ABSTRACT

Title of Dissertation:	A COMBINATORIAL STUDY OF AFFINE DELIGNE-LUSZTIG VARIETIES
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We consider affine Deligne-Lusztig varieties $X_w(b)$ and certain unions $X(\mu, b)$ in the affine flag variety of a connected reductive group. They were first introduced by Rapoport to facilitate the study of mod-p reduction of Shimura varieties and moduli spaces of shtukas. We improve upon certain existing results in the study of affine Deligne-Lusztig varieties by weakening the hypothesis to prove them. Such results include a description of generic Newton points in Iwahori double cosets in the loop group of a split reductive group, covering relations in the associated Iwahori-Weyl group, and a dimension formula for $X(\mu, b)$ in the case of a quasi-split group. As an application of the work on generic Newton point formula, we obtain a description of the dimension for $X(\mu, b)$ associated with the maximal element b in its natural range, under a mild hypothesis on μ but no further restrictions on the group.

A COMBINATORIAL STUDY OF AFFINE DELIGNE-LUSZTIG VARIETIES

by

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Dedication

To the teachers who inspire, empower, and propel us to new mathematical horizons.

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Chapter 1: Introduction

In their seminal paper [DL76], Deligne and Lusztig gave a geometric recipe to construct all the irreducible representations for finite groups of Lie type in terms of l-adic cohomology of certain algebraic varieties associated with elements of the corresponding finite Weyl group. On the other hand, the affine Deligne-Lusztig varieties - their counterpart for the affine root system - were first introduced by Rapoport [Rap05] and have found substantial application in the geometry of Shimura varieties and moduli spaces of shtukas, and therefore have been a geometric object of recurring interest in the Langlands program. In [Lan77], Langlands outlines a three-part approach to prove that the Hasse-Weil ζ-functions of Shimura varieties are related to L-functions of automorphic forms; this expectation lies at the heart of Langlands program. A key input in this approach is the description of the geometric points of special fibers of suitable integral models of Shimura varieties. A conjectural description of such mod-p points was put forth by Langlands and Rapoport in the cases of good reduction, and subsequently modified by Rapoport and Kottwitz to include cases of parahoric-level bad reduction, and the geometry of associated affine Deligne-Lusztig varieties is a key player in proving such result. For instance, information about connected components of affine Deligne-Lusztig varieties - a problem that has generated a lot of attention in the past decade and has only been proved in complete generality in a very recent preprint [GL22] - has played an important role in the version of the Langlands-Rapoport conjecture proved in [Kis17] and its subsequent strengthening in [KSZ21].

Due to the combinatorial complexity of the affine Weyl groups, several important results in the study of affine Deligne-Lusztig varieties (in the affine flag variety) in recent years have been established only under the so-called *superregularity* hypothesis. In this thesis, we weaken the superregularity hypothesis on them and sometimes eliminate it, thus strengthening these existing results. Additionally, we compute the dimension of a certain naturally occurring union $X(\mu, b)$ of affine Deligne-Lusztig varieties beyond the setting of quasi-split groups. We now proceed to explain the main results of this thesis in more detail.

1.1 Affine Deligne-Lusztig varieties in the affine flag variety

We refer to Chapter 2 for an explanation of notations and definitions. Let $\mathbf{G}$ be a reductive group over a non-archimedean local field F, with residue field κ_F and completed maximal unramified extension $\breve{F}$. Denote by σ the Frobenius morphism of $\breve{F}/F$ and choose a σ-stable Iwahori subgroup I. Then letting $\breve{G} := \mathbf{G}(\breve{F}), \breve{I} := \mathbf{I}(\breve{F})$, we have two natural decompositions of the loop group $\breve{G}$: namely,

$$\breve{G} = \coprod_{[b] \in B(\mathbf{G})} [b] = \coprod_{w \in \widetilde{W}} \breve{I} w \breve{I}.$$

Here $[b] := \{g^{-1} b \sigma(g) : g \in \breve{G}\}$ is the σ-conjugacy class of b, the so-called *Kottwitz set* $B(\mathbf{G})$ is the collection of such classes and $\widetilde{W}$ is the Iwahori-Weyl group of G. For $[b] \in B(\mathbf{G})$ and $w \in \widetilde{W}$, the associated *affine Deligne-Lusztig variety* $X_w(b)$ is a locally closed, reduced subscheme locally of finite type inside the affine flag variety $\mathcal{F}l_G$, with geometric points given by

$$X_w(b)(\bar{\kappa}_F) := \{g\breve{I} : g^{-1}b\sigma(g) \in \breve{I}w\breve{I}\} \subset \breve{G}/\breve{I}.$$

We list some of the major problems in this field below:

1. For which w, b is $X_w(b)$ nonempty?

2. If non-empty, is $X_w(b)$ equidimensional and can we give a closed formula for its dimension?

Let us briefly mention why these questions are of interest from the perspective of arithmetic geometry. Roughly speaking (at least for the Shimura varieties of Hodge type that admits a moduli interpretation), on the special fiber of a suitable integral model of the Shimura variety associated to the Shimura datum $(\mathbf{G}, \mu)$ with Iwahori level structure, there are two important stratifications: the Newton stratification, coming from grouping together abelian varieties according to the isogeny class of their p-divisible groups, and the Kottwitz-Rapoport stratification, induced from the Iwahori orbits on the associated local model; for an axiomatic approach to these stratifications pertaining to general Shimura varieties, see [HR17]. The strata are indexed by a certain subset of $B(\mathbf{G})$ in the former case, while in the latter case the index set is a certain subset of $\widetilde{W}$ governed by the Shimura datum. The affine Deligne-Lusztig varieties capture the delicate interation between these two stratifications; for instance, $X_w(b) \neq \emptyset$ if and only if the Newton stratum indexed by the element b meets the Kottwitz-Rapoport stratum indexed by the element w, cf. [Hai05, §12.3]. However, even this nonemptiness problem has not fully been resolved in complete generality; we mention the work done in [Gör+10], [GHN15], and finally [He21b] for state-of-the-art result, as well as an interesting conjecture made in [Lim23] in this direction.

An important feature of $B(\mathbf{G})$ is the poset structure on it, defined via closure relation in $\breve{G}$. Recent results of [MV20] highlight certain special elements of $\widetilde{W}$: they show that if the maximal element $[b_w]$ of $B(\mathbf{G})_w := \{[b] \in B(\mathbf{G}) : X_w(b) \neq \emptyset\}$ - which coincides with the generic σ-conjugacy class in $\breve{I}w\breve{I}$ - satisfies an explicit group-theoretic condition called *cordiality*, then the poset $B(\mathbf{G})_w$ is saturated; furthermore, all the Newton strata meeting the fixed Kottwitz-Rapoport stratum indexed by w, as well as the affine Deligne-Lusztig varieties associated with w exhibits especially well-behaved geometry. In essence, their theorem gives a condition that can be checked from knowledge of this maximal element b_w of $B(\mathbf{G})_w$, but it provides important information about the shape of the entire poset. In light of such results, it becomes important to describe b_w explicitly in a way that can be used to check whether w is cordial. By virtue of the concrete parametrization [Kot85] σ-conjugacy classes via their Kottwitz and Newton points $(\kappa, \nu) : B(\mathbf{G}) \hookrightarrow \pi_1(\mathbf{G})_\Gamma \times (X_*(T)_{\Gamma_0} \otimes \mathbb{Q})^+$, this amounts to asking the following:

Question 1.1.1. Give an explicit closed formula for the generic Newton point $\nu([b_w])$.

It turns out that such a description comes from the technical framework of the *quantum Bruhat graph*: for elements of $\widetilde{W}$ that are sufficiently far away from the walls of any chamber, such a formula was first established in [Mil21]. This combinatorial tool was introduced by Brenti, Fomin and Postnikov in [BFP99] to describe the multiplicative structure of the quantum cohomology ring of the complex flag manifold. It is obtained by augmenting the usual Hasse diagram for the Bruhat order on the finite Weyl group W by some *quantum* *edges* - certain downward edges that are labeled by the coroot associated to the reflection used to get from one vertex to the other. With this setup, one can define a certain *weight function* $\mathrm{wt} : W \times W \to \mathbb{Z}_{\geq 0}\Phi^\vee$, cf. Section 2.5. Then we have the following result, see Theorem 3.2.1.

Theorem 1.1.2. *Suppose that* $\mathbf{G}$ *is a quasi-simple split group of semi-simple rank* n*. Let* $w = xt^\lambda y$ *be an element of* $\widetilde{W}$ *with* $x, y \in W$*, and depth*$(\lambda) > \Xi_n$*, where* Ξ_n *is certain explicit linear function of* n*. Then* $\nu([b_w]) = \lambda - wt(y^{-1}, x)$*.*

Here for a dominant cocharacter λ its $\mathrm{depth}(\lambda) := \min\{\langle\alpha, \lambda\rangle : \alpha \text{ simple root}\}$ quantifies its distance from the walls of the fundamental Weyl chamber. Hence this result weakens the hypothesis on the lower bound of the depth from $O(n^2)$ in Milićević's work to $O(n)$.

The proof of Theorem 1.1.2 crucially employs the *Demazure product* and its variations on the finite Weyl group W: this comes from the product in the associated 0-Hecke algebra and induces a monoid structure on W. Using these tools, we show that the generic Newton point $\nu(b_w)$ is given by the Newton point of the maximal translation element below w, and we identify this element via simple Bruhat order considerations.

1.2 Certain union of affine Deligne-Lusztig varieties

Rapoport in [Rap05] predicted "whereas the individual affine Deligne-Lusztig varieties are very difficult to understand, the situation seems to change radically when we form a suitable finite union of them." Here the union refers to

$$X(\mu, b) := \bigcup_{w \in \mathrm{Adm}(\mu)} X_w(b),$$

where for a dominant cocharacter μ, we define its *admissible set* as $\mathrm{Adm}(\mu) := \{w \in \widetilde{W} : w \leq t^{x(\underline{\mu})} \text{ for some } x \in W\}$.

The interest in studying such a union comes from the fact that (in mixed charactersitics) they arise as the underlying reduced scheme of a formal moduli space of p-divisible groups, known as Rapoport-Zink space, cf. [RV14]. Something analogous holds in the function field case for formal moduli spaces of Shtukas, cf. [Vie18]; in this latter case, μ can be arbitrary. Rapoport-Zink spaces are local analogs of Shimura varieties, and are also related to Shimura varieties themselves by the theory of p-adic uniformization. The problem of describing the l-adic cohomology of Rapoport-Zink spaces may be traced back to Lubin-Tate theory of formal groups and has had important consequences such as the proof of the local Langlands conjecture and the local-global compatibility of the Langlands correspondence in [HT01].

As a first validation of Rapoport's expectation, let us note that the non-emptiness pattern of $X(\mu, b)$ is completely understood (as opposed to that of $X_w(b)$): settling the Kottwitz-Rapoport conjecture in [He16a], He proves that $X(\mu, b)$ is non-empty precisely when $[b]$ lies in the set $B(\mathbf{G}, \mu)$ of *neutrally acceptable σ-conjugacy classes*; this subset of $B(\mathbf{G})$ is defined by a group theoretic reformulation of Mazur's inequality between the Hodge polygon of an F-crystal and the Newton polygon of its underlying F-isocrystal.

As a crucial ingredient toward obtaining a dimension formula for $X(\mu, b)$, a useful description of sufficiently large elements in the admissible set $\mathrm{Adm}(\mu)$ in terms of quantum Bruhat graph theoretic data is given in [HY21], whenever the depth of μ is $O(n^2)$ with respect to the semisimple rank n. Based on the work on affine Bruhat order (see next section), we can relax this hypothesis on $\mathrm{depth}(\mu)$ to $O(1)$; additionally, we exploit the additivity of the admissible set to upgrade it to a necessary numerical criterion that is unconditional on μ.

Proposition 1.2.1. *Suppose that W is an irreducible Weyl group associated with an affine Weyl group $\widetilde{W}$. Let μ, λ be dominant cocharacters.*

1. *Assume that depth(μ) is bigger than a certain constant Θ_W that is at most 6 in all Cartan types, and further that $\langle \rho, \mu - \lambda \rangle < \lceil \frac{depth(\mu) - \Theta_W}{2} \rceil$. Then for any two elements $x, y \in W$, we have $xt^\lambda y \in Adm(\mu)$ if and only if $wt(x, y^{-1}) \leq \mu - \lambda$.*
2. *Assume that μ is only dominant regular, i.e. depth$(\mu) \geq 1$. Then $\langle \rho, wt(x, y^{-1}) \rangle \leq \langle \rho, \mu - \lambda \rangle$ if $xt^\lambda y \in Adm(\mu)$.*

This follows from Proposition 3.4.13 and Proposition 4.1.1. Based on Proposition 1.2.1(2), we show that certain key steps in the proof in loc. sit. of the dimension formula can be carried out differently to bypass the $O(n^2)$ superregularity constraint. Thus we establish in Section 4.1 the same dimension formula under just a regularity assumption on μ:

Theorem 1.2.2. *Suppose that $\mathbf{G}$ is a quasi-split group. Let μ be dominant regular and assume that $[b] \in B(\mathbf{G}, \mu)$. Denote by $\mathcal{O}$ the σ-conjugacy class of the longest element w_0 in W, and let $\ell_R(\mathcal{O})$ be the minimal reflection length of elements in $\mathcal{O}$.*

1. *Let us further require that the Galois average* $\mu^\diamond \geq \nu([b]) + 2\rho^\vee$. *Then*

$$\dim X(\mu, b) = \langle \rho, \underline{\mu} - \nu(b) \rangle - \frac{1}{2} def_G(b) + \frac{1}{2}(\ell(w_0) - \ell_R(\mathcal{O})).$$

2. *Furthermore, if* $\mathbf{G}$ *is split, we obtain the same result under the hypothesis that* $\mu \geq \nu([b]) + wt(w_0, 1)$ *and* $depth(\mu) > 2$.

We refer the reader to Section 3.3, Section 2.2.1 and Section 2.3.1 for relevant definitions of the notions *reflection length* ℓ_R, *Galois average* $\mu^\diamond$ and *defect* def_G, respectively. Note that the quasi-split assumption on $\mathbf{G}$ is crucial in the above theorem; unlike in the case of a single affine Deligne-Lusztig variety, here we cannot leverage Proposition 4.2.6 to deduce a dimension formula of $X(\mu, b)$ for general $\mathbf{G}$. There is no direct relation between the admissible set $\mathrm{Adm}(\mu)$ - and hence $X(\mu, b)$ - for an arbitrary reductive group and its quasi-split inner form. This adds essential difficulties in the study of $X(\mu, b)$ for non-quasi-split groups.

Nevertheless, there has been some partial success in the context of general groups. Most notably, the investigation into the structure of $X(\mu, b)$ for the element $b = b_{\min} := \min B(\mathbf{G}, \mu)$ has led to the striking observation that in certain cases, the basic locus admits a nice description as a union of classical Deligne-Lusztig varieties, the index set and the closure relations between the strata being encoded in a Bruhat-Tits building attached to the group theoretic data coming from the Shimura variety; such descriptions have been used with great success towards applications in the realm of the Langlands program, for instance in the work toward Zhang's arithmetic fundamental lemma [RTZ13], as well as the theory of non-archimedean uniformization of Shimura varieties, first pioneered by Cerednik and Drinfeld, and further developed by Rapoport–Zink [RZ96] and Howard–Pappas [HP17]. However, the dimension problem and more qualitative structural information have remained elusive for general elements of $B(\mathbf{G}, \mu)$. Standing at this juncture, it is therefore natural to look at the other extreme and ask:

Question 1.2.3. What is the dimension of $X(\mu, b)$ for $b = b_{\max} := \max B(\mathbf{G}, \mu)$?

Note that if G is quasi-split, this maximal element is just the Galois average $\mu^\diamond$; hence, the dimension is trivially zero. However, for non-quasi-split groups the description of this maximal element is more subtle and indirect, see [HN18]. However, this is related to Question 1.1.1, as $b_{\max} = \max_{x \in W} b_{t^{x\mu}}$. Utilizing a recent result in [He21a] expressing the generic Newton point in terms of the Demazure power, we can explicitly describe this maximal element for non-quasi-split groups. Furthermore, combining quantum Bruhat graph theoretic considerations with explicit computation of certain minimal length elements in Frobenius-twisted conjugacy classes of W, we prove the following in Section 4.2.5:

Theorem 1.2.4. *Assume that the image μ_{ad} of μ in $X_*(T_{ad})_{\Gamma_0}$ has depth at least 2 in every $\bar{F}$-simple factor of $\mathbf{G}_{ad}$. Let $b = b_{\max}$ be the maximal element of $B(\mathbf{G}, \mu)$. Then*

$$\dim X(\mu, b) = rk_{ss}^F \mathbf{G}^* - rk_{ss}^F \mathbf{G},$$

where $\mathbf{G}^*$ is the unique (up to isomorphism) quasi-split inner form of $\mathbf{G}$, and rk_{ss}^F stands for semisimple F-rank.

Note that if $\mathbf{G} \simeq \mathbf{G}^*$ is quasi-split, this indeed recovers the trivial case mentioned above, and as such, this theorem also serves as a geometric way to measure how far $\mathbf{G}$ is from being quasi-split. A surprising feature of the result is that this dimension does not depend on μ.

1.3 Other related work on affine Weyl groups

As is already evidenced from our discussion so far, understanding fundamental properties of the affine Weyl groups is crucial to solving problems related to affine Deligne-Lusztig varieties. Indeed, the key ingredient in Milićević's original approach to solving Question 1.1.1 was to establish that under a lower bound of order $O(n^2)$ on the depth, paths in quantum Bruhat graph encode saturated chains in the Bruhat order on the affine Weyl group. Even though we can only strengthen her result toward Question 1.1.1 to $O(n)$ depth hypothesis, we improve this result on the affine Weyl group to one with an $O(1)$ depth hypothesis. The next result follows from Theorem 3.4.1.

Theorem 1.3.1. *Suppose that $w = xt^{\lambda}y$ is an element of $\widetilde{W}$ such that depth(λ) is bigger than a certain constant (at most 6 in all Cartan types). Then we have a precise description of the cocovers of w, i.e. those elements w' such that $w' < w$ and $\ell(w') = \ell(w) - 1$: such covering relations in affine Bruhat order are in two-to-one correspondence with edges in the quantum Bruhat graph:*

$$\{\textit{cocovers of } w = xt^{\lambda}y\} \longleftrightarrow \{\textit{edges coming into } x\} \coprod \{\textit{edges going out of } y^{-1}\}.$$

The content appearing in Chapter 3, Section 4.1 and Chapter 5 corresponds to the paper [Sad23], while the arxiv preprint [Sad22] makes up the rest of Chapter 4.

Chapter 2: Preliminaries

2.1 Notations

Let $\mathbf{G}$ be a connected reductive group over a non-archimedean local field F. Let $\breve{F}$ be the completion of the maximal unramified extension of F and σ be the Frobenius morphism of $\breve{F}/F$. The residue field of F is a finite field $\mathbb{F}_q$ and the residue field of $\breve{F}$ is the algebraically closed field $\bar{\mathbb{F}}_q$. We write $\breve{G}$ for $\mathbf{G}(\breve{F})$. We use the same symbol σ for the induced Frobenius morphism on $\breve{G}$. Let S be a maximal $\breve{F}$-split torus of $\mathbf{G}$ defined over F, which contains a maximal F-split torus. Let $\mathcal{A}$ be the apartment of $\mathbf{G}_{\breve{F}}$ corresponding to $S_{\breve{F}}$. We fix a σ-stable alcove $\mathfrak{a}$ in $\mathcal{A}$, and let $\breve{\mathcal{I}} \subset \breve{G}$ be the Iwahori subgroup corresponding to $\mathfrak{a}$. Then $\breve{\mathcal{I}}$ is σ-stable.

Let T be the centralizer of S in $\mathbf{G}$. Then T is a maximal torus. We denote by N the normalizer of T in $\mathbf{G}$. The *Iwahori–Weyl group* (associated to S) is defined as

$$\widetilde{W} = N(\breve{F})/T(\breve{F}) \cap \breve{\mathcal{I}}.$$

For any $w \in \widetilde{W}$, we choose a representative $\dot{w}$ in $N(\breve{F})$; however if there is no possibility of confusion we will call the lift w too. The action σ on $\breve{G}$ induces a natural action of σ on $\widetilde{W}$, which we still denote by σ. We will sometimes identify the element $w \in \widetilde{W}$ with $w\mathbf{a}$, the (extended) alcove that one obtains as image of the base alcove $\mathbf{a}$ under w. Similarly, we may say w lies in some given chamber to express that $w\mathbf{a}$ is an alcove belonging to that chamber.

We denote by ℓ the length function on $\widetilde{W}$ determined by the base alcove $\mathfrak{a}$ and denote by $\tilde{\mathbb{S}}$ the set of simple reflections in $\widetilde{W}$. Let W_{a} be the subgroup of $\widetilde{W}$ generated by $\tilde{\mathbb{S}}$. Then W_{aff} is an affine Weyl group. Let $\Omega \subset \widetilde{W}$ be the subgroup of length-zero elements (or equivalently, the stabilizer of $\mathbf{a}$) in $\widetilde{W}$. Then

$$\widetilde{W} = W_{\mathrm{a}} \rtimes \Omega.$$

Since the length function is compatible with the σ-action, the semi-direct product decomposition $\widetilde{W} = W_{\mathrm{a}} \rtimes \Omega$ is also stable under the action of σ. Note that W_{a} is the Coxeter group associated to $\tilde{\mathbb{S}}$, and hence it comes equipped with an associated Bruhat order, which we denote by $\leq$. This is extended to $\widetilde{W}$ as follows. For two elements $\widetilde{w}_1, \widetilde{w}_2 \in \widetilde{W}$, use the above decomposition to write $\widetilde{w}_i = w_i \zeta_i$ with $w_i \in W_a, \zeta_i \in \Omega$ for $i = 1, 2$. We then declare $\widetilde{w}_1 \leq \widetilde{w}_2$ if $w_1 \leq w_2$ and $\zeta_1 = \zeta_2$. The following properties of ℓ and $\leq$ are well-known, e.g. see [Mil21, Lemma 4.1], [Kna02, exercise 23, Chapter 2] and [BB05, exercise 21, Chapter 2] respectively.

- Let $w = ut^{\lambda}v \in \widetilde{W}$, where $\lambda \in X_*(T)$ is regular dominant and $u, v \in W$. Then

$$\ell(w) = \ell(u) + \ell(t^{\lambda}) - \ell(v) = \ell(u) + \langle 2\rho, \lambda \rangle - \ell(v). \tag{2.1.1}$$

- For any element x of W, let $\mathrm{Inv}(x) = \{\alpha \in \Phi^+ : x\alpha \in -\Phi^+\}$ be its inversion set, and denote its complement by $\mathrm{Inv}(x)^c$, i.e. $\mathrm{Inv}(x)^c = \Phi^+ \setminus \mathrm{Inv}(x)$. Then for any two elements $x, y \in W$, we have

$$\ell(xy) = \ell(x)+\ell(y)-2|\mathrm{Inv}(x)\cap\mathrm{Inv}(y^{-1})| = \ell(x)-\ell(y)+2|\mathrm{Inv}(x)^c\cap\mathrm{Inv}(y^{-1})|. \quad (2.1.2)$$

- Let $w_1, w_2, v \in \widetilde{W}$ be three elements such that $\ell(w_i v) = \ell(w_i) + \ell(v)$ for $i = 1, 2$. Then

$$w_1 \geq w_2 \text{ is equivalent to } w_1 v \geq w_2 v. \quad (2.1.3)$$

Let $W = N(\breve{F})/T(\breve{F})$ be the relative Weyl group. We denote by $\mathbb{S}$ the subset of $\widetilde{\mathbb{S}}$ consisting of simple reflections generating W. We let Φ (resp. Δ) denote the set of roots (resp. simple roots) for W. We write Γ for $\mathrm{Gal}(\bar{F}/F)$, and write Γ_0 for the inertia subgroup of Γ. Then fixing a special vertex of the base alcove $\mathfrak{a}$, we have the splitting

$$\widetilde{W} = X_*(T)_{\Gamma_0} \rtimes W = \{t^\lambda w; \lambda \in X_*(T)_{\Gamma_0}, w \in W\}.$$

When considering an element $\lambda \in X_*(T)_{\Gamma_0}$ as an element of $\widetilde{W}$, we write t^λ. Note that if $\mathbf{G}$ is not quasi-split over F, then there does not exist a σ-stable special vertex in $\mathfrak{a}$ and thus the splitting $\widetilde{W} = X_*(T)_{\Gamma_0} \rtimes W$ is not σ-stable.

For an irreducible Weyl group W of rank n, we follow the labeling of roots as in [Bou02] and we usually write s_i instead of s_{α_i}, where $\Delta = \{\alpha_i : 1 \leq i \leq n\}$. Let w_0 be the longest element in W, and for $i \in [1, n]$ we let $w_{i,0}$ be the longest element of the parabolic subgroup of W corresponding to $\Delta \setminus \{\alpha_i\}$. Let ρ be the dominant weight with

$\langle \alpha^\vee, \rho \rangle = 1$ for any $\alpha \in \Delta$. Let $\{\varpi_i^\vee : 1 \le i \le n\}$ be the set of fundamental coweights. If $\varpi_i^\vee$ is minuscule, we denote the image of $t^{\varpi_i^\vee}$ under the projection $\widetilde{W} \to \Omega$ by τ_i; then conjugation by τ_i is a length preserving automorphism of $\widetilde{W}$, which we denote by $\mathrm{Ad}(\tau_i)$.

