} w_j. \tag{7.25}$$

Since $F_j^{(k)} = k \cdot F_j^{(1)}$, we also have

$$m_j^{(k)} = k \cdot m_j^{(1)}. \tag{7.26}$$

Recall the indexed lattice points

$$F_j^{(k)} \cap M = \{u_{j,0}^{(k)}, \dots, u_{j,kN_j}^{(k)}\}.$$

Since each vertex is indexed twice,

$$\#(\partial(kP) \cap M) = \sum_{j=1}^{d} \Big(\#(F_j^{(k)} \cap M) - 1 \Big) = k \sum_{j=1}^{d} N_j. \tag{7.27}$$

Also, by (7.22), (7.25), and (7.26),

$$\begin{aligned}
\sum_{u \in \partial(kP) \cap M} u &= \sum_{j=1}^{d} \left(\frac{1}{2} u_{j,0}^{(k)} + \sum_{l=1}^{kN_j - 1} u_{j,l}^{(k)} + \frac{1}{2} u_{j,kN_j}^{(k)} \right) \\
&= \sum_{j=1}^{d} \left(\frac{1}{2} u_{j,0}^{(k)} + \sum_{l=1}^{kN_j - 1} \Big(u_{j,0}^{(k)} + l \cdot w_j \Big) + \frac{1}{2} \Big(u_{j,0}^{(k)} + kN_j \cdot w_j \Big) \right) \\
&= k \sum_{j=1}^{d} N_j \left(u_{j,0}^{(k)} + \frac{kN_j}{2} w_j \right) \\
&= k \sum_{j=1}^{d} N_j m_j^{(k)} \\
&= k^2 \sum_{j=1}^{d} N_j m_j^{(1)}.
\end{aligned}$$

Combining these two equations and (7.23), we get

$$\begin{aligned}
\mathrm{Bc}_k(P) &= \frac{1}{k} \mathrm{Bc}(kP \cap M) \\
&= \frac{1}{k\#(kP \cap M)} \sum_{u \in kP \cap M} u \\
&= \frac{1}{k\left(1 + \sum\limits_{i=1}^{k} \#(\partial(iP) \cap M) \right)} \sum_{i=1}^{k} \sum_{u \in \partial(kP) \cap M} u \\
&= \frac{(k+1)(2k+1)}{6\left(1 + \frac{k(k+1)}{2} \sum\limits_{j=1}^{d} N_j \right)} \sum_{j=1}^{d} N_j m_j^{(1)},
\end{aligned}$$

and

$$\mathrm{Bc}(P) = \lim_{k \to \infty} \mathrm{Bc}_k(P) = \frac{2 \sum\limits_{j=1}^{d} N_j m_j^{(1)}}{3 \sum\limits_{j=1}^{d} N_j}.$$

Therefore

$$\mathrm{Bc}_k(P) = \frac{(k+1)(2k+1)\sum_{j=1}^{d} N_j}{4 + 2k(k+1)\sum_{j=1}^{d} N_j}\,\mathrm{Bc}(P).$$

Finally, by Definition 3.3.1, $N_j = \widetilde{\mathrm{Vol}}(F_j^{(1)})$ (The line segment $F_j^{(1)}$ has N_j fundamental parallelotopes). Then

$$\sum_{j=1}^{d} N_j = \widetilde{\mathrm{Vol}}(\partial P). \tag{7.28}$$

This concludes the proof. □

Remark 7.4.2. We record here also the proof of Theorem 7.4.1 using Ehrhart theory. Recall the proof of Theorem 7.5.3. By Theorem 7.3.4,

$$\begin{aligned} E_{P_i}(k) &= \mathrm{Vol}(P_i)k^3 + \frac{3}{2}\mathrm{Vol}(P_i)k^2 + \left(\frac{1}{2}\mathrm{Vol}(P_i) + 2\right)k + 1, \\ E_P(k) &= \mathrm{Vol}(P)k^2 + \mathrm{Vol}(P)k + 1. \end{aligned}$$

