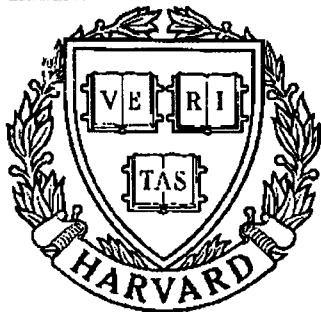


THESIS REPORT

Ph.D.



S Y S T E M S
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Studies in Robust Stability

*by L. Saydy
Advisor: E.H. Abed*

STUDIES IN ROBUST STABILITY

by

Lahcen Saydy

Dissertation submitted to the Faculty of the Graduate School
of the University of Maryland in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
1988

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Advisory Committee:

Associate Professor E. H. Abed
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Professor W. S. Levine
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Abstract

Title of Dissertation: Studies in Robust Stability

Lahcen Saydy, Doctor of Philosophy, 1988

Dissertation directed by: Dr. E. H. Abed, Associate Professor

Dr. A. L. Tits, Associate Professor

Electrical Engineering Department

In this thesis, questions in the analysis and synthesis of stability robustness properties for linear and nonlinear control systems are considered. The first part of this work is devoted to linear systems, where the emphasis is on obtaining necessary and sufficient conditions for stability of parametrized families of systems. This class of robustness problems has recently received significant attention in the literature [1]. In the second part of the thesis, questions of stabilization of nonlinear systems by feedback are considered.

Part I of this work addresses the generalized stability, i.e. stability with respect to a given domain in the complex plane, of parametrized families of linear time-invariant systems. The main contribution is the introduction and application of the new concepts of "guarding map" and "semiguarding map" for a given domain. Basically, these concepts allow one to replace the original parametrized system stability problem with a finite number of stability tests. Moreover, the tool is very powerful in that it allows the treatment of a large class of domains in the complex plane. The parametrized stability problem is completely solved

for the case of stability of a one-parameter family with respect to guarded and semiguarded domains. The primary interest in semiguarded domains arises in a process of reduction of a given multiparameter problem to one involving fewer parameters.

For the two-parameter case, we consider stability of families of matrices relative to domains with a polynomial guarding map. The first step replaces the two-parameter problem by a one-parameter stability problem relative to a new domain. The second step employs a polynomial semiguarding map for the new domain to obtain necessary and sufficient conditions for stability of the new problem. The case of three or more parameters, which involves technical questions not encountered in the one- or two-parameter case, is also considered.

In Part II, a class of nonlinear control systems for which the linear part satisfies special stabilizability conditions is considered. These conditions naturally give rise to certain nonstandard algebraic issues in linear systems. Sufficient conditions for the existence of a linear feedback control which stabilizes a given nonlinear control system within a prescribed ball of given radius (possibly infinite) are given. The feedback control is found to be robust in a certain sense against a class of modeling errors. A complete design methodology is obtained for planar systems and extended to a class of higher dimensional singularly perturbed nonlinear control systems. For these systems, nonlinear feedback laws achieving stabilization within prescribed cylindrical regions are presented.

To my wife Wassima,
who stood by me through thick and thin

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INTRODUCTION

1.1 Overview

In this thesis, we study questions in the analysis and synthesis of stability robustness properties for linear and nonlinear control systems. The first part of this work is devoted to linear systems, where the emphasis is on obtaining necessary and sufficient conditions for stability of parametrized families of systems. This class of robustness problems has recently received significant attention in the literature [1]. In the second part of the thesis, questions of stabilization of nonlinear systems by feedback are considered. The main goal is to arrive at a design procedure which allows one to realize a feedback control law achieving not only stability, but also a prespecified size requirement on the stable equilibrium point's region of asymptotic stability (RAS). In addition, robustness of the resulting controlled system with respect to certain modeling errors is also considered.

The two parts of the thesis, although seemingly addressing distinct and ostensibly unrelated problems, were indeed motivated by a common larger issue. This issue deals with a conceptual analytical/computational approach to the optimal feedback stabilization of nonlinear systems. The results of this dissertation may

thus be viewed as the first step in the study of this approach, which will be expounded on at the end of this Introduction. We now proceed to summarize the results of the thesis. More detailed summaries are given separately for Parts I and II, including discussions of the relation to previous work, in Chapters 2 and 8, respectively.

Part I of this work addresses the generalized stability, i.e. stability with respect to a given domain in the complex plane, of parametrized families of linear time-invariant systems. The main contribution is the introduction and application of the new concepts of "guarding map" and "semiguarding map" for a given domain. Basically, these concepts allow one to replace the original parametrized system stability problem with a finite number of stability tests. Moreover, the tool is very powerful in that it allows the treatment of a large class of domains in the complex plane. The parametrized stability problem is completely solved for the case of stability of a one-parameter family with respect to guarded and semiguarded domains. The primary interest in semiguarded domains arises in a process of reduction of a given multiparameter problem to one involving fewer parameters. For example, in studying stability of a two-parameter family of matrices with respect to, say, the open left-half of the complex plane, this reduction yields an equivalent stability problem for a one-parameter family, but relative to a new domain which is less amenable to analysis. The fact that the new domain is determined to be semiguarded allows one to proceed. The case of three or more parameters, which involves technical questions not encountered in the one- or two-parameter case, is also considered.

Though much of the emphasis is placed on the generalized stability of matrices, the concepts introduced in Part I apply to the stability of families of polynomials as well.

Part II of this work concerns the stabilization of nonlinear control systems. It is a fact that general methods for obtaining feedback control laws which stabilize the equilibrium point of interest of a given nonlinear control system are unavailable. Approximation techniques such as linearization around an equilibrium point are

often used in practical engineering situations. Regardless of the methods used to devise stabilizing control laws, a key step in any nonlinear control systems design is to estimate the resulting RAS. Despite the many techniques suggested in the literature, this task remains prohibitively expensive especially when it is an integral part of a synthesis effort. Notwithstanding anticipated difficulties, research efforts in the area of stabilization of nonlinear control systems within a prescribed RAS are crucially needed. Due to the complexity of this problem, it is recommended that classes of systems for which specific techniques apply be delineated, rather than attempting a more general approach.

In Part II, a class of nonlinear control systems for which the linear part satisfies special stabilizability conditions is considered. These conditions naturally give rise to certain nonstandard algebraic issues in linear systems. Sufficient conditions for the existence of a linear feedback control which stabilizes a given nonlinear control system within a prescribed ball of given radius (possibly infinite) are given. The feedback control is found to be robust in a certain sense against a class of modeling errors. A complete design methodology is obtained for planar systems and extended to a class of higher dimensional singularly perturbed nonlinear control systems. For these systems, nonlinear feedback laws achieving stabilization within prescribed cylindrical regions are presented.

1.2. Motivation

The challenges emanating from the field of nonlinear control systems design are numerous and require new techniques with CAD supporting tools. Lyapunov stability analysis has traditionally provided the main analytical tool for the analysis and synthesis of stable control systems when estimates of the region of asymptotic stability are an integral part of the problem [2]. Indeed, Lyapunov functions are instrumental in obtaining such estimates. Unfortunately, given a complex nonlinear system, it may be impossible to arrive at a useful Lyapunov function.

Consider a nonlinear control system

$$\dot{x} = F(x, u) \tag{1.1}$$

where F is a smooth vector field satisfying $F(0) = 0$. Synthesis of a stabilizing control law for this system may in principle be accomplished by the following general iterative approach. Decide on a method of estimating RAS. Choose an initial locally stabilizing design $u_0(x)$, fitting into a finite-parameter family of stabilizing controllers, and estimate the corresponding RAS. Following an optimization algorithm, decide on an updated control law $u_1(x)$ for which the RAS estimate is improved. This step involves running the RAS calculation procedure for several values of feedback gains. Continue in this way until a satisfactory RAS estimate is reached. This conceptual procedure involves two main nontrivial tasks. The first is to find a suitable optimization routine for updating the iterated feedback controls. The second, of course, is to reliably compute RAS estimates. The connection of the last of these two tasks with questions studied in Part I arises when one attempts a continuation approach to RAS estimation. At step k , denote $f(x) := F(x, u_k(x))$ and consider the stability of the parametrized family of nonlinear systems

$$\begin{aligned}\dot{x} &= (1 - \lambda)g(x) + \lambda f(x) \\ &:= h_\lambda(x)\end{aligned}\tag{1.2}$$

where $\lambda \in [0, 1]$ is the continuation parameter. Here, the vector field g is chosen such that the stability behavior of the auxiliary nonlinear system

$$\dot{z} = g(z), \quad g(0) = 0\tag{1.3}$$

is well understood (i.e., the domain of attraction is known exactly). Thus for $\lambda = 0$, $h_\lambda = g$ and the domain of attraction of System (1.2) is exactly known. For $\lambda = 1$, we obtain the system for which an RAS estimate is to be computed. The goal of the continuation method is then to use to advantage the perfect knowledge of the domain of attraction of System (1.2) at $\lambda = 0$ to infer something about that of the same system at $\lambda = 1$. This immediately raises the question of whether or not the set of “stable” vector fields is convex. Since the answer to this question is negative, it is of interest to obtain conditions ensuring stability of (1.2) for each $\lambda \in [0, 1]$, given that the vector fields g and f correspond to stable systems. One

such condition is that each member of the one-parameter family of matrices $(\frac{\partial f}{\partial x})$ denotes the Jacobian matrix of f)

$$(1 - \lambda) \frac{\partial g}{\partial x}(0) + \lambda \frac{\partial f}{\partial x}(0), \quad (1.4)$$

$\lambda \in [0, 1]$, be Hurwitz stable. This condition is one of Hurwitz stability of a one-parameter family of linear time-invariant systems.

INTRODUCTION TO PART I

In this chapter, we present some background on robust stability of linear time-invariant systems and give an overview of the contents of Part I. We then proceed to outline the essentials of the approach by treating a special case.

2.1 Background and Overview

In the analysis and synthesis of robust control systems, a fundamental problem that arises is the recognition that the mathematical model assumed for the system is always inexact, and that the parameters of the system may deviate away from their nominal values. Thus it is desirable to be able to determine to what extent a nominal system remains stable when subjected to a given class of perturbations. This is called the robust stability problem.

The specific description of the class of perturbations against which robustness is required depends, of course, on the physics and engineering of the particular plant in question. The general theory distinguishes broadly between two types of perturbations: structured and unstructured. In the latter case, perturbations are allowed to occur in “all directions” and are usually taken into account via

bounds on their norms. In the case of structured perturbations, good mathematical models are available and the plant structure is well known qualitatively but there is uncertainty regarding the numerical values of various physical parameters in the model.

After the usual simplifications, such as linearization about an equilibrium point, one ends up most often with a linear time-invariant system described by a prescribed set of differential equations for the nominal plant. There are two main approaches which have been applied to the robust stability problem for linear time-invariant systems: (i) the frequency domain approach (e.g. [3]-[9]) which is based on the transfer function representation of a system and (ii) the time domain approach (e.g. [10]-[13]).

Recently however, Kharitonov proved the following powerful result [14]: in order for every member of the family of real polynomials $\{p(s) = a_n s^n + \dots + a_1 s + a_0, a_i \in [\underline{a}_i, \bar{a}_i]\}$ to be Hurwitz stable (zeroes in the open left-half complex plane), it is necessary and sufficient that only four distinguished “corner polynomials” be Hurwitz stable. It is precisely Kharitonov’s Theorem which paved the way for the body of recent research surveyed in a paper by Barmish and DeMarco [1].

The problem of obtaining necessary and sufficient conditions for stability of polytopes of real polynomials and matrices has recently been considered by several authors (see for instance [15], [16] and [17]). For polytopes of polynomials, Bartlett, Hollot and Lin [17] showed that for such a polytope to be stable, it is enough to check that the edges of this polytope are stable. Hence, one needs only test stability of convex combinations of two polynomials. The solution of the latter problem was obtained in the case of Hurwitz stability by Bialas [15] and, in subsequent independent work, by Fu and Barmish [16]. An analogous result for Schur stability (zeroes inside the open unit disk) have also been obtained recently by Ackermann and Barmish [18]. The result in [17], a nice generalization of Kharitonov’s Theorem,¹ has unfortunately no counterpart in the case of polytopes

¹ Note that Kharitonov’s Theorem applies only to hyper-rectangular polytopes of polynomials with edges parallel to coordinate axes.

of real matrices [19]. In the particular case of the convex hull of two real matrices however, Bialas [15] and Fu and Barmish [16] derived necessary and sufficient conditions for Hurwitz stability.

In the remainder of Part I, we develop a new approach for the study of *generalized stability* of parametrized families of matrices. Similar considerations apply as well to the case of families of polynomials, although the detailed statements for the polynomial case are often omitted.

Generalized stability [8] of a matrix (polynomial) entails that its eigenvalues (zeroes) lie in a prespecified domain of the complex plane. The classical stability requirements result upon defining the domain of interest as the open left-half complex plane (continuous-time case) or the open unit disk (discrete-time case). Practical considerations relating to damping ratio, bandwidth, vehicle handling qualities, etc., are often best expressed in terms of the generalized stability formulation, with respect to a suitable domain in the complex plane.

In this first part, we introduce and apply to the generalized stability of parametrized families problem the new concepts of “guarding map” and “semi-guarding map” of a domain in the complex plane. These notions allow one to replace the problem at hand with the question of whether or not the guarding map is nonzero for all members of the family. These concepts are closely related to work of Gutman [20] and Gutman and Jury [21] on root clustering in domains of the complex plane.

Necessary and sufficient conditions are given for stability of one-parameter families relative to domains endowed with either a guarding or semiguarding polynomial map (defined below). The technique yields as a special case the result on stability of the convex hull of two matrices or polynomials mentioned above. Moreover, we solve as a special case the problem of Schur (i.e. discrete-time) stability of the convex hull of two matrices.

For the two-parameter case, we consider stability of families of matrices relative to domains with a polynomial guarding map. The first step replaces the two-parameter problem by a one-parameter stability problem relative to a new

domain. The second step employs a polynomial semiguarding map for the new domain to obtain necessary and sufficient conditions for stability of the new family.

Part I of the dissertation is organized as follows. In the remainder of this chapter, as an introduction to the techniques employed in Part I, we outline the main calculations for the case of Hurwitz stability of the convex hull of three 2×2 matrices. Chapter 3 establishes notation and provides requisite background material. The concepts of guarding and semiguarding maps are introduced in Chapter 4 where several examples are provided. In addition, a systematic procedure for constructing guarding and semiguarding maps is presented for domains with polynomial boundaries. In Chapter 5, basic results on generalized stability of parametrized families are given. These results are applied to one-parameter families of matrices in Chapter 6. Chapter 7 is devoted to the multiparameter case.

2.2 Essentials of the Approach: Calculations in a Special Case

The methodology presented here rests upon some rather technical results. However, the essentials of this methodology are basic in nature and may be more readily understood by first considering a special case. In this section, the steps of the analysis are discussed for the case of Hurwitz stability of the convex hull of three matrices. To minimize the algebra involved, this problem is considered for the case of 2×2 matrices. Parenthetically, we note that the results emanating from the discussion of this section are previously unknown and therefore interesting in their own right.

It is convenient to express the convex hull of three matrices in the equivalent form

$$A(r_1, r_2) := A_0 + r_1 A_1 + r_2 A_2, \quad r_1, r_2 \in [0, 1]. \quad (2.1)$$

We seek necessary and sufficient conditions for the matrix $A(r_1, r_2)$ to be Hurwitz stable for each pair of parameter values (r_1, r_2) with $0 \leq r_1, r_2 \leq 1$. In this case, we say that the matrix family (2.1) is Hurwitz stable.

For the matrix family (2.1) to be Hurwitz stable, it is of course necessary that, say, the matrix $A(0,0) = A_0$ be Hurwitz stable. Assume, then, that A_0 is Hurwitz stable. By continuity of the eigenvalues of a matrix in parameters, the family (2.1) is Hurwitz stable if and only if (iff) there is no pair of values (r_1, r_2) in $[0, 1] \times [0, 1]$ for which the matrix $A(r_1, r_2)$ has an eigenvalue on the imaginary axis.

To proceed further, associate to any 2×2 matrix $A = (a_{ij})$, $i, j = 1, 2$, the 3×3 matrix $\mathcal{N}(A)$ given by

$$\mathcal{N}(A) := \begin{bmatrix} 2a_{11} & 2a_{12} & 0 \\ a_{21} & a_{11} + a_{22} & a_{12} \\ 0 & 2a_{21} & 2a_{22} \end{bmatrix}. \quad (2.2)$$

The matrix $\mathcal{N}(A)$ has the interesting property that each of its eigenvalues is the pairwise sum of eigenvalues of A . More precisely,

$$\sigma(\mathcal{N}(A)) = \{2\lambda_1(A), \lambda_1(A) + \lambda_2(A), 2\lambda_2(A)\} \quad (2.3)$$

as can readily be checked. Therefore, $\mathcal{N}(A)$ is nonsingular when A is stable, but is singular when A has an eigenvalue on the imaginary axis. By these observations, and since A_0 , a member of the family (2.1), was assumed Hurwitz stable, we find that $A(r_1, r_2)$ is Hurwitz stable for all $r_1, r_2 \in [0, 1]$ iff²

$$\det \mathcal{N}(A(r_1, r_2)) \neq 0 \quad \forall r_1, r_2 \in [0, 1]. \quad (2.4)$$

Note that $\mathcal{N}(A)$ is a linear function of its matrix argument. This allows us to replace the requirement (2.4) by its equivalent

$$\det(\mathcal{N}(A_0) + r_1 \mathcal{N}(A_1) + r_2 \mathcal{N}(A_2)) \neq 0 \quad \forall r_1, r_2 \in [0, 1]. \quad (2.5)$$

Equation (2.5) is a nonsingularity condition on a two-parameter family of matrices. To replace this by a condition on a one-parameter family of matrices,

² In the general case considered in the following chapters, this observation will be associated with the notion of guarding map.

note that (2.5) holds iff³

$$\det(\mathcal{N}^{-1}(A_2)\mathcal{N}(A_0) + r_1\mathcal{N}^{-1}(A_2)\mathcal{N}(A_1) + r_2I) \neq 0 \quad (2.6)$$

for all $r_1, r_2 \in [0, 1]$. Define 3×3 matrices M_0 and M_1 by

$$M_0 = -\mathcal{N}^{-1}(A_2)\mathcal{N}(A_0) \quad (2.7)$$

$$M_1 = -\mathcal{N}^{-1}(A_2)\mathcal{N}(A_1). \quad (2.8)$$

We now have that, with the assumptions above, the family $A(r_1, r_2)$, $r_1, r_2 \in [0, 1]$, is Hurwitz stable iff $M_0 + r_1 M_1$ has no eigenvalues in $[0, 1]$ for each $r_1 \in [0, 1]$.

Denote

$$M(r_1) := M_0 + r_1 M_1. \quad (2.9)$$

Our problem has now been reduced to one of determining precisely when $M(r_1)$ has no eigenvalue in the interval $[0, 1]$ for each $r_1 \in [0, 1]$. For this to hold, it is necessary that $M_0 (= M(0))$ have no eigenvalue in $[0, 1]$, which we assume to be the case.

Since M_0 has no eigenvalue in $[0, 1]$, the only way for $M(r_1)$ to have an eigenvalue in $[0, 1]$ for some $r_1 \in [0, 1]$ is that there exist an r_1 such that either of the following three possibilities holds:

- (i) $M(r_1)$ has 0 as an eigenvalue;
- (ii) $M(r_1)$ has 1 as an eigenvalue; or
- (iii) $M(r_1)$ has a double eigenvalue in the open interval $(0, 1)$.

Conditions (i) and (ii) can be checked simply by solving for the zeroes of the polynomials $\det(M(r_1))$ and $\det(M(r_1) - I)$, respectively. On the other hand, testing condition (iii) is not as transparent, and is considered next.

Associate to any 3×3 matrix

$$M = \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix} \quad (2.10)$$

³ We have implicitly assumed $\mathcal{N}(A_2)$ is invertible. Although this is not necessary, it results in a simplified exposition.

a 3×3 matrix $Q(M)$ given by half of⁴

$$\begin{bmatrix} wy + tx + 4su + (v - r)^2 & 3tu + w(z + v - 2r) & -3sw - t(r + z - 2v) \\ 3sx + y(z + v - 2r) & wy + su + 4tx + (z - r)^2 & 3ty + s(v + r - 2z) \\ -3uy - x(z + r - 2v) & 3wx + u(v + r - 2z) & tx + su + 4wy + (z - v)^2 \end{bmatrix}$$

This matrix is interesting in the present context because its eigenvalues are given, in terms of those of M , by

$$\sigma(Q(M)) = \left\{ \frac{(\lambda_1(M) - \lambda_2(M))^2}{2}, \frac{(\lambda_1(M) - \lambda_3(M))^2}{2}, \frac{(\lambda_2(M) - \lambda_3(M))^2}{2} \right\}$$

This implies that $Q(M)$ is singular precisely when M has a repeated eigenvalue.

The following procedure summarizes the foregoing discussion. It yields necessary and sufficient conditions for Hurwitz stability of $A(r_1, r_2)$, $\forall r_1, r_2 \in [0, 1]$, under the assumption that $\mathcal{N}(A_2)$ is invertible.

- 0) Check that A_0 is Hurwitz stable.
- 1) Check that M_0 has no eigenvalues in $[0, 1]$.
- 2) Check that the polynomial $\det(M(r_1))$ has no zeroes in $[0, 1]$.
- 3) Check that the polynomial $\det(M(r_1) - I)$ has no zeroes in $[0, 1]$.
- 4) (a) Obtain all values of $r_1 \in [0, 1]$ for which $M(r_1)$ has a double eigenvalue by solving $\det Q(M(r_1)) = 0$.
 (b) For all such values, check that the corresponding double eigenvalues lie outside $(0, 1)$.

In the remainder of Part I, the technique outlined in this section is formalized and extended to the study of stability of multiparameter polynomial families of matrices relative to subsets of the complex plane of interest.

⁴ This construction is a special case of a general one, valid for matrices of arbitrary dimension, appearing in Chapter 3.

BACKGROUND FOR PART ONE

This chapter begins by establishing notation, and proceeds to a brief discussion of relevant background material on matrix algebra.

3.1. Notation

$\text{Arg}(s)$: Argument of the complex number s

\bar{s} : Conjugate of the complex number s

$\mathring{\mathbb{C}}_-$ ($\mathring{\mathbb{C}}_+$): Open left-half (right-half) complex plane

\mathcal{D}^c : Complement of set \mathcal{D}

$\overline{\mathcal{D}}$: Closure of set \mathcal{D}

$\partial\mathcal{D}$: Boundary of set \mathcal{D}

$\text{int}(\mathcal{D})$: Interior of set \mathcal{D}

I_n : Identity matrix of dimension n (also denoted I when n is clear from the context)

$\lambda_i(A)$: Eigenvalue of matrix A

$\sigma(A)$: Set of all eigenvalues of A

$\text{tr}(A)$: Trace of matrix A

Ω : Generic open subset of \mathbb{C} , symmetric about the real line

$\Xi: \mathbb{C} \setminus [0, 1]$

$\Theta: \mathbb{C} \setminus [1, \infty)$

$\mathcal{S}_n(\mathcal{D})$: Set of all $n \times n$ matrices with spectrum inside $\mathcal{D} \subset \mathbb{C}$ (also denoted $\mathcal{S}(\mathcal{D})$ when n is clear from the context)

\otimes, \oplus : Kronecker product, Kronecker sum

$A \ominus B: A \oplus (-B)$

$A \cdot B$: Bialternate product of A and B (Section 3.3)

$A^{[2]}$: Schläflian form of order 2 of matrix A (“Upper Schläflian”; Section 3.3)

$A_{[2]}$: Infinitesimal version of $A^{[2]}$ (“Lower Schläflian”; see Section 3.3)

3.2. Multivariate Polynomials

Let \mathcal{R} be a ring. Following Bose [22], we denote by $\mathcal{R}[r_1, r_2, \dots, r_k]$ the associated polynomial ring. Each element of this ring is a polynomial in the indeterminates r_1, r_2, \dots, r_k . Any such polynomial is the sum of a finite number of terms of the form

$$A_{i_1, i_2, \dots, i_k} r_1^{i_1} r_2^{i_2} \dots r_k^{i_k}$$

where i_1, i_2, \dots, i_k are nonnegative integers and the coefficients A_{i_1, \dots, i_k} are from the ring \mathcal{R} . The degree d_{r_i} of the polynomial $A(r_1, r_2, \dots, r_k)$ with respect to one of the variables r_i ($i = 1, 2, \dots, k$) is the highest exponent with which r_i occurs in the terms of this polynomial. The sum $i_1 + i_2 + \dots + i_k$ is the degree of the monomial $r_1^{i_1} r_2^{i_2} \dots r_k^{i_k}$. The largest of these sums,

$$d(A) := \max \left\{ \sum_{j=1}^k i_j : A_{i_1, i_2, \dots, i_k} \neq 0 \right\} \quad (3.1)$$

is called the degree of A . If the polynomial A has degree $d(A) = m$, then it will be denoted, for convenience, by

$$A(r_1, r_2, \dots, r_k) = \sum_{i_1 + i_2 + \dots + i_k = m} A_{i_1, i_2, \dots, i_k} r_1^{i_1} r_2^{i_2} \dots r_k^{i_k} \quad (3.2)$$

Despite the fact that polynomials with either scalar or matrix coefficients will be encountered throughout Part I, the notation in (3.2) will always be used to denote polynomial degree.

3.3. Some Tools From Matrix Algebra

Kronecker product and sum

Given square matrices A and B having dimension n_1 and n_2 , respectively, The *Kronecker product* (e.g. [23]) of A and B , denoted $A \otimes B$, is the square $n_1 n_2$ -dimensional matrix whose ij^{th} $n_2 \times n_2$ block-entry is given by $a_{ij}B$. The *Kronecker sum* $A \oplus B$ of A and B is the $n_1 n_2$ -dimensional matrix $A \otimes I_{n_2} + I_{n_1} \otimes B$. Note that $A \oplus A$ is linear in A .

The eigenvalues of $A \otimes B$ and $A \oplus B$ consist of the $n_1 n_2$ products $\lambda_i(A)\lambda_j(B)$ and $n_1 n_2$ sums $\lambda_i(A) + \lambda_j(B)$, respectively, over all ordered pairs (i, j) , $i = 1, \dots, n_1$, $j = 1, \dots, n_2$. For example, if A and B are 2×2 and 3×3 matrices with eigenvalues $\{\lambda_1, \lambda_2\}$ and $\{\alpha_1, \alpha_2, \alpha_3\}$, respectively, then

$$\sigma(A \oplus B) = \{\lambda_1 + \alpha_1, \lambda_1 + \alpha_2, \lambda_1 + \alpha_3, \lambda_2 + \alpha_1, \lambda_2 + \alpha_2, \lambda_2 + \alpha_3\}.$$

In fact, this is simply a special case of the following more general result. Let p be a complex polynomial in the variables x_1 and x_2 , given by

$$p(x_1, x_2) = \sum_{i,j=0}^{i+j=N} p_{ij} x_1^i x_2^j, \quad (3.3)$$

and consider the associated function of two complex square matrices A and B

$$P(A, B) := \sum_{i,j=0}^{i+j=N} p_{ij} A^i \otimes B^j. \quad (3.4)$$

Theorem 3.1. (Stéphanos[24]). With the notation above, the eigenvalues of $P(A, B)$ consist of the $n_1 n_2$ values $p(\lambda_i(A), \lambda_j(B))$ over all possible (ordered) pairs (i, j) , $i = 1, \dots, n_1$, $j = 1, \dots, n_2$.