2.2 The σ-conjugacy classes of $\breve{G}$

We say that two elements $b, b' \in \breve{G}$ are σ-conjugate if there is some $g \in \breve{G}$ such that $b' = gb\sigma(g)^{-1}$. Let $B(\mathbf{G})$ be the set of σ-conjugacy classes on $\breve{G}$. By the work of Kottwitz in [Kot85] and [Kot97], any σ-conjugacy class $[b]$ is determined by two invariants:

- The Kottwitz point $\kappa([b]) \in \pi_1(\mathbf{G})_\Gamma$, where $\pi_1(\mathbf{G}) := X_*(T)/\mathbb{Z}\Phi^\vee$ is the Borovoi fundamental group and $\pi_1(\mathbf{G})_\Gamma$ is the set of Γ-coinvariants in $\pi_1(\mathbf{G})$;
- The Newton point $\nu([b]) \in ((X_*(T)_{\Gamma_0,\mathbb{Q}})^+)^{\langle\sigma\rangle}$, where $X_*(T)_{\Gamma_0,\mathbb{Q}} := X_*(T)_{\Gamma_0} \otimes \mathbb{Q} = X_*(T)^{\Gamma_0} \otimes \mathbb{Q}$, and $((X_*(T)_{\Gamma_0,\mathbb{Q}})^+)^{\langle\sigma\rangle}$ is defined to be the set of $\langle\sigma\rangle$-invariants of the intersection of $X_*(T)_{\Gamma_0,\mathbb{Q}}$ with the set $X_*(T)^+_{\mathbb{Q}}$ of dominant elements in $X_*(T)_{\mathbb{Q}}$.

We denote by $\leqslant$ the dominance order on $X_*(T)^+_{\mathbb{Q}}$, i.e., for $\nu, \nu' \in X_*(T)^+_{\mathbb{Q}}$, we have $\nu \leqslant \nu'$ if and only if $\nu' - \nu$ is a non-negative (rational) linear combination of positive coroots over $\breve{F}$. The dominance order on $X_*(T)^+_{\mathbb{Q}}$ extends to a partial order on $B(\mathbf{G})$. Namely, for $[b], [b'] \in B(\mathbf{G})$, we say that $[b] \leqslant [b']$ if and only if $\kappa([b]) = \kappa([b'])$ and $\nu([b]) \leqslant \nu([b'])$.

Denote by $\mathbf{J}_b$ the σ-centralizer group of b; this is a reductive group over F with F-rational points given by

$$\mathbf{J}_b(F) = \{g \in \breve{G} \mid g^{-1} b \sigma(g) = b\}.$$

For any reductive group $\mathbf{H}$ over F, we denote by $\mathrm{rk}_F^{\mathrm{ss}}\mathbf{H}$ the semisimple F-rank of $\mathbf{H}$. The following result is implicit in [Kot06, §1.9].

Proposition 2.2.1. *Let $\mathbf{G}$ be quasi-simple over F and assume $\tau \in \Omega$. Then*

$$rk_F^{ss}\mathbf{J}_{\dot{\tau}} = \mid Ad\ \tau \circ \sigma \text{ orbits on } \widetilde{\mathbb{S}} \mid -1.$$

2.2.1 The straight σ-conjugacy classes

Note that the action of σ on $\breve{G}$ gives rise to an action on $\widetilde{W}$, still denoted by σ. The set of σ-conjugacy classes in $\widetilde{W}$ is denoted by $B(\widetilde{W}, \sigma)$.

Let $w \in \widetilde{W}, n \in \mathbb{N}$. The *$n$-th σ-twisted power of* w is defined by

$$w^{\sigma,n} = w\sigma(w)\sigma^2(w)\cdots\sigma^{n-1}(w).$$

Note that this is nothing but the image of the n-th power of $w\sigma \in \widetilde{W} \rtimes \langle\sigma\rangle$ under the quotient map $\widetilde{W} \rtimes \langle\sigma\rangle \to \widetilde{W}$, since $(w\sigma)^n = w^{\sigma,n}\sigma^n$. Then by definition, w is called a *σ-straight element* if $\ell(w^{\sigma,n}) = n\ell(w)$, for all $n \in \mathbb{N}$. A σ-conjugacy class of $\widetilde{W}$ is called straight if it contains a σ-straight element, and we denote the collection of such straight σ-conjugacy classes by $B(\widetilde{W}, \sigma)_{\mathrm{str}}$. We have a map $\Psi : B(\widetilde{W}, \sigma) \to B(\mathbf{G})$, coming from the assignment $w \to \dot{w}$. We can also define Kottwitz and Newton maps, denoted by the same symbols κ, ν resp. from $B(\widetilde{W}, \sigma)$ with the same targets as before, cf. [He14, §1.7], [Gör+10, §7.2]. The importance of the straight σ-conjugacy classes is illustrated in the following result.

Theorem 2.2.2. *[He14, Theorem 3.7] The restriction of* Ψ *induces a bijective map* $\Psi : B(\widetilde{W})_{str} \to B(\mathbf{G})$*. Moreover, We have the following commutative diagram*

$$\begin{array}{ccc} B(\widetilde{W})_{str} & \xrightarrow{\quad\Psi\quad} & B(\mathbf{G}) \\ & {\scriptstyle (\kappa,\nu)}\searrow \quad \swarrow{\scriptstyle (\kappa,\nu)} & \\ & \pi_1(\mathbf{G})_\Gamma \times ((X_*(T)_{\Gamma_0,\mathbb{Q}})^+)^{\langle\sigma\rangle} & \end{array}$$

Given $w \in \widetilde{W}$, define

$$B(\mathbf{G})_w := \{[b] \in B(\mathbf{G}) : [b] \cap \breve{I} w \breve{I} \neq \emptyset\}.$$

It is easy to see that $B(\mathbf{G})_w$ has an unique maximal element $[b_w]$, which coincides with the generic σ-conjugacy class in $\breve{I} w \breve{I}$. We will write ν_w to denote $\nu([b_w])$.

Let $\{\mu\}$ be a conjugacy class of cocharacters over $\bar{F}$. Choose μ be a dominant representative of $\{\mu\}$ and denote by $\underline{\mu}$ its image in $X_*(T)_{\Gamma_0}$.

We also have the set of *neutrally acceptable elements*, cf. [KR03]

$$B(\mathbf{G}, \mu) = \{[b] \in B(\mathbf{G}) \mid \kappa([b]) = \mu^\natural, \nu([b]) \leq \mu^\diamond\}.$$

Here $\mu^\natural$ denotes the common image of $\mu \in \{\mu\}$ in $\pi_1(\mathbf{G})_\Gamma$, and $\mu^\diamond$ denotes the average of the σ_0-orbit of $\underline{\mu}$. Here σ_0 denotes the L-action of σ, see [GHN20, definition 2.1] for details. The set $B(\mathbf{G}, \mu)$ inherits a partial order from $B(\mathbf{G})$. Since the Kottwitz map κ is constant on $B(\mathbf{G}, \mu)$, we may view it as a subset of $X_*(T)_{\Gamma_0,\mathbb{Q}}$ via the Newton map. The following description of $B(\mathbf{G}, \mu)$ is obtained in [HN18, Theorem 1.1 & Lemma 2.5]. To state the result, for each σ_0 orbit $\mathcal{O}$ on $\mathbb{S}$, we set $\varpi_{\mathcal{O}} = \sum_{i \in \mathcal{O}} \varpi_i$.

Theorem 2.2.3. *[HN18] (1) Let $v \in X_*(T)_{\Gamma_0,\mathbb{Q}}$. Then $v \in B(\mathbf{G}, \mu)$ if and only if $\sigma_0(v) = v$ is dominant, and for any σ_0-orbit $\mathcal{O}$ on $\mathbb{S}$ with $\langle v, \alpha_i \rangle \neq 0$ for each (or equivalently, some) $i \in \mathcal{O}$, we have that $\langle \mu + \sigma(0) - v, \varpi_{\mathcal{O}} \rangle \in \mathbb{Z}$ and $\langle \mu - v, \varpi_{\mathcal{O}} \rangle \geq 0$.*

(2) The set $B(\mathbf{G}, \mu)$ contains a unique maximal element.

2.3 Affine Deligne-Lusztig varieties

For $b \in \breve{G}$ and $w \in \widetilde{W}$, the associated *affine Deligne-Lusztig variety* $X_w(b)$ is a locally closed, reduced subscheme locally of finite type inside the affine flag variety $\mathcal{F}l_G$, with geometric points given by

$$X_w(b)(\bar{\kappa}_F) := \{g\breve{I} : g^{-1} b \sigma(g) \in \breve{I} w \breve{I}\} \subset \mathcal{F}l_G(\bar{\kappa}_F) = \breve{G}/\breve{I}.$$

We will also discuss certain finite unions of affine Deligne-Lusztig varieties. As before, we denote by $\underline{\mu}$ the image of a dominant cocharacter μ in $X_*(T)_{\Gamma_0}$ Following [Rap05], we then define the associated *admissible set* as

$$\mathrm{Adm}(\mu) = \{w \in \widetilde{W} : w \leq t^{x(\underline{\mu})} \text{ for some } x \in W\}.$$

We will need the following additivity property of admissible sets.

Theorem 2.3.1. *[He16a, Theorem 5.1][HH17, Theorem 1.4] Let $\mu, \mu' \in X_*(T)$ be dominant. Then we have*

$$Adm(\mu) \cdot Adm(\mu') = Adm(\mu + \mu').$$

For any $b \in \breve{G}$, we set

$$X(\mu, b) = \bigcup_{w \in \mathrm{Adm}(\mu)} X_w(b).$$

We remark that $X_w(b)$ and $X(\mu, b)$ are subschemes of the affine flag variety in the usual sense in equal characteristic, and in the sense of Zhu [Zhu17], Bhatt and Scholze [BS17] in mixed characteristic. Settling the Kottwitz-Rapoport conjecture made in [KR03] and [Rap05] about the non-emptiness pattern for $X(\mu, b)$, He proves the following result in [He16a].

Theorem 2.3.2. *[He16a, Theorem A] $X(\mu, b) \neq \emptyset$ if and only if $[b] \in B(\mathbf{G}, \mu)$.*

2.3.1 Virtual dimension of affine Deligne-Lusztig variety

Suppose that $\mathbf{G}$ is quasi-split; then σ preserves the set of finite simple roots and hence acts on W. Note that $w \in \widetilde{W}$ can be written in a unique way as $w = ut^{\lambda}v$ with λ dominant, $u, v \in W$ such that $t^{\lambda}v$ is a minimum length element in the right W-coset in $\widetilde{W}$ determined by w. In this case, we set $\eta_{\sigma}(w) = \sigma^{-1}(v)u$.

Let $b \in \breve{G}$. The *defect* of b is defined by $\mathrm{def}_{\mathbf{G}}(\mathrm{b}) = \mathrm{rank}_{\mathrm{F}}\, \mathbf{G} - \mathrm{rank}_{\mathrm{F}}\, \mathbf{J}_{\mathrm{b}}$.

Following [He14, §10.1], we then define the *virtual dimension* for $X_w(b)$ to be

$$d_w(b) = \frac{1}{2}\big(\ell(w) + \ell(\eta(w)) - \mathrm{def}_{\mathbf{G}}(\mathrm{b})\big) - \langle \rho, \nu([\mathrm{b}])\rangle.$$

The justification of defining such an expression lies in a result proved by He in [He14, Corollary 10.4] that says $\dim X_w(b) \leq d_w(b)$ whenever $\kappa(w) = \kappa([b])$. Recent work by Milićević and Viehmann in [MV20] singles out those $w \in \widetilde{W}$ for which $\dim X_w(b_w) = d_w(b_w)$

holds; these are called *cordial elements*. It is shown in loc. sit. that $B(\mathbf{G})_w$ exhibits remarkable properties for such w, see [MV20, Theorem 1.1] for a discussion of this.

2.4 Demazure product and its variations

We now discuss three operations $*, \rhd, \lhd : \widetilde{W} \times \widetilde{W} \to \widetilde{W}$. Here $*$ is the *Demazure product* and $\lhd, \rhd$ are the left and right *downward Demazure products*, respectively. We describe these operations in form of the following lemma.

Lemma 2.4.1. *[HL15, Section 2.1] Let $x, y \in \widetilde{W}$.*

1. *The subset $\{uv : u \leq x, v \leq y\}$ contains a unique maximal element, which we denote by $x * y$. Moreover, $x * y = u'y = xv'$ for some $u' \leq x$ and $v' \leq y$ and $\ell(x * y) = \ell(u') + \ell(y) = \ell(x) + \ell(v')$.*

2. *The subset $\{uy : u \leq x\}$ contains a unique minimal element, which we denote by $x \rhd y$. Moreover, $x \rhd y = u''y$ for some $u'' \leq x$ with $\ell(x \rhd y) = \ell(y) - \ell(u'')$.*

3. *The subset $\{xv : v \leq y\}$ contains a unique minimal element, which we denote by $x \lhd y$. Moreover, $x \lhd y = xv''$ for some $v'' \leq y$ with $\ell(x \lhd y) = \ell(x) - \ell(v'')$.*

The Demazure product is related to product of closure of two Iwahori double cosets in $\breve{G}$. More precisely, for any $w \in \widetilde{W}, \overline{\breve{I}w\breve{I}} = \bigcup_{w' \leq w} \breve{I}w'\breve{I}$ is a closed admissible subset of G in the sense of [He16a]. Then for any $x, y \in \widetilde{W}$, we have

$$\overline{\breve{I}x\breve{I}} \cdot \overline{\breve{I}y\breve{I}} = \bigcup_{u \leq x} \breve{I}u\breve{I} \cdot \bigcup_{v \leq y} \breve{I}v\breve{I} = \bigcup_{w \leq x * y} \breve{I}w\breve{I} = \overline{\breve{I}(x * y)\breve{I}}.$$

The above equation also implies that $*$ is an associative binary operation on $\widetilde{W}$. We will need the following results about these operations.

Lemma 2.4.2. *[He09, Lemma 2 and Lemma 3]*

1. *If $w \geq w'$, $v \leq v'$ are elements of $\widetilde{W}$, then $w \rhd v \leq w' \rhd v'$.*

2. *For any three elements $x, y, z \in \widetilde{W}$, we have $x \rhd (y \rhd z) = (x * y) \rhd z$. In other words, the action $(\widetilde{W}, *) \times \widetilde{W} \to \widetilde{W}, (x, y) \to x \rhd y$ is a left action of the monoid $(\widetilde{W}, *)$.*

2.5 Quantum Bruhat graph

We recall the quantum Bruhat graph introduced by Brenti, Fomin and Postnikov in [BFP99]. By definition, a *quantum Bruhat graph* Γ_Φ is a directed graph with

- vertices given by the elements of W;
- upward edges $w \rightharpoonup ws_\alpha$ for some $\alpha \in \Phi^+$ with $\ell(ws_\alpha) = \ell(w) + 1$;
- downward edges $w \rightharpoondown ws_\alpha$ for some $\alpha \in \Phi^+$ with $\ell(ws_\alpha) = \ell(w) - \langle 2\rho, \alpha^\vee \rangle + 1$.

Note that by [BFP99, Lemma 4.3], $\ell(s_\alpha) \leq \langle 2\rho, \alpha^\vee \rangle - 1$ for any $\alpha \in \Phi^+$, so we have $\ell(ws_\alpha) \geq \ell(w) - \ell(s_\alpha) \geq \ell(w) - \langle 2\rho, \alpha^\vee \rangle + 1$. Therefore the condition for downward edges can be rephrased to saying that

$$\ell(ws_\alpha) = \ell(w) - \ell(s_\alpha) \text{ with } \ell(s_\alpha) = \langle 2\rho, \alpha^\vee \rangle - 1.$$

Following [Len+15], we call $\alpha \in \Phi^+$ to be a *quantum root* if $\ell(s_\alpha) = \langle 2\rho, \alpha^\vee \rangle - 1$. We have the following description of quantum roots.

Lemma 2.5.1. *[Len+15, Lemma 4.2] We have that $\alpha \in \Phi^+$ is a quantum root if and only if*

1. *α is a long root, or*
2. *α is a short root, and if $\alpha = \sum_{\alpha_i \in \Delta} c_i \alpha_i$ then we have $c_i = 0$ for any long simple root α_i.*

Here for simply laced root systems we consider all roots to be long. Thus in a simply laced type, all roots are quantum. Examples of quantum root, in general, include the simple roots as well as the highest root.

We now recall some graph theoretic notions related to Γ_Φ; we refer to [Mil21, section 3.1] for more details. The weight of an upward edge is defined to be 0 and the weight of a downward edge $w \rightharpoondown ws_\alpha$ is defined to be $\alpha^\vee$. For two elements w, w', a directed path between them is defined to be concatenation of directed edges joining a sequence of vertices in Γ_Φ. The *weight of a path* in Γ_Φ is defined to be the sum of weights of the edges in the path. The length of path is defined to be the number of edges present in it. For any $x, y \in W$, we denote by $d_\Gamma(x, y)$ the minimal length among all paths in Γ_Φ from x to y. Any path between x and y affording $d_\Gamma(x, y)$ as its length is called a *shortest path* between them.

Lemma 2.5.2. *[Pos05, Lemma 1], [BFP99, Lemma 6.7] Let $x, y \in W$. Then*

1. *There exists a directed path (consisting of possibly both upward and downward edges) in Γ_Φ from x to y.*

2. *Any two shortest paths in* Γ_Φ *from* x *to* y *have the same weight, which we denote by* $wt(x, y)$.

3. *Any path in* Γ_Φ *from* x *to* y *has weight* $\geqslant wt(x, y)$.

Chapter 3: Generic Newton point and affine Bruhat order

For a split connected reductive group, an explicit description of ν_w was first derived by Milićević in [Mil21, Theorem 3.2] for elements $w \in \widetilde{W}$ with *superregular* dominant translation part. More precisely, a uniform bound is given in [Mil21, Corollary 3.3] in the quasi-simple case, which says that the description of ν_w in loc. sit. is valid whenever the following depth hypothesis is satisfied by λ:

$$depth(\lambda) \geq \begin{cases} 8\ell(w_0), & \text{if } \mathbf{G} \text{ is of classical type;} \\ 16\ell(w_0), & \text{if } \mathbf{G} \text{ is of exceptional type.} \end{cases}$$

To establish this result, Milićević first derives in [Mil21, Proposition 4.2] a characterization of the covering relations in $\widetilde{W}$, again under certain superregularity hypothesis of the following nature:

$$depth(\lambda) \geq \begin{cases} 2\ell(w_0) + 2, & \text{if } \mathbf{G} \text{ is not of type } G_2; \\ 3\ell(w_0) + 3, & \text{if } \mathbf{G} \text{ is of type } G_2. \end{cases}$$

This characterization is of independent interest, and more recently it has been utilized by He and Yu in [HY21] to deduce a partial description of the admissible sets in $\widetilde{W}$. In this chapter, we improve upon these results by weakening the superregularity hypothesis to prove them.

3.1 Some combinatorial properties

We start by proving a monotonicity property for the weight function. Let us define the function $\mathrm{wt} : W \to X_*(T)$ by $\mathrm{wt}(x) := \mathrm{wt}(x, 1)$.

Lemma 3.1.1. *Let $x_1, x_2 \in W$; if $x_1 \leq x_2$ in Bruhat order, then $wt(x_1) \leq wt(x_2)$ in dominance order.*

Proof. It suffices to show this when x_2 is a cover of x_1, i.e. $x_2 = x_1 s_\alpha$ for some $\alpha \in \Phi^+$ such that $\ell(x_2) = \ell(x_1) + 1$. We construct a path from x_1 to 1 by concatenating the upward edge $x_1 \rightharpoonup x_2$ with a path of shortest length from x_2 to 1. Since the first upward edge has weight 0, the weight of this particular path is $\mathrm{wt}(x_2)$. Appealing to Lemma 2.5.2(3), we then get $\mathrm{wt}(x_2) \geq \mathrm{wt}(x_1)$. □

We will now reduce our calculation of the maximal Newton point of an arbitrary element of $\widetilde{W}$ to that of a suitable element lying in the dominant chamber.

Proposition 3.1.2. *If λ is dominant regular, then $[b_{ut^\lambda v}] = [b_{t^\lambda (v \lhd \sigma(u))}]$.*

Proof. Note that by Equation (2.1.1), $\ell(ut^\lambda v) = \ell(u) + \ell(t^\lambda v)$, hence $ut^\lambda v = u * (t^\lambda v)$. Thus $\overline{\breve{I} u t^\lambda v \breve{I}} = \overline{\breve{I} u \breve{I}} \cdot \overline{\breve{I} t^\lambda v \breve{I}}$.

Now, recall that $\breve{I}$ is σ-stable. Therefore, if $w = w_1 w_2$ is an element with $w_1 \in \overline{\breve{I} u \breve{I}}, w_2 \in \overline{\breve{I} t^\lambda v \breve{I}}$, then $w_1^{-1} \cdot_\sigma w = w_2 \sigma(w_1)$. This shows that $\breve{G} \cdot_\sigma (\overline{\breve{I} u \breve{I}} \cdot \overline{\breve{I} t^\lambda v \breve{I}}) \subset \breve{G} \cdot_\sigma (\overline{\breve{I} t^\lambda v \breve{I}} \cdot$

$\overline{\breve{I}\sigma(u)\breve{I}}$). One obtains the other inclusion similarly. This shows that $\breve{G}\cdot_\sigma(\overline{\breve{I}u\breve{I}}\cdot\overline{\breve{I}t^\lambda v\breve{I}}) = \breve{G}\cdot_\sigma(\overline{\breve{I}t^\lambda v\breve{I}}\cdot\overline{\breve{I}u\breve{I}})$.

Hence, we conclude that $\breve{G}\cdot_\sigma\overline{\breve{I}ut^\lambda v\breve{I}} = \breve{G}\cdot_\sigma\overline{\breve{I}(t^\lambda v)*\sigma(u)\breve{I}}$.

Since $G\cdot_\sigma\overline{\breve{I}w\breve{I}}$ is the set of σ-conjugacy classes intersecting $\overline{\breve{I}w\breve{I}}$, by appealing to [Vie14, Corollary 5.6] we get that

$$[b_{ut^\lambda v}] = [b_{(t^\lambda v)*\sigma(u)}]. \tag{3.1.1}$$

Thus it suffices to show that

(a) if λ is dominant regular, then $(t^\lambda v)*u' = t^\lambda(v\lhd u')$ for elements $u',v\in W$.

By Lemma 2.4.2(2), we only need to argue the case when u' is a simple reflection $s\in\mathbb{S}$. Then this statement is equivalent to the assertion that $\ell(vs)=\ell(v)-1$ if and only if $\ell(t^\lambda vs)=\ell(t^\lambda v)+1$. This follows directly from a computation using Equation (2.1.1). Hence we are done. □

Corollary 3.1.3. *Let $x,y\in W$. Then $wt(x,y) = wt(x^{-1}\lhd y)$.*

Proof. Suppose that $\lambda\in X_*(T)^+$ is superregular. Applying the formula for maximal Newton point established in [Mil21, Theorem 3.2] to $w:=xt^\lambda y$ and $w':=t^\lambda(y\lhd x)$, we get that $\nu_w=\lambda-\mathrm{wt}(y^{-1},x)$ and $\nu_{w'}=\lambda-\mathrm{wt}((y\lhd x)^{-1})=\lambda-\mathrm{wt}(y\lhd x)$. Since $\nu_w=\nu_{w'}$ by Proposition 3.1.2, we get $\mathrm{wt}(y^{-1},x)=\mathrm{wt}(y\lhd x)$. Now replacing the pair (x,y) by (y,x^{-1}), we get the conclusion. □

We now close this section by giving another interpretation of the weight of a shortest path in the quantum Bruhat graph.

Definition 3.1.4. For an element $x \in W$, we say that $x = s_{\beta_1} \cdots s_{\beta_k}$ is a *reduced quantum reflection decomposition* if

1. $\{\beta_i : 1 \leq i \leq k\}$ is a collection of (not necessarily simple) quantum roots, i.e. $l(s_{\beta_i}) = \langle 2\rho, \beta_i^\vee \rangle - 1$.

2. $\ell(x) = \sum_{i=1}^k \ell(s_{\beta_i})$.

3. Subject to the first two conditions, k is minimal.

In this case, we say that k is the minimal reduced length of x in terms of reflections associated to quantum roots, and write $\ell_\downarrow(x) = k$.

The choice of notation is suggested from the fact that

$$\{\text{Reduced quantum reflection decompositions for } x\}$$

$$\updownarrow$$

$$\{\text{Shortest paths in } \Gamma_\Phi \text{ from } x \text{ to } 1 \text{ that uses } \textit{only downward edges}\}$$

By [MV20, Proposition 4.11], the weight of any path in the latter set is equal to $\mathrm{wt}(x)$ and $d_\Gamma(x, 1) = \ell_\downarrow(x)$. Therefore, the problem of computing $\mathrm{wt}(x)$ reduces to one of finding a suitable decomposition for x: if $x = s_{\beta_1} \cdots s_{\beta_k}$ is such a decomposition then $\mathrm{wt}(x) = \sum_{i=1}^k \beta_i^\vee$. In view of Corollary 3.1.3, we can thus reformulate $\mathrm{wt}(x, y)$ in terms of a decomposition of $x^{-1} \lhd y$.

3.2 Formula for the generic Newton point

The goal of this section is to prove the following result.

Theorem 3.2.1. *Assume that* $\mathbf{G}$ *is a quasi-simple split group of rank* n*. Let* $w = t^{\lambda}x$ *be an element of its Iwahori-Weyl group such that* λ *is dominant and depth*$(\lambda) > \Xi$*, where* Ξ *is an integer depending on* n*. Then the maximal Newton point associated to* w *is given by* $\nu_w = \lambda - wt(x)$*.*

For each Cartan type, the lower bound Ξ for the depth mentioned above is given in the following table.

Type	A_n	B_n/C_n	D_n	E_6	E_7	E_8	F_4	G_2
Ξ	$3n+1$	$6n-2$	$6n-6$	23	33	57	23	9

Remark 3.2.2. The statement of Theorem 3.2.1 seems weaker than the statement of Theorem 1.1.2 in the sense that the former gives a formula only for certain elements of $\widetilde{W}$ (associated to a quasi-simple split group $\mathbf{G}$) in the dominant chamber for whereas the latter promises to handle elements in arbitrary chambers (for arbitrary split groups). Hence let us first explain how Theorem 3.2.1 implies Theorem 1.1.2 - for quasi-simple groups; in the next remark, we further remove this condition on the group. By Proposition 3.1.2, we know that

$$\nu_{ut^{\lambda}v} = \nu_{t^{\lambda}(v \lhd \sigma(u))}.$$

Under the hypothesis on λ, this element $t^{\lambda}(v \lhd \sigma(u))$ lies in the dominant chamber. Then by Theorem 3.2.1 it follows that

$$\nu_{t^{\lambda}(v \lhd \sigma(u))} = \lambda - \mathrm{wt}(v \lhd \sigma(u)).$$

By Corollary 3.1.3 we finally deduce that $\nu_{ut^{\lambda}v} = \lambda - \mathrm{wt}(v^{-1}, \sigma(u))$. This concludes our discussion.

