

ABSTRACT

Title of dissertation: QCD RESUMMATION OF SOFT GLUONS
IN EFFECTIVE FIELD THEORY

Ahmad S. Idilbi, Doctor of Philosophy, 2006

Dissertation directed by: Professor Xiangdong Ji
Department of Physics

The main results of this thesis are the introduction of a new and systematic way to treat and resum logarithmic enhancements, to all orders in perturbation theory and to any desired logarithmic accuracy. The method developed is applied to the threshold logarithmic enhancements as well as to small transverse momentum distributions. We utilize an effective field theoretic approach to perform the resummation which turns out to be in complete agreement with the conventional approaches, though it is a much simpler one. This approach will be applied to deep inelastic scattering, Drell-Yan lepton pair production and the Standard Model Higgs production.

We have derived the functions $g_i(\alpha_s \ln \bar{N})$ for $i = 1, 2, 3$ that resums threshold logarithms up to next-to-next-to leading logarithmic accuracy. Certain quantities (the $D^{(3)}$ for the Higgs production and the Drell-yan process and the $B^{(3)}$ for deep inelastic scattering) that contribute to the resummation at the NNLL are also derived. Moreover, our method opens a door towards understanding the origin of the universality of the functions f that contribute to anomalous dimensions of the

quark and gluon form factors.

On the conceptual level, the effective approach to compute the perturbative coefficient functions of the factorization theorem(s) highlights the way these theorems could be viewed in terms of a multiple steps of integrating out momentum modes through a matching procedure. The remaining infrared divergences (at the factorization scale) are exactly the parton distribution functions.

In this work we also study exclusive processes. We will be mainly concerned with the nucleon to delta transition and how to utilize perturbative QCD to obtain a leading order factorization formula for such a process, and for similar ones. Based on this, we are able to make phenomenological predictions for certain transition form factors which can be verified experimentally.

QCD RESUMMATION OF SOFT GLUONS
IN EFFECTIVE FIELD THEORY

by

Ahmad S. Idilbi

Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2006

Advisory Committee:

Professor Xiangdong Ji, Chair/Advisor
Professor Stephen J. Wallace
Professor Thomas D. Cohen
Professor Douglas A. Roberts
Professor Zhanqing Li

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ACKNOWLEDGMENTS

I would like to begin by thanking my advisor, Prof. Xiangdong Ji, for all of his assistance over the years. I had the privilege to start interacting with Xiangdong since my very first day at University of Maryland. First by taking advanced courses in Quantum Field Theory and Nuclear physics, then by becoming his graduate research student. Both experiences were, to say the least, very stimulating and enriching.

I also wish to thank my collaborators, Dr. Feng Yuan and Prof. J. P. Ma for years of fruitful physics research.

The TQHN group at UMD is a very nice group to belong to. Interacting with Profs. Steve Wallace and Thomas Cohen is a very pleasant experience. Not to mention our marvellous secretary, Loretta Robinette to whom I am really grateful.

Finally I would like to thank my colleagues, Dr. Joydip Kundu, Dr. Dominique Toublan and my friend Yingchuan Li for their kindness and assistance.

CONTENTS

1. <i>Prelude</i>	1
2. <i>Introduction</i>	19
2.1 Basics of QCD	19
2.1.1 Factorization Theorems	22
2.1.2 Singular Contributions and the Soft Limit	29
2.2 Soft Collinear Effective Theory	35
2.3 Standard Model Higgs Production in the Large Top Quark Mass Limit	41
3. <i>Deep Inelastic Scattering and Drell-Yan In Soft Collinear Effective Theory</i>	45
3.1 Deep Inelastic Scattering	46
3.2 Drell-Yan Process	58
3.3 Summary	63
4. <i>Threshold Resummation</i>	65
4.1 Anomalous Dimensions of Effective Currents	65
4.2 The Soft Contribution	76
4.3 Comparison with Conventional Resummation	86
4.3.1 Resummed Logarithms	93
4.4 Phenomenology	99
4.5 Summary	100
5. <i>Small Transverse Momentum Resummation</i>	103
5.1 Introduction	103
5.2 Drell-Yan Production at Low Q_T	105
5.3 Matching at the Intermediate Scale μ_I	107
5.4 Resummation and Comparison with Conventional Approach	111
5.5 Higgs Production	113
5.6 Summary	116
5.7 d-Dimensional Fourier Transform	116
6. <i>Wave Distribution Amplitudes for the Δ Resonance</i>	119
6.1 Introduction	119
6.2 Covariant Decomposition	123

6.3	Twist-3	126
6.4	Twist-4	130
6.5	Twist-5 and 6	133
6.6	Fierz Transformations	136
6.7	Summary	138
7.	<i>Nucleon to Delta Transition</i>	140
7.1	Introduction	140
7.2	Kinematics and Notation	143
7.3	Leading-Order pQCD Factorization Formula for G_0	146
7.4	Phenomenology	152
7.5	Summary	155
	<i>Appendix</i>	156
A.	<i>Appendix A</i>	157
A.1	Mellin Moments	157
A.2	The Plus Distributions	158
A.3	Gamma Function	159
B.	<i>Full QCD Calculation for Drell-Yan Process</i>	161
C.	<i>Second Order Results in the Soft Limit</i>	167
D.	<i>Conformal Group</i>	171

1. PRELUDE

It is well known that there are four forces in nature: gravitational, electromagnetic, weak and strong forces (interactions). The gravitational force is well described, at least when astronomical distances are concerned, by the “General Theory of Relativity” put forward by Einstein in 1915. The other three interactions are combined in the framework of the “Standard Model” (SM) which is a culminating effort of many people [1, 2, 3, 4, 5, 6, 7, 8]. The SM consists of two sectors, the electroweak sector and the strong interaction one. It is formulated in terms of a lagrangian density and quantized fields which represent both the elementary degrees of freedom (electrons, neutrinos, quarks, etc.) and the carriers of the interactions themselves (photons, gluons and Z and W bosons).

The electroweak sector unifies the electromagnetic and the weak interactions. The first interaction is described by “Quantum Electrodynamics” (QED) which is a quantized field theory that underlies classical electromagnetism (Maxwell equations and Lorentz force law). It was first proposed by Dirac in 1927 as an attempt to quantize classical electromagnetic fields in a manner analogous to the quantization of Newtonian mechanics. The latter, as we know, was demonstrated earlier by Schrödinger and Heisenberg by proving the existence of discrete energy levels in the Hydrogen atom. It took many years to establish QED as a legitimate quantum field

theory (QFT). The main obstacle was the many infinities encountered in (higher order) calculations of physical quantities that are obviously finite (as they are experimentally measured). The effort put forth in trying to overcome this obstacle and to understand the physical origin of these infinities has resulted in the well-known “Renormalization Theory” [9]. Once proved renormalized, the founders of QED, Feynman, Schwinger and Tomonaga, won the Nobel Prize in 1965.

As we know, the electromagnetic force is a long-range one, which is mediated by a massless gauge boson, the photon. At the sub-atomic level, QED describes the interaction between electromagnetic radiation and charged particles like electrons and their anti-particles, the positrons. Within the framework of QED, precision calculations of certain physical quantities have reached unprecedented accuracies. The most famous is the anomalous magnetic moment of the electron which was calculated to 11 decimal places [10]. This calculation was later verified experimentally to 7 parts in a trillion [11]. This agreement between theory and experiment gives QED a special status among physical theories and it is considered, by many people, as the best physical theory ever formulated.

The weak interaction is mainly encountered in certain forms of radioactive decays that take place in an atomic nucleus. Processes like beta decay (where a neutron becomes a proton through the emission of an electron and anti-neutrino), electron capture (where a proton captures an electron, becomes a neutron through emission of neutrinos) or positron emission are all mediated by the weak force. Moreover, through only the weak interaction, heavy quarks (like the charm or bottom quark) can decay into lighter ones (the up or down quark) thus changing their flavor. The

same also applies for the decay of heavy leptons (like muons) into lighter ones (electrons). The carriers of this interaction are the massive W^\pm and Z gauge bosons, with masses of, roughly, 90 GeV. These particles were first discovered at CERN [12].

The unified theory of the electroweak interactions was developed in the sixties through the works of Glashow [1], Salam [5] and Weinberg [2] for which they won the Nobel Prize in 1979. The renormalizability of the electroweak theory (with its spontaneous symmetry breaking mechanism) was later proved by 't Hooft and Veltman in 1972 [13], a work for which they were also awarded the Nobel Prize in 1999.

The two guiding principles that led to the construction of the SM lagrangian density are gauge symmetry and renormalizability of the interaction. The first principle requires that the lagrangian density be invariant under a group of transformations which by themselves depend on space-time coordinate x . This is what is generically referred to as “local gauge invariance”. This requirement eventually led to the correct inclusion of the fields that represent the carriers of the interactions (the gauge bosons) into the lagrangian density. The second requirement ensures that the lagrangian contains only a finite number of unknown parameters to be determined experimentally and that one can make predictions from the theory.

The gauge group that governs the electroweak sector is the direct product: $SU_L(2) \times U_Y(1)$ where $SU_L(2)$ is the left-handed weak isospin and $U_Y(1)$ stands for hypercharge. This gauge symmetry must be broken in some way. The reason is the following: for the gauge symmetry to hold, the gauge bosons have to be massless.

On the other hand, the weak force cannot be mediated by massless particles since it is essentially a short range force (in contrast to the electromagnetic force). The way out of this came with the idea of “spontaneous symmetry breaking” [14]. With this symmetry breaking mechanism, the weak gauge bosons acquire a mass and the gauge group $SU_L(2) \times U_Y(1)$ is broken down to the electromagnetic $U_{em}(1)$ with the photon as the only massless force carrier. Another consequence of this symmetry breaking is the introduction of a massive scalar into the lagrangian, known as the “Higgs boson” which couples with the fermion fields through a Yukawa type of interaction. This particle has not been discovered yet, and it is on the “top list” of objectives for the “Large Hadron Collider” (LHC) at CERN, Geneva. We will say more about this in Chapter 2.

In this work, we are mainly interested in the third sector of the SM: the strong interaction sector described by a quantum field theory known as “Quantum Chromodynamics” (QCD). This theory was introduced as $SU_C(3)$ gauge invariant quantum field theory in 1973 [4]. The elementary degrees of freedom are the spin-1/2 quarks that interact through an exchange of spin-1 gauge bosons, the gluons. The strong interaction is responsible for binding quarks inside nucleons (protons and neutrons), thereby overcoming their electromagnetic repulsion. It is also responsible for forming the heavier nuclei, which are bound states of protons and neutrons that form the “bulk” of ordinary matter.

The subscript C in $SU_C(3)$ stands for the “color” degree of freedom [15, 16] and the number 3 designates the number of colors each quark can admit. To gain more familiarity with QCD, let us address the following issues: why quarks? and why

three colors? In an attempt to identify regularities underlying the many hadronic states already observed, Gell-Mann and Ne'eman [17] proposed, in 1961, that the observed hadronic states could be classified into (irreducible) representations of the $SU(3)$ group where 3 refers to the number of flavors (not colors). Each representation should be “filled” by hadronic states that share the same spin and are (almost) degenerate in mass. It should be mentioned that all the observed hadrons fall into two groups: mesons have an integer spin, equal to zero or 1, and baryons have half integer spin equal to $1/2$ or $3/2$. This picture was very successful, however it lacked any obvious justification. To account for this, Gell-Mann and Zweig [18], independently, introduced the idea that hadrons are not elementary objects but are made of more fundamental entities: the quarks. The proposition was that these objects are spin- $1/2$ particles. Together with their anti-particles, which have opposite electric charge but the same spin, they occupy the only two (inequivalent) fundamental representations of $SU(3)$: the $\mathbf{3}$ and $\bar{\mathbf{3}}$. Thus, there are three types, or flavors, of quarks: up, down and strange.

From elementary Quantum Mechanics, we know that adding two spin- $1/2$ states, results in either spin-0 or spin-1. Adding an additional spin- $1/2$ results in states with either spin- $1/2$ (two of them) or a spin- $3/2$. Thus it is tempting to *postulate* that the integer spin states, the mesons, are made up of a quark and anti-quark of different flavors, and the half-integer spin states, the baryons, are made up of three quarks. Moreover, the number of hadrons in each representation of $SU(3)$ can be accounted for by noticing that higher dimensional irreducible representations of the Lie group $SU(3)$ can be obtained from the fundamental ones by direct product,

i.e., $\mathbf{3} \times \bar{\mathbf{3}} = \mathbf{1} + \mathbf{8}$, and, $\mathbf{3} \times \mathbf{3} \times \mathbf{3} = \mathbf{1} + \mathbf{8} + \mathbf{8} + \mathbf{10}$. The numbers, 1, 8 and 10 are exactly what is found for the spin-0 meson octet, spin-1 meson octet, spin-1/2 baryon octet and spin-3/2 baryon decouplet.

The notion of color [15, 16] was introduced in order to explain the peculiar nature of the Δ^{++} baryon. This state constitutes of three up quarks (uuu with total charge of 2) and it has spin and isospin of 3/2. In its quantum mechanical ground state, it is quite obvious that its spatial wavefunction is symmetric under rotations, i.e., its total angular momentum is zero. Moreover, the only way to get a spin-3/2 from three spin-1/2 quarks without any contribution from angular momentum, is that all three quark spins are aligned in the same direction. The result is that we have three quarks, which are fermions, in a completely symmetric wave function. This result contradicts the Fermi-Dirac statistics, which requires a total anti-symmetric state of identical fermions. However, by introducing an extra quantum number, the color, and by assigning a different color to each one of the three up quarks, we can easily then anti-symmetrize the total state by using the totally anti-symmetric tensor, ϵ^{ijk} , where i, j, k are color indices.

The quark model was very successful in accounting for the static properties of the hadronic spectra, like mass splittings in each multiplet and ratios of magnetic moments. However, it says nothing about the dynamics that combine three quarks in a baryon or quark and anti-quark in a meson. After all, how could one explain that three positively charged up quarks combine to form a Δ^{++} baryon state? Moreover, the quarks have fractional charges with 2/3 for the up quark (in units of the proton's charge) and $-1/3$ for the down and strange quarks. However, experimentally no

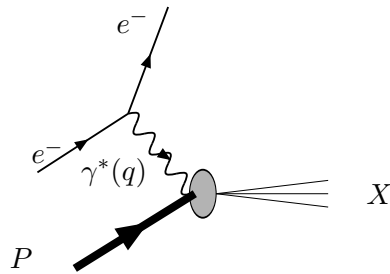


Fig. 1.1: Deep inelastic scattering

one has ever detected such charges. In this sense, all observed hadronic states are *colorless*.

Back in the sixties, and due to the above issues, the quark model was mainly conceived as a mathematical set-up and, by no means, a physical theory of the strong interactions. However, the notions introduced, like spin-1/2 quarks and the color degree of freedom, will serve (a few years later) to finally formulate QCD as the correct theory of the strong interactions. This makes the quark model as one of the most important steps in the development of QCD.

The second major progress came from a different direction. In 1969, at “Stanford Linear Accelerator” (SLAC), Bloom *et al.* performed the first experiment of electron-proton scattering, with highly energetic electrons [19]. This reaction is known as “deep inelastic scattering” (DIS) and it is shown in Fig. 1.1. The experimental data showed that the proton, P , which is being probed by a highly virtual photon, $\gamma^*(q)$, behaved as a point-like particle with no internal structure. To explain such an unexpected behavior, Feynman proposed [20] that the proton is made up of a point-like constituents (which were later on called by Bjorken as “partons”), and that the photon, due to its high virtuality, is scattered off these constituents.

In this “parton model” picture, the photon only interacts with partons at distances of order $1/Q$ where $q^2 \equiv -Q^2$ and q is the four-momentum of the photon¹, and the detailed structure of the proton at distances larger than $1/Q$ is essentially irrelevant. Moreover, further experiments at SLAC have shown that these partons were actually spin-1/2 particles. It seems that, after all, the notion of a proton made up of quarks may come to be true, and the quarks may turn out to be real physical objects.

Careful analysis of the DIS data had revealed another feature that will have rather dramatic implications on the development of QCD. Certain quantities that parameterize the DIS cross section turned out to be dependent on a single *dimensionless* variable, later to be known as the “ x of Bjorken” (denoted by x_B) instead of being a function of two independent, dimensionful quantities: energy and momentum transfers. This phenomena became to be known as “Bjorken scaling”. The fact that certain quantities *scale* was previously predicted by Callan and Gross [21] and by Bjorken [22] based on current algebra and sum rules techniques².

However, it was very difficult to understand scaling in the context of quantum field theories. Indeed, it was shown by Gross and Callan that the scaling property of Green’s functions implies that the underlying field theory has to be an “ultraviolet stable” (later to be known as “asymptotic freedom”) in the sense that the coupling constant(s), which determine the strength of the interaction, becomes vanishingly

¹ This is a simple result of the uncertainty principle.

² These notions were very popular in the sixties as an alternative theoretical tools to quantum field theories.

small at small distances or, equivalently, at high energies.

The natural question to be asked: is there any such quantum field theory? Based on the paradigm of QED, which is not an asymptotically free field theory, Gross tried to show that no such quantum field theory could possibly exist. This means that the theory of strong interactions cannot be formulated as a quantum field theory and other frameworks are to be looked for. As a first step, it was proved by Gross and Coleman [23], that no renormalizable field theory with Yukawa, scalar or abelian gauge interactions have the property of asymptotic freedom.

In order to complete this program it was necessary to investigate if the remaining, well-known, Yang-Mills non-abelian gauge theories [24] are also not asymptotically free. However, it turned out that such kind of field theories *do* have that property. This was shown by Gross and Wilczek [7] and by Politzer [6]. For this work, they shared the Nobel Prize in 2004.

More details about the discovery of asymptotic freedom could be found in a paper by Gross himself [25]. To conclude this semi “historical” introduction, we remark that the discovery of asymptotic freedom of non-abelian gauge field theories and the ability to explain scaling within such theories, paved the way for formulating QCD as the theory of the strong interactions, with quarks and gluons as its elementary fields.

Asymptotic freedom is a fundamental feature of QCD which makes it a weakly interacting quantum theory at short distances. Looking at this the opposite way, QCD becomes strongly interacting one, as it must, at large distances. To specify more accurately what is meant by “weak” and “strong” (or short and large), let

us consider the most well-known strongly interacting system: the proton. As we know, the proton is a complicated system of quarks and gluons interacting via the strong interaction. One might expect that by “bombarding” the proton with highly energetic external probes, quarks may be liberated which would allow their detection as free particles. Experimentally, though, this has never happened.

The phenomena that quarks are confined to a hadronic colorless state is known as “confinement” (or “infrared slavery”) and it suggests that at energy scales of order of a typical hadronic mass (~ 1 GeV), the strong coupling constant is strong enough to bind quarks and gluons into hadrons. We might also conclude that, at energy scales much larger than 1 GeV, asymptotic freedom makes the strong coupling constant relatively small.

The fact that QCD is a confining theory at low energies and asymptotically free at high energies has rather important consequences on how to deal, theoretically, with such a theory. As is well known, quantum field theories in four-dimensional space-time, can only be dealt with, analytically, by methods of perturbation theory. In an interacting quantum field theory, the coupling constant serves as the expansion parameter of the perturbative series. It is clear that for this expansion to make sense, the coupling constant should be small enough (compared to 1). So in order to apply perturbative methods to study QCD, we have to make sure that the strong coupling constant α_s is rather small. The asymptotic freedom property of QCD thus allows us to make use of perturbative techniques to study strong interaction processes as long as the momenta and energies involved are much larger than 1 GeV. In this thesis we will be mainly interested in such high-energy processes, also known as “hard”

ones, and the main framework applied throughout is perturbative QCD (pQCD).

However, QCD cannot be studied or applied to the vast majority of high-energy processes based only on a perturbative approach. The reason for that is the phenomenon of confinement. To illustrate this point, let us consider the DIS process, Fig. 1.1. Any cross section related to this process (whether differential or total) can be written as a product of two contributions: a leptonic one, which involves the electron-photon-electron vertex, and a hadronic one related to the proton-photon and whatever the final hadronic state, X , is. The former contribution is calculated, perturbatively, using QED Feynman rules³. However, the hadronic contribution involves matrix elements of the electromagnetic current taken between the proton and the X states. Unlike the QED case, these states (no matter what X is) are not the ones by which QCD, as a quantum field theory, is formulated. Since pQCD breaks down at energies of order of the proton mass, these matrix elements are non-perturbative and, unfortunately, our ability to calculate them analytically is very limited.

For inclusive cross sections, where we sum the contributions from all the possible final hadronic states, X , the hadronic contribution to the DIS cross section can be expressed as a matrix element of a product of two electromagnetic currents⁴ taken between proton states (i.e., the X has been summed over). The crucial point is that in certain kinematical limit⁵, and due to one of the most fundamental theo-

³ Recall that the electron and the photon are the elementary fields in the QED lagrangian.

⁴ Evaluated at two different space-time points.

⁵ For DIS it is the “Bjorken limit” where we keep $x_B = \frac{-q^2}{2P \cdot q}$ fixed and let $Q^2 \equiv -q^2$ goes to infinity. P is the four-momentum of the proton.

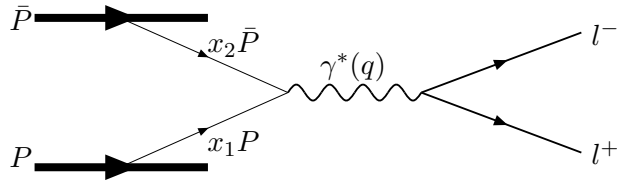


Fig. 1.2: Drell-Yan lepton pair production

rems of QCD, the *factorization theorem*, the nonperturbative input that is needed to fully determine such matrix elements (up to corrections that vanish in this limit) is related to the distribution of the partons—the quarks and gluons—inside the proton. Such distributions, however, do not depend on the specifics of the process. They are determined by the QCD dynamics which form the proton as a bound state of quarks and gluons. As such, these distributions are universal⁶. Once these distributions are extracted, say, from DIS data, they can be used to make predictions for other inclusive processes that involve (obviously) the same type of hadrons. For example, in the Drell-Yan (DY) lepton pair production process [26], depicted in Fig. 1.2, one needs to know the parton distribution of a quark inside a proton and anti-quark inside an anti-proton.

The process dependent contributions, known as the “coefficient functions” are perturbatively calculable to any desired order in the strong coupling α_s . For DIS and DY processes, these quantities will be a major focus in this thesis.

One of the well-known facts about local quantum field theories is that perturbative calculations of such elementary quantities as cross sections and decay widths usually involve infinities (or divergences) which has to be dealt with consistently

⁶ More technical discussion of the factorization theorems and parton distribution will be given in Chapter 2.

to render the calculations finite. Generally speaking, there are two kinds of divergences: ultraviolet (UV) and infrared (IR). The first type of divergences arises when the momenta of particles-that exist in a virtual quantum states-are not restricted by energy-momentum conservation relations. Quantum mechanics instructs us to sum the contributions from all possible quantum configurations. Quantitatively, this leads to integrals that diverge when contributions from infinitely large momenta are taken into account. The renormalization program [27] is designed to deal with such UV divergences, by showing, that to all orders in perturbation theory, one can absorb such infinities into a *finite* amount of quantities known as “counterterms”. Any quantum field theory for which such a program can be carried out is called “renormalizable” field theory. All interactions of the standard model have been proved renormalizable, including QCD.

The other kind of divergences, the IR ones, are not treated by the renormalization program. We will encounter many examples of such divergences later on. The important point is that both kinds of divergences leave their “fingerprints” on the perturbative series of the quantities being calculated. For the UV case, it will turn out that some mass scale, known as “renormalization scale”, has to be introduced, which our starting point, i.e, the original lagrangian, knows nothing about. Thus the (bare) parameters that appear in the lagrangian will become dependent on this new scale (upon renormalization). This is what is known as the “running” effect, and this is exactly what makes the coupling constants, and in particular α_s , scale dependent.

The IR divergences arise due to the emission of low energy or “soft” gluons.

These divergences usually lead to the appearance of logarithms of ratios of mass scales. These scales are characteristic to the process being considered. In certain kinematical limits, these scales become widely separated, thus, making the logarithms large. In perturbative calculations, these logarithms will be accompanied, obviously, by the expansion parameter which is the coupling constant. Thus the product of a small coupling constant with these large logarithms is not necessarily small (compared to 1) and the justification for perturbative expansion is endangered. These large logarithmic enhancements must be *resummed to all orders* in perturbation theory in order to make the perturbative expansion reliable. We will show explicitly that once resummed, the perturbative expansion restores the smallness of the expansion parameter. Since resummation is performed to all orders in perturbation theory, then, obviously enough, any resummed result will have much less theoretical uncertainty compared to any *fixed* order calculation. This would have a rather important phenomenological impact, as we shall see later on.

Aside from the DIS and DY processes, we will also be considering the standard model (SM) Higgs production. As we have mentioned, this massive and spin-0 particle, enters the electroweak lagrangian through the spontaneous symmetry breaking mechanism. This mechanism is one of the cornerstones of the SM. However, and despite many attempts, the Higgs particle has not been detected yet. The discovery of this type of new matter is of highest priority for the high energy physics community [28, 29].

The CERN “Large Hadron Collider” (LHC), which is a proton-proton colliding

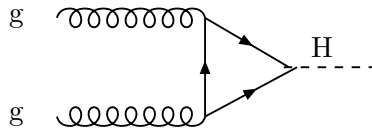


Fig. 1.3: Gluon-gluon fusion production of the Higgs particle.

machine⁷, will be the main arena at which to look for the Higgs particle. For an extensive discussion of the various production mechanisms of the Higgs particle at the LHC, we refer the reader to the review [29] and references therein. However, we will only consider the main production mechanism⁸, the *gluon-gluon fusion* through a top quark loop [30], as illustrated in Fig. 1.3. With such a production channel, it is obvious that perturbative QCD corrections have to be taken into account in order to make, as precise as possible, the theoretical predictions for the Higgs production cross section. Again, it will turn out that large logarithmic enhancements tend to spoil fixed order perturbative calculations and resummation of these logarithms is necessary.

Our discussion for the resummation of logarithmic enhancements due to soft gluon radiation for the inclusive DIS, DY and the Higgs production cross sections, will be performed within the framework of *effective field theory*. The two central notions in this approach are: *matching* and *running* [31]. As we have mentioned earlier, hard inclusive processes and, in certain kinematical limits (to be discussed later on), admit momentum scale separation. The large scale, i.e., the hard one, insures that some sort of a factorization theorem applies to the cross section formula of the

⁷ The LHC will start operating in 2007.

⁸ For Higgs mass in the range of 1 Tev, the gluon-gluon fusion process provides 50% of the total production cross section.

process considered. The lower scale arises from the kinematical limit being taken. At each scale one performs matching of an underlying (in our discussion, these theories are known) theory onto an effective one. Through this matching, one “integrates out” some degrees of freedom which are not relevant to the case at hand. The separated scales are then, pictorially speaking, linked to each other through the process of running. The running procedure accumulates all the contributions from the continuum of scales lying between the large and the lower scales. Quantitatively, this procedure is performed by solving simple, first-order ordinary differential equations, similar to the familiar “renormalization group equations” encountered in quantum field theory textbooks [32, 33, 34].

All the processes we have already mentioned are inclusive ones. However, there is another class of processes that have been studied extensively in the last three decades. These are the exclusive ones. In such processes, where a specific hadron in the final state is detected, one can study various properties of these hadronic states. Most notably are the electromagnetic transition form factors of hadrons which allow us to study charge and current distributions inside these states. However, and as in the case of inclusive ones, the confining nature of QCD highly complicates the analysis of such processes and one, usually, is enforced to work in certain “simplifying” kinematical limits.

As in the case of the high energy inclusive processes, we will be interested in the large momentum region (compared to 1 GeV), which allows perturbative treatment of certain form factors. We, again, rely on a factorization theorem for an exclusive process, which allows the factorization of long and short distance effects.

For exclusive processes, the nonperturbative, low energy quantities are known as “wave distribution amplitudes” and the short distance effects, which are perturbatively calculable, are encoded in scattering amplitudes. The above pQCD picture will be applied to study some of the electromagnetic form factors of the nucleon to the (singly-charged) delta resonance; a topic that has been studied and analyzed in many frameworks for almost three decades.

This thesis is organized as follows. In Chapter 2, we give a brief introduction to the basics of QCD, including factorization theorems for inclusive processes. The effective theories which will serve us later, the “Soft Collinear Effective Theory” (SCET) [35, 36, 37, 38] and the large top quark mass limit for the SM Higgs production [39], will be introduced. In Chapter 3 we treat the DIS and DY processes within the SCET to first order in perturbation theory.

