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Fourier Transform Inequalities With Measure Weights

by

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Abstract

Fourier transform norm inequalities, $\|\hat{f}\|_{q,\mu} \le C\|f\|_{p,v}$, are proved for measure weights μ on moment subspaces of $L^p_v(\mathbb{R}^n)$. Density theorems are established to extend the inequalities to all of $L^p_v(\mathbb{R}^n)$. In both cases the conditions for validity are computable. For $n \ge 2$, μ and v are radial, and the results are applied to prove spherical restriction theorems which include power weights $v(t) = |t|^\alpha$, $n/(p'-1) < \alpha < (p'+n)/(p'-1)$.

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Introduction

We shall prove weighted Fourier transform norm inequalities on $\mathbb{R}^n$ where the weight on the Fourier transform side is a measure, i.e., $\|\hat{f}\|_{q,\mu} \le C\|f\|_{p,v}$ for μ a measure.

There are a number of results in this area. We characterized such an inequality on $\mathbb{R}$ for $1 < p \le q < \infty$ and even weights μ and v for which $1/\mu$ and v were increasing functions on $(0,\infty)$ [BH], cf., the extension in [BHJ 1]. Using these results we proved the inequality,

$$\int |\hat{f}(\gamma)|^p |\gamma|^{p-2} \omega(1/\gamma)\, d\gamma \le C \int |f(t)|^p \omega(t)\, dt,$$

for $\mathbb{R}$ if and only if ω is a Muckenhoupt A_p weight; here $p \in (1,2]$ and ω is even on $\mathbb{R}$ and increasing on $(0,\infty)$ [BHJ 2]. This is interesting since the A_p condition is a Hilbert transform/ maximal function criterion and since our result has classical theorems of Hardy, Littlewood, Paley, and Pitt as corollaries. Further, major contributions to weighted Fourier transform norm inequalities include [JS] and [Mu 2] with an earlier theorem due to P. Knopf and Rudnick [KR] and more recent results by Sadosky and Wheeden [SW].

Generally, the above-mentioned results use rearrangement methods. These methods do not yield effective criteria for Fourier transform inequalities in the case of non-monotonic weights, and the constants C become more difficult to compute. Also these results tend to assume one or the other of such constraints as even weights, function weights, monotonic weights, or domain $\mathbb{R}$. Our goal is to construct the theory without rearrangements and with as few constraints as possible. The reasons for such a project are apparent: restriction theorems, uncertainty principle inequalities, and effective criteria to establish Fourier transform inequalities for large classes of weights. This paper gives our first results in this direction.

After setting notation in Section 0 we state a version of Hardy's inequality in Section 1 as well as verifying two useful corollaries, viz., Proposition 1.3 and Proposition 1.4.

Section 2 is devoted to Fourier transform norm inequalities on $\mathbb{R}$ with measure weights. Using the results of Section 1, Theorem 2.1 establishes our inequality on a subspace of functions with vanishing moments. A norm constant is given which is nearly sharp for some weights and which, in any case, is explicit. The weights need not be even or monotonic. Theorem 2.2 gives a general and effective density criterion to extend Theorem 2.1 to all of weighted L^p; and Theorem 2.3 combines these two results to yield the basic norm inequality on weighted L^p. The remainder of Section 2 is devoted to comments about density criteria and to checking our hypotheses in Theorem 2.3 with specific weights.

Section 3 provides some remarks about radial measures on $\mathbb{R}^n$. This material is used in Section 4 to prove the analogues in $\mathbb{R}^n$ of the results from Section 2. Theorem 4.3, corresponding to Theorem 2.3, requires both v and μ to be radial. The proofs in Section 4 are more involved than those of Section 2, but utilize the same approach. For example, the Carleson-Hunt theorem is implemented in Theorem 2.2, whereas our n-dimensional density criterion, Theorem 4.2, utilizes C. Fefferman's extension of this theorem. The final section, Section 5, contains applications of Section 4 to restriction theorems (Theorem 5.3 and Corollary 5.4) and proves results identifying a special case of one of our basic hypotheses from Sections 2 and 4, viz., (2.1) and (4.1), with a natural growth condition arising in spherical restriction theorems.

Besides condition (2.1), resp.,(4.1), which is an expected "uncertainty principle" relation between the weights v and μ, our proofs of the basic norm inequalities require another condition, (2.2), resp., (4.2), which limits the applicable pairs of weights. Each of these conditions is easy to check (there are no rearrangements); and the conditions are often satisfied, e.g., in the case μ has compact support and $v^{1-p'}$ is integrable off of a certain neighborhood of the origin. It is true, however, that the present theory does not include the case $v = 1$ because of the simple moment approach we have taken. The sequel will deal with refinements of this approach and of Hardy's inequality for non-measure weights, as well as the cases $q < p$ and $p = 1$, associated restriction theorems, and uncertainty principle inequalities.

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0. Notation

Let X be a locally compact subspace of n-dimensional Euclidean space $\mathbb{R}^n$, and let $C_c(X)$ be the vector space of complex-valued continuous functions $f: X \longrightarrow \mathbb{C}$ having compact support $\text{supp } f \subseteq X$. A measure ν on X is a linear functional defined on $C_c(X)$ satisfying $\lim_{j\to\infty} < \nu, f_j > = 0$ for every sequence $\{f_j\} \subseteq C_c(X)$ having the properties that $\lim_{j\to\infty} \|f_j\|_\infty = 0$ and $\text{supp } f_j \subseteq K$, where $K \subseteq X$ is a compact set independent of j and $\|\cdots\|_\infty$ is the usual sup-norm (on X), e.g., [Bo]. $M(X)$ is the space of measures on X and $M_+(X) = \{\nu \in M(X): \langle\nu,f\rangle \geq 0$ for all non-negative $f \in C_c(X)\}$ is the space of positive measures on X. Similarly, $M_b(X)$ is the subspace of $M(X)$ having bounded variation, i.e., the above mentioned convergence criterion on $C_c(X)$ is replaced by $(C_c(X), \|\cdots\|_\infty)$; and $M_{b+}(X)$ consists of the positive elements of $M_b(X)$. We write $\langle\nu,f\rangle = \int_X f(t)d\nu(t)$ and in case $X = \mathbb{R}^n$ we write $\langle\nu,f\rangle = \int f(t)d\nu(t)$.

For $p \in (0,\infty)$, $L^p_{loc}(\mathbb{R}^n)$ is the set of functions $f: \mathbb{R}^n \longrightarrow \mathbb{C}$ for which $|f|^p$ is locally integrable with respect to Lebesgue measure. If $\nu \in M_+(\mathbb{R}^n)$ then $L^p_\nu(\mathbb{R}^n)$ designates the set of Borel measurable functions f defined ν a.e. on $\mathbb{R}^n$ for which

$$\|f\|_{p,\nu} = (\int |f(t)|^p d\nu(t))^{1/p} < \infty.$$

There is an analogous definition of $L^p_v(\mathbb{R}^n)$, where $v \geq 0$ is a Borel measurable function not necessarily an element of $L^1_{loc}(\mathbb{R}^n)$. In fact, $L^p_v(\mathbb{R}^n) = \{f: \|f\|_{p,v} = (\int |f(t)|^p v(t)dt)^{1/p} < \infty\}$. If

$v \in L^1_{loc}(\mathbb{R}^n)$, $v \geq 0$, then "$d\nu(t) = v(t)dt$" defines a positive measure ν. Also, we write $p' = p/(p-1)$.

The <u>Fourier transform</u> $\hat{f}$ of $f \in L^1(\mathbb{R}^n)$ is the function,

$$\hat{f}(\gamma) = \int e^{-2\pi i t\cdot\gamma} f(t)dt,$$

where $\gamma \in \hat{\mathbb{R}}^n (= \mathbb{R}^n)$. Finally, χ_S designates the characteristic function of the set S.

1. <u>Hardy inequalities</u>

The following result for measures μ was observed by Sinnamon [S]. The $p = q$ and μ locally integrable case is due to Tomaselli [To] and Talenti [T]; and Muckenhoupt [Mu 1] provided new proofs of their results and also proved the $p = q$ case for measures μ. The $p \leq q$ and μ locally integrable case is due to Bradley, Kokilashvili, Maz`ja, and Andersen and Muckenhoupt, e.g., [Br;M;AM]. The $q < p$ and μ locally integrable case is due to Maz'ja (1979), Sawyer (1984), Heinig (1985), and Sinnamon (1987).

In Hardy's original inequality, $u(\gamma) = 1/|\gamma|^p$ so that "$d\mu(\gamma) = u(\gamma)d\gamma$" is not a measure. In fact, local integrability of u on a neighborhood N of the origin is not an hypothesis of the above mentioned results; and there is an analogue of <u>Theorem 1.1</u> when μ is not a measure on N.

<u>Theorem 1.1</u> Given $v \in L^1_{loc}(\mathbb{R})$, $v > 0$ a.e., and $\mu \in M_+(\mathbb{R})$. Assume $1 < p \leq q < \infty$ and $v^{1-p'} \in L^1_{loc}(\mathbb{R})$.

a. There is $C > 0$ such that for all $h \in L^1_{loc}(\mathbb{R})$, $h \geq 0$,

$$(1.1)\qquad \left(\int_{[0,\infty)} \left[\int_\gamma^\infty h(t)dt \right]^q d\mu(\gamma) \right)^{1/q} \leq C \left[\int_0^\infty h(t)^p v(t)dt \right]^{1/p}$$

if and only if

$$(1.2)\qquad B = \sup_{y>0} \left(\int_{[0,y)} d\mu(\gamma) \right)^{1/q} \left[\int_y^\infty v(t)^{1-p'}dt \right]^{1/p'} < \infty.$$

Furthermore, $B \leq C \leq B(p)^{1/q}(p')^{1/p'}$.

b. There is $C > 0$ such that for all $h \in L^1_{loc}(\mathbb{R})$, $h \geq 0$,

$$(1.3)\qquad \left(\int_{[0,\infty)} \left[\int_\gamma^\infty h(t)dt \right]^q d\mu(\gamma) \right)^{1/q} \leq C \left[\int_0^\infty h(t)^p v(t)dt \right]^{1/p}$$

if and only if

$$(1.4)\qquad B = \sup_{y>0} \left(\int_{[y,\infty)} d\mu(\gamma) \right)^{1/q} \left[\int_0^y v(t)^{1-p'}dt \right]^{1/p'} < \infty.$$

Furthermore, $B \leq C \leq B(p)^{1/q}(p')^{1/p'}$.

Remark 1.2 a. Condition (1.4), for $\mu \in M_+(\mathbb{R})$ and $v > 0$ a.e., implies that $\mu \in M_b([0,\infty))$.

b. The generalization of Theorem 1.1 from the case "$d\nu(t) = v(t)dt$" to arbitrary $\nu \in M_+(\mathbb{R})$ is vacuous. In fact, if $\nu \in M_+(\mathbb{R})$ and m is Lebesgue measure then $\nu = f + \nu_s$, where $f \in L^1_{loc}(\mathbb{R})$, $\nu_s \perp m$, and $f, \nu_s \in M_+(\mathbb{R})$ [B, Theorem 5.9]. Thus, if m is concentrated in X and ν_s is concentrated in Y with $X \cap Y = \phi$ then, considering (1.3) for example, we have $\int_0^\gamma h_1(t)dt = \int_0^\gamma h(t)dt$ for $h_1 = h\chi_X$ and $\int h_1(t)^p d\nu_s(t) = 0$, e.g.,

$$0 \le \int \chi_A \chi_X d\nu_s = \nu_s(A \cap X) = \nu_s((A\cap X)\cap Y) = \nu_s(\phi) = 0.$$

c. <u>Theorem 1.1</u> has a natural formulation if $p = 1$. In that case, $B = C$.

Since we are dealing with measures μ in <u>Theorem 1.1</u>, (1.1) and (1.3) are equivalent to the same inequalities for all non-negative $h \in L^1_{loc}(\mathbb{R})$ for which $\operatorname{supp} h \subseteq [0,\infty)$. This simple observation plays a role in the following results.

