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by C.A. Berenstein and E. Casadio Tarabusi

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THE INVERSE CONDUCTIVITY PROBLEM AND THE HYPERBOLIC X-RAY TRANSFORM

CARLOS A. BERENSTEIN AND ENRICO CASADIO TARABUSI

ABSTRACT. It is shown here how the approximate inversion algorithm of Barber & Brown for the linearized inverse conductivity problem in the disk can be interpreted exactly in terms of the X-ray transform with respect to the Poincaré metric and of suitable convolution operators.

0. INTRODUCTION

The purpose of electrical impedance imaging is to reconstruct the conductivity (or impedance) as a function on the interior of a given body from the measurements of the boundary potentials induced by boundary currents which can be varied at will. One possible approach is due to Barber and Brown [BB1], [BB2], who gave an algorithm for an approximate inversion of the linearized two-dimensional problem when the conductivity on a disk is approximately constant. Later, Santosa and Vogelius [SV] justified this by following Beylkin's generalized Radon transform method [B] to construct another approximation for the inversion. In this note we shall show the exact relations between these and the X-ray (or Radon) transform R on the hyperbolic disk, based on the exact inversion for R found in [BC1], and prepare the ground for improved inversion algorithms for the impedance problem. The complete details will be given in a forthcoming paper.

1. BACKGROUND

Let D, the unit disk of $\mathbf{C}$, be endowed with the Poincaré (hyperbolic) metric $ds^2 = 4\,|dz|^2/(1-|z|^2)^2$: this is conformal to the Euclidean metric and has constant curvature -1. The induced distance of $z \in D$ from the origin 0 is related to $|z|$ by $|z| = \tanh d(z,0)/2$. A geodesic for this metric is a diameter or the intersection with D of a circle perpendicular to ∂D. In geodesic polar coordinates $z = (\omega, r)$, with $\omega = z/|z|$ and $r = d(z,0)$, the Poincaré metric is expressed as $ds^2 = dr^2 +$

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$\sinh^2 r\, d\omega^2$, where $d\omega^2$ is the ordinary metric in ∂D. The length of a geodesic circle of radius r is $2\pi \sinh r$, while the Laplace-Beltrami operator on D is

$$\begin{aligned}\Delta_H &= \frac{(1-|z|^2)^2}{4}\Delta \\ &= \frac{\partial^2}{\partial r^2} + \coth r \frac{\partial}{\partial r} + \sinh^{-2} r \frac{\partial^2}{\partial \omega^2},\end{aligned}$$

where Δ is the Euclidean Laplacian.

Following Helgason [H1], [H2], the hyperbolic X-ray transform on D of a function f of compact support (or fast decay) is

$$Rf(\gamma) = \int_\gamma f(z)\, dm_\gamma(z) \qquad \text{for every } \gamma \in \Gamma,$$

where dm_γ is the hyperbolic arc length on γ, and G is the set of geodesics of D. The dual X-ray transform of a (say) continuous function ϕ on Γ is

$$R^*\phi(z) = \int_{\Gamma_z} \phi(\gamma)\, d\mu_z(\gamma) \qquad \text{for every } z \in D,$$

where $d\mu_z$ is the unique measure on $\Gamma_z = \{\gamma \in \Gamma \colon \gamma \ni z\}$ which is invariant by all isometries of D which keep z fixed. The transform R was inverted by Helgason in [H3], while an inversion for R^*R was proved in [BC1], based on the observation that R^*R is a convolution operator with a radial kernel, namely $1/(\pi \sinh r)$. For other properties of R, relations with the Riesz transform, and characterizations of the range see [BC3], [BC2], and the literature cited there.

2. The inverse conductivity problem

In this section we summarize the procedure of [SV]: other references for these questions are [SU], [BB1], [BB2].