In Chapter 4 we extend the results of Chapter 3. We show how to carry out the resummation (in the case of “threshold” logarithms) to arbitrary accuracies. This will be applied to all three processes: DIS, DY and the Higgs production. For the latter, we show the phenomenological implications of the resummed cross section. Remarks on some all-order (in α_s) universality relations among DY and the Higgs production will be given. In Chapter 5 we show how the resummation can be carried out for small transverse momentum distributions. This will be applied for DY and the Higgs cases.

In Chapters 6 and 7 we give our treatment for the exclusive nucleon to delta transition. A phenomenological prediction for related quantities is then introduced which can be tested at Jefferson Laboratory (JLab).

This thesis is based on the original work of Idilbi, Ji, Ma and Yuan [40, 41, 42, 43, 44].

2. INTRODUCTION

2.1 Basics of QCD

In this section we briefly review some aspects of QCD that are relevant to our future discussions. Since QCD is a relativistic quantum field theory, any discussion of it has to start from the QCD lagrangian \mathcal{L}_{QCD} . We will present only the classical version of it, i.e., the one before quantization and renormalization are carried out. A more complete treatment can be found in, e.g., [33, 45]. The QCD lagrangian for one kind of fermions is¹

$$\mathcal{L}_{\text{QCD}} = \bar{\psi}(x)(i \not{D} - m)\psi(x) - \frac{1}{4}G_{\mu\nu}^a G^{\mu\nu,a}. \quad (2.1)$$

The ψ stands for a quark field and it is a 3×1 matrix in color space. The covariant derivative is given by: $D^\mu = \partial^\mu - ig_s A^\mu$ where $A^\mu = A^{\mu a} t^a$. $A^{\mu a}$ represents a gluon field of color a and the matrices t^a are the generators of the Lie group $SU_C(3)$. The non-abelian gluon field strength tensor $G^{\mu\nu,a}$ is

$$G^{\mu\nu,a} = \partial^\mu A^{\nu,a} - \partial^\nu A^{\mu,a} + g_s f^{abc} A^{\mu,b} A^{\nu,c}. \quad (2.2)$$

¹ Unless otherwise stated, repeated indices are implicitly summed over.

The f^{abc} are the structure constants of the Lie group $SU(3)$ and they are related to the $3^2 - 1 = 8$ generators t^a through the commutation relations

$$[t^a, t^b] = if^{abc}t^c. \quad (2.3)$$

As we have mentioned, QCD admits the property of asymptotic freedom. This property actually results from the three and four gluon vertices implied by the $G^a G^a$ term in the lagrangian. These kinds of vertices are absent from the QED lagrangian and, thus, this theory is not asymptotically free. The QCD coupling constant $\alpha_s \equiv g_s^2/4\pi$ depends on the renormalization scale μ through:

$$\frac{d \ln \alpha_s(\mu^2)}{d \ln \mu^2} = -[\beta_0 a_s + \beta_1 a_s^2 + \beta_2 a_s^3 + \mathcal{O}(a_s^4)], \quad a_s \equiv \frac{\alpha_s}{4\pi}, \quad (2.4)$$

where

$$\begin{aligned} \beta_0 &= \frac{11}{3}C_A - \frac{2}{3}N_F, \\ \beta_1 &= \frac{34}{3}C_A^2 - 2C_F N_F - \frac{10}{3}C_A N_F, \\ \beta_2 &= \frac{2857}{54}C_A^3 - \frac{1415}{54}C_A^2 N_F - \frac{205}{18}C_A C_F N_F + C_F^2 N_F + \frac{79}{54}C_A N_F^2 + \frac{11}{9}C_F N_F^2. \end{aligned} \quad (2.5)$$

The color factors C_F and C_A are the Casimirs of the fundamental and adjoint representations of $SU(3)$, respectively, and we have: $C_F = 4/3$ and $C_A = 3$. N_F designates the number of independent quark fields ($N_F = 6$ in the standard model), differentiated by their flavor. With these numbers and Eq. (2.4) it is easily verified that $\alpha_s(\mu^2)$ is a decreasing function of μ^2 . Keeping only the β_0 term in Eq. (2.4), we get

$$\alpha_s(\mu^2) = \frac{\alpha_s(\mu_o^2)}{1 + (\beta_0/4\pi)\alpha_s(\mu_o^2) \ln(\mu^2/\mu_o^2)}. \quad (2.6)$$

Defining $\Lambda \equiv \Lambda_{\text{QCD}} = \mu_0 e^{-2\pi/(\beta_0 \alpha_s(\mu_0^2))}$ we have

$$\alpha_s(\mu^2) = \frac{4\pi}{\beta_0 \ln(\mu^2/\Lambda_{\text{QCD}}^2)}. \quad (2.7)$$

The parameter Λ_{QCD} determines the scale at which the coupling constant becomes large. In this region perturbation theory breaks down and nonperturbative effects take over. Based on phenomenological observations, Λ_{QCD} is of order of few hundreds of MeV, which is, roughly speaking, of the same order as the masses of light hadrons. The fact that the coupling constant becomes large (of order unity) at low energies will lead, eventually, to the phenomenon of confinement already mentioned, which means that quarks and gluons will be confined into a small volume of space-time thus forming bound states of the observed hadrons.

Quarks or gluons cannot propagate over macroscopic distances and, thus, they cannot be detected. So when trying to test and/or study the strong interactions through the QCD lagrangian, we face a situation in which the elementary degrees of freedom of the lagrangian \mathcal{L}_{QCD} are quarks and gluons (and these are exactly the quantum fields we use in perturbative calculation), however, due to confinement every practical application will include only hadronic states (protons, neutrons, pions, etc.) as a projectile beam or as a target. The QCD dynamics of these states, as we have already mentioned, are not amenable to perturbative treatment.

In quantum electrodynamics (QED), for example, the situation is completely different. The electron enters into \mathcal{L}_{QED} as an elementary fermionic field (and the photon as the gauge boson), however, it is also an asymptotic state—an actual particle that travels over macroscopic distances—and we can (and do) perform experiments

with electrons and photons. Thus, the comparison between the underlying theory (QED) and the experimental findings are, in some sense, straightforward.

The above discussion illustrates (some of) the inherent complexities of QCD. The conclusion would be that even when considering some high energy reaction, characterized by a mass scale $Q^2 \gg \Lambda_{\text{QCD}}^2$ which allows perturbative calculations to be performed, it is always the case (almost!) that some nonperturbative input will be needed to compare experimental results with QCD predictions. This input is related to the internal structure of the hadrons that participate in the reaction. As we shall argue, in the next section, this is certainly true when hadrons initiate a high energy reaction as in the case of deep inelastic scattering (DIS) and Drell-Yan (DY) processes. As we have mentioned, the way perturbative and nonperturbative contributions are organized for such processes is described quantitatively by the factorization theorems. Their content also illustrates the predictive power of perturbative QCD (pQCD) through the universality of the nonperturbative contribution.

2.1.1 Factorization Theorems

In the previous chapter we briefly discussed how the phenomenon of Bjorken scaling of DIS-related quantities² was explained by Feynman, Bjorken and others as a scattering of an electron off an “isolated” parton in the proton through the exchange of a photon with a very short wavelength. Since the QCD lagrangian was formulated, one should be able to justify the parton model picture based on

² These are called “structure functions” and will be discussed shortly.

quantum field theory arguments.

The parton model does not (and can not!) incorporate any quantum effects in the description of the reaction of DIS. The factorization theorems of QCD for inclusive processes are the field theoretic statements that, first, justify the parton model description (as a “first approximation”) and then include all the quantum correlations of a parton interacting with its surrounding (i.e., not completely isolated).

To apply a factorization theorem, the external probe has to be energetic enough so that the hadronic sub-structure is resolved. This means energies much larger than 1 GeV. Let us discuss more quantitatively the content of factorization theorems for the DIS process. In DIS process, a current of leptons l , say electrons, is scattered off a hadron h , say a proton, through an exchange of virtual gauge bosons, say a photon, with four-momentum $q = k' - k$,

$$l(k) + h(P) \rightarrow l'(k') + X , \quad (2.8)$$

where $-q^2 \equiv Q^2 \gg \Lambda_{\text{QCD}}^2$ and X stands for all unobserved hadronic states. Any (differential) cross section related to this process can be written as

$$d\sigma \propto L_{\mu\nu} W^{\mu\nu} , \quad (2.9)$$

where the leptonic tensor $L_{\mu\nu}$ is a purely electromagnetic quantity calculated from the QED lagrangian and is free from any strong interaction effects. These are encoded in the hadron tensor $W^{\mu\nu}$ which is the quantity of interest.

Following Lorentz and parity invariance of \mathcal{L}_{QCD} and defining the Bjorken variable $x_B = Q^2/2P \cdot q$, where P is the momentum of the incoming hadron, one

can parameterize $W^{\mu\nu}$ in the following form:

$$W^{\mu\nu} = - \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) W_1(x_B, Q^2) + \left(P^\mu - \frac{P \cdot q}{q^2} q^\mu \right) \left(P^\nu - \frac{P \cdot q}{q^2} q^\nu \right) W_2(x_B, Q^2). \quad (2.10)$$

In terms of W_i we can define two dimensionless quantities, known as the “structure functions”, in the following way:

$$\begin{aligned} F_1(x_B, Q^2) &= W_1(x, Q^2) , \\ F_2(x_B, Q^2) &= \nu W_2(x_B, Q^2), \quad \nu \equiv \frac{Q^2}{2x_B} , \end{aligned} \quad (2.11)$$

It is clear that the structure functions F_i are nonperturbative quantities since they are related to the matrix element of a product of two electromagnetic currents between proton states and cannot be calculated within the framework of pQCD. Working in the “Bjorken limit”, i.e., $Q^2 \rightarrow \infty$ while keeping x_B fixed, the following factorization theorem for the structure functions holds [46, 47, 48, 49],

$$F_i(x_B, Q^2) = \sum_a \int_{x_B}^1 \frac{d\xi}{\xi} G_{ia} \left(\frac{x_B}{\xi}, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right) f_{a/h}(\xi, \mu^2) + \mathcal{O} \left(\frac{\Lambda_{\text{QCD}}}{Q} \right). \quad (2.12)$$

The Q^2 -dependent coefficient function (also known as “Wilson coefficient”), G , describes the interaction between the lepton and the constituents of the proton. The variable ξ stands for the (longitudinal) momentum fraction of the proton carried by the struck parton. Since this interaction takes place at very short distances (characterized by large momentum transfers) then this function is amenable to perturbative calculations. It is expressed as an expansion in α_s and it is clearly a process-dependent quantity. The index $a = q, \bar{q}, g$ specifies the type of the struck parton (q -quark, \bar{q} -anti-quarks and g -gluons) and we sum over all possibilities.

The Q^2 -independent function $f_{a/h}(\xi, \mu^2)$ known as “parton distribution function” (PDF), stands for the distribution of a parton of type a with momentum fraction ξ inside the hadron h . It is a nonperturbative quantity and needs to be extracted experimentally. The basic message of the factorization theorem is that, in the Bjorken limit, there is an incoherence between short and long distance effects, and the DIS scattering process can be viewed as a scattering of an electron off a weakly interacting parton in the sense that quantum correlations of the parton with its surrounding are suppressed and vanish as inverse powers of Q^2 . The proof of the factorization theorem thus outlines how this statement holds to all orders in perturbation theory. To do so one has to show that the coefficient function G is finite in the massless limit (the so-called “infrared safety”) and an exact definition of the PDF function has to be introduced.

The non-interference between short and long distance effects evidently implies two things: first is that the parton distributions are insensitive to the details of the short distance reaction and thus they are process-independent. This universality has an extremely important consequence. Once $f_{a/h}$ is extracted from, e.g., a DIS experiment, it then can be used in any other hard scattering process that involves the same type of hadron, h . This is what renders pQCD predictive and useful. The second is that the coefficient function is insensitive to the details of the long distance properties of the hadron. All that matters is the kinematics of the reaction and the type of parton a . Thus, it can be calculated to any desired order in perturbation

theory through the following ‘‘partonic’’ version of the factorization theorem:

$$F_{ib}(x, Q^2) = \sum_a \int_x^1 \frac{d\xi}{\xi} G_{ia} \left(\frac{x}{\xi}, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right) f_{a/b}(\xi, \mu^2) + \mathcal{O} \left(\frac{\Lambda_{\text{QCD}}}{Q} \right), \quad (2.13)$$

where $x = \frac{Q^2}{2p \cdot q}$ and p is the parton momentum.

The quantity $f_{a/b}$ is the distribution of parton a in parton b and it can be calculated order by order in perturbation theory. The structure function F_{ib} stands for the scattering of a lepton from a parton b and can also be calculated perturbatively. Thus the coefficient function G_{ia} can be extracted from the perturbative calculation of the PDF $f_{a/b}$ and the structure functions.

It is clear that an exact definition of parton distributions is essential for any meaningful calculation in pQCD. The first gauge-invariant definition of a quark distribution in a proton was given in [50] and it reads:

$$f_{q/h}(\xi) = \frac{1}{4\pi} \int dy^- e^{-i\xi P^+ y^-} \left\langle P \left| \bar{\psi}_q(0, y^-, 0_\perp) \gamma^+ P e^{-ig_s \int_0^{y^-} dz A^+(0, z, 0_\perp) t^a} \psi_q(0) \right| P \right\rangle, \quad (2.14)$$

where for a generic four-vector l we define $l^\pm \equiv \frac{1}{\sqrt{2}}(l^0 \pm l^3)$ and write $l \equiv (l^+, l^-, l_\perp)$. The ψ_q stands for a quark field and $|P\rangle$ is a proton state. The symbol P stands for path-ordering of gluons along the line from 0 to $(0, y^-, 0_\perp)$ and the path-ordered exponential (‘‘Wilson line’’) insures gauge invariance of $f_{q/h}$ ³. By replacing the proton state $|P\rangle$ with a quark state $|q\rangle$ one gets $f_{q/q}$ which is perturbatively calculable. Definitions of anti-quark and gluon distributions in a proton or among themselves follow straightforwardly.

³ Strictly speaking, the quantities appearing in the definition of $f_{q/h}$, i.e. A^μ , ψ and g_s , should be considered as bare field and the whole object requires renormalization.

Although the PDFs, $f_{a/h}$, are nonperturbative quantities, their dependence on the renormalization scale μ is perturbatively calculable. This follows from the observation that the structure functions $F_i(x_B, Q^2)$ are physical quantities and they are independent of the scale μ . The same also applies to the partonic quantities, $F_i(x, Q^2)$. This leads to an integro-differential equation, known as Dokzshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) [51] which governs the evolution⁴ of the the PDF's with the scale μ (for more details see the discussion in [49]).

$$\mu \frac{d}{d\mu} f_{a/h}(x, \mu) = \sum_c \int_x^1 \frac{d\xi}{\xi} P_{a/c}(\xi, \alpha_s(\mu^2)) f_{c/h}\left(\frac{x}{\xi}, \mu\right). \quad (2.15)$$

The evolution kernel (also known as the ‘‘Altarelli-Parisi splitting kernel’’) $P_{a/b}(x, \alpha_s(\mu^2))$ has an expansion in $a_s \equiv \frac{\alpha_s}{4\pi}$: $\sum_{n=0} a_s^{n+1} P_{a/b}^{(n)}(x)$, i.e., they start from $\mathcal{O}(\alpha_s)$ with the well-known result for $a = b = q$,

$$\begin{aligned} P_{q/q}^{(0)}(x) &= C_F \left(\frac{4}{(1-x)_+} - 2 - 2x + 3\delta(1-x) \right) \\ &= C_F \left(\frac{2+2x^2}{(1-x)_+} + 3\delta(1-x) \right), \end{aligned} \quad (2.16)$$

where the ‘‘plus’’ distribution is defined in Appendix A.

The splitting functions $P_{a/b}$ have been calculated to $\mathcal{O}(\alpha_s)$ in [52] and to $\mathcal{O}(\alpha_s^2)$ in [53, 54, 55]. A very recent calculation [56] has extended the results to $\mathcal{O}(\alpha_s^3)$. These results will be used in Chapter 4. Knowing the PDF $f_{a/h}(\mu)$ at some energy scale, the DGLAP evolution equation allows us to determine the same function at a different scale $\mu' \neq \mu$, which is by itself of great help phenomenologically. Moreover,

⁴ Here and throughout this thesis we are only concerned with non-singlet flavor combinations, so flavor mixing is irrelevant.

it will have a very important role when discussing the resummation of enhanced logarithms.

Having discussed the definition of $f_{a/h}$ and its partonic version for $f_{a/b}$, let us now further clarify the content of the factorization theorem, Eq. (2.13). As already mentioned, perturbative calculation of the structure function F_{ia} are usually ultraviolet (UV) and infrared (IR) divergent. Working in massless QCD ($m_f \rightarrow 0$ for light flavors and no fictitious gluon mass introduced) we regularize both kinds of divergences using dimensional regularization in $d = 4 - 2\varepsilon$ and in the $\overline{\text{MS}}$ scheme.

The IR divergences are of two types: soft ones which arise from vanishing four-momentum, and collinear ones that arise when fast-moving massless partons have a vanishing relative transverse momentum (a “jet-like” configuration). This results in vanishing denominators in the amplitude of the Feynman diagram when such momentum configurations occur. Thus the integrals (either over loop momenta or over the phase space for real particle(s) production) may diverge. These divergences show up as poles in ε . Working to $\mathcal{O}(\alpha_s^n)$, IR poles up to ε^{-2n} may appear from individual diagrams which contribute to that order. The factorization theorem states that for a sufficiently inclusive cross section, the combined IR divergences from the relevant Feynman diagrams (which contribute to a given order in α_s), will cancel against the same IR divergence contained in the parton distribution, order by order in α_s and to all orders in perturbation theory. This cancellation allows us to extract the coefficient function G .

The content of the factorization theorem may seem intuitively simple, however, establishing it field theoretically is a rather complicated task. First, one has

to identify all possible momenta configurations that give rise to IR divergences for a generic Feynman diagram. This is accomplished by applying a set of equations known as “Landau equations”, supplemented by infrared power counting (which is analogous to ultraviolet power counting implemented in proofs of renormalizability of quantum field theories). Once these configurations are identified, a systematic way of subtracting the divergences arising from them is then employed. This subtraction will eventually show that all soft divergences cancel and the remaining ones originate from collinear configurations which are identified with the parton distribution function. This holds up to power corrections that vanish in the large Q^2 limit.

2.1.2 Singular Contributions and the Soft Limit

Perturbative calculations of the coefficient functions G for processes like DIS and DY often yield functions that are singular in a certain kinematical limit. These are the “plus” distributions given in Appendix A, the Dirac delta function, and terms of the form $\ln(1-x)/x$ in the case of DIS and $\ln(1-z)/z$ for DY, where $z = Q^2/2p_1 \cdot p_2$ and p_i are the momenta of the incoming partons. The singularity appears in the limit $x \rightarrow 1$ for DIS and $z \rightarrow 1$ for DY. Aside from the delta function, the singular terms only appear in the amplitudes of Feynman diagrams when the incoming or outgoing partons that initiate the hard interaction emit *real* gluons into the final state. To illustrate how these singular terms arise, we perform in Appendix B an explicit calculation for the real gluon contribution for DY process in full QCD and to first order in α_s .

These singular contributions persist to any higher order contribution in α_s . For example, in DIS and DY processes, the $\mathcal{O}(\alpha_s^k)$ calculation of the coefficient functions will involve the distributions $\mathcal{D}_i(z)$ where $i \leq 2k$. The “soft limit” is defined as the one that retains, in the final result for the coefficient functions, only the most singular contributions, i.e., the delta function and the “plus” distributions. Terms of the form $\ln^k(1-x)/x$ although are not defined for $x = 1$, however their limit, when $x \rightarrow 1$, is finite, and thus are not kept in that limit.

For the partonic deep-inelastic process, $q(p) + \gamma^*(q) \rightarrow X$, the soft limit $x \rightarrow 1$ is equivalent to vanishing invariant mass of the final hadronic state X since

$$p_X^2 = Q^2 \frac{(1-x)}{x}, \quad (2.17)$$

Thus the process becomes actually “elastic”. To leading order in α_s and for $x \neq 1$ the final state constitutes of one quark and one gluon⁵. Simple kinematical considerations show that in the soft limit, the emitted gluon can either be collinear to the outgoing quark or a soft one, in the sense that all its four-momentum components are much smaller than Q . If it is collinear to the incoming quark, then $p_X^2 \simeq Q^2$. These observations can be simply extended to diagrams with arbitrary number of real gluons. The case $x = 1$ implies no real gluon radiation into the final states and the perturbative corrections to the LO process, $q(p) + \gamma^*(q) \rightarrow q(p+q)$ are only virtual ones. This is true to arbitrary orders in α_s . For the DY process the LO partonic channel is $q(p_1) + \bar{q}(p_2) \rightarrow \gamma^*(p_1 + p_2)$ and the limit $z \rightarrow 1$ implies that only low energy, soft gluons can be emitted into the final state.

⁵ Before hadronization takes place.

The presence of singular terms of the type discussed above, known as *threshold* singularities, may endanger the perturbative expansion. One can see this by performing an integral transform⁶ to the conjugate space. The combination $\alpha_s^k \mathcal{D}_i(x)$ transforms to $\alpha_s^k \ln^{i+1} \bar{N}$ up to terms independent of \bar{N} and for large values of \bar{N} . This means that the ratio, r , of two consecutive terms in the perturbative expansion is not a small quantity (relative to 1) as must be the case for any meaningful perturbative expansion. These large logarithms have to be resummed. *Threshold resummation* is a way to reorganize the perturbative expansion in such a way that the ratio r would actually be α_s . Moreover, resummed coefficient functions reduce the theoretical uncertainty from the yet uncalculated higher order terms in the truncated perturbative expansion. This may be crucial in instances where the strong interactions constitute the main production mechanism for heavy particles (like the Higgs scalar) and/or for exploring physics beyond the standard model.

The conventional theoretical approaches [57, 58, 59] to perform threshold resummation rely mainly on establishing some sort of factorized cross section (at the kinematical region of interest) into different quantities, defined at the operator level where each one captures the physics at a relevant momentum scale. For example, for deep-inelastic scattering there is the “hard” region, where all momentum components are large and of the same order as the typical hard scale available, namely, Q . There is the “jet” scale which incorporates the effects of the outgoing collinear partons with one large momentum component, and there is the soft scale where all momentum components are small. Then by applying the well-known techniques of

⁶ Mellin convolution and its transform to the moment space \bar{N} is introduced in Appendix A.

renormalization group invariance with respect to some cleverly chosen kinematical variable(s) one obtains an evolution equation. Mellin transform is then applied to de-convolute the various contributions and to obtain a simple product of functions. In the conjugate space, the evolution equation is then solved and one obtains a resummed coefficient function where all the large logarithms of the form mentioned above, appear in an exponentials.

To illustrate the point, we write down a schematic expression for a typical result of a resummed coefficient function in moment space

$$G_N(Q^2) = g_0(\alpha_s(Q^2))e^{[\ln \bar{N} g^{(1)}(\lambda) + g^{(2)}(\lambda) + \alpha_s(Q^2) g^{(3)}(\lambda) + \dots]} , \quad (2.18)$$

where $\lambda \equiv \frac{\alpha_s(Q^2)}{4\pi} \beta_0 \ln \bar{N}$ and β_0 was introduced in Eq. (2.5). The function g_0 has no logarithms in it and it has expansion in $\alpha_s(Q^2)$. The functions $g^{(i)}(\lambda)$ resum all the large logarithms. When expanded as powers of λ , the first term in the exponent gives a sum of the form, $\alpha_s^n(Q^2) \ln^{n+1} \bar{N}$; the second term gives $\alpha_s^n(Q^2) \ln^n \bar{N}$; and from the third term we have $\alpha_s^{n+1}(Q^2) \ln^n \bar{N}$. The pattern is clear and it is followed by higher terms in the exponent. These sums are commonly called leading logarithms (LL), next-to-leading logarithms (NLL), and next-to-next-to leading logarithms (N²LL), *etc.* Explicit expressions for the functions g_0 and $g^{(i)}$ will be given in Chapter 4 for the deep-inelastic scattering, Drell-Yan and the standard model Higgs production.

Although the factorization-based methods for threshold resummation are genuinely sound and rigorous, they are nonetheless complicated—at least on the technical side. A different approach which utilizes effective field theory techniques was put forward and first applied to DIS process [60]. A threshold resummation to NLL

accuracy was performed. The specific effective theory that was implemented is the soft collinear effective theory (SCET) which will be discussed in the next section. Generally speaking, the SCET captures the physics of the full theory, i.e., QCD, when only the soft and collinear momentum modes are the essential ones. Due to the nature and origin of threshold singularities already mentioned, it is no surprise that the SCET turned out to be naturally suited to discuss the IR divergences and the logarithmic enhancements they produce in the threshold region.

The basic idea behind the effective field theoretic approach for resumming large logarithms is actually a simple one. As we mentioned briefly in Chapter 1, one must first identify the momentum scales that are relevant for the process under study and for the kinematical limit of interest. These momentum scales are in general widely separated from each other. The general methodology of the effective theories implies that at each momentum scale one needs to consider the contributions that are relevant at that scale, and contributions from physics at the remote scales are suppressed by powers of the ratios of these scales. These contributions form the so-called “matching coefficients” since they are obtained by matching the physics above and below that scale. Since, by construction, the effective theory below the matching scale has the same infrared behavior as the theory above that scale, then the infrared contributions cancel in the matching procedure and the matching coefficients incorporate only the effects of the degrees of freedom that have been integrated out, and no longer exist in the effective theory.

As for all perturbatively calculated quantities in quantum field theories, the matching coefficient depends not only on the specific scale at which they were eval-

uated, but also on the renormalization scale μ . Moreover, the dependence on μ is only logarithmic. By setting μ equal to the matching scale, all the logarithms in the matching coefficients vanish. This should hold to all orders in perturbation theory.

In quantum field theories, the matching is performed order by order in the coupling constant and at each relevant scale. Since higher order calculations in perturbation theory will lead to the introduction of the renormalization scale μ , then by exploiting this dependence we perform running of the matching coefficients between the matching scales. This running is governed by the anomalous dimensions of the effective operators in the respective theories. By doing so, we resum all the logarithmic contributions to the coefficient functions. This would be the case, provided we identify the matching scales with the relevant scales of the process: the generic hard scale Q^2 and an intermediate scale⁷ μ_I^2 which in turn, is related to the threshold region $Q^2(1-x)^n$. The parameter $n \geq 1$ depends on the process being considered. As we shall see, for DIS $n = 1$ and for DY and the Higgs production $n = 2$ and x is replaced by z .

In principle, the effective field theoretic program for resummation is actually independent of the specifics of the effective theory itself, and one can perform the resummation without referring to the SCET lagrangian (or any other effective theory lagrangian). On the practical level, this is simply due to the fact that perturbative calculations using the *full* QCD lagrangian in the soft limit has already been performed for the processes in which we are interested. Moreover, as we shall demonstrate explicitly in the next chapter, the SCET calculations reproduce exactly

⁷ It should be case that $Q^2 \gg \mu_I^2 \gg \Lambda_{\text{QCD}}^2$ for perturbation theory to apply.

the same results as those of full QCD in the soft limit.

2.2 Soft Collinear Effective Theory

In this section we will consider some aspects of the soft collinear effective theory (SCET) that will be relevant to our future discussions of hard scattering inclusive processes. The original motivations for developing this effective theory were actually related to the study of the inclusive B -meson decay channels. However, as an effective theory describing interactions of fast-moving collinear particles with soft and/or collinear ones, it can also be applied to inclusive processes mainly (but not only) in the threshold regions. It is best formulated using the “light-cone” coordinates, $l^\pm \equiv \frac{1}{\sqrt{2}}(l^0 \pm l^3)$ introduced earlier, and $l = (l^+, l^-, l_\perp)$.

Introducing the light-like four-vectors,

$$n^\mu = \frac{1}{\sqrt{2}}(1, 0, 0, -1), \quad \bar{n}^\mu = \frac{1}{\sqrt{2}}(1, 0, 0, 1), \quad n^2 = \bar{n}^2 = 0, \quad (2.19)$$

we get

$$l^+ = n \cdot l, \quad l^- = \bar{n} \cdot l, \quad l^2 = 2l^+l^- + l_\perp^2 = 2n \cdot l \bar{n} \cdot l + l_\perp^2. \quad (2.20)$$

and

$$l^\mu = (n \cdot l)\bar{n}^\mu + (\bar{n} \cdot l)n^\mu + l_\perp^\mu. \quad (2.21)$$

Let Q be again the large scale characterizing a hard process. A collinear parton (either a quark or a gluon) with large momentum component in the $+z$ direction, is assigned momentum scaling,

$$p = (p^+, p^-, p_\perp) \sim Q(1, \lambda^2, \lambda), \quad (2.22)$$

and it will be called “ \bar{n} -collinear”. λ is a small parameter that is usually determined in relation with the available lower (intermediate) scale. For our interest in summing threshold enhancements, λ will be the measure of how far we are from the threshold region and its exact value depends on the kinematics of the process. In the next chapter we fix the values of λ for the different processes we consider. For n -collinear partons, we have $p = Q(\lambda^2, 1, \lambda)$.