Remark 3.2.3. One can handle the case of a connected reductive split group $\mathbf{G}$ via a routine reduction procedure, which we discuss now. In that case, we have

$$W = W_1 \times \cdots \times W_l,$$

where W_i are irreducible Weyl groups associated to F-simple factors of $\mathbf{G}_{\mathrm{ad}}$. This gives rise to a partition of m, where m is the semisimple rank of $\mathbf{G}$. Any element x of W is of the form $(x_1, \cdots, x_l)$. Similarly, we write λ as $(\lambda_1, \cdots, \lambda_l)$. We have $w_0 = (w_{0,1}, \cdots, w_{0,l})$, where $w_{0,i}$ is the longest element of W_i. In the same way, we write the sum of positive roots as $(2\rho_1^\vee, \cdots, 2\rho_l^\vee)$ and the highest root as $(\theta_1, \cdots, \theta_l)$.

We observe that while dealing with the quasi-simple case, we introduce a restriction on depth in two stages - as found in Section 3.2.1 and Section 3.2.4. By the last paragraph we therefore need to require for each i

$$\mathrm{depth}(\lambda_i) \geq \Xi_i,$$

where Ξ_i is the lower bound given for the Weyl group W_i, to be read off from the table above. Note that Ξ_i is then some linear expression of m, depending on the type of i-th factor and the partition of m. We remark that under such hypotheses, Theorem 3.2.1 applies to each quasi-simple factor of $\mathbf{G}_{\mathrm{ad}}$ and thus we need to justify the formula for such groups.

Remark 3.2.4. Let us note that Theorem 3.2.1 has since been strengthened due to the work of multiple authors, by weakening the bound on depth of the relevant coweight. In [HN21], He and Nie prove the formula for generic Newton point under the assumption that the relevant cocharacter is of depth at least 2. Finally, Schremmer completely solves the problem for arbitrary cocharacter in [Sch22b] by removing the hypothesis on depth, albeit the description gets more complicated in this situation.

We now give a technical layout of the proof. As mentioned before, under the regularity assumption on its associated dominant coweight as in Proposition 3.1.2, we only need to handle elements $w \in \widetilde{W}$ that lie in the dominant chamber. In Proposition 3.2.6, we show that w is bigger than a certain translation element, thus giving us a lower bound for ν_w. This holds under a certain depth hypothesis proposed in Section 3.2.1, which relies on explicit calculation in Section 3.3 of $\mathrm{wt}(w_0)$ for the longest element w_0 in each Cartan type. Using this lower bound, we also show in Lemma 3.2.7 that ν_w is given by Newton point of some translation t^γ smaller than w. Note that if this translation can be shown to be lying in the dominant chamber (i.e. γ is dominant), we are practically done - as one can then multiply the inequality $t^\lambda y \geq t^\gamma$ by a large enough dominant translation element, thereby making the translation parts in the resulting elements superregular and hence suitable for applying the existing formula for ν_w in [Mil21, Theorem 3.2]. However, we cannot show this and rather bypass this problem by converting the Bruhat inequality $t^\lambda y \geq t^\gamma$ to one involving elements whose translation parts *are* dominant. This is done in Lemma 3.2.9 by applying techniques from the theory of the Demazure product on $\widetilde{W}$. Then we can carry out the aforementioned trick of artificially adding a large translation part, applying [Mil21, Theorem 3.2] and then finally subtracting it - as shown in Section 4.2.5.

3.2.1 Proposed linear bound on depth

For the rest of Section 3.2, we assume that W is an irreducible Weyl group for the root system of Cartan type X_n. We first establish an upper bound for

$$\mathcal{M} = \mathcal{M}_{X_n} := \max\{\langle \alpha_i, \mathrm{wt}(x)\rangle : \alpha_i \in \Delta, x \in W\}$$

This is based on an explicit computation of $\mathrm{wt}(w_0)$ for the longest element w_0 in each type, which we defer to the next section for the sake of continuity.

By Lemma 3.1.1, we have $\{\mathrm{wt}(x) : x \in W\} \subset \{\nu \in \mathbb{Z}\Phi^\vee : \nu \leq \mathrm{wt}(w_0)\}$. Therefore,

$$\mathcal{M} \leq \max\{\langle \alpha_i, \nu\rangle : \nu \in \mathbb{Z}\Phi^\vee, \nu \leq \mathrm{wt}(w_0), \alpha_i \in \Delta\}.$$

It is easy to see that maximum of the latter set is realized at $\nu = \kappa_l \alpha_l^\vee$, where $\kappa_l \alpha_l^\vee$ is a summand of $\mathrm{wt}(w_0)$ such that $\kappa_l = \max\{\kappa_j : \kappa_j \alpha_j^\vee \text{ is a summand of } \mathrm{wt}(w_0)\}$. Since the maximum value of $\langle \alpha_i, \alpha_l^\vee\rangle$ is 2 in every Cartan type, this shows that $\mathcal{M}_{X_n}$ is bounded above by twice the largest coefficient in the expression of $\mathrm{wt}(w_0)$ in each type X_n. This is tabulated in the list below, where we denote by $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}_{X_n}$ the upper bound obtained in this manner.

Type	A_n	B_n/C_n	D_n	E_6	E_7	E_8	F_4	G_2
$\widetilde{\mathcal{M}}$	$n+1$	$2n$	$2n$	12	16	28	12	4

3.2.2 Proof of the easier inequality

We need the following result about Bruhat order on $\widetilde{W}$.

Lemma 3.2.5. *[Rap05, Remark 3.9] Let $\lambda \in X_*(T)$ be dominant. Let $\beta \in \Phi^+$ such that $\lambda - \beta^\vee$ is dominant. Then $t^{\lambda-\beta^\vee} \le t^\lambda s_\beta \le t^\lambda$ in $\widetilde{W}$.*

We start by establishing the easier inequality.

Proposition 3.2.6. *Let $w = t^\lambda x \in \widetilde{W}$ such that $depth(\lambda) \ge \widetilde{\mathcal{M}}$. Then $w \ge t^{\lambda - wt(x)}$ and therefore $\nu_w \ge \lambda - wt(x)$.*

Proof. Suppose that $x = s_{\beta_1} \cdots s_{\beta_l}$ is a reduced quantum reflection decomposition with $\ell_\downarrow(x) = l > 1$, and therefore $\mathrm{wt}(x) = \sum_{i=1}^{l} \beta_i^\vee$. Let us first note that:

(a) $s_{\beta_1} \cdots s_{\beta_k}$ is a reduced quantum reflection decomposition for $x_k := x s_{\beta_l} \cdots s_{\beta_{k+1}}$ for all $k \le l$.

Suppose otherwise; then we can find a reduced quantum reflection decomposition of the form $x_k = s_{\gamma_1} \cdots s_{\gamma_j}$ with $j < k$. But then $x = x_k s_{\beta_{k+1}} \cdots s_{\beta_l} = s_{\gamma_1} \cdots s_{\gamma_j} s_{\beta_{k+1}} \cdots s_{\beta_l}$ satisfies length additivity:

$$\ell(x) = \ell(x_k) + \sum_{i=k+1}^{l} \beta_i^\vee = \sum_{i=1}^{j} \gamma_i^\vee + \sum_{i=k+1}^{l} \beta_i^\vee.$$

Note that all associated roots in this new decomposition of x are quantum, and it has $j + l - k < l$ factors, hence it contradicts the fact that $\ell_\downarrow(x) = l$. So statement (a) is proved.

To prove the proposition, we argue by induction on $\ell_{\downarrow}(x)$. We note our depth hypothesis ensures that $\lambda - \beta^\vee$ is dominant for all $\beta \in \Phi^+$: for $\alpha_i \in \Delta$, we have $\langle \alpha_i, \lambda - \beta^\vee \rangle = \langle \alpha_i, \lambda \rangle - \langle \alpha_i, \beta^\vee \rangle > 0$, since maximum value of $\langle \alpha_i, \beta^\vee \rangle$ equals 2 in every Cartan type, except for G_2 - in which case it equals 3, cf. [Bou02, Chapter VI, Section 1, no. 3]. Therefore Lemma 3.2.5 covers the base case, i.e. when $x = s_\beta$ for some $\beta \in \Phi^+$.

Let us now apply the induction hypothesis to $xs_{\beta_l} = s_{\beta_1} \cdots s_{\beta_{l-1}}$. Since $\mathrm{wt}(xs_{\beta_l}) = \sum_{i=1}^{l-1} \beta_i^\vee$ by virtue of statement (a), this gives

$$t^\lambda s_{\beta_1} \cdots s_{\beta_{l-1}} \geq t^{\lambda - \sum_{i=1}^{l-1} \beta_i^\vee}. \tag{3.2.1}$$

To perform the induction step, we apply a property of Bruhat order as described in Equation (2.1.3). We set

$$w_1 = t^\lambda x, w_2 = t^{\lambda - \sum_{i=1}^{l-1} \beta_i^\vee} s_{\beta_l}, v = s_{\beta_l}$$

and check the required length additivity. Note that $\lambda - \sum_{i=1}^{l-1} \beta_i^\vee$ is dominant under our depth hypothesis: for $\alpha_i \in \Delta$,

$$\langle \alpha_i, \lambda - \sum_{i=1}^{l-1} \beta_i^\vee \rangle = \langle \alpha_i, \lambda \rangle - \langle \alpha_i, \mathrm{wt}(xs_{\beta_l}) \rangle > \langle \alpha_i, \lambda \rangle - \widetilde{\mathcal{M}} \geq 0,$$

therefore the length formula in Equation (2.1.1) applies. We see that

1. $\ell(w_1 v) = \ell(t^\lambda s_{\beta_1} \cdots s_{\beta_{l-1}}) = \ell(t^\lambda) - \sum_{i=1}^{l-1} \ell(s_{\beta_i}) = \ell(t^\lambda) - \sum_{i=1}^{l} \ell(s_{\beta_i}) + \ell(s_{\beta_l}) = \ell(w_1) + \ell(v)$.
2. $\ell(w_2 v) = \ell(t^{\lambda - \sum_{i=1}^{l-1} \beta_i^\vee}) = \ell(t^{\lambda - \sum_{i=1}^{l-1} \beta_i^\vee}) - \ell(s_{\beta_l}) + \ell(s_{\beta_l}) = \ell(w_2) + \ell(v)$.

In presence of Equation (3.2.1), we therefore get $w_1 \geq w_2$, i.e. $t^\lambda x \geq t^{\lambda - \sum_{i=1}^{l-1} \beta_i^\vee} s_{\beta_l}$. Finally, we note that $\lambda - \mathrm{wt}(x) = \lambda - \sum_{i=1}^{l} \beta_i^\vee$ is dominant (by similar argument as before) and thus Lemma 3.2.5 applies to give $t^{\lambda - \sum_{i=1}^{l-1} \beta_i^\vee} s_{\beta_l} \geq t^{\lambda - \sum_{i=1}^{l} \beta_i^\vee}$. Combining these inequalities, we get $t^\lambda x \geq t^{\lambda - \mathrm{wt}(x)}$. By [Vie14, Corollary 5.6], we have

$$\nu_w = \max\{\nu(u) : u \leq w, u \in \widetilde{W}\}. \tag{3.2.2}$$

Appealing to Equation (3.2.2), we thus get $\nu_w \geq \lambda - \mathrm{wt}(x)$. This finishes the proof. □

Lemma 3.2.7. *Let $w = t^\lambda x \in \widetilde{W}$ such that depth$(\lambda) > \widetilde{\mathcal{M}}$. Then $\nu_w = \max\{\gamma^+ : t^\gamma \leq w\}$.*

Proof. By Equation (3.2.2), we can assume that $\nu_w = \nu(t^\mu z)$ for some $t^\mu z \in \widetilde{W} = X_*(T) \rtimes W$, such that $w \geq t^\mu z$. Suppose that $z \neq 1$, therefore the order of z equals $m \geq 2$. Then $\nu(t^\mu z) = (\frac{1}{m} \sum_{i=1}^{m} z^i(\mu))^+$. Note that $z \cdot \sum_{i=1}^{m} z^i(\mu) = \sum_{i=1}^{m} z^i(\mu)$, so the element $\sum_{i=1}^{m} z^i(\mu)$ lies on the wall of some Weyl chamber, cf. [Bou02, Chapter V, Section 3.3, Remark 3]. Therefore $(\frac{1}{m} \sum_{i=1}^{m} z^i(\mu))^+$ lies on the wall of $\mathcal{C}^+$, and hence ν_w is singular.

By Proposition 3.2.6, we can assume that $\nu_w = \lambda - \epsilon$, with $\epsilon \leq \mathrm{wt}(x) \leq \mathrm{wt}(w_0)$. Hence, there is some $\alpha_i \in \Delta$ such that $\langle \alpha_i, \lambda - \epsilon \rangle = 0$, and thus $\langle a_i, \lambda \rangle = \langle \alpha_i, \epsilon \rangle \leq \widetilde{M}$, which is a contradiction to hypothesis on depth(λ). Therefore, ν_w cannot be singular and hence $z = 1$. □

3.2.3 Toward computing a downward Demazure product

We need to understand the largest translation dominated by w to determine its associated generic Newton point ν_w. In view of Proposition 3.2.6, let us denote this translation element by $t^{y\gamma}$ for some $y \in W$ and $\gamma \geq \lambda - \mathrm{wt}(x)$. Of course, this would be equality if λ

is superregular. We want to leverage this result to establish such equality with some linear bound on depth here. The following lemma lays the path towards proving that.

Lemma 3.2.8. *If $\lambda - 2\rho^\vee$ is dominant, we have $t^\lambda x \geq t^{y\gamma}$ if and only if $t^{\lambda-2\rho^\vee} \geq t^{-2\rho^\vee} \rhd t^{y\gamma}$.*

Proof. Note that if $\lambda - 2\rho^\vee$ is dominant, $\ell(t^{\lambda-2\rho^\vee}) = \langle 2\rho, \lambda - 2\rho^\vee \rangle = \ell(t^\lambda) - \ell(t^{-2\rho^\vee})$; also, since $\lambda \geq \lambda - 2\rho^\vee$ are dominant coweights, $t^\lambda \geq t^{\lambda-2\rho^\vee}$, cf. [Rap05, proof of Proposition 3.5]. Hence, $t^{-2\rho^\vee} \rhd t^\lambda = t^{\lambda-2\rho^\vee}$.

Now, it suffices to show the following:

(a) If a, b are two elements in a Coxeter group and s is a simple reflection, then $a \geq b$ if and only if $s \rhd a \geq s \rhd b$.

This follows from the well known lifting property of Bruhat order. Note that we only need to consider the case when $s \rhd a = sa$, i.e. $a \geq sa$. Then there are two further cases.

1. $s \rhd b = b$: in this case, $sb \geq b$. Hence $a \geq b$ is equivalent to $sa \geq b$, and hence $s \rhd a \geq s \rhd b$.

2. $s \rhd b = sb$: in this case, $b \geq sb$. Hence $a \geq b$ is equivalent to $sa \geq sb$, and hence $s \rhd a \geq s \rhd b$.

□

In Section 3.2.4 we will focus on computing the downward Demazure product that appears in Lemma 3.2.8. Before proceeding any further to do that, let us first sketch what the answer should look like. We first apply Lemma 2.4.2 with $w = t^{-2\rho^\vee}, w' = w_0$ and $v = v' = t^{y\mu}$; if we require that λ is of large enough depth (in a sense made precise below) so that γ is dominant regular, we get $w_0 \rhd y t^\gamma y^{-1} = t^\gamma y^{-1} \geq t^{-2\rho^\vee} \rhd t^{y\gamma}$. We next apply Lemma 2.4.2 with $v' = t^{y\gamma}, v = t^{\gamma}y^{-1}$ and $w = w' = t^{-2\rho^\vee}$. If we now require λ to be of sufficiently large depth so that $\gamma - 2\rho^\vee$ is also dominant, we get $t^{-2\rho^\vee} \rhd t^{y\gamma} \geq t^{-2\rho^\vee} \rhd t^{\gamma}y^{-1} = t^{\gamma-2\rho^\vee}y^{-1}$. Combining these, we see that whenever λ has "large" depth (to be specified later) we have

$$t^{\gamma}y^{-1} \geq t^{-2\rho^\vee} \rhd t^{y\gamma} \geq t^{\gamma-2\rho^\vee}y^{-1}.$$

Therefore, it is natural to expect that this downward Demazure product depends only on y whenever λ has a "large" depth. In the next section, we quantify this depth condition and show that we indeed get the desired conclusion.

3.2.4 A technical lemma

For each irreducible Cartan type X_n, we let $\mathcal{S} = \mathcal{S}_{X_n}$ be twice the sum of all coefficients appearing in the expression of the highest root in terms of simple roots; in other words, $\mathcal{S} = \langle \theta, 2\rho^\vee \rangle$, where θ is the highest root. This quantity will become relevant for our next lemma. We record this integer for each type in the table below.

Type	A_n	B_n/C_n	D_n	E_6	E_7	E_8	F_4	G_2
$\mathcal{S}$	$2n$	$4n-2$	$4n-6$	11	17	29	11	5

Note that $\Xi = \widetilde{\mathcal{M}} + \mathcal{S}$ is the final lower bound that appears in Theorem 3.2.1. We now prove a key lemma that would help us achieve the harder inequality.

Lemma 3.2.9. *Let depth$(\lambda) > \Xi$. We continue to assume that $w = t^{\lambda}x \geq t^{y\gamma}$ such that*

$\gamma = \nu_w \geq \lambda - wt(x)$. Then there exists a coweight μ_y depending only on y such that $t^{-2\rho^\vee} \triangleright t^{y\gamma} = t^{\gamma-\mu_y} y^{-1}$.

Proof. We begin by noting that we can compute this downward Demazure product from a specific length additive decomposition of $t^{y\gamma}$. More precisely, we have the following desiderata

- (D1) a decomposition $t^{y\gamma} = a_1 t^{\mu_1} b_1 \cdot a_2 t^{\mu_2} b_2$, where μ_i is dominant and $t^{\mu_i} b_i \in {}^{\mathbb{S}}\widetilde{W}$ for $i = 1, 2$;

- (D2) $\ell(t^{y\gamma}) = \ell(a_1 t^{\mu_1} b_1) + \ell(a_2 t^{\mu_2} b_2)$;

- (D3) $a_1 t^{\mu_1} b_1$ is the largest element dominated by $t^{2\rho^\vee}$, subject to the first two conditions.

Lemma 2.4.1 asserts the existence of such decomposition, and then we have $t^{-2\rho^\vee} \triangleright t^{\gamma y} = a_2 t^{\mu_2} b_2$. We now proceed to identify these elements $a_i t^{\mu_i} b_i$ in the following three steps.

3.2.4.1 Relating relevant coweights

We first determine how the coweights associated to translation part of these elements are related. By (D1), $t^{y\gamma} = a_1 b_1 a_2 t^{a_2^{-1} b_1^{-1} \mu_1 + \mu_2} b_2$, so we have $\gamma = (a_2^{-1} b_1^{-1} \mu_1 + \mu_2)^+$, i.e. there exists some $v \in W$ such that $\gamma = v(a_2^{-1} b_1^{-1} \mu_1 + \mu_2)$. We will now show that $v = 1$.

Assume the contrary and let $\beta \in \mathrm{Inv}(v)$. Now, $\mu_2 = v^{-1}\gamma - a_2^{-1} b_1^{-1} \mu_1$ is dominant, therefore

$$\langle \beta, v^{-1}\gamma - a_2^{-1} b_1^{-1} \mu_1 \rangle \geq 0.$$

By (D3), we have $t^{2\rho^\vee} \geq a_1 t^{\mu_1} b_1$, thus $2\rho^\vee \geq \mu_1$; by a geometric characterization of dominance order, e.g. see [AB83, Lemma 12.14], this gives

$$\mu_1 = \sum_{\zeta \in W} a_\zeta \zeta \cdot 2\rho^\vee \text{ for some } a_\zeta \in \mathbb{R}_{\geq 0} \text{ such that } \sum_{\zeta \in W} a_\zeta = 1.$$

Substituting this expression of μ_1 in the pairing above and using its W-invariance, we get

$$\langle v\beta, \gamma \rangle - \sum_{\zeta \in W} a_\zeta \langle \zeta^{-1} b_1 a_2 \beta, 2\rho^\vee \rangle \geq 0.$$

Let $\beta' = -v\beta \in \Phi^+$. Note that $\zeta^{-1} b_1 a_2 \beta \geq -\theta$ for any element $\zeta \in W$, and therefore $\langle \zeta^{-1} b_1 a_2 \beta, 2\rho^\vee \rangle \geq -\langle \theta, 2\rho^\vee \rangle$. Putting all these together, we obtain

$$\langle \beta', \gamma \rangle = -\sum_{\zeta \in W} a_\zeta \langle \zeta^{-1} b_1 a_2 \beta, 2\rho^\vee \rangle \leq \sum_{\zeta \in W} a_\zeta \langle \theta, 2\rho^\vee \rangle = \mathcal{S}.$$

Now recall that $\gamma = \nu_w = \lambda - \epsilon$ is dominant with $\epsilon \leq \mathrm{wt}(x) \leq \mathrm{wt}(w_0)$. Choose a simple root $\alpha \leq \beta'$. Then we get $\langle \alpha, \gamma \rangle \leq \langle \beta', \gamma \rangle \leq \mathcal{S}$, and therefore $\langle \alpha, \lambda \rangle \leq \mathcal{S} + \langle \alpha, \epsilon \rangle \leq \mathcal{S} + \widetilde{\mathcal{M}}$. Since this yields a contradiction to the depth hypothesis of λ, we are done. Therefore $v = 1$ and $\gamma = a_2^{-1} b_1^{-1} \mu_1 + \mu_2$.

3.2.4.2 Showing that certain coweights are regular

Note that γ is regular, since for any simple root α_i we have $\langle \alpha_i, \gamma \rangle = 0 \implies \langle \alpha_i, \lambda \rangle = \langle \alpha_i, \epsilon \rangle \leq \widetilde{\mathcal{M}}$, thereby contradicting the depth hypothesis imposed on λ. Hence, knowing that the translation parts in $t^{y\gamma} = a_1 b_1 a_2 t^{a_2^{-1} b_1^{-1} \mu_1 + \mu_2} b_2$ are equal allows us to relate the finite Weyl group components from both sides. Namely, we have $y = a_1 b_1 a_2 = b_2^{-1}$. We next simplify these relations further in the following way. Let us first observe that μ_2 is regular, since otherwise we can find a simple root α such that $\langle\alpha,\gamma\rangle = \sum_{\zeta\in W} a_\zeta\langle\alpha,\zeta\cdot a_2^{-1}b_1^{-1}2\rho^\vee\rangle = \sum_{\zeta\in W}\langle b_1a_2\zeta\alpha, 2\rho^\vee\rangle \leq \sum_{\zeta\in W} a_\zeta\langle\theta,2\rho^\vee\rangle = \mathcal{S}$; but this would yield contradiction to the imposed depth condition as before.

3.2.4.3 Showing that the product describes a dominant chamber alcove

We can now show that $a_2 = 1$. Recall that $t^{-2\rho^\vee} \rhd t^{y\gamma} = a_2t^{\mu_2}b_2$. Let us rewrite $t^{-2\rho^\vee} = w_0t^{2\rho^\vee}w_0$ and apply $w_0\rhd$ to the former equation. This gives

$$w_0 \rhd \{(w_0t^{2\rho^\vee}w_0) \rhd t^{y\gamma}\} = w_0 \rhd (a_2t^{\mu_2}b_2).$$

We now apply Lemma 2.4.2(2) on the left hand side, and use the fact that μ_2 is regular in the right hand side. We get

$$\{w_0 * (w_0t^{2\rho^\vee}w_0)\} \rhd t^{y\gamma} = (w_0 \rhd a_2)t^{\mu_2}b_2.$$

Note that $w_0t^{2\rho^\vee}w_0 = w_0 * t^{2\rho^\vee}w_0$, and hence we can use associativity of operation $*$ to rewrite the element in the parenthesis on the left hand side as $w_0 * (w_0 * t^{2\rho^\vee}w_0) = (w_0 * w_0) * t^{2\rho^\vee}w_0 = w_0 * t^{2\rho^\vee}w_0 = w_0t^{2\rho^\vee}w_0 = t^{-2\rho^\vee}$. Therefore, we finally deduce

$$t^{-2\rho^\vee} \rhd t^{y\gamma} = t^{\mu_2}b_2.$$

This proves our claim.

Therefore, the relations arising from (D1) are

$$\gamma = b_1^{-1}\mu_1 + \mu_2, y = a_1 b_1 = b_2^{-1}. \tag{3.2.3}$$

Let us now utilize (D2). It says that we have length additivity in the decomposition $t^{y\gamma} = a_1 t^{\mu_1} b_1 \cdot t^{\mu_2} b_2$, i.e.

$$\langle \gamma, 2\rho \rangle = \ell(a_1) + \langle \mu_1, 2\rho \rangle - \ell(b_1) + \langle \mu_2, 2\rho \rangle - \ell(b_2).$$

By use of Equation (3.2.3), the last equation reduces to

$$\ell(a_1 b_1) = \ell(a_1) - \ell(b_1) + \langle \mu_1 - b_1^{-1}\mu_1, 2\rho \rangle. \tag{3.2.4}$$

Therefore we are led to consider the following subset of $\widetilde{W}$:

$$\Omega_y := \{at^{\mu}b \in \widetilde{W} : y = ab, at^{\mu}b \leq t^{2\rho^\vee}, \ell(ab) = \ell(a) - \ell(b) + \langle \mu - b^{-1}\mu, 2\rho \rangle\}.$$

Note that the definition of Ω_y encapsulates all information amongst a_1, b_1, μ_1 coming out of the first two items in our desiderata. By (D3), the element $a_1 t^{\mu_1} b_1$ is therefore uniquely specified as the maximal element of Ω_y. Hence, the element $b_1^{-1}\mu_1$ depends only on y. Denoting it by μ_y, we see that Equation (3.2.3) implies $\mu_2 = \gamma - \mu_y$. This finishes the proof. $\square$

3.2.5 Finishing the proof

Proof of Theorem 3.2.1. By combining Lemma 3.2.8 and Lemma 3.2.9, we observe that our standing assumption $t^\lambda x \geq t^{y\gamma}$ implies

$$t^{\lambda-2\rho^\vee} x \geq t^{\gamma-\mu_y} y^{-1}.$$

Let us now choose λ' to be large enough so that $\lambda+\lambda'$ is superregular, e.g. take $\lambda' = n\rho^\vee$ for large n. We note that

1. $\ell(t^{\lambda'} \cdot t^{\lambda-2\rho^\vee} x) = \ell(t^{\lambda+\lambda'-2\rho^\vee} x) = \langle 2\rho, \lambda+\lambda'-2\rho^\vee\rangle - \ell(x) = \langle 2\rho, \lambda'\rangle + \langle 2\rho, \lambda-2\rho^\vee\rangle - \ell(x) = \ell(t^{\lambda'}) + \ell(t^{\lambda-2\rho^\vee} x)$.

