ABSTRACT

Title of Thesis: Twisted Cohomology Groups

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We introduce the notion of a Twisted Cohomology group. Additionally, certain examples of these groups and their implications are discussed.

Twisted Cohomology Groups

by

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Thesis submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Master of Arts
2006

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ACKNOWLEDGMENTS

I would like to thank all of my professors and colleagues in the Department of Mathematics. I would especially like to thank my advisor, Professor Jonathan Rosenberg for, not only providing the opportunities to learn truly interesting mathematics, but also for his long standing support and incredible patience, particularly within these final months. I am extremely fortunate to have worked with and learned from such an extraordinary mathematician and remarkable individual. I would also like to send a special thanks to my committee members, Professor William Goldman and Professor John Millson; and to Professor Konstantina Trivisa.

I would also like to thank the administrative professionals in the Department of Mathematics, particularly Patty Woodwell and Anita Dahms.

I would also like to acknowledge University of Maryland's PROMISE program. The many workshops, mentoring program, unyielding support of Dr. Johnetta Davis and Dr. Renetta Tull and my amazing mentor Angela Grant really helped me navigate my way through the Mathematics program.

Had it not been for the special friendship and support of Shari Wiley, Chad Sprouse, Matthew Hirn and Robert Delgado, I may not have survived this journey. We may have been introduced through mathematics but our bond will last a lifetime.

Finally and supremely, I owe my undying gratitude to God and to my parents.

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Chapter 1

Introduction

Supersymmetric Quantum Mechanics is, by definition, a Quantum Mechanics equipped with a positive definite space of energy states $\mathcal{H} = \mathcal{H}^B \oplus \mathcal{H}^F$, where one can think of $\mathcal{H}^B$ as the collection of states affected by the bosonic fields and $\mathcal{H}^F$ as the collection of states affected by the fermionic fields. Supercharge operators, Q, map objects between $\mathcal{H}^B$ and $\mathcal{H}^F$ and produce a graded $Q-$complex on $\mathcal{H}$. One of the primary objectives of Supersymmetric Quantum Mechanics is the determination of Supersymmetric Ground States, or zero energy states. In particular, we seek ground states that preserve supersymmetric transformations. Often, it necessary to deform the Lagrangian, by adding a constant function, called the superpotential function. Adding this superpotential function produces a modification of the supersymmetry transformations and supercharge actions. Without adding this superpotential function, we can only describe classes of information about our ground states. The additional structure provides us with more specialized information about these ground states. This gives rise to the notion of twisted cohomology group. In particular, $\mathcal{H}$ represents an algebra of differential forms, $\Omega(M)$, and Q can be identified with a differential operator, d, on $\Omega(M)$. If $\varphi : M \to \mathbb{R}$ represents our superpotential function, our modified supercharge operator, which we'll call $\widetilde{Q}$, defines a differential d_φ on $\Omega(M)$ by letting $d_\varphi = d + d\varphi\wedge$ where d is the usual differential on our manifold. Since $\widetilde{Q}$ is related to Q by the transformation $\widetilde{Q} = e^{\varphi}Qe^{-\varphi}$, the graded complex $\widetilde{Q}-$complex is isomorphic to the graded $Q-$complex. However if $d\varphi$ is replaced by a closed $1-$form ω that is not exact, then the graded $\widetilde{Q}-$complex is actually different from the usual $Q-$ complex. We then say that the added $\omega\wedge$ is a twisting of the usual differential complex thereby producing twisted cohomology groups, the topic of this paper.

One can relate the twisted cohomology group to purchasing a used car. On the lot, two cars can seem identical. However, a scan of each car's odometer can reveal that one car has more milage than the other. Further inspection might reveal a scratch on the dashboard or a misalignment of the tires. In the end, it is determined that the two cars are, in fact, different. Similarly, the cohomology group provides a classification of various spaces while the twisting gives us a little more information. Moreover, the more complex the twisting, the more additional information can be obtained. There has been a fair amount of discussion of the twisted cohomology group for the case where the differential complex is twisted by a closed 3-form. While twisting by a 3-form is quite useful, because of a connection (via the Chern character) to twisted K-theory, this involves more advanced techniques, and so we won't go into it any further.

There are two main objectives to this paper. The first objective is to introduce a more general twisted cohomology group. That is, we will prove the existence of a twisted cohomology group where the complex is twisted is by an arbitrary closed

$(2k+1)-$form. The second objective is to provide a proof of various properties of this twisted cohomology group without using the advanced techniques used by our predecessors.

Chapter 2

Preliminaries

2.1 De Rham Cohomology

Recall that an $n-$manifold is a topological space that locally looks like $\mathbb{R}^n$. One can think of an $n-$manifold to be little pieces of $\mathbb{R}^n$ glued together by homeomorphisms. A $n-$manifold is smooth (or differentiable) if the homeomorphisms are diffeomorphisms. Suppose M is a smooth $n-$manifold with local coordinates $x_1, \ldots, x_n$. Consider the cotangent space $T^*(M)$ with local basis $dx_1, \ldots, dx_n$. We can define a bundle of algebras over M, $\bigoplus_q \Lambda^q(T^*(M))$, to be the quotient space of the tensor algebra bundle on $T^*(M)$ modulo the ideal generated by the symmetric tensors. For example,

$$\Lambda^2(T^*(M)) = T^*(M) \otimes T^*(M) / \left(dx_{\pi(1)} \otimes dx_{\pi(2)} + dx_{\pi(2)} \otimes dx_{\pi(1)} \right).$$

$\Lambda^q(T^*(M))$ has basis $\left\{ dx_\pi = dx_{\pi(1)} \wedge \cdots \wedge dx_{\pi(q)} \mid \pi(1) < \cdots < \pi(q) \right\}$ and maintains the relations:

$$dx_{\pi(i)} \wedge dx_{\pi(j)} = -dx_{\pi(i)} \wedge dx_{\pi(j)} \qquad i \neq j$$

$$dx_{\pi(i)} \wedge dx_{\pi(i)} = 0$$

Definition 1. Let $\Omega^q(M)$ denote the space of smooth sections of $\Lambda^q(T^*(M))$. A differential q-form ω is an element of $\Omega^q(M)$ and is written as

$$\omega = \sum f_\pi dx_\pi$$

where $f_\pi : M \to \mathbb{R}$ is a sufficiently smooth function on M. We then call $\Omega^q(M)$ the vector space of all differential q-forms.