“Soft“ gluons⁸ have momentum assignment,

$$p = Q(\lambda^2, \lambda^2, \lambda^2). \quad (2.23)$$

In certain applications one has also to introduce soft quarks however, these are irrelevant to our discussion since we will be working in reference frames in which the incoming partons are fast-moving in certain direction so they will be described by a collinear fields. With such momentum assignments it is clear that interactions of collinear quarks with collinear gluons (of the same n or \bar{n} type) or with soft gluons does not change its momentum assignment. This actually sounds familiar to another well-studied effective theory: the “heavy quark effective theory” where the heavy quark is treated as a classical object moving along some trajectory with constant velocity v , and the strong interactions cannot change its velocity due to its large mass. Thus the velocity v becomes a “label” of the quark field and the large momentum component is “rotated away”.

⁸ In some papers, the assignment $Q(\lambda^2, \lambda^2, \lambda^2)$, is referred to as “ultra-soft” (u-soft) and $Q(\lambda, \lambda, \lambda)$ refers to soft particles. In our applications we need to consider only one intermediate scale (besides Λ_{QCD}) so we do need two momentum assignments and $Q(\lambda^2, \lambda^2, \lambda^2)$ will be referred to as soft.

Analogously, in the SCET we decompose the momentum p of the \bar{n} -collinear quark

$$p^\mu = \tilde{p}^\mu + k^\mu , \quad (2.24)$$

where \tilde{p}^μ contains the two largest components

$$\tilde{p}^\mu = p^+ \bar{n}^\mu + p_\perp^\mu , \quad (2.25)$$

and the k^μ contains the momentum component of $\mathcal{O}(\lambda^2)$ and is known as the ‘‘residual’’ momentum. With this, one extracts the large momentum fluctuations from the full QCD quark field $\psi(x)$ by defining a new field $\psi_{\bar{n},\tilde{p}}(x)$

$$\psi(x) = \sum_{\tilde{p}} e^{-i\tilde{p}\cdot x} \psi_{\bar{n},\tilde{p}}(x). \quad (2.26)$$

The field $\psi_{\bar{n},\tilde{p}}(x)$ can be decomposed as:

$$\psi_{\bar{n},\tilde{p}}(x) = \frac{\not{\tilde{p}} \not{\tilde{p}'}}{2} \psi_{\bar{n},\tilde{p}}(x) + \frac{\not{\tilde{p}} \not{\tilde{p}'}}{2} \psi_{\bar{n},\tilde{p}}(x) \equiv \xi_{\bar{n},\tilde{p}}(x) + \eta_{\bar{n},\tilde{p}}(x) , \quad (2.27)$$

and the quark lagrangian \mathcal{L}_q becomes,

$$\begin{aligned} \mathcal{L}_q = & \sum_{\tilde{p},\tilde{p}'} e^{i(\tilde{p}-\tilde{p}')\cdot x} \left[\bar{\xi}_{\bar{n},\tilde{p}'}(x) \not{\tilde{p}'} i \bar{n} \cdot D \xi_{\bar{n},\tilde{p}}(x) + \bar{\eta}_{\bar{n},\tilde{p}'}(x) \not{\tilde{p}'} (p^+ + i n \cdot D) \eta_{\bar{n},\tilde{p}}(x) \right. \\ & \left. + \bar{\xi}_{\bar{n},\tilde{p}'}(x) (\not{p}_\perp + i \not{D}_\perp) \eta_{\bar{n},\tilde{p}}(x) + \bar{\eta}_{\bar{n},\tilde{p}'}(x) (\not{p}_\perp + i \not{D}_\perp) \xi_{\bar{n},\tilde{p}}(x) \right]. \quad (2.28) \end{aligned}$$

At this stage it is useful to invoke power counting arguments [37] which show that the collinear field ξ scales as λ that of η scales as λ^2 . Thus all terms in the lagrangian scale as λ^4 except the term $\bar{\eta} n \cdot D \eta = \eta \partial^+ \eta \sim \eta p^- \eta \sim \lambda^6$. Dropping this term means the η field is no dynamical and can be eliminated using Dirac equation.

Then one gets

$$\eta_{\bar{n},\tilde{p}} = \frac{1}{p^+ + i \bar{n} \cdot D} (\not{p}_\perp + i \not{D}_\perp) \not{\tilde{p}'} \xi_{\bar{n},\tilde{p}}(x). \quad (2.29)$$

With this we get for \mathcal{L}_q ,

$$\mathcal{L}_q = \sum_{\vec{p}, \vec{p}'} e^{i(\vec{p}-\vec{p}') \cdot x} \bar{\xi}_{\vec{n}, p'}(x) \left[\bar{n} \cdot D + (\not{p}_\perp + i \not{D}_\perp) \frac{1}{p^+ + i\bar{n} \cdot D} (\not{p}_\perp + i \not{D}_\perp) \right] \not{p} \xi_{\vec{n}, p} . \quad (2.30)$$

We want to emphasize that the gluon fields appearing in the lagrangian (through the covariant derivative, $D^\mu = \partial^\mu - ig_s A^\mu$ are, by construction, either \bar{n} -collinear or soft. The effective theory collinear gluon field $A_{\bar{n}, q}(x)$ can be defined in a similar way to that of the quark field

$$A^c = \sum_{\vec{q}} e^{-i\vec{q} \cdot x} A_{\bar{n}, q}^c(x) , \quad (2.31)$$

where A^c is the full QCD gluon field with momentum modes of the \bar{n} -collinear region: $Q(1, \lambda^2, \lambda)$. It can be shown that the power counting in terms of λ for the gluon fields (either soft or collinear) is identical to their momentum assignments,

$$\begin{aligned} A^c &= (A^{c+}, A^{c-}, A_\perp^c) \sim Q(1, \lambda^2, \lambda) , \\ A^s &= (A^{s+}, A^{s-}, A_\perp^s) \sim Q(\lambda^2, \lambda^2, \lambda^2) , \end{aligned} \quad (2.32)$$

and A^s stands for a soft gluon field. From this we see that in the first covariant derivative in Eq. (2.32) contributions from both gluon fields (A^{c-} and A^{s-}) are of the same order: λ^2 . However in the transverse covariant derivative the contribution from A_\perp^s is suppressed relative to A_\perp^c so it can be dropped. With this observation and Eq. (2.31), one gets

$$\mathcal{L}_q = \sum_{\vec{p}, \vec{p}'} e^{i(\vec{p}-\vec{p}') \cdot x} \bar{\xi}_{\vec{n}, p'}(x) \left[\bar{n} \cdot D + (\not{p}_\perp + i \not{D}_\perp^c) \frac{1}{p^+ + i\bar{n} \cdot D} (\not{p}_\perp + i \not{D}_\perp^c) \right] \not{p} \xi_{\vec{n}, p} , \quad (2.33)$$

and D^c contains only collinear gluon field.

$$\begin{aligned}
\text{---}\tilde{p}+k\text{---}\rightarrow\text{---} &= i \not{n} \frac{\bar{n}\cdot p}{\bar{n}\cdot kn\cdot p+p_{\perp}^2+i\epsilon} \\
\text{---}\rightarrow\text{---}\text{---}\text{---} &= ig_s t^a \bar{n}_{\mu} \not{n} \\
\text{---}\rightarrow\text{---}\text{---}\rightarrow\text{---} &= ig_s t^a \left(\bar{n}_{\mu} + \frac{\gamma_{\mu}^{\perp} p_{\perp}}{n\cdot p} + \frac{p'_{\perp} \gamma_{\mu}^{\perp}}{n\cdot p'} - \frac{p'_{\perp} p_{\perp}}{n\cdot p \bar{n}\cdot n\cdot p'} n_{\mu} \right) \not{n}
\end{aligned}$$

Fig. 2.1: Feynman rules for quark propagator, the vertex of collinear quark with soft gluon and collinear quark with collinear gluon.

After a few more steps and definitions which we do not go through here, the above \mathcal{L}_q can be brought into the following compact form:

$$\mathcal{L}_q = \bar{\xi}_{\bar{n}}(x) \left[i\bar{n}\cdot D + i \mathcal{D}_{\perp}^c W_{\bar{n}}(x) \frac{1}{i\bar{n}\cdot \mathcal{P}} W_{\bar{n}}^{\dagger}(x) i \mathcal{D}_{\perp}^c \right] \not{n} \xi_{\bar{n}}(x) , \quad (2.34)$$

where,

$$\xi_{\bar{n}}(x) = \sum_{\vec{p}} e^{-i\vec{p}\cdot x} \xi_{\bar{n},p} . \quad (2.35)$$

The ‘‘Wilson line’’, $W_{\bar{n}}(x)$, has its origin in the nonlocal structure of the inverse of $n\cdot D^c$ in Eq. (2.33) which by itself is remnant of integrating out the η field, and it is given by:

$$W_{\bar{n}}(x) = P \exp \left[ig_s \int_{-\infty}^x ds n\cdot A^c(sn) \right] , \quad (2.36)$$

where, as in the definition of the parton distribution function, P stands for path ordering. It can also be shown that the appearance of Wilson lines in the lagrangian is a requirement of collinear gauge invariance. The operator \mathcal{P} is known as the ‘‘label’’ operator since its action on a collinear field $\xi_{\bar{n},p}$ yields its label momentum,

$$\begin{aligned}
n\cdot \mathcal{P} \xi_{\bar{n},p} &= n\cdot p \xi_{\bar{n},p} , \\
\mathcal{P}_{\perp}^{\mu} \xi_{\bar{n},p} &= P_{\perp}^{\mu} \xi_{\bar{n},p} .
\end{aligned} \quad (2.37)$$

Some of the Feynman rules obtained from \mathcal{L}_q are given in Fig. 2.2. The complete soft collinear effective lagrangian has to include the self coupling of the gauge bosons as well as the gauge fixing terms and the ghost fields.

In the lagrangian \mathcal{L}_q , the collinear quarks and gluons are of the same collinearity type, i.e., either n or \bar{n} -collinear so there is no vertex that couples the n -collinear quark with \bar{n} gluon or vice-versa. Of course this is so by construction so that we avoid having propagators of large virtuality (of order Q^2) as would be the case otherwise. However, in certain applications one needs to consider the case when two collinear quarks or gluons are moving in (almost) opposite directions.

In the effective theory these partons can interact amongst each other through the exchange of soft partons only and the soft gluon “hidden” in the covariant derivative D supplies the required field. In this case one constructs effective operators built from fields with different collinearity subject to gauge invariance requirements. This has been done in a systematic way in [61]. One important object relevant to our discussion of DIS and DY processes is the electromagnetic current,

$$j_{eff}^\mu = \bar{\xi}_n W_n \gamma^\mu W_n^\dagger \xi_{\bar{n}}. \quad (2.38)$$

Matching this effective operator onto the full QCD analogue

$$j^\mu = \bar{\psi} \gamma^\mu \psi, \quad (2.39)$$

will be the first step towards establishing that the SCET reproduces the infrared behavior of full QCD (as it must!). By doing so we obtain the first matching coefficient needed for the resummation program. Moreover it will be shown, to first

order in α_s , that the SCET also reproduces the full QCD results with real gluon emission when the full QCD results are taken to the threshold (or the soft) region.

2.3 *Standard Model Higgs Production* *in the Large Top Quark Mass Limit*

The scalar Higgs particle is a fundamental constituent of the standard model (SM) lagrangian which results from a very specific pattern of electroweak symmetry breaking via the Higgs mechanism. This mechanism does *not* arise as a natural consequence of some underlying physical principles and it is not *a priori* clear why physics should be formulated in this way. It is really an unpleasant situation that the Higgs particle is the only particle of the SM lagrangian that has not been discovered yet. Its direct detection is a main goal of the current and future colliders⁹.

The dominant mechanism for the production of the SM Higgs particle at the Large Hadron Collider (LHC) is through a gluon-gluon fusion process, $gg \rightarrow H$, mediated, to leading order, through a triangle loop of heavy quarks as shown in Fig. 2.2 (a). The reason for this is that the coupling of the Higgs particle to a quark is proportional to the quark mass. So to a very good approximation, it would be sufficient to only consider the running of the heaviest quark in the triangle loop, namely the top quark with on-shell (pole mass) $M_t \simeq 176 \text{ GeV}$. Since the gluon fusion process is a strong interaction one, then the QCD radiative corrections have

⁹ The current lower bound on the Higgs mass is 114.1 GeV at 95% CL, which has been implied by the direct search performed at LEP [62].

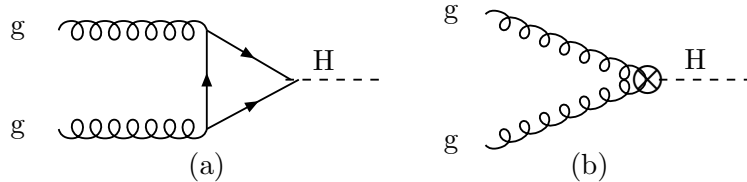


Fig. 2.2: Leading order diagram for $gg \rightarrow H$: (a) in full QCD and (b) in the effective theory.

to be taken into account. Indeed, next-to-leading order QCD corrections were found to be sizeable: 50% – 100% [39, 63].

The radiative corrections, whether real or virtual, can in principle be calculated from the QCD lagrangian. However, it turns out that one can tremendously reduce the complexity of the calculation by working within an effective theory in which $M_t \rightarrow \infty$. The resulting leading order Feynman diagram is shown in Fig. 2.2 (b). The accuracy of this limit has been analyzed in [63, 64, 65, 66] and it turned out to be less than 5% for $M_H \leq 2M_t$. Calculations of the QCD radiative corrections up to NNLO in the large top quark mass limit have been performed in [67, 68, 69]. In this limit, the reaction $gg \rightarrow H$ shares many features with the DY partonic one; $q\bar{q} \rightarrow \gamma^*$. At leading order, we have in both cases two partons that produce a color singlet particle. This means that one can simply write down a factorization theorem for the Higgs production cross section by taking $Q^2 = M_H^2$ and, of course, the parton distribution functions needed are those of a gluon inside a proton. The similarities between the two processes will persist also when radiative corrections are taken into account. This will be made more clear in Chapter 4.

The effective lagrangian for the Higgs production through gluon-gluon fusion

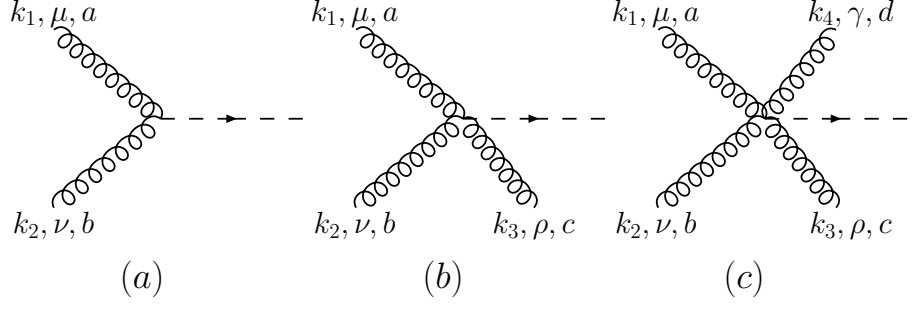


Fig. 2.3: Vertices for the scalar particle coupling to the gluons.

in the large top quark mass limit can be written as

$$\mathcal{L} = -\frac{1}{4}C_\phi(M_t, \mu) \phi G^{\mu\nu} G_{\mu\nu}(\mu) , \quad (2.40)$$

where ϕ is the scalar field, $G^{\mu\nu}$ is the gluon field strength, C_ϕ is the effective coupling, and μ is a renormalization scale. The coefficient function C_ϕ is determined by the top quark triangle loop diagram, including all the QCD corrections taken in the limit $M_t \rightarrow \infty$. It can be expressed as a product, $C_{\text{EW}}(M_t)C_T(M_t, \mu)$, where C_{EW} contains only the electroweak contributions while $C_T(M_t, \mu)$ has all the QCD corrections.

From the above effective Lagrangian, we can derive the vertex of the gluon fields and the scalar particle. We show these vertices in Fig. 2.3. Up to a common factor C_ϕ we have for Fig. 2.3(a),

$$i\delta_{ab} (k_1 \cdot k_2 g_{\mu\nu} - k_{1\nu} k_{2\mu}) . \quad (2.41)$$

Then the coupling to three gluons in Fig. 2.3(b) reads

$$-g_s f_{abc} [(k_1 - k_2)_\rho g_{\mu\nu} + (k_2 - k_3)_\mu g_{\rho\nu} + (k_3 - k_1)_\nu g_{\rho\mu}] . \quad (2.42)$$

Finally, the coupling to four gluons shown in Fig. 2.3(c) is

$$-ig_s^2 \{ f_{abe} f_{cde} (g_{\mu\rho} g_{\nu\gamma} - g_{\mu\gamma} g_{\nu\rho})$$

$$\begin{aligned}
& + f_{ace}f_{bde} (g_{\mu\nu}g_{\rho\gamma} - g_{\mu\gamma}g_{\nu\rho}) \\
& + f_{ade}f_{bce} (g_{\mu\nu}g_{\rho\gamma} - g_{\mu\rho}g_{\nu\gamma}) \}.
\end{aligned} \tag{2.43}$$

As in the case of the DY process, higher order calculations for the $gg \rightarrow H$ inclusive cross section are plagued with the singular distributions $\mathcal{D}_i(z)$ and $\delta(1-z)$, where $z = Q^2/s \equiv M_H^2/s$ and s is the invariant mass squared of the colliding partons which lead, in the limit $z \rightarrow 1$ to large logarithmic enhancement (when Mellin transformed). Resummation of these logarithms needs to be performed in order to get a coefficient function—in the factorization theorem—as accurate as possible.

It is worth mentioning that at the LHC, the invariant mass of the incoming protons $S \equiv (P_1 + P_2)^2 = (14 \text{ TeV})^2$, which is much larger¹⁰ than $Q^2 = M_H^2$. Thus the ratio $\tau_H \equiv Q^2/S$ is much smaller than 1. However the ratio τ_H is not correlated with $z \equiv Q^2/s$ in the sense that one can safely take the limit $z \rightarrow 1$ although $\tau_H \ll 1$. The main reason for this is that the parton distribution functions $f_{a/h}(x)$ (for either a quark or a gluon) are strongly vanishing in the limit $x \rightarrow 1$; thus the partons that initiate the reaction (with momenta fractions x_i) may well have: $z \equiv \frac{Q^2}{s} = \frac{Q^2}{x_1 x_2 S} \gg Q^2/S \equiv \tau_H$. Therefore the partonic threshold may well be achieved much faster than the hadronic one.

¹⁰ It is widely accepted that the Higgs mass will lie well below the 1 TeV scale.

3. DEEP INELASTIC SCATTERING AND DRELL-YAN IN SOFT COLLINEAR EFFECTIVE THEORY

In the preceding chapter, theoretical constructs such as factorization theorems, infrared (IR) divergences and large logarithmic enhancements were briefly discussed. Through considering first the deep inelastic hard scattering cross section (DIS) and then the Drell-Yan (DY) process in the threshold region (i.e., $x \rightarrow 1$ for DIS and $z \rightarrow 1$ for DY), we illustrate these constructs by working perturbatively to first order in the strong coupling constant α_s .

The relevance of the soft collinear effective theory (SCET) for inclusive processes and in the soft limit will be illustrated. We will show that the SCET is a “legitimate” effective theory of QCD in the sense that it captures the full IR behavior of this theory. More importantly, we will illustrate how the QCD corrections—real and virtual—to the leading order partonic channels for DIS and DY are related to the matching coefficients at each one of the relevant matching scales.

In this chapter we only obtain the matching coefficient for DIS and DY to first order in α_s . However, the lessons learned will allow us (in the next chapter), and *without* relying on SCET, to generalize the treatment to higher orders in α_s and to perform threshold resummation of the leading and subleading logarithms.

3.1 Deep Inelastic Scattering

Starting with the inclusive DIS process and concentrating on the strong interaction part, the reaction is,

$$P(P) + \gamma^*(q) \rightarrow X, \quad (3.1)$$

where an incoming proton P with momentum P^μ is struck by a highly virtual photon γ^* with momentum q^μ where: $-q^2 \equiv Q^2 \gg \Lambda_{\text{QCD}}^2$. X stands for all the unobserved final hadronic states. The leading order partonic channel,

$$q(p_1) + \gamma^*(q) \rightarrow q(p_2), \quad (3.2)$$

is best visualized in the Breit frame where the incoming quark is moving along the $+z$ direction with $p_1^+ \equiv \frac{1}{\sqrt{2}}(p^0 + p^3) \simeq Q$. The incoming photon and the outgoing quark are moving in the $-z$ direction with $p_2^- \equiv \frac{1}{\sqrt{2}}(p^0 - p^3) \simeq Q$. As before, letting $x = Q^2/2p_1 \cdot q$, the invariant mass of final hadronic states is

$$(p_1 + q)^2 = Q^2 \frac{1-x}{x}, \quad (3.3)$$

which vanishes in the limit $x \rightarrow 1$.

The quantity of interest to us is the structure function $F_2(x, Q^2)$ which is the partonic version of $F_2(x_B, Q^2)$ given in Eq. 2.11 and can be calculated perturbatively [70] with the expansion, $F_2(x, Q^2) = \sum_{i=0} a_s^i F_2^{(i)}$, where, $a_s = \frac{\alpha_s}{4\pi}$. The calculation of $F_2^{(1)}$ in perturbative QCD (pQCD) is straightforward and it proceeds by taking the Fourier transform of a product of the matrix elements of two electromagnetic currents between initial and final states

$$\frac{1}{8\pi} \sum_X \langle q(p_1) | j^\mu(x) | X \rangle \langle X | j^\nu(0) | q(p_1) \rangle (2\pi)^4 \delta^{(4)}(p_1 + q - p_X). \quad (3.4)$$

Then one projects out¹ F_2 by multiplying the last equation with the metric tensor $-g_{\mu\nu}$. We follow the normalization of [70] where,

$$F_2^{(0)} = \delta(1 - x). \quad (3.5)$$

As in every effective treatment one needs to identify the relevant scales. In the case we are considering it is easily seen that there are two scales. The higher scale Q^2 and the lower scale $p_X^2 = Q^2 \frac{(1-x)}{x}$. Since we are interested in the limit $x \rightarrow 1$, these two scales are well separated and an effective theoretic treatment would make sense. We now establish a relation between the intermediate scale and the expansion parameter λ in the SCET. We have already mentioned that in the region $x \rightarrow 1$, only partons which are either soft or collinear to the outgoing quark can be emitted into the final state. Using the power counting of SCET we find

$$p_X^2 = Q^2 \left[\lambda^2 + \mathcal{O}(\lambda^4) \right] , \quad (3.6)$$

where the right hand side comes from either collinear gluon to the outgoing quark with momentum $Q(\lambda^2, 1, \lambda)$ or from emission of soft gluon with momentum $Q(\lambda^2, \lambda^2, \lambda^2)$. Thus one has to identify $\lambda^2 \simeq (1 - x)$.

At the scale Q^2 , the final hadronic states may be effectively treated as a massless jet moving in the $n = (1, 0, 0, -1)$ direction. Since the structure function is essentially a product of two currents (in momentum space) then we start the matching procedure by considering the quark form factor defined as the matrix

¹ In the limit $x \rightarrow 1$ the structure function F_1 has no singularities and it is totally ignored in our discussion.

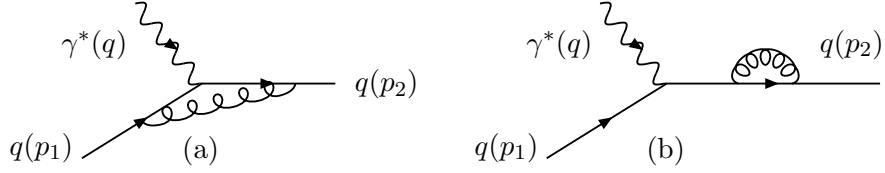


Fig. 3.1: Vertex correction and wave function renormalization in full QCD.

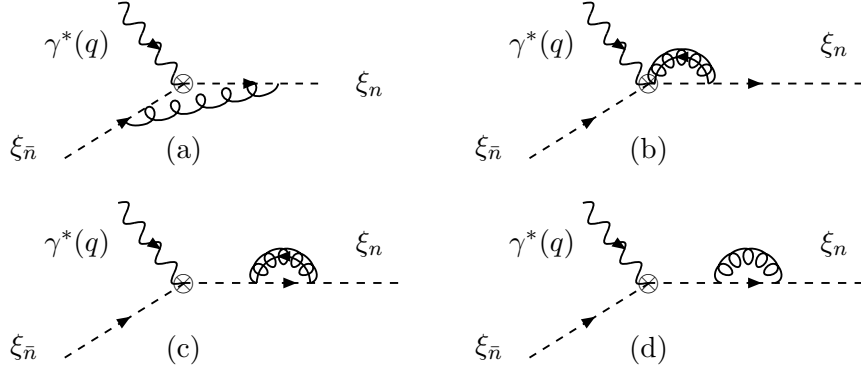


Fig. 3.2: Soft and collinear gluon contributions in SCET for electromagnetic current.

element of the full QCD current taken between quark states

$$\langle q(p_2) | j^\mu | q(p_1) \rangle \equiv i e_q (\bar{u} \gamma^\mu u) F(\alpha_s, Q^2). \quad (3.7)$$

This matrix element is matched onto the matrix element of the (gauge invariant) effective current,

$$\langle q_n | j^\mu | q_{\bar{n}} \rangle = C_{\text{DIS}}(Q^2) \langle q_{n, \tilde{p}_2} | \bar{\xi}_n W_n \gamma^\mu W_n^\dagger \xi_{\bar{n}} | q_{\bar{n}, \tilde{p}_1} \rangle, \quad (3.8)$$

where \tilde{p}_i are the label momenta for the incoming ($i = 1$) and outgoing ($i = 2$) quarks which are represented in the effective theory by $\xi_{\bar{n}}$ and ξ_n fields, respectively. The matching coefficient $C(Q^2)$ is calculated order by order in $\alpha_s(Q^2)$ and it has the expansion, $C(Q^2) = \sum_{i=0} a_s^i C^{(i)}(Q^2)$ with the obvious leading order result: $C^{(0)} = 1$.

The first nontrivial result for the matching coefficient is $C^{(1)}$. The contribu-

tions to the electromagnetic current in full QCD are given in Fig. 3.1 where we have the vertex correction in (a) and the wave function diagram² in (b). Working in Feynman gauge and in on-shell dimensional regularization (DR) in $d = 4 - 2\varepsilon$ to regularize the UV as well as the IR divergences, the contribution from the vertex diagram for massless quarks is standard and reads:

$$V = \frac{\alpha_s}{4\pi} C_F \gamma^\mu \left(\frac{Q^2}{\mu^2 e^{\gamma_E}} \right)^{-\varepsilon} \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \left[\frac{1}{\varepsilon_{\text{UV}}} - \frac{2}{\varepsilon_{\text{IR}}^2} - \frac{4}{\varepsilon_{\text{IR}}} - 8 \right], \quad (3.9)$$

where $C_F = 4/3$ and μ is the renormalization scale. This result displays both kinds of divergences. The labelling of ε specifies the kind of divergence. Using the properties of the Γ function given in Appendix A and the expansion

$$a^\varepsilon = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} (\ln a)^n, \quad (3.10)$$

one gets

$$V = \frac{\alpha_s}{4\pi} C_F \gamma^\mu \left[\frac{1}{\varepsilon_{\text{UV}}} - \frac{2}{\varepsilon_{\text{IR}}^2} + \frac{2 \ln \frac{Q^2}{\mu^2} - 4}{\varepsilon_{\text{IR}}} - \ln^2 \frac{Q^2}{\mu^2} + 3 \ln \frac{Q^2}{\mu^2} - 8 + \zeta_2 \right] \quad (3.11)$$

where $\zeta_2 = \frac{\pi^2}{6}$. The IR double pole arises when the virtual gluon propagates with a soft momentum and in a direction collinear to the outgoing quark³. The appearance of this double pole leads to the double logarithms in the last equation which are known as ‘‘Sudakov double logarithms’’.

In pure DR, calculation of the wave function diagram leads to an integral of the form,

$$I = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^4}, \quad (3.12)$$

² A diagram with gluon attached to the incoming quark leg also has to be considered.