The inverse conductivity problem in dimension 2 is modeled on D by the Neumann problem

$$\begin{cases} \operatorname{div}(\beta \operatorname{grad} u) = 0 & \text{in } D, \\ \beta \dfrac{\partial u}{\partial n} = \psi & \text{on } \partial D, \end{cases}$$

where the input ψ, the boundary current to be applied, is such that $\int_{\partial D} \psi = 0$, and the measurable output is the boundary potential $u|_{\partial D}$ (unique up to constant). The goal is the recovery of the conductivity β, which is a positive function on D (in fact it can be assumed to be bounded away from 0). As it is shown in [SV], this problem can be linearized about the conductivity $\equiv 1$, so that $\beta = 1 + \delta\beta$, thus leading to the pair of Neumann problems, with $u = U + \delta U$ (where U corresponds to the unperturbed state $\beta \equiv 1$),

$$\begin{cases} \Delta U = 0 & \text{in } D, \\ \dfrac{\partial U}{\partial n} = \psi & \text{on } \partial D, \end{cases}$$

and

$$\begin{cases} \Delta(\delta U) = -\langle \operatorname{grad} \delta\beta, \operatorname{grad} U\rangle & \text{in } D, \\ \dfrac{\partial(\delta U)}{\partial n} = -(\delta\beta)\psi & \text{on } \partial D \end{cases}$$

($\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the standard scalar product), with $\delta U = u - U$ measured on ∂D. It is natural to assume $\delta\beta = 0$ on ∂D and take $\psi = \psi_\omega$ to be a dipole at a point $\omega \in \partial D$; then the former Neumann problem, with $U = U_\omega$, becomes

$$\begin{cases} \Delta U_\omega = 0 & \text{in } D, \\ \dfrac{\partial U_\omega}{\partial n} = -\pi \dfrac{\partial \delta_\omega}{\partial \tau} & \text{on } \partial D, \end{cases}$$

where $\partial/\partial\tau$ is the counterclockwise tangential derivative, and where δ_ω is the Dirac delta function at ω. Then δU itself depends on ω also, so we denote it δU_ω, and the problem reads

$$\begin{cases} \Delta(\delta U_\omega) = -\langle \operatorname{grad} \delta\beta, \operatorname{grad} U_\omega\rangle & \text{in } D, \\ \dfrac{\partial(\delta U_\omega)}{\partial n} = 0 & \text{on } \partial D. \end{cases} \tag{2.1}$$

The function U_ω is explicitly given in [SV]:

$$U_\omega = \frac{x'}{(x')^2 + (y')^2},$$

where $x' = \langle i\omega, z\rangle$, $y' = 1 - \langle \omega, z\rangle$.

Let us introduce new independent variables

$$u = U_\omega, \qquad v = \frac{y'}{(x')^2 + (y')^2}.$$

The function $z \mapsto w = u + iv$ is a conformal mapping onto $\{\Im\mathrm{m}\, w > 1/2\}$. Let δU_ω in the new variables (u, v) be denoted by g, and $\delta\beta$ by b. Then (2.1) becomes

$$\begin{cases} \Delta g = -\dfrac{\partial b}{\partial u} & \text{in } \{\Im\mathrm{m}\, w > 1/2\}, \\ \dfrac{\partial g}{\partial v} = 0 & \text{on } \{\Im\mathrm{m}\, w = 1/2\}. \end{cases}$$

The Green function in $\{\Im\mathrm{m}\, w, \Im\mathrm{m}\, \zeta > 1/2\}$, where $\zeta = \xi + i\eta$, given by

$$G(\zeta, w) = -\frac{1}{4\pi}\left(\frac{u-\xi}{(u-\xi)^2 + (v-\eta)^2} + \frac{u-\xi}{(u-\xi)^2 + (v+\eta-1)^2}\right)$$

is the solution of the problem

$$\begin{cases} \Delta_w G = -\dfrac{\partial \delta_\zeta}{\partial u} & \text{in } \{\Im\mathrm{m}\, w > 1/2\}, \\ \dfrac{\partial G}{\partial v} = 0 & \text{on } \{\Im\mathrm{m}\, w = 1/2\}. \end{cases}$$

Therefore

$$g(w) = \int_{-\infty}^{\infty} \int_{1/2}^{\infty} b(\zeta) G(\zeta, w)\, d\xi\, d\eta + \text{constant}.$$

One possible normalization for ∂U_ω, hence for g, is to take the tangential derivative on the boundary (since the Neumann problem requires $\int_{\partial D} \delta U_\omega = 0$). So

$$\frac{\partial g}{\partial u}(u + i/2) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{1/2}^{\infty} \frac{(u-\xi)^2 - (1/2-\eta)^2}{[(u-\xi)^2 + (1/2-\eta)^2]^2} b(\zeta)\, d\xi\, d\eta. \tag{2.2}$$

The left-hand of (2.2) can be computed from the output data. Thus the linearized inverse conductivity problem leads to a family of integral equations (2.2), one for each $\omega \in \partial D$.