Observe that $\Omega^0(M) = M$ and $\Omega^1(M) = T^*M$. The wedge product of two differential forms is a bilinear map $\Omega^p(M) \times \Omega^q(M) \to \Omega^{p+q}(M)$ defined by $(\omega, \eta) \mapsto \omega \wedge \eta$ such that if $\omega = \sum f_\pi dx_\pi$ and $\eta = \sum f_\sigma dx_\sigma$,

$$\omega \wedge \eta = \sum_{\pi,\sigma} f_\pi g_\sigma dx_\pi \wedge dx_\sigma$$

Proposition 1. *Suppose $\omega \in \Omega^p(M)$, $\eta \in \Omega^q(M)$ and $\nu \in \Omega^r(M)$ are differential forms. Then the following properties are satisfied:*

1. *(Associativity)* $(\omega \wedge \eta) \wedge \nu = \omega \wedge (\eta \wedge \nu)$
2. *(Distributivity)* $\omega \wedge (\eta + \nu) = \omega \wedge \eta + \omega \wedge \nu$
3. *(Anticommutativity)* $\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega$

Example 1. Let $M = \mathbb{R}^3$ and suppose

$$\omega = xdx - ydy$$

$$\nu = zdx \wedge dy + xdy \wedge dz$$

$$\eta = zdy$$

Then $\omega \wedge \nu \wedge \eta = 0$ since

$$\begin{aligned}
\omega \wedge \nu &= (xdx - ydy) \wedge (zdx \wedge dy + xdy \wedge dz) \\
&= xdx \wedge (zdx \wedge dy + xdy \wedge dz) - ydy \wedge (zdx \wedge dy + xdy \wedge dz) \\
&= xdx \wedge zdx \wedge dy + xdx \wedge xdy \wedge dz - ydy \wedge zdx \wedge dy - ydy \wedge xdy \wedge dz \\
&= xzdx \wedge dx \wedge dy + x^2 dx \wedge dy \wedge dz + yzdx \wedge dy \wedge dy - xydy \wedge dy \wedge dz \\
&= x^2 dx \wedge dy \wedge dz
\end{aligned}$$

So

$$\omega \wedge \nu \wedge \eta = (\omega \wedge \nu) \wedge \eta = (x^2 dx \wedge dy \wedge dz) \wedge z dy = 0.$$

Definition 2. Suppose ω is a differential q-form. The exterior derivative d acts on ω in the following manner:

$$\omega = \sum f_\pi dx_\pi \quad \Longrightarrow \quad d\omega = \sum df_\pi \wedge dx_\pi$$

where $df_\pi = \sum_k \partial f_\pi / \partial x_k dx_k$.

From this definition, we see that the exterior derivative gives a linear map

$$d : \Omega^q(M) \to \Omega^{q+1}(M).$$

Proposition 2. *The exterior derivative satisfies the following properties:*

1. *if $\omega, \nu \in \Omega^q(M)$, then $d(\omega + \nu) = d\omega + d\nu$.*

2. *if $\omega \in \Omega^q(M)$ and $\eta \in \Omega^p(M)$, then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^q(\omega \wedge d\eta)$.*

3. *$d^2 = 0$.*

Proof.

1. This follows directly from Proposition 1(2) and the fact that, for any differentiable functions f and g on M, $d(f + g) = df + dg$.

In particular, if $\omega = \sum f_\pi dx_\pi$ and $\nu = \sum g_\sigma dx_\sigma$, then

$$\begin{aligned} d(\omega + \nu) &= d\left(\sum f_\pi dx_\pi + \sum g_\sigma dx_\sigma\right) \\ &= d\left(\sum_{\pi,\sigma} f_\pi dx_\pi + g_\sigma dx_\sigma\right) \\ &= \sum_\pi df_\pi \wedge dx_\pi + \sum_\sigma dg_\sigma \\ &= d\omega + d\nu \end{aligned}$$

2. Define ω as above and suppose $\eta = \sum h_\rho dx_\rho \in \Omega^p(M)$

$$\begin{aligned} d(\omega \wedge \nu) &= d\left(\sum f_\pi dx_\pi \wedge \sum h_\rho dx_\rho\right) \\ &= d\left(\sum_{\pi,\rho} f_\pi h_\rho dx_\pi \wedge dx_\rho\right) \\ &= \sum_{\pi,\rho} (d(f_\pi) \wedge h_\rho + f_\pi \wedge d(h_\rho)) \wedge (dx_\pi \wedge dx_\rho) \\ &= d\omega \wedge \eta + (-1)^p \omega \wedge (d\eta) \qquad \text{by Proposition 1.} \end{aligned}$$

3. Define ω as above. Then

$$d^2\omega = d(d\omega) = d\left(d \sum df_\pi \wedge dx_\pi\right) = d\left(\sum_k^n \frac{\partial f}{\partial x_{\pi(k)}} dx_{\pi(k)} \wedge dx_\pi\right)$$

Since $dx_i \wedge dx_i = 0$,

$$d^2\omega = \sum_{i<j}^n \frac{\partial^2 f}{\partial x_{\pi(i)} \partial x_{\pi(j)}} dx_{\pi(i)} \wedge dx_{\pi(1)} \wedge \cdots \wedge \widehat{dx}_{\pi(i)} \wedge \cdots \wedge \widehat{dx}_{\pi(j)} \wedge \cdots \wedge dx_{\pi(n)}$$

where $\widehat{dx}_{\pi(k)} = dx_{\pi(k-1)} \wedge dx_{\pi(k+1)}$.

Since f is sufficiently smooth, the mixed partials agree so, for each i and j, we obtain two copies of $\partial^2 f/(\partial x_{\pi(i)} \partial x_{\pi(j)})$. However since $dx_{\pi(i)} \wedge dx_{\pi(j)} = -dx_{\pi(j)} \wedge dx_{\pi(i)}$, these terms cancel and $d^2\omega = 0$.

$\square$

Example 2. Consider ω and ν from Example 1.

$$\begin{aligned}
d(\omega \wedge \nu) &= d\omega \wedge \nu + (-1)^2 \omega \wedge d\nu \\
&= d(xdx - ydy) \wedge (zdx \wedge dy + xdy \wedge dz) \\
&\qquad + (xdx - ydy) \wedge d(zdx \wedge dy + xdy \wedge dz) \\
&= [d(xdx) - d(ydy))] \wedge (zdx \wedge dy + xdy \wedge dz) \\
&\qquad + (xdx - ydy) \wedge [d(zdx \wedge dy) + d(xdy \wedge dz)] \\
&= (dx \wedge dx - dy \wedge dy) \wedge (zdx \wedge dy + xdy \wedge dz) \\
&\qquad + (xdx - ydy) \wedge (dz \wedge dx \wedge dy + dx \wedge dy \wedge dz) \\
&= 0 + (xdx - ydy) \wedge (2dx \wedge dy \wedge dz) \\
&= 0
\end{aligned}$$

This is an expected result since $d : \Omega^p(M) \to \Omega^{p+1}(M)$. In particular, $M = \mathbb{R}^3$ so there are no nontrivial forms of degree > 3. Since $\omega \wedge \nu$ is a 3-form, $d(\omega \wedge \nu)$ would be a 4-form. Hence $d(\omega \wedge \nu)$ must equal zero.