<u>Proposition 1.3</u> Given $v \in L^1_{loc}(\mathbb{R})$, $v > 0$ a.e., and $\mu \in M_+(\hat{\mathbb{R}})$. Assume $1 < p \le q < \infty$ and $v^{1-p'} \in L^1_{loc}(\mathbb{R})$.

a. There is $C > 0$ such that for all $h \in L^1_{loc}(\mathbb{R})$, $h \ge 0$,

$$(1.5) \qquad \left[\int \left[\int_{|t|>|\gamma|} h(t)dt\right]^q d\mu(\gamma)\right]^{1/q} \le C\left[\int h(t)^p v(t)dt\right]^{1/p}$$

if and only if

$$(1.6) \qquad B = \sup_{y>0} \left[\int_{|\gamma|<y} d\mu(\gamma)\right]^{1/q} \left[\int_{|x|>y} v(x)^{1-p'}dx\right]^{1/p'} < \infty.$$

b. If $C < \infty$ then $B \le C$. If $B < \infty$ and $\mu(\{0\}) = 0$ then $C = C_\mu$ satisfies

$$C \le 2^{1/p'} B(p)^{1/q}(p')^{1/p'},$$

and if $\mu = a\delta$, $a > 0$, then $v^{1-p'} \in L^1(\mathbb{R})$ and $C = C_\mu$ satisfies $C = B = a^{1/q}(\int v(t)^{1-p'}dt)^{1/p'}$. If $B < \infty$ and $\mu = a\delta + \eta$, where $a > 0$, $\eta \in M_+(\mathbb{R})$, and $\eta(\{0\}) = 0$, then

$$C = (C^q_{a\delta} + C^q_\eta)^{1/q}.$$

Proof. i. The case $\mu = a\delta$, $a > 0$, follows by direct calculation. If (1.5) holds let $h(t) = v(t)^{1-p'} \chi_S(t)$, where S is a compact interval. (1.5) becomes

$$a^{1/q} \int_S v(t)^{1-p'}dt \le C\left[\int_S v(t)^{1-p'}dt\right]^{1/p};$$

and, hence, by letting S vary,

$$a^{1/q} \left[\int v(t)^{1-p'}dt\right]^{1/p'} \le C.$$

The left hand side is B and so $B \le C$. If (1.6) holds then the left hand side of (1.5) is

$$a^{1/q} \int h(t)v(t)^{1/p}v(t)^{-1/p}\, dt \le a^{1/q} \|h\|_{p,v}\left[\int v(t)^{1-p'}dt\right]^{1/p'}$$

$$= B\|h\|_{p,v} ,$$

and so $C \le B$.

ii. The necessary conditions for (1.5) are, in fact, true for any $\mu \in M_+(\hat{\mathbb{R}})$. To see this, assume (1.5), fix $y > 0$, and let $h(t) = v(t)^{1-p'} \chi_S(t)$ where $S = \{t : y < |t| < Y\}$. We reduce the left hand side of (1.5) to

$$\left[\int_{|\gamma|<y} \left[\int_{|t|>|\gamma|} v(t)^{1-p'}\chi_S(t)dt\right]^q d\mu(\gamma)\right]^{1/q}$$

$$= \left[\int_{|\gamma|<y}\left[\int_S v(t)^{1-p'}dt\right]^q d\mu(\gamma)\right]^{1/q}$$

$$= \left[\int_{|\gamma|<y} d\mu(\gamma)\right]^{1/q} \int_S v(t)^{1-p'}dt \;;$$

and, hence, since $\mu \in M_+(\hat{\mathbb{R}})$, (1.5) implies

$$\left[\int_{|\gamma|<y} d\mu(\gamma)\right]^{1/q} \left[\int_S v(t)^{1-p'}dt\right]^{1/p'} \le C.$$

Letting $Y \longrightarrow \infty$ we obtain (1.6) with $B \le C$.

iii. Assume $\mu(\{0\}) = 0$ and that (1.6) holds. Take any non-negative $h \in L^1_{loc}(\mathbb{R})$. Write $\int \left[\int_{|t|>|\gamma|} h(t)dt\right]^q d\mu(\gamma)$ as

$$\int_{(0,\infty)} \left[\int_{|t|>\gamma} h(t)dt\right]^q d\mu(\gamma) + \int_{(-\infty,0)} \left[\int_{|t|>-\gamma} h(t)dt\right]^q d\mu(\gamma) =$$

$$\int_{(0,\infty)} \left[\int_{|t|>\gamma} h(t)dt\right]^q d\mu(\gamma) - \int_{(\infty,0)} \left[\int_{|t|>\gamma} h(t)dt\right]^q d\mu(-\gamma) =$$

$$\int_{(0,\infty)} \left[\int_{\gamma}^{\infty} h(t)dt + \int_{-\infty}^{-\gamma} h(t)dt\right]^q d(\mu(\gamma)+\mu(-\gamma)),$$

so that by Minkowski's inequality the left hand side of (1.5) is bounded by

$$\left[\int_{(0,\infty)} \left[\int_{\gamma}^{\infty} h(t)dt\right]^q d(\mu(\gamma)+\mu(-\gamma))\right]^{1/q} + \left[\int_{(0,\infty)} \left[\int_{\gamma}^{\infty} h(-t)dt\right]^q d(\mu(\gamma)+\mu(-\gamma))\right]^{1/q}$$

$$= I_1 + I_2.$$

The first integral of (1.6) is

$$\int_{(-y,0)} d\mu(\gamma) + \int_{(0,y)} d\mu(\gamma) = \int_{(0,y)} d(\mu(\gamma)+\mu(-\gamma)).$$

We invoke Theorem 1.1a, replacing $\mu(\gamma)$ there by $\mu(\gamma) + \mu(-\gamma)$, to obtain

$$(1.7)\qquad I_1 \le C_+\left[\int_0^\infty h(t)^p v(t)dt\right]^{1/p}$$

for all $h \in L^1_{loc}(\mathbb{R})$, $h \ge 0$, if and only if

$$(1.8)\qquad B_+ = \sup_{y>0}\left[\int_{|\gamma|<y} d\mu(\gamma)\right]^{1/q}\left[\int_y^\infty v(t)^{1-p'}dt\right]^{1/p'} < \infty.$$

By Theorem 1.1a we also have $B_+ \le C_+ \le B_+(p)^{1/q}(p')^{1/p'}$.

We again invoke Theorem 1.1a, replacing $\mu(\gamma)$ there by $\mu(\gamma) + \mu(-\gamma)$ and $v(t)$ by $v(-t)$, to obtain

$$(1.9)\qquad I_2 \le C_-\left[\int_y^\infty h(-t)^p v(-t)dt\right]^{1/p}$$

for all $h \in L^1_{loc}(\mathbb{R})$, $h \ge 0$, if and only if

$$(1.10)\qquad B_- = \sup_{y>0}\left[\int_{|\gamma|<y} d\mu(\gamma)\right]^{1/q}\left[\int_y^\infty v(-t)^{1-p'}dt\right]^{1/p'} < \infty.$$

Once again, by Theorem 1.1a, we have $B_- \le C_- \le B_-(p)^{1/q}(p')^{1/p'}$.

Since $B < \infty$ then both (1.8) and (1.10) hold, as is easily seen by positivity and by raising the various factors to the p' power; in fact, $B_\pm \le B$. Consequently, both (1.7) and (1.9) are valid so that the left hand side of (1.5) is bounded by

$$(1.11)\qquad I_1 + I_2 \le C_+\left[\int_0^\infty h(t)^p v(t)dt\right]^{1/p} + C_-\left[\int_{-\infty}^0 h(t)^p v(t)dt\right]^{1/p}.$$

We apply Holder's inequality to the right hand side of (1.11), considered as the sum $C_+D_+ + C_-D_-$, and are able to bound this right hand side by

$$(C_+^{p'} + C_-^{p'})^{1/p'} \left[\int_0^\infty h(t)^p v(t)dt + \int_{-\infty}^0 h(t)^p v(t)dt \right]^{1/p} \le$$

$$(B_+^{p'} + B_-^{p'})^{1/p'} (p)^{1/q}(p')^{1/p'} \|h\|_{p,v} \le$$

$$2^{1/p'} B(p)^{1/q}(p')^{1/p'} \|h\|_{p,v} .$$

iv. Finally, let $\mu = a\delta + \eta$. Since $B < \infty$ then $B_{a\delta} < \infty$ and $B_\eta < \infty$ by positivity, where, for example, B_η is the supremum in (1.6) for the measure η. Thus, by the previous parts of this proof,

$$\int \left[\int_{|t|>|\gamma|} h(t)dt \right]^q d\mu(\gamma) \le C_{a\delta}^q \|h\|_{p,v}^{q/p} + C_\eta^q \|h\|_{p,v}^{q/p} ,$$

and the constant is obtained. <u>q.e.d.</u>

The hypothesis, $v^{1-p'} \in L^1_{loc}(\mathbb{R})$, in <u>Proposition 1.3</u> can be weakened to assuming $v^{1-p'} \in L^1_{loc}(\mathbb{R}\setminus[-y,y])$ for each $y > 0$.

<u>Proposition 1.4</u> Given $v \in L^1_{loc}(\mathbb{R})$, $v > 0$ a.e., and $\mu \in M_+(\hat{\mathbb{R}})$. Assume $1 < p \le q < \infty$ and $v^{1-p'} \in L^1_{loc}(\mathbb{R})$.

a. There is $C > 0$ such that for all $h \in L^1_{loc}(\mathbb{R})$, $h \ge 0$,

$$\left(\int \left[\int_{|t|<|\gamma|} h(t)dt \right]^q d\mu(\gamma)\right)^{1/q} \le C\left[\int h(t)^p v(t)dt \right]^{1/p}$$

if and only if

$$B = \sup_{y>0} \left[\int_{|\gamma|>y} d\mu(\gamma) \right]^{1/q} \left[\int_{|x|<y}^{\infty} v(x)^{1-p'}dx \right]^{1/p'} < \infty.$$

b. If $C < \infty$ then $B \le C$. If $B < \infty$ and $\mu(\{0\}) = 0$ then $C = C_\mu$ satisfies

$$C \le 2^{1/p'} B(p)^{1/q} (p')^{1/p'},$$

and if $\mu = a\delta$, $a > 0$, then $v^{1-p'} \in L^1(\mathbb{R})$ and $C = C_\mu$ satisfies $C = B = a^{1/q}(\int v(t)^{1-p'} dt)^{1/p'}$. If $B < \infty$ and $\mu = a\delta + \eta$, where $a > 0$, $\eta \in M_+(\mathbb{R})$, and $\eta(\{0\}) = 0$, then

$$C = (C^q_{a\delta} + C^q_\eta)^{1/q}.$$

The proof is similar to that of Proposition 1.3 and uses Theorem 1.1b.

2. A Fourier transform norm inequality on $\mathbb{R}$

Define

$$M_0 = \{f \in L^1(\mathbb{R}) : \text{supp } f \text{ is compact and } \hat{f}(0) = 0\}.$$

Theorem 2.1 Given $v \in L^1_{loc}(\mathbb{R})$, $v > 0$ a.e., and $\mu \in M_+(\hat{\mathbb{R}})$. Assume $1 < p \le q < \infty$ and $v^{1-p'} \in L^1_{loc}(\mathbb{R} \setminus [-y,y])$ for each $y > 0$.

a. If

$$B_1 = \sup_{y>0} \left(\int_{|\gamma|<y} |\gamma|^q d\mu(\gamma) \right)^{1/q} \left(\int_{|x|<1/y} |x|^{p'} v(x)^{1-p'} dx \right)^{1/p'} < \infty \tag{2.1}$$

and

$$B_2 = \sup_{y>0} \left(\int_{|\gamma|>y} d\mu(\gamma) \right)^{1/q} \left(\int_{|x|>1/y} v(x)^{1-p'} dx \right)^{1/p'} < \infty \tag{2.2}$$

then there is $C > 0$ such that

$$\forall f \in M_0 \cap L^p_v(\mathbb{R}), \quad \|\hat{f}\|_{q,\mu} \le C \|f\|_{p,v}. \tag{2.3}$$

b. If $\mu = a\delta$, $a > 0$, then $B_1 = B_2 = 0$; and, for arbitrary $\mu \in M_+(\hat{\mathbb{R}})$, C in (2.3) can be chosen as

$$C = 2^{1+\frac{1}{p'}}(\pi B_1 + B_2)(p)^{1/q}(p')^{1/p'},$$

cf., Remark 2.4e.