³ This double pole cancels when the contribution from real gluon emission diagrams is taken into account.

where k is the loop momentum. For $d = 4 - 2\varepsilon$, the dimension of I is $-2\varepsilon \neq 0$, however, this integral is *scaleless* since it does not depend on any dimensional quantity. So in DR we always set the value of such integrals to zero. Moreover, one can show that their vanishing result is of the form,

$$I \propto \frac{1}{\varepsilon_{\text{UV}}} - \frac{1}{\varepsilon_{\text{IR}}}. \quad (3.13)$$

Let us illustrate this for the integral in Eq. (3.12). Writing

$$\frac{1}{k^4} = \frac{1}{k^2(k^2 - m^2)} - \frac{m^2}{k^4(k^2 - m^2)}, \quad (3.14)$$

we get two integrals to calculate. The first is

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(k^2 - m^2)} = \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{[xk^2 + (1-x)(k^2 - m^2)]^2}, \quad (3.15)$$

where we used Feynman parametrization

$$\frac{1}{ab} = \int_0^1 \frac{1}{[ax + (1-x)b]^2}. \quad (3.16)$$

The last integral can be calculated easily using the standard formula

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{[k^2 - m^2]^\nu} = \frac{(-1)^\nu i}{(4\pi)^{(d/2)}} \frac{\Gamma(\nu - d/2)}{\Gamma(\nu)} \left(\frac{1}{m^2}\right)^{\nu - \frac{d}{2}}, \quad (3.17)$$

which is a special case of a more general one,

$$\int \frac{d^d k}{(2\pi)^d} \frac{(k^2)^\nu}{[k^2 - m^2]^\mu} = (-1)^{\mu-\nu} i \frac{\pi^{d/2}}{(2\pi)^d} (m^2)^{\nu-\mu+\frac{d}{2}} \frac{\Gamma(\nu + \frac{d}{2})\Gamma(\mu - \nu - \frac{d}{2})}{\Gamma(\frac{d}{2})\Gamma(\mu)}. \quad (3.18)$$

Thus the integral in Eq. (3.15) becomes

$$\begin{aligned} &= \frac{i}{(4\pi)^{2-\varepsilon}} \Gamma(\varepsilon) \frac{\Gamma(1)\Gamma(1-\varepsilon)}{\Gamma(2-\varepsilon)} (m^2)^{-\varepsilon} \int_0^1 dx (1-x)^{-\varepsilon} \\ &= \frac{i}{(4\pi)^{2-\varepsilon}} \Gamma(\varepsilon) (m^2)^{-\varepsilon} \frac{1}{1-\varepsilon}. \end{aligned} \quad (3.19)$$

To get the result in the last line we used Eq. (A.14) and the properties of the Γ function, Eq. (A.12) and Eq. (A.13) given in Appendix A. At this stage we introduce the renormalization scale μ_0^2 . The integration measure $d^d k$ has dimension $d = 4 - 2\varepsilon$ so we multiply it by μ_0^2 , where μ_0 has dimension of mass, to get the correct dimension. Thus the result of the integral becomes

$$= \frac{i}{(4\pi)^{2-\varepsilon}} \Gamma(\varepsilon) \left(\frac{\mu_0^2}{m^2} \right)^\varepsilon \frac{1}{1-\varepsilon}, \quad (3.20)$$

which gives

$$= \frac{i}{4\pi^2} \left[\frac{1}{\varepsilon_{\text{UV}}} + \ln \frac{\mu^2}{m^2} + 1 \right], \quad (3.21)$$

where in $\overline{\text{MS}}$ scheme, $\mu^2 = 4\pi e^{-\gamma_E} \mu_0^2$.

The second remaining integral is calculated using Eq. (3.18),

$$\begin{aligned} -m^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^2 [k^2 - m^2]} &= i \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{1}{2^{\frac{d}{2}}} \left(\frac{\mu_0^2}{m^2} \right)^\varepsilon \frac{\Gamma(-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(2-\varepsilon)} \\ &= \frac{i}{(4\pi^2)} \left[-\frac{1}{\varepsilon_{\text{IR}}} - \ln \frac{\mu^2}{m^2} - 1 \right]. \end{aligned} \quad (3.22)$$

Combining the last result with Eq. (3.21) we get

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^4} = \frac{i}{(4\pi)^2} \left[\frac{1}{\varepsilon_{\text{UV}}} - \frac{1}{\varepsilon_{\text{IR}}} \right]. \quad (3.23)$$

Going back to the wave function diagram, its contribution is

$$I_w = -\frac{\alpha_s}{4\pi} C_F \gamma^\mu \left[\frac{1}{\varepsilon_{\text{UV}}} - \frac{1}{\varepsilon_{\text{IR}}} \right]. \quad (3.24)$$

Taking one half⁴ the contribution from the two wave function graphs and the vertex

⁴ This is a simple result of the LSZ reduction formula, where each external fermion leg is multiplied by $Z_2^{1/2}$.

correction, one gets

$$\langle q(p_2)|j^\mu|q(p_1)\rangle = \gamma^\mu \left[1 + \frac{\alpha_s}{4\pi} C_F \left(-\frac{2}{\varepsilon_{\text{IR}}^2} + \frac{2 \ln \frac{Q^2}{\mu^2} - 3}{\varepsilon_{\text{IR}}} - \ln^2 \frac{Q^2}{\mu^2} + 3 \ln \frac{Q^2}{\mu^2} - 8 + \zeta_2 \right) \right]. \quad (3.25)$$

As this result shows, all the remaining divergences in the current matrix element are IR in origin and all the UV divergences cancel. The cancellation of the UV divergences can be considered as a consequence of electromagnetic current conservation (or the ‘‘Ward’’ identity) even when the strong interaction effects are taken into account. That is, the electromagnetic coupling constant α_{em} is renormalized only by the vacuum polarization contribution of the photon propagator, and QCD effects should not contribute. Thus all vertices, such as, γqq , $\gamma q\bar{q}$, Zqq , Wqq , etc., are free from strong interaction divergences. This statement holds to all orders in perturbation theory.

The effective theory diagrams of the electromagnetic current are displayed in Fig. 3.2 and they are calculated using the SCET Feynman rules given in Fig. 2.1. However, one immediately sees that all these diagrams, including the vertex diagram, lead to a scaleless integrals in pure DR and so they vanish. Their total contribution can be inferred from the discussion in [60] and one has,

$$\langle q_n|j^\mu|q_{\bar{n}}\rangle = \gamma^\mu \left[1 + \frac{\alpha_s}{4\pi} C_F \left(\frac{2}{\varepsilon_{\text{UV}}^2} - \frac{2 \ln \frac{Q^2}{\mu^2} - 3}{\varepsilon_{\text{UV}}} - \frac{2}{\varepsilon_{\text{IR}}^2} + \frac{2 \ln \frac{Q^2}{\mu^2} - 3}{\varepsilon_{\text{IR}}} \right) \right]. \quad (3.26)$$

The fact that the IR poles in the last equation agree with those in the full QCD result, Eq. (3.25), is a consistency requirement imposed on the SCET as it must reproduce the IR behavior of the full theory. The UV poles in Eq. (3.26) are to be

cancelled by a counterterm,

$$\delta_c \equiv Z_c - 1 , \quad (3.27)$$

where the subscript c refers to the effective current. Thus, to first order in α_s we have

$$Z_c = 1 + \frac{\alpha_s}{4\pi} C_F \left[-\frac{2}{\varepsilon_{\text{UV}}^2} + \frac{2 \ln \frac{Q^2}{\mu^2} - 3}{\varepsilon_{\text{UV}}} \right] , \quad (3.28)$$

and the renormalized effective current becomes

$$\langle q_n | j^\mu | q_{\bar{n}} \rangle_{\text{ren}} = \gamma^\mu \left[1 + \frac{\alpha_s}{4\pi} C_F \left(-\frac{2}{\varepsilon_{\text{IR}}^2} + \frac{2 \ln \frac{Q^2}{\mu^2} - 3}{\varepsilon_{\text{IR}}} \right) \right] . \quad (3.29)$$

Comparing the last equation with Eq. (3.25) one gets

$$\begin{aligned} C_{\text{DIS}}^{(0)} &= 1 , \\ C_{\text{DIS}}^{(1)} \left(\frac{Q^2}{\mu^2}, Q^2 \right) &= C_F \left[-\ln^2 \frac{Q^2}{\mu^2} + 3 \ln \frac{Q^2}{\mu^2} - 8 + \zeta_2 \right] , \end{aligned} \quad (3.30)$$

which is the finite part of the full QCD result. At the scale $\mu^2 = Q^2$, we simply get

$$C_{\text{DIS}}^{(1)}(Q^2) \equiv C_{\text{DIS}}^{(1)}(1, Q^2) = C_F [-8 + \zeta_2] , \quad (3.31)$$

and all the logarithms vanish. The μ dependence of the matching coefficient is governed by the anomalous dimension of the effective current defined as

$$\frac{d \ln C_{\text{DIS}}(\mu)}{d \ln \mu} = \gamma_1(\mu) , \quad \gamma_1(\mu) \equiv \sum_{i=1} a_s^i \gamma_1^{(i)}(\mu) , \quad (3.32)$$

giving

$$\gamma_1^{(1)}(\mu) = C_F \left[4 \ln \frac{Q^2}{\mu^2} - 6 \right] . \quad (3.33)$$

The solution of Eq. (3.32) can be written as

$$C_{\text{DIS}}(Q^2/\mu_I^2, \alpha_s(\mu_I^2)) = C_{\text{DIS}}(1, \alpha_s(Q^2)) \times \exp \left(- \int_{\mu_I}^Q \gamma_1 \frac{d\mu}{\mu} \right) , \quad (3.34)$$

where μ_I is some intermediate scale below Q . This anomalous dimension also controls the μ dependence of the counterterm δ_c . This follows by noticing that the quark form factor $\langle q(p_2)|j^\nu|q(p_1)\rangle$ is a physical quantity that should not depend on the renormalization scale μ . This is so, since the original lagrangian, \mathcal{L}_{QCD} , knows nothing about this scale. This could also be argued from a different point of view. Since the electromagnetic current is UV finite, then there is no need to dimensionally regularize the UV divergences and one could set $\varepsilon_{\text{UV}} = 0$. This means that the calculation of the virtual diagrams could be performed in four dimensional space-time ($d = 4$) and the scale μ would not show up in the result of this calculation. From this we can write

$$\frac{d\langle q(p_2)|j^\nu|q(p_1)\rangle}{d\ln\mu} = 0. \quad (3.35)$$

This equation, Eqs. (3.8) and (3.32), simply give us

$$\frac{d\ln Z_c}{d\ln\mu} = -\gamma_1, \quad (3.36)$$

which is a familiar form.

At the intermediate scale, $Q^2(1-x)$, the final state invariant mass p_X^2 must be considered large compared to Λ_{QCD}^2 but far below the higher scale Q^2 . In order to have $x \neq 1$ there must be a radiation of real gluons into the final state. The complete set of diagrams in the effective theory can be found in [60] where only soft or collinear real gluons contribute. The combined result is

$$= \frac{\alpha_s}{4\pi} C_F \left\{ 4\mathcal{D}_1(x) + \left(4\ln\frac{Q^2}{\mu^2} - 3 \right) \mathcal{D}_0(x) + \delta(1-x) \left(2\ln^2\frac{Q^2}{\mu^2} - 3\ln\frac{Q^2}{\mu^2} + 7 - 6\zeta_2 \right) - \frac{1}{\varepsilon} (4\mathcal{D}_0(x) + 3\delta(1-x)) \right\}. \quad (3.37)$$

To obtain the last result, the counterterm δ_c has been added to the contribution of the real diagrams to cancel the UV divergences. However, not all the divergences cancel as is seen from the remaining pole in ε . Since it is not cancelled by the counterterm, this pole has to be IR in origin: $\varepsilon = \varepsilon_{\text{IR}}$. At this stage we need to recall our discussion of the QCD factorization theorem for the DIS process where we argued that the (UV finite) cross section is factorized into a coefficient function which is IR finite (i.e, it is finite when $\varepsilon_{\text{IR}} \rightarrow 0$) convoluted with a parton distribution function (PDF) $f_{a/b}$. In the case at hand the relevant PDF is $f_{q/q}$ (quark-in-quark). Up to first order in α_s , its renormalized expression calculated in full QCD (see, e.g., [48]) is

$$f_{q/q}(x) = \delta(1-x) - \frac{\alpha_s}{4\pi} C_F \times 2 \frac{1}{\varepsilon_{\text{IR}}} \left((1+x^2) \mathcal{D}_0(x) + \frac{3}{2} \delta(1-x) \right). \quad (3.38)$$

The pole in the last expression is due *only* to collinear gluon emission from the incoming quark and all soft divergences, which appear as a double pole in individual diagrams, cancel in the sum of contributions from real and virtual diagrams.

In the limit $x \rightarrow 1$, the expression for $f_{q/q}(x)$ reduces to that in Eq. (3.37). Thus performing matching at the intermediate scale onto a quark distribution function cancels all the ε dependence and one gets a finite matching coefficient

$$\begin{aligned} \mathcal{M}_{\text{DIS}}^{(1)}(x) = C_F \left\{ 4\mathcal{D}_1(x) + \left(4 \ln \frac{Q^2}{\mu^2} - 3 \right) \mathcal{D}_0(x) \right. \\ \left. + \delta(1-x) \left(2 \ln^2 \frac{Q^2}{\mu^2} - 3 \ln \frac{Q^2}{\mu^2} + 7 - 6\zeta_2 \right) \right\}, \quad (3.39) \end{aligned}$$

where $\mathcal{M}(x) = \sum_{i=0} a_s^i \mathcal{M}^{(i)}(x)$ and $\mathcal{M}^{(0)}(x) = \delta(1-x)$. Two remarks are in order. First, we notice that the last result is exactly equal to the finite part of the full

QCD calculation of the real diagrams in the soft limit [70], i.e., only when the plus distributions and the delta function are taken into account. Moreover, the ability to perform matching at the intermediate scale, i.e., to obtain a *finite* result when subtracting the contributions from both “sides” of the matching scale, is equivalent to the fact that the factorization theorem works, order by order in perturbation theory.

To manifestly see the large threshold logarithms, we take the Mellin transform of $\mathcal{M}_{\text{DIS}}^{(1)}(x)$. Using the formulae given in Appendix A, one gets for large \bar{N}

$$\mathcal{M}_{N,\text{DIS}}^{(1)} = C_F \left[2 \ln^2 \frac{Q^2}{\bar{N}\mu^2} - 3 \ln \frac{Q^2}{\bar{N}\mu^2} + 7 - 4\zeta_2 \right]. \quad (3.40)$$

As in the case at the higher scale, Q^2 , the matching coefficients at the specific value of the matching scale should be free of any large logarithms. For the matching at the intermediate scale, the large logarithms cancel by choosing $\mu^2 = Q^2/\bar{N}$ and one gets

$$\mathcal{M}_{N,\text{DIS}}^{(1)} = C_F [7 - 4\zeta_2]. \quad (3.41)$$

The intermediate scale, $\mu^2 = Q^2/\bar{N}$, has to be related to $Q^2(1-x) = Q^2\lambda^2$ and one can simply relate the $(1-x)$ with its conjugate variable $1/\bar{N}$ in the sense that the large \bar{N} limit is associated⁵ with $x \rightarrow 1$. Thus in moment space, \mathcal{M}_N can depend logarithmically only on the ratio $\frac{Q^2}{\bar{N}\mu^2}$. This observation persists to higher orders in perturbation theory, as we will show explicitly in the next chapter, and it should be valid to all orders in perturbation theory. Consequently, by the choice $\mu^2 = Q^2/\bar{N}$,

⁵ This is a heuristic observation and it can be argued as some kind of “uncertainty” relation between two conjugate variables: $(1-x)\bar{N} \sim 1$.

the matching coefficient at this scale is free of large logarithms to all orders in α_s .

Since the two matching coefficients depend on two different mass ratios then any *single* choice of μ^2 will certainly create large logarithms in the coefficient function due to the wide separation of the matching scales. If μ^2 is chosen close enough to the higher scale (to eliminate the $\ln^k(Q^2/\mu^2)$) then it will be far away from the intermediate scale and the logarithms of $Q^2/\bar{N}\mu^2$ will be large. Of course, the same thing occurs if μ^2 is close to the intermediate scale. This phenomenon is certainly not exclusive to the context being discussed, and it appears in almost all perturbative applications of effective field theories.

Having obtained the matching coefficients, we are now able to write down the first order QCD corrections of the coefficient function of the structure function F_2 denoted by $G^{(1),s+v}(x)$. The (s+v) stands for the combination of contributions from the virtual diagrams (which are proportional to $\delta(1-x)$) and of the contribution of real diagrams taken in the soft limit. The result is the following:

$$G_{\text{DIS}}^{s+v}(x) = \left| 1 + a_s C^{(1)} \left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right) \right|^2 \times \left(\delta(1-x) + a_s \mathcal{M}^{(1)}(x) \right) + \mathcal{O}(\alpha_s^2). \quad (3.42)$$

The square of C is due to the fact that the structure function F_2 is proportional to a product of two currents, while C is just the matching coefficient of only one current. The essence of this equation is that a cross section, calculated in a certain kinematical limit—in our case it is just the soft+virtual—can be viewed as a two step matching process which results in *multiplicative* matching coefficients. Each matching coefficient incorporates a certain contribution to the final result. As we have seen, the matching coefficient at the higher scale collects the contributions

from the virtual diagrams and the matching at the intermediate scale supplies the remaining contributions from real gluon emission diagrams.

The results for leading order (LO) and next-to-leading order (NLO) are

$$G_{\text{DIS}}^{(0)(s+v)}(x) = \delta(1-x) , \quad (3.43)$$

$$G_{\text{DIS}}^{(1)(s+v)}(x) = C_F \left[4\mathcal{D}_1(x) + \left(4 \ln \frac{Q^2}{\mu^2} - 3 \right) \mathcal{D}_0(x) + \delta(1-x) \left(3 \ln \frac{Q^2}{\mu^2} - 9 - 4\zeta_2 \right) \right] ,$$

where, $G_{\text{DIS}}^{(s+v)}(x) = \sum_{i=0} a_s^i G^{(i),(s+v)}(x)$. Since no contributions from higher order terms in perturbation theory were included yet the result of $G^{(1)}$, is a *fixed* order one as opposed to a *resummed* one. The resummed results will be presented in the next chapter.

We now turn to discuss the DY process. Our aim is to obtain the matching coefficients as in the DIS case. It will also be interesting to verify whether the SCET calculation of DY cross section also reproduces that of QCD in the soft+virtual limit, and if the factorization theorem for DY holds to first order in the strong coupling.

3.2 Drell-Yan Process

For the quark and anti-quark annihilation process into a gauge boson, say a photon,

$$q(p_1) + \bar{q}(p_2) \rightarrow \gamma^*(q) , \quad (3.44)$$

the invariant mass of the final hadronic state is

$$p_X^2 = Q^2 \left(1 + \frac{1}{z} - \frac{1}{x_1} - \frac{1}{x_2} \right) \cong Q^2 [(1-x_1)(1-x_2)] \cong Q^2 \lambda^4 , \quad (3.45)$$

where

$$z = \frac{Q^2}{s}, \quad x_1 = \frac{Q^2}{2p_1 \cdot q}, \quad x_2 = \frac{Q^2}{2p_2 \cdot q}. \quad (3.46)$$

The threshold region $z \rightarrow 1$ implies that $1 - z \sim \lambda^2$, so we get $p_X^2 \sim Q^2(1 - z)^2$. The intermediate scale in this case is then: $Q^2(1 - z)^2$. As in the case of the DIS cross section, we start the matching procedure at the higher scale Q^2 , and match the the full QCD annihilation current $q\bar{q} \rightarrow \gamma^*$ onto the SCET current. This could be accomplished by calculating the contribution from the virtual diagrams in full QCD and match the result on that of the effective theory. However, this is unnecessary due to the fact the quark form factor in the annihilation case, i.e.,

$$j^\mu = \langle 0 | j^\mu | q(p_1) \bar{q}(p_2) \rangle, \quad (3.47)$$

can be obtained from that of the scattering (DIS) case by the following observation. In the DIS case, the virtuality of the incoming photon q^2 is negative, i.e., the photon is *space-like*, and it is customary to display the results for DIS related quantities in terms of $Q^2 \equiv -q^2 > 0$. For the annihilation case, $q^2 \equiv Q^2 > 0$. This is the *time-like* region. Viewed as an analytic function of q^2 , the space-like quark form factor can be analytically continued to the time-like region through the replacement of

$$\ln \frac{Q^2}{\mu^2} \rightarrow \ln \frac{Q^2}{\mu^2} - i\pi \quad (3.48)$$

in all the logarithms that appear in the DIS results. With this substitution we get

$$\begin{aligned} \langle 0 | j^\mu | p_1 \bar{p}_2 \rangle = & \gamma^\mu \left\{ 1 + \frac{\alpha_s}{4\pi} C_F \left[-\frac{2}{\varepsilon_{\text{IR}}^2} + \frac{1}{\varepsilon_{\text{IR}}} \left(2 \ln \frac{Q^2}{\mu^2} - 3 \right) \right. \right. \\ & \left. \left. - \ln^2 \frac{Q^2}{\mu^2} + 3 \ln \frac{Q^2}{\mu^2} - 8 + \frac{7\pi^2}{6} + i\pi \left(-2 \ln \frac{Q^2}{\mu^2} + 3 \right) \right] \right\}. \end{aligned} \quad (3.49)$$

As in the case of DIS, the matching coefficient for the DY process at the scale Q^2 is the finite part of the last result,

$$C_q^{(1)}\left(\frac{Q^2}{\mu^2}\right) = C_F \left[-\ln^2 \frac{Q^2}{\mu^2} + 3 \ln \frac{Q^2}{\mu^2} - 8 + \frac{7\pi^2}{6} + i\pi \left(-2 \ln \frac{Q^2}{\mu^2} + 3 \right) \right], \quad (3.50)$$

where the complete coefficient function⁶ C_q has the already familiar expansion: $\sum_{i=0} a_s^i C_q^{(i)}$. The complex part of the matching condition will not play any role for the discussion in this chapter. However, its role will become clear when we consider contributions⁷ of order α_s^2 . Since $C_q^{(1)}$ and $C_{\text{DIS}}^{(1)}$ differ by a μ -independent term, then their μ -dependence is governed by the same quantity⁸: γ_1 .

At the intermediate scale, we consider the SCET Feynman diagrams with real gluon emission. These are displayed in Fig. 3.3.

The details of the calculation can be found in [40]. The calculation was performed using the SCET Feynman rules to calculate amplitude squared and then integrating over the phase space for the production of a massive photon and massless gluon⁹. The result reads,

$$\mathcal{I}_q(z) = \frac{\alpha_s}{4\pi} C_F \left[\frac{4}{\epsilon^2} \delta(1-z) - \frac{4}{\epsilon} \ln \frac{Q^2}{\mu^2} \delta(1-z) - \frac{8}{\epsilon} \mathcal{D}_0(z) \right] \quad (3.51)$$

⁶ Anticipating the comparison between the DY case and the Higgs production of the next chapter, we refer to DY-related quantities with subscript q (quark) and for the Higgs case with g (gluon). The DIS case will be referred to simply as DIS.

⁷ For the rest of the discussion in this chapter we refer to the real part of $C^{(1)}$ as the complete $C^{(1)}$.

⁸ Strictly speaking, the anomalous dimension for the DY case will have a complex part. This can be simply seen from Eq. (3.33) and replacing the log term according to Eq. (3.48). However this complex part will cancel in $|C_q|^2$ as could be easily seen from Eq. (3.34).

⁹ See Appendix B for details.

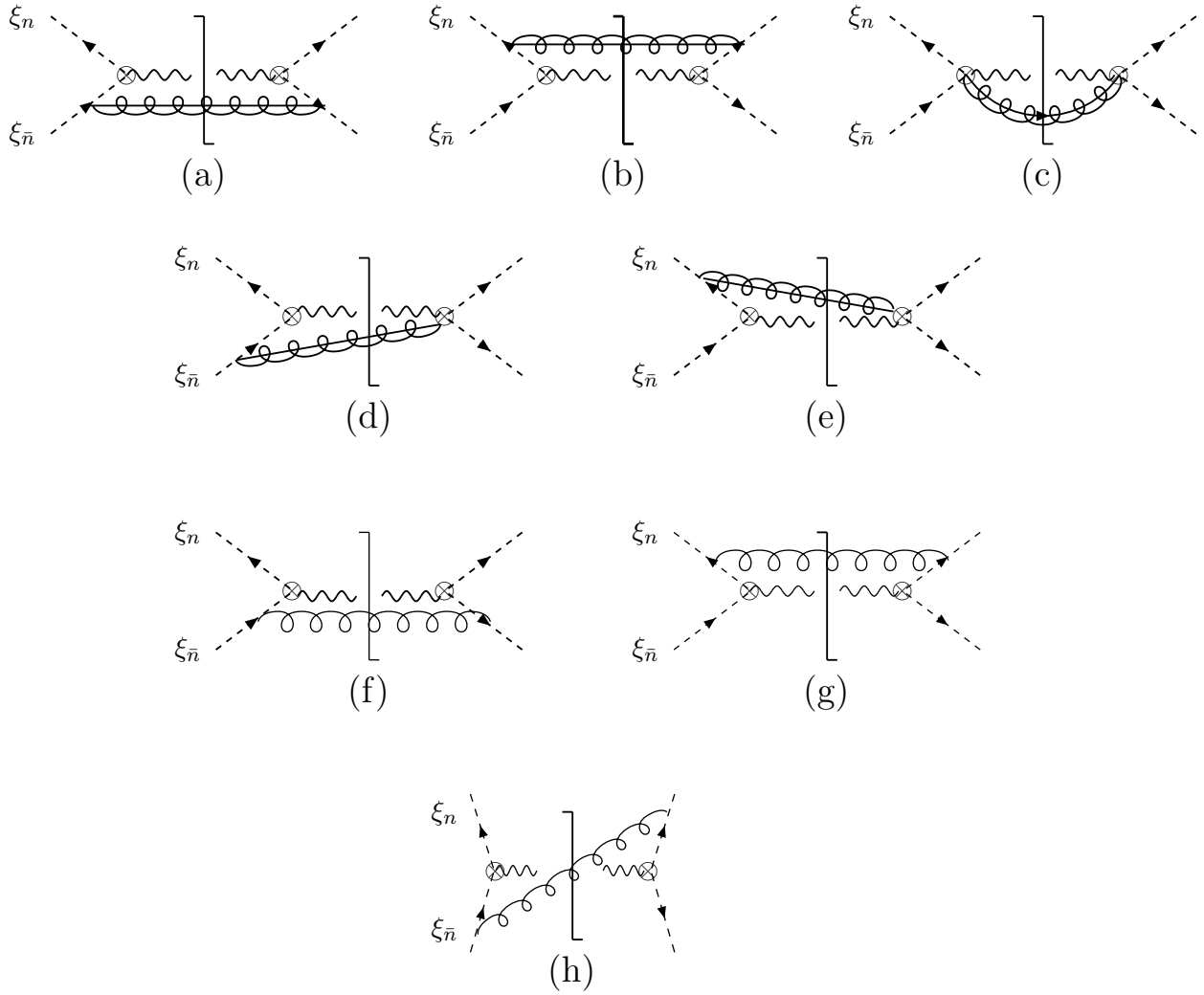


Fig. 3.3: SCET Feynman diagrams for Drell-Yan process with real gluon emission. Graphs

(a)-(e) include only collinear gluons; (f)-(h) have only soft gluons.

$$+8 \left(\frac{\ln(1-z)}{1-z} \right)_+ + 16\mathcal{D}_1(z) + 8 \ln \frac{Q^2}{\mu^2} \mathcal{D}_0(z) + \delta(1-z) \left(\ln^2 \frac{Q^2}{\mu^2} - 3\zeta_2 \right) \Big].$$

As expected, the result is identical to the full QCD calculation of the real gluon contributions to the DY differential cross section in the soft limit [70]. The ϵ -terms in Eq. (3.51) are a combination of UV and IR contributions. Let us express them in the following form:

$$\frac{\alpha_s}{4\pi} C_F \left[2 \times \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} - \frac{2}{\epsilon} \ln \frac{Q^2}{\mu^2} \right) \delta(1-z) - 2 \times \left(\frac{4}{\epsilon} \mathcal{D}_0(z) + \frac{3}{\epsilon} \delta(1-z) \right) \right].$$

These divergences will be cancelled by adding twice the counterterm, δ_c , given in Eq. (3.27) and by matching the DY cross section onto a product of two parton distributions given in Eq. (3.38).

