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On Robust Eigenvalue Location

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Abstract

The concepts of guardian and semiguardian maps were recently introduced as tools for assessing robust generalized stability of parametrized families of matrices or polynomials. Necessary and sufficient conditions were obtained for stability of parametrized families with respect to a large class of open subsets of the complex plane, namely those with which one can associate a polynomic guardian or semiguardian map. This note focusses on a class of *disconnected* subsets of the complex plane, of interest in the context of dominant pole assignment and filter design. It is first observed that the robust stability conditions originally put forth are in fact necessary and sufficient for the number of eigenvalues (matrices) or zeros (polynomials) in any given connected component to be the same for all the members of the given family. Polynomic semiguardian maps are then identified for a class of disconnected regions of interest. These maps are in fact "essentially guarding with respect to one-parameter families".

Key words. Robust stability, eigenvalue clustering, root clustering, parametrized families, dominant pole assignment, filter design.

1. Introduction

In [1,2] the concepts of guardian and semiguardian maps were introduced as tools for assessing robust generalized stability of parametrized families of real matrices or polynomials. Necessary and sufficient conditions leading to computational tests were obtained for stability of one- (resp. two-) parameter families with respect to a large class of open subsets Ω of the complex plane, namely those with which a corresponding polynomic guardian or semiguardian (resp. guardian) map can be associated. The techniques and results of [2] were subsequently extended to the case of complex matrices and polynomials [3].

This note focusses on a class of disconnected regions Ω, of interest in the context of dominant pole assignment or filter design [4,5]. In Section 2 below, after reviewing the essentials of the approach introduced in [1–3] we observe that results somewhat stronger than those given in those papers hold when the region Ω is disconnected. Then, in Section 3, polynomic semiguardian map are identified for a class of disconnected regions of interest. These maps are in fact "essentially guarding with respect to one-parameter families". Section 4 is devoted to concluding remarks.

2. Guardian and Semiguardian Maps

The guardian map approach was introduced in [1–3] as a unifying tool for the study of generalized stability of parametrized families of matrices or polynomials. We first recall some of the basic concepts used in this approach.

Definition 1. ([2]) Let $\mathcal{X}$ denote the set of all $n \times n$ complex matrices or the set of all monic polynomials of degree n with complex coefficients, and let $\mathcal{S}$ be an open subset of $\mathcal{X}$. Let ν map $\mathcal{X}$ into $\mathbb{C}$. We say that ν is *semiguarding* for $\mathcal{S}$ if the implication

$$x \in \partial\mathcal{S} \quad \Longrightarrow \quad \nu(x) = 0$$

holds. The map ν *guards* $\mathcal{S}$ if the converse implication also holds whenever $x \in \overline{\mathcal{S}}$. If ν is semiguarding for $\mathcal{S}$, any $x \in \mathcal{S}$ for which $\nu(x) = 0$ is said to be a *blind spot* for $(\nu,\mathcal{S})$. The map ν is said to be *polynomic* if it is a polynomial function of the entries (matrix case) or coefficients (polynomial case) of its argument, and of their complex conjugates. ☐

Of special interest are sets $\mathcal{S}$ of the form $\mathcal{S}(\Omega) \subset \mathcal{X}$, where $\mathcal{S}(\Omega)$ denotes the set of all complex matrices (polynomials) with eigenvalues (zeros) in Ω, a given open subset of the complex plane. In the sequel, we focus on sets of this type.

Let $\Phi := \{x(r) : r \in U\}$, $U \subset \mathbb{R}^k$ pathwise connected, be a family of $n \times n$ matrices or n^{th} order monic polynomials depending continuously on a parameter vector r and let Ω be an open subset of the complex plane. In [2], the following necessary and sufficient condition of robust stability was obtained.