Proof. Since $f \in M_0$, we have $\hat{f}(\gamma) = \int (e^{-2\pi it\gamma} - 1)f(t)dt$ and so

$$\hat{f}(\gamma) = -2i \int e^{-\pi it\gamma} \left(\frac{\sin \pi t\gamma}{\pi t\gamma}\right) \pi t\gamma f(t)dt.$$

Therefore, we find that

$$|\hat{f}(\gamma)| \le 2\pi|\gamma| \int_{\pi|t\gamma|\le 1} |tf(t)|dt + 2 \int_{\pi|t\gamma|>1} |f(t)|dt =$$

$$2\pi|\gamma| \int_{\frac{\pi}{|x|}\le\frac{1}{|\gamma|}} \left|x^{-3} f\left(\frac{1}{x}\right)\right| dx + 2 \int_{\frac{\pi}{|x|}>\frac{1}{|\gamma|}} \left|x^{-2} f\left(\frac{1}{x}\right)\right| dx .$$

Consequently, by Minkowski's inequality, we estimate

$$\left(\int |\hat{f}(\gamma)|^q d\mu(\gamma)\right)^{1/q} \le 2\pi\left(\int |\gamma|^q \left(\int_{\frac{\pi}{|x|}\le\frac{1}{|\gamma|}} \left|x^{-3} f\left(\frac{1}{x}\right)\right| dx\right)^q d\mu(\gamma)\right)^{1/q}$$

$$+ 2\left(\int \left(\int_{\frac{\pi}{|x|}>\frac{1}{|\gamma|}} \left|x^{-2} f\left(\frac{1}{x}\right)\right| dx\right)^q d\mu(\gamma)\right)^{1/q} = 2\pi J_1 + 2J_2 .$$

We first use Proposition 1.3. Let $h(t) = \left|t^{-3} f\left(\frac{1}{t}\right)\right|$ and replace $d\mu(\gamma)$ (in the proposition) by $|\gamma|^q d\mu(\gamma)$ and $v(t)$ by $|t|^{3p-2} v\left(\frac{1}{t}\right)$. Then we obtain

$$J_1 \le C_1 \left(\int \left|t^{-3} f\left(\frac{1}{t}\right)\right|^p |t|^{3p-2}\, v\left(\frac{1}{t}\right) dt\right)^{1/p} \tag{2.4}$$

for all $f \in M_0$ if

$$(2.5)\qquad \sup_{y>0}\left(\int_{|\gamma|<y/\pi}|\gamma|^q d\mu(\gamma)\right)^{1/q}\left(\int_{|t|>y/\pi}\left(|t|^{3p-2}\,v\left(\frac{1}{t}\right)\right)^{1-p'}dt\right)^{1/p'}<\infty.$$

The right hand side of (2.4) is $C_1\|f\|_{p,v}$. Note in (2.5) that $(3p-2)(1-p') = 3p-2-3pp' + 2p' + p'-p' = 3(p-pp'+p')-(2+p') = -(2+p')$. Thus, the second integral in (2.5) is

$$\left(\int_{|t|>y/\pi}|t|^{-(2+p')}v\left(\frac{1}{t}\right)^{1-p'}dt\right)^{1/p'}=\left(\int_{|x|<\pi/y}|x|^{p'}v(x)^{1-p'}dx\right)^{1/p'}.$$

Combining these observations we obtain $J_1 \le C_1\|f\|_{p,v}$ for all $f \in M_0 \cap L^p_v(\mathbb{R})$ if (2.1) holds.

Next, we use Proposition 1.4. Because of the definition of J_2 we let $h(t) = t^{-2}f\left(\frac{1}{t}\right)$ in the proposition as well as replacing $v(t)$ by $|t|^{2p-2}v\left(\frac{1}{t}\right)$. Then we have

$$(2.6)\qquad J_2 \le C_2\left(\int |t^{-2}f\left(\frac{1}{t}\right)|^p\,|t|^{2p-2}v\left(\frac{1}{t}\right)dt\right)^{1/p}$$

for all $f \in M_0$ if

$$(2.7)\qquad \sup_{y>0}\left(\int_{|\gamma|>y/\pi}d\mu(\gamma)\right)^{1/q}\left(\int_{|t|<y/\pi}\left(|t|^{2p-2}\,v\left(\frac{1}{t}\right)\right)^{1-p'}dt\right)^{1/p'}<\infty.$$

The right hand side of (2.6) is $C_2\|f\|_{p,v}$. Note in (2.7) that $(2p-2)(1-p') = 2p-2-2pp' + 2p' = -2$. Thus, the second integral in (2.7) is

$$\left(\int_{|t|<y/\pi}|t|^{-2}\,v\left(\frac{1}{t}\right)^{1-p'}dt\right)^{1/p'}=\left(\int_{|x|>\pi/y}v(x)^{1-p'}dx\right)^{1/p'}.$$

Combining these observations we obtain $J_2 \le C_2\|f\|_{p,v}$ for all $f \in M_0 \cap L^p_v(\mathbb{R})$ if (2.2) holds.

Consequently, (2.3) is obtained. The value of C in terms of B_1 and B_2 follows directly from the estimate, $\|\hat{f}\|_{q,\mu} \le 2\pi J_1 + 2J_2$, and the values of the constants in Propositions 1.3 and 1.4. q.e.d.

Theorem 2.2 Given $v \in L^r_{loc}(\mathbb{R})$ for some $r > 1$, where $v > 0$ a.e., and choose $p \in (1,\infty)$.

a. If $h \in L^p_v(\mathbb{R})'$ annihilates $M_0 \cap L^p_v(\mathbb{R})$ then h is a constant function.

b. $\overline{M_0 \cap L^p_v(\mathbb{R})} = L^p_v(\mathbb{R})$ or $L^p_v(\mathbb{R}) \subseteq L^1(\mathbb{R})$.

c. If $v^{1-p'} \not\in L^1(\mathbb{R})$ then $\overline{M_0 \cap L^p_v(\mathbb{R})} = L^p_v(\mathbb{R})$.

Proof. a. Suppose $h \in L^p_v(\mathbb{R})'$ annihilates the vector space $M_0 \cap L^p_v(\mathbb{R})$.

Let $\chi_{T/2} = \chi_{[-T/2,T/2]}$, $T > 0$, and $e_\gamma(t) = e^{2\pi i t\gamma}$. Note that $(e_\gamma f)\hat{}(\lambda) = \hat{f}(\lambda-\gamma)$ for $f \in L^1(\mathbb{R})$. We have $\hat{\chi}_{T/2}(\gamma) = T\left(\frac{\sin \pi T\gamma}{\pi T\gamma}\right)$ and so $\hat{\chi}_{T/2}(\gamma) = 0$ if $\gamma = n/T$, $n \in \mathbb{Z}\setminus\{0\}$. Therefore, $e_{n/T}\chi_{T/2} \in M_0 \cap L^p_v(\mathbb{R})$ for all $n \in \mathbb{Z}\setminus\{0\}$. Let

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} e_{n/T}(t)h(t)dt, \ n \in \mathbb{Z}.$$

Each integral is well-defined because of the elementary calculation showing that $L^p_v(\mathbb{R})' = L^{p'}_{v^{1-p'}}(\mathbb{R})$. By our hypothesis on h, $c_n = 0$ for all $n \in \mathbb{Z}\setminus\{0\}$.

Let $h_T = h$ on $[-T/2, T/2)$ and define it T-periodically on $\mathbb{R}$. The formal Fourier series of h_T is $\sum c_n e_{-n/T}(t)$, noting that $e_{-n/T}$ is T-periodic for all n.

Our next goal is to show that $h_T \in L^a_{loc}(\mathbb{R}/T\mathbb{Z})$ for some $a > 1$. If our assumption were that $v \in L^\infty_{loc}(\mathbb{R})$ instead of $v \in L^r_{loc}(\mathbb{R})$ then this fact is valid for $a = p'$ by means of the elementary estimate,

$$\int_{-T/2}^{T/2} |h(t)|^{p'} dt = \int_{-T/2}^{T/2} |h(t)|^{p'} v(t)^{1-p'} v(t)^{p'-1}\, dt$$

$$\leq K_T \int_{-T/2}^{T/2} |h(t)|^{p'} v(t)^{1-p'} dt < \infty.$$

For the more general case, $v \in L^r_{loc}(\mathbb{R})$, we proceed as follows. Let $a = rp'/(p'-1+r)$. It is easy to see that $1 < a < p'$; in fact, $rp' > p' - 1 + r$ if and only if $r(p'-1) > p'-1$ if and only if $r > 1$ and $rp' < p'(p'-1+r)$ if and only if $r < p' - 1 + r$ if and only if $p' > 1$. Set $s = p'/a$ so that $s > 1$. Consequently,

$$\int_{-T/2}^{T/2} |h(t)|^a v(t)^{-a/p} v(t)^{a/p} dt \leq$$

$$\left[\int_{-T/2}^{T/2} |h(t)|^{as} v(t)^{-as/p} dt\right]^{1/s} \left[\int_{-T/2}^{T/2} v(t)^{as'/p} dt\right]^{1/s'} =$$

$$\left[\int_{-T/2}^{T/2} |h(t)|^{p'} v(t)^{1-p'} dt\right]^{1/s} \left[\int_{-T/2}^{T/2} v(t)^{r} dt\right]^{1/s'} < \infty,$$

since

$$\frac{as'}{p} = \frac{p'}{p}\,\frac{a}{p'-a} = \frac{p'}{p}\,\frac{rp'}{p'(p'-1+r)-rp'} = r.$$

Now, because $a > 1$ we can apply the Carleson-Hunt theorem to assert that $h_T(t) = \sum c_n e_{-n/T}(t)$ a.e. on $[-T/2, T/2)$. By the properties of $\{c_n\}$ this means

$$(2.8) \qquad \forall T > 0,\ h(t) = \frac{1}{T}\int_{-T/2}^{T/2} h(u)du \text{ a.e. on } [-T/2, T/2).$$

We use (2.8) in the following way. First, $h(t) = k_{N+1}$ on $[-(N+1)/2, (N+1)/2)$ by (2.8), and hence

$$k_N = h(t) = \frac{1}{N}\int_{-N/2}^{N/2} k_{N+1} du \quad \text{a.e. on } [-N/2, N/2).$$

Thus, $k_N = k_{N+1}$ on $[-N/2, N/2)$ for each integer N; and so $h(t) = k \in \mathbb{C}$ a.e. on $\mathbb{R}$.

b. If $h(t) = k = 0$ then $\overline{M_0 \cap L^p_v(\mathbb{R})} = L^p_v(\mathbb{R})$ by the Hahn-Banach theorem.

If $h(t) = k \neq 0$ and $f \in L^p_v(\mathbb{R})$ then $|f| \in L^p_v(\mathbb{R})$, and, by the duality between $L^p_v(\mathbb{R})$ and its dual, $\int |f|(t)\overline{h(t)}dt \in \mathbb{C}$. Consequently, $\bar{k}\int |f(t)|dt \in \mathbb{C}$ and so $f \in L^1(\mathbb{R})$. We could also argue that $h(t) = k \neq 0$ implies $v^{1-p'} \in L^1(\mathbb{R})$, and so $\int |f(t)|dt < \infty$ for $f \in L^p_v(\mathbb{R})$ by Hölder's inequality.

c. Since $h \in L^{p'}_{v^{1-p'}}(\mathbb{R})$ then $h(t) = k = 0$ because $v^{1-p'} \notin L^1(\mathbb{R})$; consequently, $\overline{M_0 \cap L^p_v(\mathbb{R})} = L^p_v(\mathbb{R})$ by the Hahn-Banach theorem. <u>q.e.d.</u>

Combining <u>Theorem 2.1</u> and <u>Theorem 2.2</u> with a standard density argument, cf., [BH, p.251], we obtain –

<u>Theorem 2.3</u> Given $v \in L^r_{loc}(\mathbb{R})$ for some $r > 1$, where $v > 0$ a.e., and given $\mu \in \hat{M}_+(\mathbb{R})$. Suppose $1 < p \leq q < \infty$ and $v^{1-p'} \in L^1_{loc}(\mathbb{R}\backslash[-y,y])\backslash L^1(\mathbb{R})$ for each $y > 0$; and assume (2.1) and (2.2) are valid.

a. If $f \in L^p_v(\mathbb{R})$ then $\lim_{j\to\infty} \|f_j - f\|_{p,v} = 0$ for a sequence $\{f_j\} \subseteq M_0 \cap L^p_v(\mathbb{R})$, and $\{\hat{f}_j\}$ converges in $L^q_\mu(\hat{\mathbb{R}})$ to a function $\hat{f} \in L^q_\mu(\hat{\mathbb{R}})$. $\hat{f}$ is independent of the sequence $\{f_j\}$ and it is called the <u>Fourier transform</u> of f.

b. There is $C > 0$ such that

$$\forall f \in L^p_v(\mathbb{R}), \quad \|\hat{f}\|_{q,\mu} \leq C\|f\|_{p,v}.$$

Furthermore, C can be chosen as

$$C = 2^{1+\frac{1}{p'}}(\pi B_1 + B_2)(p)^{1/q}(p')^{1/p'}.$$

<u>Remark 2.4</u> a. Our density result, <u>Theorem 2.2</u>, is quite different in spirit and technique than that proved in [MWY, Theorem 6.19] by Muckenhoupt, Wheeden, and Young. As a particular case and for $v \in L^1_{loc}(\mathbb{R})$, they show that $\overline{M_0 \cap L^p_v(\mathbb{R})} = L^p_v(\mathbb{R})$ if

$$\lim_{j\to\infty} j^p \int_0^{1/j} v(t)dt = 0$$

(2.9)

$$\lim_{j\to\infty} \frac{1}{j^{2p}} \int_{-j}^{j} v(t)dt = 0,$$

cf., <u>Proposition 2.6.</u> (Technically, they don't use $M_0 \cap L^p_v(\mathbb{R})$ but the result is the same.)