By (7.17) and (7.18),

$$\begin{aligned} \langle \mathrm{Bc}_k(P), v_i\rangle &= \frac{E_{P_i}(k) - (2k+1)E_P(k)}{kE_P(k)} \\ &= \frac{(\mathrm{Vol}(P_i) - 2\,\mathrm{Vol}(P))k^3 + \left(\frac{3}{2}\mathrm{Vol}(P_i) - 3\,\mathrm{Vol}(P)\right)k^2 + \left(\frac{1}{2}\mathrm{Vol}(P_i) - \mathrm{Vol}(P)\right)k}{k(\mathrm{Vol}(P)k^2 + \mathrm{Vol}(P)k + 1)} \\ &= \frac{k^2 + \frac{3}{2}k + \frac{1}{2}}{\mathrm{Vol}(P)k^2 + \mathrm{Vol}(P)k + 1}(\mathrm{Vol}(P_i) - 2\,\mathrm{Vol}(P)) \\ &= \frac{(k+1)(2k+1)\,\mathrm{Vol}(P)}{2 + 2k(k+1)\,\mathrm{Vol}(P)}\langle \mathrm{Bc}(P), v_i\rangle. \end{aligned}$$

The result follows since by Proposition 7.3.4,$\mathrm{Vol}(P) = \frac{1}{2}\widetilde{\mathrm{Vol}}(\partial P)$.

Remark 7.4.3. As a consequence of Theorem 7.4.1, we see that the following are equivalent for toric del Pezzo surfaces:

1) $\mathrm{Bc}(P) = 0$;

2) $\mathrm{Bc}_k(P) = 0$ for some k;

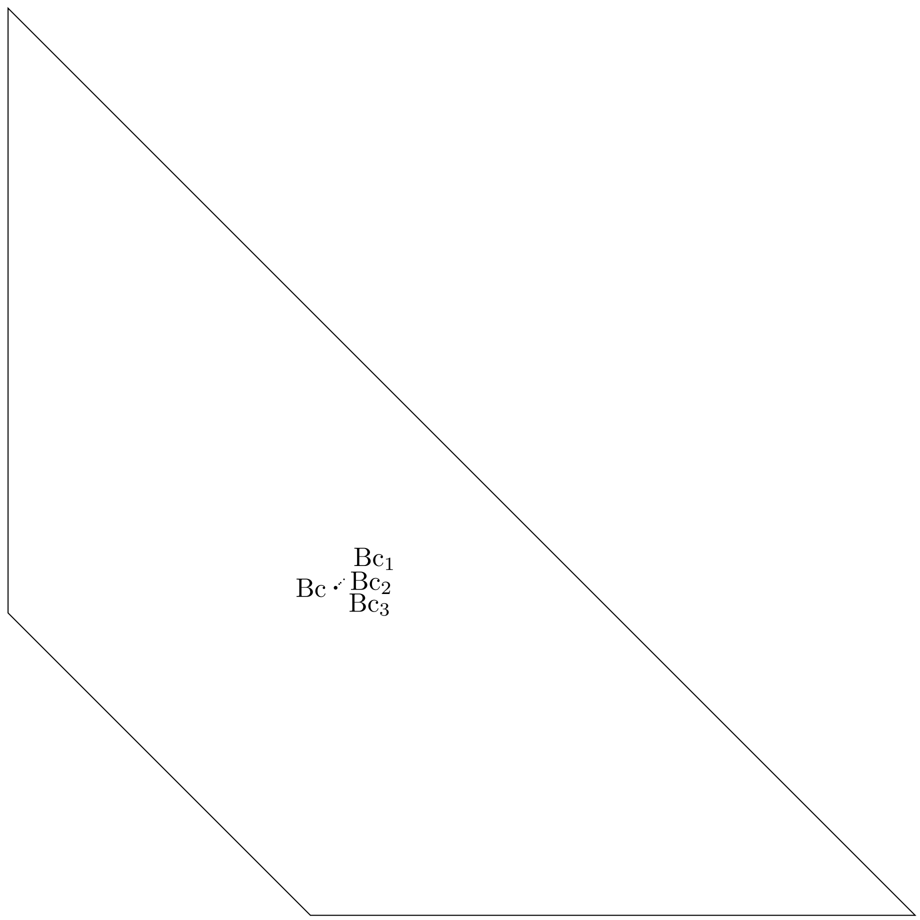

Figure 7.7: The barycenter and quantized barycenters of the polygon corresponding to $\mathbb{F}_1$. In particular, $\mathrm{Bc}_1 = (\frac{1}{9}, \frac{1}{9})$, $\mathrm{Bc}_2 = (\frac{1}{10}, \frac{1}{10})$, $\mathrm{Bc}_3 = (\frac{2}{21}, \frac{2}{21})$, and $\mathrm{Bc} = (\frac{1}{12}, \frac{1}{12})$.

3) $\mathrm{Bc}_k(P) = 0$ for all k.