The naturally graded algebra, $\Omega(M) = \bigoplus_k^n \Omega^k(M)$, together with the exterior derivative yields a $\mathbb{Z}$ graded chain complex

$$\cdots \xrightarrow{d} \Omega^{q-1}(M) \xrightarrow{d} \Omega^q(M) \xrightarrow{d} \Omega^{q+1}(M) \xrightarrow{d} \cdots$$

called the *de Rham Complex*.

Definition 3. A q-form, ω is *closed* if $d\omega = 0$. It is *exact* if $\omega = d\eta$ for some $(q-1)$-form η.

Observe that Proposition 2 (3) implies that every exact form is a closed form.

Since d increases the degree of the complex (instead of reducing the degree as usual), we can define the de Rham cohomology groups by computing the "homology" of the complex. In particular,

Definition 4. The q^{th} de Rham cohomology group is given by the quotient

$$H^q_{dR}(M) = \frac{\ker d : \Omega^q(M) \to \Omega^{q+1}(M)}{\text{im } d : \Omega^{q-1}(M) \to \Omega^q(M)} = \frac{\{\text{closed forms}\}}{\{\text{exact forms}\}}.$$

Definition 5. Suppose $\omega, \tilde{\omega} \in \Omega^q(M)$. Then ω and $\tilde{\omega}$ belong to the same cohomology class if, for some $\eta \in \Omega^{q-1}(M)$, $\omega - \tilde{\omega} = d\eta$. The class of ω is denoted $[\omega]$.

Suppose $[\omega]$ and $[\nu]$ represent classes of differential forms $\omega \in \Omega^q(M)$ and $\nu \in \Omega^p(M)$. Proposition 2(3) implies the existence of a well defined bilinear mapping

$$H^p_{dR}(M) \times H^q_{dR}(M) \to H^{p+q}_{dR}(M)$$

defined by

$$(\nu, \omega) \mapsto \nu \wedge \omega.$$

With this bilinear mapping, $H^\bullet_{dR}(M)$, the direct sum of all cohomology groups, is a graded algebra.

Example 3. Let's compute the de Rham cohomology group for $M = \mathbb{R}$.

$\ker d : \Omega^0(M) \to \Omega^1(M) = \{\varphi : \mathbb{R} \to \mathbb{R} \mid \varphi : x \mapsto \{\text{pt}\}\} = \mathbb{R}$ so $H^0(M) = \mathbb{R}$.

$\ker d : \Omega^1(M) \to \Omega^2(M) = \{f(x)dx \mid f \in C^\infty(\mathbb{R})\}$ (by Proposition 2(3)). If $\omega \in \operatorname{im} d : \Omega^0(M) \to \Omega^1(M)$, then there is a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$, such that $d\varphi = \omega$.

so $\operatorname{im} d : \Omega^0(M) \to \Omega^1(M) = \{d\varphi \mid \varphi \in C^\infty(\mathbb{R})\} = \left\{\varphi^{'}(x)dx \mid \varphi \in C^\infty(\mathbb{R})\right\}$. Since every smooth function $f : \mathbb{R} \to \mathbb{R}$ has an antiderivative φ, $H^1(M) = 0$.

Hence,

$$H^\bullet_{dR}(M) = \begin{cases} \mathbb{R} & \bullet = 0 \\ \\ 0 & \bullet > 0 \end{cases}$$

Observe that this coincides with the usual cohomology of $\mathbb{R}$. In fact,

Theorem 1.

$$H^\bullet_{dR}(M) \cong H^\bullet(M, \mathbb{R})$$

Due to Theorem 1, we can ignore the subscript dR when discussing the *de Rham* cohomology.

In particular, if M is an n-manifold, then $H^k(M) = 0$ for $k > n$. The proof of this theorem relies on the Mayer-Vietoris sequence:

2.1.1 Mayer - Vietoris sequence

The Mayer-Vietoris sequence is a very useful tool. It allows one to determine the cohomology of complicated (and not so complicated spaces) by considering smaller subspaces.

Suppose $\varphi : M \to N$ is a smooth mapping between an m-manifold M and an n-manifold N. The map φ induces a pullback map $\varphi^* : N \to M$. Recalling that $\Omega^0(M) = C^\infty(M)$ and $\Omega^0(N) = C^\infty(N)$, we can generalize this notion to discuss induced forms.

Definition 6. Suppose $f : M \to N$ is a smooth mapping. Then there is an induced form given by the mapping

$$f^* : \Omega^q(N) \to \Omega^q(M)$$

defined by

$$f^*(\sum g_\pi dy_\pi) = \sum (g_\pi \circ f) df_\pi$$

where $df_{\pi(k)} = d(y_{\pi(k)} \circ f)$.

In this case, we say that Ω^q is a contravariant functor.

Proposition 3. *Suppose $\varphi : M \to N$ and $\psi : N \to R$ are smooth mappings. Then*

1. $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$
2. *For differential forms ω and ν, $\varphi^*(\omega \wedge \nu) = \varphi^*(\omega) \wedge \varphi^*(\nu)$*
3. *For differential form ω, $d(\varphi^*(\omega)) = \varphi^*(d\omega)$*

Suppose $M = U \cup V$ where U and V are open subspaces. The contravariant functor, Ω^q, induces a short exact sequence,

$$0 \longrightarrow \Omega^q(M) \longrightarrow \Omega^q(U) \oplus \Omega^q(V) \longrightarrow \Omega^q(U \cap V) \longrightarrow 0$$

where the map $\Omega^q(U) \oplus \Omega^q(V) \to \Omega^q(U \cap V)$ is given by $\Omega^q : (\omega, \nu) \mapsto \omega|_{U \cap V} - \nu|_{U \cap V}$. Since, closed forms map to closed forms and exact forms map to exact forms, the short exact sequence induces the sequence on the cohomology groups:

$$\begin{tikzcd} \cdots \arrow[r] & H^q(M) \arrow[r] & H^q(U) \oplus H^q(V) \arrow[r] & H^q(U \cap V) \arrow[dll, "d^*"'] & \\ & H^{q+1}(M) \arrow[r] & H^{q+1}(U) \oplus \Omega^{q+1}(V) \arrow[r] & H^{q+1}(U \cap V) \arrow[r] & \cdots \end{tikzcd}$$

This sequence is called the Mayer-Vietoris sequence.

Proposition 4. *The Mayer-Vietoris sequence is exact.*

Proof. As usual, a short exact sequence of cochain complexes gives a long exact sequence in cohomology. $\square$

2.1.2 Poincaré Lemma

Definition 7. If X and Y are topological spaces, a *homotopy* is a map $H : X \times [0,1] \to Y$. Two maps $h_0, h_1 : X \to Y$ are said to be *homotopic* if there is a homotopy $H : X \times [0,1] \to Y$ such that $H(x,0) = h_0(x)$ and $H(x,1) = h_1(x)$ for each $x \in X$. If for a map $f : X \to Y$, there exists a map $g : Y \to X$ such that $gf \simeq 1_X$ and $fg \simeq 1_Y$, then X is *homotopy equivalent* to Y. We may write this equivalence as $X \simeq Y$.