The fact that we need two PDFs is a requirement of the factorization theorem for DY process—since there are two hadrons (partons) in the initial state. We also notice that, aside from divergences that are cancelled by the counterterm of the effective current, the only remaining divergences are absorbed into the PDFs. Thus the matching coefficient obtained is

$$\mathcal{M}_q^{(1)}(z) = C_F \left[16\mathcal{D}_1(z) + 8 \ln \frac{Q^2}{\mu^2} \mathcal{D}_0(z) + \delta(1-z) \left(\ln^2 \frac{Q^2}{\mu^2} - 3\zeta_2 \right) \right], \quad (3.52)$$

with the complete $\mathcal{M}(z)$ having the usual expansion in $a_s = \alpha_s/4\pi$. Taking the Mellin moments we get

$$\mathcal{M}_{N,q} = C_F \left[2 \ln^2 \frac{\bar{N}^2 \mu^2}{Q^2} + 2\zeta_2 \right], \quad (3.53)$$

To minimize the large logarithms, we need to choose $\mu = Q/\bar{N}$. This is different from the DIS case where at the intermediate scale we set $\mu = Q/\sqrt{\bar{N}}$. Thus for DY we identify $1-z \sim 1/N$.

3.3 Summary

We have illustrated so far how to obtain the matching coefficients at the two relevant scales for both DIS and DY processes to first order in α_s . With these matching coefficients we were able to express the coefficient functions that enter into the factorization theorems as a result of a two-step matching procedure. In the first step we match at the higher scale Q^2 , the full QCD current onto an effective current that “recognizes” only soft and collinear partons. By this procedure we integrate out virtualities of order Q^2 , in the sense that in the effective theory, all virtual particles have virtualities of order $Q^2\lambda^2$ with $\lambda \ll 1$. We have argued that the soft and collinear modes give rise to the complete IR behavior of the quark form factor manifested by the ε_{IR} poles in the current matrix element (whether q^2 is space-like or time-like). To NLO in α_s this has been explicitly demonstrated in [71]. This fact allows the matching procedure to be performed since these IR poles cancel when we take the difference between the full and the effective results for the current matrix elements and the UV poles cancel in the respective theories by the renormalization procedure. Thus the matching coefficients are guaranteed to be finite. They are also free of logarithms when μ^2 is set equal to Q^2 .

At the intermediate scale, we demonstrated that the renormalized (i.e., after the counterterm of the effective current has been taken into account) contribution from the SCET diagrams with real gluon emission can be matched onto the relevant parton distribution functions calculated in full QCD and taken to the threshold region ($x \rightarrow 1$ or $z \rightarrow 1$). The remainder is finite and accounts for the soft+virtual

limit of the full QCD calculation. This observation will be of much help. While in the SCET there exists only next-to-leading calculations for the inclusive processes we are interested in, in full QCD this limit has already been considered almost twenty years ago since it highly simplifies the pQCD calculations. Results for DIS, DY and gluon production in the soft+virtual limit exist to third order in α_s . We will make use of these results to extract the matching coefficients at the intermediate scale to higher orders in α_s .

Up to now we have considered only the matching coefficients at the two relevant scales. However we also need to consider the physics in between them. This is established through the concept of running and evolution equations. These evolution equations have boundary conditions which are exactly the matching coefficients calculated at the relevant scales (which are free of logarithms). By solving these equations, one usually gets exponentials that contain all the information that lies in the “gap” between the widely separated scales. We will see that this simple program gives us a resummed expressions for the relevant physical quantities and the harmful large logarithms are brought under control.

4. THRESHOLD RESUMMATION

4.1 Anomalous Dimensions of Effective Currents

In this chapter we carry out the resummation of threshold logarithms for the three inclusive processes of interest: deep inelastic scattering (DIS), Drell-Yan (DY) and the Higgs boson production in the large top quark mass limit. All three processes will be considered in the limit of large Q^2 . The starting point is the factorization theorem¹ for DY and Higgs which reads in moment space [72]:

$$\sigma_N = \sigma_0 \cdot G_N(Q) \cdot q(Q, N) \cdot q(Q, N) , \quad (4.1)$$

where σ_0 is the Born level cross section which is not relevant to our discussion and can be safely set to 1 for all processes we consider. $G_N(Q)$ and $q(Q, N)$ are the Mellin transforms of the coefficient function $G(z)$ and of the parton distribution function (PDF) of partons in hadrons, respectively. For DIS we only have one PDF in Eq. (4.1) since there is only one hadron in the initial state.

In the effective theory approach we start the matching procedure at the higher scale. As we have discussed in the previous chapter, we match the matrix element of the full QCD vector current: $J^\mu = \bar{\psi}\gamma^\mu\psi$, taken between on-shell quark² states

¹ In Eq. (4.1) we set the factorization and renormalization scales equal to Q .

² We remind the reader that we always work with massless quarks.

onto the corresponding effective theory matrix element

$$J_{\text{QCD}} = C(Q^2/\mu^2, \alpha_s(\mu^2)) J_{\text{eff}}(\mu) , \quad (4.2)$$

where $C(Q^2/\mu^2, \alpha_s(\mu^2))$ is the matching coefficient and μ is the renormalization scale of the effective current. The matrix element of the vector current gives us the quark form factor for DIS and DY processes. For the Higgs production—in the large top quark mass limit—it is the scalar current, $J = G^{\mu\nu} G_{\mu\nu}$, where $G^{\mu\nu}$ is the gluon field strength tensor introduced in Eq. (2.2).

The effective currents are μ -dependent and this dependence is controlled by the anomalous dimensions γ_1 defined as

$$\mu \frac{d \ln C_{\text{DIS}}(\mu)}{d\mu} = \gamma_1(\mu) , \quad \gamma_1(\mu) \equiv \sum_i a_s^i \gamma_1^{(i)}(\mu), \quad a_s \equiv \frac{\alpha_s}{4\pi}. \quad (4.3)$$

The matrix elements of the full currents are physical quantities and therefore they are μ -independent. This fact is related to the cancellation of the ultraviolet (UV) divergences we noticed in the previous chapter in the sense that if there are no UV divergences, one could actually perform the calculation without referring to any UV regulator, like dimensional regularization (DR). Since in DR, the μ dependence enters when regulating the UV by going to $d = 4 - 2\varepsilon$, then no μ dependence shows up in $d = 4$. From this observation and the chain rule applied to Eq. (4.2) we simply get

$$\gamma_1(\mu) = -\mu \frac{d \ln J_{\text{eff}}}{d\mu}. \quad (4.4)$$

The anomalous dimension is a function of both Q^2/μ^2 and $\alpha_s(\mu^2)$. In fact, it can be shown that it is a linear function of $\ln Q^2/\mu^2$ to all orders in perturbation

theory [73, 60] and can be expressed as

$$\gamma_1 = A(\alpha_s) \ln Q^2/\mu^2 + \tilde{B}_1(\alpha_s) \quad (4.5)$$

where A and \tilde{B}_1 have the usual expansion in a_s : $A = \sum_i a_s^i A^{(i)}$ and $\tilde{B}_1 = \sum_i a_s^i \tilde{B}_1^{(i)}$ where α_s is the renormalized coupling constant.

We have already demonstrated in the previous chapter that the quark form factor is free of UV divergences to first order in α_s and argued that this holds to all orders in perturbation theory. Thus it can be expressed as

$$F = \tilde{C} \left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right) S \left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), 1/\epsilon \right), \quad (4.6)$$

where F is defined in Eq. (3.7) and the ϵ dependence is due only to the IR poles in on-shell DR: $\epsilon = \epsilon_{\text{IR}}$. The quantity S contains all the IR poles and the coefficient function \tilde{C} is finite in the limit $\epsilon \rightarrow 0$. The function S has no finite terms

$$\lim_{\epsilon \rightarrow 0} S(1/\epsilon) = 0. \quad (4.7)$$

Both functions \tilde{C} and S have the usual expansion in a_s and Eq. (4.6) holds to all orders in α_s by construction. The function S could actually be considered as the renormalized form factor in the effective theory. Its ϵ dependence is consistent with the fact that virtual diagrams in the effective theory vanish in pure DR due to scaleless integrals and all the UV poles are removed by adding the relevant counterterms, therefore what is left is just IR poles. All these poles come with a minus sign relative to the corresponding UV ones, thus, we simply have

$$\gamma_1(\mu) = -\mu \frac{d \ln S}{d\mu}. \quad (4.8)$$

Since \tilde{C} does not contain any pole part, we can also write

$$\gamma_1(\mu) = -\mu \left. \frac{d \ln F}{d\mu} \right|_{\text{pole part}}. \quad (4.9)$$

From this point of view, the function \tilde{C} is just the matching coefficient of the full QCD current onto the effective current given in Eq. (4.2). It can be extracted to any given order in α_s from Eq. (4.6) once the perturbative full QCD calculation of F is known up to the same order. The same is true for the anomalous dimension γ_1 . In this sense one really does *not* have to rely on any effective calculation of the sort done in the previous chapter.

The quark form factor is one of the best studied quantities in pQCD [74, 75, 76] and it can be expressed as

$$\ln F(\alpha_s) = \frac{1}{2} \int_0^{Q^2/\mu^2} \frac{d\xi}{\xi} \left(K(\alpha_s(\mu^2), \epsilon) + G(1, \alpha_s(\xi\mu, \epsilon), \epsilon) + \int_\xi^1 \frac{d\lambda}{\lambda} A(\alpha_s(\lambda\mu, \epsilon)) \right), \quad (4.10)$$

where A is the anomalous dimension of the K and G functions known as the “cusp anomalous dimension”;

$$A(\alpha_s) = \mu^2 \frac{dG}{d\mu^2} = -\mu^2 \frac{dK}{d\mu^2}. \quad (4.11)$$

The functions K and G are perturbatively calculable and have the familiar expansion in a_s . By construction, the function K contains only IR poles

$$K = \sum_{n=1}^{\infty} \frac{K_n(\alpha_s)}{\epsilon^n}, \quad K_n(\alpha_s) = \sum_{m=n}^{\infty} a_s^m k_n^{(m)}, \quad (4.12)$$

and it has no explicit dependence on the renormalization scale μ . This fact and Eq. (4.11) allows the reconstruction of the entire function by knowing the perturbative expansion of the anomalous dimension: $A = \sum_i a_s^i A^{(i)}$.

The function G contains the finite ‘‘hard’’ part of the form factor. It is finite in the limit $\varepsilon \rightarrow 0$ and expanded as: $G(1, \alpha_s, \varepsilon) = \sum_i a_s^i G^{(i)}(\varepsilon)$. Thus $G^{(i)}$ and $A^{(i)}$ completely determine $\ln F$. Moreover, the anomalous dimension A is exactly the same one as in Eq. (4.5).

The anomalous dimensions A_q for the DY and DIS and A_g for the Higgs production have been calculated up to $\mathcal{O}(\alpha_s^3)$ [56]:

$$\begin{aligned}
A_{(q,g)}^{(1)} &= 4C_{(q,g)} , \\
A_{(q,g)}^{(2)} &= 8C_F C_{(q,g)} \left[\left(\frac{67}{18} - \zeta_2 \right) C_A - \frac{5}{9} N_F \right] , \\
A_{(q,g)}^{(3)} &= 16C_{(q,g)} \left[C_A^2 \left(\frac{245}{24} - \frac{67}{9} \zeta_2 + \frac{11}{6} \zeta_3 + \frac{11}{5} \zeta_2^2 \right) + C_F N_F \left(\frac{-55}{24} + 2\zeta_3 \right) \right. \\
&\quad \left. + C_A N_F \left(-\frac{209}{108} + \frac{10}{9} \zeta_2 - \frac{7}{3} \zeta_3 \right) + N_F^2 \left(-\frac{1}{27} \right) \right] , \tag{4.13}
\end{aligned}$$

where $C_{(q,g)} = C_F$ for the quark form factor and C_A for the gluon form factor. In this sense, A is universal. The coefficients $G^{(i)}$ have been also calculated in [78] up to $i = 3$:

$$\begin{aligned}
G_{(q,g)}^{(1)} &= 2(B_{2,(q,g)}^{(1)} - \delta_g \beta_0) + f_{(q,g)}^{(1)} + \epsilon \tilde{G}_{(q,g)}^{(1)} + \epsilon^2 \tilde{\tilde{G}}_{(q,g)}^{(1)} , \\
G_{(q,g)}^{(2)} &= 2(B_{2,(q,g)}^{(2)} - 2\delta_g \beta_1) + f_{(q,g)}^{(2)} + \beta_0 \tilde{G}_{(q,g)}^{(1)} + \epsilon \tilde{G}_{q,g}^{(2)} , \\
G_{(q,g)}^{(3)} &= 2(B_{2,(q,g)}^{(3)} - 3\delta_g \beta_2) + f_{(q,g)}^{(3)} + \beta_1 \tilde{G}_{(q,g)}^{(1)} + \beta_0 \left[\tilde{G}_{(q,g)}^{(2)} - \beta_0 \tilde{\tilde{G}}_{(q,g)}^{(1)} \right] , \tag{4.14}
\end{aligned}$$

and δ_g is zero for quark and 1 for gluon. The ‘tilde’ is defined as

$$\tilde{F} = \frac{1}{\varepsilon} [F - F(\varepsilon = 0)]. \tag{4.15}$$

The B ’s are the coefficient in front of the delta function $\delta(x - 1)$ in Altarelli-Parisi splitting function and they have also been calculated up to $\mathcal{O}(\alpha_s^3)$ [56]:

$$B_{2,q}^{(1)} = 3C_F ,$$

$$\begin{aligned}
B_{2,q}^{(2)} &= 4C_F C_A \left(\frac{17}{24} + \frac{11}{3}\zeta_2 - 3\zeta_3 \right) - 4C_F N_F \left(\frac{1}{12} + \frac{2}{3}\zeta_2 \right) + 4C_F^2 \left(\frac{3}{8} - 3\zeta_2 + 6\zeta_3 \right) , \\
B_{2,q}^{(3)} &= 16C_A C_F N_F \left(\frac{5}{4} - \frac{167}{54}\zeta_2 + \frac{1}{20}\zeta_2^2 + \frac{25}{18}\zeta_3 \right) \\
&\quad + 16C_A C_F^2 \left(\frac{151}{64} + \zeta_2\zeta_3 - \frac{205}{24}\zeta_2 - \frac{247}{60}\zeta_2^2 + \frac{211}{12}\zeta_3 + \frac{15}{2}\zeta_5 \right) \\
&\quad - 16C_A^2 C_F \left(\frac{1657}{576} - \frac{281}{27}\zeta_2 + \frac{1}{8}\zeta_2^2 + \frac{97}{9}\zeta_3 - \frac{5}{2}\zeta_5 \right) \\
&\quad - 16C_F N_F^2 \left(\frac{17}{144} - \frac{5}{27}\zeta_2 + \frac{1}{9}\zeta_3 \right) \\
&\quad - 16C_F^2 N_F \left(\frac{23}{16} - \frac{5}{12}\zeta_2 - \frac{29}{30}\zeta_2^2 + \frac{17}{6}\zeta_3 \right) \\
&\quad + 16C_F^3 \left(\frac{29}{32} - 2\zeta_2\zeta_3 + \frac{9}{8}\zeta_2 + \frac{18}{5}\zeta_2^2 + \frac{17}{4}\zeta_3 - 15\zeta_5 \right) , \tag{4.16}
\end{aligned}$$

for quarks and

$$\begin{aligned}
B_{2,g}^{(1)} &= \frac{11}{3}C_A - \frac{2}{3}N_F , \\
B_{2,g}^{(2)} &= 4C_A N_F \left(-\frac{2}{3} \right) + 4C_A^2 \left(\frac{8}{3} + 3\zeta_3 \right) + 4C_F N_F \left(-\frac{1}{2} \right) , \\
B_{2,g}^{(3)} &= 16C_A C_F N_F \left(-\frac{241}{288} \right) + 16C_A n_f^2 \frac{29}{288} - 16C_A^2 N_F^2 \left(\frac{233}{288} + \frac{1}{6}\zeta_2 + \frac{1}{12}\zeta_2^2 + \frac{5}{3}\zeta_3 \right) \\
&\quad + 16C_A^3 \left(\frac{79}{32} - \zeta_2\zeta_3 + \frac{1}{6}\zeta_2 + \frac{11}{24}\zeta_2^2 + \frac{67}{6}\zeta_3 - 5\zeta_5 \right) \\
&\quad + 16C_F N_F^2 \frac{11}{144} + 16C_F^2 N_F \frac{1}{16} , \tag{4.17}
\end{aligned}$$

for gluons. The quantities $f^{(i)}$ are universal and are given by

$$\begin{aligned}
f_{(q,g)}^{(1)} &= 0 , \\
f_{(q,g)}^{(2)} &= C_{(q,g)} C_A \left[\frac{808}{27} - \frac{22}{3}\zeta_2 - 28\zeta_3 \right] + C_{(q,g)} N_F \left[-\frac{112}{27} + \frac{8}{3}\zeta_2 \right] , \\
f_{(q,g)}^{(3)} &= C_{(q,g)} C_A^2 \left[\frac{136781}{729} - \frac{12650}{81}\zeta_2 - \frac{1361}{3}\zeta_3 + \frac{352}{5}\zeta_2^2 + \frac{176}{3}\zeta_2\zeta_3 + 192\zeta_5 \right] \\
&\quad + C_{(q,g)} C_A N_F \left[-\frac{11842}{729} + \frac{2828}{81}\zeta_2 + \frac{728}{27}\zeta_3 - \frac{96}{5}\zeta_2^2 \right] + C_{(q,g)} C_F N_F \left[-\frac{1771}{27} \right. \\
&\quad \left. + 4\zeta_2 + \frac{304}{9}\zeta_3 + \frac{32}{5}\zeta_2^2 \right] + C_{(q,g)} N_F^2 \left[-\frac{2080}{729} - \frac{40}{27}\zeta_2 + \frac{112}{27}\zeta_3 \right] . \tag{4.18}
\end{aligned}$$

Writing, in the usual, $\gamma_1 = \sum_i a_s^i \gamma_1^{(i)}$, we then have

$$\gamma_{1,(q,g)}^{(i)} = A_{(q,g)}^{(i)} \ln Q^2/\mu^2 + B_{1,(q,g)}^{(i)} + 2i\delta_g \beta_{i-1}. \quad (4.19)$$

Comparing with Eq. (4.5) we see that: $\tilde{B}^{(i)} = B^{(i)} + 2i\delta_g \beta_{i-1}$. The $B_{1,(q,g)}$ are related to $B_{2,(q,g)}$ through the relation:

$$B_{1,(q,g)}^{(i)} = -2B_{2,(q,g)}^{(i)} - f_{(q,g)}^{(i)}. \quad (4.20)$$

Below, we show how to get the last relation up to $i = 3$; however, we remark that it should work to all orders in perturbation theory³. Considering the quark form factor (although the treatment applies as well to the gluon case), the starting point is the expansion of the form factor in terms of the strong coupling constant.

Dropping for simplicity the subscript q , we have

$$F^b = 1 + (a_s^b)\lambda^{-\varepsilon} F^{(1)} + (a_s^b)^2 \lambda^{-2\varepsilon} F^{(2)} + (a_s^b)^3 \lambda^{-3\varepsilon} F^{(3)} + \mathcal{O}(\alpha_s^4), \quad (4.21)$$

where $\lambda \equiv \frac{Q^2}{\mu^2}$, and a_s^b is the bare coupling constant which is related to the renormalized one, a_s , through the relation:

$$a_s^b = a_s \left[1 - \frac{\beta_0}{\varepsilon} a_s + \left(\frac{\beta_0^2}{\varepsilon^2} - \frac{\beta_1}{2\varepsilon} \right) a_s^2 \right] + \mathcal{O}(\alpha_s^4). \quad (4.22)$$

Taking the logarithm of F^b we get

$$\begin{aligned} \ln F^b &= (a_s^b)\lambda^{-\varepsilon} F^{(1)} + (a_s^b)^2 \lambda^{-2\varepsilon} \left(F^{(2)} - \frac{1}{2}(F^{(1)})^2 \right) \\ &\quad + (a_s^b)^3 \lambda^{-3\varepsilon} \left(F^{(3)} - F^{(1)} F^{(2)} + \frac{1}{3}(F^{(1)})^3 \right). \end{aligned} \quad (4.23)$$

³ Since we interested to obtain the anomalous dimension γ_1 , we did not supply explicit expressions for $G^{(i)}$ and the tilde functions in Eq. (4.15) since they do not contribute to γ_1 .

Substituting Eq. (4.22) in Eq. (4.23) we get

$$\begin{aligned}
\ln F &= a_s \lambda^{-\varepsilon} F^{(1)} + a_s^2 \left[-\frac{\beta_0}{\varepsilon} \lambda^{-\varepsilon} F^{(1)} + \lambda^{-2\varepsilon} \left(F^{(2)} - \frac{1}{2} (F^{(1)})^2 \right) \right] \\
&+ a_s^3 \left[\left(\frac{\beta_0^2}{\varepsilon^2} - \frac{\beta_1}{\varepsilon} \right) \lambda^{-\varepsilon} F^{(1)} - \frac{2\beta_0}{\varepsilon} \lambda^{-2\varepsilon} \left(F^{(2)} - \frac{1}{2} (F^{(1)})^2 \right) \right. \\
&\left. + \lambda^{-3\varepsilon} \left(F^{(3)} - F^{(1)} F^{(2)} + \frac{1}{3} (F^{(1)})^3 \right) \right]. \tag{4.24}
\end{aligned}$$

The expansion coefficients F_i can be expressed in terms of the $A^{(i)}$ of Eq. (4.13)

and the $G^{(i)}$ of Eq. (4.15) in the following form [77]:

$$\begin{aligned}
F^{(1)} &= -\frac{1}{2} \frac{1}{\varepsilon^2} A^{(1)} - \frac{1}{2} \frac{1}{\varepsilon} G^{(1)}, \\
F^{(2)} &= \frac{1}{8} \frac{1}{\varepsilon^4} (A^{(1)})^2 + \frac{1}{8} \frac{1}{\varepsilon^3} A^{(1)} (2G^{(1)} - \beta_0) + \frac{1}{8} \frac{1}{\varepsilon^2} ((G^{(1)})^2 - A^{(2)} - 2\beta_0 G^{(1)}) - \frac{1}{4} \frac{1}{\varepsilon} G^{(2)}, \\
F^{(3)} &= -\frac{1}{48} \frac{1}{\varepsilon^6} (A^{(1)})^3 - \frac{1}{16} \frac{1}{\varepsilon^5} (A^{(1)})^2 (G^{(1)} - \beta_0) - \frac{1}{144} \frac{1}{\varepsilon^4} A^{(1)} (9(G^{(1)})^2 - 9A^{(2)} - 27\beta_0 G^{(1)} \\
&+ 8\beta_0^2) - \frac{1}{144} \frac{1}{\varepsilon^3} (3(G^{(1)})^3 - 9A^{(2)} G^{(1)} - 18A^{(1)} G^{(2)} + 4\beta_1 A^{(1)} - 18\beta_0 (G^{(1)})^2 \\
&+ 16\beta_0 A^{(2)} + 24\beta_0^2 G^{(1)}) + \frac{1}{72} \frac{1}{\varepsilon^2} (9G^{(1)} G^{(2)} - 4A^{(3)} - 6\beta_1 G^{(1)} - 24\beta_0 G^{(2)}) \\
&- \frac{1}{6} \frac{1}{\varepsilon} G^{(3)}. \tag{4.25}
\end{aligned}$$

We now substitute these terms in Eq. (4.24). Displaying only the simple poles⁴ in ε , we get after many cancellations,

$$\begin{aligned}
&a_s \frac{1}{2\varepsilon} \left[A^{(1)} \ln \lambda - G^{(1)} \right] + (a_s)^2 \frac{1}{4\varepsilon} \left[A^{(2)} \ln \lambda - G^{(2)} + \beta_0 \frac{1}{\varepsilon} G^{(1)} \right] \\
&+ (a_s)^3 \frac{1}{6\varepsilon} \left[A^{(3)} \ln \lambda - G^{(3)} + \frac{1}{\varepsilon} \beta_0 G^{(2)} + \frac{1}{\varepsilon} \beta_1 G^{(1)} - \frac{1}{\varepsilon^2} \beta_0^2 G^{(1)} \right]. \tag{4.26}
\end{aligned}$$

The terms in the square brackets are exactly the expansion coefficients $\gamma_1^{(i)}$ of γ_1 . This can be seen as follows. Taking (minus) the derivative of $\ln F$ with respect

⁴ The terms in the square brackets are finite for $\varepsilon = 0$.

to $\ln \mu$ and using,

$$\frac{da_s}{d \ln \mu} = -2\varepsilon a_s + \mathcal{O}(\alpha_s^2) , \quad (4.27)$$

in $d = 4 - 2\varepsilon$, then the finite result is just the anomalous dimension. Comparing the non-logarithmic parts of $\gamma_1^{(i)}$ with Eq. (4.15), we immediately get up to $i = 3$, the desired relation given in Eq. (4.20).

We remark here that the anomalous dimension γ_1 could have been calculated from the matching coefficient itself by taking the logarithm of Eq. (4.6)

$$\ln F = \ln C \left(\frac{Q^2}{\mu^2}, \alpha_s \right) + \ln S \left(\frac{Q^2}{\mu^2}, \varepsilon \right) , \quad (4.28)$$

where we used the fact that $\tilde{C} = C$. Subtracting away all the ε terms from this relation, one then gets

$$\gamma_1 = \frac{d}{d \ln \mu} \{ \ln F |_{\text{finite part}} \} , \quad (4.29)$$

to any desired order in α_s . This method is straightforward but rather tedious, however, it completely agrees with the previous one.