b. Suppose $v^{1-p'} \in L^1_{loc}(\mathbb{R})$ for even v and assume supp μ is not compact; if (2.2) holds then $\underline{v^{1-p'} \in L^1(\mathbb{R})}$ and $\mu \in M_b(\hat{\mathbb{R}})$. In particular, we can not determine that $M_0 \cap L^p_v(\mathbb{R}) = L^p_v(\mathbb{R})$ from Thoerem 2.2, noting that $L^p_v(\mathbb{R}) \subseteq L^1(\mathbb{R})$ when $v^{1-p'} \in L^1(\mathbb{R})$.

c. The weight condition in [BH] for "$d\mu(\gamma) = u(\gamma)d\gamma$", u and v even, and u and $1/v$ decreasing on $(0,\infty)$ is that $(u,v) \in F(p,q)$, i.e.,

$$F(p,q) \qquad \sup_{y\to 0} \left[\int_0^y u(\gamma)d\gamma \right]^{1/q} \left[\int_0^{1/y} v(x)^{1-p'}dx \right]^{1/p'} < \infty.$$

Using the given monotonicity it is easy to see that (2.1) is a consequence of $F(p,q)$. We have no such expectation for (2.2); in fact, $F(p,q)$ is valid and (2.2) fails for $u(\gamma) = 1/|\gamma|^{\alpha}$, $v(x) = |x|^{\alpha}$, $p = q = 2$, and $0 \le \alpha < 1$.

d. If $\mu \in M_+(\hat{\mathbb{R}})$ and (2.2) holds then $\mu \in M_{b+}(\hat{\mathbb{R}})$. However, if $\mu \in M_{b+}(\hat{\mathbb{R}})$ and $v^{1-p'} \in L^1_{loc}(\mathbb{R}\setminus[-y,y])$ for each $y > 0$ we can not necessarily conclude that (2.2) holds. On the other hand, (2.2) is obtained for $\mu \in M_{b+}(\hat{\mathbb{R}})$ and $v^{1-p'} \in L^1(\mathbb{R})$ or for $\mu \in M_{b+}(\hat{\mathbb{R}})$ with compact support $K \subseteq [-y_1,y_1]$ and $v^{1-p'} \in L^1(\mathbb{R}\setminus[-1/y_1,1/y_1])$.

e. The Fourier transform defined in Theorem 2.3a is the usual Fourier transform when the latter exists on $L^p_v(\mathbb{R})$. However, it provides an extension of the Fourier transform on other $L^p_v(\mathbb{R})$. As a trivial example, but one which explains the constants in Theorem 2.1b, let $\mu = \delta$. Then (2.3) becomes $|\hat{f}(0)| \le C\|f\|_{p,v}$ for $f \in M_0$. Even more, $B_1 = B_2 = 0$ implies $C = 0$ in this case; but this causes no problem since $f \in M_0$. If $v(t) = |t|^p$

then $v(t)^{1-p'} \notin L^1(\mathbb{R})$ so that Theorem 2.2 applies; but the unique continuous extension $L^p_v(\mathbb{R}) \longrightarrow L^q_\delta(\hat{\mathbb{R}})$ of the well-defined Fourier transform map $M_0 \cap L^p_v(\mathbb{R}) \longrightarrow L^q_\delta(\hat{\mathbb{R}})$ is nothing more than the 0-function, cf., Example 2.5c.

Example 2.5 a. If $u(\gamma) = \gamma^{-2}$, $v(x) = x^2$, and $p = q = 2$, then (2.1) and (2.2) are satisfied, whereas $\mu \notin M(\hat{\mathbb{R}})$ for "$d\mu(\gamma) = u(\gamma)d\gamma$" since $u \notin L^1_{loc}(\hat{\mathbb{R}})$. This does not allow us to apply Theorem 1.1 as it is stated.

b. If $u(\gamma) = e^{-|\gamma|}$, $v(x) = e^{|x|}$, and $1 < p \le q < \infty$, then all the conditions of Theorem 2.1 are satisfied. In fact, the conclusion (2.3) is expected since $L^q_u(\hat{\mathbb{R}})$ is "large" and $L^p_v(\mathbb{R})$ is "small". It is clear that (2.9) fails whereas $L^p_v(\mathbb{R}) \subseteq L^1(\mathbb{R})$.

c. Given $v(t) = |t|^{1+\varepsilon}$ and $\mu = \Sigma'\left(1/|n|^{1+\varepsilon}\right)\delta_n$ for fixed $\varepsilon \in (0,2)$, and let $p = q = 2$. The conditions of Theorem 2.3 are satisfied. Clearly, $v^{-1} \in L^1(\mathbb{R}\backslash[-y,y])\backslash L^1(\mathbb{R})$ for each $y > 0$; and

$$B_1 \le \begin{cases} \dfrac{2}{2-\varepsilon}\,(2^{2-\varepsilon}-1)^{1/2}, & 0 < \varepsilon < 1 \\[2ex] \dfrac{2}{2-\varepsilon}, & 1 \le \varepsilon < 2 \end{cases}$$

and $B_2 \le 2/\varepsilon$. (For computations, note that $2^{2-\varepsilon}-1 \le 3-\varepsilon$.) Consequently,

$$\Sigma' \frac{|\hat{f}(n)|^2}{|n|^{1+\varepsilon}} \le 2^5(\pi B_1+B_2)^2 \int |f(t)|^2|t|^{1+\varepsilon}dt.$$

By direct construction, it is easy to see that the Fourier transform map $M_0 \cap L^2_v(\mathbb{R}) \longrightarrow L^2_\mu(\hat{\mathbb{R}})$ extends to $L^2_v(\mathbb{R})$ in a non-trivial way, cf., Remark 2.4e.

Because of Theorem 2.2 and (2.9) we give the following application of Hardy's inequality.

Proposition 2.6 Given $p \in (1,\infty)$ and $v \in L^1_{loc}(\mathbb{R})$, $v > 0$ a.e. If

$$\lim_{j\to\infty} \frac{1}{j^p} \int_{-j}^{j} v(t)dt = 0$$

then $v^{1-p'} \notin L^1(\mathbb{R})$.

Proof. Taking $\mu = \delta$ and any $q \geq p$ we apply Theorem 1.1a to obtain $v^{1-p'} \in L^1(\mathbb{R})$ if and only if $\int h(t)dt \leq C\|h\|_{p,v}$ for all non-negative $h \in L^1_{loc}(\mathbb{R})$, where C is independent of h. Thus, if $v^{1-p'} \in L^1(\mathbb{R})$ then

$$\forall h = \chi_j,\ j > 0,\ (2j)^p \leq C^p \int_{-j}^{j} v(t)dt,$$

and the result follows. q.e.d.

3. Remarks about measures on $\mathbb{R}^n$

Example 3.1 If $\mu \in M(\hat{\mathbb{R}}^n)$ then $\mu(\{0\})$ is well-defined by $\mu(\{0\}) = \lim_{j\to\infty} \langle\mu,\varphi_j\rangle$, where $\varphi_j \in C_c(\hat{\mathbb{R}}^n)$, supp $\varphi_j \subseteq B(0,1/j)$ (the closed ball of radius $1/j$ centered at the origin), $\varphi_j = 1$ on a neighborhood of $0 \in \hat{\mathbb{R}}^n$, and $\|\varphi_j\|_\infty = 1$. To see this, first observe that

$$\left|\int(\varphi_j(\gamma)-\varphi_k(\gamma))d\mu(\gamma)\right| \leq 2 \int_{B(0,1/j)\setminus\{0\}} d|\mu|(\gamma),$$

where $k > j$. The right hand side tends to 0 as $j \longrightarrow \infty$ since $\cap(B(0,1/j)\setminus\{0\}) = \phi$, $B(0,1/(j+1))\setminus\{0\} \subseteq B(0,1/j)\setminus\{0\}$, and $|\mu| \in$

$M_+(\hat{\mathbb{R}}^n)$. ($|\mu|$ is defined as $<|\mu|,\varphi> = \sup\{|\int\psi d\mu|: |\psi| \le \varphi\}$ where $\varphi \in C_c(\hat{\mathbb{R}}^n)$ is non-negative; the extension of $|\mu|$ as an element of $M_+(\hat{\mathbb{R}}^n)$ is routine.) Thus, $\{<\mu,\varphi_j>\}$ is a Cauchy sequence and the limit exists. Any such sequences $\{\varphi_j\}$ or $\{\psi_j\}$ yield the same limit since $|A-B| \le |A-<\mu,\varphi_j>| + |<\mu,\varphi_j-\psi_j>| + |B-<\mu,\psi_j>|$ and since $\lim_{j\to\infty} <\mu,\varphi_j-\psi_j> = 0$ as in the above estimate.

<u>Example 3.2</u> For $\varphi \in L^1(\hat{\mathbb{R}}^n)$ or for measurable non-negative functions φ on $\hat{\mathbb{R}}^n$ the polar coordinates change of variable formula is

$$(3.1) \qquad \int \varphi(\gamma)d\gamma = \int_{\Sigma_{n-1}} \int_0^\infty \rho^{n-1}\varphi(\rho\theta)d\rho d\sigma_{n-1}(\theta),$$

where $\gamma \in \hat{\mathbb{R}}^n\backslash\{0\}$ has the representation $\gamma = \rho\theta$ for $\rho > 0$ and $\theta \in \Sigma_{n-1}$, the unit sphere of $\hat{\mathbb{R}}^n$, and where σ_{n-1} is $(n-1)$-dimensional area measure on $\hat{\mathbb{R}}^n$. Note that, even though σ_{n-1} is not σ-finite on $\hat{\mathbb{R}}^n$, it is a bounded measure on Σ_{n-1}; and so, by Fubini's theorem, the integral on the right can be written in either order [Sm, pp.389 ff.]. If μ is the restriction of σ_{n-1} to Σ_{n-1}, then we shall also denote μ by σ_{n-1}, and, in this case, supp $\sigma_{n-1} = \Sigma_{n-1}$.

Take $n > 1$. $SO(n)$ is the non-commutative "special orthogonal" group of proper rotations. $S \in SO(n)$ is a real $n \times n$ matrix whose transpose S^τ is also its inverse S^{-1} and whose determinant det S is 1. A function φ on $\hat{\mathbb{R}}^n$ is <u>radial</u> if $\varphi(S\gamma) = \varphi(\gamma)$ for all $S \in SO(n)$.

Definition 3.3 $\mu \in M(\hat{\mathbb{R}}^n)$, $n > 1$, is *radial* if $S\mu = \mu$ for all $S \in SO(n)$, where $S\mu$ is defined as

$$\forall \varphi \in C_c(\hat{\mathbb{R}}^n),\ \langle S\mu, \varphi\rangle = \langle \mu(\gamma), \varphi(S\gamma)\rangle.$$

If "$d\mu(\gamma) = u(\gamma)d\gamma$", i.e., μ is identified with $u \in L^1_{loc}(\hat{\mathbb{R}}^n)$, then $(Su)(\gamma) = u(S^{-1}\gamma)$ for $S \in SO(n)$; in fact, $\int (Su)(\gamma)\varphi(\gamma)d\gamma = \int u(\gamma)\varphi(S\gamma)d\gamma = \int u(S^{-1}\gamma)\varphi(\gamma)d\gamma$, where the second equality follows since the Jacobian of any rotation is 1.

Proposition 3.4 Given $\mu \in M(\hat{\mathbb{R}}^n)$ and assume $\mu(\{0\}) = 0$. If μ is radial then there is a unique measure $\nu \in M(0,\infty)$ such that for all radial functions $\varphi \in C_c(\hat{\mathbb{R}}^n)$,

$$\langle \mu, \varphi\rangle = \omega_{n-1} \int_{(0,\infty)} \rho^{n-1}\varphi(\rho)d\nu(\rho), \tag{3.2}$$

where $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the unit sphere Σ_{n-1} of $\hat{\mathbb{R}}^n$.

Formula (3.2) extends to the radial elements of $L^1_\mu(\hat{\mathbb{R}}^n)$ by Labesgue's theorem.

Proof. Given a sequence $\{\psi_j : j = 1,2,\cdots\} \subset C_c(\hat{\mathbb{R}}^n)$ of non-negative functions having the properties, $\int \psi_j(\gamma)d\gamma = 1$ and $\text{supp}\ \psi_j \subset B(0,1/j)$ for each j. Then, for any $\varphi \in C_c(\hat{\mathbb{R}}^n)$, a standard approximate identity argument shows that $\lim_{j\to\infty} \langle \mu*\psi_j, \varphi\rangle = \langle \mu, \varphi\rangle$, where $\mu * \psi_j$ is a continuous function.