Bialternate product

Let A and B be $n \times n$ matrices. To introduce the bialternate product of A and B , we first establish some notation. Let V^n be the $\frac{1}{2}n(n-1)$ -tuple consisting of pairs of integers (p, q) , $p = 2, 3, \dots, n$, $q = 1, \dots, p-1$, listed lexicographically. That is,

$$V^n = [(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3), \dots, (n, n-1)]. \quad (3.5)$$

Denote by V_i^n the i^{th} entry of V^n . Denote

$$f((p, q); (r, s)) = \frac{1}{2} \left(\begin{vmatrix} a_{pr} & a_{ps} \\ b_{qr} & b_{qs} \end{vmatrix} + \begin{vmatrix} b_{pr} & b_{ps} \\ a_{qr} & a_{qs} \end{vmatrix} \right) \quad (3.6)$$

where the dependence of f on A and B is kept implicit for simplicity. The *bialternate product* $A \cdot B$ of A and B is a $\frac{1}{2}n(n-1)$ -dimensional square matrix whose ij^{th} entry is given by⁵

$$(A \cdot B)_{ij} = f(V_i^n; V_j^n). \quad (3.7)$$

Define

$$\Psi(A, A) := \sum_{p, q} \psi_{pq} A^p \cdot A^q, \quad (3.8)$$

and denote the eigenvalues of the $n \times n$ matrix A by $\lambda_1, \dots, \lambda_n$.

Theorem 3.2. (Stéphanos [24]). With the notation above, the eigenvalues of $\Psi(A, A)$ are the $\frac{1}{2}n(n-1)$ values

$$\psi(\lambda_i, \lambda_j) := \frac{1}{2} \sum_{p, q} \psi_{pq} (\lambda_i^p \lambda_j^q + \lambda_i^q \lambda_j^p), \quad i = 2, \dots, n; \quad j = 1, \dots, i-1. \quad (3.9)$$

For example, if A is 3×3 then $\sigma(A \cdot A) = \{\lambda_1 \lambda_2, \lambda_1 \lambda_3, \lambda_2 \lambda_3\}$. In contrast, note that in this case $\sigma(A \otimes A) = \{\lambda_1^2, \lambda_1 \lambda_2, \lambda_1 \lambda_3, \lambda_2 \lambda_1, \lambda_2^2, \dots, \lambda_3^2\}$. As another example, it is easily checked (e.g. [20], [21]) that if

$$Q(A) = (A^2 \cdot I - A \cdot A) \quad (3.10)$$

⁵ As far as the properties discussed below are concerned, the particular ordering of V^n is immaterial. In the literature, it is typically left unspecified ([25], [20] and [21]).

for an $n \times n$ matrix A with eigenvalues $\lambda_1, \dots, \lambda_n$, then

$$\sigma(Q(A)) = \left\{ \frac{(\lambda_1 - \lambda_2)^2}{2}, \dots, \frac{(\lambda_1 - \lambda_n)^2}{2}, \frac{(\lambda_2 - \lambda_3)^2}{2}, \dots, \frac{(\lambda_{n-1} - \lambda_n)^2}{2} \right\}. \quad (3.11)$$

Schläflian Forms

The Schläflian forms,⁶ discussed next, have spectral properties akin to those of the Kronecker product and sum with the advantage of reduced dimensionality. Let $x = (x_1, \dots, x_n)^T$ and $p \geq 2$ be an integer. Denote by $x^{[p]}$ the N_p^n -dimensional vector ($N_p^n := \binom{n+p-1}{p}$) formed by the lexicographic listing of all linearly independent terms of the form

$$x_1^{p_1} \dots x_n^{p_n}, \quad \sum_{\substack{i=1 \\ p_i \geq 0}}^n p_i = p. \quad (3.12)$$

For a given $n \times n$ matrix A , the associated (*upper*) *Schläflian matrix* of order p [27], [28], [26] and [29], denoted $A^{[p]}$, is the N_p^n -dimensional square matrix defined by the implicit relationship

$$(Ax)^{[p]} = A^{[p]}x^{[p]}, \quad \forall x \in \mathbb{R}^n. \quad (3.13)$$

The related form $A_{[p]}$ (the “lower Schläflian matrix”) is defined as follows. Consider the equation $\dot{x} = Ax$ for $x \in \mathbb{R}^n$. Then $A_{[p]}$ is defined as the coefficient matrix in the equation

$$\frac{dx^{[p]}}{dt} = A_{[p]}x^{[p]}. \quad (3.14)$$

It is a simple exercise to show that $A_{[p]}$ may be alternatively defined in terms of $A^{[p]}$ by

$$A_{[p]} := \lim_{h \rightarrow 0} \frac{1}{h} [(I_n + hA)^{[p]} - I_{N_p^n}]. \quad (3.15)$$

In other words, $A_{[p]}$ is the Gateaux derivative of the nonlinear map $A \mapsto A^{[p]}$ evaluated at the identity I_n and acting on A . As such, $A_{[p]}$ is linear in A .

⁶ These are also referred to as power transformations [26].

Note. Strictly speaking, the Schläflian forms defined here are slightly different from those used in [28], [26] and [29] where the $x^{[p]}$ vector is defined by

$$C(p_1, p_2, \dots, p_n) x_1^{p_1} \dots x_n^{p_n}, \quad \sum_{\substack{i=1 \\ p_i \geq 0}}^n p_i = p$$

where the $C(p_1, p_2, \dots, p_n)$ are normalizing constants chosen so that the property $\|x^{[p]}\| = \|x\|^p$ holds. While this property is a desirable one for the applications sought in the aforementioned papers, it is not necessary in our case where only the spectral properties of the resulting Schläflian matrices are of interest. It is indeed a simple exercise to check that the matrices obtained here are similarly related to those derived using the latter definition for $x^{[p]}$.

The next result is essentially the same as results in [28] and [26].

Theorem 3.3. The eigenvalues of $A_{[p]}$ (resp. $A^{[p]}$) consist of the N_p^n sums (products) over *distinct* unordered index sets of the form

$$\lambda_{i_1}(A) + \dots + \lambda_{i_p}(A) \quad (\text{resp. } \lambda_{i_1}(A) \times \dots \times \lambda_{i_p}(A)). \quad (3.16)$$

In contrast, recall that the eigenvalues of the Kronecker sum $A \oplus A$ consist of the n^2 sums $\lambda_i(A) + \lambda_j(A)$ over ordered pairs (i, j) . In the light of Theorem 3, it is clear that $\sigma(A \oplus A) = \sigma(A_{[2]})$ (not counting multiplicities). Hence the $\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}$ lower Schläflian matrix $A_{[2]}$ may be viewed as a redundancy-free version of the $n^2 \times n^2$ matrix $A \oplus A$, as far as the eigenvalues are concerned. Because of this, $A_{[2]}$ may be used to advantage, instead of $A \oplus A$, in the application of some of the results presented in Part I. A similar statement clearly holds for $A \otimes A$ and $A^{[2]}$ as well.

GUARDING AND SEMIGUARDING MAPS

In this chapter, the concepts of guarding and semiguarding maps, relative to a given domain of the complex plane, are introduced. These concepts play a key role in subsequent developments. Several examples and some important properties are presented. A systematic procedure for constructing guarding and semiguarding maps for a whole class of domains of interest is also presented.

4.1 Definitions and Examples

4.1.1 Guarding Maps

The concept of guarding map will first be defined with reference to sets of square matrices. The degree of generality in Definition 4.1 will not be required in this dissertation however. A specialization of this concept tailored for our purposes is given in Definition 4.2.

Definition 4.1. Let \mathcal{F} and \mathcal{S} be subsets of $\mathbb{C}^{n \times n}$ such that the set $\mathcal{S} \cap \mathcal{F}$ is relatively open in \mathcal{F} . Let $\nu : \mathcal{F} \rightarrow \mathbb{C}$ be a continuous map. Say that ν *guards* \mathcal{S} *with respect to* \mathcal{F} if, for all $A \in \mathcal{F} \cap \overline{\mathcal{S}}$, the equivalence

$$A \in \partial\mathcal{S} \iff \nu(A) = 0$$

holds.

Throughout this dissertation, our main interest lies in sets of $n \times n$ real matrices which are stable relative to a given open subset of the complex plane. That is, subsets $S(\Omega)$ of $\mathbb{R}^{n \times n}$ which are given by

$$S(\Omega) = \{A \in \mathbb{R}^{n \times n} : \sigma(A) \subset \Omega\}, \quad (4.1)$$

where Ω is a given domain of the complex plane. At times, we choose to employ the notation $S_n(\Omega)$, in which the dimension n is explicit. Note that $S(\Omega)$ is an open subset of $\mathbb{R}^{n \times n}$ since eigenvalues of matrices are continuous functions of their arguments.

The definition above of guarding maps for sets of matrices in $\mathbb{C}^{n \times n}$ is of a sufficiently general nature to be useful in many problems not addressed in this dissertation. Indeed, there are stability questions which lead to sets of matrices which are not characterizable by stability relative to any domain in the complex plane. One such question is that of strict aperiodicity of a matrix wherein all its eigenvalues are real and distinct [25].

For the purposes of this dissertation, however, we introduce the following notion of “guarding map for a domain in the complex plane.” This notion is useful when the discussion centers around sets of matrices which are defined by being stable relative to given domains in the complex plane.

Definition 4.2. Let Ω be an open subset of the complex plane and \mathcal{F} a family of $n \times n$ real matrices. Let $\nu : \mathcal{F} \rightarrow \mathbb{C}$ be a continuous map. Say that the map ν *guards* Ω with respect to \mathcal{F} if, for all $A \in \mathcal{F} \cap \overline{S_n(\Omega)}$, the equivalence

$$A \in \partial S_n(\Omega) \iff \nu(A) = 0. \quad (4.2)$$

holds. In this case, we also say that Ω is *guarded* by ν with respect to \mathcal{F} . If a map ν guards a domain Ω with respect to $\mathcal{F} = \mathbb{R}^{n \times n}$ for each n , we say simply that ν guards Ω .

Definition 4.3. A guarding map is said to be *polynomial* if it is a polynomial function of the entries of its argument.

We now give some simple examples of guarding maps and the associated guarded domains. In each of these examples, n denotes a positive integer.

Examples A.

- A1) $\nu : A \mapsto \det(A)$ guards $\mathbb{C} \setminus \{0\}$ with respect to (w.r.t.) $\mathbb{R}^{n \times n}$, for any n .
- A2) $\nu : A \mapsto \det(A \oplus A)$ guards $\overset{\circ}{\mathbb{C}}_-$ w.r.t $\mathbb{R}^{n \times n}$, for any n . This follows from the property (mentioned above) that the spectrum of $A \oplus A$ consists of all pairwise sums of eigenvalues of A . Note that ν guards $\overset{\circ}{\mathbb{C}}_+$ as well. Another such guarding map ν is given by $\nu(A) = \det(A_{[2]})$.
- A3) $\nu : A \mapsto \det(A) \operatorname{tr}(A)$ guards $\overset{\circ}{\mathbb{C}}_-$ w.r.t. $\mathbb{R}^{2 \times 2}$.
- A4) Given $\beta > 0$, the map $\nu : A \mapsto \det[(A + i\beta I) \ominus (A - i\beta I)]$ guards the domain $\Omega^\beta := \{s : |\Im m(s)| < \beta\}$ (see Figure 4.1). Indeed, $\nu(A) = 0$ iff A has two eigenvalues $\lambda_1 = x + iy_1$ and $\lambda_2 = x + iy_2$ such that $y_2 - y_1 = 2\beta$. Therefore, if $A \in \overline{\mathcal{S}(\Omega^\beta)}$ then $\nu(A) = 0$ iff A has some eigenvalues on $\partial\Omega^\beta$, i.e., $A \in \partial\mathcal{S}(\Omega^\beta)$.
- A5) Given $\theta_0 \in [+ \frac{\pi}{2}, \pi)$, $\nu : A \mapsto \det(e^{i\theta_0} A \ominus e^{-i\theta_0} A)$ guards the sector $\Omega_{\theta_0} := \{s : |\operatorname{Arg}(s)| > \theta_0\}$ (see Figure 4.2). Again, $\nu(A) = 0$ iff A has eigenvalues $\lambda_k = r_k e^{i\theta_k}$, $k = 1, 2$ such that $r_1 e^{i\theta_1} e^{i\theta_0} - r_2 e^{i\theta_2} e^{-i\theta_0} = 0$, i.e., iff $r_1 = r_2$ and $\theta_2 - \theta_1 = 2\theta_0 \pmod{2\pi}$. Since $\frac{\pi}{2} \leq \theta_0 < \pi$, this says that for all $A \in \overline{\mathcal{S}(\Omega_{\theta_0})}$, $\nu(A) = 0$ iff A has at least one pair of eigenvalues on $\partial\Omega_{\theta_0}$ or a single eigenvalue at 0. Note that $\Omega_{\theta_0} = \overset{\circ}{\mathbb{C}}_-$ for $\theta_0 = \frac{\pi}{2}$.
- A6) Given $\rho > 0$, the map $\nu : A \mapsto \det(A \otimes A - \rho^2 I \otimes I)$ guards $B(\rho) := \{s : s\bar{s} < \rho^2\}$. Here too, ν guards the complement $\operatorname{int}(B^c(\rho))$ as well. Another guarding map for $B(\rho)$ is given by $\nu(A) = \det(A^{[2]} - \rho^2 I^{[2]})$.

It turns out that (interiors of) *complements* of guarded domains are of particular interest. For $\overset{\circ}{\mathbb{C}}_-$ and the open disk, Examples A2 and A6 show that the given maps which guard these domains also guard the interior of their complements. However, this is not the case in general, as the following examples illustrate.

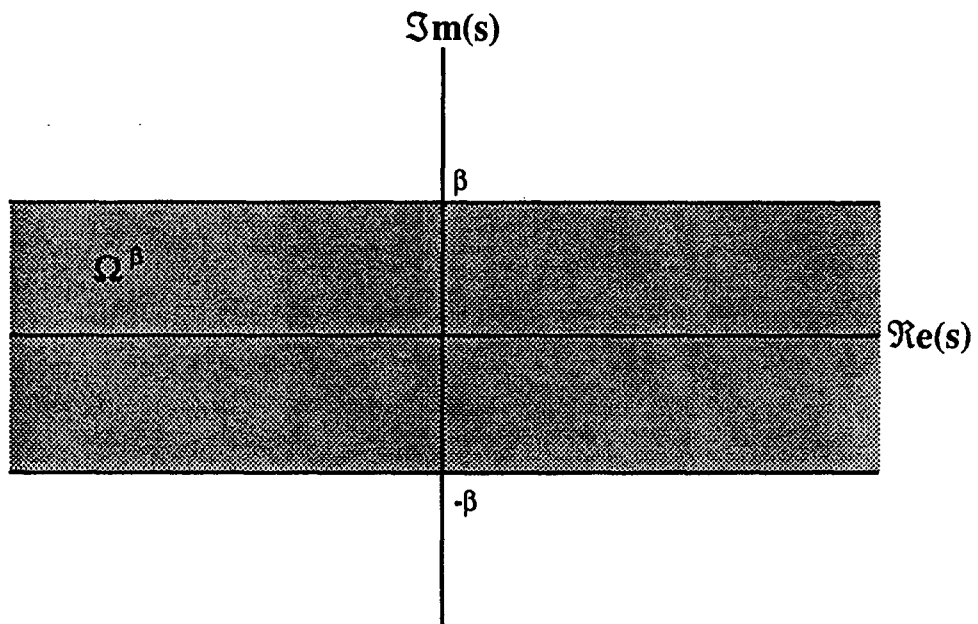


Figure 4.1: Stability domain for Example A4

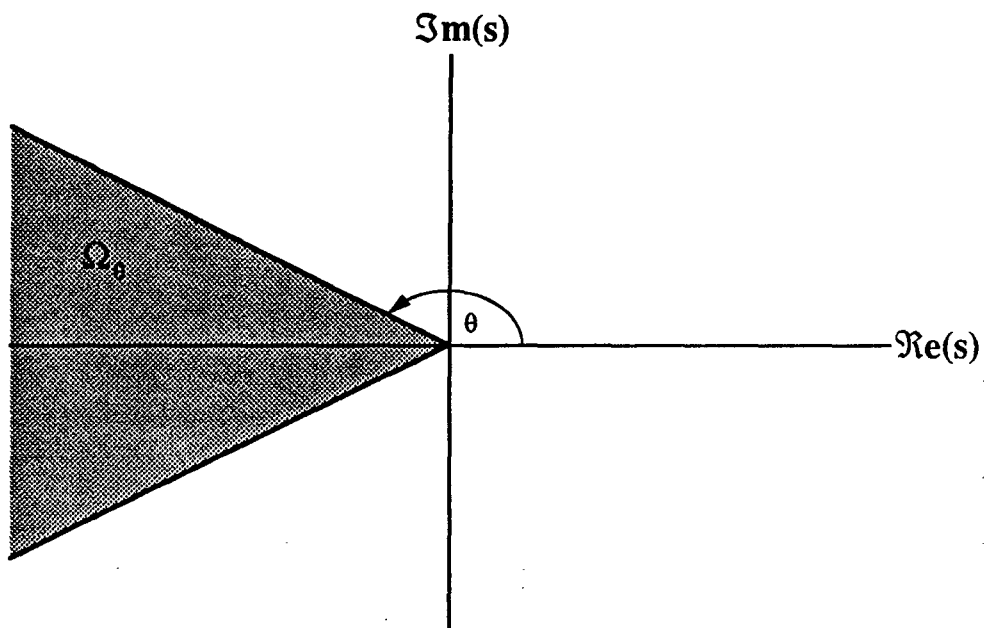


Figure 4.2: Stability domain for Example A5

Examples B.

- B1) Given $\beta > 0$, the set $\Omega = \{s : |\Im m(s)| > \beta\}$ is not guarded by $\nu : A \mapsto \det [(A + i\beta I) \ominus (A - i\beta I)]$ w.r.t. $\mathbb{R}^{n \times n}$ if $n > 3$. Indeed, one can easily construct a matrix $A \in S(\Omega)$ with a pair of eigenvalues $\lambda_1 = x + iy_1$, $\lambda_2 = x + iy_2$ such that $y_2 - y_1 = 2\beta$. For instance, take $A = \text{diag}(J_1, J_2)$ where

$$J_1 = \begin{bmatrix} x & y_1 \\ -y_1 & x \end{bmatrix}, \quad J_2 = \begin{bmatrix} x & y_1 + 2\beta \\ -(y_1 + 2\beta) & x \end{bmatrix}$$

and $y_1 > \beta$. Clearly, this cannot be done if $n \leq 3$, and Ω is guarded by ν w.r.t. both $\mathbb{R}^{2 \times 2}$ and $\mathbb{R}^{3 \times 3}$.

- B2) For $\theta_0 \in (\frac{\pi}{2}, \pi)$, the interior of the complement of the sector Ω_{θ_0} , namely, $\Omega = \{s : |\text{Arg}(s)| < \theta_0\}$, is not guarded by $\nu : A \mapsto \det(e^{i\theta_0} A \ominus e^{-i\theta_0} A)$ w.r.t. $\mathbb{R}^{n \times n}$ for any $n \geq 2$. Indeed, if $A \in S(\Omega)$ has eigenvalues $e^{i\theta}$ and $e^{-i\theta}$ where $\theta = \pi - \theta_0 < \theta_0$, then one eigenvalue of $e^{i\theta_0} A \ominus e^{-i\theta_0} A$ is given by $e^{i\theta_0} e^{i\theta} - e^{-i\theta_0} e^{-i\theta} = e^{i\pi} - e^{-i\pi} = 0$, implying $\nu(A) = 0$ although A is a stable matrix.
- B3) For $\theta_0 \in [\frac{\pi}{4}, \frac{\pi}{2})$, the map ν in Example B2 guards the set $\Omega := \{s : \theta_0 < |\text{Arg}(s)| < \pi - \theta_0\}$ w.r.t. $\mathbb{R}^{n \times n}$ for any $n \geq 2$. This fact is ascertained by paralleling the reasoning of Example A5 and noting that

$$\pi - 2\theta_0 = \max\{(\theta_2 - \theta_1) : \theta_1, \theta_2 \in (\theta_0, \pi - \theta_0)\} \leq 2\theta_0.$$

If $\theta_0 \in (0, \frac{\pi}{4})$, then one can easily check that ν does not guard Ω w.r.t. $\mathbb{R}^{n \times n}$ for any $n > 3$.

A common feature of the sets of Examples B1-B3 is that they fail to satisfy the sufficiency part of (4.2). For these sets, $\nu(A) = 0$ does not imply that $A \in \partial S(\Omega)$, even under the assumption $A \in \overline{S(\Omega)}$. The converse of the last statement does hold for these sets however. The next subsection is devoted to such maps (“semi-guarding maps”) and the associated domains (“semiguarded domains”). Later in Section 4.3, we find that a large class of domains in the complex plane (boundaries of which are defined by polynomials) possess polynomial semiguarding maps.

4.1.2 Semiguarding Maps

The following generalization of the concept of guarding maps will prove useful in the development to follow.

Definition 4.4. Let \mathcal{F} , \mathcal{S} and ν be as in Definition 4.1, with ν not identically zero. The map ν is said to be *semiguarding for \mathcal{S} with respect to \mathcal{F}* if, for all $A \in \mathcal{F} \cap \overline{\mathcal{S}}$, the implication

$$A \in \partial\mathcal{S} \Rightarrow \nu(A) = 0$$

holds. A matrix $A \in \mathcal{F} \cap \mathcal{S}$ for which $\nu(A) = 0$ is said to be a *blind spot*.

The analogue of Definition 4.2 for semiguarded domains now follows.

Definition 4.5. Let Ω , \mathcal{F} and ν be as in Definition 4.2, with ν not identically zero. The map ν is said to be *semiguarding for Ω with respect to \mathcal{F}* if, for all $A \in \mathcal{F} \cap \overline{\mathcal{S}(\Omega)}$, the implication

$$A \in \partial\mathcal{S}(\Omega) \Rightarrow \nu(A) = 0$$

holds. If ν is semiguarding for Ω w.r.t. $\mathbb{R}^{n \times n}$ for each n , then we say that ν is semiguarding for Ω . A matrix $A \in \mathcal{F} \cap \mathcal{S}(\Omega)$ for which $\nu(A) = 0$ is said to be a *blind spot* for $(\Omega, \nu, \mathcal{F})$.

In the light of this definition, a guarding map for a given domain Ω is simply a semiguarding map for which the corresponding set of blind spots is empty.

The maps considered in Examples B1-B2 above are semiguarding maps for the associated domains. A blind spot for the triplet $(\Omega, \nu, \mathcal{F})$ of Example B2 is any matrix in \mathcal{F} which has at least one eigenvalue in the mirror image (w.r.t. the imaginary axis) of $\partial\Omega \setminus \{0\}$ (dashed lines in Figure 4.3). Other examples, fundamental to the study of multiparameter families of matrices, are given next.

Examples C.

C1) Let α, β be finite real numbers, with $\alpha \neq \beta$. Then the set $\Omega = \mathbb{C} \setminus [\alpha, \beta]$ is semiguarded by the map.

$$\nu : A \mapsto \det(A^2 \cdot I - A \cdot A) \det((A - \alpha I)(A - \beta I)) \quad (4.3)$$

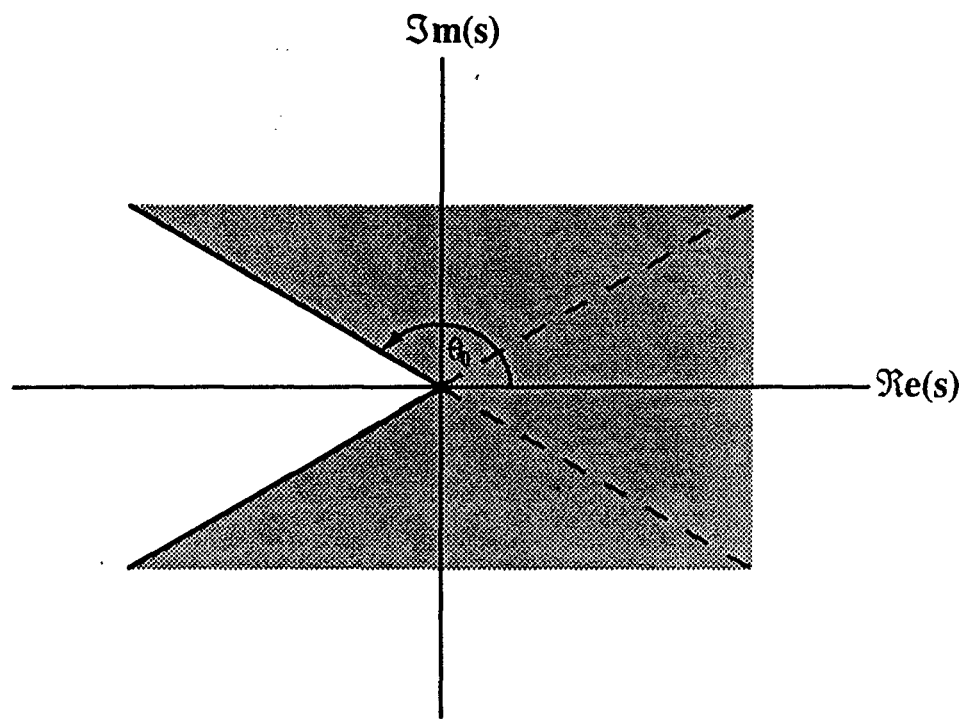


Figure 4.3: Eigenvalue location for the blind spots of Example B2

(A proof may be found in Section 4.3.) In previous examples, the map ν (guarding or semiguarding) took the form

$$\nu(A) = \det \mathcal{N}(A) \quad (4.4)$$

for some polynomial map \mathcal{N} of A . An efficient way in which the map (4.3) of the present example can be written in the form (4.4) is as follows. If $n = 3$, the matrices $(A^2 \cdot I - A \cdot A)$ and $((A - \alpha I)(A - \beta I))$ are each of dimension $\frac{n(n-1)}{2} = 3$ and $n = 3$, respectively, so that we can take $\mathcal{N}(A)$ to be their product. In case $n > 3$, then $\frac{n(n-1)}{2} > n$, and $\mathcal{N}(A)$ may now be taken as the product of $(A^2 \cdot I - A \cdot A)$ and

$$\begin{bmatrix} (A - \alpha I)(A - \beta I) & 0 \\ 0 & I \end{bmatrix}$$

where the identity matrix appearing in the lower right position has dimension $\frac{n(n-1)}{2} - n$. The case $n = 2$ can be handled in an analogous fashion.