Using Eq. (4.6) and the known expressions for the (renormalized) quark form factor [79] and for the Higgs production [80, 81] to second order in α_s we can write down the matching coefficient C up to the same order. As we have already discussed in the previous chapter, the time-like form factor for DY (and also for the Higgs production), contains a complex part, which means that C and S will also have complex parts. To the accuracy we are interested in, we need to keep the complex part only of $C^{(1)}$ as in Eq. (3.50). Let us illustrate the derivation of $C_g^{(2)}$. Similar analysis applies for $C_q^{(2)}$ and $C_{\text{DIS}}^{(2)}$. To second order in α_s the gluon form factor is

given by:

$$\begin{aligned}
F_g(Q^2) = & 1 + (-1)^{-\varepsilon} a_s \left(\frac{Q^2}{\mu^2} \right)^{-\varepsilon} C_A \left[-\frac{2}{\varepsilon^2} + \zeta(2) - \varepsilon \left(2 - \frac{14}{3} \zeta(3) \right) \right. \\
& + \varepsilon^2 \left(-6 + \frac{47}{20} \zeta^2(2) \right) \left. \right] + (-1)^{2\varepsilon} a_s^2 \left(\frac{Q^2}{\mu^2} \right)^{-2\varepsilon} \left[C_A^2 \left\{ \frac{2}{\varepsilon^4} - \frac{11}{6\varepsilon^3} \right. \right. \\
& - \left(\frac{67}{18} + \zeta(2) \right) \frac{1}{\varepsilon^2} - \left(-\frac{68}{27} - \frac{11}{2} \zeta(2) + \frac{25}{3} \zeta(3) \right) \frac{1}{\varepsilon} \\
& + \left. \left. \frac{5861}{162} + \frac{67}{6} \zeta(2) + \frac{11}{9} \zeta(3) - \frac{21}{5} \zeta^2(2) \right\} \right. \\
& + N_F C_A \left\{ \frac{1}{3\varepsilon^3} + \frac{5}{9\varepsilon^2} - \left(\frac{26}{27} + \zeta(2) \right) \frac{1}{\varepsilon} - \frac{808}{81} - \frac{5}{3} \zeta(2) - \frac{74}{9} \zeta(3) \right\} \\
& + N_F C_F \left\{ -\frac{1}{\varepsilon} - \frac{67}{3} + 16 \zeta(3) \right\} \left. \right] \\
& - (-1)^\varepsilon a_s \left(\frac{1}{\varepsilon} \right) \left(\frac{Q^2}{\mu^2} \right)^{-\varepsilon} \beta_0 C_A \left[-\frac{2}{\varepsilon^2} + \zeta(2) - \varepsilon \left(2 - \frac{14}{3} \zeta(3) \right) \right]. \quad (4.30)
\end{aligned}$$

It should be noted that the terms proportional to ε and ε^2 in $F_g^{(1)}$ are important in the derivation below. Expanding Eq. (4.6) up to $\mathcal{O}(\alpha_s^2)$ we get

$$F_g(Q^2) = 1 + a_s [C_g^{(1)} + S_g^{(1)}] + a_s^2 [C_g^{(2)} + C_g^{(1)} S_g^{(1)} + S_g^{(2)}]. \quad (4.31)$$

Working order by order in α_s we have

$$\begin{aligned}
C_g^{(1)}(Q^2/\mu^2) &= C_A \left[-\ln^2 \left(\frac{Q^2}{\mu^2} \right) + 7\zeta_2 + 2i\pi^2 \ln \left(\frac{Q^2}{\mu^2} \right) \right], \\
\text{Re}[C_g^{(2)}(Q^2/\mu^2)] &= C_A^2 \left[\frac{1}{2} \ln^4 \left(\frac{Q^2}{\mu^2} \right) + \frac{11}{9} \ln^3 \left(\frac{Q^2}{\mu^2} \right) - \left(\frac{67}{9} - 17\zeta_2 \right) \ln^2 \left(\frac{Q^2}{\mu^2} \right) \right. \\
&+ \left(\frac{80}{27} - \frac{88}{3} \zeta_2 - 2\zeta_3 \right) \ln \left(\frac{Q^2}{\mu^2} \right) + \frac{5105}{162} + \frac{335}{6} \zeta_2 - \frac{143}{9} \zeta_3 + \frac{125}{10} \zeta_2^2 \left. \right] \\
&+ C_A N_F \left[-\frac{2}{9} \ln^3 \left(\frac{Q^2}{\mu^2} \right) + \frac{10}{9} \ln^2 \left(\frac{Q^2}{\mu^2} \right) + \left(\frac{52}{27} + \frac{16}{3} \zeta_2 \right) \ln \left(\frac{Q^2}{\mu^2} \right) \right. \\
&- \left. \frac{916}{81} - \frac{25}{3} \zeta_2 - \frac{46}{9} \zeta_3 \right] \\
&+ C_F N_F \left[2 \ln^2 \left(\frac{Q^2}{\mu^2} \right) - \frac{67}{6} + 8\zeta_3 \right]. \quad (4.32)
\end{aligned}$$

Including the already known result from Chapter 3 for $C_{\text{DIS}}^{(1)}$, Eq.(3.30) one

finds

$$\begin{aligned}
C_{\text{DIS}}^{(1)}\left(\frac{Q^2}{\mu^2}, Q^2\right) &= C_F \left[-\ln^2\left(\frac{Q^2}{\mu^2}\right) + 3 \ln\left(\frac{Q^2}{\mu^2}\right) - 8 + \zeta_2 \right], \\
C_{\text{DIS}}^{(2)}(Q^2/\mu^2) &= C_F^2 \left[\frac{1}{2} \left(\ln^2\left(\frac{Q^2}{\mu^2}\right) - 3 \ln\left(\frac{Q^2}{\mu^2}\right) + 8 - \zeta_2 \right)^2 \right. \\
&\quad \left. + \left(\frac{3}{2} - 12\zeta_2 + 24\zeta_3 \right) \ln\left(\frac{Q^2}{\mu^2}\right) - \frac{1}{8} + 29\zeta_2 - 30\zeta_3 - \frac{44}{5}\zeta_2^2 \right] \\
&\quad + C_F N_F \left[-\frac{2}{9} \ln^3\left(\frac{Q^2}{\mu^2}\right) + \frac{19}{9} \ln^2\left(\frac{Q^2}{\mu^2}\right) - \left(\frac{209}{27} + \frac{4}{3}\zeta_2 \right) \ln\left(\frac{Q^2}{\mu^2}\right) \right. \\
&\quad \left. + \frac{4085}{324} + \frac{23}{9}\zeta_2 + \frac{2}{9}\zeta_3 \right] \\
&\quad + C_F C_A \left[\frac{11}{9} \ln^3\left(\frac{Q^2}{\mu^2}\right) + \left(2\zeta_2 - \frac{233}{18} \right) \ln^2\left(\frac{Q^2}{\mu^2}\right) \right. \\
&\quad \left. + \left(\frac{2545}{54} + \frac{22}{3}\zeta_2 - 26\zeta_3 \right) \ln\left(\frac{Q^2}{\mu^2}\right) - \frac{51157}{648} - \frac{337}{18}\zeta_2 + \frac{313}{9}\zeta_3 + \frac{44}{5}\zeta_2^2 \right]. \tag{4.33}
\end{aligned}$$

For the DY case we use the replacement of $\ln\left(\frac{Q^2}{\mu^2}\right)$ with $\ln\left(\frac{Q^2}{\mu^2}\right) - i\pi$.

The solution of the evolution equation for the matching coefficient C , Eq. (4.3),

and as a generalization of Eq (3.34), is written as:

$$C(Q^2/\mu_I^2, \alpha_s(\mu_I^2)) = C(1, \alpha_s(Q^2)) \times \exp\left(-\int_{\mu_I}^Q \gamma_1 \frac{d\mu}{\mu}\right) \tag{4.34}$$

where $C(1, \alpha_s(Q^2)) \equiv C(\alpha_s(Q^2))$ is just the non-logarithmic part of $C\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right)$.

It should be noted that when performing the integration over μ in the last equation, the dependence of the running coupling constant α_s on μ has to be taken into account. As we have argued in the previous chapter, the scale μ_I is to be identified with the scale $Q(1-x) \sim Q/\sqrt{N}$ for DIS and with $Q(1-z) \sim Q/\bar{N}$ for the DY and Higgs cases. The exponential in Eq. (4.34) will be supplemented by another two more exponentials which together will completely resum all the threshold logarithms.

4.2 The Soft Contribution

In Chapter 3 we have shown that the matching coefficient at the intermediate scale equals the finite part of the full QCD computation of the relevant partonic cross section with real gluon emission in the soft limit, in which one keeps only the plus distributions and the Dirac delta function. Based on this observation, we can write down the matching coefficients for DIS, DY and the Higgs production to the same order in α_s up to which the calculation in full QCD have already been performed. It should be noted that higher order calculations of partonic cross sections for the above-mentioned processes, have been calculated in full QCD while keeping the *full* dependence on the kinematical variables x and z . However, since we are primarily interested in the threshold region, we consider only the soft limit.

In the literature, though, it is common practice to combine the contributions from the virtual diagrams⁵ and those with real gluon emission, which comprises to the soft+virtual contribution. For DIS one consults Refs. [82, 83] and for the DY process, they can be found in Ref. [79]. For the Higgs production we refer to Refs. [80, 81]. For simplicity of reference we have gathered in Appendix C the results for the coefficient function G for the three processes in the soft+virtual limit up to second order in α_s together with their Mellin transform G_N for large values of N .

To obtain the matching coefficient $\mathcal{M}_N = \sum_i a_s^i \mathcal{M}_N^{(i)}$ to any desired order, we

⁵ Based on four-momentum conservation, the contribution of virtual diagrams to a cross section is always proportional to a delta function.

start from the following relation for DIS and DY:

$$G_N^{s+v} \left(\frac{Q^2}{\mu^2}, \bar{N}, \alpha_s(\mu^2) \right) = \left| C \left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right) \right|^2 \times \mathcal{M}_N \left(\frac{Q^2}{\mu^2}, \bar{N}, \alpha_s(\mu^2) \right), \quad (4.35)$$

which is a generalization of Eq. (3.42) to arbitrary higher orders in α_s . For the Higgs case, the above relation has to be modified and the right hand side has to be multiplied by C_ϕ^2 which comes from the effective lagrangian of the large top quark mass limit which also has an expansion in a_s . Aside from this extra factor, the treatment below is identical for all three processes⁶.

As we discussed earlier, this relation is just the result of a two-step matching at the two relevant scales—a procedure that results in multiplicative matching coefficients. It should be noted that writing a cross section in this form is not a new one; see, e.g., Eq. (2.25) in Ref. [84]. Expanding the above formula to third⁷ order in α_s we get

$$\begin{aligned} G_N^{(1),s+v} &= 2\text{Re}[C^{(1)}] + \mathcal{M}_N^{(1)}, \\ G_N^{(2),s+v} &= |C^{(1)}|^2 + 2\text{Re}[C^{(2)}] + 2\text{Re}[C^{(1)}]\mathcal{M}_N^{(1)} + \mathcal{M}_N^{(2)}, \\ G_N^{(3),s+v} &= 2\text{Re}[C^{(1)}C^{(2)*}] + 2\text{Re}[C^{(3)}] + |C^{(1)}|^2\mathcal{M}_N^{(1)} + 2\text{Re}[C^{(1)}]\mathcal{M}_N^{(2)} \\ &\quad + 2\text{Re}[C^{(2)}]\mathcal{M}_N^{(1)} + \mathcal{M}_N^{(3)}. \end{aligned} \quad (4.36)$$

These results are consistent with those of Ref. [85]. In the notation of that paper, the \mathcal{S}_i stands for pure real emission of n gluons into the final state. By “pure” we mean Feynman diagrams without any virtual vertex corrections. \mathcal{F}_i stands for the form factor type of diagrams, i.e., with virtual vertex corrections. These are

⁶ This will be the case until we discuss the anomalous dimension $\tilde{\gamma}_1$.

⁷ The third order result will not be used in our work. We include it here for completeness.

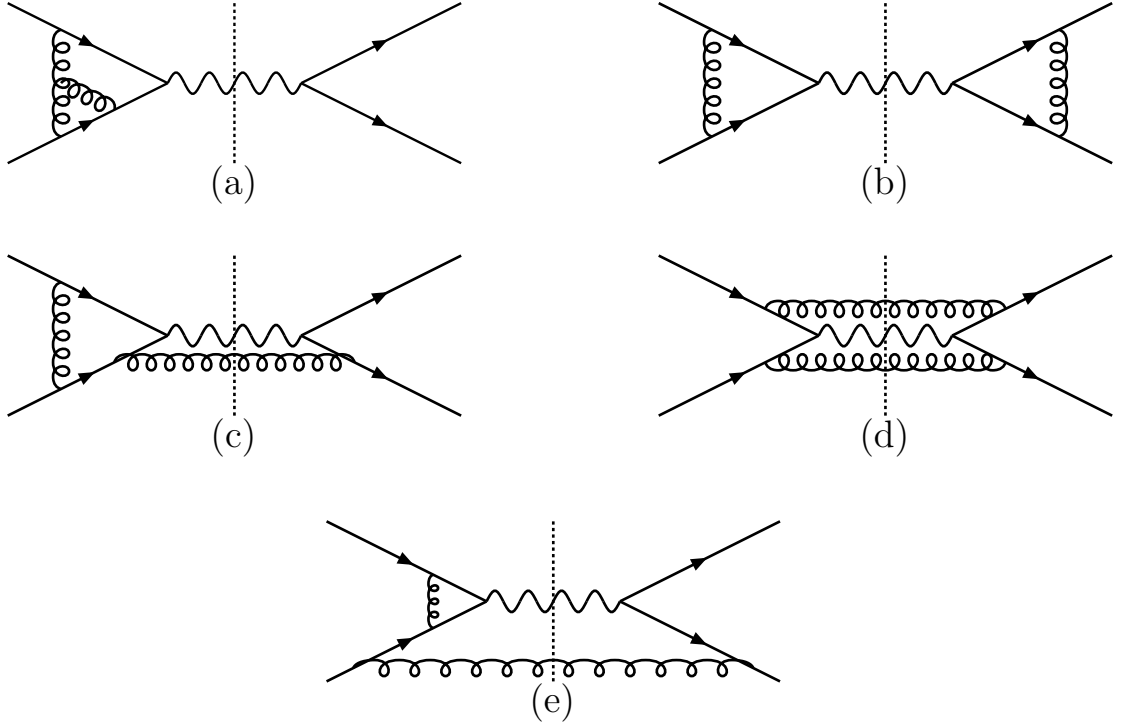


Fig. 4.1: Some of the Feynman diagrams that contribute to the Drell-Yan process to $\mathcal{O}(\alpha_s^2)$.

exactly our $\mathcal{M}^{(i)}$ and $C^{(i)}$, respectively. The various terms on the right-hand side of Eq. (4.36) have a simple interpretation in terms of Feynman diagrams contributing in the soft+virtual limit.

Let us consider some of the Feynman diagrams that contribute to $G_N^{(2),s+v}$ given in Fig. (4.1) in the case of the DY process. Graph (a) and its hermitian conjugate—which is not displayed—are purely virtual and contribute to the $2\text{Re}[C^{(2)}]$ term. Graph (b) contributes to the $|C^{(1)}|^2$ term and graph (d) contributes to the $\mathcal{M}_N^{(2)}$ term. The term $2\text{Re}[C^{(1)}]\mathcal{M}_N^{(1)}$ gets contributions only from diagram (e) while the contribution from diagram (c) vanishes in the soft+virtual limit. The reason for this is that loop momentum which enters into the quark line between the two

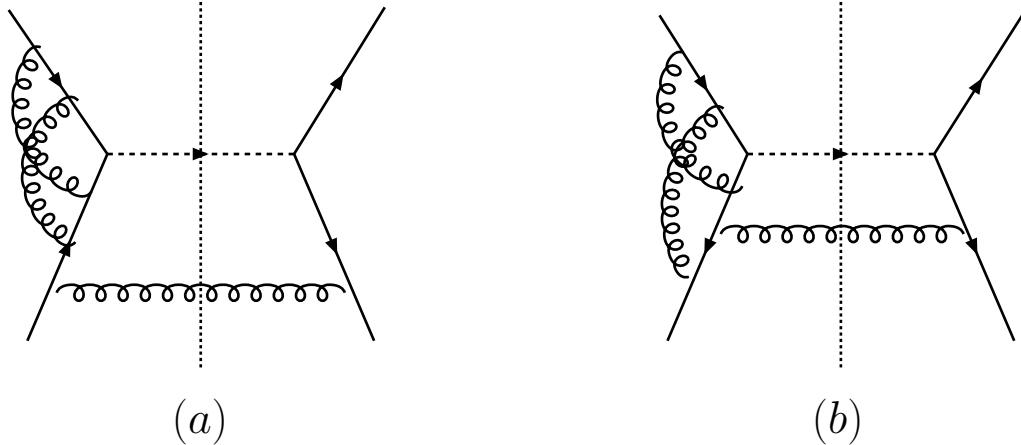


Fig. 4.2: Feynman diagram that contributes to the Drell-Yan process to $\mathcal{O}(\alpha_s^3)$. Graph (b) does not contribute in the soft limit.

quark-gluon-quark vertices “protects” the contribution of this diagram from being IR divergent even if the real gluon emitted is soft. This is not the case for diagram (e), where the momentum of the virtual gluon does not enter into the quark propagator and the contribution from such a diagram factorizes into the product $C^{(1)}\mathcal{M}^{(1)}$.

This observation persists to higher orders as can be easily seen from the contributions to $G_N^{(3),s+v}$. For example, in the DY case, only graph (a) in Fig. 4.2 contributes, and for the Higgs production it is only graph (a) in Fig. 4.3. Each one of them contributes to the relevant $\text{Re}[C^{(2)}]\mathcal{M}_N^{(1)}$ term in $G_N^{(3),s+v}$.

Since the matching coefficients C has been written down for DIS, DY and Higgs up to second order, we can extract the \mathcal{M} up to the same order by using Eq. (4.36) and the expressions for $G_N^{(i),s+v}$ given in Appendix C. The calculation is straightforward and the results are

$$\mathcal{M}_{N,\text{DIS}}^{(1)} = C_F [2L^2 + 3L + 7 - 4\zeta_2] ,$$

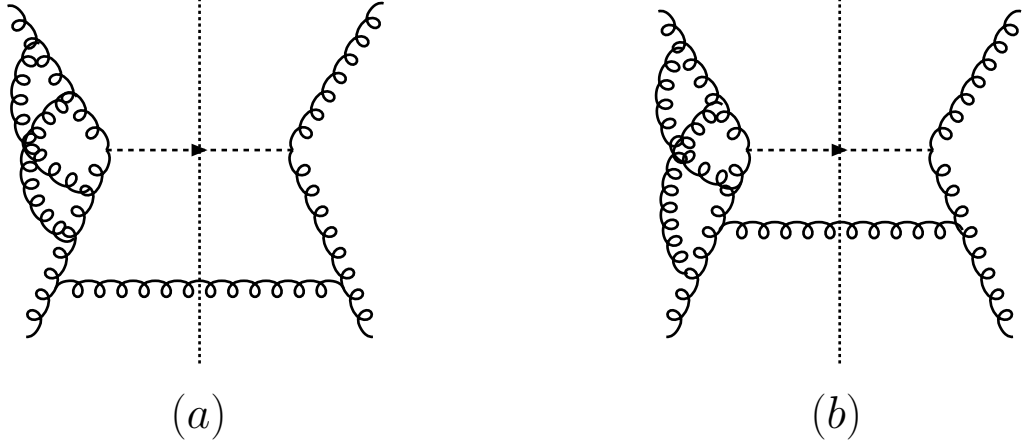


Fig. 4.3: Feynman diagram that contributes to the Higgs production to $\mathcal{O}(\alpha_s^3)$. Graph (b)

does not contribute in soft limit.

$$\begin{aligned}
\mathcal{M}_{N,\text{DIS}}^{(2)} &= C_F^2 \left[2L^4 + 6L^3 + \left(\frac{37}{2} - 8\zeta_2 \right) L^2 + \left(\frac{45}{2} - 24\zeta_2 + 24\zeta_3 \right) L \right] \\
&+ C_F C_A \left[\frac{22}{9} L^3 + \left(\frac{367}{18} - 4\zeta_2 \right) L^2 + \left(-\frac{3155}{54} + \frac{22}{3}\zeta_2 + 40\zeta_3 \right) L \right] \\
&- C_F N_F \left[\frac{4}{9} L^3 + \frac{29}{9} L^2 + \left(\frac{4}{3}\zeta_2 - \frac{247}{27} \right) L \right] \\
&+ C_F^2 \left[\frac{205}{5} - \frac{97}{2}\zeta_2 - 6\zeta_3 + \frac{122}{5}\zeta_2^2 \right] + C_F C_A \left[\frac{53129}{648} - \frac{155}{6}\zeta_2 - \frac{37}{5}\zeta_2^2 \right] \\
&+ C_F N_F \left[-\frac{4057}{324} + \frac{13}{3}\zeta_2 \right], \tag{4.37}
\end{aligned}$$

where $L = \ln \frac{\mu^2 \bar{N}}{Q^2}$. For DY we get

$$\begin{aligned}
\mathcal{M}_{N,q}^{(1)} &= C_F \left[2\tilde{L}^2 + 2\zeta_2 \right], \\
\mathcal{M}_{N,q}^{(2)} &= C_F^2 \left(\frac{1}{2} \right) \left[2\tilde{L}^2 + 2\zeta_2 \right]^2 + C_A C_F \left[\frac{22}{9}\tilde{L}^3 + \left(\frac{134}{9} - 4\zeta_2 \right) \tilde{L}^2 + \left(\frac{808}{27} - 28\zeta_3 \right) \tilde{L} \right] \\
&- C_F N_F \left[\frac{4}{9}\tilde{L}^3 + \frac{20}{9}\tilde{L}^2 + \frac{112}{27}\tilde{L} \right] \\
&+ C_F C_A \left[\frac{2428}{81} + \frac{67}{9}\zeta_2 - \frac{22}{9}\zeta_3 - 12\zeta_2^2 \right] \\
&+ C_F N_F \left[-\frac{328}{81} - \frac{10}{9}\zeta_2 + \frac{4}{9}\zeta_3 \right], \tag{4.38}
\end{aligned}$$

where $\tilde{L} = \ln \frac{\mu^2 \bar{N}^2}{Q^2}$, and for the Higgs case

$$\begin{aligned}
\mathcal{M}_{N,g}^{(1)} &= C_A [2\tilde{L}^2 + 2\zeta_2] , \\
\mathcal{M}_{N,g}^{(2)} &= C_A^2 \left(\frac{1}{2}\right) [2\tilde{L}^2 + 2\zeta_2]^2 + C_A C_A \left[\frac{22}{9} \tilde{L}^3 + \left(\frac{134}{9} - 4\zeta_2\right) \tilde{L}^2 + \left(\frac{808}{27} - 28\zeta_3\right) \tilde{L} \right] \\
&\quad - C_A N_F \left[\frac{4}{9} \tilde{L}^3 + \frac{20}{9} \tilde{L}^2 + \frac{112}{27} \tilde{L} \right] \\
&\quad + C_A C_A \left[\frac{2428}{81} + \frac{67}{9} \zeta_2 - \frac{22}{9} \zeta_3 - 12\zeta_2^2 \right] \\
&\quad + C_A N_F \left[-\frac{328}{81} - \frac{10}{9} \zeta_2 + \frac{4}{9} \zeta_3 \right] .
\end{aligned} \tag{4.39}$$

For all three processes we also have⁸ $G_N^{(0)} = 1$. The results for $\mathcal{M}_{N,\text{DIS}}^{(1)}$ and for $\mathcal{M}_{N,q}^{(1)}$ are of course the same as those in Chapter 3.

The matching coefficients should be free from large logarithms when μ is set equal to the matching scale. This is certainly true for $C^{(2)}$ for all of the three processes. The results obtained for $\mathcal{M}_N^{(2)}$ also have this feature. For DIS we choose the intermediate scale $\mu = \mu_I = Q/\sqrt{\bar{N}}$, and for DY and Higgs it is $\mu = \mu_I = q/\bar{N}$. With these choices, all the logarithms vanish in $\mathcal{M}_N^{(2)}$. These observations are valid to all orders in perturbation theory [60], so we can write

$$\mathcal{M}_N \left(\frac{Q^2}{\mu^2}, \bar{N}, \alpha_s(\mu^2) \right) = \mathcal{M}_N \left(\ln \left(\frac{Q^2}{\bar{N}^p \mu^2} \right), \alpha_s(\mu^2) \right) , \tag{4.40}$$

and for $\mu^2 = \mu_I^2 \equiv \frac{Q^2}{\bar{N}^p}$ we have

$$\mathcal{M}_N \left(\ln \left(\frac{Q^2}{\bar{N}^p \mu_I^2} \right), \alpha_s(\mu_I^2) \right) = \mathcal{M}_N(\alpha_s(\mu_I^2)) , \tag{4.41}$$

where $p = 1$ for DIS and 2 for DY and Higgs.

Another important feature arises when comparing between the DR and Higgs

⁸ Since $G^{(0)}(x) = \delta(1-x)$ and the Mellin transform of a delta function is just 1.

processes. The color structure of $\mathcal{M}_{N,(q,g)}^{(2)}$ has an ‘abelian’ part⁹ and ‘non-abelian’ one. The abelian parts are proportional to $C_{(q,g)}^2$ (where $C_{(q,g)} = C_F$ for the DY case and C_A for the Higgs case). The non-abelian part includes all the remaining terms. The basic observation is that one could get the $\mathcal{M}_{N,g}$ from that of $\mathcal{M}_{N,q}$ (or viceversa) by replacing C_F^2 with C_A^2 for the abelian part, and by replacing an overall factor of C_F with C_A to the non-abelian part. This is also true for the first order terms, $\mathcal{M}_{N,(q,g)}^{(1)}$ where the contribution is only abelian.¹⁰ The reason for this can be explained by considering the various Feynman diagrams that contribute in the soft limit for the real gluon emission in both processes. It would be interesting to prove to all orders in perturbation theory that in the soft limit, both quantities, $\mathcal{M}_{N,q}$ and $\mathcal{M}_{N,g}$ are related by color factor replacements of the kind discussed above. We believe it should be the case. Moreover, this would also give a proof of why the quantities $f_{(q,g)}^{(i)}$ of Eq. (4.18) are universal. This will be discussed below.

We need now to consider the μ dependence of the PDFs, $f_{q/q}$ for DY and DIS and $f_{g/g}$ for the Higgs production. As we have discussed in Chapter 2, this is governed by the DGLAP evolution equation. In moment space the relevant quantity is the anomalous dimension γ_2^N defined as

$$\mu \frac{d}{d\mu} f_{a/a}(\alpha_s(\mu^2), N) = -\gamma_{2,a}^N f_{a/a}(\alpha_s(\mu^2), N) , \quad a = q, g , \quad (4.42)$$

⁹ By ‘abelian’ we mean a contribution from a Feynman diagram which also exists in the $U(1)$ gauge invariant QED. The difference between the contributions of these diagrams in QED and QCD is just a color factor C_F^n , n being the order of α_s .

¹⁰ Although in QED there is, of course, no three-photon vertex, we still call the contribution for the Higgs case as abelian.

where $f_{a/a}(\alpha_s(\mu^2), N)$ is the PDF of parton a in parton a in moment space. It should be noted that these functions have no explicit μ -dependence and so the anomalous dimension is only a function of $\alpha_s(\mu^2)$ (and, of course, N). Moreover, it can be shown to all orders in perturbation theory [60] that the anomalous dimension γ_2^N is a linear function of $\ln \bar{N}$:

$$\gamma_{2,(q,g)}^N = A_{(q,g)} \ln \bar{N}^2 - 2B_{2,(q,g)}, \quad \gamma_{2,(q,g)}^N = \sum_{i=1} a_s^i \gamma_{2,(q,g)}^{(i),N}, \quad (4.43)$$

where $A_{(q,g)}$ and $B_{2,(q,g)}$ are given in Eq. (4.13) and (4.16). The fact that the logarithmic parts of γ_1 and γ_2^N have the same coefficient, A , will be discussed below.

Letting

$$I_2 = 2 \int_{\mu_I}^Q \frac{d\mu}{\mu} \gamma_{2,(q,g)}^N, \quad (4.44)$$

we get from Eq. (4.34) and (4.42):

$$G_{N,(q,g)}(Q) = |C_{(q,g)}(\alpha_s(Q^2))|^2 e^{I_1(Q,\mu_I)} \times \mathcal{M}_{N,(q,g)}(\alpha_s(\mu_I^2)) e^{I_2(\mu_I,Q)} \quad (4.45)$$

where

$$I_1 = -2 \int_{\mu_I}^Q \frac{d\mu}{\mu} \tilde{\gamma}_1, \quad (4.46)$$

and

$$\tilde{\gamma}_1 = \gamma_1 - 2i\delta_g \beta_{i-1}. \quad (4.47)$$

The last relation is a result of the μ dependence of C_ϕ in the Higgs case. This μ -dependence is governed by anomalous dimension which we denote by γ_T following the notation of [43]. There it was shown that¹¹

$$\gamma_T = a_s[-2\beta_0] + a_s^2[-4\beta_1], \quad (4.48)$$

¹¹ See also the discussion in Chapter 5.

so the conclusion is that the only effect of this anomalous dimension, when combined with anomalous dimension of the matching coefficient at the scale Q^2 is to cancel the β_i terms in γ_1 for the Higgs case. For DIS we need to consider the running of only one PDF; thus, the last exponential, e^{I_2} becomes $e^{1/2I_2}$. In Eq. (4.45) we have set $\mu_F = Q$. We should also remember that the μ_I for DIS is different than that of the DY and Higgs cases.

We notice that in Eq. (4.45) the two matching coefficients are calculated at the respective scales, Q^2 and μ_I^2 , as it should be in order to cancel the logarithms appearing in them. However, this makes the coupling constant dependent on two different scales. This can be remedied by exploiting the renormalization equation for α_s , and we can write,

$$\mathcal{M}_N(\alpha_s(\mu_I^2)) = \mathcal{M}_N(\alpha_s(Q^2)) e^{I_3} \quad (4.49)$$

where

$$I_3 = 2 \int_{\mu_I}^Q \frac{d\mu}{\mu} \beta(\alpha_s(\mu^2)) \frac{d \ln \mathcal{M}_N}{d \ln \alpha_s(\mu^2)} \equiv - \int_{\mu_I^2}^{Q^2} \frac{d\mu^2}{\mu^2} \Delta B, \quad (4.50)$$

and the QCD β -function is $\beta(a_s) = -d \ln \alpha_s / d \ln \mu^2 = \beta_0 a_s + \beta_1 a_s^2 + \dots$, with $\beta_0 = 11C_A/3 - 2N_F/3$ and $\beta_1 = \frac{34}{3}C_A^2 - 2C_F N_F - \frac{10}{3}C_A N_F$.

Therefore we write

$$G_N(Q) = \mathcal{F}(\alpha_s(Q)) e^{I(\lambda, \alpha_s(Q))}, \quad (4.51)$$

where

$$\mathcal{F} = |C(\alpha_s(Q^2))|^2 \mathcal{M}_N(\alpha_s(Q^2)), \quad I \equiv I_1 + I_2 + I_3. \quad (4.52)$$

\mathcal{F} depends only on $\alpha_s(Q^2)$. It is understood that in the Higgs case, the term $C_\phi^2(\alpha_s(Q^2))$ multiplies the right hand side of the last equation.