Definition 8. A space is said to be *contractible* if it is homotopy equivalent to a point.

Lemma 2. *If two manifolds have the same homotopy type, then they have the same cohomology groups.*

Proof. It is important to first make use of another Lemma:

Lemma 3. *Homotopic maps induce the same maps on cohomology.*

In particular, suppose M and N are manifolds and let $h_0, h_1 : M \to N$ with $h_0 \simeq h_1$. Then $h_0^* = h_1^*$.

Proof. Let $H : M \times \mathbb{R} \to N$ be a homotopy map with $H(m, 0) = h_0$ and $H(m, 1) = h_1$. Suppose ι_0 and ι_1 represent the embeddings of M as $M \times \{0\}$ and $M \times \{0\}$ into $M \times \mathbb{R}$, respectively, and consider the projection map $\pi : M \times \mathbb{R} \to M$. Since, $\pi \circ \iota_k = \mathrm{Id}$, $(k = 1, 2)$, $\iota_k^* \circ \pi^* = \mathrm{Id}$ on the chain level; but, since $\iota_k \circ \pi \neq \mathrm{Id}$, $\pi^* \circ \iota_k^* \neq \mathrm{Id}$. However, $\pi^* \circ \iota_k^* \simeq \mathrm{Id}$ on the chain level. To illustrate this, we can introduce a degree lowering homotopy operator, $K : \Omega^p(M \times \mathbb{R}) \to \Omega^{p-1}(M \times \mathbb{R})$. If $\eta \in \Omega^q(M \times \mathbb{R})$, we can write η in the form

$$\eta(x, t) = \sum_I f_I(x, t)\, dx_I \wedge dt + \sum_J g_J(x, t)\, dx_J.$$

Here the $f_I(x, t)$ and $g_J(x, t)$ are smooth real-valued functions on $M \times \mathbb{R}$, x is a local coordinate on M, I runs over multi-indices of length $q - 1$, J runs over multi-indices of length q, and t is the usual coordinate on $\mathbb{R}$. Then K is given by

$$K\eta(x, t) = \sum_I \left(\int_0^t f_I(x, s)\, ds \right) dx_I$$

(note that the dx_J terms are all killed). From Bott and Tu [1], it follows that the operator K satisfies the identity

$$\pm(K \circ d - d \circ K) = 1 - \pi^* \circ \iota_0^*$$

So $1 = \pi^* \circ \iota_0^*$ on cohomology. Similarly, we also have that $1 = \pi^* \circ \iota_1^*$ on cohomology and

$$\pi^* \iota_1^* = \pi^* \iota_0^* \quad .$$

By composing with the induced map H^*,

$$\pi^* \circ \iota_1^* \circ H^* = \pi^* \circ \iota_0^* \circ H^* \Longrightarrow$$

$$\pi^* \circ (H \circ \iota_1)^* = \pi^* \circ (H \circ \iota_0)^*$$

Since $H \circ \iota_0 = H(m, 0) = h_0$ and $H \circ \iota_1 = H(m, 1) = h_1$,

$$\pi^* \circ (H \circ \iota_1)^* = \pi^* \circ (H \circ \iota_0)^* \Longrightarrow$$

$$\pi^* \circ h_1^* = \pi^* \circ h_0^*$$

Therefore, $h_1^*[\nu] = h_0^*[\nu]$ for any closed form $\nu \in \Omega^p(N)$. □

Now, let $f : M \to N$ and $g : N \to M$ where f and g are homotopy inverses of one another. That is, $gf \simeq 1_M$ and $fg \simeq 1_N$. Then

$$(gf)^* = f^* g^* = 1 : H^\bullet(M) \to H^\bullet(M) \qquad \text{and}$$

$$(fg)^* = g^* f^* = 1 : H^\bullet(N) \to H^\bullet(N).$$

So $f^* : H^\bullet(N) \to H^\bullet(M)$ is the inverse to $g^* : H^\bullet(M) \to H^\bullet(N)$ and the cohomology groups agree. □

Proposition 5. *(Poincaré Lemma) If M is contractible, then $H^\bullet(M) = H^\bullet(\{pt\})$. In particular, every closed form is exact.*

Proof. That $H^\bullet(M) = H^\bullet(\{\mathrm{pt}\})$ follows immediately from Lemma 2. □

Example 4. Let $U \subseteq \mathbb{R}^n$ be a *star shaped* set. That is, U is an open subset of $\mathbb{R}^n$ such that if $x \in U$, then there is a line segment joining x, to the "origin," $x_0 \in U$.

By defining $h_0 : U \to \{x_0\}$ by $x \mapsto x_0$ and $h_1 : \{x_0\} \to U$ by $x_0 \mapsto x_0$, we see that $U \simeq \{x_0\}$ so U is contractible. In particular, $\mathbb{R}^n$ is a star shaped set so

$$H^\bullet(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \bullet = 0 \\ \\ 0 & \text{otherwise} \end{cases}$$

Proposition 6. *If a manifold, M, is of finite type, then its cohomology is finite dimensional.*

Proof. Suppose M has dimension n. Since M is of finite type, then M has a nonempty good cover, i.e., a finite open cover $\{U_\alpha\}_{\alpha \in \{1,\ldots,m\}}$ of size m, with each $U_\alpha \in \{U_\alpha\}$ and every nonempty finite intersection, $U_{\alpha_1} \cap \cdots \cap U_{\alpha_j}$, diffeomorphic to $\mathbb{R}^n$.

We can proceed by induction on m.

If $m = 1$, then by Proposition 5, $H^\bullet(M)$ is finite dimensional. Assume then, that $H^\bullet(M)$ is finite dimensional for all good covers of size k and suppose M has a good cover of size $k + 1$. Let $U = U_1 \cup \cdots \cup U_k$ and $V = U_{k+1}$ so $M = U \cup V$.

By the Mayer-Vietoris sequence,

$$\cdots \longrightarrow H^q(U \cap V) \xrightarrow{d^*} H^{q+1}(M) \xrightarrow{r} H^{q+1}(U) \oplus \Omega^{q+1}(V) \longrightarrow \cdots$$

so

$$H^{q+1}(M) \cong \ker r \oplus \text{im } r \cong \text{im } d^* \oplus \text{im } r$$

and $H^{q+1}(M)$ is finite dimensional since $H^{q+1}(U)$, $H^{q+1}(V)$ and $H^q(U \cap V)$ are all finite dimensional (note: that $H^q(U \cap V)$ is finite dimensional follows from the induction hypothesis since $U \cap V = (U_1 \cap U_{m+1}) \cup \cdots \cup (U_m \cap U_{m+1})$ is a finite good cover of size k). Hence, $H^\bullet(M)$ is finite dimensional. $\square$

2.2 Cohomology with local coefficients

In the discussion thus far, we have relied on the fact that our manifold, M is simply connected. But what happens when M is not simply connected? This question leads us to the discussion of a cohomology group with local coefficients, $H^\bullet(M, \mathcal{L})$. Before describing the nature of $H^\bullet(M, \mathcal{L})$, lets first introduce some necessary vocabulary.