C2) Consider again the domain Ω of Example C1, but now with $\alpha = -\infty$ and β finite. That is, $\Omega = \mathbb{C} \setminus (-\infty, \beta]$. In this case, Ω is semiguarded by the map

$$\nu : A \mapsto \det(A^2 \cdot I - A \cdot A) \det(A - \beta I). \quad (4.5)$$

4.1.3 Extension to Families of Polynomials

The family of (monic) real *polynomials* of degree n is isomorphic to the family of real $n \times n$ *companion matrices*. That this is true may be seen by noting that the polynomial $p(s) := s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$ and the characteristic polynomial of the companion matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \vdots & \vdots & \vdots & \vdots & 0 & 1 \\ -a_n & \vdots & \vdots & \vdots & \vdots & -a_2 & -a_1 \end{bmatrix}$$

are identical. Stability of the polynomial p is therefore equivalent to that of the matrix A . Consequently, stability questions for polynomials may be rephrased as a stability question for companion matrices. A direct but inefficient approach involves using the guarding map for $\mathring{\mathcal{C}}_-$ w.r.t. $n \times n$ companion matrices given in Example A1. The size of the test matrix involved would be $O(n^2)$. In the special case of companion matrices, a guarding map involving test matrices of size n may be constructed via *Orlando's formula* [30], [15]:

$$\det \mathcal{H}(A) = (-1)^{\frac{n(n-1)}{2}} \left(\frac{1}{2}\right)^n \prod_{1 \leq i < k \leq n} (x_i + x_k).$$

Here, x_1, \dots, x_n are the zeroes of $p(s)$, and $\mathcal{H}(A)$ is given by

$$\mathcal{H}(A) = \begin{bmatrix} a_1 & a_3 & a_5 & . & . & . & . & . & 0 \\ 1 & a_2 & a_4 & . & . & . & . & . & 0 \\ 0 & a_1 & a_3 & . & . & . & . & . & 0 \\ 0 & 1 & a_2 & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & a_{n-3} & a_{n-1} & 0 \\ 0 & . & . & . & . & . & . & a_{n-2} & a_n \end{bmatrix}.$$

Note that $\mathcal{H}(A)$ is the Hurwitz matrix associated with A , and as such is $n \times n$ and affine in A . From Orlando's formula, it is clear that $\mathring{\mathcal{C}}_-$ is guarded by the map $\nu : A \mapsto \det \mathcal{H}(A)$ w.r.t. the family of $n \times n$ real companion matrices.

An analogous construction for the study of Schur stability follows readily from results in [31] and [18]: Define the $(n-1) \times (n-1)$ matrix $D(A)$ by

$$D(A) = \begin{bmatrix} 1 & a_{n-1} & a_{n-2} & . & . & a_3 & a_2 - a_0 \\ 0 & 1 & a_{n-1} & . & . & a_4 - a_0 & a_3 - a_1 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & -a_0 & -a_1 & . & . & a_n - a_{n-4} & a_{n-1} - a_{n-3} \\ -a_0 & -a_1 & -a_2 & . & . & -a_{n-3} & a_n - a_{n-2} \end{bmatrix}.$$

Then we have the following expression for $\det D(A)$:

$$\det D(A) = \prod_{\substack{k=1 \\ i < k}}^n (1 - z_i z_k).$$

Clearly, $\det D(A)$ vanishes whenever A has a pair of conjugate eigenvalues on the unit circle. Note however that $\det D(A)$ does not necessarily vanish if A only has 1 (or -1) as eigenvalue. Taking this into account, we conclude that the unit disk is guarded by $\nu : A \mapsto \det D(A) \det(I - A^2)$ w.r.t. the family of $n \times n$ real companion matrices.

4.2 Generating New Guarded and Semiguarded Domains

The next proposition states properties which provide means for the construction of new guarded and semiguarded domains from known ones.

Proposition 4.1. Let \mathcal{F} be a subspace of $\mathbb{R}^{n \times n}$.

- (i) Let Ω be guarded (resp. semiguarded) by ν w.r.t. \mathcal{F} . Then $-\Omega := \{-s : s \in \Omega\}$ is guarded (resp. semiguarded) by $\nu_- : A \mapsto \nu(-A)$ w.r.t. \mathcal{F} .
- (ii) Let Ω be guarded (resp. semiguarded) by ν w.r.t. \mathcal{F} . Then $\Omega^{(\alpha)} := \{s + \alpha : s \in \Omega\}$, $\alpha \in \mathbb{R}$, is guarded (resp. semiguarded) by $\nu_\alpha : A \mapsto \nu(A - \alpha I)$ w.r.t. \mathcal{F} .
- (iii) Let Ω_1 and Ω_2 be guarded (resp. semiguarded) by ν_1 and ν_2 w.r.t. \mathcal{F} , respectively and suppose $\Omega_1 \cap \Omega_2 \neq \emptyset$. Then $\Omega_1 \cap \Omega_2$ is guarded (resp. semiguarded) by $\nu : A \mapsto \nu_1(A) \nu_2(A)$ w.r.t. \mathcal{F} .

Two examples illustrating the application of Proposition 1 are presented next.

Examples D.

- D1) To ensure an adequate step response, it is often desirable in the design of compensators for linear control systems that the eigenvalues (poles) of the closed-loop system be in the domain $\Omega_1 := \{s : \Re(s) < -\sigma ; |\Arg(s)| > \theta\}$, for some $\sigma > 0$ and $\theta \in (\frac{\pi}{2}, \pi)$ (see Figure 4.4). Letting Ω_θ denote the set in Example A5, and using the notation of Proposition 1, it is seen that $\Omega_1 = \Omega_\theta \cap \mathring{\mathcal{C}}_-^{(-\sigma)}$. Since guarding maps for Ω_θ and $\mathring{\mathcal{C}}_-^{(-\sigma)}$ are available, it follows from Proposition 1 that Ω_1 is guarded by the map ν_1 given by

$$\nu_1(A) = \det(e^{i\theta} A \ominus e^{-i\theta} A) \det((A + \sigma I) \oplus (A + \sigma I)). \quad (4.6)$$

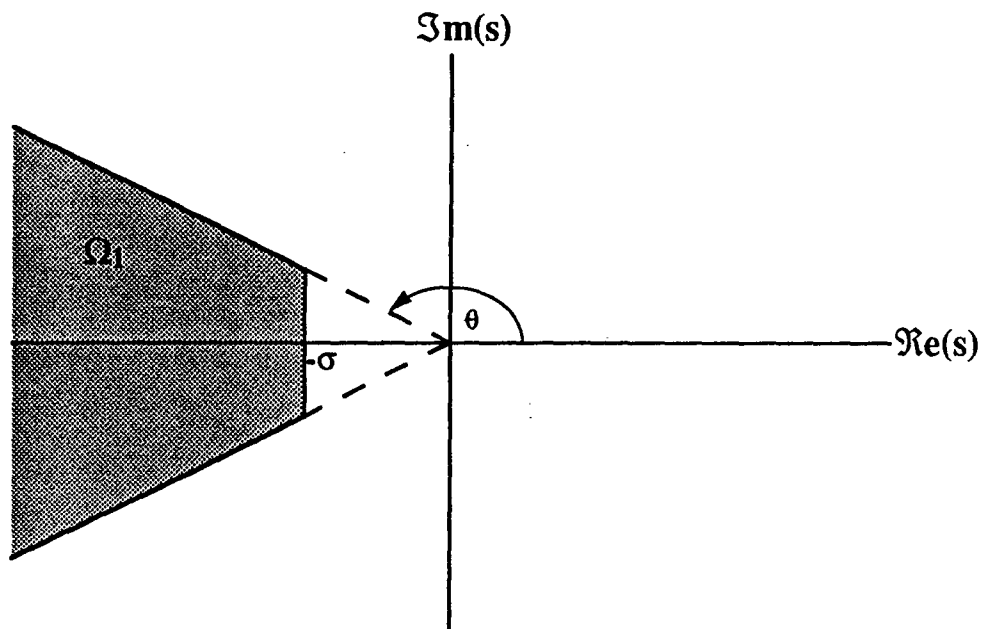


Figure 4.4: Stability domain for Example D1

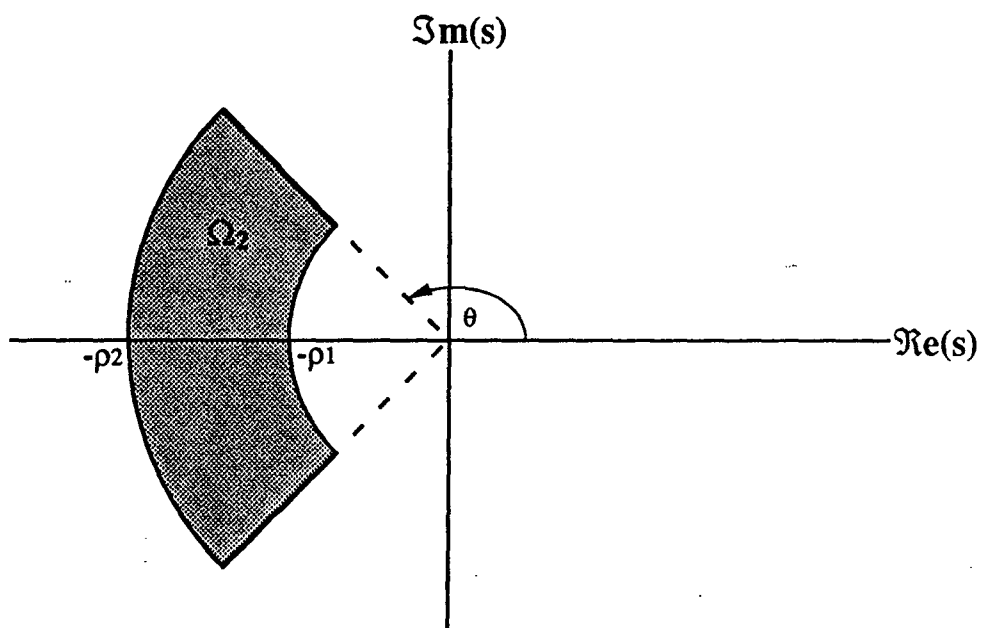


Figure 4.5: Stability domain for Example D2

D2) The domain $\Omega_2 := \{s : \rho_1 < |s| < \rho_2 ; |\text{Arg}(s)| > \theta\}$, where $\theta \in (\frac{\pi}{2}, \pi)$ and ρ_1 and ρ_2 are two positive real numbers (see Figure 4.5), arises in aircraft controller design ([32], p. 394). To apply Proposition 4.1, write

$$\Omega_2 = \Omega_\theta \cap B(\rho_1) \cap \text{int}(B^c(\rho_2))$$

where $B(\rho)$ is as in Example A6. Thus Ω_2 is guarded by the map ν_2 given by

$$\nu_2(A) = \det(e^{j\theta} A \ominus e^{-j\theta} A) \det(A^{[2]} - \rho_1^2 I^{[2]}) \det(A^{[2]} - \rho_2^2 I^{[2]}). \quad (4.7)$$

Since any convex domain (symmetric w.r.t. to the real axis, symmetric for short) with polygonal boundary may be generated from the two basic domains Ω^β and Ω_θ , using the three basic operations in Proposition 4.1 we have the following subsidiary result.

Proposition 4.2. Any (symmetric) convex domain with polygonal boundary is guarded by a polynomial map. Moreover, Proposition 4.1 can be used to construct a guarding map.

4.3 Maps for Domains with Polynomial Boundaries

In this section, we construct guarding and semiguarding maps for a whole class of subsets of the complex plane. Specifically, these maps are constructed for domains whose boundaries are given by a polynomial equation $p(x, y) = 0$ where x and y denote real and imaginary parts, respectively.

Denote

$$\Omega = \{s = x + iy : p(x, y) < 0\} \quad (4.8)$$

where

$$p(x, y) = \sum_{k, \ell=0}^{k+2\ell=N} p_{k\ell} x^k y^{2\ell}, \quad (4.9)$$

is a real polynomial of degree $d(p) = N$. The fact that we focus on real matrices is accounted for by considering polynomials containing only even powers of y . Thus only domains symmetric w.r.t. the real axis are considered.

Associate with p the real valued polynomial

$$\begin{aligned} q(\lambda, \bar{\lambda}) &= p\left(\frac{\lambda + \bar{\lambda}}{2}, \frac{\lambda - \bar{\lambda}}{2i}\right) \\ &= \sum_{k, \ell=0}^{k+2\ell=N} p_{k\ell} (-1)^\ell \left(\frac{1}{2}\right)^{k+2\ell} (\lambda + \bar{\lambda})^k (\lambda - \bar{\lambda})^{2\ell}. \end{aligned} \quad (4.10)$$

Rewrite (4.10) as

$$q(\lambda, \bar{\lambda}) = \sum_{k, \ell=0}^{k+2\ell=N} q_{k\ell} \lambda^k \bar{\lambda}^\ell \quad (4.11)$$

where the coefficients $q_{k\ell}$ are real.

With this notation, Ω and $\partial\Omega$ have the alternative expressions

$$\Omega = \{\lambda \in \mathbb{C} : q(\lambda, \bar{\lambda}) < 0\} \quad (4.12)$$

$$\partial\Omega = \{\lambda \in \mathbb{C} : q(\lambda, \bar{\lambda}) = 0\} \quad (4.13)$$

Consider the mapping $\mathcal{N} : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n^2 \times n^2}$ given by

$$\mathcal{N}(A) := \sum_{k, \ell} q_{k\ell} A^k \otimes A^\ell. \quad (4.14)$$

Theorem 1 implies that with $\sigma(A) = \{\lambda_1, \dots, \lambda_n\}$,

$$\sigma(\mathcal{N}(A)) = \{q(\lambda_i, \lambda_j) : i, j = 1, \dots, n\}. \quad (4.15)$$

Now suppose that $A \in \partial S(\Omega)$. Then some eigenvalue of A satisfies $\lambda_i \in \partial\Omega$, i.e. $q(\lambda_i, \bar{\lambda}_i) = 0$. It then follows from (4.15) that $\mathcal{N}(A)$ is singular ($\det \mathcal{N}(A) = 0$). We obtain the following propositions.

Proposition 4.3. Assume that ν is not identically zero. Then the map

$$\nu : A \mapsto \det \sum_{k, \ell} q_{k\ell} A^k \otimes A^\ell \quad (4.16)$$

is semiguarding for Ω .

Proposition 4.4. The map (4.16) guards Ω iff q satisfies the condition⁷

$$q(\lambda, \bar{\lambda}) < 0 \text{ and } q(\mu, \bar{\mu}) < 0 \Rightarrow q(\lambda, \mu) \neq 0. \quad (\text{Property } \mathcal{G})$$

Maps such as (4.16) involve determinants of matrices the size of which increases rapidly as n does. Alternative formulas, based on the bialternate product, exist which involve matrices of dimension $\frac{n(n-1)}{2}$, which is approximately half that of $\mathcal{N}(A)$ for large n .

Consider the mapping Q from $\mathbb{R}^{n \times n}$ to $\mathbb{R}^{\frac{n(n-1)}{2} \times \frac{n(n-1)}{2}}$ given by

$$Q(A) = \sum_{k, \ell} q_{k\ell} A^k \cdot A^{2\ell}. \quad (4.17)$$

Theorem 3.2 implies that

$$\sigma(Q(A)) = \left\{ \sum_{k, \ell} q_{k\ell} (\lambda_i^k \lambda_j^\ell + \lambda_i^\ell \lambda_j^k) / 2 : i = 1, \dots, n-1; j = i+1, \dots, n \right\}. \quad (4.18)$$

Note that

$$\sigma(Q(A)) = \{q(\lambda_i, \lambda_j) : i = 1, \dots, n-1; j = i+1, \dots, n\}. \quad (4.19)$$

(Compare with Eq. (4.15).) This follows from (4.18) and the fact (implied by (4.10)) that

$$q(\lambda, \mu) = q(\mu, \lambda) = \frac{1}{2}(q(\lambda, \mu) + q(\mu, \lambda)). \quad (4.20)$$

Proposition 4.5. Suppose that $\det Q(A)$ is not identically zero.

(a) If $\partial\Omega \cap \mathbb{R} = \emptyset$, then the map

$$\nu : A \mapsto \det Q(A) \quad (4.21)$$

⁷ “ Ω -transformability” [21].

is semiguarding for Ω .

- (b) Let $\partial\Omega \cap \mathbb{R} = \cup_{i=1}^{\xi} [\alpha_i, \beta_i]$ with $\alpha_1 \leq \beta_1 < \dots < \alpha_i \leq \beta_i < \dots < \alpha_{\xi} \leq \beta_{\xi}$.

Denote $\mathcal{P}(A) = \det \left(\prod_{i=1}^{\xi} (A - \alpha_i I)(A - \beta_i I) \right)$ where, by convention, the factor $(A - \alpha_i I)$ (resp. $(A - \beta_i I)$) is omitted when α_i (resp. β_i) is $-\infty$ (resp. $+\infty$). Then the mapping

$$\nu : A \mapsto \det Q(A) \det \mathcal{P}(A) \quad (4.22)$$

is semiguarding for Ω .

Proof. Let $A \in \mathbb{R}^{n \times n} \cap \partial\mathcal{S}(\Omega)$:

- (a) If A has an eigenvalue $\lambda \in \partial\Omega$ then $\bar{\lambda}$, also an eigenvalue of A , is distinct from λ . Therefore $q(\lambda, \bar{\lambda}) = 0 \in \sigma(Q(A))$ by virtue of (4.17). Hence $\nu(A) = 0$.
- (b) Let $\lambda \in \partial\Omega$ be an eigenvalue of A . That is, $q(\lambda, \bar{\lambda}) = 0$. If $\lambda \notin \mathbb{R}$, then $\bar{\lambda}$ is also an eigenvalue of A , distinct from λ . Consequently, $q(\lambda, \bar{\lambda}) = 0 \in \sigma(Q(A))$ and $\nu(A) = 0$. If $\lambda \in \mathbb{R} \cap \partial\Omega$, then $\lambda \in [\alpha_i, \beta_i]$ for some $i \in \{1, \dots, \xi\}$. Since by assumption $A \in \partial\Omega$, then it is the limit of a sequence of matrices $\{A_k\}$ with each $A_k \in \mathcal{S}(\Omega)$. It follows that there is a $j \in \{1, \dots, n\}$ such that $\lambda = \lim_{k \rightarrow \infty} \lambda_j(A_k)$. If $\lambda \in (\alpha_i, \beta_i)$ ($\beta_i > \alpha_i$ assumed) then there is a positive K such that for all $k > K$

$$\lambda_j(A_k) \in \mathbb{C} \setminus \mathbb{R} \quad \text{and} \quad \lambda = \lim_{k \rightarrow \infty} \lambda_j(A_k).$$

Since $\{A_k\}$ is a sequence of real matrices, we have that $\lambda = \lim_{k \rightarrow \infty} \bar{\lambda}_j(A_k)$ as well. Consequently, λ must be an eigenvalue of A of multiplicity at least 2. By virtue of (4.17), $q(\lambda, \bar{\lambda}) = q(\lambda, \lambda) \in \sigma(Q(A))$, i.e. $\nu(A) = 0$. If $\alpha_i \neq \beta_i$ and λ is either α_i or β_i , then λ might be a simple eigenvalue of A , in which case $q(\lambda, \lambda) = 0$ is no longer an eigenvalue of $Q(A)$. The case when α_i and β_i are finite and $\lambda = \alpha_i = \beta_i$ is handled similarly. The reason for the introduction of the second factor $\det \mathcal{P}(A)$ in the expression (4.22) for ν should now be clear.

Remark 4.1. Proposition 4.4 applies for the maps of Proposition 4.5 as well.

We have exhibited semiguarding maps for domains with polynomial boundaries. Determining whether or not these maps are also guarding requires further investigation. One needs to check whether or not the polynomial q satisfies Property \mathcal{G} . Sufficient conditions for Property \mathcal{G} were obtained by Gutman and Jury [21] for the cases in which the degree of polynomial p (or q) is 1, 2, 3 or 4. Another result in this direction is given next.

Proposition 4.6. Suppose that

$$q_{kk} \geq 0, \quad \forall k \geq 1, \quad (4.23)$$

$$q_{k\ell} = 0, \quad \forall k \neq \ell, \quad k\ell \neq 0. \quad (4.24)$$

Then q satisfies Property \mathcal{G} .

Proof. Proceeding by contradiction, assume that for some pair (λ, μ) , $q(\lambda, \mu) = 0$, $q(\lambda, \bar{\lambda}) < 0$ and $q(\mu, \bar{\mu}) < 0$. Since the coefficients of q are real, we also have $q(\bar{\lambda}, \bar{\mu}) = 0$. Set

$$w := q(\lambda, \bar{\lambda}) + q(\mu, \bar{\mu}).$$

Thus $w < 0$ and

$$\begin{aligned} w &= q(\lambda, \bar{\lambda}) + q(\mu, \bar{\mu}) - (q(\lambda, \mu) + q(\bar{\lambda}, \bar{\mu})) \\ &= \sum_{k,\ell} q_{k\ell} (\lambda^k \bar{\lambda}^\ell + \mu^k \bar{\mu}^\ell - (\lambda^k \mu^\ell + \bar{\lambda}^k \bar{\mu}^\ell)). \end{aligned}$$

From (4.10) and (4.11), we have

$$q_{k0} = q_{0k}, \quad \forall k \geq 1. \quad (4.25)$$

It now follows from (4.24) and (4.25) that

$$\begin{aligned} w &= \sum_{k=1} q_{k0} (\lambda^k + \mu^k - (\lambda^k + \bar{\lambda}^k) + \bar{\lambda}^k + \bar{\mu}^k - (\mu^k + \bar{\mu}^k)) \\ &\quad + \sum_{k=1} q_{kk} (\lambda^k \bar{\lambda}^k + \mu^k \bar{\mu}^k - (\lambda^k \mu^k + \bar{\lambda}^k \bar{\mu}^k)). \end{aligned}$$

The first summation yields zero. Under assumption (4.23), the second summation is nonnegative, as can be seen by noting that, with $\alpha = r_1 e^{j\theta_1}$ and $\beta = r_2 e^{j\theta_2}$,

$$\begin{aligned}\alpha\bar{\alpha} + \beta\bar{\beta} - (\alpha\beta + \bar{\alpha}\bar{\beta}) &= r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 + \theta_2) \\ &\geq (r_1 - r_2)^2.\end{aligned}$$

This contradicts the fact that $w < 0$.

□

Example for Proposition 4.6. Let $\Omega = \{s = x + iy : x + y^2 < 0\}$. Here $p(x, y) = x + y^2$ and

$$q(\lambda, \mu) = \frac{1}{2}\lambda + \frac{1}{2}\mu + \frac{1}{2}\lambda\mu - \frac{1}{4}\lambda^2 - \frac{1}{4}\mu^2$$

It follows from Proposition 4.6 that q satisfies Property \mathcal{G} , and therefore that Ω is guarded by both

$$\nu : A \mapsto \det \mathcal{N}(A) \text{ and } \nu : A \mapsto \det Q(A) \det(A).$$

Here,

$$\begin{aligned}\mathcal{N}(A) &= \frac{1}{2}(A \otimes I + I \otimes A) + \frac{1}{2}A \otimes A - \frac{1}{4}(A^2 \otimes I + I \otimes A^2) \\ &= \frac{1}{2}(A \oplus A) - \frac{1}{4}(A \ominus A)^2,\end{aligned}\tag{4.26}$$

and

$$\begin{aligned}Q(A) &= \frac{1}{2}(A \cdot I + I \cdot A) + \frac{1}{2}A \cdot A - \frac{1}{4}(A^2 \cdot I + I \cdot A^2) \\ &= A \cdot I + \frac{1}{2}A \cdot A - \frac{1}{2}I \cdot A^2.\end{aligned}\tag{4.27}$$

**BASIC RESULTS ON GENERALIZED STABILITY
OF PARAMETRIZED MATRICES**

In this chapter, basic results on stability of parametrized matrices relative to guarded and semiguarded domains are given.

Let $r = (r_1, \dots, r_k) \in U$, where U is a connected subset of \mathbb{R}^k , and let $A(r) \in \mathcal{F}$ be a real $n \times n$ matrix which depends continuously on the parameter vector r where \mathcal{F} is a given family of matrices in $\mathbb{R}^{n \times n}$. Given a domain Ω and an associated guarding or semiguarding map ν w.r.t. \mathcal{F} , we seek basic conditions for stability of the family $A(r)$, $r \in U$, relative to Ω . Throughout this section, reference to \mathcal{F} will be omitted for convenience.

5.1. Guarded Domain Case

Proposition 5.1. Let Ω be guarded by the map ν and assume that $A(r^0) \in \mathcal{S}(\Omega)$ for some $r^0 \in U$. Then

$$A(r) \in \mathcal{S}(\Omega) \quad \text{for all } r \in U \quad \Longleftrightarrow \quad \nu(A(r)) \neq 0 \quad \text{for all } r \in U \quad (5.1)$$

Proof. Proceeding by contradiction, suppose that $A(r^1) \notin \mathcal{S}(\Omega)$ for some $r^1 \in U$. By virtue of the connectedness of U , there exists a curve $\{r(t) : t \in [t_0, t_1]\}$ within

U , such that $r(t_0) = r^0$ and $r(t_1) = r^1$. Now consider $A(r(t))$ as t increases from t_0 . Since $A(r^0) \in \mathcal{S}(\Omega)$ and $\mathcal{S}(\Omega)$ is open, it follows that there is a $t^* \in (t_0, t_1]$ such that

$$A(r(t^*)) \in \partial\mathcal{S}(\Omega).$$

This implies that there is an $r^* \in U$ (namely $r^* = r(t^*)$) such that

$$A(r^*) \in \partial\mathcal{S}(\Omega).$$

Since ν guards Ω , we conclude that

$$\nu(A(r^*)) = 0.$$

This proves sufficiency. Necessity follows in a similar fashion from the definition of guardedness of Ω . □

Remark 5.1. The sufficiency part still holds if ν is only semiguarding for Ω . More specifically, if Ω is semiguarded by ν and $A(r^0) \in \mathcal{S}(\Omega)$ for some $r^0 \in U$, then $\nu(A(r)) \neq 0$ for all $r \in U$ implies that $A(r) \in \mathcal{S}(\Omega)$ for all $r \in U$.

5.2. Semiguarded Domain Case

The next proposition is the analogue of Proposition 5.1 for semiguarded domains. Its proof is similar to the one given above and is omitted.