This strengthens Corollary 7.1.3 that holds in all dimensions. Moreover, this resolves Problem 7.0.5 in dimension $n = 2$ showing the answer is "no". In addition, (7.20) tells us that the quantized barycenter $\mathrm{Bc}_k(P)$ lies on a certain line.

Remark 7.4.4. For higher dimensions, it is harder to derive such a formula this way, and in general Bc_k and Bc are not collinear. See Example 7.7.1. In fact, for a polytope of dimension 3 or higher, it has faces of dimension 2 or higher, whose shapes and lattice points are more difficult to characterize.

7.5 Asymptotics of δ_k-invariants

In this section we study the asymptotics of δ_k-invariant using the results of the previous sections.

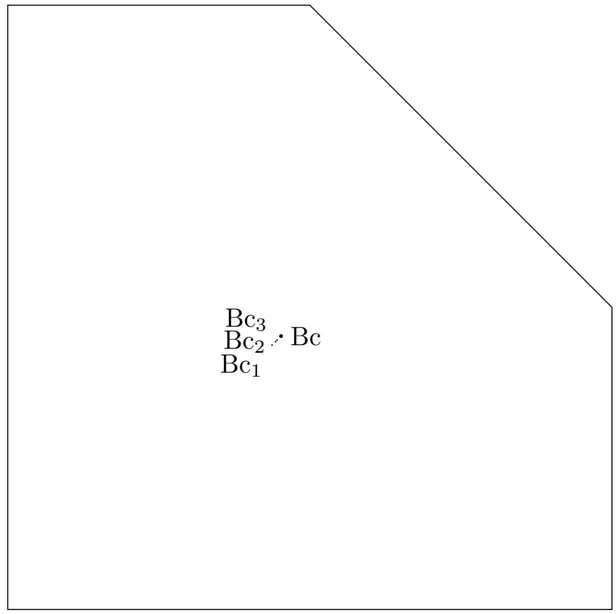

Figure 7.8: The barycenter and quantized barycenters of the polygon corresponding to the blow-up of $\mathbb{P}_1 \times \mathbb{P}_1$. In particular, $\mathrm{Bc}_1 = (-\frac{1}{8}, -\frac{1}{8})$, $\mathrm{Bc}_2 = (-\frac{5}{44}, -\frac{5}{44})$, $\mathrm{Bc}_3 = (-\frac{14}{129}, -\frac{14}{129})$, and $\mathrm{Bc} = (-\frac{2}{21}, -\frac{2}{21})$.

Theorem 7.5.1. *If $\delta_k(-K_X) = 1$ for sufficiently large k, then $\delta_k(-K_X) = 1$ for all k.*

Proof. By (7.1), $\delta_k(-K_X) = 1$ if and only if $\mathrm{Bc}_k(P) = 0$, where P is the polytope associated to X. By Corollary 7.1.3, $\mathrm{Bc}_k(P) = 0$ for all k. □

Proof of Corollary 7.0.11. By Proposition 7.1.1 and Theorem 3.4.1,

$$\begin{aligned}
\delta_k^{-1} &= \max_{i=1,\dots,d} (\langle \mathrm{Bc}_k(P), v_i \rangle + b_i) \\
&= \max_{i=1,\dots,d} \left(\langle \mathrm{Bc}(P), v_i \rangle + b_i + \frac{\widetilde{\mathrm{Vol}}(\partial P)}{2\,\mathrm{Vol}(P)} \langle (\widetilde{\mathrm{Bc}}(\partial P) - \mathrm{Bc}(P)), v_i \rangle k^{-1} + O(k^{-2}) \right) \\
&= \delta^{-1} + \frac{\widetilde{\mathrm{Vol}}(\partial P)}{2\,\mathrm{Vol}(P)} \max_{i \in I} \langle \widetilde{\mathrm{Bc}}(\partial P) - \mathrm{Bc}(P), v_i \rangle k^{-1} + O(k^{-2}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\delta_k - \delta &= \frac{\delta^2(\delta^{-1} - \delta_k^{-1})}{\delta \delta_k^{-1}} \\
&= \frac{\delta^2(\delta^{-1} - \delta_k^{-1})}{1 - \delta(\delta^{-1} - \delta_k^{-1})} \\
&= \delta^2 \left(\frac{1}{\delta} - \frac{1}{\delta_k} \right) + O(k^{-2})
\end{aligned}$$

$$= -\delta^2 \frac{\widetilde{\mathrm{Vol}}(\partial P)}{2\,\mathrm{Vol}(P)} \max_{i\in I} \langle \widetilde{\mathrm{Bc}}(\partial P) - \mathrm{Bc}(P), v_i \rangle k^{-1} + O(k^{-2}).$$