2.2.1 Sheaves and Presheaves

Definition 9. Suppose R is a PID (recall that a PID is an integral domain with the property that all ideals are principal) and let M be a topological space. A *Sheaf*, $\mathcal{S}$ of $R-$modules over M is a topological space $\mathcal{S}$ with a map $\pi : \mathcal{S} \to M$ such that the following conditions hold:

1. π is a local homeomorphism of $\mathcal{S}$ onto M;
2. $\pi^{-1}(m)$ is an $R-$module for each $m \in M$; and
3. composition laws are continuous in the topology on $\mathcal{S}$.

Definition 10. A presheaf $\mathcal{P} = (S_U, \rho_{U,V})$ on M is a collection of $R-$modules, S_U for each open set $U \subset M$ and homomorphisms $\rho_{U,V} : S_U \to S_V$ for each inclusion $U \subseteq V$ of open sets in M such that the following conditions are satisfied.

1. $\rho_{U,U} = \mathrm{id}_{S_U}$

2. If $U \subseteq V \subseteq W$, then $\rho_{U,W} = \rho_{U,V} \circ \rho_{V,W}$

If $U \subset M$, the functor $\Omega^q : U \to \{\ q\text{−forms on } U\ \}$ is an example of a presheaf.

There is a natural identification between sheaves and presheaves. If $\mathcal{S}$ is a sheaf, then for each open set $U \subset M$, one can associate an R−module, $\Gamma(\mathcal{S}, U)$ of sections of $\mathcal{S}$ over U. Conversely, each presheaf has a unique associated sheaf, called the "sheafification" $\mathcal{S}$. If $\mathcal{P} = (S_U, \rho_{U,V})$ is a presheaf on M, then the stalk of $\mathcal{S}$ over m is $\varinjlim S_U$, the limit taken over finer and finer neighborhoods of m.

For example, if N is a manifold, the set of germs of C^∞ functions on M is a sheaf. The corresponding presheaf attaches $C^\infty(U)$ to each open set U.

Definition 11. Suppose $\mathcal{S}$ and $\mathcal{S}'$ are two sheaves over M with projections π and π' respectively. A sheaf homomorphism is a map $\psi : \mathcal{S} \to \mathcal{S}'$ such that $\pi' \circ \psi = \pi$.

Definition 12. A sheaf $\mathcal{S}$ over M is said to be fine if it has partitions of unity. That is, for any open cover $\{U_i\}$ of M, there is a family of endomorphisms such that

1. $\mathbf{supp}(\varphi_i) \subseteq U_i$; and

2. $\sum_i \varphi_i = \mathrm{id}$

Definition 13. A sheaf, $\mathcal{S}$, on M is locally constant if for any $m \in M$, there is a neighborhood N of m such that the restriction of $\mathcal{S}$ to N is isomorphic to a product $N \times G$, where the coefficient module G is given the discrete topology.

2.2.2 $H^\bullet(M, \mathcal{L})$

Suppose M is a manifold that is not simply connected, with universal cover $\widetilde{M}$ and covering map π. If $\mathcal{L}$ is a representation space of π, then there is a cohomology group of M with local coefficients in $\mathcal{L}$, denoted $H^\bullet(M, \mathcal{L})$. Note that in the special case where $\mathcal{L} = \mathbb{C}$, with the trivial representation, $H^\bullet(M, \mathcal{L})$ is just the usual cohomology group, $H^\bullet(M, \mathbb{C})$.

$H^\bullet(M, \mathcal{L})$ can be described as the sheaf cohomology of the locally constant sheaf $\mathcal{L}$ whose local sections can be identified with the locally constant functions on $\widetilde{M}$ with values in $\mathcal{L}$ that transform according to the representation $f(g \cdot m) = \sigma(g) \cdot f(m)$.

Chapter 3

Twisted Cohomology Group

In Chapter 2, we introduced the notion of cohomology with local coefficients, $H^\bullet(M, \mathcal{L})$. One can think of the twisted cohomology group as a generalization of $H^\bullet(M, \mathcal{L})$. In particular, suppose ω is a closed (not necessarily exact!) differential $(2k+1)$-form on a manifold M. A twist on the usual cohomology can be constructed from the space of differential forms, $\Omega^*(M)$, by defining a differential $d_\omega = d + \omega\wedge$ where d is the usual exterior derivative $d : \Omega^q(M) \to \Omega^{q+1}(M)$. Noting that, for any differential form η,

$$d_\omega^2 \eta = d_\omega(d\eta + \omega \wedge \eta) = d^2\eta + \omega \wedge d\eta + d\omega \wedge \eta - \omega \wedge d\eta + \omega \wedge \omega \wedge \eta = 0,$$

we see that d_ω induces a $\mathbb{Z}_{2k}$ graded differential complex:

$$\cdots \xrightarrow{d_\omega} \Omega^{j \bmod 2k}(M) \xrightarrow{d_\omega} \Omega^{j+1 \bmod 2k}(M) \xrightarrow{d_\omega} \Omega^{j+2 \bmod 2k}(M) \xrightarrow{d_\omega} \cdots$$

Observe that if $k = 0$, d_ω induces a $\mathbb{Z}$ graded complex and if $k = 1$, d_ω induces a $\mathbb{Z}_2$ graded complex. Additionally, for $k \geq 1$, the action by d_ω sends forms of odd degree to forms of even degree and forms of even degree to forms of odd degree. We then define the twisted cohomology group to be the quotient group

$$H_\omega^\bullet(M) = \frac{\ker d_\omega : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)}{\text{im } d_\omega : \Omega^{\bullet-1}(M) \to \Omega^\bullet(M)}$$

Proposition 7. *$H_\omega(M)$ forms a group under addition and satisfies the following functorial properties:*

1. *(Normalization) If* $\omega = 0$*, then* $H^\bullet_\omega(M) = H^\bullet(M)$

2. *(Cup Product) If* ω *and* $\omega^{'}$ *are two odd forms, there is a cup product homomorphism*

$$H^p_\omega(M) \otimes H^q_{\omega^{'}}(M) \to H^{p+q}_{\omega+\omega^{'}}(M)$$

3. *(Naturality) If* $f : M \to N$ *is a smooth map, then there is a homomorphism*

$$f^* : H^p_\omega(N) \to H^p_{f^*\omega}(M)$$

Proof.