Proposition 5.2. Let Ω be semiguarded by ν and assume that $A(r^0) \in \mathcal{S}(\Omega)$ for some $r^0 \in U$. Then the equivalence

$$A(r) \in \mathcal{S}(\Omega) \text{ for all } r \in U \iff A(r) \in \mathcal{S}(\Omega) \text{ for all } r \in U_{\text{cr}} \quad (5.2)$$

holds, where

$$U_{\text{cr}} := \{r \in U : \nu(A(r)) = 0\}.$$

Proposition 5.2 asserts that for the infinite family of real matrices $\{A(r) : r \in U\}$ to be stable relative to Ω , it suffices to check that the family of matrices

$\{A(r) : r \in U_{\text{cr}}\}$ is stable. In other words, to establish that the family $\{A(r) : r \in U\}$ is stable relative to Ω , one has to check that every matrix $A(r)$, $r \in U$, for which $\nu(A(r)) = 0$ is indeed a blind spot, in the sense of Definition 4.5. In cases where U_{cr} is a finite set, Proposition 5.2 therefore provides a tool for asserting the stability of the family

Remark 5.2. The assumption $A(r^0) \in \mathcal{S}(\Omega)$ for some $r^0 \in U$ appearing in Proposition 5.2 is, strictly speaking, required only in the case $U_{\text{cr}} = \emptyset$. Thus, Proposition 5.2 may be replaced by the following two statements: (i) Let Ω be semiguarded by ν and let $U_{\text{cr}} = \emptyset$. Then the equivalence

$$A(r) \in \mathcal{S}(\Omega) \text{ for all } r \in U \iff A(r^0) \in \mathcal{S}(\Omega) \text{ for some } r^0 \in U$$

holds. (ii) Let Ω be semiguarded by ν and let $U_{\text{cr}} \neq \emptyset$. Then the equivalence

$$A(r) \in \mathcal{S}(\Omega) \text{ for all } r \in U \iff A(r) \in \mathcal{S}(\Omega) \text{ for all } r \in U_{\text{cr}}$$

holds.

GENERAL ONE-PARAMETER CASE

In this chapter, we investigate the generalized stability of one-parameter families of matrices. Necessary and sufficient conditions are given for both guarded and semiguarded domains. In addition, an expression for the largest range of parameter variations for which a given parametrized family of matrices is stable is presented. These results are applied to the special cases of convex combinations of two matrices or polynomials.

6.1 Necessary and Sufficient Conditions for Generalized Stability

In this section, we derive necessary and sufficient conditions for stability of a one-parameter family of matrices

$$A(r) = A_0 + rA_1 + \cdots + r^m A_m , \quad (6.1)$$

$r \in [0, 1]$,⁸ relative to a given domain $\Omega \subset \mathbb{C}$. In (6.1), A_k , $k = 1, \dots, m$, are given $n \times n$ real matrices. In the remainder of this section, \mathcal{A} denotes $\{A(r) : r \in [0, 1]\}$.

⁸ In fact, the discussion of the one-parameter case applies for r constrained to lie in *any* interval (not necessarily compact).

The case in which a polynomial guarding map for Ω is available is considered first. We then consider the case in which a polynomial semiguarding map is available.

Evidently, when $m = 1$ and $\Omega = \overset{\circ}{\mathbb{C}}_-$, this problem reduces to that of Hurwitz stability of the convex hull of two real matrices, which was solved independently by Bialas [15] and Fu and Barmish [16].

Let ν be a polynomial guarding map for Ω . Then $\nu(A(r))$ is a polynomial in r of degree $s \leq m \cdot d(\nu(A))$, where $d(\nu(A))$ is the degree of $\nu(A)$, viewed as a multivariate polynomial in the entries of A . If the family \mathcal{A} is nominally stable relative to Ω , i.e. $A(r^0) \in \mathcal{S}(\Omega)$ for some $r^0 \in [0, 1]$, then $\nu(A(r))$ is not identically zero. In this case we may write

$$\nu(A(r)) = \sum_{i=0}^s r^i \nu_i(A_0, \dots, A_m). \quad (6.2)$$

Note that $\nu_0(A_0, \dots, A_m) = \nu(A(0)) = \nu(A_0)$. For simplicity, denote

$$\nu_i := \nu_i(A_0, \dots, A_m). \quad (6.3)$$

From Proposition 5.1, it now follows that the family \mathcal{A} is stable relative to Ω iff (i) A_0 is stable relative to Ω (i.e., $A_0 \in \mathcal{S}(\Omega)$), and (ii) the univariate polynomial $\nu(A(r))$ has no zeroes in $[0, 1]$. The following theorem is merely a restatement of this fact in terms of a stability condition on a related matrix with respect to a new domain in \mathbb{C} . The usefulness of this step will become apparent in our study of stability of two-parameter families of matrices.

Theorem 6.1. Let ν be a polynomial guarding map for Ω , and ν_i , $i = 1, \dots, s$, ($s \geq 2$), be as in Eqs. (6.2), (6.3). Let $A_0 = A(0) \in \mathcal{S}(\Omega)$. Then $A(r) \in \mathcal{S}(\Omega)$ for each $r \in [0, 1]$ iff the matrix $B(A_0, \dots, A_m) \in \mathcal{S}(\Xi)$, where $\Xi := \mathbb{C} \setminus [0, 1]$, and

$B(A_0, \dots, A_m)$ is given by

$$B(A_0, \dots, A_m) := \begin{bmatrix} 0 & 1 & 0 & . & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & 1 & 0 \\ 0 & . & . & . & . & . & 0 & 1 \\ -\frac{\nu_0}{\nu_s} & . & . & . & . & . & . & -\frac{\nu_{s-1}}{\nu_s} \end{bmatrix} \quad (6.4)$$

Applying Theorem 6.1 assumes availability of a guarding map $\nu(A)$ explicitly in the form of a polynomial in the entries of A . However, examples considered in Chapter 4, as well as results on a whole class of domains given in therein, reveal that guarding and semiguarding maps often occur naturally in the form

$$\nu(A) = \det \mathcal{N}(A). \quad (6.5)$$

Here, \mathcal{N} is a polynomial mapping defined on $\mathbb{R}^{n \times n}$. A result analogous to Theorem 6.1, but not requiring expansion of the determinant (6.5), is now formulated.

Let the polynomial mapping $\mathcal{N}(A)$ have degree $N := d(\mathcal{N})$. With $A(r)$ as in Eq. (6.1), we may rewrite

$$\mathcal{N}(A(r)) = \sum_{i=0}^q r^i \mathcal{N}_i(A_0, \dots, A_m) \quad (6.6)$$

where $q \leq mN$ is the degree of $\mathcal{N}(A(r))$ in the parameter r . Note that

$$\mathcal{N}_0(A_0, \dots, A_m) = \mathcal{N}(A_0). \quad (6.7)$$

In the sequel, \mathcal{N}_i denotes $\mathcal{N}_i(A_0, \dots, A_m)$ for $i = 0, 1, \dots, q$.

Theorem 6.2. Let Ω be guarded by a map ν of the form (6.5), and let $A_0 \in \mathcal{S}(\Omega)$. Then $A(r) \in \mathcal{S}(\Omega)$ for all $r \in [0, 1]$ iff $M(A_0, \dots, A_m) \in \mathcal{S}(\Theta)$ where

$$M(A_0, \dots, A_m) = \begin{bmatrix} 0 & I & . & . & 0 \\ \vdots & \vdots & & \vdots & \\ . & . & . & 0 & I \\ -M_0 & . & . & . & -M_{q-1} \end{bmatrix} \quad (6.8)$$

with

$$M_i = \mathcal{N}^{-1}(A_0)\mathcal{N}_{q-i}, \quad i = 0, \dots, q-1 \quad (6.9)$$

if $q \geq 2$ and

$$M(A_0, \dots, A_m) = -\mathcal{N}^{-1}(A_0)\mathcal{N}_1 \quad (6.10)$$

if $q = 1$. Here $\Theta = \mathbb{C} \setminus [1, \infty)$.

Proof. From Proposition 5.1, we have $A(r) \in \mathcal{S}(\Omega)$ for all $r \in [0, 1]$ iff

$$\nu(A(r)) \neq 0 \text{ for all } r \in [0, 1]. \quad (6.11)$$

Since $A_0 \in \mathcal{S}(\Omega)$ and ν guards Ω , it follows that $\nu(A_0) = \det \mathcal{N}(A_0) \neq 0$. Therefore \mathcal{N}_0 is invertible. Thus (6.5) and (6.6) imply

$$\nu(A(r)) = \det \mathcal{N}_0 \det(I + rM_{q-1} + \dots + r^q M_0). \quad (6.12)$$

This implies that $\nu(A(r))$ is nonvanishing for all $r \in [0, 1]$ iff

$$\chi(\mu) := \det(\mu^q I + \mu^{q-1} M_{q-1} + \dots + \mu M_1 + M_0) \neq 0 \quad (6.13)$$

for all $\mu \in [1, \infty)$, where $\mu := \frac{1}{r}$. Since $\chi(\mu)$ is the characteristic polynomial of $M(A_0, \dots, A_m)$ if $q > 1$ (also of $-M_{q-1}$ if $q = 1$), we have that $A(r) \in \mathcal{S}(\Omega)$ for all $r \in [0, 1]$, iff $M(A_0, \dots, A_m)$ has no eigenvalues in $[1, \infty)$. □

Thus the stability of $M(A_0, \dots, A_m)$ relative to Θ is necessary and sufficient for stability of the family \mathcal{A} relative to the guarded domain Ω . In the case where ν is merely semiguarding for Ω and the matrix $M(A_0, \dots, A_m)$ is well-defined, the condition $M(A_0, \dots, A_m) \in \mathcal{S}(\Xi)$ remains sufficient for stability of the family \mathcal{A} , but is no longer necessary. (In the formulation of Theorem 6.1, an analogous statement holds, with M replaced by B .) Specifically, if $M(A_0, \dots, A_m)$ has no eigenvalue in $[1, \infty)$ and ν is semiguarding for Ω , then the family \mathcal{A} is stable relative to Ω . However, the test is inconclusive if $M(A_0, \dots, A_m)$ has an eigenvalue in $[1, \infty)$.

Let then Ω be semiguarded by a map ν of the form $\nu(A) = \det \mathcal{N}(A)$ and suppose that $\nu(A_0) \neq 0$. The matrix $M(A_0, \dots, A_m)$ given by (6.8) (or (6.10)) is well defined since $\nu(A_0) = \det(\mathcal{N}(A_0)) \neq 0$. Define the *critical subset* of the spectrum of $M(A_0, \dots, A_m)$ by

$$\Sigma_{\text{cr}} := \sigma(M(A_0, \dots, A_m)) \cap [1, \infty). \quad (6.14)$$

In case $\Sigma_{\text{cr}} \neq \emptyset$, denote this set by $\{\mu_1, \dots, \mu_\ell\}$. Since $\mu = \frac{1}{r}$, the set $U_{\text{cr}} = \{r \in [0, 1] : \nu(A(r)) = 0\}$ is in this case given by $\{\mu_1^{-1}, \dots, \mu_\ell^{-1}\}$. Proposition 5.2 now yields the following.

Theorem 6.3. Let Ω be semiguarded by a map ν of the form (6.5) and suppose that $\nu(A_0) \neq 0$. If $\Sigma_{\text{cr}} = \emptyset$, i.e. $M(A_0, \dots, A_m)$ has no eigenvalues in $[1, \infty)$, then the family \mathcal{A} is stable relative to Ω iff $A_0 \in \mathcal{S}(\Omega)$. If, however, $\Sigma_{\text{cr}} = \{\mu_1, \dots, \mu_\ell\} \neq \emptyset$, then the family \mathcal{A} is stable relative to Ω iff

$$A(\mu_i^{-1}) \in \mathcal{S}(\Omega), \quad i = 1, \dots, \ell. \quad (6.15)$$

Remark 6.1. A result analogous to Theorem 6.3 may be obtained with ν a polynomial map rather than being specifically of the form (6.5). The matrix $B(A_0, \dots, A_m)$ then plays the role of $M(A_0, \dots, A_m)$, and the assumption $\nu(A_0) \neq 0$ is then no longer relevant.

6.2 Maximal Interval of Stability

Let Ω be an open subset of the complex plane guarded by ν_Ω and consider again the parametrized family of matrices

$$A(r) = A_0 + rA_1 + \dots + r_m A^m \quad (6.16)$$

where now the range of the real parameter r is not specified. In this section, we assume that A_0 is stable relative to Ω and seek the largest range of parameter values for which the family (6.16) is stable. Specifically, we seek the largest open interval (r_{\min}, r_{\max}) containing 0 such that

$$A(r) \in \mathcal{S}(\Omega) \quad \text{for each } r \in (r_{\min}, r_{\max}). \quad (6.17)$$

Note that since $\mathcal{S}(\Omega)$ is open and $A(r)$ is continuous in r , $A_0 \in \mathcal{S}(\Omega)$ guarantees the existence of an open interval containing 0, say $(-\epsilon, \epsilon)$, such that

$$A(r) \in \mathcal{S}(\Omega) \quad \text{for all } r \in (-\epsilon, \epsilon).$$

We use the following notation: If R is a square matrix, denote by $\lambda_{\min}^+(R)$ the smallest positive eigenvalue of R with the convention $\lambda_{\min}^+(R) = +\infty$ when R has no positive eigenvalue. Similarly, denote the largest negative eigenvalue of R by $\lambda_{\max}^-(R)$ and set $\lambda_{\max}^-(R) = -\infty$ if R has no negative eigenvalue.

The next result gives a closed form formula for r_{\min} and r_{\max} .

Theorem 6.4. Let ν , Ω and $B(A_0, \dots, A_m)$ be as in Theorem 6.1 and assume that $A_0 \in \mathcal{S}(\Omega)$. Then the largest open interval containing 0 for which

$$A(r) \in \mathcal{S}(\Omega) \quad \text{for all } r \in (r_{\min}, r_{\max})$$

is specified by

$$r_{\min} = \lambda_{\max}^-(B(A_0, \dots, A_m)) \tag{6.18}$$

$$r_{\max} = \lambda_{\min}^+(B(A_0, \dots, A_m)). \tag{6.19}$$

Proof. Let $\underline{r} < 0$ and $\bar{r} > 0$ be any two given real numbers. Then $A(r)$ is stable relative to Ω if and only if the matrix $B(A_0, \dots, A_m)$ given by (6.4) has no eigenvalues in (\underline{r}, \bar{r}) . On the other hand, by construction the interval $(\lambda_{\max}^-(B(A_0, \dots, A_m)), \lambda_{\min}^+(B(A_0, \dots, A_m)))$ is the largest open interval containing 0 in which $B(A_0, \dots, A_m)$ has no (real) eigenvalues.⁹ The result follows. □

6.3 Applications to the Convex Hull of Two Matrices or Polynomials

As an application of the results in Section 6.1, we consider Hurwitz and Schur stability of the convex hull of two real matrices or polynomials. For the former

⁹ Note that $B(A_0, \dots, A_m)$ is nonsingular since $A_0 \in \mathcal{S}(\Omega)$ and Ω is guarded by ν .

problem, a known result is obtained [15], [16]. For the latter, the result on polynomials appeared recently in the literature [18] while the result on matrices obtained below appears, to the author's knowledge, for the first time here. It is worth pointing out that generalized stability criteria for the convex hull of two polynomials, though interesting in their own right, are very instrumental in dealing with more general polytopes of polynomials. Recall indeed that Bartlett, Hollot and Lin [17] showed that in order to check the stability of a polytope, it suffices to check its edges.

Given two $n \times n$ real matrices A_0 and A_1 , the convex hull $\text{co}(A_0, A_1)$ of A_0 and A_1 is the set of matrices

$$\begin{aligned} A(r) &= (1-r)A_0 + rA_1 \\ &= A_0 + r(A_1 - A_0), \end{aligned} \tag{6.20}$$

with $r \in [0, 1]$. Similarly, given two real n th order monic polynomials p_0 and p_1 , the convex hull $\text{co}(p_0, p_1)$ of p_0 and p_1 is the set of polynomials

$$\begin{aligned} p(r) &= (1-r)p_0 + rp_1 \\ &= p_0 + r(p_1 - p_0), \end{aligned} \tag{6.21}$$

6.3.1 First application: Hurwitz stability

Let $\Omega = \mathring{\mathbb{C}}_-$, which we know to be semiguarded by $\nu : A \mapsto \det \mathcal{N}(A)$ w.r.t. $\mathbb{R}^{n \times n}$, where $\mathcal{N}(A)$ may denote either $A_{[2]}$ or $A \oplus A$.

Corollary 6.1. Assume A_0 is Hurwitz stable. Then $\text{co}(A_0, A_1)$ is Hurwitz stable iff $\mathcal{N}^{-1}(A_0)\mathcal{N}(A_1)$ has no eigenvalues in $(-\infty, 0]$. Here, $\mathcal{N}(A)$ denotes either $A_{[2]}$ or $A \oplus A$.

Proof. Since \mathcal{N} is linear,

$$\begin{aligned} \nu(A(r)) &= \det \mathcal{N}(A_0 + r(A_1 - A_0)) \\ &= \det (\mathcal{N}(A_0) + r\mathcal{N}(A_1 - A_0)). \end{aligned}$$

Hence $\mathcal{N}_0 = \mathcal{N}(A_0)$ and $\mathcal{N}_1 = \mathcal{N}(A_1 - A_0)$, in the notation of Theorem 6.2. Applying Theorem 6.2 with $q = 1$ yields that $\text{co}(A_0, A_1)$ is Hurwitz iff $M(A_0, A_1 -$

$A_0) = -\mathcal{N}^{-1}(A_0)\mathcal{N}(A_1 - A_0) \in \mathcal{S}(\Theta)$, i.e. has no eigenvalue in the interval $[1, \infty)$. Finally, $-\mathcal{N}_0^{-1}\mathcal{N}(A_1 - A_0) = I - \mathcal{N}^{-1}(A_0)\mathcal{N}(A_1)$, and the result follows. \square

Clearly, Corollary 6.1 may be applied to the the convex hull of two polynomials p_0 and p_1 , with, A_0 and A_1 being the companion matrices associated with p_0 and p_1 , respectively. The size of the resulting test matrix would then be $O(n^2)$. However, by taking into account the specific structure of the problem at hand, it is possible to obtain test matrices the size of which is $O(n)$. In the case of n th order real (monic) polynomials, or, equivalently, $n \times n$ real companion matrices, we have indeed seen that $\Omega = \overset{\circ}{\mathcal{C}}_-$ is guarded by $\nu : A \mapsto \det \mathcal{H}(A)$, where $\mathcal{H}(A)$ denotes the $n \times n$ Hurwitz matrix associated with the companion matrix A (see Section 4.1.3).

Corollary 6.2. Assume that the polynomial p_0 is Hurwitz stable. Then $\text{co}(p_0, p_1)$ is Hurwitz stable if and only if the $n \times n$ matrix $\mathcal{H}^{-1}(p_0)\mathcal{H}(p_1)$ has no eigenvalues in $(-\infty, 0]$. Here, $\mathcal{H}(p)$ denotes the Hurwitz matrix associated with the polynomial p .

Proof. The proof of Corollary 6.2 follows along the same lines in the proof of Corollary 6.1 by noting that the equality

$$\mathcal{H}(A_0 + r(A_1 - A_0)) = \mathcal{H}(A_0) + r(\mathcal{H}(A_1) - \mathcal{H}(A_0))$$

still holds even if \mathcal{H} is only affine. \square

6.3.2 Second application: Discrete time (Schur) stability

Let Ω be the ball of radius $\rho > 0$, i.e. the set $\{s : |s| < \rho\}$. From Example A6, Ω is guarded by $\nu : A \mapsto \det \mathcal{N}(A)$ where $\mathcal{N}(A)$ may be taken as either $A \otimes A - \rho^2 I \otimes I$ or $A^{[2]} - \rho^2 I^{[2]}$. Although the latter map is preferable from a computational point of view, the former is used here for convenience.

Denoting $A_1 - A_0$ by \tilde{A}_1 , we have that

$$\begin{aligned}\mathcal{N}(A(r)) &= \mathcal{N}(A_0 + r\tilde{A}_1) \\ &= (A_0 \otimes A_0) - \rho^2 I \otimes I + r \left[A_0 \otimes \tilde{A}_1 + \tilde{A}_1 \otimes A_0 \right] + r^2 \tilde{A}_1 \otimes \tilde{A}_1 \\ &=: \mathcal{N}_0(A_0, \tilde{A}_1) + r\mathcal{N}_1(A_0, \tilde{A}_1) + r^2\mathcal{N}_2(A_0, \tilde{A}_1).\end{aligned}\tag{6.22}$$

We now apply Theorem 6.2 with $\mathcal{N}_0, \mathcal{N}_1$ and \mathcal{N}_2 as in Eq. (6.22) and \tilde{A}_1 identified with A_1 .

Corollary 6.3. Assume all eigenvalues of A_0 have magnitude less than ρ . Then the same is true for any matrix in $\text{co}(A_0, A_1)$ iff $M(A_0, \tilde{A}_1)$ has no eigenvalues in $[1, \infty)$, where

$$M(A_0, \tilde{A}_1) = \begin{bmatrix} 0 & I \\ -\mathcal{N}_0^{-1}\mathcal{N}_2 & -\mathcal{N}_0^{-1}\mathcal{N}_1 \end{bmatrix}\tag{6.23}$$

and $\mathcal{N}_0, \mathcal{N}_1$ and \mathcal{N}_2 are as in (6.22).

In the case of Schur stability of $\text{co}(p_0, p_1)$, we make the additional assumption that p_1 is Schur stable, for simplicity. For real companion matrices (hence real polynomials), we have seen that the unit disk is guarded by (see Section 4.1.3).

$$\nu : A \mapsto \det D(A) \det(I - A^2).\tag{6.24}$$

Corollary 6.4. Suppose that p_0 and p_1 are Schur stable. Then $\text{co}(p_0, p_1)$ is Schur stable if and only if the matrix $D^{-1}(p_0)D(p_1)$ has no eigenvalues in $(-\infty, 0)$. Here, the matrix $D(p)$ is the one defined in Section 4.1.3.

Proof. Let A_0 and A_1 be the companion matrices associated with p_0 and p_1 , respectively. Proceeding as before, we have that $\text{co}(p_0, p_1)$ is Schur stable for all $r \in (0, 1)$ iff

$$\det D(A(r)) \neq 0 \quad \text{and} \quad \det(I - A^2(r)) \neq 0,\tag{6.25}$$

for all $r \in (0, 1)$. The last requirement of Eq. (6.25) indicates that $A(r)$ (resp. $p(r)$) has no eigenvalue (resp. zero) ± 1 for all $r \in (0, 1)$. As argued in [31] and [18], that is always guaranteed under the assumptions made. Specifically, if p_0 and

p_1 are Schur table, then $p_0(1)$ and $p_1(1)$ are both strictly positive. It follows that $(1-r)p_0(1) + rp_1(1) > 0$ for all $r \in (0, 1)$. Proceeding in a similar fashion for -1 , we obtain $(1-r)p_0(-1) + rp_1(-1) < 0$ for all $r \in (0, 1)$. Therefore, we only need to deal with the first part of Eq. (6.25). The result then follows by following the same steps in the proof of Corollary 6.3.

□

STABILITY OF MULTIPARAMETER FAMILIES

This chapter is devoted to the stability of families of matrices depending on more than one parameter. Both the two- and three-parameter case relative to guarded domains are considered. Examples illustrating the method are presented for the two-parameter case.

7.1 Two-Parameter Families

In this section, we consider stability of *two-parameter* families of matrices relative to a domain Ω endowed with a polynomial guarding map ν_Ω which we assume to be real.¹⁰ The matrix families we study are of the general form

$$\begin{aligned} A(r_1, r_2) &= \sum_{i_1, i_2=0}^{i_1+i_2=m} r_1^{i_1} r_2^{i_2} A_{i_1, i_2} \\ &= A_{0,0} + r_1 A_{1,0} + r_2 A_{0,1} + \cdots + r_2^m A_{0,m} \end{aligned} \tag{7.1}$$

¹⁰ All of the relevant examples considered thus far are of this type. In particular, Examples A4 and A5, for which complex-valued guarding maps were given, admit real-valued guarding maps (see Section 4.3).

with each $A_{i,j} \in \mathbb{R}^{n \times n}$ and $(r_1, r_2) \in [0, 1] \times [0, 1]$.¹¹

Assume that $A(0, 0) = A_{0,0} \in \mathcal{S}(\Omega)$. Proposition 3 then implies $A(r_1, r_2) \in \mathcal{S}(\Omega)$ for all $r_1, r_2 \in [0, 1]$ iff

$$\nu_\Omega(A(r_1, r_2)) \neq 0 \quad (7.2)$$

for all $(r_1, r_2) \in [0, 1] \times [0, 1]$. Since both ν_Ω and $A(r_1, r_2)$ are polynomial in their arguments, we may write

$$\nu_\Omega(A(r_1, r_2)) = \sum_{i_1, i_2=0}^{i_1+i_2=s} r_1^{i_1} r_2^{i_2} \nu_{i_1, i_2} \quad (7.3)$$

where s is the degree of this bivariate polynomial and the $\nu_{i,j}$ are scalar coefficients. We have

$$\nu_{0,0} = \nu_\Omega(A(0, 0)).$$

For simplicity of notation, let $\nu_\Omega(r_1, r_2)$ denote $\nu_\Omega(A(r_1, r_2))$. Note that $\nu_\Omega(r_1, r_2)$ is not identically zero since $\nu_{0,0} \neq 0$ by virtue of the assumption $A_{0,0} \in \mathcal{S}(\Omega)$ and the fact that Ω is guarded by ν_Ω .

Before proceeding to the general case, we first eliminate the (albeit simple) special case $s = 0$. This corresponds to $\nu_\Omega(r_1, r_2) \equiv \nu_\Omega(0, 0)$ and is thus nonzero for any $r_1, r_2 \in [0, 1]$. Hence, \mathcal{A} is stable relative to Ω .

Consider now the case $s \geq 1$. To proceed, assume that at least one coefficient among $\nu_{s,0}$ and $\nu_{0,s}$ in the expansion (7.3) is nonzero. Without loss of generality, let $\nu_{0,s} \neq 0$. We may now rewrite (7.3) in the form of a univariate polynomial in r_2 :

$$\nu_\Omega(r_1, r_2) = \alpha_0(r_1) + \alpha_1(r_1)r_2 + \cdots + \alpha_{s-1}(r_1)r_2^{s-1} + \alpha_s r_2^s \quad (7.4)$$

where each coefficient $\alpha_i(r_i)$, $i = 0, \dots, s-1$ is a polynomial in r_1 , and $\alpha_s = \nu_{0,s}$ is independent of r_1 by assumption.

¹¹ No generality is lost by taking the parameters to lie in $[0, 1]$: any compact intervals may be considered.

Motivated by the one-parameter case, form the r_1 -dependent companion matrix

$$B(r_1) := \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & . & . & . & . & 1 & 0 \\ 0 & . & . & . & . & 0 & 1 \\ -\frac{\alpha_0(r_1)}{\alpha_s} & -\frac{\alpha_1(r_1)}{\alpha_s} & . & . & . & . & -\frac{\alpha_{s-1}(r_1)}{\alpha_s} \end{bmatrix}. \quad (7.5)$$

For any r_1 , the eigenvalues of $B(r_1)$ coincide with the zeroes of $\nu_\Omega(r_1, r_2)$ viewed as a univariate polynomial in r_2 . Therefore, the bivariate polynomial $\nu_\Omega(r_1, r_2)$ does not vanish for $0 \leq r_1, r_2 \leq 1$ iff

$$B(r_1) \in \mathcal{S}(\Xi) \text{ for each } r_1 \in [0, 1] \quad (7.6)$$

(recall that $\Xi = \mathcal{C} \setminus [0, 1]$).