Since δ_k is the minimum of finitely many rational functions, for sufficiently large k, there is a rational function that dominates the rest. Indeed, consider the asymptotic expansion and take the one whose coefficient dominates in the lexicographic order. Therefore δ_k is a rational function of k when k is sufficiently large. □

Remark 7.5.2. Recall

$$\delta_k = \min_{i=1,\dots,d} \frac{1}{\langle \mathrm{Bc}_k(P), v_i \rangle + b_i}.$$

Since $\mathrm{Bc}_k(P)$ has an asymptotic expansion by Corollary 7.1.3, for each $i = 1, \dots, d$,

$$\frac{1}{\langle \mathrm{Bc}_k(P), v_i \rangle + b_i}$$

has an asymptotic expansion. However, $\mathrm{argmin}_{i=1,\dots,d} (\langle \mathrm{Bc}_k(P), v_i \rangle + b_i)^{-1}$ is not necessarily constant for all k. Therefore we can only deduce that there is an expansion that works for sufficiently large k, but not necessarily all k.

Theorem 7.5.3. *Let X be Fano. Then*

$$\delta_k(-K_X) = \delta(-K_X) - \frac{1}{2k}\delta(-K_X)(1 - \delta(-K_X)) + O(k^{-2}).$$

Proof. By Theorem 7.3.5,

$$\begin{aligned}
\delta_k^{-1} &= \max_{i=1,\dots,d} \langle \mathrm{Bc}_k(P), v_i \rangle + 1 \\
&= \max_{i=1,\dots,d} \langle \mathrm{Bc}(P), v_i \rangle \left(1 + \frac{1}{2k}\right) + 1 + O(k^{-2}) \\
&= \delta^{-1} + \frac{1}{2k}\big(\delta^{-1} - 1\big) + O(k^{-2}).
\end{aligned}$$

Therefore

$$\delta_k = \frac{\delta}{1 + \frac{1}{2k}(1-\delta) + O(k^{-2})}$$
$$= \delta - \frac{1}{2k}\delta(1-\delta) + O(k^{-2}).$$

$\square$

Lemma 7.5.4. *For a smooth del Pezzo surface X,*

$$h^0(X, -K_X) = 1 + K_X^2.$$

In particular, if X is toric with associated polytope P, then

$$\widetilde{\mathrm{Vol}}(\partial P) = K_X^2.$$

Proof. Since $-K_X$ is ample, by Kodaira Vanishing Theorem $\chi(\mathcal{O}_X(-K_X)) = h^0(X, -K_X)$ and by Riemann–Roch [56, p. 472] $\chi(\mathcal{O}_X(-K_X)) = \chi(\mathcal{O}_X) + K_X^2 = h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X) + K_X^2 = 1 - h_{\bar\partial}^{0,1} + h_{\bar\partial}^{0,2} + K_X^2 = 1 + K_X^2$.

By (7.23), (7.27) and (7.28), $h^0(X, -K_X) = \#(P \cap M) = 1 + \sum_{j=1}^d N_j = 1 + \widetilde{\mathrm{Vol}}(\partial P)$. $\square$

Theorem 7.5.5. *Let X be a smooth toric del Pezzo surface. Then*

$$\delta_k(-K_X) = \left(1 + \frac{(k+1)(2k+1)K_X^2}{4 + 2k(k+1)K_X^2}\big((\delta(-K_X))^{-1} - 1\big)\right)^{-1}, \tag{7.29}$$

In particular,

$$\frac{\delta_k(-K_X)}{\delta(-K_X)} = 1 - \frac{1}{2k}(1 - \delta(-K_X)) + O\big(k^{-2}\big).$$