Theorem 1. ([2]) Let ν be a semiguardian map for $\mathcal{S}(\Omega)$ and assume that the family Φ is nominally stable, i.e., $x(r^0) \in \mathcal{S}(\Omega)$ for some $r^0 \in U$. Then the whole family Φ is stable relative to Ω if and only if $x(r) \in \mathcal{S}(\Omega)$ for all $r \in U$ for which $\nu(x(r)) = 0$. If ν guards $\mathcal{S}(\Omega)$, then Φ is stable relative to Ω if and only if $\nu(x(r)) \neq 0$ for all $r \in U$. $\square$

Suppose now that Ω is the union of several connected components. Clearly, continuity of $x(\cdot)$ implies that if the entire family Φ is stable relative to Ω then, in fact, for any $r \in U$, $x(r)$ has the same number of eigenvalues or zeros (counting multiplicity) in each component of Ω as $x(r^0)$ does. Specifically let $\mathcal{P} = \{\Omega_i : i = 1, \ldots, \ell\}$, with $\ell \geq 1$, be a family of finitely many connected components of Ω, let $\mathbf{n} = (n_1, \ldots, n_\ell)$ be an ℓ-tuple of nonnegative integers, and denote by $\mathcal{S}(\Omega, \mathcal{P}, \mathbf{n}) \subset \mathcal{S}(\Omega)$ the set of all complex matrices (polynomials) of $\mathcal{S}(\Omega)$ having exactly n_i eigenvalues (zeros) in Ω_i, $i = 1, \ldots, \ell$. The following holds.

Theorem 2. Let ν be a semiguardian map for $\mathcal{S}(\Omega)$ and suppose that $x(r^0) \in \mathcal{S}(\Omega, \mathcal{P}, \mathbf{n})$ for some $r^0 \in U$. Then $\Phi \subset \mathcal{S}(\Omega, \mathcal{P}, \mathbf{n})$ if and only if $x(r) \in \mathcal{S}(\Omega, \mathcal{P}, \mathbf{n})$ for all $r \in U$ for which $\nu(x(r)) = 0$. If ν guards $\mathcal{S}(\Omega)$ and $\mathcal{P}$ entirely covers Ω, this holds if and only if $\nu(x(r)) \neq 0$ for all $r \in U$. $\square$

In [2,3] it is shown that for a large class of regions of interest a polynomic guardian or semiguardian map is available. In such case, if r is a scalar parameter and x depends polynomially on r, Theorems 1 and 2 provide tests involving the computation of the zeros of a univariate polynomial, namely $\nu(x(\cdot))$. If ν guards $\mathcal{S}(\Omega)$, the test amounts to verifying that $\nu(x(\cdot))$ has no zeros in a specific real interval, and this can be done by means of Sturm sequences (see, e.g., [6]).

3. Constructing Semiguardian Maps for a Class of Disconnected Regions.

For the sake of simplicity of exposition, we now focus on families of matrices. Similar results hold for the case of polynomials. We also assume that x depends polynomially on r.

Consider an open set $\Omega \subset \mathbb{C}$ such that a polynomic semiguardian map $\nu : \mathbb{C}^{n\times n} \to \mathbb{C}$ is available for $\mathcal{S}(\mathbb{C}\backslash\partial\Omega)$, i.e., such that

$$\nu(A) = 0 \quad \text{whenever} \quad A \in \partial\mathcal{S}(\mathbb{C}\backslash\partial\Omega), \tag{1}$$

or, equivalently,

$$\nu(A) = 0 \quad \text{whenever} \quad \sigma(A) \cap \partial\Omega \neq \emptyset. \tag{2}$$

Note that, since

$$\partial\mathcal{S}(\Omega) \subset \partial\mathcal{S}(\mathbb{C}\backslash\partial\Omega), \tag{3}$$

(1) implies that ν is semiguarding for $\mathcal{S}(\Omega)$, but not conversely; also note that (1) neither implies nor is implied by guardedness of $\mathcal{S}(\Omega)$ by ν. In [2] and [3], semiguardian maps are obtained for $\mathcal{S}(\Omega)$ whenever Ω can be expressed in the form

$$\Omega = \{s : \ \pi(\Re e(s), \Im m(s)) < 0\}, \tag{4}$$

with $\pi(\cdot,\cdot)$ a bivariate polynomial. It is a simple exercise to verify that all these maps are also semiguarding for $\mathcal{S}(\mathbb{C}\backslash\partial\Omega)$. In particular, if Ω is a disk of radius $\rho > 0$ centered at $c \in \mathbb{C}$, i.e.,

$$\Omega = \{s : \ |s - c| < \rho\}, \tag{5}$$

$\mathcal{S}(\mathbb{C}\backslash\partial\Omega)$ is semiguarded by the polynomic map