Consequently, a stability question for a two-parameter family of matrices has been reduced to a similar question for a related one-parameter family relative to the specific domain Ξ . Note, however, that unlike Ω , only a *semiguarding* map is available for Ξ (see Example C1).¹²

From Example C1, the map ν_Ξ given by

$$\nu_\Xi(A) = \det(A^2 \cdot I - A \cdot A) \det(A(A - I)) \quad (7.7)$$

is semiguarding for Ξ . An application of Proposition 5.2 now yields that $B(r_1) \in \mathcal{S}(\Xi)$ for all $r_1 \in [0, 1]$ iff $B(0) \in \mathcal{S}(\Xi)$ and

$$B(r_1) \in \mathcal{S}(\Xi) \text{ for all } r_1 \in U_{\text{cr}} \quad (7.8)$$

where

$$U_{\text{cr}} = \{r_1 \in [0, 1] : \nu_\Xi(B(r_1)) = 0\}. \quad (7.9)$$

¹² We do not know at this time whether or not a polynomial guarding map for Ξ exists.

From (7.7), it is clear that ν_{Ξ} is a polynomial mapping in A . Hence $\nu_{\Xi}(B(r_1))$ is a polynomial in the parameter r_1 , which we assume not to be identically zero. Thus the set U_{cr} is finite.

For the case in which U_{cr} is empty, we have, by Remark 5.2, that $B(r_1) \in \mathcal{S}(\Omega)$ for all $r_1 \in [0, 1]$ iff $B(0) \in \mathcal{S}(\Xi)$. Hence, in the current setting, \mathcal{A} is stable relative to Ω iff $B(0) \in \mathcal{S}(\Xi)$.

Suppose, on the other hand, that $U_{\text{cr}} =: \{\mu_1, \dots, \mu_{\ell}\}$ where the μ_i belong to $[0, 1]$. The requirement (7.8) then yields that $B(\mu_i) \in \mathcal{S}(\Xi)$, $i = 1, \dots, \ell$ is necessary and sufficient for stability of the family \mathcal{A} relative to Ω .

7.2 The Case of Three or More Parameters

The stability of two-parameter families of matrices relative to a *semiguarded* domain, discussed next, exhibits the essential difficulties of the three-parameter case relative to a guarded domain.

Let \mathcal{A} denote the family $\{A(r_1, r_2) : r_1, r_2 \in [0, 1]\}$ where $A(r_1, r_2)$ is as in Eq. (7.1). Assume that $A(0, 0) = A_{0,0} \in \mathcal{S}(\Omega)$ where Ω is now assumed to be semiguarded by a real polynomial ν_{Ω} . From Proposition 5.2, we have that \mathcal{A} is stable relative to Ω iff

$$A(r_1, r_2) \in \mathcal{S}(\Omega) \text{ for all } (r_1, r_2) \in U_{\text{cr}}^{(0)} \quad (7.10)$$

where

$$U_{\text{cr}}^{(0)} := \{(r_1, r_2) \in [0, 1] \times [0, 1] : \nu_{\Omega}(A(r_1, r_2)) = 0\} \quad (7.11)$$

With the notation of Section 7.1, we obtain by proceeding in a similar fashion that

$$U_{\text{cr}}^{(0)} = \{(r_1, r_2) \in [0, 1] \times [0, 1] : r_2 \in \sigma(B(r_1))\} \quad (7.12)$$

where the companion matrix $B(r_1)$ is as in Eq. (7.5) ($s \geq 2$ assumed for simplicity).

Applying Proposition 5.2 a second time with the assumption $B(0) \in \mathcal{S}(\Xi)$ yields that $B(r_1) \in \mathcal{S}(\Xi)$ for all $r_1 \in [0, 1]$ iff

$$B(r_1) \in \mathcal{S}(\Xi) \text{ for all } r_1 \in U_{\text{cr}}^{(1)} \quad (7.13)$$

where

$$U_{\text{cr}}^{(1)} := \{r_1 \in [0, 1] : \nu_{\Xi}(B(r_1)) = 0\}. \quad (7.14)$$

Clearly $\nu_{\Xi}(B(r_1))$ is a real polynomial which we assume to be not identically zero (this is the case if, for example, $B(0)$ is not a blind spot). Thus $U_{\text{cr}}^{(1)}$ is finite. If $U_{\text{cr}}^{(1)} = \emptyset$ then $B(r_1) \in \mathcal{S}(\Xi)$ for all $r_1 \in [0, 1]$. This in turn implies that $U_{\text{cr}}^{(0)} = \emptyset$ and hence that \mathcal{A} is stable relative to Ω . Assume now that $U_{\text{cr}}^{(1)} =: \{\mu_1, \dots, \mu_{\ell}\} \neq \emptyset$. Again, if $B(\mu_i) \in \mathcal{S}(\Xi)$, $i = 1, \dots, \ell$ then \mathcal{A} is stable relative to Ω . Thus the case of interest is when not all $B(\mu_i)$, $i = 1, \dots, \ell$ are blind spots. Let us then assume, without loss of generality, that $B(\mu_i) \in \mathcal{S}(\Xi)$, $i = 1, \dots, \ell$. Notice that the μ_i are precisely *all* the values of r_1 in $[0, 1]$ for which some eigenvalue of $B(r_1)$ enters the interval $[0, 1]$.

To each μ_i , associate $\{\lambda_1^i, \dots, \lambda_{j_i}^i\}$, the set of all eigenvalues in $[0, 1]$ of the matrix $B(\mu_i)$ and define the set

$$\Sigma_{\text{cr}}^{(0)} := \{(\mu_i, \lambda_j^i) : i = 1, \dots, \ell, j = 1, \dots, j_i\}. \quad (7.15)$$

The set $\Sigma_{\text{cr}}^{(0)}$ is clearly a subset of $U_{\text{cr}}^{(0)}$. Difficulties arise upon finding that $A(r_1, r_2) \in \mathcal{S}(\Omega)$ for all $(r_1, r_2) \in \Sigma_{\text{cr}}^{(0)}$. Indeed, this need not imply that \mathcal{A} is stable relative to Ω since $\Sigma_{\text{cr}}^{(0)}$ is, in general, a proper subset of $U_{\text{cr}}^{(0)}$.

7.3 Applications

In this section, we apply some of the results obtained in Section 7.1 to two examples. The first example deals with Schur stability, and the second with Hurwitz stability.

7.3.1. Example on Schur Stability

We apply the procedure described in Chapter 7 to the stability of the family \mathcal{A} of matrices

$$A(r_1, r_2) = \begin{bmatrix} -r_1 & r_1 - r_2 \\ r_1 & r_2 \end{bmatrix},$$

$r_1, r_2 \in [0, 1]$, relative to the unit disk. It is straightforward to check that this family is unstable, and that only two members of the family are Schur unstable, namely those corresponding to $(r_1, r_2) = (0, 1)$ or $(1, 1)$. As an illustration, this example is now tackled using the techniques presented in Chapter 7.

Note that $A(0, 0) = 0$ is Schur stable. From Example A6 of Chapter 4, a guarding map ν_Ω for the unit disk is

$$\nu_\Omega(A) := \det(A^{[2]} - I^{[2]})$$

where $A^{[2]}$ is given, for a 2×2 matrix A , by

$$\begin{bmatrix} a_{11}^2 & 2a_{11}a_{12} & a_{12}^2 \\ a_{11}a_{21} & a_{11}a_{22} + a_{12}a_{21} & a_{12}a_{22} \\ a_{21}^2 & 2a_{21}a_{22} & a_{22}^2 \end{bmatrix}.$$

We obtain

$$\nu_\Omega(A(r_1, r_2)) = r_2^2 - 1 - 2r_2r_1 + (r_2^2 + 2)r_1^2 - 2r_2r_1^3 + 2r_1^4 - r_1^6$$

where the highest power of r_1 has coefficient equal to -1 (nonzero and independent of r_2). We may therefore proceed to form the matrix

$$B(r_2) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ r_2^2 - 1 & -2r_2 & r_2^2 + 2 & -2r_2 & 2 & 0 \end{bmatrix}.$$

Checking Schur stability of the family \mathcal{A} now reduces to checking whether or not $B(r_2) \in \mathcal{S}(\Xi)$ for all $r_2 \in [0, 1]$. Here, a semiguarding map for Ξ is given by

$$\nu_\Xi(A) = \det Q(A) \det(A(A - I)).$$

For the example at hand, the critical set is given by

$$\begin{aligned} U_{\text{cr}} &= \{r_2 \in [0, 1] : \nu_\Xi(B(r_2)) = 0\} \\ &= \{1\}. \end{aligned}$$

It follows that $B(r_2) \in \mathcal{S}(\Xi)$ for all $r_2 \in [0, 1]$ iff $B(r_2) \in \mathcal{S}(\Xi)$ for all $r_2 \in U_{\text{cr}}$, i.e. iff $B(1) \in \mathcal{S}(\Xi)$. The eigenvalues of $B(1)$ are $-i$, i , -2 , 0 and 1 . Therefore $B(1) \notin \mathcal{S}(\Xi)$ implying that the family \mathcal{A} is not Schur stable. It is indeed easily seen that for (r_1, r_2) given by the critical pairs $(0, 1)$ and $(1, 1)$, $A(r_1, r_2)$ is Schur unstable.

7.3.2. Example on Hurwitz Stability

In this example, $\Omega = \mathring{\mathbb{C}}_-$ and

$$A(r_1, r_2) = \begin{bmatrix} -1 + 5r_1r_2 + 3r_2^2 & 2 - 7r_1r_2 - r_2^2 \\ -2 + 2r_2^2 & -1 - r_1r_2 \end{bmatrix}.$$

Note that

$$A_{0,0} = A(0, 0) = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$$

is Hurwitz stable. A semiguarding map for Ω is given by

$$\nu_{\Omega}(A(r_1, r_2)) = \det A_{[2]}(r_1, r_2).$$

Proceeding as in the previous example, we obtain a test matrix B depending on the parameter r_1 . In this case, $U_{\text{cr}} = \{\frac{1}{4}, \frac{2}{5}, 1\}$. For $r_1 = \frac{1}{4}$, $B(r_1)$ has two eigenvalues in the interval $[0, 1]$, one of which is $\frac{2}{3}$, implying that $A(\frac{1}{4}, \frac{2}{3})$ is Hurwitz unstable (its eigenvalues are $\pm \frac{\sqrt{301}}{18}$). In fact in this case $B(0)$ is unstable relative to Ξ .

INTRODUCTION TO PART II

Part II of this dissertation is devoted to the stabilization of nonlinear control systems. Given a nonlinear control model

$$\dot{x} = f(x, u), \quad t \geq 0 \tag{8.1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a smooth mapping satisfying $f(0, 0) = 0$, we seek to find a state feedback control law $u(x)$ (with $u(0) = 0$) which stabilizes (the 0 equilibrium of the) closed-loop system

$$\dot{x} = f(x, u(x)), \quad t \geq 0. \tag{8.2}$$

This would implicitly guarantee the existence of a neighborhood of the origin, called a region of asymptotic stability (RAS for short), such that all trajectories of the closed-loop system starting within that neighborhood asymptotically converge to the origin.

The *size* of the RAS is usually not stated as an *explicit* control objective. The reason for this is the lack of systematic analytical tools for the synthesis of feedback control laws achieving specifications on the RAS. The importance of obtaining such tools is clear, and has been emphasized in [33].

The traditional approach based solely on linearization at an operating point is often considered unreliable from a stability point of view and can yield unsatisfactory performance, especially when the system is highly nonlinear and undergoes large motions. An alternative design method consists in repeated testing of the performance of the closed-loop system for each of a set of possible stabilizing control laws. Since approximation of the obtained RAS is often very difficult, each of these tests typically involves many simulations of the closed-loop dynamics, and the method is hence very costly [34].

The local nature of feedback stabilization of nonlinear control systems is considered to be a serious restriction for engineering applications. In practice, techniques for determining whether a nonlinear control system can be stabilized within a prescribed region of asymptotic stability are crucially needed.

Ideally, the problem of stabilization with a prescribed RAS may be stated as follows: Given an open connected region $\mathcal{D} \subset \mathbb{R}^n$, does there exist a smooth feedback control $u(x)$ such that the origin of (8.2) is asymptotically stable, with \mathcal{D} being the corresponding domain of attraction (i.e. the largest RAS). Analytically speaking, this problem may be stated in terms of Zubov's equation ([35]) as that of finding a smooth feedback $u(x)$, and positive definite functions $V, \phi : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that the following holds

$$\sum_{i=1}^n \frac{\partial V}{\partial x_i}(x) f_i(x, u(x)) = -\phi(x), \quad x \in \mathcal{D} \quad (8.3)$$

with $V(x) \rightarrow \infty$ as $x \rightarrow \partial\mathcal{D}$ or $|x| \rightarrow \infty$.

Besides being extremely untractable this setting is not practically motivated. In control systems design one is rather interested in synthesizing a feedback control law which stabilizes the given nonlinear system, guaranteeing in addition that a "sufficiently large" prescribed region of the state space (dictated by practical considerations) lies within the resulting domain of attraction.

Despite many recent advances in some qualitative concepts of nonlinear control theory, few techniques exist for the control of systems described by nonlinear

mathematical models. In practice, linear systems-based methodologies are still the most widely used. These methodologies almost invariably use a combination of linear control theory and Lyapunov methods to achieve the required stabilization task.

Let us rewrite System (8.1) in the more suggestive form

$$\dot{x} = Ax + Bu + h(x, u) \quad (8.4)$$

where

$$A = \frac{\partial f}{\partial x} \big|_{(0,0)} \quad (8.5)$$

$$B = \frac{\partial f}{\partial u} \big|_{(0,0)} \quad (8.6)$$

and h represents higher order terms. Then it is well known that if the linear part of (8.4) is controllable, i.e.

$$\text{rank}[B, AB, \dots, A^{n-1}B] = n, \quad (8.8)$$

then there exists a linear state feedback $u(x) = Kx$, $K \in \mathbb{R}^{m \times n}$, such that the zero solution of the closed-loop system

$$\dot{x} = (A + BK)x + h(x, Kx) \quad (8.7)$$

is asymptotically stable. This guarantees, as we just mentioned, the existence of a neighborhood (depending here on the feedback gain matrix K) containing the origin with the property that it is attracting. An interesting feature of this technique, commonly referred to as pole assignment, is that, except for the single input case, the stabilizing feedback gain K is not unique. A question of interest is then to what extent can one exploit this non-uniqueness toward controlling the size of resulting RAS's? The goal of Part II is to precisely delineate a class of nonlinear systems for which favorable answers to such questions are obtainable.

In the next chapter, requisite background material is presented. In Chapter 10, sufficient conditions are obtained for the existence of a *linear* feedback stabilizing an equilibrium point of a given nonlinear system with the resulting region of

asymptotic stability (RAS) containing a ball of given radius. Conditions for global stabilization are also given. Feedback stabilization is achieved while satisfying a certain robustness property. Synthesis of the desired feedback control laws rests on the solution of certain nonstandard questions in linear systems (Chapter 12). These questions are addressed successfully for the case of planar systems (Chapter 11), for which a complete design methodology is achieved. In Chapter 13, the results of Chapter 11 are extended to the design of a two-time scale feedback stabilization of a class of singularly perturbed control systems within cylindrical RAS's. Examples and simulations illustrating the method are presented.

BACKGROUND FOR PART II

This chapter presents requisite background material for the remaining chapters. After a short description of the concept of eigenstructure assignment in linear systems in Section 9.1, we proceed to recall some basic results from Lyapunov theory, including a criterion of Krasovskii, in Section 9.2. In Section 9.3, a result of Chow and Kokotovic for singularly perturbed nonlinear control systems is given.

9.1 Eigenstructure Assignment

Given a linear time-invariant (LTI) control system

$$\dot{x} = Ax + Bu \tag{9.1}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and a set of desired closed-loop eigenvalues $\{\lambda_1, \dots, \lambda_n\}$; the pole assignment problem in linear systems concerns the ability of finding a state feedback law $u(x) = Kx$, $K \in \mathbb{R}^{m \times n}$, such that the eigenvalues of the closed-loop matrix $A + BK$ coincide with $\lambda_1, \dots, \lambda_n$. Conditions under which such a property holds are well known [36]. It is also well known that when these conditions are met, the specification of a desired set of eigenvalues does not uniquely determine the feedback gain matrix K , except in the single input case.

The eigenstructure assignment approach to linear control systems design consists in exploiting the degrees of freedom that are available in pole assignment to obtain well conditioned closed-loop systems ([37], [38], [39]). It turns out that the design freedom available beyond pole assignment is that of selecting a corresponding set of eigenvectors from appropriate vector spaces. To see what these are, simply note that if $(A + BK)v_i = \lambda_i v_i$, then $v_i \in \mathcal{F}(\lambda_i)$, where

$$\mathcal{F}(\lambda_i) := \{x \in \mathbb{R}^n : (\lambda_i I - A)x \in \mathcal{R}(B)\} \quad (9.2)$$

where \mathcal{R} denotes the range space of matrix B .

A desirable property of any control design is that the eigenvalues of the closed-loop matrix be insensitive to perturbations inherent to the model. It is well known in numerical analysis that the sensitivity of the eigenvalues of a non-defective $n \times n$ matrix M (i.e. M has n linearly independent eigenvectors) depends on the condition number given by [40]

$$\kappa(V) := \|V\| \|V^{-1}\| \quad (9.3)$$

where V is a matrix of eigenvectors of M . The closer the condition number to the value 1 (its minimal value), the lower the sensitivity. It is a fact [41] that the minimal value 1 is achieved when the matrix M is normal (i.e. satisfies $M^T M = M M^T$) thus making the class of normal matrices the least sensitive of all.

The problem of designing a feedback gain which minimizes the condition number of the closed-loop matrix is studied in [38], where it is referred to as robust eigenstructure assignment.

Let $\{\lambda_1, \dots, \lambda_n\}$ be a self-conjuguate set of desired closed-loop eigenvalues. If $\{v_1, \dots, v_n\}$ is a set of eigenvectors of $A + BK$ corresponding to $\{\lambda_1, \dots, \lambda_n\}$, then $v_i \in \mathcal{F}(\lambda_i)$, $i = 1, \dots, n$, where $\mathcal{F}(\lambda)$ is given by (9.2). Conversely, if it is possible to select n linearly independent vectors v_1, \dots, v_n , such that $v_i \in \mathcal{F}(\lambda_i)$, $i = 1, \dots, n$, then there exists a feedback gain K such that the closed-loop matrix $A + BK$ has

$\lambda_1, \dots, \lambda_n$ as its eigenvalues with v_1, \dots, v_n being the corresponding eigenvectors. To see this, note that $v_i \in \mathcal{F}$, $i = 1, \dots, n$, implies that

$$(\lambda_i I - A)v_i = Bg_i, \quad i = 1, \dots, n \quad (9.4)$$

for some m -dimensional vectors g_1, \dots, g_n . Rewriting (9.4) as

$$Av_i + Bg_i = \lambda_i v_i, \quad i = 1, \dots, n, \quad (9.5)$$

it is seen that K is obtained by solving the equations

$$Kv_i = g_i, \quad i = 1, \dots, n. \quad (9.6)$$

Setting $V = [v_1, \dots, v_n]$ and $G = [g_1, \dots, g_n]$, we obtain

$$K = GV^{-1}. \quad (9.7)$$

By selecting the vectors v_1, \dots, v_n in such a way that the condition

$$\bar{\lambda}_i = \lambda_j \implies \bar{v}_i = v_j$$

holds, it can be shown that K can be taken to be a real matrix.

In the light of this discussion, it is clear that if one were interested in rendering the closed-loop matrix normal, then one would need to be able to select n orthogonal vectors v_1, \dots, v_n from $\mathcal{F}(\lambda_1), \dots, \mathcal{F}(\lambda_n)$, respectively. This issue will be pursued further in subsequent chapters.

9.2 Lyapunov Stability of Nonlinear Systems

This section is devoted to the concept of Lyapunov stability of nonlinear autonomous systems. These are systems described by a nonlinear ordinary differential equation of the form

$$\dot{x}(t) = f(x(t)), \quad t \geq 0 \quad (9.8)$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a Lipschitz continuous mapping. A vector x_e is said to be an equilibrium point of (9.8) if $f(x_e) = 0$. Here, it is assumed (without loss of generality) that $x_e = 0$, i.e. the origin is an equilibrium point for (9.8).

Lyapunov stability is concerned with trajectories of (9.8) starting near an equilibrium point. Roughly speaking, the equilibrium point is stable if arbitrarily small perturbations of the initial state about the equilibrium position result in arbitrarily small perturbations of the corresponding trajectories. This is made more precise in the following definition where $|\cdot|$ denotes a norm in \mathbb{R}^n and $x(t, x_0)$ denotes the trajectory of (9.8) starting at x_0 at time $t = 0$.

Definition 9.1. The equilibrium point 0 is (Lyapunov) stable if for each $\epsilon > 0$, there is a $\delta > 0$ such that

$$|x_0| < \delta \implies |x(t, x_0)| < \epsilon, \quad \forall t \geq 0.$$

The equilibrium point is said to be asymptotically stable if it is stable and there exists a $\delta_0 > 0$ such that

$$|x_0| < \delta_0 \implies |x(t, x_0)| \longrightarrow 0 \text{ as } t \longrightarrow \infty. \quad (9.9)$$

It is said to be globally asymptotically stable if the right hand side of (9.9) holds for every initial state x_0 in \mathbb{R}^n .

A continuous function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be a locally positive definite function (l.p.d.f.) if it satisfies the following conditions: (i) $V(0) = 0$ and (ii) $V(x) > 0$ for all $x \neq 0$ in a neighborhood of the origin. It is said to be a positive definite function (p.d.f.) if, in addition, Condition (ii) holds for all $x \in \mathbb{R}^n$ and, in addition, $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$.

If V is continuously differentiable, let \dot{V} denote the derivative of V along trajectories of (9.8), i.e.

$$\dot{V} = (\nabla V(x))^T f(x). \quad (9.10)$$

The following classical theorem [42] gives a sufficient condition for the equilibrium point 0 of System (9.8) to be asymptotically stable.

Theorem 9.1. The equilibrium point 0 is asymptotically stable if there exist a continuously differentiable l.p.d.f. $V(x)$ such that $-\dot{V}$ is also an l.p.d.f. V is then said to be a Lyapunov function for the nonlinear system (9.8). The equilibrium point is globally asymptotically stable if V and $-\dot{V}$ above are positive definite function.

Suppose that the mapping f is continuously differentiable and let A denote the Jacobian of f evaluated at the equilibrium point 0, i.e.

$$A = \frac{\partial f}{\partial x}(0). \quad (9.11)$$

Then the nonlinear system (9.8) may be written as

$$\dot{x} = Ax + h(x) \quad (9.12)$$

where $h(x) := f(x) - Ax$ denotes the higher order terms and satisfies

$$\lim_{|x| \rightarrow 0} \frac{|h(x)|}{|x|} = 0. \quad (9.13)$$

The system

$$\dot{x} = Ax \quad (9.14)$$

is referred to as the linearization of System (9.8) around the equilibrium point 0.

Since asymptotic stability is, in nature, a local concept, it is of interest to know under what circumstances will it be possible to infer stability (or instability) conclusions of the equilibrium point of (9.8) based on the corresponding linearization. The next theorem deals with this question.

Theorem 9.2. If all the eigenvalues of matrix A have negative real parts, then the equilibrium point 0 of System (9.8) is asymptotically stable. If at least one eigenvalue of A has a positive real part, then 0 is unstable.

For the ‘critical’ case in which A has at least one eigenvalue on the imaginary axis but none in the right-half plane, the linearization above is insufficient to determine stability.

A region of asymptotic stability (RAS) of the equilibrium point 0 (assumed to be asymptotically stable) is any nonempty subset S of \mathbb{R}^n , containing 0, such that

$$\forall x_0 \in S, \quad \lim_{t \rightarrow \infty} |x(t, x_0)| = 0. \quad (9.15)$$

The largest such set, denoted \mathcal{D} , is called the domain of attraction of the equilibrium point 0.

Estimating the domain of attraction of an asymptotically stable equilibrium point is a key issue in engineering design and has been an area of research for decades (see [29], [43] and references therein). A commonly used theorem for obtaining regions of asymptotic stability is given next.

Theorem 9.3. Let $V(x)$ be a continuously differentiable l.p.d.f. Assume that $S(R) := \{x \in \mathbb{R}^n : V(x) < R\}$ is bounded, $V(x) > 0$ and $\dot{V}(x) < 0$ for all $x \in S(R) \setminus 0$. Then the equilibrium point 0 is asymptotically stable and $S(R)$ is an RAS for 0.

The next Theorem, due to Krasovskii, will be needed in our consideration of singularly perturbed control systems in Chapter 13. Let the matrix $J(x)$ be given by

$$J(x) := \frac{\partial f}{\partial x}(x). \quad (9.16)$$

Theorem 9.4. (Krasovskii's Criterion [44])

If there exists a positive definite matrix $P(x)$ and a positive scalar r such that the eigenvalues of the matrix

$$\frac{1}{2}(J^T(x)P(x) + P(x)J(x)) \quad (9.17)$$

are bounded above by a fixed negative number c_0 for all $x \in B(r)$, then the equilibrium point of (9.8) is asymptotically stable.

Following [45], we say that the scalar valued function $V(x)$ is a Lyapunov function of the Krasovskii type for System (9.8) if there is an $r > 0$ such that

$$V(x) = f^T(x)P(x)f(x), \quad (9.18)$$

where $P(x)$ is positive definite, differentiable with respect to x , and $\dot{V}(x)$ is negative definite for all $x \in B(r)$. For future use, note that

$$\dot{V}(x) = f^T(x)N(x)f(x) \quad (9.19)$$

where

$$N(x) = J(x)^T P(x) + P(x)J(x) + \sum_{j=1}^n P_{x_j}(x)f_j(x). \quad (9.20)$$

Here, a subscript x_j indicates differentiation with respect to x_j and $f_j(x)$ denotes the j th component of f .

Finally, given a nonlinear control system of the form

$$\dot{x} = f(x, u) \quad (9.21)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a smooth mapping satisfying $f(0, 0) = 0$, x and u are the state and control vectors, respectively; we say that the feedback law $u(x) = \phi(x)$ (with $\phi(0) = 0$) stabilizes (9.2.12) if the equilibrium point 0 of the (closed-loop) system

$$\dot{x} = f(x, \phi(x)) \quad (9.22)$$

is asymptotically stable.