1. If $\omega = 0$, then $d_\omega = d + \omega = d$ and

$$H^\bullet_\omega(M) = \frac{\ker d_\omega : \Omega^{\text{even}}(M) \to \Omega^{\text{odd}}(M)}{\text{im } d_\omega : \Omega^{\text{odd}}(M) \to \Omega^{\text{even}}(M)} = \frac{\ker d : \Omega^{\text{even}}(M) \to \Omega^{\text{odd}}(M)}{\text{im } d : \Omega^{\text{odd}}(M) \to \Omega^{\text{even}}(M)} = H^\bullet(M).$$

2. If suffices to show that if $d_\omega = 0$ and $d_{\omega^{'}} = 0$, where $\omega \in \Omega^p(M)$ and $\omega^{'} \in \Omega^q(M)$, then $d_{\omega+\omega^{'}} = 0$. In particular,

$$d_\omega\alpha = 0 \iff d\alpha = -\omega \wedge \alpha \qquad \text{and} \qquad d_{\omega^{'}}\beta = 0 \iff d\beta = -\omega \wedge \beta$$

Then

$$\begin{aligned} d(\alpha \wedge \beta) &= d\alpha \wedge \beta + (-1)^{\deg(\alpha)}\alpha \wedge d\beta \\ &= -\omega \wedge \alpha \wedge \beta + (-1)^{\deg(\alpha)+1}\alpha \wedge \omega^{'} \wedge \beta \\ &= -\omega \wedge \alpha \wedge \beta - \omega^{'} \wedge \alpha \wedge \beta \\ &= -(\omega + \omega^{'}) \wedge (\alpha \wedge \beta) \end{aligned}$$

So $d_{\omega+\omega^{'}} = 0$.

3.

$$\begin{aligned} f^* d_\omega \eta &= f^*(d + \omega \wedge)\eta \\ &= f^*(d\eta) + f^*(\omega \wedge \eta) \\ &= d(f^*\eta) + (f^*\omega) \wedge (f^*\eta) \\ &= d_{f^*\omega}\tilde{\eta} \qquad \text{where } \tilde{\eta} = f^*(\eta) \end{aligned}$$

□

Corollary 4. *$H^\bullet_\omega(M)$ is a $\mathbb{Z}_{2k}$ graded module over $H^\bullet(M)$.*

Proof. By Proposition 8(1), the cup product homomorphism in Proposition 8(2) defines an action

$$H^p(M) \otimes H^q_\omega(M) \to H^{p+q}_\omega(M)$$

whose module properties follow directly from the usual wedge product properties. Since $d_\omega^2 = 0$, the module $H^q_\omega(M)$ is $\mathbb{Z}_{2k}$ graded. □

Theorem 5. *If ω and $\omega^{'}$ are closed $(2k+1)$ forms in the same de Rham class, then $H^\bullet_\omega(M) \cong H^\bullet_{\omega'}(M)$.*

Proof. Since $[\omega] = [\omega^{'}]$, then, for some $\eta \in \Omega^{2k}(M)$, $\omega^{'} - \omega = d\eta$. It is then sufficient to show that e^η conjugates d_ω to $d_{\omega'}$. In particular, we wish to show that

$$e^{-\eta} d_\omega e^\eta = d_{\omega'}.$$

Suppose $\nu \in \Omega^q(M)$. Then

$$\begin{aligned}
e^{-\eta} d_\omega e^\eta \nu =& e^{-\eta}(d + \omega \wedge) e^\eta \nu \\
=& e^{-\eta} d(e^\eta \wedge \nu) + \omega \wedge \nu \\
=& e^{-\eta} d(e^\eta) \wedge \nu + e^{-\eta} e^\eta \wedge d\nu + \omega \wedge \nu \\
=& e^{-\eta} e^\eta d\eta \wedge \nu + d\nu + \omega \wedge \nu \qquad (\text{since } d(e^\eta) = e^\eta d\eta) \\
=& (d\eta \wedge + d + \omega \wedge)\nu \\
=& ((\omega' - \omega) \wedge + d + \omega \wedge)\nu = (d + \omega')\nu = d_{\omega'}\nu.
\end{aligned}$$

□

3.1 Twisting by a one form

A special case to note is when $k = 0$. As mentioned in the introduction, twisting by a one-form arises naturally in the study of Supersymmetry.

Define a map

$$p_\omega : \pi_1(M) \to \mathbb{R} \qquad \text{by}$$

$$\gamma \mapsto \int_\gamma \omega.$$

By Stokes' Theorem, p_ω is a homomorphism and so defines a local coefficient system $\mathcal{L}$. We identify $\mathcal{L}$ with the sheaf of germs of C^∞ functions f such that $df = -\omega f$. Note that when $\omega \equiv 0$, this is just the constant sheaf $\mathbb{R}$.

Theorem 6. *Suppose M is a differentiable manifold and ω a closed one form. Then*

$$H^\bullet_\omega(M) \cong H^\bullet(M, \mathcal{L})$$

Proof. Note that

$$0 \longrightarrow \mathcal{L} \longrightarrow \Omega^0(M) \xrightarrow{d+\omega} \Omega^1(M) \xrightarrow{d+\omega} \Omega^2(M) \xrightarrow{d+\omega} \cdots$$

is a fine resolution of $\mathcal{L}$, since the $\Omega^j(M)$ are fine sheaves (see Warner [9]) and by construction of $\mathcal{L}$, we have exactness at $\Omega^0(M)$. So the cohomology groups for the fine resolution $\mathcal{L}$ coincide with the twisted cohomology groups for the 1-form ω. That is,

$$H^\bullet_\omega(M) \cong H^\bullet(M, \mathcal{L}).$$

$\square$

Example 5. Let $M = S^1 = \mathbb{R}/\mathbb{Z}$. Let's compute the twisted cohomology group $H^\bullet_\omega(M)$.

First, observe that, since $\omega \in \Omega^1(M)$, $\omega = h(x)dx$ for $h(x) \in C^\infty(M)$. Lift $h(x)$ to a periodic function on $\mathbb{R}$. Then if $[\omega] = 0$, $\int_0^1 h(x)dx = 0$.

Now,

$$\ker\left(d_\omega : \Omega^0(M) \to \Omega^1(M)\right) = \left\{f \in C^\infty(M) \mid d_\omega f = 0\right\}.$$

The condition $d_\omega f = 0$ says that $f'(x) = -h(x)f(x)$ or $f(x) = f(0)e^{\int h(x)}$. But if $f \in C^\infty(M)$ then $f(1) = f(0)$ so

$$e^{-\int_0^1 h(x)dx} = 1 \Longrightarrow \int_0^1 h(x)dx = 0.$$

Therefore, if ω is exact, $\ker d_\omega = \mathbb{R} \cdot e^{\int h(x)dx}$ and $H^0_\omega(M) \cong \mathbb{R}$; and if ω is not exact, $\ker d_\omega \equiv 0$ and $H^0_\omega(M) = 0$.