$$\nu : \quad A \mapsto \det((A - cI) \otimes (A - cI)^H - \rho^2 I \otimes I) \tag{6}$$

and if Ω is of the form

$$\Omega = \{s : \ \Re e(s) < \alpha\} \tag{7}$$

for some $\alpha \in \mathbb{R}$, $\mathcal{S}(\mathbb{C}\backslash\partial\Omega)$ is semiguarded by the polynomic map

$$\nu : \quad A \mapsto \det((A - \alpha I) \oplus (A - \alpha I)^H). \tag{8}$$

Here superscript H denotes the conjugate transpose, $\otimes$ and $\oplus$ the Kronecker product and sum respectively. Maps for arbitrary polygonal regions are similarly available.

The following result combines elementary domains into patterns.

Theorem 3. Let $\nu_1, \ldots, \nu_\ell$ be maps and $\Omega_1, \ldots, \Omega_\ell \subset \mathbb{C}$, not necessarily disjoint, be such that, for $i = 1, \ldots, \ell$, ν_i is semiguarding for $\mathcal{S}(\mathbb{C} \backslash \partial\Omega_i)$. Then $\mathcal{S}(\cup\Omega_i)$ is semiguarded by $\nu = \prod \nu_i$.

Proof. It suffices to show that, for $A \in \partial\mathcal{S}(\cup\Omega_i)$, there is i such that $\nu_i(A) = 0$. The latter follows directly from (2) and the observation that, if $A \in \partial\mathcal{S}(\cup\Omega_i)$, then A has some eigenvalue on the boundary of one of the Ω_i's, i.e., $\sigma(A) \cap \partial\Omega_i \neq \emptyset$. ☐

In particular, the union of circles of radius ρ_i centered at c_i (e.g. a Butterworth filter pattern) is semiguarded by

$$\nu : A \mapsto \prod_i \det((A - c_i I) \otimes (A - c_i I)^H - \rho_i^2 I \otimes I). \tag{9}$$

A semiguardian map corresponding to domains such as the one depicted in Figure 1, of interest in dominant pole placement, may similarly be obtained by combining maps (8) and (9). If the maps ν_i of Theorem 3 are polynomic (as in all examples considered so far) and the Ω_i's are pairwise disjoint, this theorem, together with Theorem 2, provides a necessary and sufficient condition for the robust eigenvalue location problem. In the case of a scalar parameter r, this yields a computational test.

It should be clear that, except in some degenerate cases, the semiguardian map ν of Theorem 3 is not *guarding* for $\mathcal{S}(\cup\Omega_i)$, and thus that, in applying Theorem 2, the zeros of $\nu(A(\cdot))$ must actually be computed. Of help however is the fact that, if the Ω_i's are pairwise disjoint and guarded by the ν_i's, then $\prod \nu_i$ is "essentially guarding with respect to one-parameter families" for $\mathcal{S}(\cup\Omega_i)$ in the sense of the following definition.

Definition 2. Let ν be a polynomic semiguardian map for $\mathcal{S}$. Then ν is *essentially guarding with respect to k-parameter families* if the corresponding set of blind spots has real codimension at least $k + 1$. ☐

If ν is essentially guarding with respect to one-parameter families and the Ω_i's are pairwise disjoint and r is a scalar parameter, then, generically, $\Phi \subset \mathcal{S}(\Omega)$ if and only if $\nu(A(r)) \neq 0$ for all $r \in U$. Thus, if $\nu(A(\cdot))$ has some zero in U, it is "almost sure" that $\Phi \not\subset \mathcal{S}(\Omega)$ (and thus $\Phi \not\subset \mathcal{S}(\Omega, \mathcal{P}, \mathbf{n})$).

4. Discussion

A test has been obtained for the confinement of the eigenvalues of all the matrices (zeros of all the polynomials) of a given parametrized family in specific connected components of a subset of the complex plane. This result follows as a simple application of the concepts of guardian and semiguardian maps. Although Theorem 2 does not hold if the Ω_i's are not pairwise disjoint, it is readily verified that, in such case, the map provided by Theorem 3 is still semiguarding for $\mathcal{S}(\cup\Omega_i, \mathcal{P}, \mathbf{n})$, where $\mathcal{P} = (\Omega_1, \ldots, \Omega_\ell)$ and $\mathbf{n}$ is any ℓ-tuple of nonnegative integers. However, this map is no longer essentially guarding.