Proof. By Theorem 7.4.1,

$$\delta_k(-K_X)^{-1} - 1 = \max_{i=1,\ldots,d} \langle \mathrm{Bc}_k(P), v_i \rangle$$

$$
\begin{aligned}
&= \frac{(k+1)(2k+1)K_X^2}{4+2k(k+1)K_X^2} \max_{i=1,\dots,d} \langle \mathrm{Bc}(P), v_i \rangle \\
&= \frac{(k+1)(2k+1)K_X^2}{4+2k(k+1)K_X^2} \big(\delta(-K_X)^{-1} - 1\big).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\delta_k(-K_X) &= \left(1 + \frac{(k+1)(2k+1)K_X^2}{4+2k(k+1)K_X^2}\big((\delta(-K_X))^{-1} - 1\big)\right)^{-1} \\
&= \left(1 + \frac{1+\frac{3}{2k}+O(k^{-2})}{1+\frac{1}{k}+O(k^{-2})}\big((\delta(-K_X))^{-1} - 1\big)\right)^{-1} \\
&= \left((\delta(-K_X))^{-1} + \frac{1}{2k}\big((\delta(-K_X))^{-1} - 1\big)O(k^{-2})\right)^{-1} \\
&= \left(1 - \frac{1}{2k}(1-\delta(-K_X)) + O\big(k^{-2}\big)\right)\delta(-K_X),
\end{aligned}
$$

as claimed. □

7.6 Relation to test configurations

In this section, we explain how the coefficients of the asymptotic expansion relate to the higher Donaldson–Futaki invariants.

Definition 7.6.1.

- A test configuration for (X, L) is a triple $(\mathcal{X}, \mathcal{L}, \sigma)$, where σ is a $\mathbb{C}^*$-action on the pair $(\mathcal{X}, \mathcal{L})$, with a $\mathbb{C}^*$-equivariant projective morphism $\pi : \mathcal{X} \to \mathbb{C}$ such that $\mathcal{L}$ is π-relatively very ample, i.e., π lifts to $\mathcal{X} \hookrightarrow \mathbb{P}^r_{\mathbb{C}^1} = \operatorname{Proj}\mathbb{C}[t][Z_0, \dots, Z_r]$ which pulls $\mathcal{O}(1)$ back to $\mathcal{L}$, and each fiber over $t \in \mathbb{C}^* \subseteq \mathbb{C}$, denoted by $(\mathcal{X}_t, \mathcal{L}_t)$, is isomorphic to (X, L).
- We say that a test configuration is a product configuration if $\mathcal{X} \cong X \times \mathbb{C}$, and a trivial configuration if the $\mathbb{C}^*$ action on $\mathcal{X} \cong X \times \mathbb{C}$ is trivial on the first factor.
- The $\mathbb{C}^*$-action on $\mathcal{L}$ induces a $\mathbb{C}^*$-action on $H^0(\mathcal{X}_0, \mathcal{L}_0^k)$. Let w_k denote the sum of the weights of this action.

- Define the Donaldson–Futaki coefficients to be the coefficients of the expansion

$$\frac{w_k}{kH^0\big(\mathcal{X}_0, \mathcal{L}_0^k\big)} = \sum_{i=0}^{\infty} DF_i(\mathcal{X}, \mathcal{L}, \sigma)k^{-i}.$$

7.6.1 Construction of the rooftop test configuration

We construct a product test configuration

$$(\mathcal{X}, \mathcal{L}, \sigma_{q(v)}) \tag{7.30}$$

as follows. Recall the pair $(\overline{\mathcal{X}}, \overline{\mathcal{L}})$ constructed in (7.7). The projection $M_{\mathbb{R}} \times \mathbb{R} \to \mathbb{R}$ corresponds to an equivariant map $\pi : \overline{\mathcal{X}} \to \mathbb{P}^1$ (see Figure 7.2). The subfan consisting of cones generated by $(v_1, 0), \dots, (v_d, 0), (0, 1)$ (i.e., the cones on or above the hyperplane $M_{\mathbb{R}} \times \{0\}$) is the product of the fan of X and the fan of $\mathbb{C}$, and hence corresponds to the toric manifold $\pi^{-1}(\mathbb{C}) \cong X \times \mathbb{C}$. Similarly, the subfan consisting of cones generated by $(v_1, 0), \dots, (v_d, 0), (v, -1)$ (i.e., the cones on or below the hyperplane $M_{\mathbb{R}} \times \{0\}$) corresponds to $\pi^{-1}(\mathbb{P}^1 \setminus \{0\}) \cong X \times \mathbb{C}$. Let $\mathcal{X}$ denote the latter open submanifold and $\mathcal{L}$ the restriction of $\overline{\mathcal{L}}$ over $\mathcal{X}$. Note that this product structure implies that each fiber of π is a copy of (X, L).