In the next section we will show explicitly that the exponent I is a function of $\lambda = \beta_0 \ln \bar{N} \alpha_s(Q^2)$ and $\alpha_s(Q^2)$. Equation (4.51) shows that all the harmful large N logarithms are totally contained in the exponential. Comparing Eq. (4.51) with Eq. (4.35), we can easily see that $\mathcal{F}(\alpha_s)$ is just the non-logarithmic part of $G_N(\alpha_s(Q))$ and of $G_N^{(s+v)}$.

Since the cross section σ_N in Eq. (4.1) is independent of the intermediate scale μ , then from Eq. (4.45) and the definitions of γ_1 and γ_2 we get the following relation for DY and Higgs:

$$\frac{d \ln M_{N,(q,g)}(\alpha_s(\mu), \tilde{L})}{d \ln \mu} = [2\gamma_2 - 2\tilde{\gamma}_1]_{(q,g)} = 2[A\tilde{L} + f]_{(q,g)}, \quad (4.53)$$

from which we get

$$\left. \frac{d \ln M_{N,(q,g)}(\alpha_s(\mu^2), \tilde{L})}{d \ln \mu} \right|_{\mu=\mu_I} = 2f_{(q,g)}(\alpha_s(\mu_I^2)), \quad \mu_I = \frac{Q}{N}, \quad (4.54)$$

where $A_{(q,g)}$ are given in Eq. (4.13) and $f_{(q,g)}$ are given in Eq. (4.18). Here we see the intimate relation between the universality of $f_{(q,g)}$ and the color structure relation between DY and Higgs already mentioned. The last equation also shows the consistency between the requirement of vanishing logarithms at μ_I in \mathcal{M}_N , and the fact that the same $A_{(q,g)}$ appears in $\gamma_{1,(q,g)}$ and $\gamma_{2,(q,g)}$. For DIS we get

$$\frac{d \ln M_{\text{DIS}}(\alpha_s(\mu^2), L)}{d \ln \mu} = [\gamma_2 - 2\gamma_1]_q = 2[AL + f]_q, \quad (4.55)$$

from which we obtain

$$\left. \frac{d \ln M_{N,\text{DIS}}(\alpha_s(\mu), L)}{d \ln \mu} \right|_{\mu=\mu_I} = 2f_q(\alpha_s(\mu_I^2)), \quad \mu_I = \frac{Q}{\sqrt{N}}. \quad (4.56)$$

4.3 Comparison with Conventional Resummation

In this section we want to illustrate the equivalence of the effective field theoretic approach and the existing ones. We will start by showing this for DY and Higgs, then we will turn to the DIS case.

One of the well-known forms to express the coefficient function for DY and Higgs in moment space is the following [87],

$$G_N(Q^2) = g_0(\alpha_s(Q^2))e^{I_\Delta} \Delta C(\alpha_s(Q^2)) \quad (4.57)$$

where we have normalized the Born term to 1. The g_0 has the expansion form: $g_0 = \sum_i a^i g_{0i}$. The term ΔC has the only role of cancelling the non-logarithmic contributions appearing in the exponent. These contributions arise from the various ζ -terms in the Mellin transform of the plus distributions and we will consider them in the next section. Equation (4.45) shows, again, an expression for the coefficient function in which all the logarithms appear in an exponent which is multiplied by a non-logarithmic function.

The exponential term I_Δ (commonly referred to as Sudakov exponential) is given by:

$$I_\Delta = \int_0^1 dz \frac{z^{N-1} - 1}{1-z} \left[2 \int_{Q^2}^{(1-z)^2 Q^2} \frac{d\mu^2}{\mu^2} A(\alpha_s(\mu^2)) + D(\alpha_s((1-z)^2 Q^2)) \right], \quad (4.58)$$

where, as already mentioned, we set $\mu_F^2 = Q^2$. The quantities, g_0 , A and D have the usual expansion in a_s and they are already known up to $\mathcal{O}(\alpha_s^3)$ [88]. The A is identical to the logarithmic coefficient in γ_1 and γ_2 . It is our aim to relate these quantities with those that appear in G_N . For this we follow the procedure outlined

in Appendices A, B and C of [72].

The integral in I_Δ can be rewritten in terms of the already defined I_1 , I_2 and I_3 :

$$I \equiv I_1 + I_2 + I_3 = I_\Delta + \ln \Delta C(\alpha_s(Q^2)) , \quad (4.59)$$

where the coefficient function ΔC does not depend on $\mu_I \sim Q/\bar{N}$. To work out the above relation, we first use the following relation:

$$z^{N-1} - 1 = -\tilde{\Gamma} \left(1 - \frac{\partial}{\partial \ln \bar{N}} \right) \theta \left(1 - z - \frac{1}{\bar{N}} \right) + \mathcal{O}(1/\bar{N}) , \quad (4.60)$$

where

$$\tilde{\Gamma} \left(1 - \frac{\partial}{\partial \ln \bar{N}} \right) = 1 - \Gamma_2 \left(\frac{\partial}{\partial \ln \bar{N}} \right) \left(\frac{\partial}{\partial \ln \bar{N}} \right)^2 , \quad (4.61)$$

with

$$\Gamma_2(\epsilon) = \frac{1}{\epsilon^2} [1 - e^{-\gamma_E \epsilon} \Gamma(1 - \epsilon)] = -\frac{1}{2} \zeta_2 - \frac{1}{3} \zeta_3 \epsilon - \frac{9}{40} \zeta_2^2 \epsilon^2 + o(\epsilon^3). \quad (4.62)$$

In Eq. (4.60) we used $(\partial/\partial \ln N)f(\ln \bar{N}) = (\partial/\partial \ln \bar{N})f(\ln \bar{N})$ for arbitrary function f .

After some algebra, I_Δ can be expressed as

$$I_\Delta = -\tilde{\Gamma} \left(1 - \frac{\partial}{\partial \ln \bar{N}} \right) \left\{ \int_{Q^2/\bar{N}^2}^{Q^2} \frac{d\mu^2}{\mu^2} \left[A(\alpha_s(\mu^2)) \ln \frac{Q^2}{\mu^2} + \frac{1}{2} D(\mu^2) \right] + \int_{Q^2}^{Q^2/\bar{N}^2} \frac{d\mu^2}{\mu^2} A(\alpha_s(\mu^2)) \ln \bar{N}^2 \right\} , \quad (4.63)$$

The double derivative acting on the curly bracket gives

$$\Gamma_2 \left(\frac{\partial}{\partial \ln \bar{N}} \right) \left[\frac{\partial}{\partial \ln \bar{N}} D(\alpha_s(Q^2/\bar{N}^2)) - 4A(\alpha_s(Q^2/\bar{N}^2)) \right] \quad (4.64)$$

To compare with the exponent $I = I_1 + I_2 + I_3$ we express it in the form

$$I_1 + I_2 + I_3 = - \left\{ \int_{Q^2/\bar{N}^2}^{Q^2} \frac{d\mu^2}{\mu^2} \left[A(\alpha_s(\mu^2)) \ln \frac{Q^2}{\mu^2} + (B_1 + \Delta B + 2B_2) \right] + \int_{Q^2}^{Q^2/\bar{N}^2} \frac{d\mu^2}{\mu^2} A(\alpha_s(\mu^2)) \ln \bar{N}^2 \right\}, \quad (4.65)$$

Comparing the last equation with I_Δ we get

$$\begin{aligned} & - \int_{Q^2/\bar{N}^2}^{Q^2} \frac{d\mu^2}{\mu^2} (B_1 + \Delta B + 2B_2)(\alpha_s(\mu^2)) \\ & = \Gamma_2 \left(\frac{\partial}{\partial \ln \bar{N}} \right) \left[\frac{\partial}{\partial \ln \bar{N}} D(\alpha_s(Q^2/\bar{N}^2)) - 4A(\alpha_s(Q^2/\bar{N}^2)) \right] \\ & - \frac{1}{2} \int_{Q^2/\bar{N}^2}^{Q^2} \frac{d\mu^2}{\mu^2} D(\alpha_s(\mu^2)) + \ln \Delta C(\alpha_s(Q^2)). \end{aligned} \quad (4.66)$$

At this stage it proves useful to follow [72] and replace the derivative $\partial/\partial \ln \bar{N}$ with ∂_{α_s} where

$$\partial_{\alpha_s} \equiv -2 \frac{d\alpha_s(\mu^2)}{d \ln \mu^2} \frac{\partial}{\partial \alpha_s} = -2\beta(\alpha_s) \alpha_s \frac{\partial}{\partial \alpha_s}. \quad (4.67)$$

This is so since

$$\left(\frac{\partial}{\partial \ln \bar{N}} \right) f(\alpha_s(Q^2/\bar{N}^2)) = \partial_{\alpha_s} f(\alpha_s(Q^2/\bar{N}^2)), \quad (4.68)$$

where f is an arbitrary function. The equality given in Eq. (4.66) holds to all values of \bar{N} , thus for $\bar{N} = 1$ we get

$$\ln \Delta C(\alpha_s(Q^2)) = -\Gamma_2(\partial_{\alpha_s}) \left[\partial_{\alpha_s} D(\alpha_s(Q^2/\bar{N}^2)) - 4A(\alpha_s(Q^2/\bar{N}^2)) \right] \Big|_{\bar{N}=1}. \quad (4.69)$$

Applying one more $\partial/\partial \ln \bar{N} = \partial_{\alpha_s}$ on both sides of Eq. (4.66) we obtain

$$2(B_1 + \Delta B + 2B_2)(\alpha_s(\mu^2)) = D(\alpha_s(\mu^2)) + \partial_{\alpha_s} \Gamma_2(\partial_{\alpha_s}) [4A - \partial_{\alpha_s} D](\alpha_s(\mu^2)). \quad (4.70)$$

The last equation enables us to calculate the $D^{(i)}$. To do so, and to see more clearly the content of the last result, let us expand both sides up to $\mathcal{O}(\alpha_s^4)$. First

we work out the expansion of the ΔB term. From Eq. (4.50) we get

$$\begin{aligned}\Delta B_{(q,g)}^{(0)} &= \Delta B_{(q,g)}^{(1)} = 0, \\ \Delta B_{(q,g)}^{(2)} &= -\beta_0 M_{N,(q,g)}^{(1)}, \\ \Delta B_{(q,g)}^{(3)} &= -\beta_0 \left[2M_{N,(q,g)}^{(2)} - \left(\mathcal{M}_{N,(q,g)}^{(1)} \right)^2 \right] - \beta_1 \mathcal{M}_{N,(q,g)}^{(1)},\end{aligned}\quad (4.71)$$

$$\begin{aligned}\Delta B_{(q,g)}^{(4)} &= -\beta_0 \left[3\mathcal{M}_{N,(q,g)}^{(3)} - 3\mathcal{M}_{N,(q,g)}^{(1)} \mathcal{M}_{N,(q,g)}^{(2)} + \left(\mathcal{M}_{N,(q,g)}^{(1)} \right)^3 \right] \\ &\quad - \beta_1 \left[2\mathcal{M}_{N,(q,g)}^{(2)} - \left(\mathcal{M}_{N,(q,g)}^{(1)} \right)^2 \right] - \beta_2 \mathcal{M}_{N,(q,g)}^{(1)}.\end{aligned}\quad (4.72)$$

Noticing that $B_{1,(q,g)}^{(i)} + 2B_{2,(q,g)}^{(i)} = -f_{(q,g)}^{(i)}$ and using the expansion of Γ_2 , we get

$$\begin{aligned}D_{(q,g)}^{(0)} &= D_{(q,g)}^{(1)} = 0, \\ D_{(q,g)}^{(2)} &= -2f_{(q,g)}^{(2)} + 2\Delta B_{(q,g)}^{(2)} + 4\beta_0 \zeta_2 A_{(q,g)}^{(1)}, \\ D_{(q,g)}^{(3)} &= -2f_{(q,g)}^{(3)} + 2\Delta B_{(q,g)}^{(3)} + 4\zeta_2 \beta_1 A_{(q,g)}^{(1)} + 8\zeta_2 \beta_0 A_{(q,g)}^{(2)} + \frac{32}{3} \zeta_3 \beta_0^2 A_{(q,g)}^{(1)}, \\ D_{(q,g)}^{(4)} &= -2f_{(q,g)}^{(4)} + 2\Delta B_{(q,g)}^{(4)} + 12\zeta_2 \beta_0 A_{(q,g)}^{(3)} + 8\zeta_2 \beta_1 A_{(q,g)}^{(2)} + 32\zeta_3 \beta_0^2 A_{(q,g)}^{(2)} \\ &\quad + \frac{80}{3} \zeta_3 \beta_0 \beta_1 A_{(q,g)}^{(1)} + \frac{216}{5} \zeta_2^2 \beta_0^3 A_{(q,g)}^{(1)} - 12\zeta_2 \beta_0^2 D_{(q,g)}^{(2)}.\end{aligned}\quad (4.73)$$

From the last two equations we see that in order to get $D^{(k)}$, the only same order information needed is $f^{(k)}$. All the quantities needed to calculate $D^{(2)}$ and $D^{(3)}$ are known and we get

$$D_{(q,g)}^{(2)} = C_{(q,g)} \left\{ C_A \left(-\frac{101}{27} + \frac{11}{3} \zeta_2 + \frac{7}{2} \zeta_3 \right) + N_F \left(\frac{14}{27} - \frac{2}{3} \zeta_2 \right) \right\}, \quad (4.74)$$

and

$$\begin{aligned}D_{(q,g)}^{(3)} &= C_{(q,g)} C_A^2 \left[-\frac{594058}{729} + \frac{98224}{81} \zeta_2 + \frac{40144}{27} \zeta_3 - \frac{2992}{15} \zeta_2^2 - \frac{352}{3} \zeta_2 \zeta_3 - 384 \zeta_5 \right] \\ &\quad + C_{(q,g)} C_A N_F \left[\frac{125252}{729} - \frac{29392}{81} \zeta_2 - \frac{2480}{9} \zeta_3 + \frac{736}{15} \zeta_2^2 \right]\end{aligned}$$

$$\begin{aligned}
& +C_{(q,g)}C_F N_F \left[\frac{3422}{27} - 32\zeta_2 - \frac{608}{9}\zeta_3 - \frac{64}{5}\zeta_2^2 \right] \\
& +C_{(q,g)}N_F^2 \left[-\frac{3712}{729} + \frac{640}{27}\zeta_2 + \frac{320}{27}\zeta_3 \right] , \tag{4.75}
\end{aligned}$$

where $C_{(q,g)} = C_F$ for the DY case and C_A for the Higgs case. The above results agree with the recent calculation in [85, 89, 90].

The non-logarithmic contribution

$$\mathcal{F}_{(q,g)}(Q^2) = \sum_i a^i \mathcal{F}_{(q,g)}^{(i)} = |C(Q^2)|^2 \mathcal{M}_N(Q^2) , \tag{4.76}$$

can be calculated from the already known results for $C_{(q,g)}^{(i)}(Q^2)$ and $\mathcal{M}_{N,(q,g)}^{(i)}(\alpha_s(Q^2))$, or we can simply read them from the well-known results for $G^{i,s+v}(Q^2)$ through Eq. (4.35) and Eq. (4.36)

$$\begin{aligned}
\mathcal{F}_q^{(1)} &= 16C_F(\zeta_2 - 1) , \\
\mathcal{F}_q^{(2)} &= C_F^2 \left[\frac{511}{4} - 198\zeta_2 - 60\zeta_3 + \frac{552}{5}\zeta_2^2 \right] \\
&+ C_F C_A \left[-\frac{1535}{12} + \frac{376}{3}\zeta_2 + \frac{604}{9}\zeta_3 - \frac{92}{5}\zeta_2^2 \right] \\
&+ C_F N_F \left[\frac{127}{6} - \frac{64}{3}\zeta_2 + \frac{8}{9}\zeta_3 \right] . \tag{4.77}
\end{aligned}$$

For the Higgs case we have

$$\begin{aligned}
\mathcal{F}_g^{(1)} &= 16\zeta_2 C_A \\
\mathcal{F}_g^{(2)} &= C_A^2 \left[93 + \frac{1072}{9}\zeta_2 - \frac{308}{9}\zeta_3 + 92\zeta_2^2 \right] \\
&+ C_A C_F \left[-\frac{1535}{12} + \frac{376}{3}\zeta_2 + \frac{604}{9}\zeta_3 - \frac{92}{5}\zeta_2^2 \right] \\
&+ C_A N_F \left[-\frac{80}{3} - \frac{160}{9}\zeta_2 + \frac{88}{9}\zeta_3 \right] + C_F N_F \left[-\frac{67}{3} + 16\zeta_3 \right] . \tag{4.78}
\end{aligned}$$

The above results agree with the g_{01} and g_{02} in [85].

The γ_E terms in the results of [85] are due to the use of N instead of \overline{N} as in our case. It is very simple to also reproduce these γ_E -terms. We also notice that their results for the g_{0i} do not include the contribution from the non-logarithmic terms in I_Δ .

For the DIS case there are essentially two major differences. The first is that the D term in I_Δ is zero to all orders in α_s [91, 92]. This can be understood intuitively, as the D term essentially arises from the interaction of the two incoming hadrons in the DY and Higgs production. The second one comes from the ‘‘jet function’’ (the B -term in the equation below) which encodes the effects of collinear gluon emission from the outgoing parton. This term does not contribute to the DY or Higgs production since, in those processes, there is no collinear jet of hadrons in the final state.

So for DIS we start from the following expression for the exponent in the coefficient function $C_N(Q^2)$:

$$I_{\text{DIS}} = \int_0^1 dz \frac{z^{N-1}}{1-z} \left[\int_{Q^2}^{(1-z)Q^2} \frac{d\mu^2}{\mu^2} A(\alpha_s(\mu^2)) + B_q(\alpha_s((1-z)Q^2)) \right], \quad (4.79)$$

where again we set $\mu_F^2 = Q^2$. We now follow the same procedure as for the DY case and we get

$$I_\Delta = -\tilde{\Gamma} \left(1 - \frac{\partial}{\partial \ln \overline{N}} \right) \left\{ \int_{Q^2/\overline{N}}^{Q^2} \frac{d\mu^2}{\mu^2} \left[A(\alpha_s(\mu^2)) \ln \frac{Q^2}{\mu^2} + B_q(\mu^2) \right] + \int_{Q^2}^{Q^2/\overline{N}} \frac{d\mu^2}{\mu^2} A(\alpha_s(\mu^2)) \ln \overline{N} \right\}. \quad (4.80)$$

Our result for DIS reads

$$I_1 + I_2 + I_3 = - \left\{ \int_{Q^2/\overline{N}}^{Q^2} \frac{d\mu^2}{\mu^2} \left[A(\alpha_s(\mu^2)) \ln \frac{Q^2}{\mu^2} + (B_1 + \Delta B + B_2) \right] \right.$$

$$+ \int_{Q^2}^{Q^2/\bar{N}} \frac{d\mu^2}{\mu^2} A(\alpha_s(\mu^2)) \ln \bar{N} \Big\}. \quad (4.81)$$

Noticing that

$$\left(\frac{\partial}{\partial \ln \bar{N}} \right) f(\alpha_s(Q^2/\bar{N})) = \frac{1}{2} \partial_{\alpha_s} f(\alpha_s(Q^2/\bar{N})), \quad (4.82)$$

we finally get for the DIS case

$$(B_1 + \Delta B + B_2)(\alpha_s(\mu^2)) = B_q(\alpha_s(\mu^2)) + \frac{1}{2} \partial_{\alpha_s} \Gamma_2 \left(\frac{1}{2} \partial_{\alpha_s} \right) \left[A - \frac{1}{2} \partial_{\alpha_s} B_q \right] (\alpha_s(\mu^2)). \quad (4.83)$$

Expanding the above we obtain

$$\begin{aligned} B_q^{(0)} &= 0, \\ B_q^{(1)} &= -B_{2,q}^{(1)}, \\ B_q^{(2)} &= -B_{2,q}^{(2)} - f_q^{(2)} + \Delta B_q^{(2)} + \frac{1}{2} \zeta_2 \beta_0 A_q^{(1)}, \\ B_q^{(3)} &= -B_{2,q}^{(3)} - f_q^{(3)} + \Delta B_q^{(3)} + \beta_0 \zeta_2 A^{(2)} + \frac{1}{2} \beta_1 \zeta_2 A^{(1)} + \frac{2}{3} \beta_0^2 \zeta_3 A^{(1)}, \end{aligned} \quad (4.84)$$

from which we get the $B_q^{(i)}$. Up to third order we have

$$\begin{aligned} B_q^{(1)} &= -3C_F \\ B_q^{(2)} &= C_F^2 \left[-\frac{3}{2} + 12\zeta_2 - 24\zeta_3 \right] + C_F C_A \left[-\frac{3155}{54} + \frac{44}{3} \zeta_2 + 40\zeta_3 \right] \\ &\quad + C_F N_F \left[\frac{247}{27} - \frac{8}{3} \zeta_2 \right], \\ B_q^{(3)} &= C_F^3 \left[-\frac{29}{2} - 18\zeta_2 - 68\zeta_3 - \frac{288}{5} \zeta_2^2 + 32\zeta_2 \zeta_3 + 240\zeta_5 \right] \\ &\quad + C_A C_F^2 \left[-46 + 287\zeta_2 - \frac{712}{3} \zeta_3 - \frac{272}{5} \zeta_2^2 - 16\zeta_2 \zeta_3 - 120\zeta_5 \right] \\ &\quad + C_A^2 C_F \left[-\frac{599375}{729} + \frac{32126}{81} \zeta_2 + \frac{21032}{27} \zeta_3 - \frac{652}{15} \zeta_2^2 - \frac{176}{3} \zeta_2 \zeta_3 - 232\zeta_5 \right] \\ &\quad + C_F^2 N_F \left[\frac{5501}{54} - 50\zeta_2 + \frac{32}{9} \zeta_3 \right] + C_F N_F^2 \left[-\frac{8714}{729} + \frac{232}{27} \zeta_2 - \frac{32}{27} \zeta_3 \right] \\ &\quad + C_A C_F N_F \left[\frac{160906}{729} - \frac{9920}{81} \zeta_2 - \frac{776}{9} \zeta_3 + \frac{208}{15} \zeta_2^2 \right]. \end{aligned} \quad (4.85)$$

These results agree with those in Ref. [88]. Similar to the cases of DY and Higgs, we get after simple calculation the \mathcal{F}_{DIS} which reads as

$$\begin{aligned}
\mathcal{F}_{\text{DIS}}^{(1)} &= 16C_F(-9 - 2\zeta_2) \\
\mathcal{F}_{\text{DIS}}^{(2)} &= C_F^2 \left[\frac{331}{8} + \frac{111}{2}\zeta_2 - 66\zeta_3 + \frac{4}{5}\zeta_2^2 \right] \\
&\quad + C_F C_A \left[-\frac{5465}{72} - \frac{1139}{18}\zeta_2 + \frac{464}{9}\zeta_3 + \frac{51}{5}\zeta_2^2 \right] \\
&\quad + C_F N_F \left[\frac{457}{36} + \frac{85}{9}\zeta_2 + \frac{4}{9}\zeta_3 \right]. \tag{4.86}
\end{aligned}$$

Again these results agree with g_{01}^{DIS} and g_{02}^{DIS} .

4.3.1 Resummed Logarithms

For a better understanding of the content of Eq. (4.51), the integrations must be performed using the already known results in the a_s expansion of γ_1 and γ_2 . This equation can be brought in the following form,

$$I_1 + I_2 + I_3 = \int_{Q^2/\bar{N}^2}^{Q^2} \frac{d\mu^2}{\mu^2} \left[A_{(q,g)}(\alpha_s(\mu^2)) \ln \frac{\mu^2 \bar{N}^2}{Q^2} - (\Delta B_{(q,g)} - f_{(q,g)}) \right], \tag{4.87}$$

where we have used the expressions for γ_1 and γ_2 given in Eq. (4.3) and Eq. (4.43) respectively. To perform the integrations, the μ dependence of the running coupling constant has to be taken into account. We recall from Chapter 2 that the running of the coupling constant is determined perturbatively in α_s (see Eq. (2.4)). Up to $\mathcal{O}(\alpha_s^3)$ the solution of that equation is given by

$$\begin{aligned}
\alpha_s(\mu^2) &= \frac{\alpha_s(Q^2)}{l} \left\{ 1 - \frac{\alpha_s(Q^2) b_1}{l} \frac{b_1}{b_0} \ln l + \left(\frac{\alpha_s(Q^2)}{l} \right) \left[\frac{b_1^2}{b_0^2} (\ln^2 l - \ln l + l - 1) \right. \right. \\
&\quad \left. \left. - \frac{b_2}{b_0} (\ln l - 1) \right] + \mathcal{O}(\alpha_s(Q^2)) \right\}, \tag{4.88}
\end{aligned}$$

where $l = 1 + b_0 \alpha_s(Q^2) \ln \mu^2 / Q^2$ and $b_i = \frac{1}{(4\pi)^{i+1}} \beta_i$. Explicit expressions for β_i up to $i = 2$ are given in Eq. (2.5).

Let us start the computation by considering the contribution of the $A_{(q,g)}^{(1)}$ term.

Changing the integration variable from μ^2 to l , one simply gets

$$\begin{aligned}
&= \frac{A_{(q,g)}^{(1)}}{4\pi b_0} \int_{1-2\lambda}^1 \frac{dl}{l} \left\{ 1 - \alpha_s(Q^2) \frac{b_1 \ln l}{b_0 l} + \left(\frac{\alpha_s(Q^2)}{l} \right)^2 \left[\frac{b_1^2}{b_0^2} [\ln^2 l - \ln l + l - 1] \right. \right. \\
&\quad \left. \left. - \frac{b_2}{b_0} (\ln l - 1) \right] \right\} \left(2 \ln \bar{N} + \frac{l-1}{b_0 \alpha_s(Q^2)} \right), \tag{4.89}
\end{aligned}$$

where $\lambda \equiv b_0 \alpha_s(Q^2) \ln \bar{N}$.

The last equation includes a pattern that repeats itself when other contributions are included. Accepting as a working rule that $\ln \bar{N} \sim (1/\alpha_s(Q^2))$ then the last two terms give rise to comparable contributions, however, inside the curly brackets, we have an expansion in $\alpha_s(Q^2)$. Thus the hierarchy is manifest. Carrying out the integrals in Eq. (4.89) is a simple matter and we obtain

$$\begin{aligned}
&= \ln \bar{N} \left\{ \frac{A_{(q,g)}^{(1)}}{4\pi b_0} \left[\frac{2\lambda + (1-2\lambda) \ln(1-2\lambda)}{\lambda} \right] \right\} + \frac{A_{(q,g)}^{(1)} b_1}{4\pi b_0^3} \left[2\lambda + \ln(1-2\lambda) \right. \\
&\quad \left. + \frac{1}{2} \ln^2(1-2\lambda) \right] + \alpha_s(Q^2) \frac{A_{(q,g)}^{(1)} b_1^2}{4\pi b_0^4} \left[2\lambda^2 + 2\lambda \ln(1-2\lambda) + \frac{1}{2} \ln^2(1-2\lambda) \right] \frac{1}{1-2\lambda}. \tag{4.90}
\end{aligned}$$

Expanding the λ -terms in the last equation by using the relation,

$$\ln(1-x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n}, \tag{4.91}$$

one easily gets a sum of the form $\alpha_s^n(Q^2) \ln^{n+1} \bar{N}$ arising from the first term, $\alpha_s^n(Q^2) \ln^n \bar{N}$ from the second term, and $\alpha_s^{n+1} \ln^n \bar{N}$ from the last term. These are commonly called leading logarithms (LL), next-to-leading logarithms (NLL) and next-to-next-to leading logarithms (NNLL). Higher logarithmic accuracies follow easily.