Recently, the Edge Theorem [7] has been extended to many disconnected regions, including those considered here [5,8]. Combined with these extensions, our techniques yield a computational test for the case of polytopes of polynomials.

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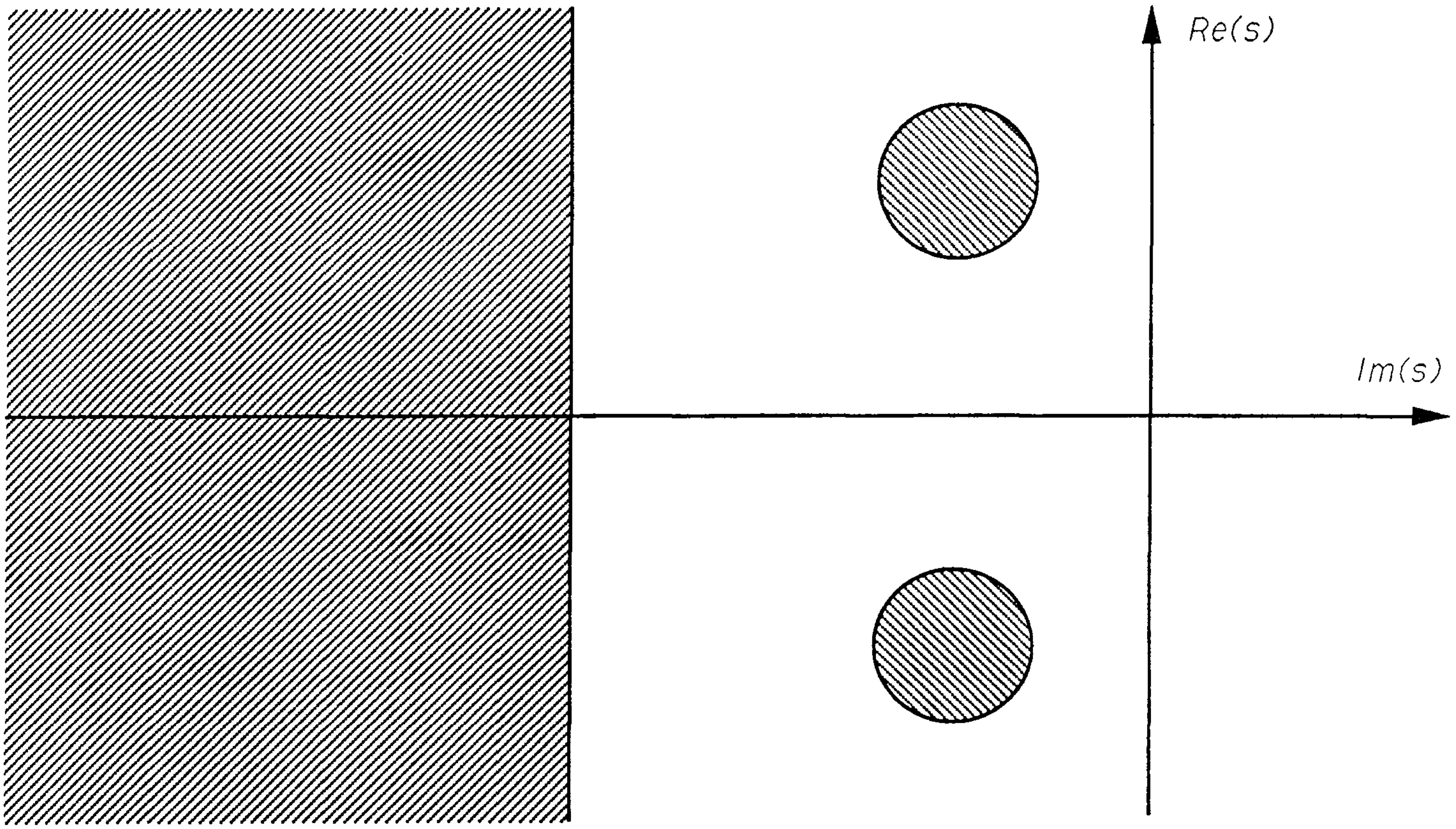

Figure 1. Dominant eigenvalue assignment configuration