The lattice point $(0, -1) \in N \times \mathbb{Z} = \mathrm{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^{n+1})$ induces a $\mathbb{C}^*$-action expressed on the torus $(\mathbb{C}^*)^{n+1} = (\mathbb{C}^*)^n \times \mathbb{C}^*$ (the first factor is the open-orbit in X considered as a toric variety) as

$$\sigma(t) : z \mapsto \sigma(t).z := \big(z_1, \dots, z_n, t^{-1}z_{n+1}\big), \quad t \in \mathbb{C}^*.$$

By definition of a toric variety (both X and $X \times \mathbb{C}$ are such) it is enough to specify the action on the open orbit and it will extend uniquely to $X \times \mathbb{C}$. Of course, it is not obvious what the action does at $t = 0$ as $1/t$ blows up; this shows in passing that what we construct is not a trivial, but rather a product, test configuration (this test configuration is trivial if and only if $\overline{\mathcal{X}} = X \times \mathbb{P}^1$, which is precisely when $v = 0$). We can lift it to a $\mathbb{C}^*$-action on the line bundle $\overline{\mathcal{L}}$ by (recall $q = q(v)$ defined in (7.4))

$$\sigma_{q(v)}(t) : (z, f) \mapsto \big(\sigma(t).z, t^{-q}f\big).$$

More precisely, for a section $s_{u'} \in H^0(\overline{\mathcal{X}}, \overline{\mathcal{L}})$ given by $z \mapsto z^{u'}$ under the standard local trivialization over $(\mathbb{C}^*)^{n+1}$, where $u' \in M \times \mathbb{Z}$,

$$\begin{aligned}\sigma_{q(v)}(t)\Big(z, z^{u'}\Big) &= \Big(\big(z_1, \ldots, z_n, t^{-1} z_{n+1}\big), t^{-q} z_1^{u'_1} \cdots z_n^{u'_n} z_{n+1}^{u'_{n+1}}\Big) \\ &= \Big(\big(z_1, \ldots, z_n, t^{-1} z_{n+1}\big), t^{u'_{n+1}-q} z_1^{u'_1} \cdots z_n^{u'_n} \big(t^{-1} z_{n+1}\big)^{u'_{n+1}}\Big) \\ &= \Big(\sigma_t(z), t^{u'_{n+1}-q} \sigma_t(z)^{u'}\Big).\end{aligned}$$

Thus,

$$\sigma_{q(v)}(t)(s_{u'}) = t^{u'_{n+1}-q} s_{u'} \circ \sigma_t,$$

i.e., a section $s_{u'}$ has weight $u'_{n+1} - q$.

We define this triple $(\mathcal{X}, \mathcal{L}, \sigma_{q(v)})$ to be the product test configuration (7.30). This test configuration is also studied in [92, Remark 2.4], [39, Proposition 2.2.2].

7.6.2 Quantized barycenters and higher Donaldson–Futaki invariants

Theorem 7.6.2. *Let $(\mathcal{X}, \mathcal{L}, \sigma_v)$ be the product test configuration* (7.30) *constructed in §7.6.1. Then*

$$\langle \mathrm{Bc}_k(P), v\rangle = \sum_{i=0}^{\infty} DF_i\big(\mathcal{X}, \mathcal{L}, \sigma_{q(v)}\big) k^{-i}.$$

Proof. Consider the lattice points on the facet

$$\big\{(u, \langle u, v\rangle + q) \in M_{\mathbb{R}} \times \mathbb{R} \,\big|\, u \in P \cap k^{-1} M\big\}.$$

They correspond to the torus invariant sections of $\overline{\mathcal{L}}^k$ that do not vanish along $\mathcal{X}_0 = \mathcal{D}_{d+2}$ [49, p. 61]. For $u \in P \cap k^{-1} M$, let s_u denote the section of $\overline{\mathcal{L}}^k$ corresponding to $(u, \langle u, v\rangle + q)$, which has weight $k\langle u, v\rangle$. It follows that the section $s_u|_{\mathcal{L}_0^k}$ on $\mathcal{L}_0^k$ also has weight $k\langle u, v\rangle$. Since $\{s_u|_{\mathcal{L}_0^k}\}_{u \in P \cap k^{-1} M}$ spans $H^0(\mathcal{X}_0, \mathcal{L}_0^k)$, taking the sum of all their weights, we get

$$\frac{w_k}{kH^0\big(\mathcal{X}_0,\mathcal{L}_0^k\big)}=\frac{1}{kE_P(k)}\sum_{u\in P\cap k^{-1}M}k\langle u,v\rangle=\langle \mathrm{Bc}_k(P),v\rangle.$$

This concludes the proof. □

Combining this with Corollary 7.2.7, we obtain an expression for the higher Donaldson–Futaki invariants of this test configuration.