Next, assume each ψ_j is radial and take $S \in SO(n)$. We compute $(S(\mu*\psi_j))(\gamma) = \int \psi_j(S^{-1}\gamma - \lambda)d\mu(\lambda)$ and $(\mu*\psi_j))(\gamma) = ((S\mu)*\psi_j)(\gamma) = \int \psi_j(\gamma - S\lambda)d\mu(\lambda)$, where the second fact uses the assumption $S\mu = \mu$. Since ψ_j is radial, $\psi_j(S^{-1}\gamma-\lambda) = \psi_j(S(S^{-1}\gamma-\lambda))$ $= \psi_j(\gamma-S\lambda)$. Thus, $S(\mu*\psi_j) = \mu * \psi_j$, i.e., each $\mu * \psi_j$ is radial.

Set $\Psi_j = \mu*\psi_j$. Since Ψ_j is radial we compute

$$\langle\mu*\psi_j,\varphi\rangle = \omega_{n-1} \int_0^\infty \rho^{n-1}\Psi_j(\rho)\varphi(\rho)d\rho, \tag{3.3}$$

for all radial $\varphi \in C_c(\hat{\mathbb{R}}^n)$ by means of (3.1) and the fact that $(\mu*\psi_j)\varphi \in L^1(\hat{\mathbb{R}}^n)$. Now, consider the locally compact space $X = (0,\infty)$, the function space $C_c(X)$, and the linear subspace $\mathcal{E} = \{\Phi\in C_c(X):\Phi(\rho) = \rho^{n-1}\varphi(\rho)$, for some radial $\varphi \in C_c(\hat{\mathbb{R}}^n)\}$; note that each such φ vanishes in a neighborhood of $0 \in \hat{\mathbb{R}}^n$; i.e., φ is radial and $\varphi \in C_c(\hat{\mathbb{R}}^n\backslash\{0\})$. Define $\nu:\mathcal{E} \longrightarrow \mathbb{C}$ as $\langle\nu,\Phi\rangle = \lim_{j\to\infty} \int_X \Psi_j(\rho)\Phi(\rho)d\rho$. This limit exists by (3.3) and the weak $*$ convergence of $\{\mu*\psi_j\}$ to μ; and, in fact, $\omega_{n-1}\langle\nu,\Phi\rangle = \langle\mu,\varphi\rangle$ where $\Phi(\rho) = \rho^{n-1}\varphi(\rho)$. Clearly ν is linear on $\mathcal{E}$. Next, let the sequence $\{\Phi_k:\Phi_k(\rho) = \rho^{n-1}\varphi_k(\rho)\} \subseteq \mathcal{E}$ have the properties that $\lim_{k\to\infty} \|\Phi_k\|_\infty = 0$ and supp $\Phi_k \subseteq K$, where $K \subseteq X$ is a compact set. Then $\lim_{k\to\infty} \omega_{n-1}\langle\nu,\Phi_k\rangle = \lim_{k\to\infty} \langle\mu,\varphi_k\rangle = 0$ since $\mu \in M(\hat{\mathbb{R}}^n)$, $\lim_{k\to\infty} \|\varphi_k\|_\infty = 0$, and supp $\varphi_k \subseteq \{\theta K:\theta\in\Sigma_{n-1}\}$ (a compact set in $\hat{\mathbb{R}}^n$). Consequently, by the Hahn-Banach theorem, ν extends to a measure on $C_c(X)$ which we also denote by ν.

For a given radial $\varphi \in C_c(\hat{\mathbb{R}}^n)$, let $\{\varphi_j\} \subseteq C_c(\hat{\mathbb{R}}^n\backslash\{0\})$ and compact K have the properties that supp φ_j, supp $\varphi \subseteq K$, $B(0,1/j) \subseteq K$, $\varphi_j = \varphi$ on $K\backslash B(0,1/j)$, and $\|\varphi_j\|_\infty \le \|\varphi\|_\infty$. Then

$$\lim_{j\to\infty} \langle\mu,\varphi_j\rangle = \langle\mu,\varphi\rangle. \tag{3.4}$$

In fact,

$$|\langle\mu,\varphi-\varphi_j\rangle| = \Big|\int_{B(0,1/j)} (\varphi-\varphi_j)(\gamma)d\mu(\gamma) + \int_{K\backslash B(0,1/j)} (\varphi-\varphi_j)(\gamma)d\mu(\gamma)\Big|$$

$$= \Big|\int_{B(0,1/j)} (\varphi-\varphi_j)(\gamma)d\mu(\gamma)\Big| \le 2\|\varphi\|_\infty \int_{B(0,1/j)} d|\mu|(\gamma);$$

and the last term tends to zero since $|\mu| \in M_+(\hat{\mathbb{R}}^n)$ and $\{B(0,1/j)\}$ a decreasing sequence imply $\lim_{j\to\infty} |\mu|(B(0,1/j)) = |\mu|(\cap B(0,1/j))$ $= |\mu|(\{0\})$, where $|\mu|(\{0\}) = 0$ by the definition of $|\mu|$ in terms of μ and by the assumption $\mu(\{0\}) = 0$.

If for a given radial $\varphi \in C_c(\hat{\mathbb{R}}^n)$ we define $\Phi_j(\rho) = \rho^{n-1}\varphi_j(\rho)$, with φ_j as in (3.4), then, because $\langle\mu,\varphi_j\rangle = \omega_{n-1}\langle\nu,\Phi_j\rangle$, (3.4) yields the relation

$$\langle\mu,\varphi\rangle = \lim_{j\to\infty}\lim_{k\to\infty} \omega_{n-1}\int_0^\infty \Psi_k(\rho)\Phi_j(\rho)d\rho$$

$$= \lim_{j\to\infty} \omega_{n-1}\int_{(0,\infty)} \rho^{n-1}\varphi_j(\rho)d\nu(\rho).$$

We denote this last term by $\omega_{n-1}\int_{(0,\infty)} \rho^{n-1}\varphi(\rho)d\nu(\rho)$ since its value $\langle\mu,\varphi\rangle$ is independent of the sequence $\{\varphi_j\}$.

Finally, we prove the uniqueness of ν. Suppose ν_1 and ν_2 both give rise to (3.2).

If $\varphi \in C_c(\hat{\mathbb{R}}^n\backslash\{0\})$ we see that $\nu_1 = \nu_2$ on $\mathcal{E}$. Also $\nu_1-\nu_2$ is a continuous linear functional on $\mathcal{E}(K) = \{\Phi\in\mathcal{E}: \text{supp }\Phi \subseteq K,$ compact$\}$ and so it extends to a bounded measure ν on $C_c(K)$ having the same norm. Therefore, since $\nu_1 = \nu_2$ on $\mathcal{E}(K)$,

$\nu = \nu_1 - \nu_2$ is the zero measure on $C_c(K)$. It follows that $\nu_1 = \nu_2$ on $C_c(X)$ because $C_c(X) = \cup\{C_c(K) : K \subseteq X\}$. q.e.d.

Example 3.5. The assumption, $\mu(\{0\}) = 0$, is required in Proposition 3.4. To see this let $\mu = \delta$ and for simplicity of calculation take $n = 2$ and $\psi_j(\gamma) = (j^2/\pi)\chi_{B(0,1/j)}(\gamma)$. Then, for radial $\varphi \in C_c(\hat{\mathbb{R}}^2)$,

$$\varphi(0) = \langle\delta,\varphi\rangle = \lim_{j\to\infty} \langle\mu * \psi_j,\varphi\rangle =$$

$$\lim_{j\to\infty} \langle\psi_j,\varphi\rangle = \lim_{j\to\infty} \omega_1 \int_0^\infty \rho\psi_j(\rho)\varphi(\rho)d\rho,$$

and, of course, the right hand side is also seen to be $\varphi(0)$ by direct computation. The measure ν on $(0,\infty)$ must be 0 since, by definition of $\{\psi_j\}$, its support is forced to be the origin. Even if ν had $[0,\infty)$ as its domain it is forced to have the form $a\delta$. In either case the formula (3.2) fails when $\varphi(0) \neq 0$, e.g.,

$$0 \neq \varphi(0) = \langle\delta,\varphi\rangle \quad \text{and} \quad \omega_{n-1} \int_0^\infty \rho^{n-1}\varphi(\rho)d\delta(\rho) = 0,$$

where it does not matter if the domain of integration in the integral is $(0,\infty)$ or $[0,\infty)$.

4. A Fourier transform norm inequality on $\mathbb{R}^n$

Define

$$M_0(n) = \{f \in L^1(\mathbb{R}^n) : \text{supp } f \text{ is compact and } \hat{f}(0) = 0\}.$$

Theorem 4.1. Given radial $v \in L^1_{loc}(\mathbb{R}^n)$, $v > 0$ a.e., and radial $\mu \in M_+(\hat{\mathbb{R}}^n)$, $\mu(\{0\}) = 0$. Let $\nu \in M_+((0,\infty))$ denote the measure on

$(0,\infty)$ corresponding to μ (as in Proposition 3.4). Assume $1 < p \le q < \infty$ and $v^{1-p'} \in L^1_{loc}(\mathbb{R}^n \backslash B(0,y))$ for each $y > 0$. If

$$(4.1) \quad B_1 = \sup_{y>0} \left[\int_{(0,y)} \rho^{n-1+q} d\nu(\tfrac{\rho}{\pi}) \right]^{1/q} \left[\int_0^{1/y} r^{n-1+p'} v(r)^{1-p'} dr \right]^{1/p'} < \infty$$

and

$$(4.2) \quad B_2 = \sup_{y>0} \left[\int_{(y,\infty)} \rho^{n-1} d\nu(\tfrac{\rho}{\pi}) \right]^{1/q} \left[\int_{1/y}^{\infty} r^{n-1} v(r)^{1-p'} dr \right]^{1/p'} < \infty ,$$

then there is $C > 0$ such that for all $f \in M_0(n) \cap L^p_v(\mathbb{R}^n)$

$$(4.3) \qquad \|\hat{f}\|_{q,\mu} \le C\|f\|_{p,v}.$$

Furthermore, C can be chosen as

$$C = 2\omega_{n-1}^{\frac{1}{q}+\frac{1}{p'}} \pi^{-(n-1)/q} (p)^{1/q} (p')^{1/p'} (B_1+B_2).$$

The notation "$d\nu\left(\frac{\rho}{\pi}\right)$" signifies "$\frac{1}{\pi}\, \eta\left(\frac{\rho}{\pi}\right) d\rho$" in the case "$d\nu(\rho) = \eta(\rho)d\rho$".

Proof. Since $f \in M_0(n)$,

$$\hat{f}(\gamma) = -2i \int e^{-\pi i t\cdot\gamma} \sin(\pi t\cdot\gamma) f(t)dt.$$

If $\pi|t||\gamma| \le 1$ then $|\sin \pi t\cdot\gamma|/(\pi|t||\gamma|) \le 1$ since

$$\left|\frac{\pi t\cdot\gamma}{\pi|t||\gamma|} \; \frac{\sin \pi t\cdot\gamma}{\pi t\cdot\gamma}\right| \le 1.$$

Therefore, for a fixed $\gamma \ne 0$,

$$|\hat{f}(\gamma)| \le 2\pi|\gamma| \int_{\pi|t||\gamma|\le 1} |t||f(t)|dt + 2\int_{\pi|t||\gamma|>1} |f(t)|dt,$$

where the terms on the right hand side are radial functions. Consequently, by Minkowki's inequality, we estimate

$$\left(\int |\hat{f}(\gamma)|^q d\mu(\gamma)\right)^{1/q} \le 2\pi\left(\int \Big| \int_{|t|\le \frac{1}{\pi|\gamma|}} |t||f(t)|dt\Big|^q |\gamma|^q d\mu(\gamma)\right)^{1/q}$$

$$+ 2\left(\int \Big| \int_{|t|> \frac{1}{\pi|\gamma|}} |f(t)|dt\Big|^q d\mu(\gamma)\right)^{1/q} = 2\pi\ J_1 + 2J_2.$$

We use (3.1) to estimate J_1. Let $y = \pi|\gamma|$ and calculate

$$\int_{|t|\le\frac{1}{y}} |t||f(t)|dt = \int_{\Sigma_{n-1}} \int_0^{1/y} r^n |f(r\theta)|dr\ d\sigma_{n-1}(\theta)$$

$$= \int_{\Sigma_{n-1}} \int_y^{\infty} s^{-(n+2)} \left|f\left(\frac{\theta}{s}\right)\right| ds d\sigma_{n-1}(\theta)$$

$$= \int_y^{\infty} r^{-(n+2)} \left(\int_{\Sigma_{n-1}} \left|f\left(\frac{\theta}{r}\right)\right| d\sigma_{n-1}(\theta)\right) dr.$$