9.3 Singularly Perturbed Control Systems

9.3.1 Singular Perturbations

Singularly perturbed systems are systems modeled by differential equations in which the high order derivatives of some states appear with a small coefficient. A standard model for singularly perturbed control systems is ([46])

$$\dot{x} = f(x, z, u, \epsilon, t) \quad (9.23)$$

$$\epsilon \dot{z} = g(x, z, u, \epsilon, t) \quad , \quad t \geq t_0 \quad (9.24)$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^p$, $u \in \mathbb{R}^m$ and ϵ is a small positive parameter.

By formally setting $\epsilon = 0$ in (9.23)-(9.24), one obtains the quasi-steady-state equation

$$g(x, z, u, 0, t) = 0. \quad (9.25)$$

Assuming that Eq. (9.25) possesses a solution

$$\bar{z} = \phi(\bar{x}, \bar{u}, t), \quad (9.26)$$

supposed to be unique for simplicity, and substituting \bar{z} for z in (9.23) yields the *reduced order model*

$$\dot{\bar{x}} = f(\bar{x}, \phi(\bar{x}, \bar{u}, t), \bar{u}, 0, t). \quad (9.27)$$

By a slight abuse of notation, System (9.27), also referred to as the *slow subsystem* for obvious reasons, is usually written as

$$\dot{\bar{x}} = f(\bar{x}, \bar{u}, t) \quad (9.28)$$

Here, the bar over a variable denotes its slow part.

Thus by setting $\epsilon = 0$ in the full system (9.23)-(9.24), the dimension of the state space is reduced from $n + p$ to n . Reduced order modeling is common engineering practice. Small parasitic quantities are often neglected, resulting sometimes in oversimplified models which may yield erroneous results.

The role of singular perturbation techniques is to provide a means by which to legitimize *ad hoc* simplifications of dynamic models. The simplification resulting from setting $\epsilon = 0$ amounts to neglecting the fast dynamics of System (9.23)-(9.24). To describe the effects of such a simplification, a key step in singular perturbation techniques consists in studying, in addition to the slow subsystem (9.28), the so-called *boundary-layer* (or *fast*) *subsystem* given by

$$\frac{d\hat{z}}{d\tau} = g(x(0), \hat{z}(\tau), u, 0, t_0) \quad (9.29)$$

where $\tau := \frac{t-t_0}{\epsilon}$ represents a “stretched” time scale.

9.3.2 Composite Feedback Control Laws

The control of systems of the form (9.23)-(9.24) has received a great deal of attention, resulting, in particular, in a two-time scale procedure. This procedure consists in synthesizing a composite control law for the full system based on separate control design for the slow and fast subsystems (see for instance [45], [47] and [48]).

In the remainder of this section, we outline one such a result, due to Chow and Kokotovic [45]. Consider the singularly perturbed nonlinear control system

$$\dot{x} = f(x) + F(x)z + B_1(x)u \quad (9.30)$$

$$\epsilon \dot{z} = g(x) + G(x)z + B_2(x)u \quad (9.31)$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^p$, $u \in \mathbb{R}^m$, the matrices F, G, B_1, B_2 are of appropriate dimensions, and where ϵ is the small singular perturbation parameter.

Next, System (9.30)-(9.31) is separated into the two lower order slow and fast subsystems. Here, Eq. (9.25) takes the form

$$g(\bar{x}) + G(\bar{x})\bar{z} + B_2(\bar{x})\bar{u} = 0. \quad (9.32)$$

Assuming $G(\bar{x})$ is nonsingular, we obtain

$$\bar{z} = -G^{-1}(\bar{x})(g(\bar{x}) + B_2(\bar{x})\bar{u}). \quad (9.33)$$

Eq. (9.33) thus yields the slow subsystem or the reduced order system of (9.30)-(9.31), namely

$$\dot{\bar{x}} = a(\bar{x}) + B(\bar{x})\bar{u} \quad (9.34)$$

Here,

$$a(\bar{x}) := f(\bar{x}) - F(\bar{x})G^{-1}(\bar{x})g(\bar{x}) \quad (9.35)$$

and

$$B(\bar{x}) := B_1(\bar{x}) - F(\bar{x})G^{-1}(\bar{x})B_2(\bar{x}). \quad (9.36)$$

To derive the fast subsystem, we assume that the slow variables are constant in the boundary layer; i.e., $\dot{\bar{z}} = 0$ and $x = \bar{x} = \text{constant}$. Defining $z_f = z - \bar{z}$ and $u_f = u - \bar{u}$ and subtracting (9.32) from (9.31) yields the fast subsystem

$$\frac{dz_f}{d\tau} = G(\bar{x})z_f + B_2(\bar{x})u_f, \quad (9.37)$$

where τ is the fast time scale.

Systems (9.30)-(9.31), (9.34) and (9.37) are assumed to satisfy (some of) the following conditions for all x, \bar{x} in a closed subset D of \mathbb{R}^n .

C1: The vector fields f, g and the matrices F, G, B_1, B_2 are bounded and differentiable with respect to x , and the unique solution of $f(x) = 0$ and $g(x) = 0$ is $x^* = 0$.

C2: G is nonsingular and

$$\text{rank}[B_2, GB_2, \dots, G^{p-1}B_2] = p \quad (9.38)$$

C3: There exists a control law $k(\bar{x})$ with $k(0) = 0$ such that the closed-loop system

$$\dot{\bar{x}} = a(\bar{x}) + B(\bar{x})k(\bar{x}) \quad (9.39)$$

possesses a Lyapunov function of Krasovskii type.

C4: System (9.39) possesses a Lyapunov function $v(\bar{x})$ guaranteeing asymptotic stability of the equilibrium $\bar{x} = 0$. Furthermore, D is a level set for v ; that is, $D = \{x \in \mathbb{R}^n : v(x) \leq c_0\}$ for some $c_0 > 0$.

Condition C2 guarantees the existence of a fast control u_f of the form $u_f(\bar{x}, z_f) = L(\bar{x})z_f$ such that $\Re(\lambda_i(G + B_2L)) \leq \sigma$ for a fixed $\sigma < 0$. Conditions C3 and C4 both guarantee that $\bar{u}(\bar{x}) = k(\bar{x})$ is a stabilizing feedback law for the slow subsystems.

Based on the control laws designed for the slow and fast subsystems, the composite control law

$$u(x, z) = (I + H(x)G^{-1}(x)B_2(x))k(x) + L(x)z + L(x)G^{-1}(x)g(x) \quad (9.40)$$

is proposed for the full system.

Theorem 9.5. (Chow and Kokotovic [45].) Let D_1 be a closed set in the interior of D and E be a bounded subset of \mathbb{R}^p . If Conditions C1-C3 are satisfied, then there exists $\epsilon^* > 0$ such that for all $\epsilon \in (0, \epsilon^*]$, the feedback control $u(x, z)$ given by (9.40) stabilizes the origin of the full system (9.30)-(9.31). Furthermore, the set $D \times E$ is a subset of the corresponding domain of attraction.

In case only condition C4 holds instead of Condition C3, one obtains a (theoretically) weaker result. Under Conditions C1, C2 and C4, the theorem below states that the feedback control given by (9.40) “practically” stabilizes the full system.

Theorem 9.6. (Chow and Kokotovic [45].) If only Conditions C1, C2 and C4 hold, then there exists $\epsilon^* > 0$ such that for all $\epsilon \in (0, \epsilon^*]$, the control law given by (9.40) steers every trajectory of System (9.30)-(9.31) starting within $D \times E$, to a sphere centered at the origin, whose radius is $O(\epsilon)$.

**SUFFICIENT CONDITIONS FOR STABILIZATION
WITH PRESCRIBED RAS**

In this chapter, sufficient conditions are obtained for the existence of a linear feedback which stabilizes the origin of a given nonlinear system with the resulting domain of attraction containing a ball of radius R (possibly infinite), centered at the origin.

We consider nonlinear multi-input control systems of the form

$$\dot{x} = f(x) + Bu \tag{10.1}$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is analytic over \mathbb{R}^n and satisfies $f(0) = 0$, and B is an $n \times m$ matrix. Notice that this model is not overly restrictive since a more general model $\dot{z} = F(z, v)$ may always be put in the form above by letting x be the augmented state $(z, v)^T$ and taking $u = \dot{v}$. It is convenient to rewrite the model (10.1) in the equivalent form

$$\dot{x} = Ax + Bu + h(x), \tag{10.2}$$

where $A := \frac{\partial F}{\partial z}(0)$ and $h(x)$ represents higher order terms.

It is appropriate at this point to outline related work by Bacciotti [49], which introduced and considered the so-called “potentially global stabilizability” prob-

lem. System (10.2) is said to be potentially globally stabilizable if, given any $R > 0$, there exists a matrix $K = K(R)$ such that the feedback $u(x) = Kx$ stabilizes the origin of (10.2) and the resulting domain of attraction contains a ball of radius R centered at the origin.

We have seen that under controllability assumptions on the linear part (A, B) , it is possible to assign the eigenvalues of the closed-loop matrix $A+BK$ to arbitrary locations in the open left-half complex plane. The main result in [49], which was recognized by the author as being erroneous in [50], is: “A sufficient condition for System (10.2) to be potentially globally stabilizable is that the linear part (A, B) of (10.2) be controllable.” The following simple counterexample to this assertion was given in [50]: The system

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1 x_2 \\ \dot{x}_2 &= u.\end{aligned}$$

This system clearly has a controllable linear part. However, the line $x_1 = 1$ is an invariant set regardless of the choice of the control u . Therefore, trajectories starting at points for which $x_1 \geq 1$ cannot be driven to the origin by an appropriate choice of the control u .

The property of arbitrary pole assignability does not imply that of potentially global stabilizability, except of course, in case B is a square nonsingular matrix (in fact the nonlinearity f may be cancelled in this case).

Before presenting the main results of this chapter, we establish notation and give some definitions. With S a subset of \mathbb{C} , $\Re(S)$ denotes the set $\{\Re(s) : s \in S\}$. For a real matrix M , $[M]_s$ and $[M]_{ss}$ denote its symmetric and skew-symmetric parts, respectively, i.e.

$$\begin{aligned}[M]_s &:= \frac{1}{2}(M + M^T), \\ [M]_{ss} &:= \frac{1}{2}(M - M^T).\end{aligned}$$

The spectrum of M is denoted by $\sigma(M)$. For x a vector in \mathbb{R}^n , $|x|$ denotes its Euclidean norm. Denote by $B(R)$ the open ball in \mathbb{R}^n of radius R centered

at the origin. By \mathbb{R}_- and $\overset{\circ}{\mathbb{C}}_-$, we intend $(-\infty, 0)$ and the open left-half of the complex plane, respectively. For S a given set, S^n denotes the Cartesian product $S \times S \times \cdots \times S$ (n times).

Two definitions relating to stabilizability of linear systems are now introduced. Let A and B be real matrices of dimensions $n \times n$ and $n \times m$, respectively.

Definition 10.1. Say that the pair (A, B) is *symmetrically stabilizable*¹⁴ if there exists $K \in \mathbb{R}^{m \times n}$ such that $\sigma([A + BK]_s) \subset \mathbb{R}_-^n$. For Δ a nonempty subset of \mathbb{R}_-^n , the pair (A, B) is said to be *symmetrically stabilizable within Δ* if for all $\Lambda \in \Delta$, there exists $K \in \mathbb{R}^{m \times n}$ such that $\sigma([A + BK]_s) = \Lambda$.

Definition 10.2. Say that the pair (A, B) is *normally stabilizable* if there exists $K \in \mathbb{R}^{m \times n}$ such that $\sigma(A + BK) \subset \overset{\circ}{\mathbb{C}}_-^n$ with $A + BK$ a normal matrix. Let $\Delta \subset \overset{\circ}{\mathbb{C}}_-^n$ be nonempty. Say that (A, B) is *normally stabilizable within Δ* if for all $\Lambda \in \Delta$, there exists $K \in \mathbb{R}^{m \times n}$ such that $\sigma(A + BK) = \Lambda$ with $A + BK$ normal.

It is a simple exercise to show that if (A, B) is normally stabilizable within Δ then it is symmetrically stabilizable within $\Re(\Delta)$.

As we have mentioned earlier, the methodology which will be used in the sequel uses a combination of Lyapunov's method and results from linear systems. As such, precise algebraic properties of the higher order terms $h(x)$ will not be used explicitly. Rather, only information relating to the "magnitude" of h will be of interest. To give a more precise meaning to what is meant by magnitude, let us introduce the following space:

$$\mathcal{H} := \{h : \mathbb{R}^n \rightarrow \mathbb{R}^n, \text{ analytic over } \mathbb{R}^n : h(0) = 0, \frac{\partial h}{\partial x}(0) = 0\}. \quad (10.3)$$

Clearly $(\mathcal{H}, +, \cdot)$ is a vector space. Let R be a fixed positive number. We endow $(\mathcal{H}, +, \cdot)$ with the following two functions:

¹⁴ Recall that Hurwitz stability of $[M]_s$ implies that of M .

(i) $\|\cdot\|_R : \mathcal{H} \rightarrow \mathbb{R}_+$ given by

$$\|h\|_R = \sup_{\substack{x \in B(R) \\ x \neq 0}} \frac{|h(x)|}{|x|}$$

(ii) $\|\cdot\| : \mathcal{H} \rightarrow \mathbb{R}_+ \cup \{\infty\}$ given by

$$\|h\| = \sup_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{|h(x)|}{|x|}$$

Notice that $\lim_{|x| \rightarrow 0} \frac{|h(x)|}{|x|} = 0$ and as such $\|\cdot\|$ is well defined. We allow the possibility $\|h\| = \infty$ with the understanding that inequalities such as $\|h_1 + h_2\| \leq \|h_1\| + \|h_2\|$ are to be interpreted in the obvious way when $\|h_1\|$ or $\|h_2\|$ is infinite.

We have that $(\mathcal{H}, +, \cdot)$ with either $\|\cdot\|_R$ or $\|\cdot\|$ is a normed vector space. This can be seen easily, noting that analyticity of the elements of \mathcal{H} implies that a function which vanishes within $B(R)$ must also vanish everywhere.

The two norms $\|\cdot\|_R$ and $\|\cdot\|$ naturally lead to defining two type of balls in \mathcal{H} , namely, for any $R > 0$,

$$B_R(\rho) := \{h \in \mathcal{H} : \|h\|_R < \rho\}$$

and

$$B(\rho) := \{h \in \mathcal{H} : \|h\| < \rho\}$$

where ρ is a given positive number.

Theorem 10.1. Fix $R > 0$ and let (A, B) be symmetrically stabilizable within $\Delta \subset (-\infty, -\|h\|_R)^n$ (resp. $(-\infty, -\|h\|)^n$). Then the nonlinear control system (10.2) is stabilizable within $B(R)$ (resp. globally stabilizable) using linear state feedback.

The proof of this theorem relies on Proposition 10.1, given next. Let

$$\dot{x} = F(x), \quad F(0) = 0 \tag{10.4}$$

where F is analytic over \mathbb{R}^n . Let the null solution of (2) be asymptotically stable in the sense of Lyapunov and denote the associated domain of attraction by \mathcal{D}^x . Consider a change of coordinates $z = Q^T x$ where Q is an *orthogonal* matrix (i.e., Q satisfies $Q^T Q = Q Q^T = I$). In the new coordinates,

$$\dot{z} = \tilde{F}(z) \quad (10.5)$$

where

$$\tilde{F}(z) := Q^T F(Qz).$$

Clearly, the origin is also asymptotically stable for Eq. (10.5). The sets \mathcal{D}^x and \mathcal{D}^z are in general different. However, we can use the fact that orthogonal transformations preserve norms and angles to obtain the following proposition.

Proposition 10.1. The largest Euclidean balls in \mathcal{D}^x and \mathcal{D}^z are identical.

Consequently, for each $R > 0$,

$$B(R) \subset \mathcal{D}^x \iff B(R) \subset \mathcal{D}^z.$$

Proof of Theorem 10.1. Let $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \Delta$. Since (A, B) is symmetrically stabilizable within Δ , there is a feedback gain matrix K such that

$$\sigma([A + BK]_s) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}.$$

With $u(x) = Kx$, we obtain the closed loop system

$$\dot{x} = (A + BK)x + h(x). \quad (10.6)$$

By writing the closed-loop matrix $A + BK$ as the sum of its symmetric and skew-symmetric part, (10.6) becomes

$$\dot{x} = [A + BK]_{ss}x + [A + BK]_s x + h(x). \quad (10.7)$$

Since $[A + BK]_s$ is symmetric, it can be diagonalized using an orthogonal transformation. Let Q be such a transformation and define new coordinates $z = Q^T x$. Then z satisfies

$$\dot{z} = Gz + Dz + \tilde{h}(z) \quad (10.8)$$

where

$$D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n),$$

$$G = Q^T[A + BK]_{ss}Q$$

and

$$\tilde{h}(z) = Q^T h(Qz).$$

Now consider the Lyapunov function candidate

$$V(z) = \frac{1}{2} |z|^2$$

and note that $B(R)$ is a level set for V . The derivative of V along trajectories of (10.8) is

$$\dot{V}(z) = z^T G z + z^T D z + z^T \tilde{h}(z). \quad (10.9)$$

The first term in (10.9) vanishes since G is skew-symmetric. Therefore,

$$\begin{aligned} \dot{V}(z) &\leq \sum_{i=1}^n \lambda_i z_i^2 + |z| |\tilde{h}(z)| \\ &\leq \max_{1 \leq i \leq n} (\lambda_i) |z|^2 + \|\tilde{h}\|_R |z|^2 \\ &= \left(\max_{1 \leq i \leq n} (\lambda_i) + \|\tilde{h}\|_R \right) |z|^2 \end{aligned} \quad (10.10)$$

for all $z \in B(R)$. It follows since $\|\tilde{h}\|_R = \|h\|_R$ that

$$\dot{V}(z) \leq \left(\max_{1 \leq i \leq n} (\lambda_i) + \|h\|_R \right) |z|^2 \quad (10.11)$$

for all $z \in B(R)$.

Noting that $\Delta \subset (-\infty, -\|h\|_R)^n$, we have that $\dot{V}(z) < 0$ for all nonzero $z \in B(R)$. Theorem 8.3 now implies $B(R) \subset \mathcal{D}^z$. In view of Proposition (10.1), an analogous statement also holds for Eq. (10.6). This proves the first assertion of Theorem 10.1. The second assertion follows similarly from the observation

$$\begin{aligned} \dot{V}(z) &\leq \left(\max_{1 \leq i \leq n} (\lambda_i) + \|h\| \right) |z|^2 \\ &< 0 \end{aligned}$$

for all $z \in \mathbb{R}^n$, $z \neq 0$. This proves global asymptotic stability. □

Note 10.1. Note that the linear feedback claimed to exist in Theorem 10.1 guarantees that $\overline{B(R')}$ is also an RAS for $R' > R$, R' close enough to R . Note also that Theorem 10.1 holds under a normal stabilizability assumption.

Robustness of the stabilization property of Theorem 10.1 with respect to perturbations in the nonlinear terms $h(x)$ is now considered. The higher order terms $h(x)$ do not affect asymptotic stability of the null solution of a hyperbolic system (linearization with no imaginary eigenvalues). In our framework, we note that for $u(x) = Kx$ a linearly stabilizing feedback, the null solution of $\dot{x} = (A + BK)x + h(x)$ is asymptotically stable for all $h \in \mathcal{H}$. However, the domain of attraction does indeed depend on variations in h . The next result states that the linear feedback $u(x) = Kx$ in Theorem 10.1 is robust to variations in h . Specifically, the assertion is that $B(R)$ is guaranteed to be within the domain of attraction for each member of a family of systems each of whose linear parts is $\dot{x} = (A + BK)x$.

Theorem 10.1 (Robustness Form). Let $R > 0$ be fixed. Suppose that (A, B) is symmetrically stabilizable within $\Delta \subset \mathbb{R}_+^n$, and let

$$\alpha := \sup_{\Lambda \in \Delta} \max_{1 \leq i \leq n} (\Lambda_i).$$

If $h \in \mathcal{B}_R(|\alpha|)$ (resp. $\mathcal{B}(|\alpha|)$), then the nonlinear control system $\dot{x} = Ax + Bu + h(x)$ is stabilizable within $B(R)$ (resp. globally stabilizable) using linear state feedback.

Under the foregoing assumption, this asserts the existence of a feedback gain matrix $K \in \mathbb{R}^{m \times n}$ for which the associated domain of attraction contains $B(R)$, for each $h \in \mathcal{B}_R(|\alpha|)$. No $h \in \mathcal{B}_R(|\alpha|)$ results in a domain of attraction not entirely containing $B(R)$.

PLANAR SYSTEMS

In this chapter, we specialize the results of Chapter 10 to obtain a complete design methodology for the case of planar systems. This includes an investigation of the normal stabilizability question for these systems.

Consider the planar nonlinear control system

$$\dot{x} = Ax + bu + h(x) \quad (11.1)$$

where $A \in \mathbb{R}^{2 \times 2}$, $b \in \mathbb{R}^2 \setminus \{0\}$ ¹⁵ and h denotes the higher order terms, i.e. $h \in \mathcal{H}$.

Define the sets Δ and $\Delta(A, b)$ by

$$\Delta := \{ (\lambda_1, \lambda_2) \in \mathbb{R}_- \times \mathbb{R}_- : \lambda_1 \neq \lambda_2 \text{ and } \lambda_1, \lambda_2 \notin \sigma(A) \} \quad (11.2)$$

and

$$\Delta(A, b) = \{ (\lambda_1, \lambda_2) \in \Delta : \lambda_1 \lambda_2 - \nu(A, b)(\lambda_1 + \lambda_2) + \mu(A, b) = 0 \}, \quad (11.3)$$

¹⁵ If we let B be a nonzero 2×2 matrix, then it is either nonsingular, in which case the stabilization problem becomes trivial, or of rank one. The latter case is equivalent to considering B to be a vector b in \mathbb{R}^2 .

where

$$\nu(A, b) := \frac{b^T \text{Adj}(A)b}{|b|^2} \quad (11.4)$$

and

$$\mu(A, b) := \frac{|\text{Adj}(A)b|^2}{|b|^2}. \quad (11.5)$$

Here, $\text{Adj}(A)$ denotes the adjugate matrix of A .

The defining equation in (11.3) is that of a hyperbola which may be equivalently characterized by

$$\lambda_2 = \frac{\nu(A, b)\lambda_1 - \mu(A, b)}{\lambda_1 - \nu(A, b)}. \quad (11.6)$$

For the next result, we assume that the hyperbola (11.6) is nondegenerate, i.e. it is not a horizontal line. It will be seen shortly that this assumption amounts to (A, b) being controllable.

Theorem 11.1. Assume that (A, b) is controllable. Then the pair (A, b) is normally stabilizable within $\Delta(A, b)$ if and only if $\nu(A, b) < 0$. Furthermore, given any set of desired closed-loop eigenvalues $(\lambda_1, \lambda_2) \in \Delta(A, b)$, the corresponding normally stabilizing feedback gain is given by

$$k = [1 \ 1] [(\lambda_1 I - A)^{-1}b \ (\lambda_2 I - A)^{-1}b]^{-1}. \quad (11.7)$$

Proof. First, we show that $\Delta(A, b) \neq \emptyset$. From Eq. (11.6), we obtain

$$\frac{\partial \lambda_2}{\partial \lambda_1} = \frac{\mu(A, b) - \nu^2(A, b)}{(\lambda_1 - \nu(A, b))^2}. \quad (11.8)$$

Define the controllability matrices C, C_a by $C := [b \ Ab]$, $C_a := [b \ \text{Adj}(A)b]$ and note that $\det(C_a) = -\det(C) \neq 0$. Then it easily follows that

$$\det(C_a^T C_a) = (\det(C_a))^2 = |b|^4 (\mu(A, b) - \nu^2(A, b)) > 0. \quad (11.9)$$

and hence that λ_2 is a monotonically strictly increasing function of λ_1 . A quick sketch of the plot of λ_2 as a function of λ_1 convinces us that $\Delta(A, b) \cap \mathbb{R}_-^2 \neq \emptyset$

precisely when $\nu(A, b) < 0$. The sketch just referred to is also useful in finding pairs $(\lambda_1, \lambda_2) \in \Delta(A, b)$. To find such a pair, we may simply pick a value λ_1 in $(\nu(A, b), 0) \setminus \sigma(A)$, and then use Eq. (11.6) to compute the corresponding value of λ_2 .

Next, we show that for $(\lambda_1, \lambda_2) \in \Delta(A, b)$, the vectors

$$v_i = (\lambda_i I - A)^{-1} b, \quad i = 1, 2$$

are orthogonal. It is equivalent to show that the vectors w_1 and w_2 are orthogonal, where

$$w_i = \chi_A(\lambda_i) v_i, \quad i = 1, 2$$

and $\chi_A(s)$ denotes the characteristic polynomial of A . (Recall that $\lambda_1, \lambda_2 \notin \sigma(A)$.) We have

$$\begin{aligned} w_1^T w_2 &= b^T (\text{Adj}(\lambda_1 I - A))^T \text{Adj}(\lambda_2 I - A) b \\ &= b^T \left(\lambda_1 \lambda_2 I - \lambda_1 \text{Adj}(A) - \lambda_2 \text{Adj}^T(A) + \text{Adj}^T(A) \text{Adj}(A) \right) b \\ &= |b|^2 \lambda_1 \lambda_2 - (\lambda_1 + \lambda_2) b^T \text{Adj}(A) b + |\text{Adj}(A) b|^2. \end{aligned}$$

Since $(\lambda_1, \lambda_2) \in \Delta(A, b)$, it follows that $w_1^T w_2 = 0$.

We now show that v_1 and v_2 are eigenvectors of $A + bk$ corresponding to λ_1 and λ_2 , respectively. Let $V = [v_1 \ v_2]$. Then $k = [1 \ 1]V^{-1}$ and

$$\begin{aligned} [(A + bk)v_1 \ (A + bk)v_2] &= (A + bk)V = AV + bkV \\ &= [Av_1 + b \ Av_2 + b]. \end{aligned}$$

On the other hand, $Av_i + b = (A(\lambda_i I - A)^{-1} + I) b = \lambda_i v_i$ for $i = 1, 2$. Therefore

$$(A + bk)v_i = \lambda_i v_i, \quad i = 1, 2,$$

i.e., $\sigma(A + bk) = \{\lambda_1, \lambda_2\}$. Since the eigenvectors v_1 and v_2 are orthogonal, we obtain that $A + bk$ is a normal matrix (in fact symmetric since λ_1 and λ_2 are

real).

□

Theorem 11.1 states that if (A, b) is controllable and $\nu(A, b) < 0$, then every pair $(\lambda_1, \lambda_2) \in \Delta(A, b)$ may be assigned via linear feedback while achieving the normality requirement. It is not necessary however that a pair (A, b) be controllable for it to be normally stabilizable. We shall return to this case in Chapter 12.

Remarks 11.1.