Corollary 7.6.3. *Assume X is smooth. Let $(\mathcal{X},\mathcal{L},\sigma_v)$ be the product test configuration* (7.30) *constructed in §7.6.1. Then*

$$DF_j\big(\mathcal{X},\mathcal{L},\sigma_{q(v)}\big)=\sum_{i=0}^{j}c'_{n+1-j+i}\left(\frac{\delta_{i0}}{a_n}+\sum_{\ell=1}^{i}\frac{(-1)^\ell}{a_n^{\ell+1}}\sum_{\sum_{m=1}^{\ell}i_m=i-\ell}\ \prod_{m=1}^{\ell}a_{n-i_m-1}\right),$$

where a_j and c'_j are defined in Corollary 7.2.7.

7.7 Examples

7.7.1 A 3-dimensional reflexive example with non-collinear quantized barycenters

Example 7.7.1. The Fano 3-fold $No.3.29$ has a toric structure with the primitive elements of the rays of its fan being the columns of the matrix [110, p. 43]

$$\begin{pmatrix}1&0&0&1&0&-1\\0&1&0&-1&-1&-1\\0&0&1&0&0&-1\end{pmatrix}.$$

Then a computer calculation shows

$$\mathrm{Bc}_1(P)=(\frac{6}{28},\frac{-12}{28},\frac{3}{28}),$$

$$\mathrm{Bc}_2(P) = (\frac{51}{260}, \frac{-99}{260}, \frac{24}{260})$$
$$\mathrm{Bc}_3(P) = (\frac{201}{1071}, \frac{-387}{1071}, \frac{93}{1071}).$$

It follows that Bc_k are not on a line. This means that $a_i(P) \neq 0$ for some $i > 1$.

7.7.2 Virtual rooftop polytopes and verification of the terms in the expansion

Example 7.7.2. Consider $X = \mathbb{P}^1$ with $L = -K_X \sim \{0\} + \{\infty\}$. Then $P = [-1, 1]$. For $v \in \mathbb{Z}$, let $q = |v| + 1$. Then

$$P_{v,q} = \{(u, h) \in [-1, 1] \times \mathbb{R}_+ \,|\, h \leq v \cdot u + q\}.$$

This corresponds to $\overline{\mathcal{X}} = \mathbb{F}_{|v|}$ with $\overline{\mathcal{L}} = E + H + qF_\infty$, where E is the $-|v|$-curve, H the torus invariant $|v|$-curve, and F_∞ the fiber over $\infty \in \mathbb{P}^1$. See Figure 7.9.

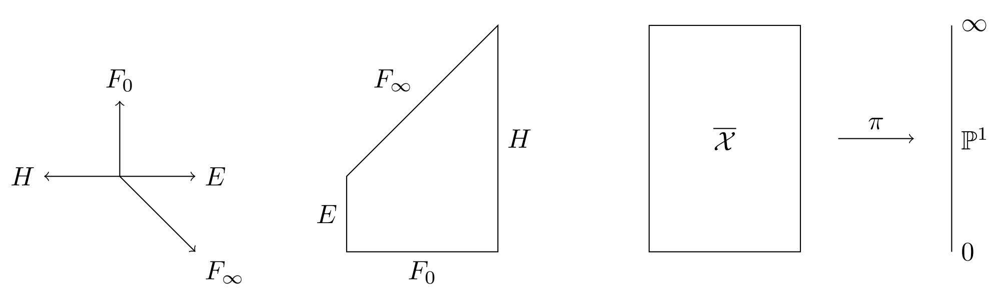

Figure 7.9: The fan and the polytope $P_{v,q}$ of the pair $(\overline{\mathcal{X}}, \overline{\mathcal{L}})$.

Example 7.7.3. Let $X = \mathbb{P}^2$ with $L = -K_X$. We have $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_3 = (-1, -1)$. Let us verify Corollary 7.2.7 with $v = (1, 0)$.