Therefore, by this calculation and <u>Proposition 3.4</u>,

$$J_1 = \left(\int \Big| \int_{\pi|\gamma|}^{\infty} r^{-(n+2)} \left(\int_{\Sigma_{n-1}} \left|f\left(\frac{\theta}{r}\right)\right| d\sigma_{n-1}(\theta)\right) dr\Big|^q |\gamma|^q d\mu(\gamma)\right)^{1/q}$$

(4.5)
$$= \left(\omega_{n-1} \int_{(0,\infty)} s^{n-1+q} \left(\int_{\pi s}^{\infty} r^{-(n+2)} \left(\int_{\Sigma_{n-1}} \left|f\left(\frac{\theta}{r}\right)\right| d\sigma_{n-1}(\theta)\right)^q d\nu(s)\right)^{\frac{1}{q}}.$$

Let $h(r)$ be the integrand whose domain is $(\pi s,\infty)$ in the last term of (4.5), replace $d\mu$ in <u>Theorem 1.1a</u> by $s^{n-1+q} d\nu(s)$, and make the change of variable $\gamma = \pi s$. Thus, by this theorem,

$$J_1 \le C_1 \omega_{n-1}^{1/q}\ \pi^{-(n-1+q)/q} \left(\int_0^{\infty} h(r)^p V(r)dr\right)^{1/q}$$

if

$$B_1 = \sup_{y>0} \left\{ \int_{(0,y)} \gamma^{n-1+q} d\nu\left(\frac{\gamma}{n}\right) \right\}^{1/q} \left\{ \int_y^\infty V(t)^{1-p'} dt \right\}^{1/p'} < \infty .$$

We now calculate V so that the inequality,

$$\int_0^\infty r^{-(n+2)p} \left(\int_{\Sigma_{n-1}} \left| f\left(\frac{\theta}{r}\right) \right| d\sigma_{n-1}(\theta) \right)^p V(r)dr \le \omega_{n-1}^{p/p'} \|f\|_{p,v}^p ,$$

is valid. The quantity $\|f\|_{p,v}^p$ on the right hand side of (4.6) is

$$\int_{\Sigma_{n-1}} \int_0^\infty r^{n-1} |f(r\theta)|^p v(r\theta) dr d\sigma_{n-1}(\theta)$$

$$= \int_0^\infty r^{n-1} \int_{\Sigma_{n-1}} |f(r\theta)|^p v(r\theta) d\sigma_{n-1}(\theta) dr$$

$$= \int_0^\infty s^{-(n+1)} \int_{\Sigma_{n-1}} \left| f\left(\frac{\theta}{s}\right) \right|^p v\left(\frac{\theta}{s}\right) d\sigma_{n-1}(\theta) ds$$

$$= \int_0^\infty s^{-(n+2)p} \left(\int_{\Sigma_{n-1}} \left| f\left(\frac{\theta}{s}\right) \right|^p d\sigma_{n-1}(\theta) \right) s^{(n+2)p-(n+1)} v\left(\frac{1}{s}\right) ds,$$

where we have used the hypothesis that v is radial. Comparing this last term with the left hand side of (4.6) we set

$$V(s) = v\left(\frac{1}{s}\right) s^{(n+2)p-(n+1)} ,$$

and we must show

$$\left\{ \int_0^\infty r^{-(n+2)p} \left\{ \int_{\Sigma_{n-1}} \left| f\left(\frac{\theta}{r}\right) \right| d\sigma_{n-1}(\theta) \right\}^p V(r)dr \right\}^{1/p} \le$$

$$(4.7) \qquad \omega_{n-1}^{1/p'} \left(\int_0^\infty r^{-(n+2)p} \left(\int_{\Sigma_{n-1}} \left| f\left(\frac{\theta}{r}\right) \right|^p d\sigma_{n-1}(\theta) \right) V(r)dr \right)^{1/p}$$

in order to prove (4.6) for this function V. To this end we temporarily write (4.7) as

$$\left(\int_0^\infty \left(\int_{\Sigma_{n-1}} g(r,\theta)d\sigma_{n-1}(\theta)\right)^p d\eta(r)\right)^{1/p} \le$$

(4.7′)

$$\omega_{n-1}^{1/p'}\left(\int_0^\infty \int_{\Sigma_{n-1}} g(r,\theta)^p d\sigma_{n-1}(\theta)d\eta(r)\right)^{1/p} .$$

By (generalized) Minkowski's inequality with $p > 1$, e.g., [HLP, Theorem 202], the left hand side of (4.7′) is dominated by

$$\int_{\Sigma_{n-1}} \left(\int_0^\infty g(r,\theta)^p d\eta(r)\right)^{1/p} d\sigma_{n-1}(\theta);$$

and so we need only show that

$$\int_{\Sigma_{n-1}} G(\theta)^{1/p} d\sigma_{n-1}(\theta) \le \omega_{n-1}^{1/p'}\left(\int_{\Sigma_{n-1}} G(\theta)d\sigma_{n-1}(\theta)\right)^{1/p}$$

and this is a consequence of the estimate,

$$\int_{\Sigma_{n-1}} G(\theta)^{1/p} d\sigma_{n-1}(\theta) \le \left(\int_{\Sigma_{n-1}} d\sigma_{n-1}(\theta)\right)^{1/p'}\left(\int_{\Sigma_{n-1}} G(\theta)d\sigma_{n-1}(\theta)\right)^{1/p} .$$

Thus, (4.6) is valid for $V(s) = v\left(\frac{1}{s}\right)s^{(n+2)p-(n+1)}$. Recall that the left hand side of (4.6) is $\int_0^\infty h(r)^p V(r)dr$ and so, by our application of <u>Theorem 1.1a</u> and definition of V, we obtain

$$(4.8) \qquad J_1 \le C_1\, \omega_{n-1}^{\frac{1}{q}+\frac{1}{p'}}\, n^{-(n-1+q)/q}\|f\|_{p,v}$$

when $B_1 < \infty$, where $B_1 \le C_1 \le B_1(p)^{1/q}(p')^{1/p'}$. Note that

$$B_1 = \sup_{y>0} \left[\int_{(0,y)} \rho^{n-1+q} d\nu\left(\frac{\rho}{\pi}\right)\right]^{1/q} \left[\int_y^\infty v\left(\frac{1}{s}\right)^{1-p'} s^{[(n+2)p-(n+1)](1-p')} ds\right]^{1/p'}$$

(4.9)

$$= \sup_{y>0} \left[\int_{(0,y)} \rho^{n-1+q} d\nu\left(\frac{\rho}{\pi}\right)\right]^{1/q} \left[\int_0^{1/y} r^{n-1+p'} v(r)^{1-p'} dr\right]^{1/p'}$$

since $(n+2)p-(n+1) = (n+1)(p-1) + p$ and $1-p' = -1/(p-1)$ implies $[(n+2)p-(n+1)](1-p') = -(n+1+p')$.

We now use (3.1) to estimate J_2. Let $y = \pi|\gamma|$ and calculate

$$\int_{|t|>\frac{1}{y}} |f(t)|dt = \int_{\Sigma_{n-1}} \int_{1/y}^\infty r^{n-1} |f(r\theta)| dr\, d\sigma_{n-1}(\theta) =$$

$$\int_{\Sigma_{n-1}} \int_0^y s^{-(n+1)} \left|f\left(\frac{\theta}{s}\right)\right| ds d\sigma_{n-1}(\theta) = \int_0^y r^{-(n+1)} \left[\int_{\Sigma_{n-1}} \left|f\left(\frac{\theta}{r}\right)\right| d\sigma_{n-1}(\theta)\right] dr.$$

Therefore, by this calculation and Proposition 3.4,

$$J_2 = \left[\int \left| \int_0^{\pi|\gamma|} r^{-(n+1)} \left[\int_{\Sigma_{n-1}} \left|f\left(\frac{\theta}{r}\right)\right| d\sigma_{n-1}(\theta)\right] dr \right|^q d\mu(\gamma)\right]^{1/q}$$

(4.10)

$$= \left[\omega_{n-1} \int_{(0,\infty)} s^{n-1} \left[\int_0^{\pi s} r^{-(n+1)} \left[\int_{\Sigma_{n-1}} \left|f\left(\frac{\theta}{r}\right)\right| d\sigma_{n-1}(\theta)\right] dr\right]^q d\nu(s)\right]^{\frac{1}{q}}.$$

Let $h(r)$ be the integrand whose domain is $(0,\pi s)$ in the last term of (4.5), replace $d\mu$ in Theorem 1.1b by $s^{n-1}d\nu(s)$, and make the change of variable $\gamma = \pi s$. Thus, by this theorem,

$$J_2 \le C_2 \omega_{n-1}^{1/q} \pi^{-(n-1)/q} \left[\int_0^\infty h(r)^p V(r) dr\right]^{1/q}$$

if

$$B_2 = \sup_{y>0} \left\{ \int_{(y,\infty)} \gamma^{n-1} d\nu\left(\frac{\gamma}{\pi}\right) \right\}^{1/q} \left[\int_0^y V(t)^{1-p'} dt \right]^{1/p'} < \infty .$$

We now calculate V so that the inequality,

$$(4.11) \qquad \int_0^\infty r^{-(n+1)p} \left[\int_{\Sigma_{n-1}} \left| f\left(\frac{\theta}{r}\right) \right| d\sigma_{n-1}(\theta) \right]^p V(r)\,dr \le \omega_{n-1}^{p/p'} \|f\|_{p,v}^p ,$$

is valid. The quantity $\|f\|_{p,v}^p$ on the right hand side of (4.11) is

$$\int_0^\infty s^{-(n+1)p} \left[\int_{\Sigma_{n-1}} \left| f\left(\frac{\theta}{s}\right) \right|^p d\sigma_{n-1}(\theta) \right] s^{(n+1)p-(n+1)} v\left(\frac{1}{s}\right) ds$$

by a calculation similar to that after (4.6) where, once again, we have used the hypothesis that v is radial. Comparing this term with the left hand side of (4.11) we set

$$V(s) = v\left(\frac{1}{s}\right) s^{(n+1)(p-1)} ,$$

and we must show

$$(4.12) \qquad \left\{ \int_0^\infty r^{-(n+1)p} \left[\int_{\Sigma_{n-1}} \left| f\left(\frac{\theta}{r}\right) \right| d\sigma_{n-1}(\theta) \right]^p V(r)\,dr \right\}^{1/p} \le$$

$$\omega_{n-1}^{1/p'} \left\{ \int_0^\infty r^{-(n+1)p} \left[\int_{\Sigma_{n-1}} \left| f\left(\frac{\theta}{r}\right) \right|^p d\sigma_{n-1}(\theta) \right] V(r)\,dr \right\}^{1/p}$$

in order to prove (4.11) for this function V. (4.12) follows by the same argument as that given after (4.7'). Consequently, (4.11) is valid for $V(s) = v\left(\frac{1}{s}\right) s^{(n+1)(p-1)}$. Recall that the left hand side of (4.11) is $\int_0^\infty h(r)^p V(r)\,dr$ and so, by our application of

Theorem 1.16 and definition of V, we obtain

$$(4.13) \qquad J_2 \le C_2\, \omega_{n-1}^{\frac{1}{q}+\frac{1}{p'}} \pi^{-(n-1)/q} \|f\|_{p,v}$$

when $B_2 < \infty$, where $B_2 \le C_2 \le B_2(p)^{1/q}(p')^{1/p'}$. Note that

$$B_2 = \sup_{y>0} \left[\int_{(y,\infty)} \rho^{n-1} d\nu\left[\frac{\rho}{\pi}\right]\right]^{1/q} \left[\int_0^y v\left[\frac{1}{s}\right]^{1-p'} s^{(n+1)(p-1)(1-p')} ds\right]^{1/p'}$$

$$= \sup_{y>0} \left[\int_{(y,\infty)} \rho^{n-1} d\nu\left[\frac{\rho}{\pi}\right]\right]^{1/q} \left[\int_{1/y}^{\infty} r^{n-1} v(r)^{1-p'} dr\right]^{1/p'}.$$

Combining our estimates, we have

$$\|\hat{f}\|_{q,\mu} \le 2\omega_{n-1}^{\frac{1}{q}+\frac{1}{p'}} \pi^{-(n-1)/q}(C_1+C_2)\|f\|_{p,v}$$

when $B_1 + B_2 < \infty$. By the above mentioned bounds on C_j in terms of B_j we obtain the desired bound for C. q.e.d.

The analogue of Theorem 2.2 is true, and the proof proceeds as follows.