1. It is easily shown that by allowing complex eigenvalues in Δ and $\Delta(A, b)$, one obtains one additional pair of assignable eigenvalues; namely $\nu(A, b) \pm i|\det(C)|$. Thus the set of all distinct assignable eigenvalues not in $\sigma(A)$ is essentially real.
2. The condition $\nu(A, b) < 0$ implies that $\det[b \text{ Adj}(A)b] \neq 0$, hence (A, b) controllable since $\det[b \text{ Adj}(A)b] = -\det[A \text{ Adj}(A)b]$, in all but the case when b and $\text{Adj}(A)b$ are of opposite directions.
3. Note that it is not possible to force *both* eigenvalues to be arbitrarily large. Clearly, this is a consequence of the normality requirement. It can be shown that in order for a pair (A, B) to be “arbitrarily” normally stabilizable (normally stabilizable with arbitrarily negative assignable closed-loop eigenvalues) it is necessary and sufficient that $\text{rank}(B) = n$, n being the size of A .

The next theorem is a direct consequence of Theorems 10.1 and 11.1. Let $R > 0$ be a fixed number.

Theorem 11.2. Assume that $\nu(A, b) < 0$. If $h \in \mathcal{B}_R(|\nu(A, b)|)$ (resp. $\mathcal{B}(|\nu(A, b)|)$) then there exists a linear feedback $u(x) = kx$ such that (the origin of) the closed-loop system $\dot{x} = (A + bk)x + h(x)$ is asymptotically stable within $B(R)$ (resp. globally asymptotically stable). Furthermore, for any desired closed-loop eigenvalues $\lambda_1 \in (\nu(A, b), -\|h\|_R) \setminus \sigma(A)$ (resp. $(\nu(A, b), -\|h\|) \setminus \sigma(A)$) and λ_2 given by (11.6), the feedback gain k is given by (11.7).

The analogue of Theorem 10.1 (Robustness Form) in the two-dimensional case

follows by taking α to be any number strictly between $\nu(A, b)$ and 0. The desired feedback k is obtained in a manner identical to that outlined in Theorem 11.2, with $-\|h\|_R$ (resp. $-\|h\|$) replaced by α .

Two examples are now presented to illustrate application of Theorem 11.2.

Example 11.1. Let $R = 1$ and consider the system

$$\begin{aligned}\dot{x}_1 &= -\frac{3}{2}x_1 + x_2 + x_1^2 \\ \dot{x}_2 &= x_2 + u - x_2^2.\end{aligned}$$

The origin of the unforced system is unstable since $\sigma(A) = \{-\frac{3}{2}, 1\}$. It is easily checked that (A, b) is controllable, $\nu(A, b) = -\frac{3}{2}$, $\|h\|_{R=1} = 1 < |\nu(A, b)|$ and $\mu(A, b) = \frac{13}{4}$. By picking λ_1 in $(-\frac{3}{2}, -\|h\|_R) \setminus \sigma(A) = (-\frac{3}{2}, -1)$, say $\lambda_1 = -\frac{5}{4}$, we get from (11.6) that $\lambda_2 = -\frac{11}{2}$ and from (11.7) that $k = [1 \quad -\frac{25}{4}]$. The closed-loop system is

$$\begin{aligned}\dot{x}_1 &= -\frac{3}{2}x_1 + x_2 + x_1^2 \\ \dot{x}_2 &= x_1 - \frac{21}{4}x_2 - x_2^2.\end{aligned}$$

Simulations of the closed-loop system for various initial conditions, shown in Figure 11.1, corroborate the fact that $B(1)$ is contained in the actual domain of attraction. Note that some initial conditions in the immediate vicinity of $B(1)$ lead to instability (e.g., $x_0 = (1.4, 0)$ and $(1.2, 1.2)$).

Example 11.2. We consider globally stabilizing the system

$$\begin{aligned}\dot{x}_1 &= -\frac{3}{2}x_1 + x_2 \\ \dot{x}_2 &= -x_1 + x_2 + u + \sin(x_1).\end{aligned}$$

Here, A and b are as in Example 11.1, $h(x) = [0, \sin(x_1) - x_1]^T$ and $\|h\| = 1.217$. Proceeding as in Example 11.1, we obtain by choosing $\lambda_1 = -\frac{5}{4}$ that the closed-

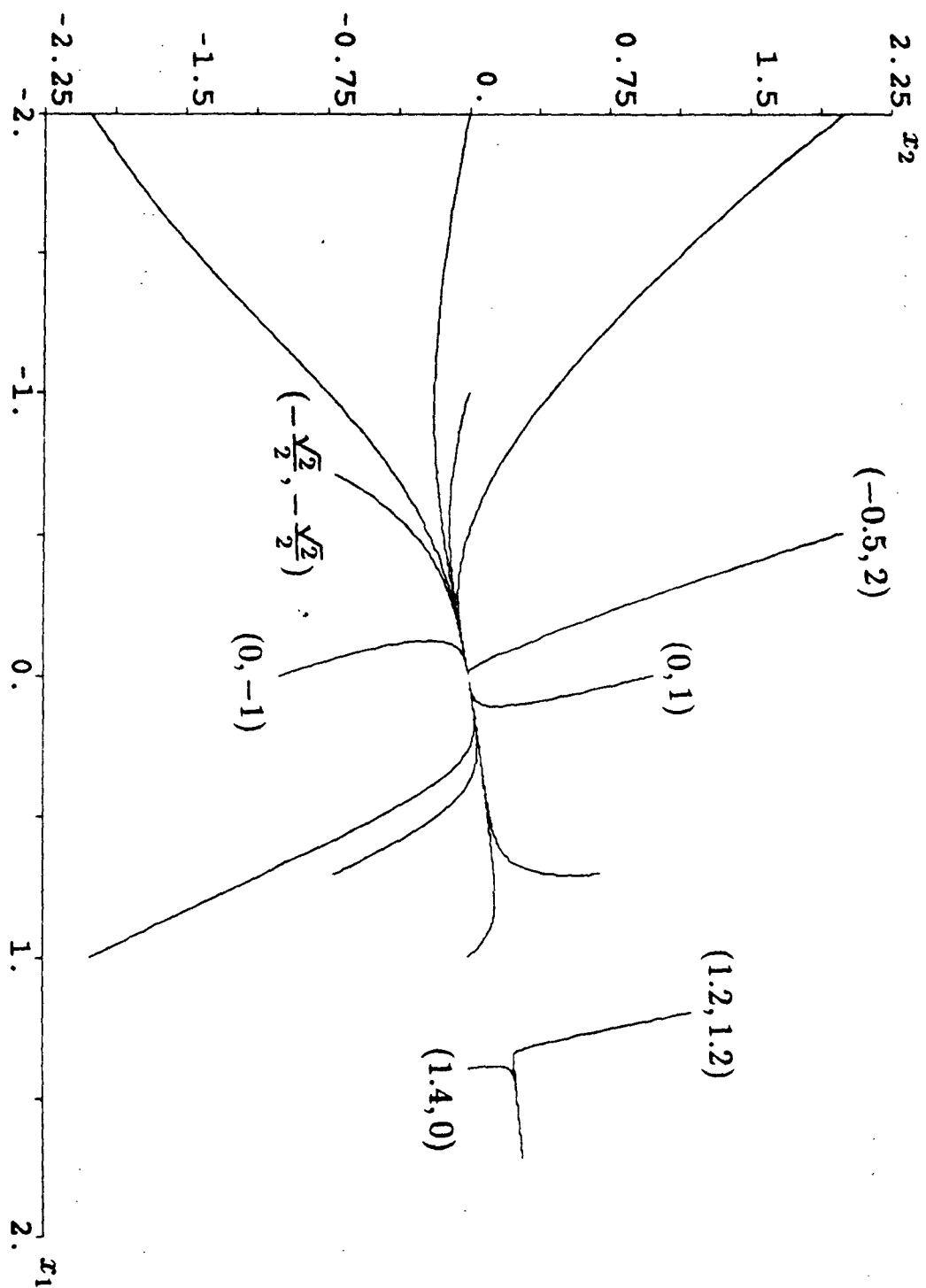


Figure 11.1: Closed loop trajectories for Example 11.1.

loop system is

$$\dot{x}_1 = -\frac{3}{2}x_1 + x_2$$

$$\dot{x}_2 = -\frac{21}{4}x_2 + \sin(x_1).$$

With the Lyapunov function $V(x) = x_1^2 + x_2^2$, we find that $\dot{V}(x) < -x_1^2 - \frac{17}{2}x_2^2 < 0$ for all $x \neq 0$. Therefore, the null solution of the closed-loop system is globally asymptotically stable, as predicted by Theorem 11.2.

NORMAL AND SYMMETRIC
STABILIZABILITY OF LTI
CONTROL SYSTEMS

In Chapter 10, sufficient conditions for stabilizability within a prescribed Euclidean ball were given. These conditions were based on the assumption that the underlying linear system is normally (or symmetrically) stabilizable. In Chapter 11, necessary and sufficient conditions for normal stabilizability within the set $\Delta(A, b)$ were given for planar systems. Here, we investigate this question for multidimensional linear time-invariant (LTI) control systems.

The normal stabilizability problem for LTI control systems falls in the general category of robust eigenstructure assignment [38]. In [38], the authors describe numerical methods for determining robust, or well-conditioned, solutions to the problem of pole assignment by linear state feedback. Using the degrees of freedom that are available for choosing a stabilizing feedback gain K , an algorithmic approach for selecting such a gain while minimizing the condition number of the closed-loop matrix is presented. Since the best conditioned matrices are normal matrices (their condition number achieves the minimum value 1), it is seen that

the work in [38] is indeed a numerical approach to the normal stabilizability problem. There is unfortunately, to the author's knowledge, no analytical results in this area and a complete answer to this problem remains unknown.

Consider the LTI control system

$$\dot{x} = Ax + Bu \quad (12.1)$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ is, without loss of generality, of full rank $= m$.

For simplicity, we focus on the following problem, which it turns out contains essentially the same difficulties as the general problem of normal stabilizability. Given a set of desired negative closed-loop eigenvalues $\Lambda = \{\lambda_1, \dots, \lambda_n\}$, we seek conditions under which there exists a feedback gain $K \in \mathbb{R}^{m \times n}$ such that

- (i) $\sigma(A + BK) = \Lambda$
- (ii) $A + BK$ symmetric.

Note that in the case $m = n$, i.e., B square nonsingular, this problem has a trivial solution; thus, it is implicitly assumed that $m < n$. Since B is full rank, then the QR -factorization ([41]) allows one to write $B = QR$ where R is a nonsingular upper triangular $m \times m$ matrix and Q is an $n \times m$ matrix and such that $Q^T Q = I_m$ (i.e. the columns of Q form an orthonormal system in \mathbb{R}^n). By operating a change of basis in the input space if necessary, we may therefore assume, without loss of generality, that B is such that $B^T B = I_m$. Denote $D = \text{diag}(\lambda_1, \dots, \lambda_n)$.

Theorem 12.1. There exists $K \in \mathbb{R}^{m \times n}$ such that (i) and (ii) hold if and only if the equations

$$(I - BB^T)(AQ - QD) = 0 \quad (12.2)$$

$$Q^T Q - I = 0 \quad (12.3)$$

have a solution Q . In this case a feedback gain K ensuring satisfaction of (i) and (ii) is given by

$$K = B^T(QDQ^T - A) \quad (12.4)$$

Proof. Following [38], assume (i), (ii) has a solution K . Let then Q be any orthogonal matrix which diagonalizes $(A + BK)$. It follows that

$$BK = QDQ^T - A, \quad (12.5)$$

and upon multiplying by B^T , that

$$K = B^T(QDQ^T - A). \quad (12.6)$$

Therefore,

$$QDQ^T = A + BK = A + BB^T(QDQ^T - A). \quad (12.7)$$

Eq. (12.2) is obtained by multiplying both sides of (12.7) on the right by Q .

Conversely, if Q satisfies (12.2)-(12.3), then by writing (12.2) as

$$AQ - BB^T(AQ - QD) = QD, \quad (12.8)$$

and multiplying by Q^T , we obtain

$$A + BB^T(QDQ^T - A) = QDQ^T. \quad (12.9)$$

By taking K to be $B^T(QDQ^T - A)$, this says that the closed loop matrix $A + BK$ is diagonalizable using the orthogonal matrix Q and is hence symmetric.

□

Denote by \mathcal{L} the linear mapping

$$\mathcal{L}: X \mapsto (I - BB^T)(AX - XD). \quad (12.10)$$

An alternate interpretation of Eqs. (12.2)-(12.3) is the following: the problem (i)-(ii) has a solution if and only if $\ker(\mathcal{L})$ contains an orthogonal matrix. Here, $\ker(\mathcal{L})$ denotes the kernel or null space of \mathcal{L} , i.e. the vector space

$$\{X \in \mathbb{R}^{n \times n} : \mathcal{L}(X) = 0\}.$$

If $\dim \ker(\mathcal{L}) = \ell$, then the problem under consideration amounts to finding constants $\alpha_1, \alpha_2, \dots, \alpha_\ell$, such that the matrix $\sum \alpha_i V_i$ is orthogonal, where the V_i are such that $\text{span}\{V_1, V_2, \dots, V_\ell\} = \ker(\mathcal{L})$.

Proposition 12.1. The mapping \mathcal{L} has a nontrivial kernel, i.e. $\ker(\mathcal{L}) \neq \{0\}$, and

$$\ell := \dim \ker(\mathcal{L}) \geq nm. \quad (12.11)$$

Moreover, equality holds in (12.11) if and only if $\lambda_i \notin \sigma(A)$, $i = 1, \dots, n$.

Proof. First of all note that $\lambda = 1$ is an eigenvalue of the matrix BB^T since it is an eigenvalue of $B^TB = I_m$. Consequently, the mapping \mathcal{L} is singular and, therefore, it has a nontrivial kernel. Next, by writing the equation $\mathcal{L}(X) = 0$ in vector form, it is seen that

$$\ell = \dim \ker(\mathcal{L}) = \dim \ker(L), \quad (12.12)$$

where L is the $n^2 \times n^2$ matrix given by

$$L = (I \otimes (I - BB^T))(D \oplus A) \quad (12.13)$$

where \oplus and \ominus are the Kronecker sum and difference defined in Chapter 3. It follows that

$$\ell \geq \dim \ker(I \otimes C), \quad (12.14)$$

where $C := I - BB^T$, with equality holding if and only if the matrix $D \ominus A$ is nonsingular, i.e. (see Section 3.3, Part I), if and only if $\lambda_i \notin \sigma(A)$, $i = 1, \dots, n$. Using the fact that for any two matrices M and N

$$\text{rank}(M \otimes N) = \text{rank}(M) \text{rank}(N), \quad (12.15)$$

we obtain that

$$\dim \ker(I \otimes C) = n^2 - n \text{rank}(C) \quad (12.16)$$

$$= n(n - \text{rank}(C)) \quad (12.17)$$

$$= n \dim \ker(I - BB^T) \quad (12.18)$$

$$= n m. \quad (12.19)$$

□

As a simple example, let $\lambda_1 = -\frac{1}{2}$, $\lambda_2 = -3$,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We obtain $\ell = 2$ and

$$\ker(\mathcal{L}) = \text{span}\{V_1, V_2\}$$

$$V_1 = \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}; \quad V_2 = \begin{bmatrix} -\frac{1}{2} & 0 \\ 1 & 0 \end{bmatrix}.$$

The matrix with coordinates $(\frac{\sqrt{5}}{5}, 2\frac{\sqrt{5}}{5})^T$ in this basis solves (12.2)-(12.3) and yields $k = [-\frac{5}{2}, 0]$.

Obtaining conditions on A, B and D for which Eqs. (12.2)-(12.3) have a solution turns out to be a difficult task. The next proposition gives a sufficient condition for the case in which A is symmetric. It is worth pointing out that, as far as the application of this proposition to the stabilization with prescribed RAS is concerned, the matrix A need not be symmetric as we will see shortly. Let b_1, b_2, \dots, b_m denote the column vectors of B and b_{m+1}, \dots, b_n be chosen such that $\{b_1, \dots, b_n\}$ is an orthonormal set. Let $\sigma(A) = \{\alpha_1, \dots, \alpha_n\}$, where the α_i 's are real.

Proposition 12.2. Assume that $A^T = A$ and that $A^T b_k = \alpha_k b_k$, $k = m + 1, \dots, n$. Then for every $\lambda \in \mathbb{R} \setminus \{\alpha_{m+1}, \dots, \alpha_n\}$, there is a $K \in \mathbb{R}^{m \times n}$ such that

$$(i) \quad \sigma(A + BK) = \{\lambda, \dots, \lambda, \alpha_{m+1}, \dots, \alpha_n\}$$

$$(ii) \quad (A + BK) \text{ symmetric}$$

Proof. It is enough to show that there exists an orthonormal set $\{v_1, \dots, v_n\}$ such that

$$v_i \in \mathcal{F}(\lambda), \quad i = 1, \dots, m \tag{12.20}$$

$$v_k \in \mathcal{F}(\alpha_k), \quad k = m + 1, \dots, n, \tag{12.21}$$

where

$$\mathcal{F}(\beta) := \{x \in \mathbb{R}^n : (\beta I - A)x \in \mathcal{R}(B)\}. \tag{12.22}$$

If such a set exists and (12.20)-(12.21) hold, i.e.

$$(\lambda I - A)v_i = Bg_i, \quad i = 1, \dots, m \quad (12.23)$$

$$(\alpha_k I - A)v_k = Bg_k, \quad k = m + 1, \dots, n \quad (12.24)$$

for some g_i 's, then K is given by

$$K = [g_1, \dots, g_n][v_1, \dots, v_n]^{-1}. \quad (12.25)$$

Since $\lambda \notin \sigma(A)$, then $\mathcal{F}(\lambda)$ is an m -dimensional space, and

$$\mathcal{F}(\lambda) = (\lambda I - A)^{-1}\mathcal{R}(B). \quad (12.26)$$

Therefore, we can always select $\{v_1, \dots, v_m\}$, orthonormal, such that $v_i \in \mathcal{F}(\lambda)$, $i = 1, \dots, m$ (e.g. by operating a Gram-Schmidt procedure on $\{(\lambda I - A)^{-1}b_i, i = 1, \dots, m\}$). Set $v_k = b_k$, $k = m + 1, \dots, n$. Then $\{v_{m+1}, \dots, v_n\}$ is itself an orthonormal set and it only remains to show that $v_i^T v_k = 0$, $i = 1, \dots, m$, $k = m + 1, \dots, n$ and $v_k \in \mathcal{F}(\alpha_k)$, $k = m + 1, \dots, n$.

Since

$$(\lambda I - A)v_i \in \mathcal{R}(B) \quad (12.27)$$

and

$$\mathcal{R}(B) = (\ker(B^T))^\perp = \text{span}^\perp\{v_k, k = m + 1, \dots, n\}, \quad (12.28)$$

it follows that

$$\begin{aligned} 0 &= v_k^T(\lambda I - A)v_i = \lambda v_k^T v_i - (A^T v_k)^T v_i \\ &= (\lambda - \alpha_k)v_k^T v_i \end{aligned} \quad (12.29)$$

By assumption, $\lambda \neq \alpha_k$, $k = m + 1, \dots, n$, hence $v_k^T v_i = 0$. That $v_k \in \mathcal{F}(\alpha_k)$, $k = m + 1, \dots, n$, may be seen by noting that $(\alpha_k I - A)v_k = (\alpha_k I - A^T)v_k = 0 \in \mathcal{R}(B)$.

□

Remark 12.1.

- 1) Since $v_k = b_k$, $k = m + 1, \dots, n$, it follows from the fact $A^T v_k = \alpha_k b_k$ and $B^T v_k = 0$, $k = m + 1, \dots, n$, that $\alpha_{m+1}, \dots, \alpha_n$ are the uncontrollable modes

of (A, B) . If these are stable, then Proposition 12.2 states that there always exists a feedback gain K which will make $(A + BK)$ stable and symmetric.

2) To apply the results of Chapter 10 to a nonlinear control system of the form

$$\dot{x} = Ax + Bu + h(x), \quad (12.30)$$

where, as usual, h denotes the higher order terms, it is not necessary that A be symmetric. Rather, Proposition 12.2 should be applied to the pair $([A]_s, B)$, where $[\]_s$ denotes the symmetric part of a matrix. To see this, recall that the asymptotic stability of the origin was proved in the main theorem of Chapter 10 using a quadratic Lyapunov V . As a result, only $[A]_s$ contributes in \dot{V} . In the case Proposition 12.2 applies to (12.30), the largest uncontrollable eigenvalue then dictates the size of the maximal ball of asymptotic stability which can be achieved.

Consider for example the nonlinear control system

$$\dot{x}_1 = -2x_1 + 2x_2 + 3x_3 - x_1x_2 - u_1 \quad (12.31)$$

$$\dot{x}_2 = -2x_3 - x_2^2 + u_2 \quad (12.32)$$

$$\dot{x}_3 = -x_1 - 2x_3 + x_1^2 \quad (12.33)$$

In this case,

$$A = \begin{bmatrix} -2 & 2 & 3 \\ 0 & 0 & -2 \\ -1 & 0 & -2 \end{bmatrix},$$

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and $h(x) = [-x_1x_2, -x_2^2, x_1^2]^T$. Here, the unforced nonlinear system is unstable ($0.445 \in \sigma(A)$). To find a stabilizing control for the nonlinear system (12.31)-(12.33), we apply Proposition 12.2 to the pair $([A]_s, B)$. In this example,

$$[A]_s = \begin{bmatrix} -2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & -2 \end{bmatrix}$$

and, in the notation of Proposition 12.2, $b_3 = [1, 0, 1]^T$. Clearly, $[A]_* b_3 = -b_3$ and we are in a position to apply Proposition 12.2 with $\alpha_3 = -1$. Proposition 12.2 states that given any real number λ , there is a feedback gain K such that $A + BK$ is symmetric and $\sigma(A + BK) = \{\lambda, \lambda, -1\}$. Letting $\lambda = -2$ and operating a Gram-Schmidt procedure on the vectors $(\lambda I - A)^{-1} b_1$ and $(\lambda I - A)^{-1} b_2$ (which span $\mathcal{F}(\lambda)$), we obtain

$$v_1 = \begin{bmatrix} -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix}, \quad v_2 = \begin{bmatrix} -\frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \end{bmatrix}.$$

Selecting $v_3 = [\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}]^T$, we have

$$g_1 = \begin{bmatrix} \frac{2\sqrt{2}}{2} \\ \frac{2}{2} \\ 0 \end{bmatrix}, \quad g_2 = \begin{bmatrix} -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \\ \sqrt{6} \end{bmatrix} \quad \text{and} \quad g_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, $K = GV^{-1}$ is given by

$$K = \begin{bmatrix} -\frac{1}{2} & 1 & \frac{1}{2} \\ -1 & -2 & 1 \end{bmatrix}.$$

With the feedback control $u(x) = Kx$, we obtain the closed-loop system

$$\dot{x}_1 = -\frac{3}{2}x_1 + x_2 + \frac{5}{2}x_3 - x_1x_2 \tag{12.34}$$

$$\dot{x}_2 = -x_1 - 2x_2 - x_3 - x_2^2 \tag{12.35}$$

$$\dot{x}_3 = -\frac{3}{2}x_1 + x_2 - \frac{3}{2}x_3 + x_1^2 \tag{12.36}$$

It can be verified that $[A + BK]_*$ has the eigenvalues $-2, -2, -1$. Applying Theorem 10.1, we obtain by noting that for this example, $\|h\|_R \leq R$, that the closed-loop system (12.34)-(12.36) is asymptotically stable within any ball of radius $R < 1$.

**TWO-TIME SCALE FEEDBACK STABILIZATION
WITHIN A CYLINDRICAL REGION**

In this chapter, we extend results obtained in previous chapters to a class of multidimensional singularly perturbed nonlinear control systems. More specifically, we investigate the stabilization with prescribed RAS problem for singularly perturbed nonlinear control systems with planar slow subsystems. In the analysis, the composite control methodology of Chow and Kokotovic [45] is combined with the results of Chapter 11 on planar systems.

13.1 Problem Setting

For simplicity, we consider single input models of the form

$$\dot{x} = f(x) + Fz + b_1 u \quad (13.1)$$

$$\epsilon \dot{z} = g(x) + Gz + b_2 u \quad (13.2)$$

where $x \in \mathbb{R}^2, z \in \mathbb{R}^p$; F, G, b_1, b_2 are constant matrices of appropriate dimensions; f, g are analytic and satisfy $f(x_e) = 0$ and $g(x_e) = 0$ uniquely for $x_e = 0$.

Given a fixed radius $R > 0$ and a bounded subset E of \mathbb{R}^p , our goal is to synthesize a feedback control law $u(x, z)$ which, for ϵ small enough, stabilizes the

origin of System (13.1)-(13.2) and guarantees, in addition, that

$$x(t, x_0), z(t, z_0) \longrightarrow 0 \text{ as } t \longrightarrow +\infty \text{ for all } (x_0, z_0) \in B(R) \times E.$$

The fact that a prescribed RAS in the form of a cylindrical region, rather than a (more restrictive) ball in \mathbb{R}^{p+2} , is sought, is clearly a consequence of the fast subsystem living in \mathbb{R}^p .

We assume G to be a nonsingular matrix and the pair (G, b_2) controllable. By formally setting $\epsilon = 0$ in System (13.1)-(13.2), we obtain

$$z = -G^{-1}(g + b_2 u). \quad (13.3)$$

The slow and the fast subsystems are respectively given by

$$\dot{x} = a(x) + bu \quad (13.4)$$

and

$$\dot{z} = Gz + b_2 u, \quad (13.5)$$

where, with a slight abuse of notation, x, z are used to denote the slow and fast states \bar{x}, \bar{z} and “.” denotes differentiation in both the slow and the fast time scales. In Eq. (13.4), $a(x)$ and b are given by

$$a(x) = f(x) - FG^{-1}g(x) \quad (13.6)$$

$$b = b_1 - FG^{-1}b_2. \quad (13.7)$$

Write

$$a(x) = Ax + h(x) \quad (13.8)$$

where

$$A = f_x(0) - FG^{-1}g_x(0) \quad (13.9)$$

$$h(x) = h^f(x) - FG^{-1}h^g(x) \quad (13.10)$$

and

$$h^f(x) := f(x) - f_x(0)x \quad (13.11)$$

$$h^g(x) := g(x) - g_x(0)x. \quad (13.12)$$

13.2 Stabilizing Control Laws

A control law stabilizing the full system (13.1)-(13.2) within the region $B(R) \times E$ is now presented. Define

$$\nu = \frac{b^T \text{Adj}(A)b}{|b|^2} \quad (13.13)$$

and

$$\mu = \frac{|\text{Adj}(A)b|^2}{|b|^2}. \quad (13.14)$$

We make the following assumptions:

A1: $\nu < 0$

A2: $\|h\|_R < |\nu|$

A3: $\max_{x \in \overline{B(R)}} \lambda^{\max}([h_x(x)]_s) < |\nu|$

Theorem 13.1. Under the foregoing assumptions there exists $\epsilon^* > 0$ such that for all $\epsilon \in (0, \epsilon^*]$, the following holds: There exist $k \in \mathbb{R}^{1 \times 2}, L \in \mathbb{R}^{1 \times p}$ such that the feedback law

$$u(x, z) = kx + Lz + LG^{-1}(g(x) + b_2 kx) \quad (13.15)$$

stabilizes the origin of System (13.1)-(13.2) within $B(R) \times E$. Furthermore, k and L may be readily computed using Theorem 11.2 and standard linear control formulas, respectively.