Let D_1, D_2, D_3 be the corresponding toric divisors. Geometrically, they correspond to virtual polytopes $P_{D_i} = P_{-K_X+D_i} - P_{-K_X}$. We have $L = -K_X = D_1 + D_2 + D_3$. By Corollary 7.2.7,

$$a_2 = \frac{L^2}{2} = \frac{9}{2},$$
$$a_1 = \frac{L^2}{2} = \frac{9}{2},$$
$$a_0 = \frac{L^2 + \sum_{i<j} D_i \cdot D_j}{12} = 1.$$

For divisors in (7.8), we have $\overline{\mathcal{L}}' = -K_{\mathcal{X}/\mathbb{P}^1} = \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3$, and

$$\mathcal{D}_2 \sim \mathcal{D}_3, \qquad \mathcal{D}_1^3 = -2, \qquad \mathcal{D}_1^2 \cdot \mathcal{D}_2 = -1, \qquad \mathcal{D}_1 \cdot \mathcal{D}_2^2 = 0, \qquad \mathcal{D}_2^3 = 1.$$

By Corollary 7.2.7,

$$\begin{aligned} c_3' &= \frac{\overline{\mathcal{L}}'^3}{6} = 0, \\ c_2' &= \frac{\overline{\mathcal{L}}'^3}{4} = 0, \\ c_1' &= \frac{\overline{\mathcal{L}}' \cdot \left(\overline{\mathcal{L}}'^2 + \sum_{1 \le i < j \le d} \mathcal{D}_i \cdot \mathcal{D}_j \right)}{12} = 0. \end{aligned}$$

Therefore,

$$\langle \mathrm{Bc}_k(P), v \rangle = \frac{\sum_{j=0}^{n} c_{j+1}' k^j}{\sum_{j=0}^{n} a_j k^j} = 0.$$

7.7.3 Three non-stabilizing quantized barycenter examples

Example 7.7.4. Batyrev–Selivanova symmetry of P [7] implies $\mathrm{Bc}_k(P) = 0$ for all $k \in \mathbb{N}$, and is furthermore equivalent to it when $n \le 6$ [62]. This relies on Nill–Paffenholz's exhaustive computer search showing there are exactly three toric Fano manifolds X_1, X_2, X_3 of dimension up to 8 whose associated polytopes have $\mathrm{Bc} = 0$ but are not symmetric in the sense of Batyrev–Selivanova [80, Proposition 2.1].

X_1 is 7-dimensional. Let P_1 be the associated polytope. Then a computer calculation shows

$$\begin{aligned} \mathrm{Bc}_1(P_1) &= \frac{16}{2257}(-1,-1,-1,1,1,1,2), \\ \mathrm{Bc}_2(P_1) &= \frac{60}{27121}(-1,-1,-1,1,1,1,2), \\ \mathrm{Bc}_3(P_1) &= \frac{2744}{2579721}(-1,-1,-1,1,1,1,2). \end{aligned} \tag{7.31}$$

It is interesting to note that our results give a refinement of a theorem of Ono–Sano–Yotsutani [82, Theorem 1.6], who showed that $(X_1, -kK_{X_1})$ is not Chow semistable for k large enough. Chow stability of

$(X_1, -kK_{X_1})$ is equivalent to existence of k-anticanonical balanced metrics (see [85, Theorem 4], [121, Theorem 3.2], [75, Theorem 0.1]), so this means Ono–Sano–Yotsutani showed that X_1 does not admit k-anticanonical balanced metrics for k large enough. But in fact from the above computation and Corollary 7.1.3, $\delta_k(-K_{X_1}) < 1$ for all k except at most 7 values of k. Thus, by [90, Theorem 2.3], $(X_1, -kK_{X_1})$ is actually not Chow semistable for all but at most 7 values of k.

Next, $X_2 = X_1 \times \mathbb{P}^1$. So $P_2 = P_1 \times [-1, 1]$. Then $\mathrm{Bc}_k(P_2) = (\mathrm{Bc}_k(P_1), 0)$ (see (7.31)).

Finally, X_3 is 8-dimensional. Let P_3 be the associated polytope. Then a computer calculation shows

$$\begin{aligned}
\mathrm{Bc}_1(P_3) &= \frac{32}{5459}(-1, -1, -1, 1, 1, 1, 1, 2), \\
\mathrm{Bc}_2(P_3) &= \frac{580}{321787}(-1, -1, -1, 1, 1, 1, 1, 2).
\end{aligned}$$

Note that in these three examples, the $\mathrm{Bc}_k(P_i)$ we computed are colinear. This is actually true for all k and is because while P_i is not Batyrev–Selivanova symmetric, there is a 1-dimensional subspace that is invariant under $\mathrm{Aut}\, P$. We explain this in more detail in [62].

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