If $\alpha \in \text{im } d_\omega : \Omega^0(M) \to \Omega^1(M)$, then for some $f \in C^\infty(M)$, $\alpha = d_\omega f$. Additionally, since $M = \mathbb{R}/\mathbb{Z}$, we can consider behavior on the universal cover $\widetilde{M} = \mathbb{R}$. Since

$\alpha \in \Omega^1(M)$, we can denote $\alpha = g(x)dx$ for some $g \in C^\infty(\mathbb{R})$. So

$$\begin{aligned} d_\omega f =& g\,dx \qquad \Longleftrightarrow \\ (d+\omega)f =& g\,dx \qquad \Longleftrightarrow \\ f' + hf = g. \end{aligned}$$

Solving the first order linear differential equation, we see that

$$f(x) = e^{-\int h dx} \int g \cdot e^{\int h dx} dx + Ce^{-\int h dx}.$$

Letting $H(x) = e^{\int h(x)dx}$, we see that

$$H(0) = e^{\int_0^0 h(x)dx} = 1$$

and

$$f(x) = \frac{1}{H(x)} \left[\int_0^x g(t) \cdot H(t)dt + C \right].$$

Note that

$$f(0) = \frac{1}{H(0)} \left[\int_0^0 g(t) \cdot H(t)dt + C \right] = C$$

So

$$f(x) = \frac{1}{H(x)} \left[\int_0^x g(t) \cdot H(t)dt + f(0) \right]. \tag{3.1}$$

So α is exact precisely when a solution to (3.1) descends to M; in particular, if (3.1) has a periodic solution. It is then necessary to consider two cases:

- Case I: $[\omega] = 0$ (ω is exact): Again, $\int_0^1 h(x)dx = 0$ so $H(1) = H(0) = 1$ so, from (3.1), we have

$$\begin{aligned} f(x) =& \frac{1}{H(x)} \left[\int_0^x g(t) \cdot H(t)dt + f(0) \right] \qquad \text{and} \\ f(1) =& \int_0^1 g(t) \cdot H(t)dt + f(0) \end{aligned}$$

Since $f(x) \in C^\infty(M)$, $f(1) = f(0)$, so α is exact with respect to the twisted differential on M if and only if $\int_0^1 g(x) \cdot H(x)dx = 0$. So, $H^1_\omega \cong \mathbb{R}$.

- Case II: $[\omega] \neq 0$ (ω is not exact):

  If ω is not exact, then $H(1) = A \neq 1 = H(0)$.

$$\begin{aligned} f(x) =& \frac{1}{H(x)} \left[\int_0^x g(t) \cdot H(t)dt + f(0) \right] \qquad \text{and} \\ f(1) =& \frac{1}{A} \left[\int_0^1 g(t) \cdot H(t)dt + f(0) \right] \end{aligned}$$

  Again, since $f(0) - f(1) = 0$ and $f \in C^\infty(M)$,

$$\begin{aligned} f(0) = \frac{1}{A} \left[\int_0^1 g(x) \cdot H(x) + f(0) \right] dx \qquad \Longleftrightarrow \\ f(0) = \frac{1}{A-1} \int_0^1 g(x) \cdot H(x)dx. \end{aligned}$$

  So there is exactly one periodic solution and α is exact with respect to the twisted differential.

Hence, if ω is exact,

$$H^\bullet_\omega = \begin{cases} \mathbb{R} & \bullet = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

If ω is not exact, then

$$H^\bullet_\omega = 0,$$

which agrees with the usual cohomology with local coefficients of S^1.

3.2 An Example with $k > 0$

Suppose $M = S^{2k+1}$ with $k \geq 1$ and $\omega \in \Omega^{2k+1}(M)$. Let us calculate the twisted cohomology group $H^\bullet_\omega$. If $[\omega] = 0$, then, by Theorem 5,

$$H^{\text{even}}_\omega(M) \cong H^0(M) = \mathbb{R} \qquad \text{and}$$

$$H^{\text{odd}}_\omega(M) \cong H^{2k+1}(M) = \mathbb{R}.$$

However, if $[\omega] \neq 0$, then $H^{\text{even}}_\omega = 0$. We can see this by first supposing that $\eta \in \Omega^{\text{even}}(M)$. Then $\eta = \eta_0 + \eta_2 + \cdots + \eta_{2k}$ where $\deg(\eta_a) = a$. If η is $d_\omega-$closed, then $d_\omega \eta = 0$ so

$$(d + \omega\wedge)\eta = d\eta_0 + d\eta_2 + \cdots + d\eta_{2k} + \omega \wedge \eta_0 + \omega \wedge \eta_2 + \cdots + \omega \wedge \eta_{2k} = 0.$$

For dimensional reasons, this implies that $d\eta_{2j} = 0$ for $j \in \{0, \ldots, k-1\}$. Additionally, $d\eta_0 = 0 \Longrightarrow \eta_0 = C$ for some $C \in \mathbb{R}$ so it follows that

$$d\eta_{2k} + \omega \wedge \eta_0 = 0 \qquad \Longleftrightarrow \qquad d\eta_{2k} = -C\omega.$$

Since $[\omega] \neq 0$, $C = \eta_0 = 0 \Longrightarrow d\eta_{2k} = 0$ (otherwise, ω would be exact). Since $H^{\deg \eta_a}(M) = 0$, the η_a are exact so there exist $\nu_b \in \Omega^{\text{odd}}(M)$ such that $\eta_{2j} = d\nu_{2j-1}$ for $j \in \{1, \ldots, k\}$. Hence,

$$\eta = \eta_0 + \eta_2 + \cdots + \eta_{2k} = d\nu_1 + d\nu_3 + \cdots + d\nu_{2k-1} = d(\nu_1 + \nu_3 + \cdots + \nu_{2k+1}) = d\nu.$$

Since $\nu \in \Omega^{\text{odd}}(M)$, for dimensional reasons, $d\nu = d_\omega \nu$. So every closed form is exact in the twisted differential complex and $H^{\text{even}}_\omega(M) = 0$.

Similarly suppose that $\nu \in \Omega^{\text{odd}}(M)$. Then $\nu = \nu_1 + \nu_3 + \cdots + \nu_{2k+1}$ where $\deg(\nu_a) = a$. If ν is $d_\omega-$closed, then $d_\omega \nu = 0$ so

$$(d + \omega\wedge)\nu = d\nu_1 + d\nu_3 + \cdots + d\nu_{2k+1} + \omega \wedge \nu = 0.$$

For dimensional reasons, $\omega \wedge \nu = 0$ and $d\nu_{2j+1} = 0$ for $j \in \{0, \ldots, k\}$.