2. $\ell(t^{\lambda'} \cdot t^{\gamma-\mu_y} y^{-1}) = \ell(t^{\lambda'+\gamma-\mu_y} y^{-1}) = \langle 2\rho, \lambda'+\gamma-\mu_y\rangle - \ell(y^{-1}) = \langle 2\rho, \lambda'\rangle + \langle 2\rho, \gamma - \mu_y\rangle - \ell(y^{-1}) = \ell(t^{\lambda'}) + \ell(t^{\gamma-\mu_y} y^{-1})$.

Hence, we can apply Equation (2.1.3) in this situation, and get

$$t^{\lambda+\lambda'-2\rho^\vee} x \geq t^{\gamma+\lambda'-\mu_y} y^{-1}.$$

Now, we apply Lemma 3.2.8 again to deduce from the previous equation

$$t^{\lambda+\lambda'} x \geq t^{y(\gamma+\lambda')}.$$

Therefore, by Equation (3.2.2) we have $\nu_{t^{\lambda+\lambda'}x} \geq \nu_{t^{y(\gamma+\lambda')}}$. Since the formula for the maximal Newton point in [Mil21, Theorem 3.2] applies to elements of $\widetilde{W}$ having $\lambda+\lambda'$ as its dominant translation part, we get that $\lambda+\lambda'-\mathrm{wt}(x) \geq \gamma+\lambda'$, whence $\lambda-\mathrm{wt}(x) \geq \gamma = \nu_w$. Combining this with Proposition 3.2.6, we get $\nu_w = \lambda - \mathrm{wt}(x)$. This finishes the proof. □

3.2.6 A remark about Bruhat order on $\widetilde{W}$

Proposition 3.2.10. *Suppose that* $\mathbf{G}$ *is a quasi-simple split group. Assume depth*$(\lambda) > \Xi$*. Then* $ut^{\lambda}v \geq t^{y\gamma}$ *for some dominant* γ *implies that* $t^{\lambda}(v \lhd u) \geq t^{\gamma}$*.*

Proof. Note that $ut^{\lambda}v \geq t^{y\gamma}$ implies $\nu_{ut^{\lambda}v} \geq \nu_{t^{y\gamma}}$ by Equation (3.2.2). By Theorem 3.2.1 and Corollary 3.1.3, we then have $\lambda - \mathrm{wt}(v \lhd u) \geq \gamma$. By Proposition 3.2.6, we have that $t^{\lambda}(v \lhd u) \geq t^{\lambda-\mathrm{wt}(v\lhd u)}$. Since $\lambda - \mathrm{wt}(v \lhd u), \gamma$ are both dominant, we also have that $t^{\lambda-\mathrm{wt}(v\lhd u)} \geq t^{\gamma}$, cf. [Rap05, proof of Proposition 3.5]. Putting together the last two inequalities, we finally get $t^{\lambda}(v \lhd u) \geq t^{\gamma}$. □

Remark 3.2.11. Note that if $u = 1$, this says that $t^{\lambda}v \geq t^{y\gamma}$ implies $t^{\lambda}v \geq t^{\gamma}$. In other words, if some W-conjugate of a dominant translation element lies below an element in the dominant chamber, so does the dominant translation element itself. We point out that in the presence of Proposition 3.2.6 and the fact that maximal Newton point formula is available for elements with superregular translation part, this is equivalent to Theorem 3.2.1. Hence, an independent proof of Proposition 3.2.10 would help us get rid of the bound imposed in Section 3.2.4; similarly, an improved estimate for $\mathcal{M}$ would weaken the final depth hypothesis required in Theorem 3.2.1.

3.3 Weight of the longest element

In this section, we give a reduced quantum reflection decomposition for the longest element w_0 in Weyl group W of each irreducible Cartan type. For an element $x \in W$, the *reflection length* of x is the smallest number l such that x can be written as a product of l reflections in W. We denote by $\ell_R(x)$ the reflection length of x. Note that in general $\ell_R(x) \leq \ell_{\downarrow}(x)$, and strict inequality can occur. We enumerate reflection length for w_0 for all the Cartan types below.

Type	A_n	B_n/C_n	D_n	E_6	E_7	E_8	F_4	G_2
$\ell_R(w_0)$	$\lceil \frac{n}{2} \rceil$	n	$2\lfloor \frac{n}{2} \rfloor$	4	7	8	4	2

We compute $\mathrm{wt}(w_0)$ by exhibiting suitable decomposition of w_0 in each type, and on the way we see that $\ell_R(w_0) = \ell_{\downarrow}(w_0)$. We first write down expression of the longest element affording its reflection length; for the classical types, we extract this from [Bla09], and for the exceptional ones we can check this directly by hand or using a computer algebra system such as [TheYY]. Once we find these decompositions, we see that they only involve reflections corresponding to quantum roots; furthermore, these expressions do satisfy the length additivity condition as well. Taken together, these observations establish that we have found reduced quantum reflection decomposition for w_0 in these types. Finally, we add up the coroots associated to the reflections appearing in each such decompositions and deduce $\mathrm{wt}(w_0)$ from that. See Remark 3.3.1 for an alternative recipe for computing $\mathrm{wt}(w_0)$.

(a) **Type** A_n: here

$$w_0 = \begin{cases} s_{\alpha_1+\cdots+\alpha_{2k}} s_{\alpha_2+\cdots+\alpha_{2k-1}} \cdots s_{\alpha_k+\alpha_{k+1}}, & \text{if } n = 2k; \\ s_{\alpha_1+\cdots+\alpha_{2k+1}} s_{\alpha_2+\cdots+\alpha_{2k}} \cdots s_{\alpha_{k+1}}, & \text{if } n = 2k+1. \end{cases}$$

Hence,

$$\mathrm{wt}(w_0) = \begin{cases} \alpha_1^\vee + 2\alpha_2^\vee + \cdots + (k-1)\alpha_{k-1}^\vee + k\alpha_k^\vee + k\alpha_{k+1}^\vee + (k-1)\alpha_{k+2}^\vee & \\ + \cdots + \alpha_{2k}^\vee, & \text{if } n = 2k; \\ \alpha_1^\vee + 2\alpha_2^\vee + \cdots + k\alpha_k^\vee + (k+1)\alpha_{k+1}^\vee + k\alpha_{k+2}^\vee & \\ + \cdots + \alpha_{2k+1}^\vee, & \text{if } n = 2k+1. \end{cases}$$

(b) **Type** B_n: here

$$w_0 = \begin{cases} s_{\alpha_1+2\alpha_2+\cdots+2\alpha_{2k}} s_{\alpha_1} s_{\alpha_3+2\alpha_4+\cdots+2\alpha_{2k}} s_{\alpha_3} \cdots s_{\alpha_{2k-1}+2\alpha_{2k}} s_{\alpha_{2k-1}}, & \text{if } n = 2k; \\ s_{\alpha_1+2\alpha_2+\cdots+2\alpha_{2k+1}} s_{\alpha_1} s_{\alpha_3+2\alpha_4+\cdots+2\alpha_{2k+1}} s_{\alpha_3} \cdots s_{\alpha_{2k-1}+2\alpha_{2k}+2\alpha_{2k+1}} s_{\alpha_{2k+1}}, & \text{if } n = 2k+1. \end{cases}$$

Hence,

$$\mathrm{wt}(w_0) = \begin{cases} 2\alpha_1^\vee + 2\alpha_2^\vee + 4\alpha_3^\vee + 4\alpha_4^\vee + \cdots + 2(k-1)\alpha_{2k-3}^\vee + 2(k-1)\alpha_{2k-2}^\vee & \\ +2k\alpha_{2k-1}^\vee + k\alpha_{2k}^\vee, & \text{if } n = 2k; \\ 2\alpha_1^\vee + 2\alpha_2^\vee + 4\alpha_3^\vee + 4\alpha_4^\vee + \cdots + 2k\alpha_{2k-1}^\vee + 2k\alpha_{2k}^\vee + (k+1)\alpha_{2k+1}^\vee, & \text{if } n = 2k+1. \end{cases}$$

(c) **Type** C_n: here $w_0 = s_{2\alpha_1+\cdots+2\alpha_{n-1}+\alpha_n} s_{2\alpha_2+\cdots+2\alpha_{n-1}+\alpha_n} \cdots s_{2\alpha_{n-1}+\alpha_n} s_{\alpha_n}$, and thus

$$\mathrm{wt}(w_0) = \alpha_1^\vee + 2\alpha_2^\vee + \cdots + n\alpha_n^\vee.$$

(d) **Type** D_n: here

$$w_0 = \begin{cases} s_{\alpha_1+2\alpha_2+\cdots+2\alpha_{2k-2}+\alpha_{2k-1}+\alpha_{2k}} s_{\alpha_1} s_{\alpha_3+2\alpha_4+\cdots+2\alpha_{2k-2}+\alpha_{2k-1}+\alpha_{2k}} & \\ s_{\alpha_3} \cdots s_{\alpha_{2k-3}+2\alpha_{2k-2}+\alpha_{2k-1}+\alpha_{2k}} s_{\alpha_{2k-3}} s_{\alpha_{2k}}, & \text{if } n = 2k; \\ s_{\alpha_1+2\alpha_2+\cdots+2\alpha_{2k-2}+\alpha_{2k-1}+\alpha_{2k}} s_{\alpha_1} s_{\alpha_3+2\alpha_4+\cdots+2\alpha_{2k-2}+\alpha_{2k-1}+\alpha_{2k}} & \\ s_{\alpha_3} \cdots s_{\alpha_{2k-3}+2\alpha_{2k-2}+2\alpha_{2k-1}+\alpha_{2k}+\alpha_{2k+1}} s_{\alpha_{2k-1}+\alpha_{2k}+\alpha_{2k+1}} s_{\alpha_{2k-1}}, & \text{if } n = 2k+1. \end{cases}$$

Hence,

$$\mathrm{wt}(w_0) = \begin{cases} 2\alpha_1^\vee + 2\alpha_2^\vee + 4\alpha_3^\vee + 4\alpha_4^\vee + \cdots + (2k-2)\alpha_{2k-3}^\vee & \\ +(2k-2)\alpha_{2k-2}^\vee + k\alpha_{2k-1}^\vee + k\alpha_{2k}^\vee, & \text{if } n = 2k; \\ 2\alpha_1^\vee + 2\alpha_2^\vee + 4\alpha_3^\vee + 4\alpha_4^\vee + \cdots + (2k-2)\alpha_{2k-3}^\vee + (2k-2)\alpha_{2k-2}^\vee & \\ +2k\alpha_{2k-1}^\vee + k\alpha_{2k}^\vee + k\alpha_{2k+1}^\vee, & \text{if } n = 2k+1. \end{cases}$$

(e) **Type** E_6: here $w_0 = s_{\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6} s_{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6} s_{\alpha_3+\alpha_4+\alpha_5} s_{\alpha_4}$, and thus

$$\mathrm{wt}(w_0) = 2\alpha_1^\vee + 2\alpha_2^\vee + 4\alpha_3^\vee + 6\alpha_4^\vee + 4\alpha_5^\vee + 2\alpha_6^\vee.$$

(f) **Type** E_7: here $w_0 = s_{2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7} s_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7} s_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5} s_{\alpha_2}$

$s_{\alpha_3}s_{\alpha_5}s_{\alpha_7}$, and thus

$$\mathrm{wt}(w_0) = 2\alpha_1^\vee + 5\alpha_2^\vee + 6\alpha_3^\vee + 8\alpha_4^\vee + 7\alpha_5^\vee + 4\alpha_6^\vee + 3\alpha_7^\vee.$$

(g) **Type** E_8: here $w_0 = s_{2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+3\alpha_7+2\alpha_8}s_{2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7}$ $s_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7}s_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5}s_{\alpha_2}s_{\alpha_3}s_{\alpha_5}s_{\alpha_7}$, and thus

$$\mathrm{wt}(w_0) = 4\alpha_1^\vee + 8\alpha_2^\vee + 10\alpha_3^\vee + 14\alpha_4^\vee + 12\alpha_5^\vee + 8\alpha_6^\vee + 6\alpha_7^\vee + 2\alpha_8^\vee.$$

(h) **Type** F_4: here $w_0 = s_{2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4}s_{\alpha_2+2\alpha_3+2\alpha_4}s_{\alpha_2+2\alpha_3}s_{\alpha_2}$, and thus

$$\mathrm{wt}(w_0) = 2\alpha_1^\vee + 6\alpha_2^\vee + 4\alpha_3^\vee + 2\alpha_4^\vee.$$

(i) **Type** G_2: here $w_0 = s_{3\alpha_1+2\alpha_2}s_{\alpha_1}$, and thus

$$\mathrm{wt}(w_0) = 2\alpha_1^\vee + 2\alpha_2^\vee.$$

Remark 3.3.1. We note that the weight of longest element found above appears in a rather different context, cf. [Lus18]. In that paper, Lusztig points out that the coroots appearing as a summand of $\mathrm{wt}(w_0)$ as above form a so-called *cascade* (terminology due to Kostant). He lists out these coroots and their sums in loc. sit. section 1.2 and section 1.8. In fact, a map $x \mapsto r_x$ is defined from the set of involutions $\mathcal{I}_W$ in an irreducible Weyl group W in loc. sit. and this is crucially used in constructing certain lifts of involutions in associated reductive groups. We compare this map defined on the set of involutions with our weight function in Chapter 5.

3.4 Covering relation in Iwahori-Weyl group

The main result of this section characterizes the covering relation of Bruhat order for most of the elements in $\widetilde{W}$.

Theorem 3.4.1. *Suppose that* $\mathbf{G}$ *is a split reductive group and* $w = ut^{\lambda}v$ *be an element of* $\widetilde{W}$ *such that depth*(λ) *is bigger than a certain constant. Let* $r_{\beta} = t^{mu\alpha^{\vee}}s_{u\alpha}$ *be an affine reflection for some positive root* α *and integer* m*, and let* $w' = r_{\beta}w$*. Then* $w \gtrdot w'$ *is a covering relation, i.e.* $w \geq w'$ *and* $\ell(w) = \ell(w') + 1$*, if and only if one of the following conditions holds:*

1. $m = 0$ *and* $\ell(us_{\alpha}) = \ell(u) - 1$*; in this case,* $w' = us_{\alpha}t^{\lambda}v$*.*

2. $m = 1$ *and* $\ell(us_{\alpha}) = \ell(u) + \langle 2\rho, \alpha^{\vee}\rangle - 1$*; in this case,* $w' = us_{\alpha}t^{\lambda-\alpha^{\vee}}v$*.*

3. $m = \langle \alpha, \lambda\rangle$ *and* $\ell(s_{\alpha}v) = \ell(v) + 1$*; in this case,* $w' = ut^{\lambda}s_{\alpha}v$*.*

4. $m = \langle \alpha, \lambda\rangle - 1$ *and* $\ell(s_{\alpha}v) = \ell(v) - \langle 2\rho, \alpha^{\vee}\rangle + 1$*; in this case,* $w' = ut^{\lambda-\alpha^{\vee}}s_{\alpha}v$*.*

In fact, it suffices to show the following result instead; see the remarks following the theorem below.

Theorem 3.4.2. *Suppose that* $\widetilde{W}$ *is an Iwahowi Weyl group of a quasi-simple split reductive group. Let* $t^{m\alpha^{\vee}}s_{\alpha}$ *be an affine reflection, with* $(\alpha, m) \in \Phi^{+} \times \mathbb{Z}$ *and assume* $w = t^{\lambda}y$ *is an* *element of* $\widetilde{W}$ *such that*

$$depth(\lambda) \geq \begin{cases} 3, & \textit{if } W \textit{ is of simply laced type;} \\ 4, & \textit{if } W \textit{ is of non-simply laced type but not of type } G_2\textit{;} \\ 6, & \textit{if } W \textit{ is of type } G_2. \end{cases} \tag{3.4.1}$$

Then $w' := t^{m\alpha^\vee} s_\alpha w$ *is a cocover of* w[1] *if and only if one of the following holds.*

1. $m = 1$ *and* $\ell(s_\alpha) = \langle 2\rho, \alpha^\vee \rangle - 1$*; in this case,* $w' = s_\alpha t^{\lambda - \alpha^\vee} y$.

2. $m = \langle \alpha, \lambda \rangle$ *and* $\ell(s_\alpha y) = \ell(y) + 1$*; in this case,* $w' = t^\lambda s_\alpha y$.

3. $m = \langle \alpha, \lambda \rangle - 1$ *and* $\ell(s_\alpha y) = \ell(y) - \langle 2\rho, \alpha^\vee \rangle + 1$*; in this case,* $w' = t^{\lambda - \alpha^\vee} s_\alpha y$.

Remark 3.4.3. We explain how Theorem 3.4.2 implies Theorem 3.4.1. Let us first note that we can again reduce to the case of irreducible factors as in Remark 3.2.3. Note that if λ is regular, $\ell(ut^\lambda v) = \ell(u) + \ell(t^\lambda v)$ by Equation (2.1.1). Choose reduced expressions $u = s_{i_1} \cdots s_{i_l}$ and $t^\lambda v = s_{j_1} \cdots s_{j_k} \zeta$, where $s_{i_p}, s_{j_q} \in \mathbb{S}$ for $1 \leq p \leq l, 1 \leq q \leq k$ and $\zeta \in \Omega$. Then $ut^\lambda v = s_{i_1} \cdots s_{i_l} s_{j_1} \cdots s_{j_k} \zeta$ is a reduced expression, and a cocover of $ut^\lambda v$ must be of the form of either $s_{i_1} \cdots \widehat{s_{i_m}} \cdots s_{i_l} s_{j_1} \cdots s_{j_k} \zeta$ for some $p \in [1, l]$ or $s_{i_1} \cdots s_{i_l} s_{j_1} \cdots \widehat{s_{j_n}} \cdots s_{j_k} \zeta$ for some $q \in [1, k]$. In other words, any cocover of $ut^\lambda v$ is obtained by

- (C1) either multiplying a cocover of u with $t^\lambda v$, or
- (C2) multiplying u with a cocover of $t^\lambda v$.

[1]for two elements w, w' of $\widetilde{W}$, we say that w' is a *cocover* of w if w is a cover of w', i.e. $w > w'$ and $\ell(w') = \ell(w) - 1$. We also shorthand these last two conditions by writitng $w \gtrdot w'$.

The procedure listed in (C1) is easy to describe. A cocover of u is of the form us_α for some $\alpha \in \Phi^+$ such that $\ell(us_\alpha) = \ell(u) - 1$. This corresponds to case (1) in theorem B. We claim that the three other cases there, i.e. (2)-(4), come from the procedure described in (C2), and hence they correspond to those listed in Theorem 3.4.2. Clearly, the third and fourth case in theorem B corresponds respectively to the second and third case in Theorem 3.4.2. Finally, suppose that $w' := s_a t^{\lambda - \alpha^\vee} v$ is a cocover of $w := t^\lambda v$ such that uw' is a cocover of uw. By Equation (2.1.1),

$$\ell(uw') = \ell(us_a) + \ell(t^{\lambda - \alpha^\vee}) - \ell(v) = \ell(us_a) + \langle 2\rho, \lambda - \alpha^\vee \rangle - \ell(v).$$

We use dominance of $\lambda - \alpha^\vee$ in the last equality; this is true as $\langle \alpha_i, \lambda - a^\vee \rangle = \langle \alpha_i, \lambda \rangle - \langle \alpha_i, \alpha^\vee \rangle \geq 0$ by our earlier discussion about the maximum value of $\langle \alpha_i, \alpha^\vee \rangle$. Note that

$$\ell(us_\alpha) \leq \ell(u) + \ell(s_a) \leq \ell(u) + \langle 2\rho, \alpha^\vee \rangle - 1. \tag{3.4.2}$$

Since $\ell(ut^\lambda v) = \ell(u) + \langle 2\rho, \lambda \rangle - \ell(v)$, the cocover condition $uw \gtrdot uw'$ gives

$$\begin{aligned} \ell(u) + \langle 2\rho, \lambda \rangle - \ell(v) - 1 &= \ell(us_a) + \langle 2\rho, \lambda - \alpha^\vee \rangle - \ell(v) \\ &\leq \ell(u) + \langle 2\rho, \alpha^\vee \rangle - 1 + \langle 2\rho, \lambda - \alpha^\vee \rangle - \ell(v). \end{aligned}$$

Note that

$$\ell(u) + \langle 2\rho, \lambda \rangle - \ell(v) - 1 = \ell(us_a) + \langle 2\rho, \lambda - \alpha^\vee \rangle - \ell(v) \leq \ell(u) + \langle 2\rho, \alpha^\vee \rangle - 1 + \langle 2\rho, \lambda - \alpha^\vee \rangle - \ell(v).$$

Hence we deduce that all of the inequalities in Equation (3.4.2) must be equality, and thus we get $\ell(us_\alpha) = \ell(u) + \langle 2\rho, \alpha^\vee \rangle - 1$. This is exactly the situation listed in the second item in theorem B.

Remark 3.4.4. We note that in a recent preprint [Sch22a][proposition 4.5], Schremmer gives several equivalent criteria that describes the covering relation, without any assumption on the depth of the relevant coweight.

We will only focus on proving the necessity of the above conditions. For the sufficiency part, the argument in [Mil21, Section 4.1] toward the end of proposition 4.2 can be applied.

We divide our discussion of the proof in the following subsections.

3.4.1 Some useful inequalities

In this subsection, we lay out estimate of some relevant quantities that we shall use in the next subsection. We shall resume the assumption on the depth of λ stated in Theorem 3.4.2 throughout our discussion after the first lemma below. Note that we can assume throughout that $w \in W_a$, i.e. the length zero component of w is $1 \in \Omega$.

Lemma 3.4.5. *Let $w = t^\lambda y$ be an element of $\widetilde{W}$ with λ dominant and $w \in {}^{\mathbb{S}}\widetilde{W}$. Suppose $w \gtrdot w'$. Then we must have $w' = t^{m\alpha^\vee} s_\alpha w$ for some affine reflection $t^{m\alpha^\vee} s_\alpha$ with $1 \leq m \leq \langle \alpha, \lambda \rangle$.*

Proof. Let $w = s_{i_1} \cdots s_{i_l}$ be a reduced expression, where $s_{i_j} \in \tilde{\mathbb{S}}$. This describes a reduced gallery from $\mathbf{a}$ to $w\mathbf{a}$ via the chain of adjacent alcoves $\mathbf{a} \to s_{i_1}\mathbf{a} \to \cdots \to s_{i_1} \cdots s_{i_l}\mathbf{a}$. Since the gallery is reduced, it cannot cross any hyperplane twice - hence all the alcoves lie in the dominant chamber. Let r_j be the affine reflection with respect to the common wall between $\mathbf{a}_{j-1} := s_{i_1} \cdots s_{i_{j-1}} \mathbf{a}$ and $\mathbf{a}_j := s_{i_1} \cdots s_{i_j} \mathbf{a}$, that is $r_j = s_{i_1} \cdots s_{i_{j-1}} s_{i_j} s_{i_{j-1}} \cdots s_{i_1}$.

Now suppose that $w' = s_{i_1} \cdots \widehat{s_{i_k}} \cdots s_{i_l}$ is a cocover; as before, let $\{\mathbf{a}'_j : 1 \leq j \leq l-1\}$ be the collection of alcoves describing the reduced gallery corresponding to this expression of w' and let r'_j be the associated affine reflections. Then we see that $r_j = r'_j$ and $\mathbf{a}_j = \mathbf{a}'_j$ for $1 \leq j \leq k-1$, and the rest of the gallery for w' is the reflection of portion of the gallery for w from $\mathbf{a}_{k+1}$ onward with respect to the hyperplane corresponding to r_k. This is because for $j \geq k+1$ we have

$$\mathbf{a}'_{j-1} = s_{i_1} \cdots s_{i_{k-1}} s_{i_{k+1}} \cdots s_{i_j} \mathbf{a} = s_{i_1} \cdots s_{i_{k-1}} s_{i_k} s_{i_{k-1}} \cdots s_{i_1} (s_{i_1} \cdots s_{i_j} \mathbf{a}) = r_k(\mathbf{a}_j).$$

In particular, $w' = r_k w$.

Therefore, to create a cocover of w, we must reflect $w\mathbf{a}$ with respect to some hyperplane $H_{\alpha-m}$ lying between $\mathbf{a}$ and itself. Since the number of hyperplanes between the alcove $w\mathbf{a}$ and the wall H_α is $\langle \alpha, \lambda \rangle$, this restricts the possibility of m to asserted values above. □

Now, $w' = t^{m\alpha^\vee} s_\alpha w = s_\alpha t^{\lambda - m\alpha^\vee} y$. By Lemma 3.4.5, the coweight associated to the translation part of w' belongs to following list of coweights:

$$\lambda - \alpha^\vee, \lambda - 2\alpha^\vee, \cdots, \lambda - (\langle \alpha, \lambda \rangle - 1)\alpha^\vee = s_\alpha(\lambda - \alpha^\vee), \lambda - \langle \alpha, \lambda \rangle \alpha^\vee = s_\alpha(\lambda). \qquad (3.4.3)$$

Let us now define

$$k = k_{\lambda,\alpha} := \max\{m : \lambda - m\alpha^\vee \in \overline{\mathcal{C}^+}, 1 \leq m \leq \langle \alpha, \lambda \rangle, m \in \mathbb{Z}\}.$$

For a chamber $\mathcal{C}_x$, we denote its closure by $\overline{\mathcal{C}_x}$. We now give a lower bound for k.