Proof. Under Assumptions A1-A2, Theorem (11.2) (together with Note 10.1) guarantees the existence of a feedback gain k , a radius $R' > R$ such that the feedback control $u(x) = kx$ stabilizes the origin of the slow system (13.4), (13.6)-(13.7) within $\overline{B(R')}$.

With $u(x) = kx$, the slow subsystem becomes

$$\begin{aligned} \dot{x} &= (A + bk)x + h(x) \\ &:= a_k(x) \end{aligned} \quad (13.16)$$

Consider the Lyapunov function

$$V(x) = \frac{1}{2} a_k^T(x) a_k(x) \quad (13.17)$$

Then, for system (13.16)

$$\dot{V}(x) = a_k^T(x) ([A + bk]_s + [h_x(x)]_s) a_k(x) \quad (13.18)$$

$$\leq (\lambda^{\max}([A + bk]_s) + \lambda^{\max}([h_x(x)]_s)) |a_k(x)|^2 \quad (13.19)$$

Therefore, $\dot{V}(x) < 0$ for all $x \in \overline{B(R')}$ if

$$\lambda^{\max}([h_x(x)]_s) < -\lambda^{\max}([A + bk]_s) \quad \forall x \in \overline{B(R')} \quad (13.20)$$

Since $A + bk$ is symmetric and $\lambda^{\max}(A + bk) \in (\nu, -\|h\|_R)$, it follows that (13.20) holds if

$$\lambda^{\max}([h_x(x)]_s) < -\nu \quad \forall x \in \overline{B(R')} \quad (13.21)$$

which is implied by A3 for $R' - R$ sufficiently small.

Thus, $V(x)$ is a Lyapunov function of the Krasovskii type and the result follows by using Theorem 9.5 with $D = \overline{B(R')}$ and L any standard stabilizing gain for the fast subsystem (13.5). □

In case Assumption A3 does not hold for a given desired radius R , the theorem below yields an alternative result. Its proof follows from Theorem 9.6 and the fact that (see proof of Theorem 10.1) $v(x) = |x|^2$ is also a Lyapunov function for (13.16), implying that condition C4 holds since any $B(c_0)$ is a level set for v , for any $c_0 > 0$.

Theorem 13.2. If all the foregoing assumptions, but A3, hold, then there exists $\epsilon^* > 0$ such that for all $\epsilon \in (0, \epsilon^*]$, the following holds: There exist $k \in \mathbb{R}^{1 \times 2}, L \in \mathbb{R}^{1 \times p}$ such that the feedback law

$$u(x, z) = kx + Lz + LG^{-1}(g(x) + b_2 kx) \quad (13.22)$$

steers every trajectory of system (13.1)-(13.2) starting within $B(R) \times E$ to a sphere centered at the origin, whose radius is $O(\epsilon)$.

Example 13.1

Consider the singularly perturbed nonlinear control system

$$\dot{x}_1 = -2x_1 + x_2 + \frac{1}{2}x_2^2 \quad (13.23)$$

$$\dot{x}_2 = z - \frac{1}{2}x_2^2 + 2u \quad (13.24)$$

$$\epsilon \dot{z} = \frac{1}{2}(x_1^2 + x_2^2) + z - u \quad (13.25)$$

and let $R = 3$ be the prescribed radius. We thus have

$$f(x) = \begin{bmatrix} -2x_1 + x_2 + \frac{1}{2}x_2^2 \\ -\frac{1}{2}x_2^2 \end{bmatrix}; \quad F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad b_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$g(x) = \frac{1}{2}(x_1^2 + x_2^2); \quad G = 1, b_2 = -1.$$

Note that System (13.23)-(13.25) is unstable for every $\epsilon > 0$.

The slow and the fast subsystem are given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}x_1^2 \\ \frac{1}{2}x_2^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (13.26)$$

$$:= Ax + h(x) + bu \quad (13.27)$$

and

$$\dot{z} = z - u \quad (13.28)$$

We also have

$$\sigma(A) = \{-3, 1\}$$

$$\nu = -2; \quad \mu = 5;$$

and

$$\|h\|_R \leq \frac{R}{2} = \frac{3}{2}$$

Following the procedure in Theorem 11.2 for constructing the feedback gain k , we let $\lambda_1 \in (-2, -\frac{3}{2}) \setminus \sigma(A)$ and compute λ_2 according to Eq. 11.2 For $\lambda_1 = -\frac{7}{4}$, we obtain $\lambda_2 = -6$. The feedback gain for the slow subsystem then obtains from Eq. 11.3 and is

$$k = [-2, -\frac{23}{4}].$$

Also, for $L = 2$, the feedback $u(z) = Lz$ stabilizes the fast subsystem.

The composite control of Eq (13.22), which is in this case given by

$$u(x, z) = 2x_1 + \frac{23}{4}x_2 + 2z + x_1^2 + x_2^2, \quad (13.29)$$

yields the closed-loop system

$$\dot{x}_1 = -2x_1 + x_2 + \frac{1}{2}x_2^2 \quad (13.30)$$

$$\dot{x}_2 = 4x_1 + \frac{23}{2}x_2 + 5z + 2x_1^2 + \frac{3}{2}x_2^2 \quad (13.31)$$

$$\epsilon \dot{z} = -2x_1 - \frac{23}{4}x_2 - z - \frac{1}{2}(x_1^2 + x_2^2) \quad (13.32)$$

Simulated trajectories of the closed-loop system (13.30)-(13.32) for various initial conditions and parameter values are given at the end of this dissertation.

**CHAPTER
FOURTEEN**

CONCLUSION

The main contribution of Part I of this dissertation is the introduction and application of a new conceptual tool for the analysis of generalized stability of parametrized families of matrices and polynomials. Polynomial guarding and semiguarding maps were shown to exist for a large variety of domains of practical interest of the complex plane. A methodology for establishing necessary and sufficient conditions for stability of one- and two-parameter families with respect to such domains has been expounded. Previously known results on Hurwitz stability of the convex hull of two matrices and polynomials were obtained as a special case. New results of a similar nature were also obtained.

The implementation of the methodology described in Part I of this dissertation is clearly better suited for symbolic manipulations. A program, using the symbolic language MACSYMA, has indeed been implemented to serve as a tool for analyzing the stability of parametrized families of real matrices relative to domains with polynomial boundaries. The program has obvious limitations but works reasonably well for families of matrices of moderate size and up to two parameters.

Despite the computational complexity which is bound to seriously affect the

implementation of the methods developed in Part I for the case of more than two parameters, further investigation of the general multiparameter case is warranted for its theoretical value. Analytical tools for robust stability assessment of uncertain linear state space models are indeed a prerequisite for a better understanding of how to control these models.

In Part II, sufficient conditions for stabilizability of nonlinear systems with a region of asymptotic stability containing a prescribed ball in \mathbb{R}^n have been presented. Under a symmetric stabilizability condition on the system linearization, it was shown that there is a linear stabilizing controller, and that the closed-loop system stability is robust to certain model perturbations. Necessary and sufficient conditions for normal stabilizability of a two-dimensional linear time-invariant system were obtained. These facilitated identification of a closed-form formula for a stabilizing feedback gain k which guarantees stabilization of a class of multi-dimensional singularly perturbed control systems within a prescribed cylindrical region.

The issue of normal (or symmetric) stabilizability raised in Part II which played a determinant role in our treatment of the stabilization with prescribed RAS problem is an issue with clear robustness implications and needs to be investigated further.

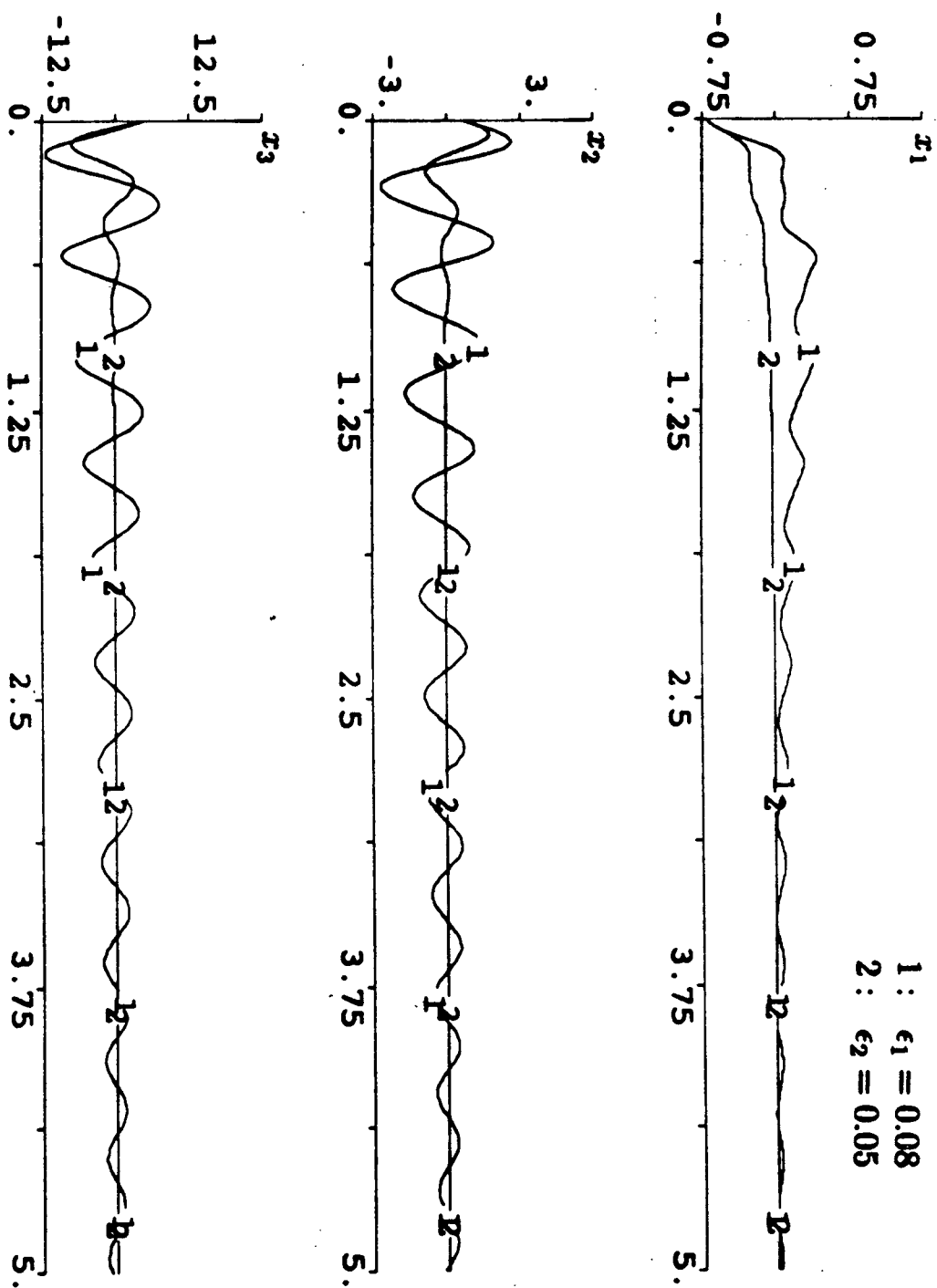


Figure 13.1 : Closed loop behavior for Example 13.1, $x_0 = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 5)$.

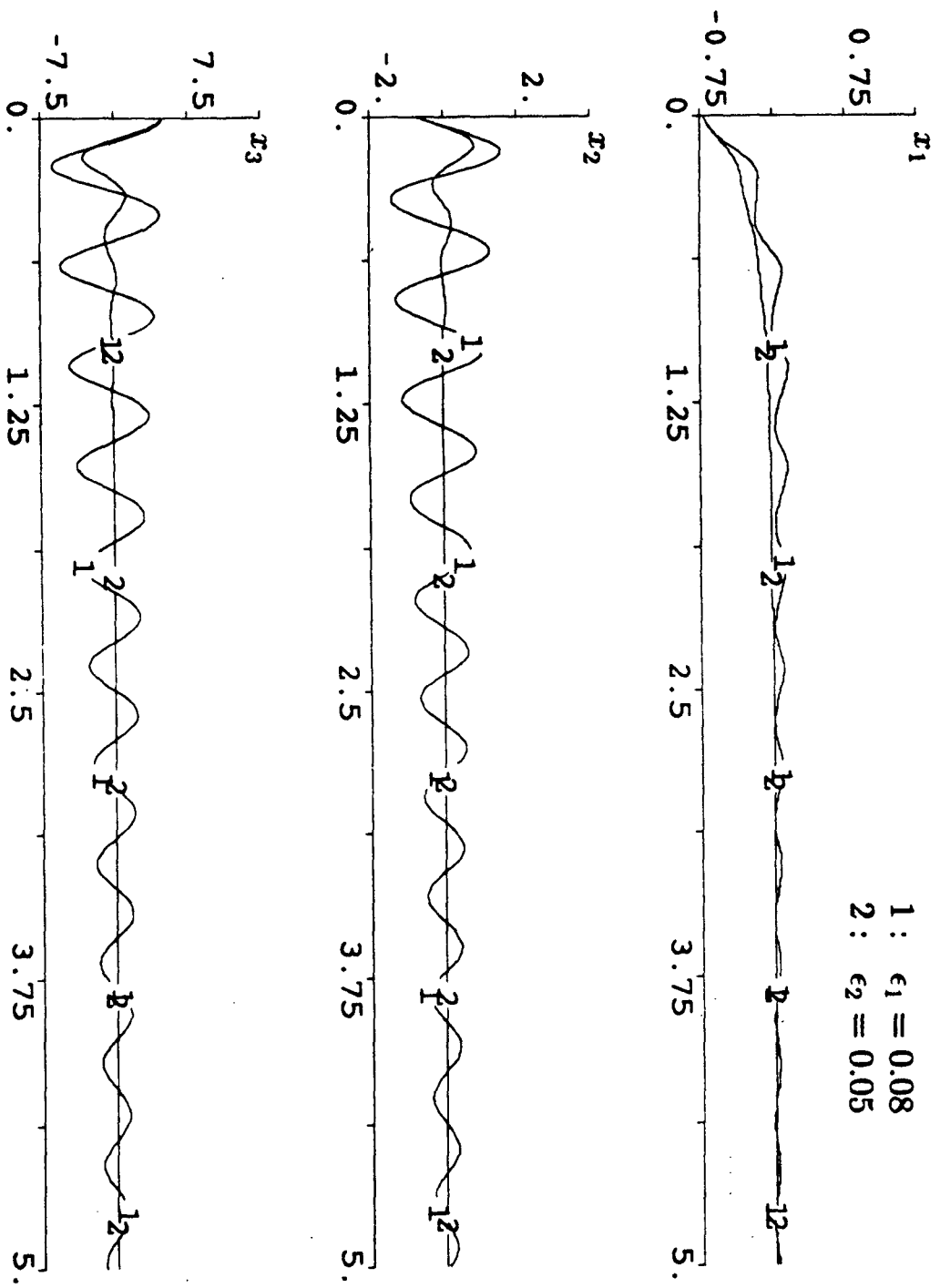


Figure 13.2: Closed loop behavior for Example 13.1, $x_0 = (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 5)$.

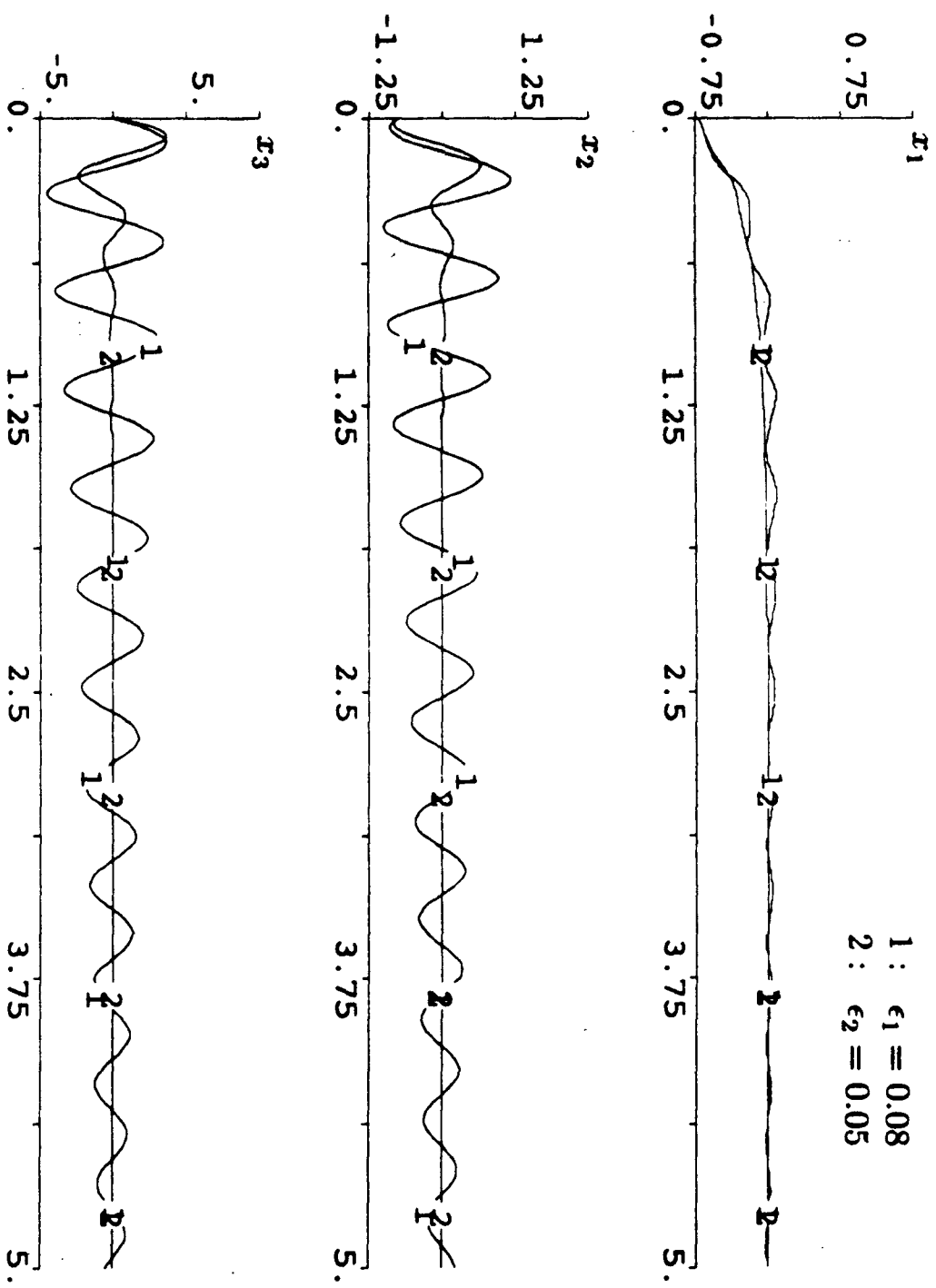


Figure 13.3: Closed loop behavior for Example 13.1, $x_0 = (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0)$.

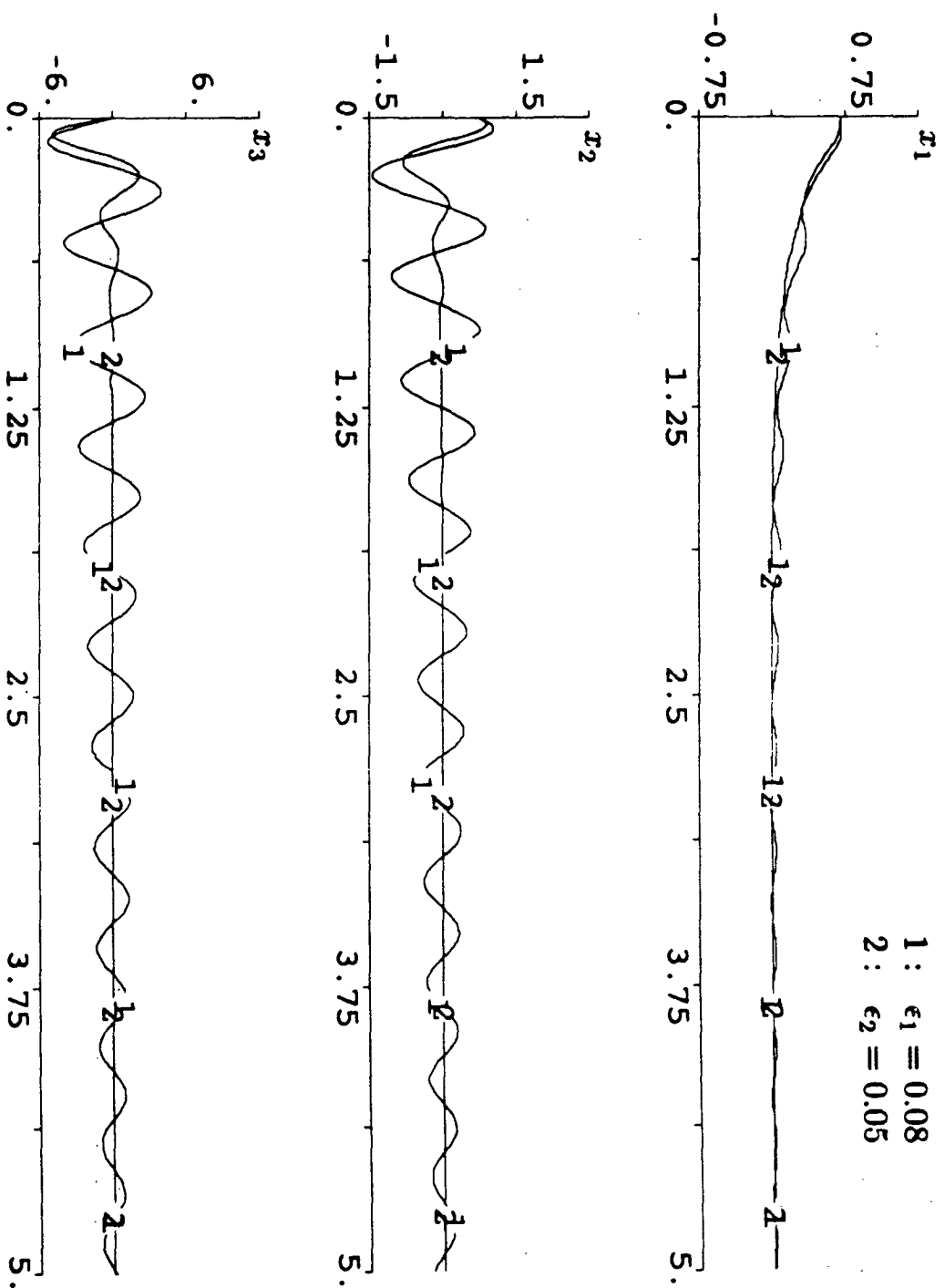


Figure 13.4: Closed loop behavior for Example 13.1, $x_0 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$.

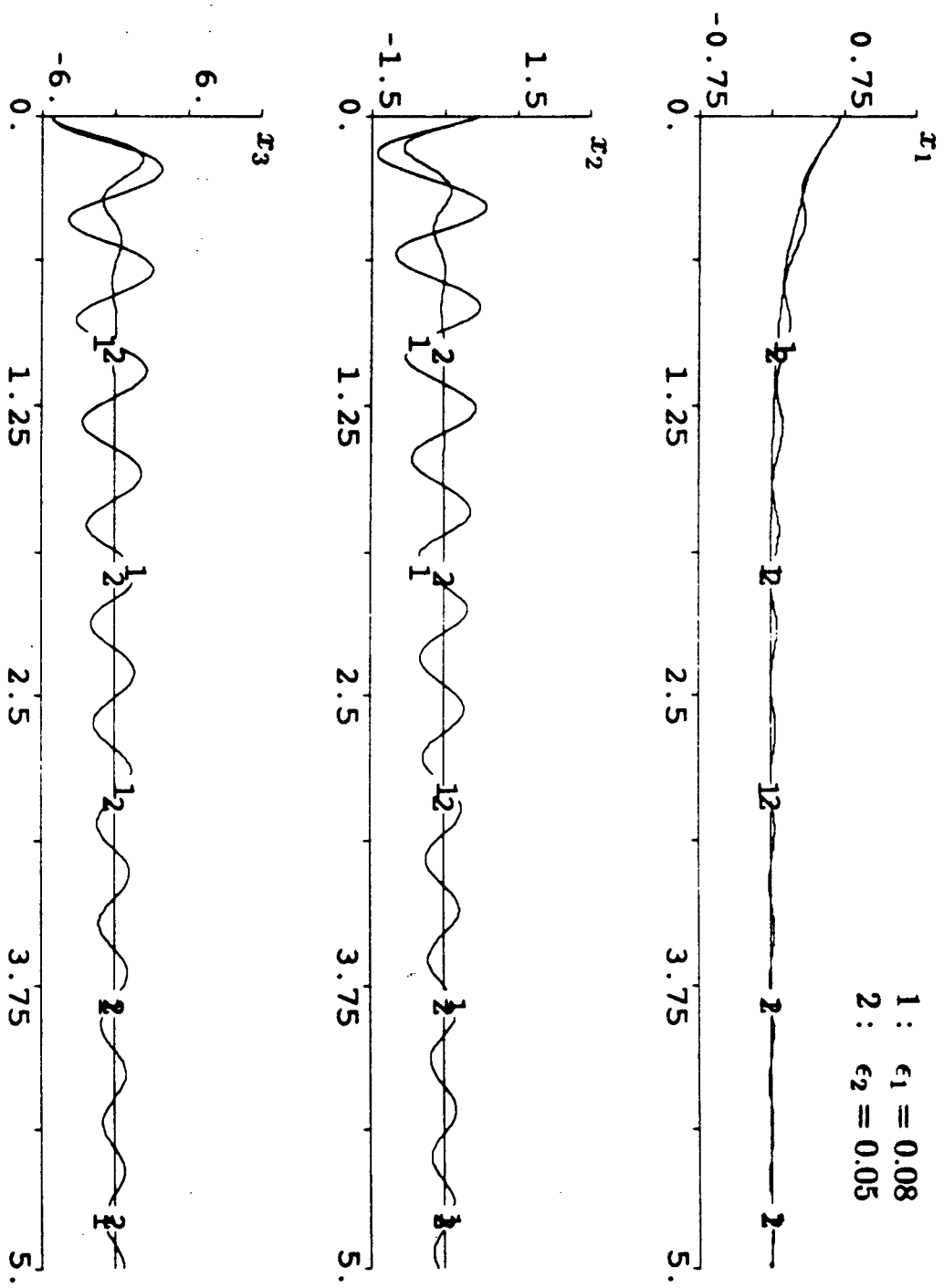


Figure 13.5: Closed loop behavior for Example 13.1, $x_0 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -5)$.

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