Since $H^{\deg \nu_a}(M) = 0$, for $a \in \{1, \ldots, 2k-1\}$, the corresponding ν_a are exact so there exist $\mu_b \in \Omega^{\text{even}}(M)$ such that $\nu_{2j+1} = d\mu_{2j}$ for $j \in \{0, \ldots, k-1\}$. Now, since $[\omega]$ is a generator of $H^{2k+1}(M)$, $\nu_{2k+1} - \mu_0 \wedge \omega = C\omega + d\mu_{2k}$, i.e., $\nu_{2k+1} = (C + \mu_0\wedge)\omega + d\mu_{2k}$ for some constant $C \in \mathbb{R}$ and $\mu_0, \mu_{2k} \in \Omega^{\text{even}}(M)$. So

$$\begin{aligned}
\nu = \nu_1 + \nu_3 + \cdots + \nu_{2k+1} &= d\mu_0 + d\mu_2 + \cdots + d\mu_{2k} + (C + \mu_0\wedge)\omega \\
&= d(\mu_0 + \mu_2 + \cdots + \mu_{2k-2}) + (C + \mu_0\wedge)\omega + d\mu_{2k} \\
&= d_\omega(\mu + C).
\end{aligned}$$

So if $[\omega] \neq 0$, $H^{\text{odd}}_\omega(M) = 0$.

Hence, for $[\omega] \neq 0$

$$H^{\text{even}}_\omega(M) = H^{\text{odd}}_\omega(M) = 0$$

3.3 The double complex

The $\mathbb{Z}_{2k}$ graded twisted differential complex can actually be viewed as coming from a double complex. In particular, by definition of the exterior derivative, we have a $\mathbb{Z}$ graded complex

$$\cdots \xrightarrow{d} \Omega^{q-1}(M) \xrightarrow{d} \Omega^q(M) \xrightarrow{d} \Omega^{q+1}(M) \xrightarrow{d} \cdots$$

and since ω is a closed odd form, we also have a complex

$$\cdots \xrightarrow{\omega\wedge} \Omega^{q-2k-1}(M) \xrightarrow{\omega\wedge} \Omega^{q}(M) \xrightarrow{\omega\wedge} \Omega^{q+2k+1}(M) \xrightarrow{\omega\wedge} \cdots$$

Together, these two complexes produce a double complex with $\Omega^{p+q(2k+1)}(M)$ in the (p, q) position:

$$\begin{tikzcd}
 & & q & & & & \\
 & \vdots & \vdots \arrow[u] & \vdots & \vdots & & \\
\cdots \arrow[r, "d"] & \Omega^{4k+1}(M) \arrow[u, "\omega\wedge"'] \arrow[r, "d"] & \Omega^{4k+2}(M) \arrow[u, "\omega\wedge"'] \arrow[r, "d"] & \Omega^{4k+3}(M) \arrow[u, "\omega\wedge"'] \arrow[r, "d"] & \Omega^{4k+4}(M) \arrow[u, "\omega\wedge"'] \arrow[r, "d"] & \cdots & \\
\cdots \arrow[r, "d"] & \Omega^{2k}(M) \arrow[u, "\omega\wedge"'] \arrow[r, "d"] & \Omega^{2k+1}(M) \arrow[u, "\omega\wedge"'] \arrow[r, "d"] & \Omega^{2k+2}(M) \arrow[u, "\omega\wedge"'] \arrow[r, "d"] & \Omega^{2k+3}(M) \arrow[u, "\omega\wedge"'] \arrow[r, "d"] & \cdots & \\
 & & \Omega^{0}(M) \arrow[u, "\omega\wedge"'] \arrow[r, "d"] & \Omega^{1}(M) \arrow[u, "\omega\wedge"'] \arrow[r, "d"] & \Omega^{2}(M) \arrow[u, "\omega\wedge"'] \arrow[r, "d"] & \cdots \arrow[r] & p
\end{tikzcd}$$

To show that the twisted differential complex arises from this double complex, it is sufficient to show that the actions d and $\omega\wedge$ anticommute. This condition is satisfied since, for any differential form η, $d(\omega \wedge \eta) = d\omega \wedge \eta - \omega \wedge d\eta = -\omega \wedge d\eta$. Note that the *anti*commuting property is necessary since $d : \Omega^p(M) \to \Omega^{p+1}(M)$, $\omega\wedge : \Omega^p(M) \to \Omega^{p+2k+1}(M)$.

3.3.1 Spectral Sequences

Definition 14. A spectral sequence is a sequence of differential groups $\{E_k, d_k\}$ such that $E_k = H(E_{k-1})$.

Spectral Sequences are extremely useful for computing cohomology and homology groups. By considering the twisted cohomology group as the cohomology of the total complex of a double complex, we can obtain the following results involving spectral sequences:

Theorem 7. *There is a spectral sequence converging to $H^\bullet_\omega(M)$, with E_1 term $H^\bullet_{deR}(M)$ and differential d_1 given by cup product with the de Rham class of ω.*

Proof. This is immediate from §III.14 of Bott and Tu [1]. □

Corollary 8. *If M has finite type, then the twisted cohomology groups $H^\bullet_\omega(M)$ are finite-dimensional, with* $\dim H^\bullet_\omega(M) \leq \dim H^\bullet_{deR}(M)$.

Proof. Since $E_k = H(E_{k-1})$ with $E_1 = H^\bullet_{\mathrm{deR}}(M)$, $\dim E_k \leq \dim E_{k-1}$ for each k so $\dim H^\bullet_\omega(M) \leq \dim H^\bullet_{\mathrm{deR}}(M)$. □

Corollary 9. *Assume M has finite type. Then the Euler characteristic of the twisted cohomology,* $\dim H^{even}_\omega(M) - \dim H^{odd}_\omega(M)$*, agrees with the usual Euler characteristic $\chi(M)$.*

Proof. This follows from the Euler-Poincaré principle, which says that the homology of a chain complex of finite type has the same Euler characteristic as the original complex. Thus,

$$\chi(H^\bullet_\omega(M)) = \chi(E_\infty) = \chi(E_1) = \chi(H^\bullet_{\mathrm{deR}}(M)) = \chi(M).$$

□

BIBLIOGRAPHY

[1] Bott, R. and Tu, L *Differential Forms in Algebraic Topology* (Springer-Velag, New York, NY, 1982).

[2] Bouwknegt, P., Carey, A., Mathai, V., Murray, M. and Stevenson, D. *Twisted K Theory and K-Theory of bundle gerbes* Comm. Math. Phys. **228** (2002), 17-45, [arXiv: hep-th/0106194].

[3] Bouwknegt, P., Evslin, J. and Mathai, V. *T-Duality: Topology Change from H-flux* Comm. Math. Phys. **249** (2004), 383-415, [arXiv: hep-th/0306062].

[4] Bredon, G. *Topology and Geometry* (Springer-Verlag, New York, NY, 1993).

[5] Darling, R.W.R. *Differential Forms and Connections* (Cambridge University Press, Cambridge, UK, 1994).

[6] Guillemin, V and Pollack, A. *Differential Topology* (Prentice-Hall, Englewood Cliffs, NJ, 1974).

[7] Munkres, J.R. *Elements of Algebraic Topology* (Benjamin/Cummings, Menlo Park, CA 1984).

[8] Vafa, C and Zaslow, E Eds. *Mirror Symmetry* (American Mathematical Society, Providence, RI, 2003).

[9] Warner, F.W. *Foundations of Differentiable Manifolds and Lie Groups* (Scott Foresman and Company, Glenview, IL 1971).