Lemma 3.4.6. *We have that*

$$k \geq \begin{cases} 1, & \text{if } W \text{ is of simply laced type;} \\ 2, & \text{if } W \text{ is of non-simply laced type.} \end{cases}$$

Proof. By the definition of k, there is a real number $\widetilde{m}$ with $k \leq \widetilde{m} < k+1$ such that $\lambda - \widetilde{m}\alpha^\vee$ lies on at least one of the walls of $\mathcal{C}^+$. It follows that for some simple root α_i, we have $\langle \alpha_i, \lambda - \widetilde{m}\alpha^\vee \rangle = 0$. Therefore, $\widetilde{m} = \frac{\langle \alpha_i, \lambda \rangle}{\langle \alpha_i, \alpha^\vee \rangle}$ - hence giving $k = \lfloor \frac{\langle \alpha_i, \lambda \rangle}{\langle \alpha_i, \alpha^\vee \rangle} \rfloor$. Recalling the estimate about maximum value of $\langle \alpha_i, \alpha^\vee \rangle$ in each Cartan type, we get the desired bound on k from the depth condition on λ. □

Our next lemma estimates the length of the translation elements corresponding to the coweights in the list (6.1). This proof closely follows a part of the proof of [Mil21, Proposition 4.2]. We include it here for completeness' sake.

Lemma 3.4.7. *Suppose that for some integer $m \in [1, \langle \alpha, \lambda \rangle]$, $\lambda - m\alpha^\vee$ lies in the closure of a chamber different from $\mathcal{C}^+$ or $\mathcal{C}_{s_\alpha}$. Then $\ell(t^{\lambda - m\alpha^\vee}) \leq \ell(t^{\lambda - k\alpha^\vee})$.*

Proof. Following [LS10], define the function $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ by linearly extending the function $\widetilde{f} : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ defined by

$$\widetilde{f}(m) = \ell(t^{\lambda - m\alpha^\vee}).$$

More precisely, f is the function associated to the graph obtained by joining $\widetilde{f}(m)$ and $\widetilde{f}(m+1)$ by the line passing through them for every $m \in \mathbb{Z}$. It is easy to see that f is a convex function, cf. [LS10], proof of proposition 4; in fact, it is a piece-wise linear function, and is given by a single expression linear in m as long as $\lambda - m\alpha^\vee$ is in the same chamber. Since $\lambda, \lambda - \alpha^\vee$ are both dominant due to the imposed depth hypothesis, we see that $f(1) = \langle 2\rho, \lambda - \alpha^\vee \rangle, f(\langle \alpha, \lambda \rangle) = \langle 2\rho, \lambda \rangle$.

We now show that f is decreasing around 1 and increasing around $\langle \alpha, \lambda \rangle$. For example, if $m \in (0, \frac{3}{2})$ and $\alpha_i \in \Delta$ then by our earlier discussion about maximum value of $\langle \alpha_i, \alpha^\vee \rangle$ we have

$$\langle \alpha_i, \lambda - m\alpha^\vee \rangle = \langle \alpha_i, \lambda \rangle - m\langle \alpha_i, \alpha^\vee \rangle \geq \begin{cases} \text{depth}(\lambda) - 3, & \text{if } W \text{is not of type } G_2; \\ \text{depth}(\lambda) - \frac{9}{2}, & \text{if } W \text{is of type } G_2. \end{cases}$$

Hence, our depth hypothesis ensures that $\langle \alpha_i, \lambda - m\alpha^\vee \rangle \geq 0$. Therefore, $\lambda - m\alpha^\vee$ is in $\overline{\mathcal{C}^+}$ whenever $m \in (0, \frac{3}{2})$, and thus

$$f(m) = \langle 2\rho, \lambda - m\alpha^\vee \rangle = \langle 2\rho, \lambda \rangle - m\langle 2\rho, \alpha^\vee \rangle \tag{3.4.4}$$

is clearly decreasing in this neighbourhood. Similarly, we note that if $m \in (\langle \alpha, \lambda \rangle - \frac{1}{2}, \langle \alpha, \lambda \rangle + \frac{1}{2})$ and $\alpha_i \in \Delta$ we have

$$\langle \alpha_i, s_\alpha(\lambda - m\alpha^\vee) \rangle = \langle \alpha_i, \lambda \rangle - (\langle \alpha, \lambda \rangle - m)\langle \alpha_i, \alpha^\vee \rangle \geq \begin{cases} \text{depth}(\lambda) - 1, & \text{if } W \text{is not of type } G_2; \\ \text{depth}(\lambda) - \frac{3}{2}, & \text{if } W \text{is of type } G_2. \end{cases}$$

Again, our depth hypothesis ensures that $\langle \alpha_i, s_\alpha(\lambda - m\alpha^\vee)\rangle > 0$. Hence $\lambda - m\alpha^\vee \in \mathcal{C}_{s_\alpha}$ whenever $m \in (\langle \alpha, \lambda\rangle - \frac{1}{2}, \langle \alpha, \lambda\rangle + \frac{1}{2})$, and thus

$$f(m) = \langle 2\rho, s_a(\lambda - m\alpha^\vee)\rangle = \langle 2\rho, \lambda\rangle - (\langle \alpha, \lambda\rangle - m)\langle 2\rho, \alpha^\vee\rangle = \langle 2\rho, s_a(\lambda)\rangle + m\langle 2\rho, \alpha^\vee\rangle \quad (3.4.5)$$

is clearly increasing in the neighbourhood.

Since the intervals $(0, \frac{3}{2})$ and $(\langle \alpha, \lambda\rangle - \frac{1}{2}, \langle \alpha, \lambda\rangle + \frac{1}{2})$ are disjoint, by convexity of f we infer the global shape of the graph of f. Namely, $f(m)$ steadily decreases and is defined by Equation (3.4.4) as long as $\lambda - m\alpha^\vee \in \overline{\mathcal{C}^+}$; as $\lambda - m\alpha^\vee$ traverses through other chambers with value of m increasing, it (weakly) decreases further until it reaches local minimum, then it starts increasing; and finally, once $\lambda - m\alpha^\vee$ enters $\overline{\mathcal{C}_{s_\alpha}}$, $f(m)$ is defined by Equation (3.4.5) and steadily increases. Therefore, if we define

$$k' = k'_{\lambda,\alpha} := \min\{m : \lambda - m\alpha^\vee \in \overline{\mathcal{C}_{s_\alpha}}, 1 \leq m \leq \langle \alpha, \lambda\rangle, m \in \mathbb{Z}\},$$

we must have $f(m) \leq \max\{(f(k), f(k')\}$ for all $m \in [k, k'] \cap \mathbb{Z}$. Since $\lambda - k\alpha^\vee = s_\alpha(\lambda - k'\alpha^\vee)$ by symmetry, we have $f(k) = f(k') = \ell(t^{\lambda - k\alpha^\vee})$. Hence we are done. □

Recall from Lemma 3.4.5 that $m \leq \langle \alpha, \lambda\rangle$. We now define $z \in W$ to be such that $\lambda - m\alpha^\vee \in \overline{\mathcal{C}_z}$. Note that if $\lambda - m\alpha^\vee$ is singular, z is not uniquely specified by this condition. However, the precise choice of z would be immaterial in what follows, and henceforth fix one such z satisfying the above condition. Let us denote the left and right weak Bruhat order by $\prec_{\text{left}}$ and $\prec_{\text{right}}$ respectively.

Lemma 3.4.8. *We have $z \prec_{right} s_\alpha$. As a consequence, $\ell(s_\alpha z) + \ell(z) = \ell(s_\alpha)$.*

Proof. It suffices to show that $\mathrm{Inv}(z^{-1}) \subset \mathrm{Inv}(s_\alpha)$, because this is equivalent to $z^{-1} \prec_{\mathrm{left}} s_\alpha$, which in turn is equivalent to the claim above.

We start by noting that for $\beta \in \mathrm{Inv}(z^{-1})$, we have $\langle \beta, \lambda - m\alpha^\vee \rangle \leq 0$. This follows from rewriting $\langle \beta, \lambda - m\alpha^\vee \rangle = \langle z^{-1}\beta, z^{-1}(\lambda - m\alpha^\vee) \rangle$ and noting that $z^{-1}(\lambda - m\alpha^\vee)$ is a dominant coweight and $z^{-1}\beta \in -\Phi^+$. Therefore we get $\langle \beta, \lambda \rangle \leq m\langle \beta, \alpha^\vee \rangle$. Since λ dominant regular, $\langle \beta, \lambda \rangle > 0$ and hence $\langle \beta, \alpha^\vee \rangle > 0$ as well. Dividing out both side of the inequality by $\langle \beta, \alpha^\vee \rangle$, we get $m \geq \frac{\langle \beta, \lambda \rangle}{\langle \beta, \alpha^\vee \rangle}$. Combining this with Lemma 3.4.5, we obtain

$$\langle \alpha, \lambda \rangle \geq \frac{\langle \beta, \lambda \rangle}{\langle \beta, \alpha^\vee \rangle}.$$

But then $\langle \beta, \alpha^\vee \rangle \langle \alpha, \lambda \rangle \geq \langle \beta, \lambda \rangle$, whence $\langle s_\alpha(\beta), \lambda \rangle = \langle \beta - \langle \beta, \alpha^\vee \rangle \alpha, \lambda \rangle \leq 0$. Since λ is dominant regular, this gives $s_\alpha(\beta) \in -\Phi^+$, hence $\beta \in \mathrm{Inv}(s_\alpha)$. Thus we have shown $\mathrm{Inv}(z^{-1}) \subset \mathrm{Inv}(s_\alpha)$.

Finally, let us note that

$$\ell(s_\alpha z) + \ell(z) = \ell(s_\alpha) - \ell(z) + 2|\mathrm{Inv}(s_\alpha)^c \cap \mathrm{Inv}(z^{-1})| + \ell(z) = \ell(s_\alpha) + 2|\mathrm{Inv}(s_\alpha)^c \cap \mathrm{Inv}(z^{-1})| = \ell(s_\alpha).$$

This finishes the proof. $\square$

3.4.2 Two distinct possibilities

In this subsection we further pin down the possible values of z. Recall that $z^{-1}(\lambda - m\alpha^\vee)$ is dominant; let us temporarily denote this coweight by μ. Let $J \subset \Delta$ be such that $\mathrm{Stab}(\mu) = W_J$, the associated standard parabolic subgroup. Abbreviate $\zeta = z^{-1}y$.

By standard fact about Coxeter groups, ζ has an unique factorization as $\zeta = \zeta_J \zeta^J$ with $\zeta_J \in W_J$ and $\zeta^J \in {}^J W$. Note that

$$w' = s_\alpha t^{\lambda - m\alpha^\vee} y = s_\alpha z t^\mu z^{-1} y = s_\alpha z t^\mu \zeta_J \zeta^J = s_\alpha z \zeta_J t^\mu \zeta^J. \tag{3.4.6}$$

Lemma 3.4.9. *The alcove $t^\mu \zeta^J \mathbf{a}$ is in the dominant chamber.*

Proof. Define $\bar{\omega} := \frac{1}{n} \sum_{i=1}^n \frac{\omega_i^\vee}{n_i}$, where n_i is the coefficient of α_i in the highest root θ. Since $\mu + \zeta^J \bar{\omega}$ is centroid of the alcove $t^\mu \zeta^J \mathbf{a}$, we can make the following observation: *$t^\mu \zeta^J \mathbf{a}$ is in the dominant chamber if and only if $\mu + \zeta^J \bar{\omega}$ is dominant.*

We now show that the latter statement holds true. Note that if $\alpha_i \in J$, then $\langle \alpha_i, \mu + \zeta^J \bar{\omega} \rangle = \langle (\zeta^J)^{-1} \alpha_i, \bar{\omega} \rangle > 0$, since by definition $s_i \zeta^J > \zeta^J$, or equivalently $(\zeta^J)^{-1} s_i > (\zeta^J)^{-1}$, thereby giving $(\zeta^J)^{-1} \alpha_i \in \Phi^+$. Now let $\alpha_i \in \Delta \setminus J$, then $\langle \alpha_i, \mu \rangle \geq 1$; since $(\zeta^J)^{-1} \alpha_i \geq -\theta$, we have $\langle \alpha_i, \mu + \zeta^J \bar{\omega} \rangle \geq 1 + \langle -\theta, \bar{\omega} \rangle = 1 - 1 = 0$. This proves the claim. □

Applying the previous lemma, we get from Equation (3.4.6)

$$\ell(w') = \ell(s_\alpha z \zeta_J) + \langle 2\rho, \mu \rangle - \ell(\zeta^J).$$

Therefore the cocover condition gives

$$\langle 2\rho, \lambda \rangle - \ell(y) - 1 = \ell(s_\alpha z \zeta_J) + \langle 2\rho, \mu \rangle - \ell(\zeta^J).$$

In other words, we get

$$\ell(t^\lambda) - \ell(t^\mu) = 1 + \ell(s_\alpha z \zeta_J) + \ell(y) - \ell(\zeta^J). \tag{3.4.7}$$

Write $y = z \cdot z^{-1}y = z\zeta_J\zeta^J$. Note that Equation (2.1.2) then gives

$$\ell(y) = \ell(z \cdot \zeta_J\zeta^J) = \ell(z) + \ell(\zeta_J) + \ell(\zeta^J) - 2|\mathrm{Inv}(z) \cap \mathrm{Inv}((\zeta^J)^{-1}\zeta_J^{-1})|.$$

Similarly,

$$\ell(s_a z\zeta_J) = \ell(s_\alpha z) - \ell(\zeta_J) + 2|\mathrm{Inv}(s_\alpha z)^c \cap \mathrm{Inv}(\zeta_J^{-1})|$$

Hence we can rewrite the right hand side of Equation (3.4.7) as

$$1 + \ell(s_\alpha z) + \ell(z) + 2\{|\mathrm{Inv}(s_\alpha z)^c \cap \mathrm{Inv}(\zeta_J^{-1})| - |\mathrm{Inv}(z) \cap \mathrm{Inv}((\zeta^J)^{-1}\zeta_J^{-1})|\}.$$

Since $\zeta_J^{-1} \prec_{\text{left}} (\zeta^J)^{-1}\zeta_J^{-1}$, we have that $\mathrm{Inv}(\zeta_J^{-1}) \subset \mathrm{Inv}((\zeta^J)^{-1}\zeta^{-1})$. Combining this with the fact that $|A| - |B| \leq |A \setminus B|$ for two sets A, B, we therefore conclude that

$$\ell(t^\lambda) - \ell(t^\mu) \leq 1 + \ell(s_\alpha z) + \ell(z) + 2|\mathrm{Inv}(s_\alpha z)^c \cap \mathrm{Inv}(z)^c \cap \mathrm{Inv}(\zeta_J^{-1})|. \tag{3.4.8}$$

Lemma 3.4.10. *We have $Inv(s_\alpha z)^c \cap Inv(z)^c \cap Inv(\zeta_J^{-1}) = \emptyset$.*

Proof. The argument here is similar to the proof of Lemma 3.4.8. Suppose $\beta \in \mathrm{Inv}(s_\alpha z)^c \cap \mathrm{Inv}(z)^c \cap \mathrm{Inv}(\zeta_J^{-1}) \subset \mathrm{Inv}(s_\alpha z)^c \cap \mathrm{Inv}(z)^c \cap \Phi_J^+$. Thus $\langle \beta, z^{-1}(\lambda - m\alpha^\vee)\rangle = 0$, hence $\langle z\beta, \lambda\rangle =$

$m\langle z\beta, \alpha^\vee\rangle$. Since $z\beta$ is a positive root, both sides are positive and

$$m = \frac{\langle z\beta, \lambda\rangle}{\langle z\beta, \alpha^\vee\rangle} \leq \langle \alpha, \lambda\rangle.$$

Therefore, $\langle z\beta, \lambda\rangle - \langle z\beta, \alpha^\vee\rangle\langle \alpha, \lambda\rangle = \langle z\beta, s_\alpha(\lambda)\rangle \leq 0$. Since $\beta \in \mathrm{Inv}(s_\alpha z)^c$, we have $s_\alpha(z\beta) \in \Phi^+$. Hence we have $\langle z\beta, s_\alpha(\lambda)\rangle = \langle s_\alpha(z\beta), \lambda\rangle \leq 0$, but that is a contradiction since λ is dominant regular. This shows that the purported set must be empty and we are done. □

Therefore, combining Equation (3.4.8) with Lemma 3.4.8 and Lemma 3.4.10 gives

$$\ell(t^\lambda) - \ell(t^\mu) \leq 1 + \ell(s_\alpha). \tag{3.4.9}$$

Now, suppose that $z \neq 1, s_\alpha$. By Lemma 3.4.7 this gives

$$\ell(t^\lambda) - \ell(t^\mu) \geq \ell(t^\lambda) - \ell(t^{\lambda - k\alpha^\vee}) = k\langle 2\rho, \alpha^\vee\rangle.$$

Combining this with Equation (3.4.9), we get

$$k\langle 2\rho, \alpha^\vee\rangle \leq 1 + \ell(s_\alpha) \leq \langle 2\rho, \alpha^\vee\rangle$$

This gives $k = 1$. For the non-simply laced types, this is a contradiction with previously established lower bound in Lemma 3.4.6. Therefore in such cases, we get that $z \in \{1, s_\alpha\}$.

Before going forward, we make the following observation about a simply laced root system:

$$\text{if } \langle \beta, \alpha^\vee \rangle = 2 \text{ for a fixed coroot } \alpha, \text{ then } \beta = \alpha.$$

This can be checked directly in type A_n and D_n, where the positive roots are given by $\{e_i - e_j : 1 \leq i < j \leq n\}$ and $\{e_i \pm e_j : 1 \leq i < j \leq n\} \cup \{e_i + e_n : 1 \leq i < n\}$ respectively. Since root systems of type E_6, E_7 arise as subsystem of type E_8, we just give an argument for root system of type E_8 to conclude. Note that for root system of type E_8, the positive roots are of two kinds - given by $\{\pm e_i + e_j : 1 \leq i < j \leq 8\}$, and $\{\frac{1}{2}(e_8 + \sum_{i=1}^{7}(-1)^{\nu(i)} e_i)$: $\sum_{i=1}^{7} \nu(i) \in 2\mathbb{Z}\}$. We can easily see that pairing between a root γ from the first set with a coroot $\alpha^\vee$ corresponding to elements of either sets cannot be 2 unless $\gamma = \alpha$. The remaining possibility is $\langle \frac{1}{2}(e_8 + \sum_{i=1}^{7}(-1)^{\nu_1(i)} e_i), \frac{1}{2}(e_8 + \sum_{i=1}^{7}(-1)^{\nu_2(i)} e_i) \rangle = 2$, but that gives $\langle e_8 + \sum_{i=1}^{7}(-1)^{\nu_1(i)} e_i, e_8 + \sum_{i=1}^{7}(-1)^{\nu_2(i)} e_i \rangle = 8$ - therefore forcing $\nu_1 = \nu_2$.

We now resume the discussion about estimating k and restrict ourselves to the simply laced types. Recall that $k = 1$. This means that $\lambda - \alpha^\vee \in \overline{\mathcal{C}^+}$, but $\lambda - 2\alpha^\vee \notin \overline{\mathcal{C}^+}$; hence there exists $\beta \in \Phi^+$ such that $\langle \beta, \lambda - 2\alpha^\vee \rangle \leq 0$. By the depth condition, this yields $3 \leq \langle \beta, \lambda \rangle \leq 2\langle \beta, \alpha^\vee \rangle$. Therefore $\langle \beta, \alpha^\vee \rangle = 2$, and $\langle \beta, \lambda \rangle$ is equal to either 3 or 4. But this gives $\beta = \alpha$, and hence $\langle \alpha, \lambda \rangle$ is either 3 or 4. If $\langle \alpha, \lambda \rangle = 3$, the relevant coweights are $\lambda - \alpha^\vee, \lambda - 2\alpha^\vee = s_\alpha(\lambda - \alpha^\vee), \lambda - 3\alpha^\vee = s_\alpha(\lambda)$; the first one is in $\mathcal{C}^+$ by the depth condition, and the last two are in $\mathcal{C}_{s_\alpha}$. If $\langle \alpha, \lambda \rangle = 4$, then the relevant coweights are $\lambda - \alpha^\vee, \lambda - 2\alpha^\vee, \lambda - 3\alpha^\vee = s_\alpha(\lambda - \alpha^\vee), \lambda - 4\alpha^\vee = s_\alpha(\lambda)$; the first one is in $\mathcal{C}^+$, the last two are in $\mathcal{C}_{s_\alpha}$ and $\lambda - 2\alpha^\vee$ lies on the wall H_α, so it lies in both $\overline{\mathcal{C}^+}$ and $\overline{\mathcal{C}_{s_\alpha}}$.

We summarize the content of this subsection as follows.

Proposition 3.4.11. *If $s_\alpha t^{\lambda - m\alpha^\vee} y$ is a cocover of $t^\lambda y$, then $\lambda - m\alpha^\vee$ is either in $\overline{\mathcal{C}^+}$ or $\overline{\mathcal{C}_{s_\alpha}}$.*

3.4.3 Finishing the proof

Proof of Theorem 6.1. We deal with the two cases found above.

1. When $\lambda - m\alpha^\vee$ is in $\overline{\mathcal{C}^+}$: in this case, Equation (3.4.9) gives us

$$\langle 2\rho, \lambda\rangle - \langle 2\rho, \lambda - m\alpha^\vee\rangle \leq 1 + \ell(s_\alpha) \leq \langle 2\rho, \alpha^\vee\rangle.$$

Therefore we get $m \leq 1$; hence $m = 1$ and $1 + \ell(s_\alpha) = \langle 2\rho, \alpha^\vee\rangle$. This corresponds to the first case in Theorem 3.4.2.

2. When $\lambda - m\alpha^\vee$ is in $\overline{\mathcal{C}_{s_\alpha}}$: in this case, Equation (3.4.9) gives us

$$\langle 2\rho, \lambda\rangle - \langle 2\rho, s_\alpha(\lambda - m\alpha^\vee)\rangle \leq 1 + \ell(s_\alpha). \tag{3.4.10}$$

Since $\ell(s_a) \leq \langle 2\rho, \alpha^\vee\rangle - 1$, we thus get

$$(\langle \alpha, \lambda\rangle - m)\langle 2\rho, \alpha^\vee\rangle \leq \langle 2\rho, \alpha^\vee\rangle.$$

Therefore we get $\langle \alpha, \lambda\rangle - m \leq 1$. Since $m \leqslant \langle \alpha, \lambda\rangle$, this means that either $m = \langle \alpha, \lambda\rangle - 1$ or $m = \langle \alpha, \lambda\rangle$.

Suppose first that it is the former case. Then substituting this value of m in Equation (3.4.10) yields $\ell(s_\alpha) = \langle 2\rho, \alpha^\vee\rangle - 1$. Note that $\lambda - m\alpha^\vee = s_\alpha(\lambda - \alpha^\vee)$ is regular (since $\lambda - \alpha^\vee$ is regular dominant due to the depth hypothesis) and thus $J = \emptyset$; substituting $\mu = s_\alpha(\lambda - \alpha^\vee), z = s_\alpha, \zeta_J = 1, \zeta^J = s_\alpha y$ in Equation (3.4.7) therefore produces $\langle 2\rho, \alpha^\vee \rangle = 1 + \ell(y) - \ell(s_\alpha y)$, whence we get $\ell(s_\alpha y) = \langle 2\rho, \alpha^\vee \rangle - 1 - \ell(y) = \ell(s_\alpha) - \ell(y)$. This corresponds to the third case in Theorem 3.4.2.

Now, if it is the latter case we can carry out a similar substitution in Equation (3.4.7) to get $0 = 1 + \ell(y) - \ell(s_\alpha y)$; Equation (3.4.10) does not yield any information in this case. This gives $\ell(s_a y) = \ell(y) + 1$ and hence it corresponds to the second case in Theorem 3.4.2. This completes the proof.

□

Remark 3.4.12. We know a posteriori that only the first and the last two choices from the string of coweights in Equation (3.4.3) are viable - so we must necessarily put a depth condition on λ to ensure that $\lambda - \alpha^\vee$ is dominant; in that case, the first coweight is in the dominant chamber and the last two are in $\mathcal{C}_{s_\alpha}$. In other words, the least stringent hypothesis on λ under which one can expect to prove a result like this would be that $\lambda - \alpha^\vee$ is *dominant*. The depth condition that we impose above for the simply laced types is almost as weak an assumption as this, in the sense that we require $\lambda - \alpha^\vee$ to be *dominant regular*. Calculations in small rank seem to provide evidence for our this speculation. In other words, we suspect that the depth hypothesis can be lowered to 2 uniformly in all the cases.

3.4.4 Admissible subsets of $\widetilde{W}$

A useful description of the admissible set is given in [HY21] in terms of weight of minimal length paths in the quantum Bruhat graph. This relies on the characterization of covering relation in $\widetilde{W}$ as established in [Mil21, Proposition 4.2] and hence as such it necessarily brings into picture the superregularity condition, cf. [HY21, Proposition 3.3]. Since we can strengthen the result about covering relation, we automatically get the following improvement in the description of the admissible set.

Proposition 3.4.13. *Suppose that W is an irreducible Weyl group. Assume that*

$$depth(\mu) \geq \begin{cases} 3, & \text{if } W \text{ is of simply laced type;} \\ 4, & \text{if } W \text{ is of non-simply laced type but not of type } G_2. \\ 6, & \text{if } W \text{ is of type } G_2. \end{cases}$$

Let λ be dominant and assume that

$$\langle \rho, \mu - \lambda \rangle < \begin{cases} \lceil \frac{depth(\mu)-3}{2} \rceil, & \text{if } W \text{ is of simply laced type;} \\ \lceil \frac{depth(\mu)-4}{2} \rceil, & \text{if } W \text{ is of non-simply laced type but not of type } G_2; \\ \lceil \frac{depth(\mu)-6}{2} \rceil, & \text{if } W \text{ is of type } G_2. \end{cases}$$

Then $xt^{\lambda}y \in \mathrm{Adm}(\mu)$ if and only if $wt(x, y^{-1}) \leq \mu - \lambda$.

The proof of this proposition is identical to the argument made in [HY21], cf. section 3.3 and proof of proposition 3.3 in there and hence we do not repeat it here. The additional ingredient is Theorem 3.4.2, and the conditions on μ and λ are a direct reflection of that.