As in Theorem 2.2a we let $h \in L_v^p(\mathbb{R}^n)' = L^{p'}_{v^{1-p'}}(\mathbb{R}^n)$ annihilate $M_0(n) \cap L_v^p(\mathbb{R}^n)$ and then check that $c_{n_1,\cdots,n_n} = 0$ for each $(n_1,\cdots,n_n) \in \mathbb{Z}^n\setminus\{0\}$, where

$$c_{n_1,\cdots,n_n} = \frac{1}{T^n} \int_{C_{T(n)}} \prod_{j=1}^{n} e_{n_j/T}(t_j)h(t)dt$$

and $C_{T(n)} = [-T/2,T/2]\times\cdots\times[-T/2,T/2]$ for fixed $T > 0$. (This generalization from $\mathbb{R}$ to $\mathbb{R}^n$ uses the function $\chi_{T(n)/2}(t) =$

$\prod_{j=1}^{n} \chi_{T_j/2}(t_j)$, $T_j = T > 0$ and $t = (t_1,\cdots,t_n)$; and, consequently,

$$\hat{\chi}_{T(n)/2}(\gamma) = T^n \prod_{j=1}^{n} \left(\frac{\sin \pi T\gamma_j}{\pi T\gamma_j}\right), \quad \gamma = (\gamma_1,\cdots,\gamma_n)$$

and

$$\forall (n_1,\cdots,n_n) \in \mathbb{Z}^n\setminus\{0\}, \ \left[\prod_{j=1}^{n} e_{n_j/T}\right]\chi_{T(n)/2} \in M_0(n) \cap L^p_v(\mathbb{R}^n).)$$

Next, let $h_{T(n)} = h$ on $C_{T(n)}$ and extend it periodically to $\mathbb{R}^n$. As in Theorem 2.2 we can show that $h_{T(n)} \in L^a(\mathbb{R}^n/T\mathbb{Z}^n)$, where $a = rp'/(p'-1+r)$ and our hypothesis is that $v \in L^r_{loc}(\mathbb{R}^n)$ for some $r > 1$. Because $a > 1$ we can assert that

$$\lim_{m\to\infty} \sum_{-m}^{m}\cdots\sum_{-m}^{m} c_{n_1,\cdots,n_n} e_{-n_1/T}(t_1)\cdots e_{-n_n/T}(t_n)$$

$$= h_{T(n)}(t) \quad \text{a.e. on } C_{T(n)} .$$

This result is due to C. Fefferman [F] and is a consequence of the Carleson-Hunt theorem, though not by iteration (or induction) as might be expected but by the proper decomposition of $\mathbb{Z}^n$. Therefore, since each of the coefficients except $c_{0,\cdots,0}$ vanishes, we obtain

(4.14) $$\forall\ T > 0,\ h(t) = \frac{1}{T^n}\int_{C_{T(n)}} h(u)du \quad \text{a.e. on } C_{T(n)}.$$

We use (4.14) in precisely the same way we used (2.8). As a result, we have proved -

Theorem 4.2 Given $v \in L^r_{loc}(\mathbb{R}^n)$ for some $r > 1$, where $v > 0$ a.e., and choose $p \in (1,\infty)$.

a. If $h \in L^p_v(\mathbb{R}^n)'$ annihilates $M_0(n) \cap L^p_v(\mathbb{R}^n)$ then h is a constant function.

b. $\overline{M_0(n) \cap L^p_v(\mathbb{R}^n)} = L^p_v(\mathbb{R}^n)$ or $L^p_v(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$.

c. If $v^{1-p'} \notin L^1(\mathbb{R}^n)$ then $\overline{M_0(n) \cap L^p_v(\mathbb{R}^n)} = L^p_v(\mathbb{R}^n)$.

Combining Theorem 4.1 and Theorem 4.2 we obtain -

Theorem 4.3 Given radial $v \in L^r_{loc}(\mathbb{R}^n)$ for some $r > 1$, where $v > 0$ a.e., and given radial $\mu \in M_+(\hat{\mathbb{R}}^n)$ for which $\mu(\{0\}) = 0$. Suppose $1 < p \le q < \infty$ and $v^{1-p'} \in L^1_{loc}(\mathbb{R}^n \backslash B(0,y)) \backslash L^1(\mathbb{R}^n)$ for each $y > 0$; and assume conditions (4.1) and (4.2) are valid.

a. If $f \in L^p_v(\mathbb{R}^n)$ then $\lim_{j\to\infty} \|f_j - f\|_{p,v} = 0$ for a sequence $\{f_j\} \subseteq M_0(n) \cap L^p_v(\mathbb{R}^n)$, and $\{\hat{f}_j\}$ converges in $L^q_\mu(\hat{\mathbb{R}}^n)$ to a function $\hat{f} \in L^q_\mu(\hat{\mathbb{R}}^n)$. $\hat{f}$ is independent of the sequence $\{f_j\}$ and it is called the Fourier transform of f.

b. There is $C > 0$ such that

$$\forall f \in L^p_v(\mathbb{R}^n), \quad \|\hat{f}\|_{q,\mu} \le C\|f\|_{p,v} ;$$

and C can be chosen as

$$C = 2\omega_{n-1}^{\frac{1}{q}+\frac{1}{p'}} \pi^{-(n-1)/q} (p)^{1/q} (p')^{1/p'} (B_1 + B_2).$$

5. Restriction theorems and the F(p,q,n) condition

Definition 5.1 a. Notationally, set $\Sigma_{n-1}(\rho) = \{\gamma \in \hat{\mathbb{R}}^n : |\gamma| = \rho\}$, and let μ_ρ be the restriction of σ_{n-1} to $\Sigma_{n-1}(\rho)$. μ_ρ is the positive measure corresponding to a uniformly distributed mass on the sphere $\Sigma_{n-1}(\rho)$ with surface or (n-1)-dimensional density (= mass divided by surface area) equal to 1.

b. Fix $\rho > 0$ and let $\chi = \chi_{[-1/2,1/2)}$. $\delta(|\gamma|-\rho) \in M_{b+}(\hat{\mathbb{R}}^n)$ is the $\sigma(M_b(\hat{\mathbb{R}}^n), C_0(\hat{\mathbb{R}}^n))$ limit,

$$\lim_{\tau \to 0} \frac{1}{\tau} \chi\left[\frac{|\gamma|-\rho}{\tau}\right],$$

and $\delta(|\gamma|-\rho) = \mu_\rho$. The mass of $\delta(|\gamma|-\rho)$ is "$\int \delta(|\gamma|-\rho)d\gamma$" =

$$\lim_{\tau\to 0} \frac{1}{\tau} \int \chi\left[\frac{|\gamma|-\rho}{\tau}\right] d\gamma = \lim_{\tau\to 0} \frac{\omega_{n-1}}{\tau} \int_0^\infty \beta^{n-1} \chi\left[\frac{\beta-\rho}{\tau}\right] d\beta$$

$$= \lim_{\tau\to 0} \frac{\omega_{n-1}}{\tau} \int_{p-\frac{\tau}{2}}^{p+\frac{\tau}{2}} \beta^{n-1} d\beta = \lim_{\tau\to 0} \frac{\omega_{n-1}}{n\tau} \sum_{k=0}^{n} (1-(-1)^k) \begin{bmatrix} n \\ k \end{bmatrix} \rho^{n-k} \left[\frac{\tau}{2}\right]^k$$

$$= \lim_{\tau\to 0} \frac{\omega_{n-1}}{n\tau} 2 \begin{bmatrix} n \\ 1 \end{bmatrix} \rho^{n-1} \left[\frac{\tau}{2}\right] = \rho^{n-1} \omega_{n-1}.$$

Consequently, one easily checks that the surface density of $\delta(|\gamma|-\rho) = 1$. For example, if $n = 2$ then this calculation gives "$\int \delta(|\gamma|-\rho)d\gamma$" $= 2\pi\rho$; and since the length of $\Sigma_1(\rho)$ is $2\pi\rho$ we see that the linear density of $\delta(|\gamma|-\rho)$ is 1.

c. The measure ν on $(0,\infty)$ (from Proposition 3.4) corresponding to μ_ρ is δ_ρ; and if $\varphi \in C_c(\hat{\mathbb{R}}^n)$ is radial then $\int \varphi(\gamma) d\mu_\rho(\gamma) = \omega_{n-1} <\delta_\rho(\beta), \beta^{n-1}\varphi(\beta)>$. To see this, first note that $\mathrm{supp}\, \nu = \{\rho\}$ since $\int \varphi(\gamma)d\mu_\rho(\gamma) = 0$ if $\mathrm{supp}\, \varphi \cap \mathrm{supp}\, \mu_\rho = \phi$. Thus, $\nu = c\delta_\rho$. Then letting $\varphi = 1$ and applying part b we have $\rho^{n-1}\omega_{n-1} = \int d\mu_\rho(\gamma) = \omega_{n-1} <c\delta_\rho(\beta), \beta^{n-1}> = c\rho^{n-1}\omega_{n-1}$; and so $c = 1$.

Definition 5.2 Given radial $v \in L^1_{loc}(\mathbb{R}^n)$, $v \geq 0$, and suppose $1 < p \leq q < \infty$. The (L^p_v, L^q) spherical restriction property with constant $C(p,q,\rho)$ holds for $\mathbb{R}^n$ if there is a subspace $M \subseteq L^1(\mathbb{R}^n) \cap L^p_v(\mathbb{R}^n)$ for which $\bar{M} = L^p_v(\mathbb{R}^n)$ and if

$$\forall \rho > 0 \quad \exists C(p,q,\rho) \quad \forall f \in M,$$

$$\left(\int_{\Sigma_{n-1}(\rho)} |\hat{f}(\gamma)|^q d\sigma_{n-1}(\gamma) \right)^{1/q} \leq C(p,q,\rho) \|f\|_{p,v},$$

cf., [St, pp.108-109].

Theorem 5.3 Given radial $v \in L^1_{loc}(\mathbb{R}^n)$, $v > 0$ a.e., and suppose $1 < p \leq q < \infty$. Assume $v^{1-p'} \in L^1_{loc}(\mathbb{R}^n \setminus B(0,y))$ for each $y > 0$ and set

$$C(p,q,\rho) = 2\omega_{n-1}^{\frac{1}{q}+\frac{1}{p'}} (p)^{1/q}(p')^{1/p'} \rho^{\frac{n-1}{q}} \left[\rho\pi \left(\int_0^{1/(\rho\pi)} r^{n-1+p'} v(r)^{1-p'} dr \right)^{1/p'} + \left(\int_{1/(\rho\pi)}^{\infty} r^{n-1} v(r)^{1-p'} dr \right)^{1/p'} \right].$$

Then for all $\rho > 0$ and for all $f \in M_0(n) \cap L^p_v(\mathbb{R}^n)$,

$$\left(\int_{\Sigma_{n-1}(\rho)} |\hat{f}(\gamma)|^q d\sigma_{n-1}(\gamma) \right)^{1/q} \leq C(p,q,\rho) \|f\|_{p,v}.$$

Proof. The proof is a direct application of Theorem 4.1. If ν is the measure on $(0,\infty)$ corresponding to μ_ρ then, by Definition 5.1c, $\nu = \delta_\rho$. In particular,

(5.1) $$\forall y < \rho\pi, \quad \int_{(0,y)} \beta^{n-1+q} d\nu\left(\frac{\beta}{\pi}\right) = 0$$

and

$$(5.2) \qquad \forall y > \rho\pi, \quad \int_{(y,\infty)} \beta^{n-1} d\nu\left[\frac{\beta}{\pi}\right] = 0;$$

for example, $y > \rho\pi$ and $\beta > y$ imply $\beta/\pi > \rho$ and so "$\delta_\rho(\beta/\pi) = 0$" since $\delta_\rho = 0$ on (ρ,∞).

Let $B_j = \sup_{y>0} B_j(y)$, $j = 1,2$, in order to apply Theorem 4.1.
By (5.1), $B_1 = \sup_{y\ge\rho\pi} B_1(y)$, and, for $y \ge \rho\pi$,

$$B_1(y) \le \left(\int_{(0,\infty)} \beta^{n-1+q} d\nu\left[\frac{\beta}{\pi}\right]\right)^{1/q} \left[\int_0^{1/y} r^{n-1+p'} v(r)^{1-p'} dr\right]^{1/p'} =$$

$$(\rho\pi)^{(n-1+q)/q} \left[\int_0^{1/y} r^{n-1+p'} v(r)^{1-p'} dr\right]^{1/p'}.$$

Thus, we have

$$B_1 \le (\rho\pi)^{(n-1+q)/q} \left[\int_0^{1/(\rho\pi)} r^{n-1+p'} v(r)^{1-p'} dr\right]^{1/p'}.$$

By (5.2), $B_2 = \sup_{y\le\rho\pi} B_2(y)$, and, for $y \le \rho\pi$,

$$B_2(y) \le (\rho\pi)^{(n-1)/q} \left[\int_{1/y}^{\infty} r^{n-1} v(r)^{1-p'} dr\right]^{1/p'}.$$

Thus, we have

$$B_2 \le (\rho\pi)^{(n-1)/q} \left[\int_{1/(\rho\pi)}^{\infty} r^{n-1} v(r)^{1-p'} dr\right]^{1/p'}.$$

The result in obtained by substituting these bounds for the B_j into Theorem 4.1. q.e.d.