3. Convolution operators and the hyperbolic X-ray transform

In trying to explain the ad hoc solution by Barber and Brown [BB1], [BB2] of the inverse conductivity problem, Santosa and Vogelius used in [SV] the method of the Beylkin generalized Radon transform [B] to construct an approximation to the inversion of (2.2). Their approach consisted in introducing a transform that integrates along the geodesics of D (which happen to be the level curves of U_ω, for $\omega \in \partial D$) after multiplication by an exponential factor: the main problem being that it distinguished the orientation of the geodesics, while physical considerations indicate that it is not the case. At any rate, their questions led us in [BC1] to consider the inversion of R^*R since it was an approximate solution to the problem. The main point of the present note is to show that (2.1) with the equivalent collection of integral equations (2.2) can be exactly formulated in terms of hyperbolic geometry and X-ray transform.

Let $\gamma(\omega, s)$ be the geodesic determined by the equation $\{U_\omega = s\}$, and χ the function on Γ given by

$$\chi(\gamma(\omega, s)) = \frac{\partial(\delta U_\omega)}{\partial \tau}(z(\omega, s)),$$

where $\gamma(\omega, s)$ has extreme points ω and $z(\omega, s) \in \partial D$; the function χ is given by the data (as proposed by Barber and Brown).

Proposition 3.1. *We have*

$$\chi = R(A * \delta\beta),$$

where A is the radial convolution kernel

$$A(r) = \frac{1}{4\pi}(3 \cosh^{-4} r - \cosh^{-2} r). \quad \square$$

The Barber-Brown backprojection was the dual X-ray transform R^* (see Section 1), so that the exact equation for the Barber-Brown inversion is

$$R^*\chi = R^*R(A * \delta\beta). \tag{3.1}$$

Since the composition product R^*R is a radial convolution operator, it commutes with the convolution by A. Recalling (cf. [BC1]) that $T = (1/4\pi)\Delta_H(1 - \coth r)$ is the exact convolution inverse of R^*R, where Δ_H is the Laplace-Beltrami operator on D for the Poincaré metric, we have

$$T(r) \approx A(r) \qquad \text{for } r \to \infty;$$

on the other hand, T is singular at $r = 0$ but A is not. The Barber-Brown procedure $R^*\chi \approx \delta\beta$ can be understood now by replacing A by T in (3.1). The proof of the proposition consists in analyzing the commutator of R with an arbitrary radial convolution operator. The symbol of A (with respect to Helgason's spherical Fourier transform [H2]) is easily computed. Setting $\Gamma(a \pm b) = \Gamma(a+b)\,\Gamma(a-b)$, where Γ is the Euler Gamma function, recalling that $\hat{\Delta}_H(\lambda) = -\lambda^2 - 1/4$, and using [GR, formula 7.132.7] and the functional relation $\Gamma(x+1) = x\,\Gamma(x)$ we get

$$\begin{aligned}\hat{A}(\lambda) &= 2\pi \int_0^\infty A(r)\, P_{i\lambda-1/2}(\cosh r)\, \sinh r\, dr \\ &= -\frac{1}{2\sqrt{\pi}}\left[\Gamma\left(\frac{3}{4} \pm \frac{i\lambda}{2}\right) - 2\Gamma\left(\frac{7}{4} \pm \frac{i\lambda}{2}\right)\right] \\ &= -\frac{1}{4\sqrt{\pi}}\,\hat{\Delta}(\lambda)\,\Gamma\left(\frac{3}{4} \pm \frac{i\lambda}{2}\right),\end{aligned}$$

where P_ν is the associated Legendre function. The determination of the convolution inverse of A is still an open question.

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Department of Mathematics and Systems Research Center of the University of Maryland, College Park, Maryland 20742, U.S.A.

E-mail address: cab@math.umd.edu, carlos@src.umd.edu

Dipartimento di Matematica "G. Castelnuovo", Università di Roma "La Sapienza", Piazzale A. Moro 2, 00185 Roma, Italy
E-mail address: casadio@itnvax.cineca.it, casadio@itncisca.bitnet