Chapter 4: A dimension formula for $X(\mu, b)$

The primary goal of this chapter is to investigate the dimension of the union $X(\mu, b)$ of affine Deligne-Lusztig varieties. Such a geometric object is the group-theoretic model for the Newton stratum associated with the element $[b]$ lying $B(\mathbf{G}, \mu)$. We remark that in the situations where the He-Rapoport axioms in [HR17] hold, this sought-after dimension is equal to the dimension of the associated Newton stratum minus the dimension of the corresponding central leaf, see [He16a, §2.12]. The said union of affine Deligne-Lusztig varieties can be defined purely in terms of group-theoretic data without any reference to Shimura varieties, and as before we resort to this latter setup. In Section 4.1, we obtain a dimension formula in the case of a quasi-split group, improving an earlier work of [HY21].

Let us now place ourselves in the context of a general group $\mathbf{G}$, and let $b_{\min}$ be the unique minimal (equivalently, basic) element of $B(\mathbf{G}, \mu)$. Then the work of Görtz, He and Rapoport in [GHR22] gives characterizations of when $X(\mu, b_{\min})$ can be of minimal dimension zero, or of maximal dimension $\langle 2\rho, \mu \rangle$; the model cases of such phenomena are the Lubin-Tate case and the Drinfeld case, respectively. Let us also remark that in [GHN20], Görtz, He and Nie provide a sharp lower bound for the dimension of $X(\mu, b_{\min})$, and they are also able to classify completely when this becomes an equality. Besides these results, there is no dimension formula available for $X(\mu, b)$, nor is there any conjectural description.

Standing at this juncture, it is therefore natural to investigate the dimension problem for the other extremal element of $B(\mathbf{G}, \mu)$, namely $b_{\max}$. In the rest of this chapter, we compute this dimension with a mild depth hypothesis on $\underline{\mu}$.

4.1 Dimension in the quasi-split case

In this section, we show that the dimension formula provided in [HY21, Theorem 6.1] holds in the absence of the superregularity condition imposed there. Our argument closely follows the proof scheme established in [HY21]. Therefore, we replace most proofs with references to the corresponding arguments made in loc. sit. and only point out how to bypass certain steps that will enable us to drop the superregularity condition. We first prove an enhanced version of [HY21, Proposition 4.4]. First, we set $d_{\mathrm{Adm}(\mu)}(b) = \max_{w \in \mathrm{Adm}(\mu)} d_w(b)$.

Proposition 4.1.1. *Assume that* $\mathbf{G}$ *is quasi-split. Let* $\underline{\mu}$ *be regular. Then*

$$d_{Adm(\mu)}(b) = \langle \rho, \underline{\mu} - \nu([b]) \rangle - \frac{1}{2}\mathrm{def}_{\mathbf{G}}(b) + \frac{1}{2}\ell(w_0) - \frac{1}{2}\min\{d_\Gamma(x, \sigma(x)w_0) : x \in W\}. \quad (4.1.1)$$

Proof. We first show that

(i) if $xt^{\underline{\lambda}}y \in \mathrm{Adm}(\mu)$, then $xt^{\underline{\lambda}+\underline{\mu}'}y \in \mathrm{Adm}(\mu+\mu')$ for any dominant $\underline{\mu}'$.

We write $xt^{\underline{\lambda}+\underline{\mu}'}y = xt^{\underline{\lambda}}yy^{-1}t^{\underline{\mu}'}y$, and note that $y^{-1}t^{\underline{\mu}'}y \in \mathrm{Adm}(\mu')$ by definition. Hence the claim follows from an application of Theorem 2.3.1.

Now we argue that statement (a) in the course of the proof in loc. sit. holds true in the absence of the superregularity condition. In other words, we need to show that

(ii) if $xt^{\underline{\lambda}}y \in \mathrm{Adm}(\mu)$ where $\underline{\mu}$ is regular, then $\langle \rho, \mathrm{wt}(x, y^{-1}) \rangle \leq \langle \rho, \underline{\mu} - \underline{\lambda} \rangle$.

Let $xt^{\underline{\lambda}}y \in \mathrm{Adm}(\mu)$. Choose $\underline{\mu}'$ to be superregular, e.g. $\mu' = n\rho^\vee$ for large enough n. Apply the statement in (i) to obtain $xt^{\underline{\lambda}+\underline{\mu}'}y \in \mathrm{Adm}(\mu+\mu')$. Now, $\underline{\mu}+\underline{\mu}'$ is superregular and thus statement (a) in loc. sit. applies to give us

$$\langle \rho, \mathrm{wt}(x, y^{-1})\rangle \leq \langle \rho, (\underline{\mu}+\underline{\mu}') - (\underline{\lambda}+\underline{\mu}')\rangle = \langle \rho, \underline{\mu}-\underline{\lambda}\rangle.$$

This finishes the proof of (ii). Therefore, one can use this enhanced version of statement (a) to get the established upper bound (i.e. the expression in the right hand side of Equation (4.1.1)) for $d_{\mathrm{Adm}}(\mu)$ by resorting to the same proof method in loc. sit.

In the remaining part of the proof, the authors construct an explicit $w \in \mathrm{Adm}(\mu) \cap \mathcal{C}_x$ for some $x \in W$ and argue that

$$d_w(b) = \langle \rho, \underline{\mu} - \nu([b])\rangle - \frac{1}{2}\mathrm{def}_{\mathbf{G}}(b) + \frac{1}{2}\ell(w_0) - \frac{1}{2}d_\Gamma(x, \sigma(x)w_0). \tag{4.1.2}$$

To prove this, they employ the description of $\mathrm{Adm}(\mu)$ established in proposition 3.3 in loc. sit. and hence it relies on the superregularity hypothesis. However, one can bypass this simply by choosing a different candidate. Namely, let $w = xt^{\mu}(\sigma(x)w_0)^{-1}$ for some $x \in W$ such that $\sigma(x)w_0 \geq x$. Let us first explain how this can be obtained using [HY21, Theorem 5.1].

Let $\mathcal{O}$ be the σ-conjugacy class of w_0 and define $\ell_R(\mathcal{O}) = \min\{\ell_R(w) : w \in \mathcal{O}\}$. Then the aforementioned theorem asserts that for any finite Coxeter group W and a length preserving graph automorphism on W, we have that

$$\ell(w_0) - \ell_R(\mathcal{O}) = 2\max\{\ell(x) : x \leq \sigma(x)w_0\}.$$

Therefore, the existence of our required element is equivalent to the assertion that $\ell(w_0) > \ell_R(\mathcal{O})$. The latter statement is true in all types arising from quasi-split groups, cf. section 5.1 in loc. sit.

Then we have $w \leq \sigma(x)w_0 t^{\mu}(\sigma(x)w_0)^{-1}$ by a combination of Equation (2.1.1) and Equation (2.1.3). This is where we use that μ is regular. Hence, $w \in \mathrm{Adm}(\mu)$. As has been shown in loc. sit., it now follows from definition that Equation (4.1.2) holds true for this element. This completes the proof that Equation (4.1.1) holds for dominant regular μ. □

Let us record an immediate consequence of the above result.

Corollary 4.1.2. *Suppose that* $\underline{\mu}$ *is regular. Then*

$$\dim X(\mu, b) \leq \langle \rho, \underline{\mu} - \nu([b]) \rangle - \frac{1}{2}\mathrm{def}_{\mathbf{G}}(b) + \frac{1}{2}\ell(w_0) - \frac{1}{2}\min\{d_\Gamma(x, \sigma(x)w_0) : x \in W\}.$$

Proof of Theorem 1.2.2[(1).] We note that once the formula for $d_{\mathrm{Adm}(\mu)}(b)$ in Equation (4.1.1) is established, the remaining part of [HY21] focuses on showing

$$\min\{d_\Gamma(x, \sigma(x)w_0) : x \in W\} = \ell_R(\mathcal{O}).$$

However, this is a calculation done purely on the finite Weyl group W, and the superregularity condition does not appear in establishing this.

Finally, the characterization of admissible sets is used once more in the proof of the theorem in section 6 in loc. sit. to show that $xt^{\mu}w_0\sigma(x)^{-1} \in \mathrm{Adm}(\mu)$ for certain element $x \in W$ satisfying $\sigma(x)w_0 \geq x$. However, this follows directly as we have shown above. Hence, the superregularity hypothesis on μ can be avoided completely. This finishes the proof. $\square$

Remark 4.1.3. We remark that [HY21, Remark 6.2] gives an example where μ is singular and W is of type C_4, and they point out that the dimension formula fails in this case. Therefore, the depth hypothesis on $\underline{\mu}$ is optimal, subject to the other condition. However, we do not know if the formula for $d_{\mathrm{Adm}}(\mu)$ given in Proposition 4.1.1 is valid in the absence of regularity condition on μ.

When **G** is split, we can prove the following statement.

Theorem 4.1.4. *The same assertion as above holds under the hypothesis that μ has depth at least 3 and $\mu \geq \nu([b]) + wt(w_0, 1)$.*

Note that $\mathrm{wt}(w_0, 1)$ is substantially smaller than $2\rho^{\vee}$, cf. Section 3.3; for instance, in type A_{2n}, we have $\mathrm{wt}(w_0, 1) = \varpi_n^{\vee} + \varpi_{n+1}^{\vee}$. We remark that unlike our above proof of the dimension formula in the quasi-split case, the proof of Theorem 4.1.4 is completely independent of the arguments in [HY21].

Basically, the restriction $\mu^{\diamond} \geq \nu(b) + 2\rho^{\vee}$ enters in the proof of [HY21][Theorem 6.1] because the dimension for the affine Deligne-Lusztig variety associated to $w := xt^{\mu}w_0\sigma(x)^{-1}$ - the element chosen in the proof of Theorem 6.1. in loc. sit. - can be asserted to be equal to $d_w(b)$ by [He21a] only under that stated restriction. Note that it is then shown for this element, we have $d_w(b) = d_{\mathrm{Adm}(\mu)}(b)$ - by which one concludes that $\dim X(\mu, b) = d_{\mathrm{Adm}(\mu)}(b)$.

In other words, for this particular choice of w, the associated affine Deligne-Lusztig variety has a known dimension by design, and also makes it to the top dimensional component of $X(\mu, b)$.

Proof. Instead of working with the element w as above, let us consider the element $w' := w_0 t^{\mu - \mathrm{wt}(w_0,1)}$. We claim that

(a) $w' \in \mathrm{Adm}(\mu)$.

Indeed, proceeding as in the proof of Proposition 3.2.6 and utilizing the decomposition of w_0 in Section 3.3, we have

$$t^{\mu - \mathrm{wt}(w_0,1)} \leq t^{\mu} w_0, \text{ and hence } w' \leq w_0 t^{\mu} w_0.$$

This last inequality needs depth at least 3 to ensure that $\mu - \mathrm{wt}(w_0, 1)$ is dominant regular, as well the coweights appearing in the intermediate steps (arising from application of the proof technique in Proposition 3.2.6) are dominant. Hence, $w' \in \mathrm{Adm}(\mu)$.

Now, we claim that

(b) $d_{w'}(b) = d_{\mathrm{Adm}(\mu)}(b)$.

To that end, let us compute

$$\begin{aligned} d_{w'}(b) &= \frac{1}{2}\{\ell(w') + \ell(w_0) - \mathrm{def}(b) - \langle 2\rho, \nu([b])\rangle\} \\ &= \frac{1}{2}\{\ell(w_0) + \langle 2\rho, \mu - \mathrm{wt}(w_0, 1)\rangle + \ell(w_0) - \mathrm{def}(b) - \langle 2\rho, \nu([b])\rangle\} \\ &= \frac{1}{2}\{\langle 2\rho, \mu - \nu(b)\rangle - \mathrm{def}(b)\} + \{\ell(w_0) - \langle \rho, \mathrm{wt}(w_0, 1)\rangle\}. \end{aligned}$$

The claim then would follow if we can show that the part in the second curly braces is equal to $\ell(w_0) - \ell_R(w_0)$. Note that Equation (5.0.2) gives $\langle 2\rho, \mathrm{wt}(w_0, 1)\rangle = \ell(w_0) + d(w_0, 1)$, but the explicit decomposition of w_0 in Section 3.3 shows that $d(w_0, 1) = \ell_R(w_0)$. Combining, we have $\ell(w_0) - \langle \rho, \mathrm{wt}(w_0, 1)\rangle = \ell(w_0) - \frac{1}{2}(\ell(w_0) + \ell_R(w_0)) = \frac{1}{2}\{\ell(w_0) - \ell_R(w_0)\}$.

Finally, we have that

(c) $\dim X_{w'}(b) = d_{w'}(b)$, whenever $X_{w'}(b) \neq \emptyset$.

By [MV20, Theorem 1.2] every element in the antidominant chamber is cordial. Therefore w' is cordial, and then above claim follows from [MV20, Corollary 3.17].

Finally, note further that the basic element in $B(\mathbf{G})$ satisfying $\kappa(b) = \kappa(w')$ is indeed the minimal element of $B(G)_{w'}$ by [GHN16][Theorem B]. Hence, $B(G)_{w'} = \{[b] : [b] \leq [b_{w'}]\}$. By [He21b, Theorem 4.2], we have $\nu([b_{w'}]) = \mu - \mathrm{wt}(w_0, 1)$. By Equation (3.2.2), this in turn enforces the condition $\mu \geq \nu([b]) + \mathrm{wt}(w_0, 1)$ in order to ensure that $X_{w'}(b) \neq \emptyset$. We are done. □

4.2 Expressing $b_{\max}$ via generic Newton point

In the rest of this chapter, we focus on the dimension problem for $X(\mu, b)$ associated to the maximal element $b = b_{\max}$ of $B(\mathbf{G}, \mu)$. We first identify $b_{\max}$ as the maximum of generic σ-conjugacy classes associated with the maximal translation elements in the μ-admissible set. Then we proceed to express such generic σ-conjugacy classes in two distinct ways, first in terms of σ-twisted Demazure power and then via a reduction to the quasi-split case. Such considerations allow us to express the dimension of individual affine Deligne-Lusztig varieties associated to such generic σ-conjugacy classes in terms of certain statistics on the quantum Bruhat graph. Further analysis in Section 4.2.5 shows that $\dim X(\mu, b_{\max})$ lies between the length and the reflection length of certain minimal length elements of some Frobenius-twisted conjugacy classes in the finite Weyl group. We then show that such minimal length elements are, in fact, partial Coxeter elements via a case-by-case calculation in Section 4.3, and hence their length (which is equal to their reflection length) matches the asserted dimension in Theorem 1.2.4, thereby finishing the proof. Throughout the rest of this chapter, we make use of the following strengthened formula for the generic Newton point (for quasi-split groups) as established in [HN21].

Theorem 4.2.1. *[HN21, Proposition 3.1] Suppose that* $\mathbf{G}$ *is quasi-split and adjoint over* F*. Let* $x, y \in W$ *and* $\underline{\mu} \in X_*(T)^+_{\Gamma_0}$ *be such that depth*$(\underline{\mu}) \geq 2$*. Then* $\nu_{xt^{\underline{\mu}}y}$ *is the average of the* σ*-orbit of* $\underline{\mu} - wt(y^{-1}, \sigma(x))$*.*

4.2.1 Dimension of a generic affine Deligne-Lusztig variety

Following [He21a], we define the *n-th σ-twisted Demazure power of* w by setting

$$w^{*\sigma,n} = w * \sigma(w) * \sigma^2(w) * \cdots * \sigma^{n-1}(w).$$

We need the following result, which on the one hand describes the dimension of a single affine Deligne-Lusztig variety associated with generic σ-conjugacy class, and on the other hand quantifies such generic σ-conjugacy class in terms of σ-twisted Demazure power. For the first equality below, see [He16a, Theorem 2.23] and [MV20, Lemma 3.2], whereas for the second equality, see [He21a, Theorem 0.1].

Theorem 4.2.2. *Let* $w \in \widetilde{W}$*. Then* $\dim X_w(b_w) = \ell(w) - \langle 2\rho, \nu_w \rangle = \ell(w) - \lim_{n\to\infty} \frac{\ell(w^{*\sigma,n})}{n}$.

Note that there can be elements $w \in \widetilde{W}$ that are not pure translations but still satisfy $b_w = b_{\max}$; this can be seen e.g. using the Deligne-Lusztig reduction method, see [He14, Proposition 4.2]. However, if w contributes to top dimensional components - i.e., $\dim X(\mu, b_{\max}) = \dim X_w(b_{\max})$ - then w is necessarily a translation element of the form $t^{x\underline{\mu}}$ for some $x \in W$, by the first equality in Theorem 4.2.2.

Now, again by an application of the first equality in Theorem 4.2.2 we see that $\dim X_{t^{x\underline{\mu}}}(b_{t^{x\underline{\mu}}})$ is a decreasing function of $b_{t^{x\underline{\mu}}}$; therefore, finding top dimensional $X_w(b_{\max})$ inside $X(\mu, b_{\max})$ boils down to understanding elements $x \in W$ such that $\dim X_{t^{x\underline{\mu}}}(b_{t^{x\underline{\mu}}})$ is minimized, and this minimum value of the dimension is indeed the dimension of $X(\mu, b_{\max})$. We approach the problem of computing $\dim X_{t^{x\underline{\mu}}}(b_{t^{x\underline{\mu}}})$ below in two different ways. In the rest of this chapter, we omit the underline for simplicity and write μ instead of $\underline{\mu}$ throughout.

4.2.2 A standard reduction

Let $\mathbf{G}$ be a connected reductive group over F, and let $\mathbf{G}_{\text{ad}}$ be its adjoint group. Let T_{ad} be the image of T in $\mathbf{G}_{\text{ad}}$, and denote by μ_{ad} the image of μ in $X_*(T_{\text{ad}})_{\Gamma_0}$. Similarly for any $b \in \breve{G}$, we denote by b_{ad} its image in $\breve{G}_{\text{ad}}$. By [Kot97, Proposition 4.10], we then have an isomorphism of posets

$$B(\mathbf{G}, \mu) \to B(\mathbf{G}_{\text{ad}}, \mu_{\text{ad}}), \text{ via } [b] \mapsto [b_{\text{ad}}].$$

Next, we have a decomposition $\mathbf{G}_{\text{ad}} \simeq \mathbf{G}_1 \times \cdots \times \mathbf{G}_r$, where each $\mathbf{G}_i$ is adjoint and simple over F. Write $\mu = (\mu_1, \cdots, \mu_r)$, where μ_i is a dominant coweight of $\mathbf{G}_i$. This way we can identify

$$B(\mathbf{G}_{\mathrm{ad}}, \mu) = \prod B(\mathbf{G}_i, \mu_i).$$

We also have

$$\dim X^{\mathbf{G}}(\mu, b) = \dim X^{\mathbf{G}_{\mathrm{ad}}}(\mu_{\mathrm{ad}}, b_{\mathrm{ad}}) = \sum \dim X^{\mathbf{G}_i}(\mu_{\mathrm{ad},i}, b_{\mathrm{ad},i}).$$

Hence, it suffices to focus on adjoint F-simple groups. We work with the absolute local Dynkin diagram (i.e., the affine Dynkin diagram attached to $\mathbf{G}$ over $\breve{F}$), together with the diagram automorphism induced by Frobenius.

We can describe such groups in terms of tuples $(\widetilde{W}, \sigma)$, where $\widetilde{W}$ is the associated Iwahori-Weyl group with its finite root datum being irreducible of adjoint type, and $\sigma = \mathrm{Ad}\ \tau \circ \sigma_0$ is a length preserving automorphism of $\widetilde{W}$ with $\tau \in \Omega, \sigma_0(\mathbb{S}) = \mathbb{S}$. Thus $\mathrm{Ad}\tau$ (resp. $\mathrm{Ad}\sigma_0$) corresponds to a symmetry of the affine (resp. finite) Dynkin diagram. More concretely, τ can be taken to be $\tau_i = t^{\varpi_i^\vee} w_{i,0} w_0$ whenever $\varpi_i^\vee$ is a minuscule fundamental coweight, and σ_0 can be nontrivial only when W is of type $A_n, D_n (n \geq 5)$ and E_6. Throughout Section 4.2 and Section 4.4, we denote by ζ the finite Weyl group part of τ, i.e. $\zeta = \zeta_i := w_{i,0} w_0$ for some minuscule fundamental coweight $\varpi_i^\vee$ - often suppressing the label i as well. We henceforth assume that $\mathrm{depth}(\mu) \geq 2$ throughout this chapter.

4.2.3 via Demazure power

Here we express $\langle 2\rho, \nu_{t^{x(\mu)}} \rangle$ explicitly using Demazure power of $t^{x(\mu)}$. We will apply the following result to do so.

Theorem 4.2.3. *[HN21, Proposition 3.3] Let $x_1t^{\mu_1}y_1, x_2t^{\mu_2}y_2$ be two elements of $\widetilde{W}$ such that the depth of μ_1, μ_2 is at least 2. Then $\mu_1 + \mu_2 - wt(y_1^{-1}, x_2)$ is dominant, and*

$$x_1t^{\mu_1}y_1 * x_2t^{\mu_2}y_2 = x_1t^{\mu_1+\mu_2-wt(y_1^{-1},x_2)}y_2.$$

We also need the following results, taken from [HY21, §4.2] and [HN21, §2.5].

Lemma 4.2.4. *Let $x, y \in W$. Then*

1. *We have $\langle wt(a,b), \alpha\rangle \leq 2$, for any simple root α.*
2. *$\ell(y) - \ell(x) = d(x,y) - \langle 2\rho, wt(x,y)\rangle$.*

To ease notational burden, we separately discuss the case with trivial σ_0 first. Suppose that $o(\zeta) = m$. Then we have

$$\begin{aligned}(t^{x\mu})^{*_\sigma, m+1} &= (xt^{\mu}x^{-1}) * (\zeta xt^{\mu}x^{-1}\zeta^{-1}) * \cdots * (\zeta^{m-1}xt^{\mu}x^{-1}\zeta^{1-m}) * (xt^{\mu}x^{-1}) \\ &= xt^{\mu_1}x^{-1}, \text{ with } \mu_1 := (m+1)\mu - \sum_{i=0}^{m-1} \mathrm{wt}(\zeta^i x, \zeta^{i+1}x).\end{aligned}$$

Here the second equality follows from a repeated application of Theorem 4.2.3. It is now easy to see that for any positive integer k, we have

$$(t^{x\mu})^{*_\sigma, km+1} = xt^{\mu_k}x^{-1}, \text{ with } \mu_k := (km+1)\mu - k\sum_{j=0}^{m-1} \mathrm{wt}(\zeta^j x, \zeta^{j+1}x).$$

Note that μ_k is dominant under the depth hypothesis on μ and we have $\frac{1}{km+1}\ell((t^{x\mu})^{*_\sigma, km+1}) =$

$\langle 2\rho, \mu\rangle - \frac{k}{km+1}\langle 2\rho, \sum\limits_{j=0}^{m-1} \mathrm{wt}(\zeta^j x, \zeta^{j+1}x)\rangle$. Since the limit in Theorem 4.2.2 exists, we deduce

$$\lim_{n\to\infty} \frac{1}{n}\ell((t^{x\mu})^{*_\sigma, n}) = \lim_{k\to\infty} \frac{1}{km+1}\ell((t^{x\mu})^{*_\sigma, km+1}) = \langle 2\rho, \mu\rangle - \frac{1}{m}\langle 2\rho, \sum_{j=0}^{m-1} \mathrm{wt}(\zeta^j x, \zeta^{j+1}x)\rangle.$$

Thus by Theorem 4.2.2, $\dim X_{t^{x\mu}}(b_{t^{x\mu}}) = \frac{1}{m}\langle 2\rho, \sum\limits_{j=0}^{m-1} \mathrm{wt}(\zeta^j x, \zeta^{j+1}x)\rangle$.

Now we consider the general case, where σ_0 may be nontrivial. Then

$$\sigma^j(t^{x\mu}) = \mathrm{Ad}((\zeta\sigma_0)^j)(t^{x\mu})) = \mathrm{Ad}(\zeta^{\sigma_0, j})(t^{\sigma_0^j(x\mu)}),$$

where we recall $\zeta^{\sigma_0, j} = \zeta\sigma_0(\zeta)\sigma_0^2(\zeta)\cdots\sigma_0^{j-1}(\zeta)$. Let m' denote the order of $\zeta\sigma_0$ inside $W \rtimes \langle\sigma_0\rangle$; since the group is adjoint, this is also the order of σ as an automorphism of the subset of translation elements in $\widetilde{W}$.

Then a similar calculation as above shows for any positive integer k,

$$(t^{x\mu})^{*_\sigma, km'+1} = xt^{\mu'_k}x^{-1}, \text{ with } \mu'_k = \mu + k\sum_{j=0}^{m'-1} \sigma_0^j(\mu) - k\sum_{j=0}^{m'-1} \mathrm{wt}(\zeta^{\sigma_0, j}\sigma_0^j(x), \zeta^{\sigma_0, j+1}\sigma_0^{j+1}(x)).$$

In other words, the dominant translation part of $(t^{x\mu})^{*_\sigma, km'+1}$ is

$$\mu'_k := \mu + km'\mu^\diamond - k\sum_{j=0}^{m'-1} \mathrm{wt}(\zeta\sigma_0)^j(x), (\zeta\sigma_0)^{j+1}(x)). \tag{4.2.1}$$

Noting that $\langle 2\rho, \mu\rangle = \langle 2\rho, \sigma_0^j(\mu)\rangle$ as 2ρ is $\langle\sigma_0\rangle$-invariant, we then proceed in the same way and conclude $\dim X_{t^{x\mu}}(b_{t^{x\mu}}) = \frac{1}{m'}\langle 2\rho, \sum\limits_{j=0}^{m'-1} \mathrm{wt}(\zeta\sigma_0)^j(x), (\zeta\sigma_0)^{j+1}(x))\rangle$.

Rewriting the dimension formula obtained above using Lemma 4.2.4, we deduce the following.