This result can also be proved using a more classical form of Theorem 4.1 where μ_ρ and ν are replaced by approximants such as defined in Definition 5.1b. Then standard real variable methods including Fatou's lemma and the fundamental theorem of calculus (Lebesgue's differentiation theorem) yield the result.

Corollary 5.4 Given $v(r) = r^{\alpha}$ and $1 < p \le q < \infty$. Assume

$$\frac{n}{p'-1} < \alpha < \frac{p'+n}{p'-1} .$$

Then the (L^p_v, L^q) spherical restriction property with constant

$$C(p,q,\rho) = 2\omega_{n-1}^{\frac{1}{q}+\frac{1}{p'}} (p)^{1/q} (p')^{1/p'} \rho^{\frac{n-1}{q}} (\rho\pi)^{-[n+\alpha(1-p')/p'} \left[\left(\frac{(-1)}{n+\alpha(1-p')} \right)^{1/p'} + \left(\frac{1}{n+\alpha(1-p')+p'} \right)^{1/p'} \right]$$

holds for $\mathbb{R}^n$.

Proof. a. For this weight v the integrals used to define $C(p,q,\rho)$ (in Theorem 5.3) are

$$\int_0^{1/(\rho\pi)} r^{n-1+p'+\alpha-\alpha p'} dr = \frac{-1}{n+\alpha+p'(1-\alpha)} \left(\frac{1}{\rho\pi} \right)^{n+\alpha+p'(1-\alpha)}$$

and

$$\int_{1/(\rho\pi)}^{\infty} r^{n-1+\alpha-\alpha p'} dr = \frac{-1}{n+\alpha(1-p')} \left(\frac{1}{\rho\pi} \right)^{n+\alpha(1-p')} ,$$

respectively, where the first integral requires $n + \alpha + p'(1-\alpha) > 0$ and the second requires $n + \alpha(1-p') < 0$. Combining these inequalities gives the stated interval of α values.

b. It remains to check the local integrability hypothesis and to find the appropriate dense subspace M. First, $v^{1-p'} \in L^1_{loc}(\mathbb{R}^n \backslash B(0,y))$ for each $y > 0$ since

$$(5.3)\qquad \int_{B(0,y)^{\sim}\cap B(0,b)} |x|^{\alpha(1-p')}dx = \omega_{n-1}\int_y^b r^{n-1+\alpha(1-p')}dr < \infty.$$

Second, set $M = M_0(n) \cap L_v^p(\mathbb{R}^n)$. Clearly, $v \in L^{\infty}_{loc}(\mathbb{R}^n)$ so that we need only check that $v^{1-p'} \notin L^1(\mathbb{R}^n)$ in order to apply Theorem 4.2. The non-integrability is immediate since the right hand integral of (5.3) with $y = 0$ and $b = \infty$ is

$$\lim_{r\to 0+} \frac{1}{|n+\alpha(1-p')|} r^{n+\alpha(1-p')} = \infty$$

since $n + \alpha(1-p') < 0$. q.e.d.

Zygmund [Z] was among the first to verify the spherical restriction property for the case $v = 1$.

Definition 5.5a. Given $v \in L^1_{loc}(\mathbb{R}^n)$, $v \geq 0$, and $\mu \in M_+(\hat{\mathbb{R}}^n)$; and assume $p > 1$ and $q \geq 1$. The pair μ, v satisfies the $F(p,q,n)$ condition, written $(\mu, v) \in F(p,q,n)$, if

$$F(p,q,n)\qquad B = \sup_{y>0}\left(\int_{B(0,y)} d\mu(\gamma)\right)^{1/q}\left(\int_{B(0,1/y)} v(x)^{1-p'}dx\right)^{1/p'} < \infty,$$

cf., Remark 2.4c for the 1-dimensional even case. If μ and v are radial with $\mu(\{0\}) = 0$ then

$$B = \omega_{n-1}^{\frac{1}{q}+\frac{1}{p'}} \sup_{y>0}\left(\int_{(0,y]} \gamma^{n-1}d\nu(\gamma)\right)^{1/q}\left(\int_0^{1/y} t^{n-1}v(t)^{1-p'}dt\right)^{1/p'}.$$

b. If $1 < p \leq q < \infty$ and $(\mu, v) \in F(p,q,n)$ for "$d\mu(\gamma) = u(\gamma)d\gamma$", where $u \in L^1_{loc}(\hat{\mathbb{R}}^n)$ and $v \in L^1_{loc}(\mathbb{R}^n)$ are radial and where $u(|\gamma|)$ and $1/v(|\gamma|)$ are decreasing on $(0,\infty)$, then $\|\hat{f}\|_{q,u} \le C\|f\|_{p,v}$, C being independent of f, e.g., [H; JS; Mu2]. (Strictly speaking, this result requires that the intervals $(0,y]$ and $(0,1/y]$ in Definition 5.5a be modified in terms of the volume of the unit n-sphere. However, for most weights the $F(p,q,n)$ condition yields the result as stated.)

Now consider the growth condition

$$\forall \rho > 0, \quad \mu\{\rho < |\gamma| \le 2\rho\} \le A\rho^{a}, \tag{5.4}$$

where $\mu \in M_+(\hat{\mathbb{R}}^n)$ and $\mu(\{0\}) = 0$ and where $a(p,q,n) = a = qn/p'$ for $p > 1$ and $q \ge 1$. If $n \ge 2$, $p > 1$, and $q = \left(\frac{n-1}{n+1}\right)p'$ then

$$\frac{n(n-1)}{n+1} = \frac{nq}{p'} . \tag{5.5}$$

In particular, $a(p,q,1)$ gives non-zero meaning to the left hand side of (5.5) for the case $n = 1$.

Proposition 5.6 Given radial $\mu \in M_+(\hat{\mathbb{R}}^n)$, $n \ge 2$, for which $\mu(\{0\}) = 0$. $(\mu,1) \in F(p,q,n)$, where $p > 1$ and $q \ge 1$, if and only if the inequality (5.4) is satisfied.

Proof. Note that

$$\mu\{\rho<|\gamma|\le 2\rho\} = \omega_{n-1} \int_{(\rho,2\rho]} r^{n-1}d\nu(\gamma)$$

and

$$\left(\int_0^{1/y} x^{n-1}dx\right)^{q/p'} = n^{-q/p'}y^{-nq/p'} .$$

Suppose $(\mu,1) \in F(p,q,n)$. Then

$$\mu\{\rho<|\gamma|\le 2\rho\} \le B^q\,\omega_{n-1}^{-q/p'}\left(\int_0^{1/(2\rho)} t^{n-1}dt\right)^{-q/p'} = B^q\omega_{n-1}^{-q/p'}n^{q/p'}2^{nq/p'}\rho^{nq/p'},$$

and so we obtain the inequality (5.4) with $A = B^q(n2^n/\omega_{n-1})^{q/p'}$.

For the converse, let $B = \sup_{y>0} B(y)$ so that

$$B(y)^q = \omega_{n-1}^{1+\frac{q}{p'}} n^{-q/p'}y^{-nq/p'}\int_{(0,y]}\gamma^{n-1}d\nu(\gamma) =$$

$$(ny^n/\omega_{n-1})^{-q/p'}\sum_{j=0}^{\infty}\mu\left\{\frac{y}{2^{j+1}} < |\gamma| \le \frac{y}{2^j}\right\} \le$$

$$A(ny^n/\omega_{n-1})^{-q/p'}\sum_{j=0}^{\infty}\left(\frac{y}{2^{j+1}}\right)^{nq/p'} = \frac{A\omega_{n-1}^{q/p'}}{n^{q/p'}(2^{nq/p'}-1)} = B^q.$$

q.e.d.

The following is a consequence of Proposition 5.6 and Definition 5.5b.

Corollary 5.7 Given radial $u \in L^1_{loc}(\hat{\mathbb{R}}^n)$, $n \ge 2$ and $u \ge 0$ (with corresponding radial $\mu \in M_+(\hat{\mathbb{R}}^n)$ defined by "$d\mu(\gamma) = u(\gamma)d\gamma$") and suppose μ satisfies (5.4). Assume $\mu(|\beta|)$ is a decreasing function on $(0,\infty)$.

a. If $1 < p \le q < \infty$ then there is $C > 0$ such that

(5.6) $$\forall f \in L^p(\mathbb{R}^n),\ \|\hat{f}\|_{q,\mu} \le C\|f\|_p.$$

b. If

$$1 < p \le 2n/(n+1) \quad\text{and}\quad q = \left(\frac{n-1}{n+1}\right)p'$$

then $p \le q$ and so part a applies.

The proof is clear except for noting, in part b, that, for q so defined, $q \geq p$ if and only if $\left(\frac{n-1}{n+1}\right) \frac{p}{p-1} \geq p$ if and only if $2n/(n+1) \geq p$.

As an example for Corollary 5.7, let $u(|\beta|) = |\beta|^{-\frac{2n}{n+1}}$. Clearly, $u \in L^1_{loc}(\hat{\mathbb{R}}^n)$, and therefore it defines a positive (in fact, $u \geq 0$) measure μ for which $\mu(\{0\}) = 0$. Also, $(\mu,1) \in \mathbf{F}(p,q,n)$ since $q = \left(\frac{n-1}{n+1}\right) p'$.

Remark 5.8. Assuming (5.4), Christ [C] proved (5.6) for radial measures $\mu \in M_+(\hat{\mathbb{R}}^n)$, $n > 2$, in the range $1 < p \leq 2(n+1)/(n+3)$ and $q = ((n-1)/(n+1))p'$. This can be compared with Corollary 5.7 where we are restricted to decreasing functions u but where the range of values p is larger (clearly, $2n/(n+1) > 2(n+1)/(n+3)$). Christ also showed that (5.4) is a necessary condition for (5.6).

The condition,

$$(5.7) \quad \sup_{y>0} \left(\int_{B(0,y)^\sim} |\gamma|^{-q} d\mu(\gamma) \right)^{1/q} \left(\int_{B(0,1/y)^\sim} |x|^{-p'} v(x)^{1-p'} dx \right)^{1/p'} < \infty,$$

also arises in Fourier transform norm inequalities. It corresponds to (4.2) in the same way that $(\mu,v) \in \mathbf{F}(p,q,n)$ corresponds to (4.1). If $n = 1$, Proposition 5.6 and (5.7) lead to the following relationship for non-symmetric $\mu \in M_+(\hat{\mathbb{R}})$.

Proposition 5.9 Given $\mu \in M_+(\hat{\mathbb{R}})$ for which $\mu(\{0\}) = 0$. $(\mu,1) \in \mathbf{F}(p,q,1)$, where $p > 1$ and $q \geq 1$, if and only if (5.4) is satisfied. Also, (5.4) implies (5.7) for $v = 1$, and, so, (5.7) for $v = 1$ is a consequence of the hypothesis $(\mu, 1) \in F(p,q,1)$.

<u>Proof</u>. Assume $(\mu,1) \in F(p,q,1)$. Then

$$\mu\{\rho < |\gamma| \leq 2\rho\} = \int_{\rho<|\gamma|\leq 2\rho} d\mu(\gamma) \leq B^q \left(\int_{B(0,1/(2\rho))} dx \right)^{-q/p'} = B^q \rho^{q/p'}$$

and so we obtain (5.4) as in the first part of <u>Proposition 5.6</u>.

For the converse, since $\mu(\{0\}) = 0$ we have

$$B(y)^q = \left(\frac{2}{y}\right)^{q/p'} \sum_{j=0}^{\infty} \mu\left\{\frac{y}{2^{j+1}} < |\gamma| \leq \frac{y}{2^j}\right\} ;$$

so we obtain $F(p,q,1)$ from (5.4) as in the second part of

<u>Proposition 5.6</u>.

Finally, we show that (5.4) implies (5.7) when $v = 1$. In fact,

$$\left(\int_{B(0,1/y)^{\sim}} |x|^{-p'} \right)^{q/p'} \int_{B(0,y)^{\sim}} |\gamma|^{-q} d\mu(\gamma) \leq$$

$$\left(\frac{2}{p'-1}\right)^{q/p'} y^{q/p} \sum_{j=0}^{\infty} \int_{2^j y \leq |\gamma| \leq 2^{j+1} y} |\gamma|^{-q} d\mu(\gamma) \leq$$

$$\left(\frac{2}{p'-1}\right)^{q/p'} y^{q/p} \sum_{j=0}^{\infty} (2^j y)^{-q} A(2^j y)^{q/p'} =$$

$$\frac{A2^q}{(p'-1)^{q/p'}} \frac{1}{(2^{q/p}-1)} .$$

<u>q.e.d.</u>