Proposition 4.2.5. *Let* $\mathbf{G}$ *be an adjoint* F*-simple group and* $x \in W$ *be an element in its Weyl group. Suppose that* $\sigma = Ad(\tau\sigma_0)$ *is the Frobenius, with* $\tau \in \Omega$ *and* $\sigma_0(\mathbb{S}) = \mathbb{S}$*. Let* ζ *be the finite Weyl group component of* τ*, and assume that depth*$(\mu) \geq 2$*. Finally, let* $o_{tr}(\sigma)$ *denote the order of* σ *as an element of* $\mathrm{Aut}(X_*(T)_{\Gamma_0})$*. Then*

$$\dim X_{t^{x\mu}}(b_{t^{x\mu}}) = \frac{1}{o_{tr}(\sigma)} \sum_{j=0}^{o_{tr}(\sigma)-1} d((\zeta\sigma_0)^j(x), (\zeta\sigma_0)^{j+1}(x)).$$

4.2.4 via reduction to quasi-split case

Recall that $b_{\max} = \max_{x \in W} b_{t^{x(\mu)}}$. For computation of the generic Newton point, we can further pass on to a quasi-split group in the following way. For an adjoint F-simple group $\mathbf{G}$ as in the previous subsection, let $\mathbf{G}^*$ be its quasi-split inner form; then the Frobenius action for $\mathbf{G}^*$ is given by $\sigma_{\mathbf{G}^*} = \sigma_0$. By [GHN16, §2.4], we can identify $B(\mathbf{G}) = B(\mathbf{G}^*)$ via $[b] \to [b\dot{\tau}]$, and this bijection preserves the partial order defined via the closure relations in $\breve{G} = \breve{G}^*$. Hence, restricting this map to a natural bijection $B(\mathbf{G})_w = B(\mathbf{G}^*)_{w\tau}$ for any $w \in \widetilde{W}(\mathbf{G}) = \widetilde{W}(\mathbf{G}^*)$, we then get $\nu^{\mathbf{G}}([b_w]) = \nu^{\mathbf{G}^*}([b_w\dot{\tau}]) = \nu^{\mathbf{G}^*}([b_{w\tau}])$.

We now give alternative (in fact, simpler) formula for the dimension by relating it to the quasi-split case.

Proposition 4.2.6. *In the setup of Proposition 4.2.5, we have* $\dim X_{t^{x\mu}}(b_{t^{x\mu}}) = d(\zeta^{-1}x, \sigma_0(x))$*.*

Proof. Assume first that $\mathrm{depth}(\mu) \geq 3$. Note that $t^{x\mu}\tau = t^{x\mu}t^{\varpi_i^\vee}w_{i,0}w_0 = xt^{\mu + x^{-1}\varpi_i^\vee}x^{-1}w_{i,0}w_0$.

Here $\mu + x^{-1}\varpi_i^\vee$ is dominant with depth at least 2: for any simple root α, we have

$$\langle \mu + x^{-1}\varpi_i^\vee, \alpha\rangle = \langle \mu, \alpha\rangle + \langle \varpi_i^\vee, x\alpha\rangle \geq \langle \mu, \alpha\rangle + \langle \varpi_i^\vee, -\theta\rangle = \langle \mu, \alpha\rangle - 1 \geq 2.$$

Hence, by Theorem 4.2.1 we have $\nu^{\mathbf{G}}(b_{t^{x\mu}}) = \nu^{\mathbf{G}^*}([xt^{\mu+x^{-1}\varpi_i^\vee}x^{-1}\zeta]) = (\mu + x^{-1}\varpi_i^\vee - \mathrm{wt}(\zeta^{-1}x, \sigma_0(x)))^\diamond$, giving us

$$\dim X_{t^{x\mu}}(b_{t^{x\mu}}) = \langle 2\rho, \mu - \nu(b_{t^{x(\mu)}})\rangle = \langle 2\rho, \mathrm{wt}(\zeta^{-1}x, \sigma_0(x)) - x^{-1}\varpi_i\rangle. \tag{4.2.2}$$

Note that we used $\langle\sigma_0\rangle$-invariance of 2ρ above. Now using the length formula, we get

$$\langle 2\rho, \mu\rangle = \ell(t^{x(\mu)}) = \ell(t^{x(\mu)}\tau) = \ell(xt^{\mu+x^{-1}\varpi_i^\vee}x^{-1}\zeta) = \ell(x) + \langle 2\rho, \mu + x^{-1}\varpi_i^\vee\rangle - \ell(x^{-1}\zeta),$$

giving us $\langle 2\rho, x^{-1}\varpi_i^\vee\rangle = \ell(x^{-1}\zeta) - \ell(x)$. Plugging this in Equation (4.2.2) and applying Lemma 4.2.4, we then get

$$\dim X_{t^{x\mu}}(b_{t^{x\mu}}) = \langle 2\rho, \mathrm{wt}(\zeta^{-1}x, \sigma_0(x))\rangle - \ell(x^{-1}\zeta) + \ell(x) = d(\zeta^{-1}x, \sigma_0(x)).$$

Combining Corollary 4.2.7 and Proposition 4.2.5, we see that this last formula is valid even under the weaker hypothesis that $\mathrm{depth}(\mu) \geq 2$. Hence, we are done. □

Let us record an immediate corollary of Proposition 4.2.6 and Proposition 4.2.5.

Corollary 4.2.7. *Let $x \in W$. Suppose that σ_0 is an automorphism of W with $\sigma_0(\mathbb{S}) = \mathbb{S}$, and $\zeta = \zeta_i = w_{i,0}w_0$ for some minuscule fundamental coweight $\varpi_i^\vee$. Finally, let $o_{tr}(\sigma) =$*

$\min\{j : \zeta^{\sigma_0,j} = 1\}$. *Then*

$$\frac{1}{o_{tr}(\sigma)} \sum_{j=0}^{o_{tr}(\sigma)-1} d((\zeta\sigma_0)^j(x), (\zeta\sigma_0)^{j+1}(x)) = d(\zeta^{-1}x, \sigma_0(x)).$$

4.2.5 Finishing the proof

As an consequence of the dimension formulas in the preceeding sections, we obtain the following estimate.

Proposition 4.2.8. *In the setup of Proposition 4.2.5, we have*

$$\ell(x^{-1}\zeta\sigma_0(x)) \geq \dim X_{t^{x\mu}}(b_{t^{x\mu}}) \geq \ell_R(x^{-1}\zeta\sigma_0(x)).$$

Proof. We can prove it using the dimension formula established in either Proposition 4.2.5 or Proposition 4.2.6, but we choose to resort to the latter for notational ease. Then the assertion follows from the easy observation:

(a) for any two elements $a, b \in W$, we have $\ell(a^{-1}b) \geq d(a, b) \geq \ell_R(a^{-1}b)$.

Taking any reduced expression for $a^{-1}b = s_{i_1} \cdots s_{i_k}$ and following the edges determined by the simple roots $\alpha_{i_k}, \cdots, \alpha_{i_1}$ in order, we obtain a path from a to b which has exactly $\ell(a^{-1}b)$ edges. This proves the first inequality. For the other one, let $d = d(a, b)$ and choose a shortest path $a \to ar_1 \to ar_1r_2 \cdots \to ar_1 \cdots r_d = b$, where r_i's are not necessarily simple roots. Then $a^{-1}b = r_1 \cdots r_d$ is a decomposition of $a^{-1}b$ into reflections, proving the other inequality. □

Let $\xi \in W$. Then we denote the σ_0 conjugacy class of ξ in W by $\mathcal{O}_{\sigma_0}(\xi)$. We define

$\ell_R(\mathcal{O}_{\sigma_0}(\xi)) = \min\{\ell_R(w) : w \in \mathcal{O}_{\sigma_0}(\xi)\}$. We can define $\ell(\mathcal{O}_{\sigma_0}(\xi))$ similarly.

Proof of Theorem 1.2.4. Since $\dim X(\mu, b_{\max}) = \min_{x\in W} \dim X_{t^{x\mu}}(b_{t^{x\mu}})$, we see from Proposition 4.2.8

$$\ell(\mathcal{O}_{\sigma_0}(\zeta)) \geq \dim X(\mu, b_{\max}) \geq \ell_R(\mathcal{O}_{\sigma_0}(\zeta)).$$

By Lemma 4.3.1 proved in Section 4.3, we have $\ell(\mathcal{O}_{\sigma_0}(\zeta)) = \ell_R(\mathcal{O}_{\sigma_0}(\zeta)) = |\sigma_0\backslash\widetilde{\mathbb{S}}| - |\sigma\backslash\widetilde{\mathbb{S}}|$, hence it equals the dimension. By Proposition 2.2.1, $|\sigma_0\backslash\widetilde{\mathbb{S}}| - |\sigma\backslash\widetilde{\mathbb{S}}| = \mathrm{rk}_F^{\mathrm{ss}}\mathbf{G}^* - \mathrm{rk}_F^{\mathrm{ss}}\mathbf{G}$, and we are done. □

Remark 4.2.9. As a byproduct of our discussion, we obtain that

$$\ell_R((O_{\sigma_0}(\zeta)) = \min_{x\in W} d(\zeta^{-1}x, \sigma_0(x))$$

for the element $\zeta = w_{i,0}w_0$, whenever $\varpi_i^\vee$ is minuscule. Contrast this with in [HY21, Theorem 5.1] that says

$$\ell_R(\mathcal{O}_{\sigma_0}(w)) = \min_{x\in W} d(x, \sigma_0(x)w)$$

for the element $w = w_0$. It would be an interesting problem to explore connection between reflection length and distance in the quantum Bruhat graph for arbitrary elements of W.

4.3 Minimal length elements in certain σ_0-conjugacy class

For simplicity, in the rest of this chapter, we will say σ_0-orbits in $\widetilde{\mathbb{S}}$ instead of $\mathrm{Ad}(\sigma_0)$-orbits, likewise for $\mathrm{Ad}(\sigma)$ etc. The goal of this section is to prove the following result via a case-by-case analysis.

Lemma 4.3.1. *Suppose that* $(W, \mathbb{S})$ *is an irreducible finite Coxeter system associated to a Weyl group* W *and let* $\widetilde{\mathbb{S}}$ *be the associated set of affine simple roots. Let* σ_0 *be a diagram automorphism of* $(W, \mathbb{S})$ *and* $\tau = t^{\varpi_i^\vee} w_{i,0} w_0 \in \widetilde{W}$*, where* $\varpi_i^\vee$ *is a minuscule fundamental coweight. Then there exists a subset* $J \subset \mathbb{S}$ *of cardinality* $|\sigma_0 \backslash \widetilde{\mathbb{S}}| - |\tau\sigma_0 \backslash \widetilde{\mathbb{S}}|$ *such that the set* $\mathcal{O}_{\min,\sigma_0}(w_{i,0}w_0)$ *of minimal length elements in* $\mathcal{O}_{\sigma_0}(w_{i,0}w_0)$ *consists of Coxeter elements of* W_J*. This subset* J *is unique, unless* $(W, \varpi_i^\vee)$ *is of the form* $({}^2D_{2n}, \varpi_{2n-1}^\vee)$ *or* $({}^2D_{2n}, \varpi_{2n}^\vee)$*.*

Proof. As our calculation below shows, the set J is either empty or W_J has only one Coxeter element in the cases where σ_0 is nontrivial. As for the cases where σ_0 is trivial, we only have to show some Coxeter element from asserted W_J belongs to $\mathcal{O}_{\min,\sigma_0}(w_{i,0}w_0)$ - since all such Coxeter elements are conjugate to each other.

We follow the labelling of [Bou02]. To simplify notation, in the exceptional types we may simply write $s_{ij\cdots}$ for $s_i s_j \cdots$. For $1 \leq a \leq b \leq n$, we abbreviate $s_{[a,b]} = s_a s_{a+1} \cdots s_b$. Below we group together the cases according to their absolute local Dynkin diagram.

1. Type A_n: Here all the fundamental coweights are minuscule and τ_i acts on $\widetilde{\mathbb{S}}$ by i-step rotations, thus the number of τ_i-orbits on $\widetilde{S}$ are $\frac{|\widetilde{\mathbb{S}}|}{|o(\mathrm{Ad}\ \tau_i)|} = \frac{n+1}{\gcd(n+1,i)}$.

   Here $\zeta_1 = s_{[1,n]}$ is a Coxeter element for W and $\zeta_i = \zeta_1^i$ for $i \geq 2$. Set $\kappa_i = \frac{n+1}{\gcd(n+1,i)}$ and let m be the largest integer smaller than $\frac{n}{\kappa_i}$, i.e. $m = \lfloor \frac{n}{\kappa_i} \rfloor$. Then we have

$$J_i = \{1, \cdots, \kappa_i - 1\} \cup \{\kappa_{i+1}, \cdots, 2\kappa_i - 1\} \cup \cdots \cup \{m\kappa_i + 1, \cdots, n\}.$$

2. Type ${}^2A_n$: The nontrivial diagram automorphism σ_0 of $\mathbb{S}$ is conjugation by w_0, hence minimal length elements in $\mathcal{O}_{\sigma_0}(w_{i,0}w_0)$ are maximal length elements in $\mathcal{O}_{\mathrm{id}}(w_{i,0})$

multiplied with w_0 on the right. The orbits of σ_0 on $\widetilde{\mathbb{S}}$ are

$$\begin{cases} \{0\}, \{i, n+1-i\} \text{ for } 1 \leq i \leq \frac{n}{2} & \text{, if } n \text{ is even;} \\ \{0\}, \{\frac{n+1}{2}\}, \{i, n+1-i\} \text{ for } 1 \leq i \leq \frac{n-1}{2} & \text{, if } n \text{ is odd.} \end{cases}$$

In this case, $\tau_i\sigma_0$-orbits on $\widetilde{\mathbb{S}}$ are $\{0, i\}, \{j, i-j\}$ for $0 < j < i$, and $\{i+j, n-j\}$ for $0 < j < n-i$. We have that $J = \emptyset$ if n is even or i is even, and $J = \{\frac{n+1}{2}\}$ if both n, i are odd.

3. Type B_n: Here the only minuscule coweight is $\varpi_1^\vee$, so $\tau = \tau_1$; its orbits are $\{0, 1\}, \{i\}$ for $2 \leq i \leq n$. We have $\zeta_1 = s_{[1,n-1]} s_n s_{[1,n-1]}^{-1}$, thus $J = \{n\}$.

4. Type C_n: Here the only minuscule coweight is $\varpi_n^\vee$, so $\tau = \tau_n$; its orbits are

$$\begin{cases} \{i, n+1-i\}, \text{ for } 0 \leq i \leq \frac{n-1}{2} & \text{, if } n \text{ is odd;} \\ \{\frac{n}{2}\}, \{i, n+1-i\}, \text{ for } 0 \leq i \leq \frac{n}{2} - 1 & \text{, if } n \text{ is even.} \end{cases}$$

We have $\zeta_n = s_n s_{[n-1,n]} \cdots s_{[2,n]} s_{[1,n]}$ and $J = \{i : 1 \leq i \leq n, i \text{ is odd}\}$.

5. Type D_n: Here we have three minuscule coweights: $\varpi_1^\vee, \varpi_{n-1}^\vee, \varpi_n^\vee$. There is an outer diagram automorphism of D_n permuting the last two coweights. Thus it suffices to consider the case where $\tau = \tau_1$ or τ_n.

Case 1: $\tau = \tau_1$. The τ-orbits on $\widetilde{\mathbb{S}}$ are $\{0, 1\}, \{n-1, n\}$ and $\{i\}$ for $2 \leq i \leq n-2$. We have $\zeta = \zeta_1 = s_{[1,n-2]} s_{[1,n]}^{-1}$ and $J = \{n-1, n\}$.

Case 2: $\tau = \tau_n$; then there are two sub-cases.

- n is odd: The τ-orbits on $\widetilde{\mathbb{S}}$ are $\{0, n, 1, n-1\}$ and $\{i, n-i\}$ for $2 \leq i \leq \frac{n-1}{2}$. Here $\zeta = \zeta_n = s_n s_{[n-2,n-1]} s_{[n-3,n-2]} s_n \cdots s_n s_{[3,n-1]} s_{[2,n-2]} s_n s_{[1,n-1]}$ and $J = \{i : 1 \leq i \leq n-2, i \text{ is odd}\} \cup \{n-1, n\}$.

- n is even: The τ-orbits on $\widetilde{\mathbb{S}}$ are $\{i, n-i\}$ for $0 \leq i \leq \frac{n}{2}$. Here $\zeta = \zeta_n = s_n s_{[n-2,n-1]} s_{[n-3,n-2]} s_n \cdots s_n s_{[2,n-1]} s_{[1,n-2]} s_n$ and

$$J = \begin{cases} \{i : 1 \leq i \leq n, i \text{ is odd}\}, & \text{, if } n \text{ is congruent to 0 mod 4;} \\ \{i : 1 \leq i \leq n-2, i \text{ is odd}\} \cup \{n\} & \text{, if } n \text{ is congruent to 2 mod 4.} \end{cases}$$

6. Type ${}^2D_n$: The nontrivial diagram automorphism σ_0 of $\mathbb{S}$ here swaps the vertices labeled $n-1, n$; the orbits on $\widetilde{\mathbb{S}}$ are therefore $\{n-1, n\}, \{i\}$ for $0 \leq i \leq n-2$. As explained before, we have the following three cases.

   Case 1: $\tau = \tau_1$. The $\tau\sigma_0$-orbits on $\widetilde{\mathbb{S}}$ are $\{0, 1\}$ and $\{i\}$ for $2 \leq i \leq n$. One may directly check that $1 \in \mathcal{O}_{\sigma_0}(\zeta_1)$ and hence $J = \emptyset$.

   Case 2: $\tau = \tau_n$; then there are two sub-cases.

   - n is odd: The $\tau\sigma_0$-orbits on $\widetilde{\mathbb{S}}$ are $\{i, n-i\}$ for $0 \leq i \leq \frac{n-1}{2}$, and $J = \{i : 1 \leq i \leq n-2, i \text{ is odd}\}$.

   - n is even: The $\tau\sigma_0$-orbits on $\widetilde{\mathbb{S}}$ are $\{0, 1, n-1, n\}$ and $\{i, n-i\}$ for $2 \leq i \leq \frac{n}{2}$, and the choices for J are $\{i : 1 \leq i \leq n, i \text{ is odd}\}$ and $\{i : 1 \leq i \leq n-2, i \text{ is odd}\} \cup \{n\}$.

7. Type ${}^3D_4$: Without loss of generality, we may assume that σ_0 is the outer diagram automorphism on D_4 sending $1 \to 3 \to 4 \to 1$. As $\langle\sigma_0\rangle$ acts transitively on $\{1, 3, 4\}$,

it suffices to consider the case where $\tau = \tau_1$. In that case, the orbits of $\tau\sigma_0$ on $\widetilde{\mathbb{S}}$ are $\{0, 1, 4\}, \{2\}, \{3\}$. Finally, one directly checks that $1 \in \mathcal{O}_{\sigma_0}(\zeta_1)$, whence $J = \emptyset$.

8. Type E_6: There are two minuscule coweights: $\varpi_1^\vee, \varpi_6^\vee$. The unique outer diagram automorphism of E_6 permutes these two coweights. Thus it suffices to consider the case where $\tau = \tau_1$; its orbits on $\widetilde{\mathbb{S}}$ are $\{0, 1, 6\}, \{2, 3, 5\}, \{4\}$. We have $\zeta = \zeta_1 = s_{1345624534132456}$, and $J = \{1, 3, 5, 6\}$.

9. Type ${}^2E_6$: As explained above, it suffices to consider the case where $\tau = \tau_1$. Here σ_0 acts on $\widetilde{\mathbb{S}}$ by swapping $1 \leftrightarrow 6, 3 \leftrightarrow 5$, so its orbit on $\widetilde{\mathbb{S}}$ are $\{0\}, \{1, 6\}, \{2\}, \{3, 5\}, \{4\}$. Also, the orbits of $\tau\sigma_0$ on $\widetilde{\mathbb{S}}$ are $\{0, 1\}, \{2, 3\}, \{4\}, \{5\}, \{6\}$. Finally, one directly checks that $1 \in \mathcal{O}_{\sigma_0}(\zeta_1)$, whence $J = \emptyset$.

10. Type E_7: There is a unique minuscule coweight: $\varpi_7^\vee$, so $\tau = \tau_7$. In this case, τ-orbits on $\widetilde{\mathbb{S}}$ are $\{0, 7\}, \{1, 6\}, \{3, 5\}, \{2\}, \{4\}$. Here $\zeta = \zeta_7 = \zeta = s_{765432456713456245341324567}$, and $J = \{2, 5, 7\}$.

□

Remark 4.3.2. In type A_n, the elements ζ_i are regular elements of the Weyl group. In that case, the asserted conjugacy relations follow more conceptually from [Spr74, theorem 4.2(4)].

4.4 Explicit description of the Newton point of $b_{\max}$

Note that the existence of $b_{\max}$ as in Theorem 2.2.3 was asserted in [HN18] via reduction to a similar statement on the associated Iwahori-Weyl group, and then they further reduce it to treat the superbasic case and conclude it from there. However, this is somewhat indirect and does not give an explicit formula for the maximal element. Based on our calculation, in this section we list out $\nu(b_{\max})$ for the cases treated in Lemma 4.3.1.

4.4.1

Recall that the calculation in Section 4.2 combined with Theorem 4.2.2 gives us

$$\langle 2\rho, \nu_{t^{x\mu}} \rangle = \langle 2\rho, \mu^{\diamond} - \mathrm{av}_{\sigma}(x) \rangle, \tag{4.4.1}$$

where we set $\mathrm{av}_{\sigma}(x) := \frac{1}{o_{\mathrm{tr}}(\sigma)} \sum_{i=0}^{o_{\mathrm{tr}}(\sigma)-1} \mathrm{wt}((\zeta\sigma_0)^i(x), (\zeta\sigma_0)^{i+1}(x))$ However, one can upgrade this.

Proposition 4.4.1. *Suppose that we are in the setup of Proposition 4.2.5. Let us further assume that $\nu_{t^{x\mu}}$ is regular. Then we have*

$$\nu_{t^{x\mu}} = \mu^{\diamond} - av_{\sigma}(x).$$

Lemma 4.4.2. *Let $w \in \widetilde{W}$. Then there is a straight element $w_s \in \widetilde{W}$ such that $w_s \leq w$ and $\nu(w_s) = \nu_w$.*

Proof. This essentially follows from the arguments in [He21a, section 2.2]. □

Proof of Proposition 4.4.1. Let $w' = t^{x(\mu)}$ and choose a straight element w_s corresponding to $w := w'\tau$ as given in Lemma 4.4.2. By the properties of the Demazure product and by definition of straight elements,

$$w_s^{\sigma_0,n-1} \leq w_s^{\sigma_0,n} = w_s^{*\sigma_0,n} \leq w^{*\sigma_0,n} \text{ for all } n \in \mathbb{N}. \tag{4.4.2}$$

Note that $w'^{*\sigma,n} = w^{*\sigma_0,n}$ whenever $(\tau\sigma_0)^n = 1$, i.e. $o_{\mathrm{tr}}(\sigma) \mid n$. Furthermore, for such n we have $\sigma^n(t^{x\mu}) = t^{x\mu}, \sigma_0^n = 1$, and hence

$$w^{*\sigma_0,n+1} = (w'\tau)^{*\sigma_0,n} * w'\tau = (t^{x\mu})^{*\sigma,n} * t^{x\mu}\tau = (t^{x\mu})^{*\sigma,n+1}\tau. \tag{4.4.3}$$

Let us recall the following well-known fact.

(a) Let $\mathrm{pr} : \widetilde{W} \to X_*(T)_{\Gamma_0}$ be the projection map, which sends any element w to the unique dominant coweight λ with $w \in Wt^\lambda W$. Assume that $w_1, w_2 \in \widetilde{W}$ such that $w_1 \leq w_2$ in Bruhat order. Then $(\mathrm{pr}(w_1))^+ \leq (\mathrm{pr}(w_2))^+$ in the dominance order.

Recall now from Section 4.2.3 that $o_{\mathrm{tr}}(\sigma) \mid o(\sigma)$; let $o(\sigma) = \Xi o_{\mathrm{tr}}(\sigma)$. Then the calculation in Section 4.2.3, combined with Equation (4.4.3), shows that for $n = ko(\sigma) + 1$ (with $k \in \mathbb{Z}_{\geq 0}$), we have

$$w^{*\sigma_0,n} = xt^{\mu''_k}x^{-1}\tau = x^{\mu''_k + x^{-1}\varpi_i^\vee}x^{-1}\zeta, \text{ with } \mu''_k = \mu + ko(\sigma)\mu^\diamond - k\Xi \sum_{j=0}^{o_{\mathrm{tr}}(\sigma)-1} \mathrm{wt}(\zeta\sigma_0)^i(x), (\zeta\sigma_0)^{i+1}(x)).$$

Since $\nu(w_s)$ is assumed to be regular, we have by [He14, proposition 2.4] that $w_s = yt^\gamma\sigma_0(y)^{-1}$ for some $y \in W$. Using this in Equation (4.4.2) with $n = ko(\sigma) + 1$ and then applying fact (a), we thus obtain

$$ko(\sigma)\nu(w_s) \leq \mu + x^{-1}\varpi^\vee + ko(\sigma)\mu^\diamond - k\Xi \sum_{j=0}^{o_{\mathrm{tr}}(\sigma)-1} \mathrm{wt}(\zeta\sigma_0)^i(x), (\zeta\sigma_0)^{i+1}(x)).$$

This gives

$$\nu_w = \nu(w_s) \leq \frac{1}{ko(\sigma)}(\mu + x^{-1}\varpi^\vee) + \mu^\diamond - \frac{1}{o_{\mathrm{tr}}(\sigma)} \sum_{j=0}^{o_{\mathrm{tr}}(\sigma)-1} \mathrm{wt}(\zeta\sigma_0)^i(x), (\zeta\sigma_0)^{i+1}(x)).$$

Letting k to be sufficiently large and noting that Newton points have bounded denominator, we obtain $\nu_w \leq \mu^\diamond - \frac{1}{o_{\mathrm{tr}}(\sigma)} \sum\limits_{j=0}^{o_{\mathrm{tr}}(\sigma)-1} \mathrm{wt}(\zeta\sigma_0)^i(x), (\zeta\sigma_0)^{i+1}(x))$. Hence by Equation (4.4.1), we are done.

□

4.4.2

To describe $\nu(b_{\max}) = \max\limits_{x \in W} \nu_{t^{x\mu}}$, it suffices by Proposition 4.4.1 to tabulate

$$\mu^\diamond - \nu(b_{\max}) = \min_{x \in W} \mathrm{av}_\sigma(x), \tag{4.4.4}$$

for the cases discussed in Lemma 4.3.1. Note that $(\zeta\sigma_0)^j(x))^{-1}(\zeta\sigma_0)^{j+1}(x) = \sigma_0^j(x^{-1}\zeta\sigma_0(x))$. By a combination of [Sad23, corollary 3.3] and [Sad23, lemma 3.1], we have $\mathrm{wt}(\zeta\sigma_0)^j(x), (\zeta\sigma_0)^{j+1}(x)) \leq \mathrm{wt}(\sigma_0^j(x^{-1}\zeta\sigma_0(x)), 1)$, whence

$$\mu^\diamond - \nu(b_{\max}) \leq \min_{x \in W} \frac{1}{o_{\mathrm{tr}}(\sigma)} \sum_{j=0}^{o_{\mathrm{tr}}(\sigma)-1} \mathrm{wt}(\sigma_0^j(x^{-1}\zeta\sigma_0(x)), 1).$$

Thus $\mu^\diamond = \nu(b_{\max})$ whenever $1 \in \mathcal{O}_{\min,\sigma_0}(\zeta)$. One can read off such cases from the list given in the proof of Lemma 4.3.1; those cases can also be directly seen to be corresponding to quasi-split groups. For the remaining cases, the proof of Theorem 1.2.4 shows that the minimum in Equation (4.4.4) is realized at some/any element $x \in W$ such that $x^{-1}\zeta\sigma_0(x)$ is a partial Coxeter element as described in Lemma 4.3.1. This way, one can evaluate $\mathrm{av}_\sigma(x)$ at such an element x to get the answer.

Alternatively, we can describe this minimum by applying the criterion in Theorem 2.2.3. Let us call $\Xi_\sigma = \min_{x \in W} \mathrm{av}_\sigma(x) = \mu^\diamond - \nu(b_{\max})$. Note that, when $\sigma = \mathrm{Ad}(\tau_i)$, the condition in Theorem 2.2.3 translates to

$$\Xi_\sigma = \min\{\xi : \langle \varpi_i^\vee + \xi, \varpi_j \rangle \in \mathbb{Z}, \langle \xi, \varpi_j \rangle \geq 0, \text{ for all } j\}.$$

Thus, we can simply use expressions of fundamental coweights in terms of simple coroots coming from the inverse of the Cartan matrix and find this quantity. For a real number m, we set $\{m\} = m - \lfloor m \rfloor$.

1. Type A_n: For $1 \leq i \leq n$, we have $\varpi_i^\vee = \sum_{j=1}^{n} a_{ij}\alpha_j^\vee$, where

$$a_{ij} = \min(i,j) - \frac{ij}{n+1} = \min(i,j) - \lfloor \frac{ij}{n+1} \rfloor - \{\frac{ij}{n+1}\}. \tag{4.4.5}$$

   Hence, $\Xi_\sigma = \sum_{j=1}^{n} \{\frac{ij}{n+1}\}\alpha_j^\vee$.

2. Type B_n: Here $\varpi_1^\vee = \sum_{j=1}^{n-1} \alpha_j^\vee + \frac{1}{2}\alpha_n^\vee$, whence $\Xi_\sigma = \frac{1}{2}\alpha_n^\vee$.

3. Type C_n: Here $\varpi_n^\vee = \sum_{j=1}^{n} \frac{j}{2}\alpha_j^\vee$, whence $\Xi_\sigma = \sum_{j \text{ odd}, 1 \leq j \leq n} \frac{1}{2}\alpha_j^\vee$.

4. Type D_n: Here the relevant coweights are $\varpi_1^\vee$ and $\varpi_n^\vee$.

   We have $\varpi_1^\vee = \sum_{j=1}^{n-2} \alpha_j^\vee + \frac{1}{2}(\alpha_{n-1}^\vee + \alpha_n^\vee)$, thus $\Xi_\sigma = \frac{1}{2}(\alpha_{n-1}^\vee + \alpha_n^\vee)$.

   On the other hand, $\varpi_n^\vee = \sum_{j=1}^{n-2} \frac{j}{2}\alpha_j^\vee + \frac{n-2}{4}\alpha_{n-1}^\vee + \frac{n}{4}\alpha_n^\vee$, whence

$$\Xi_\sigma = \begin{cases} \sum_{j \text{ odd}, 1\le j\le n-2} \frac{1}{2}\alpha_j^\vee + (1-\{\frac{n-2}{4}\})\alpha_{n-1}^\vee + (1-\{\frac{n}{4}\})\alpha_n^\vee, & \text{if } n \text{ is odd;} \\ \sum_{j \text{ odd}, 1\le j\le n-2} \frac{1}{2}\alpha_j^\vee + (1-\{\frac{n-2}{4}\})\alpha_{n-1}^\vee, & \text{if } n \text{ is congruent to 0 mod 4;} \\ \sum_{j \text{ odd}, 1\le j\le n-2} \frac{1}{2}\alpha_j^\vee + (1-\{\frac{n}{4}\})\alpha_n^\vee, & \text{if } n \text{ is congruent to 2 mod 4.} \end{cases}$$

5. Type E_6: Here $\varpi_1^\vee = \frac{1}{3}(4\alpha_1^\vee + 3\alpha_2^\vee + 5\alpha_3^\vee + 6\alpha_4^\vee + 4\alpha_5^\vee + 2\alpha_6^\vee)$, thus $\Xi_\sigma = \frac{1}{3}(2\alpha_1^\vee + \alpha_3^\vee + 2\alpha_5^\vee + 4\alpha_6^\vee)$.

6. Type E_7: Here $\varpi_7^\vee = \frac{1}{2}(2\alpha_1^\vee + 3\alpha_2^\vee + 4\alpha_3^\vee + 6\alpha_4^\vee + 5\alpha_5^\vee + 4\alpha_6^\vee + 3\alpha_7^\vee)$, thus $\Xi_\sigma = \frac{1}{2}(\alpha_2^\vee + \alpha_5^\vee + \alpha_7^\vee)$.

Finally, we have the following remaining cases where σ_0 is non-trivial:

1. Type ${}^2A_n$: As per our discussion above, we can assume that both n, i are odd. It is easy to see using Equation (4.4.5) that $\varpi_j + \varpi_{n+1-j} \in \mathbb{Z}_{\geq 0}\Delta$ for all $1 \leq j < \frac{n+1}{2}$. Then by Theorem 2.2.3, $\Xi_\sigma = \min\{\xi : \langle \varpi_i^\vee + \xi, \varpi_{\frac{n+1}{2}}\rangle \in \mathbb{Z}, \langle \xi, \varpi_{\frac{n+1}{2}}\rangle \geq 0, \sigma_0(\xi) = \xi\}$, whence again by Equation (4.4.5) applied to $j = \frac{n+1}{2}$ we get $\Xi_\sigma = \frac{1}{2}\alpha_{\frac{n+1}{2}}^\vee$.

2. Type ${}^2D_n$: As per our discussion above, we only need to deal with the case when $\tau = \tau_n$. Considerations similar to the case of type D_n shows that if $\Xi_\sigma = \sum_{j=1}^{n} x_j\alpha_j^\vee$, then $x_j = \frac{1}{2}$ for $1 \leq j \leq n-2$; to find the rest of the coefficients, we have to use the

conditions:

$$\sigma_0(\Xi_\sigma) = \Xi_\sigma, \langle \Xi_\sigma, \varpi_{n-1} + \varpi_n \rangle \geq 0, \langle \varpi_n^\vee + \Xi_\sigma, \varpi_{n-1} + \varpi_n \rangle \in \mathbb{Z}.$$

This in turn yields $x_{n-1} = x_n, x_{n-1} + x_n \geq 0$ and $x_{n-1} + x_n + \frac{2n-2}{4} \in \mathbb{Z}$. From this, we deduce

$$\Xi_\sigma = \begin{cases} \sum_{j \text{ odd}, 1 \leq j \leq n-2} \frac{1}{2}\alpha_j^\vee, \text{ if } n \text{ is odd;} \\ \sum_{j \text{ odd}, 1 \leq j \leq n-2} \frac{1}{2}\alpha_j^\vee + \frac{1}{4}(\alpha_{n-1}^\vee + \alpha_n^\vee), \text{ if } n \text{ is even.} \end{cases}$$

Chapter 5: Some remarks about the weight function in type A_n

In this chapter, we explore the connection between the function wt and the notion of cascades. The goal of this chapter is to prove the following result.

Proposition 5.0.1. *Let W be an irreducible Weyl group. Then $wt(x) = r_x$ for any involution x in W if and only if W is of type A_n.*

Here $r_x \in \mathbb{Z}\Phi^\vee$ is associated to x as per the construction in [Lus18, Section 1.8], which we recall in the next section.

5.0.1 Definition of r_x

Let us briefly summarize the setup from [Lus18], for more details we refer to section 1.1 and 1.8 in loc. sit. Let W be an irreducible Weyl group W and we denote the set of involutions in it by $\mathcal{I}_W$. For an element $x \in \mathcal{I}_W$, set $Y_x = \{\lambda \in X_*(T)_\mathbb{R} : x(\lambda) = -\lambda\}$ and define $\Phi_x^\vee = \Phi^\vee \cap Y_x$. One can similarly define (X_x, Φ_x) using the action of W on $X^*(T)_\mathbb{R}$. It is then proved in loc. sit. that $(X_x, Y_x, \Phi_x, \Phi_x^\vee)$ forms a root system. Furthermore, $\Phi_x^{\vee+} = \Phi_x^\vee \cap \Phi^{\vee+}$ is a set of positive coroots for this system.

Then one inductively defines the following subsets of X_x. Namely, let $\mathcal{E}_{x,1}$ be the set of maximal elements in Φ_x^+ with respect to the usual dominance order. Then for $i \geq 2$, let

$\mathcal{E}_{x,i}$ to be the maximal elements of

$$\{\alpha \in \Phi_x^+ : \langle \alpha, \beta^\vee \rangle = 0 \text{ for all } \beta \in \mathcal{E}_{x,1} \cup \cdots \cup \mathcal{E}_{x,i-1}\}.$$

Finally, one defines $\mathcal{E}_x^\vee = \bigcup_{i\geq 1}\{\alpha^\vee : \alpha \in \mathcal{E}_{x,i}\}$. Using this set, the following element of $\mathbb{Z}\Phi^\vee$ is defined in loc. sit.

$$r_x = \sum_{\beta \in \mathcal{E}_x^\vee} \beta.$$

The following result gives a characterization of r_x constructed in this way.

Theorem 5.0.2. *[Lus18, Theorem 0.2] There is a unique map $\mathcal{I}_W \to \mathbb{Z}\Phi$, $x \mapsto r_x$ such that (i)-(iii) below hold.*

(i) $r_1 = 0$, $r_{s_\alpha} = \alpha^\vee$ for any $\alpha \in \Delta$;

(ii) for any $x \in \mathcal{I}_W$ and $\alpha \in \Delta$ such that $s_\alpha x \neq x s_\alpha$, we have $s_\alpha(r_x) = r_{s_\alpha x s_\alpha}$;

(iii) for any $x \in \mathcal{I}_W$ and $\alpha \in \Delta$ such that $s_a x = x s_a$, we have $r_{s_\alpha x} = r_x + \mathcal{N}\alpha^\vee$ where $\mathcal{N} \in \{-1, 0, 1\}$.

If in addition G is simply laced we have $\mathcal{N} \in \{-1, 1\}$.

5.0.2 The notion of W-depth

We now recall some feature about another function defined on Coxeter groups that will be relevant for us in the next section. For a Coxeter system $(W, \mathbb{S})$ and a set of positive roots Φ^+ in it, [BB05, Section 4.6] defines *W-depth*[1] of an element $\beta \in \Phi^+$ by $\mathrm{dp}(\beta) := \min\{k : x(\beta) \in -\Phi^+ \text{ for some } x \in W \text{ with } \ell(x) = k\}$.

[1]In fact, this is called *depth* in [BB05], [Bag+16] and [PT15]; we alter the terminology here to avoid any confusion with the notion of depth of a coweight defined earlier.

It is a classically known fact that one can use this to give a partial order on the set of roots, cf. [BB05]. In [PT15], Petersen and Tenner extend this concept to define the following function on W (still denoted by dp)

$$\mathrm{dp}(x) := \min\{\sum_i \mathrm{dp}(\beta_i) : x = s_{\beta_1} \cdots s_{\beta_k}, \beta_i \in \Phi^+\}.$$

It is easy to see that $\mathrm{dp}(s_\beta) = \mathrm{dp}(\beta) = \frac{1}{2}(\ell(s_\beta) + 1)$ for any positive root β. Hence we have that $\mathrm{dp}(s_\beta) \leq \langle \rho, \beta^\vee \rangle$, and equality occurs if and only if β is a quantum root. In particular, if x is an element of a Weyl group of simply laced type, we have

$$\mathrm{dp}(x) = \min\{\sum_i \langle \rho, \beta_i^\vee \rangle : x = s_{\beta_1} \cdots s_{\beta_k}, \beta_i \in \Phi^+\}. \tag{5.0.1}$$

Following [Bag+16], we say that *the W-depth of x is realized by a reduced factorization of x* if there exists an expression $x = s_{\beta_1} \cdots s_{\beta_k}$ with $\beta_i \in \Phi^+$, such that $\ell(x) = \sum_{i=1}^{k} \ell(s_{\beta_i})$ and $\mathrm{dp}(x) = \sum_{i=1}^{k} \mathrm{dp}(s_{\beta_i})$.

We also recall the notion of *reduced reflection length*[2] ℓ_{red} introduced in loc. sit. Essentially, its definition is parallel to Definition 3.1.4, without the quantum root proviso - i.e. it is defined using a decomposition that satisfies only the last two conditions in that definition.

It is proved in [PT15] that the W-depth of every element in a classical finite Coxeter group is realized by reduced factorization. This observation plays a central role in the following result.

[2]In the same vein, we alter the terminology ℓ_R used in loc. sit. for the definition of this length function and call it ℓ_{red}.

Proposition 5.0.3. *[Bag+16, Proposition 6.6] If the W-depth of x is realized by a reduced factorization then $dp(x) = \frac{1}{2}(\ell(x) + \ell_{red}(x))$; in particular, this equality holds whenever W is a classical finite Coxeter group.*

5.0.3 Relation between wt and dp

Lemma 5.0.4. *Let W be an irreducible Weyl group of type A_n or D_n. Then we have $\langle \rho, wt(x) \rangle = dp(x)$ for any element $x \in W$.*

Proof. Note that for any irreducible Weyl group W and for any two elements $x, y \in W$, the following holds true by Lemma 4.2.4:

$$\ell(y) = \ell(x) - \langle 2\rho, \mathrm{wt}(x, y) \rangle + d_\Gamma(x, y).$$

Letting $y = 1$, we get

$$\langle \rho, \mathrm{wt}(x) \rangle = \frac{1}{2}(\ell(x) + d_\Gamma(x, 1)) = \frac{1}{2}(\ell(x) + \ell_\downarrow(x)) \tag{5.0.2}$$

Since all roots are quantum in a group of simply laced type, we have $\ell_\downarrow(x) = \ell_{\mathrm{red}}(x)$. Now, we get the desired conclusion appealing to Proposition 5.0.3. □

Remark 5.0.5. Note that in general we have $\langle \rho, \mathrm{wt}(x) \rangle \geq \mathrm{dp}(x)$ just by appealing to the definition. If W is an irreducible Weyl group of non-simply laced type, the existence of an element $x \in W$ with $\ell_\downarrow(x) > \ell_{\mathrm{red}}$ ensures that strict inequality do occur. We refer to Remark 5.0.9 for explicit example of such elements. As noted in [Bag+16, Question 6.5], it is hard to verify by a computer whether the W-depth of all elements are realized by reduced factorization, even for a group of type E_7. Hence it is not clear if Lemma 5.0.4 holds true for the groups of type E_6, E_7, E_8 as well.

5.0.4 Weight of an involution

It is classically known that if x is an involution in an irreducible Weyl group of rank n, then it can be written as a product of $\ell_R(x)$ commuting reflections, where $\ell_R(x) \leq n$. This fact is needed in the statement of Proposition 5.0.7 below. We also need the following lemma.

Lemma 5.0.6. *[Rap05, Lemma 3.3] Let P be a W-stable convex polygon in $X_*(T)_{\mathbb{R}}$. Let $v \in \mathbf{a}$ and assume that w_1, w_2 are two elements of $\widetilde{W}$ such that $w_1 \leq w_2$. Then we have $w_1 v - v \in P$, if $w_2 v - v \in P$.*

In our case, we are going to let P to be $P_\zeta := \mathrm{Conv}(W\zeta)$ for certain dominant ζ; this is the convex hull of points in its W-orbit.

Proposition 5.0.7. *Suppose that W is a Weyl group of type A_n. Let $x \in W$ be an involution. Suppose that $x = s_{\beta_1} \cdots s_{\beta_k}$, where $k \leq n$ and $\{\beta_i : 1 \leq i \leq k\}$ is a set of orthogonal positive roots. Then*

$$wt(x) = \sum_{i=1}^{k} \beta_i^\vee.$$

Proof. Let us first note that it suffices to show $\mathrm{wt}(x) \geq \sum_{i=1}^{k} \beta_i^\vee$. Indeed, we then have

$$\langle \rho, \mathrm{wt}(x) \rangle \geq \langle \rho, \sum_{i=1}^{k} \beta_i^\vee \rangle.$$

By Lemma 5.0.4 and Equation (5.0.1), the above must be an equality. Therefore, writing

$\mathrm{wt}(x) - \sum_{i=1}^{k} \beta_i^\vee = \sum_{\alpha \in \Delta} n_\alpha \alpha^\vee$ with $n_\alpha \in \mathbb{Z}_{\geq 0}$, we get $\sum_{\alpha \in \Delta} n_\alpha = 0$, thereby showing that $n_\alpha = 0$ for each $\alpha \in \Delta$. Hence we have the desired equality.

Let us choose $\lambda \in X_*(T)$ such that $\mathrm{depth}(\lambda) \geq 2n$. By Proposition 3.2.6, we then have $t^\lambda x \geq t^{\lambda - \mathrm{wt}(x)}$. An easy computation shows that

$$t^\lambda x \varpi_j^\vee - \varpi_j^\vee = \lambda - \sum_{i=1}^{k} \langle \beta_i, \varpi_j^\vee \rangle \beta_i^\vee = \lambda - \sum_{\beta_i \geq \alpha_j} \beta_i^\vee.$$

We set $\zeta_j := \lambda - \sum_{\beta_i \geq \alpha_j} \beta_i^\vee$. Note that at most n coroots are subtracted in the expression of ζ_j, and since the maximum value of $\langle \alpha, \beta_i^\vee \rangle$ is 2 for any simple root α, the depth hypothesis on λ ensures that ζ_j is dominant. Applying Lemma 5.0.6, we get that $\lambda - \mathrm{wt}(x) \in P_{\zeta_j}$. Since $\lambda - \mathrm{wt}(x)$ is dominant, we conclude that $\lambda - \mathrm{wt}(x) \leq \zeta_j$, i.e.

$$\mathrm{wt}(x) \geq \sum_{\beta_i \geq \alpha_j} \beta_i^\vee. \tag{5.0.3}$$

Since Equation (5.0.3) is valid for any $j = 1, \cdots, n$, we get that

$$\mathrm{wt}(x) \geq \bigvee_{j=1}^{n} \Big(\sum_{\beta_i \geq \alpha_j} \beta_i^\vee \Big),$$

where $\bigvee$ stands for the join operation. It is easy to see that that this join is equal to $\sum_{i=1}^{k} \beta_i^\vee$. Hence we are done. □

Remark 5.0.8. We note that [Bag+16, Example 2.12] gives an example of such an element x where the conclusions of Proposition 5.0.7 and Proposition 5.0.1 fail. Namely, let W be a Weyl group of type D_4. Then

$$x = s_4 s_2 s_3 s_1 s_2 s_4 s_2$$

is an involution such that $\ell_R(x) = 3$. The only way to write x as a product of 3 commuting reflections is $x = s_{\alpha_1+\alpha_2+\alpha_4} s_{\alpha_2+\alpha_3+\alpha_4} s_2$, and thus by [Lus18, Section 1.8, statement (a)] we conclude that r_x is the sum of coroots corresponding to these roots. Hence if $\mathrm{wt}(x) = r_x$, we would get $\mathrm{dp}(x) = 7$ by Lemma 5.0.4. However, [Bag+16, Theorem 2.9] shows that $\mathrm{dp}(x) = 6$.

Since the root systems of E_n for $n = 6, 7, 8$ all contain a subsystem isomorphic to that of D_4, the conclusions of Proposition 5.0.7 and Proposition 5.0.1 fail in these cases as well. By our discussion about the computation of r_x, we also see that $\mathrm{wt}(x) \neq r_x$.

Remark 5.0.9. We now give some explicit examples of elements in irreducible Weyl groups of non-simply laced type, for which the conclusions of Proposition 5.0.1 and Lemma 5.0.4 fail.

(i) Let W be a Weyl group of type C_3 and let x be the reflection element in W corresponding to a non-quantum root $\beta = \alpha_1 + 2\alpha_2 + \alpha_3$, i.e. $x = s_2 s_3 s_1 s_2 s_3 s_1 s_2$. We claim that

(a) $x = s_{2\alpha_2+\alpha_3} s_1 s_{2\alpha_2+\alpha_3}$ is a reduced quantum reflection decomposition of x.

Indeed, this is a length additive decomposition using only quantum roots. We note that $\ell_\downarrow(x) \neq 2$ since the right hand side of Equation (5.0.2) must be an integer; since β is a non-quantum root, we cannot have $\ell_\downarrow(x) = 1$. Hence, $\ell_\downarrow(x) = 3 > 1 = \ell_{\mathrm{red}}(x)$. However, note that in this case $\mathrm{wt}(x) = (\alpha_2^\vee + \alpha_3^\vee) + \alpha_1^\vee + (\alpha_2^\vee + \alpha_3^\vee)$, which does match with $r_x = \beta^\vee = \alpha_1^\vee + 2\alpha_2^\vee + 2\alpha_3^\vee$.

Now, considering W to be the Weyl group associated to a root system of type B_3, let x' be the reflection element corresponding to a non-quantum root $\alpha_1 + \alpha_2 + \alpha_3$. Direct computation shows that there is only one way to write x' as a product of 3 reflections in a length additive manner: $x' = s_1 s_{\alpha_2+\alpha_3} s_1$, but this uses a non-quantum root $\alpha_2 + \alpha_3$. Hence, its reduced quantum reflection decomposition is $x = s_1 s_2 s_3 s_2 s_1$, thereby giving $\ell_\downarrow(x) = 5 > 1 = \ell_{\text{red}}(x)$

(ii) Now, let W be a Weyl group of type B_4 and pick the element $y = s_4 s_3 s_4 s_2 s_3 s_4 s_1 s_2 s_3 s_4 s_2$. We claim that

(b) $y = s_{\alpha_3+2\alpha_4} s_2 s_3 s_1 s_{\alpha_2+\alpha_3+2\alpha_4}$ is a reduced quantum reflection decomposition of y.

Indeed, this is a length additive decomposition using only quantum roots. Direct computation shows that there are only two ways to express y as a product of 3 reflections; namely, we have $y = s_{\alpha_1+\alpha_2+\alpha_3+2\alpha_4} s_{\alpha_2+2\alpha_3+2\alpha_4} s_2$ and $y = s_{\alpha_1+\alpha_2+\alpha_3+2\alpha_4} s_{\alpha_2+\alpha_3+\alpha_4} s_{\alpha_3+\alpha_4}$, but none of these are length additive decomposition. This rules out the possibility $\ell_\downarrow(y) = 3$. Since y is not a reflection, we deduce that the claim is true. In this case, $\text{wt}(y) = \alpha_1^\vee + 2\alpha_2^\vee + 3\alpha_3^\vee + 2\alpha_4^\vee$. An easy computation about the -1-eigenspace of y shows that $r_x = \alpha_1^\vee + 3\alpha_2^\vee + 3\alpha_3^\vee + 2\alpha_4^\vee$.

Since the root system of type B_4 occurs as a subsystem of type F_4, this example works in that case as well.

(iii) Finally, for a Weyl group of type G_2 we let $z = s_2 s_1 s_2$. This is the reflection element of W corresponding to a non-quantum root $\gamma = \alpha_1 + \alpha_2$. We have $\ell_\downarrow(z) = 3 > 1 = \ell_{\text{red}}(z)$. Also, here $\text{wt}(z) = \alpha_1^\vee + 2\alpha_2^\vee$ and $r_z = \gamma^\vee = \alpha_1^\vee + 3\alpha_2^\vee$.

In view of the above remarks, we only need to justify that $\mathrm{wt}(x) = r_x$ for any involution x in a Weyl group of type A_n.

Proof of Proposition 8.1. We need to show that the properties (i)-(iii) in Theorem 5.0.2 are satisfied by wt as well. Note that (i) is trivially true. To check (ii), let us choose a decomposition $x = s_{\beta_1} \cdots s_{\beta_k}$ as described in Proposition 5.0.7. Then $s_\alpha x s_\alpha = s_{s_\alpha(\beta_1)} \cdots s_{s_\alpha(\beta_k)}$ is a decomposition of similar kind for $s_\alpha x s_\alpha$. By Proposition 5.0.7, we have $r_{s_\alpha x s_\alpha} = \sum_{i=1}^{k} s_\alpha(\beta_i^\vee) = s_\alpha(r_x)$.

Finally, we argue that property (iii) is satisfied by wt for any $x \in W$, where W is any irreducible Weyl group. Suppose first that $s_\alpha x > x$; if $s_\alpha x < x$, we can swap the roles of x and $s_\alpha x$ in the argument below. Then we can form a path from $s_\alpha x$ to 1 by concatenating the edge $s_a x \rightharpoondown x$ with a path from x to 1 of shortest length that only uses downwards edges, whence $\alpha^\vee + \mathrm{wt}(x) \geq \mathrm{wt}(s_\alpha x)$. By Lemma 3.1.1, we also have $\mathrm{wt}(s_\alpha x) \geq \mathrm{wt}(x)$. Combining these two inequalities, we get that $\mathrm{wt}(s_\alpha x)$ is either $\mathrm{wt}(x)$ or $\mathrm{wt}(x) + \alpha^\vee$.

This completes the proof. □

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