Consider now the contribution from $A_{(q,g)}^{(2)}$. Similar to the $A^{(1)}$ we get,

$$= \frac{A_{(q,g)}^{(2)}}{(4\pi)^2 b_0} \int_{1-2\lambda}^1 \frac{dl}{l^2} \alpha_s(Q^2) \left[1 - 2\alpha_s(Q^2) \frac{b_1}{b_0} \frac{\ln l}{l} + \mathcal{O}(\alpha_s^3(Q^2)) \right] \left(2 \ln \bar{N} + \frac{l-1}{b_0 \alpha_s(Q^2)} \right), \quad (4.92)$$

so we see that $A_{(q,g)}^{(2)}$ does not contribute to the LL but starts from NLL. This contribution is

$$= -\frac{A_{(q,g)}^{(2)}}{(4\pi)^2 b_0^2} [2\lambda + \ln(1-2\lambda)] - \alpha_s(Q^2) \frac{A_{(q,g)}^{(2)} b_1}{(4\pi)^2 b_0^3} [2\lambda + 2\lambda^2 + \ln(1-2\lambda)]. \quad (4.93)$$

From $A_{(q,g)}^{(3)}$ we get

$$= \frac{A_{(q,g)}^{(3)}}{(4\pi)^3 b_0} \int_{1-2\lambda}^1 \frac{dl}{l^3} \alpha_s^2(Q^2) [1 + \mathcal{O}(\alpha_s(Q^2))] \left(2 \ln \bar{N} + \frac{l-1}{b_0 \alpha_s(Q^2)} \right), \quad (4.94)$$

which is a NNLL contribution:

$$= \alpha_s(Q^2) \frac{A_{(q,g)}^{(3)}}{(4\pi)^3 b_0^2} \frac{2\lambda}{1-2\lambda}. \quad (4.95)$$

The contribution from the term $\Delta B^{(i)} - f^{(i)}$ term start at NNLL accuracy since this term vanishes for $i = 0, 1$. From Eq. (4.73) we have: $\Delta B_{(q,g)}^{(2)} - f_{(q,g)}^{(2)} = (1/2)(D^{(2)} - 4\beta_0 \zeta_2 A_{(q,g)}^{(1)})$. The contribution of this term gives

$$= -\frac{1}{(4\pi^2)} \frac{1}{b_0 \alpha_s(Q^2)} [\Delta B_{(q,g)}^{(2)} - f_{(q,g)}^{(2)}] \int_{1-2\lambda}^1 \frac{dl}{l^2} \alpha_s^2(Q^2), \quad (4.96)$$

which is a NNLL contribution:

$$= \alpha_s(Q^2) \frac{1}{(4\pi)^2 b_0} [4\beta_0 \zeta_2 A_{(q,g)}^{(1)} - D_{(q,g)}^{(2)}] \frac{\lambda}{1-2\lambda}. \quad (4.97)$$

Writing the sum of all contributions already obtained in the form,

$$\ln \bar{N} g_{(q,g)}^{(1)} + g_{(q,g)}^{(2)} + \alpha_s(Q^2) g_{(q,g)}^{(3)}, \quad (4.98)$$

we then have the following expressions:

$$\begin{aligned}
g_{(q,g)}^{(1)}(\lambda) &= \frac{A_{(q,g)}^{(1)}}{4\pi b_0} \left[\frac{2\lambda + (1-2\lambda)\ln(1-2\lambda)}{\lambda} \right], \\
g_{(q,g)}^{(2)}(\lambda) &= -\frac{A_{(q,g)}^{(2)}}{(4\pi)^2 b_0^2} [2\lambda + \ln(1-2\lambda)] + \frac{A_{(q,g)}^{(1)} b_1}{4\pi b_0^3} \left[2\lambda + \ln(1-2\lambda) + \frac{1}{2} \ln^2(1-2\lambda) \right], \\
g_{(q,g)}^{(3)}(\lambda) &= \left[\frac{4\zeta_2 A_{(q,g)}^{(1)}}{4\pi} - \frac{D_{(q,g)}^{(2)}}{(4\pi)^2 b_0} \right] \frac{\lambda}{1-2\lambda} + \frac{A_{(q,g)}^{(1)} b_1^2}{4\pi b_0^3} \left[2\lambda + 2\lambda \ln(1-2\lambda) \right. \\
&\quad \left. + \frac{1}{2} \ln^2(1-2\lambda) \right] + \frac{A_{(q,g)}^{(1)} b_2}{4\pi b_0^3} \left[2\lambda + \ln(1-2\lambda) + \frac{2\lambda^2}{1-2\lambda} \right] + \frac{2A_{(q,g)}^{(3)}}{(4\pi)^3 b_0^2} \frac{\lambda^2}{1-2\lambda} \\
&\quad - \frac{A_{(q,g)}^{(2)} b_1}{(4\pi)^2 b_1^3} [2\lambda + 2\lambda^2 + \ln(1-2\lambda)] \frac{1}{1-2\lambda}. \tag{4.99}
\end{aligned}$$

The above functions sum the large logarithms to LL, NLL and N²LL, respectively. It is straightforward to get also the $\alpha_s^2 g^{(4)}$ which resums the N³LL. It will contain contributions from $A_{(q,g)}^{(i)}$ up to $i = 4$ and from $D_{(q,g)}^{(2)}$ and $D_{(q,g)}^{(3)}$. The yet un-calculated quantity $A_{(q,g)}^{(4)}$ is the only missing piece to complete the resummation up to N³LL accuracy. The above results for $g^{(i)}$ agree with the ones in [72, 86]. We remind the reader that we have set the factorization scale and the renormalization scale equal to Q^2 and the γ_E dependence is hidden in \overline{N} throughout. The analysis for the DIS case can be performed similarly and one also gets agreement with the known results.

It is illustrative to compare the above derivation of $g^{(i)}$ with those of Refs. [72, 86]. Let us consider the following integral which contributes to NLL:

$$\frac{-2A^{(1)}}{4\pi b_0} \int_0^1 dz \frac{z^N - 1}{1-z} \ln \left[1 - \frac{b_1 \alpha_s(Q^2)}{b_0} \frac{\ln[1 + b_0 \alpha_s(Q^2) \ln(1-z)^2]}{1 + b_0 \alpha_s(Q^2) \ln(1-z)^2} \right]. \tag{4.100}$$

One way to perform this integral in the large N limit is to make the replacement,

$$z^N - 1 \rightarrow -\theta(1-z-1/\overline{N}). \tag{4.101}$$

With this, the integral becomes

$$-\frac{2A^{(1)}b_1\alpha_s(Q^2)}{4\pi b_0^2} \int_0^{1-1/N} \frac{dz}{1-z} \frac{\ln[1+2b_0\alpha_s(Q^2)\ln(1-z)]}{1+2b_0\alpha_s(Q^2)\ln(1-z)}. \quad (4.102)$$

Define $t = 1 + 2b_0\alpha_s(Q^2)\ln(1-z)$ and perform the integral. The result is

$$\frac{A^{(1)}b_1}{4\pi b_0^3} \times \frac{1}{2} \ln^2(1-2\lambda). \quad (4.103)$$

A different way to perform the integral is to use

$$b^x = 1 + x \ln b + \mathcal{O}(x^2), \quad (4.104)$$

where b is a real number (which is not 0 or 1). With this we have

$$\frac{\ln[1+a\ln(1-z)]}{1+a\ln(1-z)} = \lim_{\epsilon \rightarrow 0} \left\{ \frac{1}{\epsilon} \left[(1+a\ln(1-z))^{-1+\epsilon} - (1+a\ln(1-z))^{(-1)} \right] \right\}, \quad (4.105)$$

where $a = 2b_0\alpha_s(Q^2)$. Let $\eta = \frac{b_1}{b_0}\alpha_s(Q^2)$; then we obtain

$$\ln \left[1 - \eta \frac{\ln[1+a\ln(1-z)]}{1+a} \right] = \ln \left\{ 1 - \frac{\eta}{\epsilon} [1+a\ln(1-z)]^{-1+\epsilon} + \frac{\eta}{\epsilon} [1+a\ln(1-z)]^{-1} \right\}, \quad (4.106)$$

where we have thrown away terms that vanish in the limit $\epsilon \rightarrow 0$. Expanding in η as a small parameter, the logarithm becomes

$$= -\frac{\eta}{\epsilon} [1+a\ln(1-z)]^{-1+\epsilon} + \frac{\eta}{\epsilon} [1+a\ln(1-z)]^{-1}, \quad (4.107)$$

so there are two integrals to calculate.

The first one is

$$\begin{aligned} & -\frac{2A^{(1)}}{4\pi b_0} \left(-\frac{\eta}{\epsilon} \right) \int_0^1 dz \frac{z^N - 1}{1-z} \frac{1}{[1+a\ln(1-z)]^{1-\epsilon}} \\ & = -\frac{2A^{(1)}}{4\pi b_0} \left(\frac{\eta}{\epsilon} \right) \left(\sum_{i=0}^{\infty} \frac{\Gamma(1-\epsilon+i)}{\Gamma(1-\epsilon)} a^i \times \underbrace{S_{1,1,\dots,1}_{i+1}(N)} \right), \end{aligned} \quad (4.108)$$

where $S_{m_1, m_2, \dots, m_k}(N)$ are the harmonic sums [93]. And

$$S_{\underbrace{1, 1, \dots, 1}_{i+1}}(N) = \sum_{j_1=1}^N \frac{1}{j_1} \sum_{j_2=1}^{j_1} \frac{1}{j_2} \cdots \sum_{j_{i+1}=1}^{j_i} \frac{1}{j_{i+1}}. \quad (4.109)$$

The second integral

$$\begin{aligned} & -\frac{2A^{(1)}}{4\pi b_0} \left(+\frac{\eta}{\epsilon} \right) \int_0^1 dz \frac{z^N - 1}{1 - z} \frac{1}{[1 + a \ln(1 - z)]} \\ & = \frac{2A^{(1)}}{4\pi b_0} \left(\frac{\eta}{\epsilon} \right) \left(\sum_{i=0}^{\infty} \frac{\Gamma(1+i)}{\Gamma(1)} a^i \times S_{\underbrace{1, 1, \dots, 1}_{i+1}}(N) \right). \end{aligned} \quad (4.110)$$

Combine both results and use

$$S_{\underbrace{1, 1, \dots, 1}_{i+1}}(N) = \frac{1}{(i+1)!} [\theta_{i+1,1} \ln^{i+1} \bar{N}] + \mathcal{O}(\ln^{i-1} \bar{N}), \quad (4.111)$$

where $\theta_{i,j} = 1$ for $i \geq j$ and zero otherwise. We finally get

$$\frac{A^{(1)} b_1}{4\pi b_0^3} \sum_{k=1}^{\infty} \left[(k-1)! + \theta_{k,2} \sum_{i=0}^{k-2} k(k-1) \cdots (k-i)(k-i-2) \right] \frac{(2\lambda)^{k+1}}{(k+1)!}. \quad (4.112)$$

The last result is exactly the expansion of

$$\frac{A^{(1)} b_1}{4\pi b_0^3} \left[\frac{1}{2} \ln^2(1 - 2\lambda) \right] \quad (4.113)$$

in powers of λ . The last result is part of the contribution of $A^{(1)}$ to the function $g^{(1)}$ as can be seen from Eq. (4.99).

In the above derivation we have neglected contributions of the form $\mathcal{O}(\ln^{i-1} N)$.

When these terms are included, they give rise to non-logarithmic terms in I_{Δ} . This comes from the following:

$$\begin{aligned} S_{\underbrace{1, 1, \dots, 1}_{i+1}}(N) & = \frac{1}{(i+1)!} [\theta_{i+1,1} \ln^{i+1} \bar{N} + \frac{1}{2} \theta_{i+1,2} i(i+1) \zeta_2 \ln^{i-1} \ln \bar{N}] \\ & + \mathcal{O}(\ln^{i-3} \bar{N}). \end{aligned} \quad (4.114)$$

For $i = 1$ we have

$$S_{1,1}(N) = \frac{1}{2} \left[\ln^2 \bar{N} + \zeta_2 \right], \quad (4.115)$$

which is the Mellin transform of $\mathcal{D}^{(1)}(z)$ in the large N limit. These ζ -terms are cancelled by the $\Delta C(\alpha_s(Q^2))$ which is given in Eq. (4.69).

To illustrate this point, let us consider the following contribution to $g_{q,g}^{(1)}$

$$\begin{aligned} & \left(\frac{A_{q,g}^{(1)}}{4\pi} \right) \left(\frac{2}{b_0} \right) \int_0^1 dz \frac{z^N - 1}{1 - z} \ln[1 + a \ln(1 - z)] \\ &= \left(\frac{A_{q,g}^{(1)}}{4\pi} \right) \left(\frac{2}{b_0} \right) \left\{ \ln \bar{N} \left[\frac{2\lambda + (1 - 2\lambda) \ln(1 - 2\lambda)}{2\lambda} \right] + \frac{a}{2} \zeta_2 + \frac{a^2}{3} \zeta_3 + \mathcal{O}(\alpha_s^3) \right\}, \end{aligned} \quad (4.116)$$

which is calculated along the same lines as the previous ones. Expanding $\Delta C(\alpha_s(Q^2))$ to $\mathcal{O}(\alpha_s^2)$ we get,

$$\begin{aligned} \Delta C^{(0)} &= 0, \\ \Delta C^{(1)} &= -2\zeta_2 A_{q,g}^{(1)}, \\ \Delta C^{(2)} &= -2\zeta_2 A_{q,g}^{(2)} - 2\zeta_2 \left(A^{(1)} \right)^2 - \frac{8}{3} \zeta_3 \beta_0 A^{(1)}. \end{aligned} \quad (4.117)$$

The terms proportional to $A_{q,g}^{(1)}$ exactly cancel those in Eq. (4.116).

4.4 Phenomenology

In this section we discuss some of the phenomenological impacts of resummed results on the production cross section for the Higgs scalar at the TEVATRON and the LHC and in the large top quark mass limit. Extensive discussion can be found in [72, 80, 81]. The inclusion of the $D_g^{(3)}$ and its phenomenological effect was discussed in [85]. Quoting Ref. [85], the total cross section for Higgs production as function of

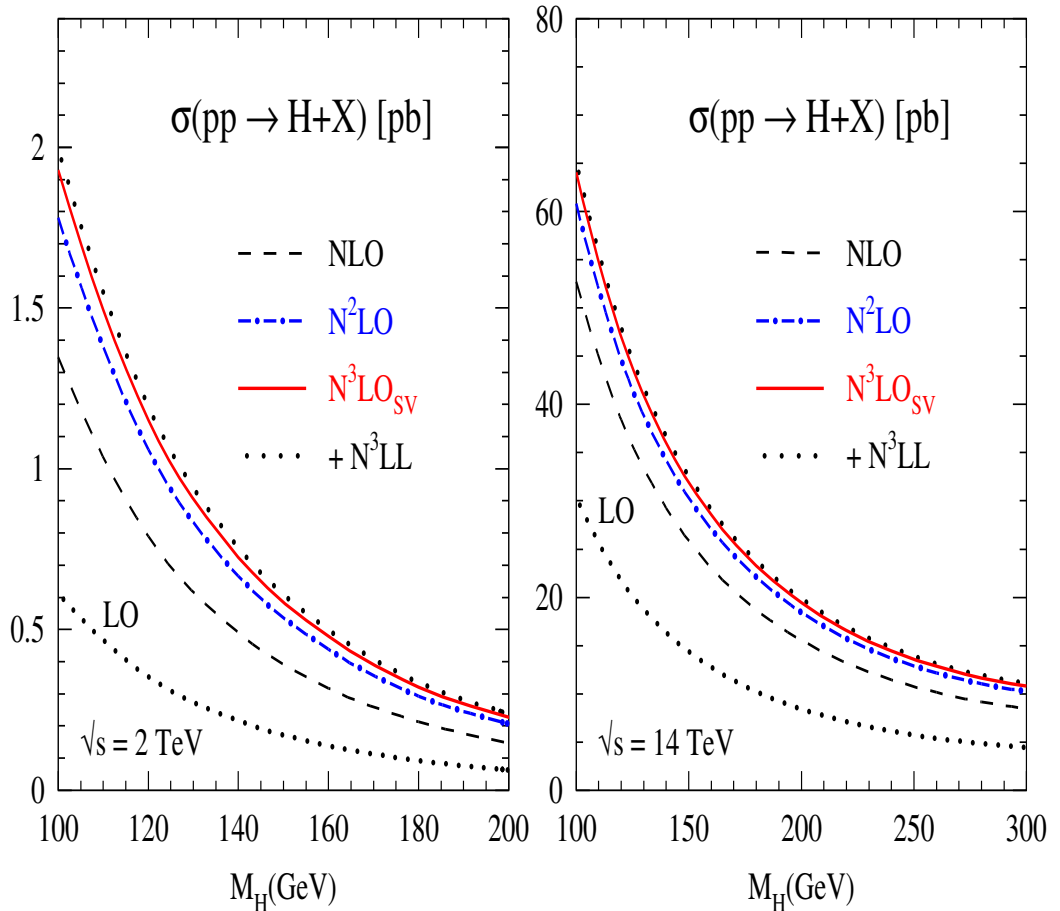


Fig. 4.4: Higgs Boson cross section at the TEVTRON and the LHC.

the Higgs mass is plotted in Fig. 4.4 . The PDF used are the MRST2002 [94] and the strong coupling constant is $\alpha_s = \alpha_s(M_Z) = 0.114$. The above plots, obtained by setting the renormalization and factorization scales to be: $\mu_r = \mu_f = M_H = Q$, show that when the $D_g^{(3)}$ term is taken into account, there is an increase of the cross section by about 10% at the TEVATRON and 5% at the LHC.

4.5 Summary

Threshold resummation of logarithmic enhancements due to soft gluon radiation has been performed using the methodology of effective field theory. This

Moreover, by varying the renormalization scale μ_r , with μ_f fixed, one can assess the uncertainty from the yet uncalculated higher order contributions. It turns out that by inclusion of (the known) higher order, fixed and resummed [72, 80, 81] contributions, the scale dependence decreases as expected. Again quoting [85], we

show this in Fig. 4.5 where μ_f is taken as M_H .

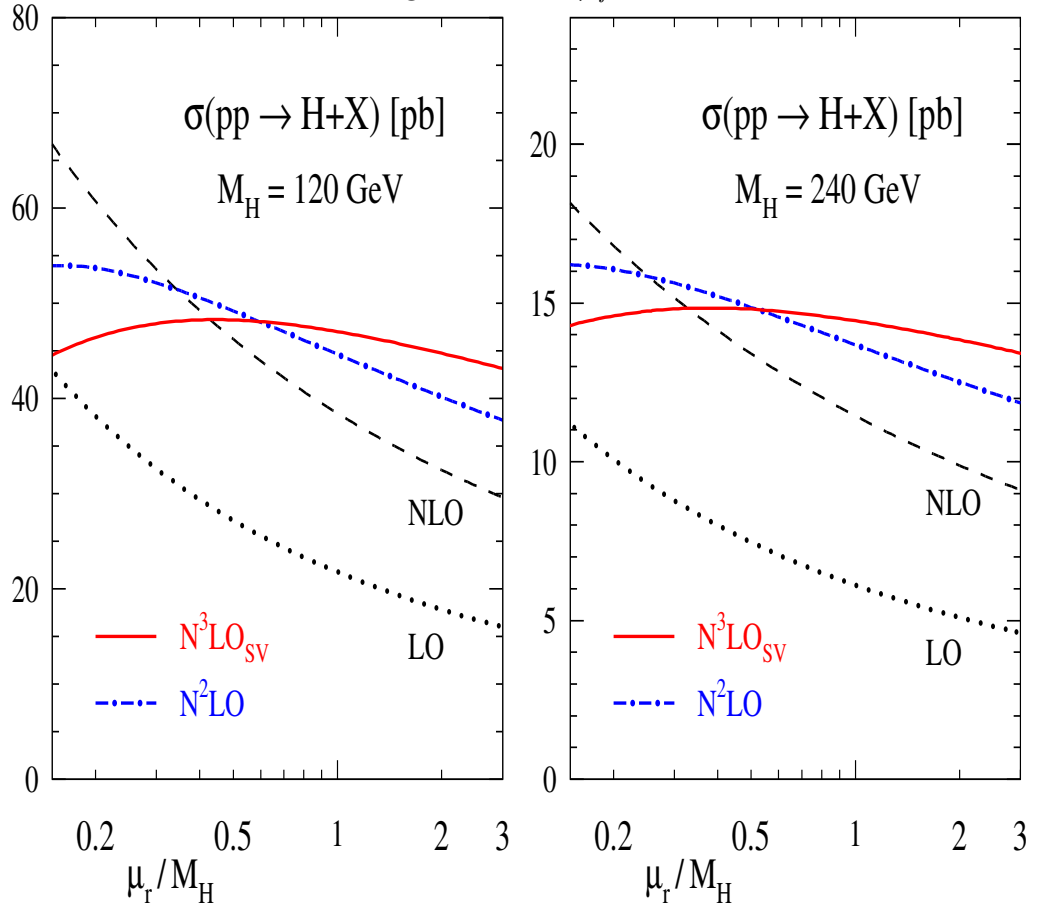


Fig. 4.5: Renormalization scale dependence of fixed order calculations for the Higgs production at the LHS.

method works to any desired sub-leading logarithmic accuracy and it is completely equivalent to the more conventional, factorization-based, techniques. This has been illustrated to all three inclusive processes we considered: DIS, DY and the SM Higgs production.

Conceptually and technically, however, this approach is much less complicated and it is physically more coherent than other ones. Working perturbatively in moment space (and for large values of N) we saw that one does *not* need to introduce any additional nonperturbative quantities (other than the conventional PDFs) as it is usually the case in the traditional approaches. All the quantities needed to get the resummed coefficient functions are straightforwardly obtained from fixed order calculations of the form factors (which supply the $C^{(i)}$ and the $\gamma_1^{(i)}$), the Altarelli-Parisi splitting kernels (which supply the $\gamma_2^{(i)}$) and the cross section for real gluon emission in the soft limit (from which we get the $\mathcal{M}^{(i)}$).

On the conceptual level we emphasize the fact that the mere ability to perform matching (i.e., to obtain finite matching coefficients and without IR poles) of full QCD onto the relevant effective theory at the higher scale and onto (a product of) PDF at the intermediate scale, is equivalent to the statement of the factorization theorem, order by order in α_s . This could be considered as an “effective” realization to the content of the factorization theorem. And as we have seen, the running between the matching scales allows the resummation of the large logarithms.

The effective approach can be easily extended to other cases that admit scale separation and some form of factorized cross section. This will be explicitly shown in the next chapter for the case of small transverse momentum distributions.

5. SMALL TRANSVERSE MOMENTUM RESUMMATION

5.1 *Introduction*

In the previous two chapters we have discussed the application of the effective field theoretic approach to resum the large logarithmic enhancements of soft gluons (in Mellin space) in the threshold region. It is our aim to apply the same approach to deal with yet another kind of large logarithms that may also endanger the perturbative expansion of certain physical quantities. We will consider processes, like Drell-Yan and the standard model Higgs production, where the transverse momentum of the final states is measured and we will concentrate on the region of low transverse momentum [95, 96]. For the transverse momentum distribution, the rigorous theoretical study in QCD started with the classical work on semi-inclusive processes in e^+e^- annihilation by Collins and Soper in [97], where a factorization was proved based on the transverse momentum dependent (TMD) parton distributions and fragmentation functions [98]. The resummation of TMD large logarithms was performed by solving the relevant energy evolution equation. This approach was later applied to the DY process in [99], where a general and systematic analysis of the factorization and resummation were performed. This latter procedure became to be known as “Collins-Soper-Sterman” (CSS) resummation formalism. The

resummed formulas are used for many processes with the relevant coefficients extracted by comparing between the expansion of the resummed expressions and the fixed-order calculations [100].

In general the resummation formula for these processes can be written in the following form, taking the DY process as an example [99],

$$\frac{d\sigma}{dQ^2 dy dQ_T^2} = \sigma_0 \left[\int \frac{d^2\vec{b}}{(2\pi)^2} e^{-i\vec{Q}_T \cdot \vec{b}} W(b, Q^2, x_1, x_2) + Y(Q_T, Q^2, x_1, x_2) \right], \quad (5.1)$$

where $\sigma_0 = 4\pi^2\alpha_{\text{em}}^2/(9SQ^2)$ represents the Born-level cross section for DY production. The variables used here are standard: Q^2 is the invariant mass of the DY pair; Q_T is the observed transverse momentum relative to the beam axis; $y = (1/2)\ln[(Q^0 + Q^3)/(Q^0 - Q^3)]$ is the rapidity; $S = (P_1 + P_2)^2$ is the center-of-mass energy squared with P_i the momentum of the incoming hadrons. The variable x_1 and x_2 are the equivalent parton fractions, $x_1 = \sqrt{Q^2/S} e^y$ and $x_2 = \sqrt{Q^2/S} e^{-y}$. The W term contains the most singular contributions at small Q_T , resummed to all orders in perturbation theory. The second term $Y(Q_T, Q^2)$ represents the regular part of a fixed-order calculation for the cross section, which becomes important when the transverse momentum Q_T is on the order of $\sqrt{Q^2}$.

The resummation of logarithms in the W term is performed in the conjugate space of the impact parameter b . This is analogous to the resummation in threshold limit which is performed in the Mellin space. The study of the resummed W term for DY and Higgs production using an effective field theory approach was performed in Refs. [101, 43]. The main result for DY process can be expressed in the following

form

$$\begin{aligned}
W(b, Q^2, x_1, x_2) &= \sum_{qq'} C^2(Q^2, \alpha_s(Q^2)) e^{-\mathcal{S}(Q^2, \mu_I^2)} \\
&\times \left(C_q \otimes f_{q/q} \right) (x_1, b, Q, \mu_I^2) \times \left(C_{q'} \otimes f_{q'/q'} \right) (x_2, b, Q, \mu_I^2)
\end{aligned} \tag{5.2}$$

where $f_{q/q}$ and $f_{q'/q'}$ are the parton distributions and/or fragmentation functions related to the processes studied, and the notation \otimes stands, as usual, for convolution. Two matching coefficients appear in the above formula: one is $C(Q^2, \alpha_s(Q^2))$ connecting between the matrix element of full QCD current and the effective theory analogue at the scale Q^2 ; the other is the coefficient function C_q obtained by calculating the processes in SCET at an intermediate scale $\mu_I^2 \sim Q_T^2$. The exponential suppression form factor arises from the anomalous dimension of the effective current in SCET

$$\mathcal{S}(Q^2, \mu_I^2) = \int_{\mu_I}^Q \frac{d\mu}{\mu} 2\gamma_1(\mu^2, \alpha_s(\mu^2)) \tag{5.3}$$

We have already mentioned that the anomalous dimension is identical for DIS and DY processes, and depends only on the effective theory operators. Moreover, the same γ_1 controls the threshold and the low transverse momentum resummations. The matching coefficient $C(Q^2, \alpha_s(Q^2))$, on the other hand, is process dependent (but independent of threshold or transverse-momentum resummation). All the large double logarithms are included in the Sudakov form factor \mathcal{S} .

5.2 Drell-Yan Production at Low Q_T

As we have done for the threshold resummation in the effective approach, we consider DY production of lepton pair with finite transverse momentum Q_T in two

steps, assuming $Q \gg Q_T \gg \Lambda_{\text{QCD}}$. In the first step, one integrates out all loops with virtuality of order Q to get an effective theory in which there are only collinear and soft modes. The collinear modes have virtuality of order Q_T . In the second step, one integrates out the collinear modes with virtuality Q_T . In this case, the theory is matched onto the ordinary QCD without external hard scales. The soft physics is now included in the parton distribution functions (PDF).

Let us consider the first step: integrating out modes of virtuality of order Q . At low Q_T , the most singular contribution in DY comes from the form-factor type of diagrams, in which the quark and antiquark first radiate soft gluons, followed by an annihilation vertex decorated with loop corrections. If the gluon radiations are attached to the loops, the soft gluon limit does not give rise to any infrared singularity, and hence diagrams yield higher-order contributions in Q_T/Q . Therefore, one needs to consider only the form-factor type of diagrams in studying the virtual corrections. This is exactly identical to the case $z = 1$. Thus to integrate out the hard modes (where all gluon momenta are of order Q) from the theory, we follow the same steps as in the threshold limit case and match the full QCD current onto the (gauge invariant) SCET current, obtaining the matching coefficient that encodes the hard contributions. Therefore, the the matching coefficients at the higher scale Q^2 and the anomalous dimension γ_1 are the same as in the threshold resummation case. Recalling the result from Chapter 3, Eq. (3.50), we have for DY process

$$C^{(1)}(\mu^2, \alpha_s(\mu^2)) = - \left[\ln^2 \frac{Q^2}{\mu^2} - 3 \ln \frac{Q^2}{\mu^2} + 8 - 7\zeta_2 \right], \quad (5.4)$$

from which we get,

$$C(Q^2, \alpha_s(Q^2)) = 1 + \frac{\alpha_s(Q^2)}{4\pi} C^{(1)} = 1 - \frac{\alpha_s(Q^2)}{4\pi} C_F (8 - 7\zeta_2). \quad (5.5)$$

And for the anomalous dimension of the quark form factor we recall

$$\gamma_1^{(1)} = C_F \left[4 \ln \frac{Q^2}{\mu^2} - 6 \right], \quad (5.6)$$

where all perturbative quantities have the usual expansion in $a_s = \frac{\alpha_s}{4\pi}$.

In the second step, we calculate the SCET cross section. One needs to compute only the real contributions since the virtual diagrams are scaleless in the effective theory and vanish in on-shell dimensional regularization (DR). At this stage the TMD cross section thus obtained, is the same as the one in full QCD taken to the relevant kinematical limit. This will be shown explicitly to first order in α_s . Again we note that this is identical to our previous observations related to the Q_T integrated cross section $\frac{d\sigma}{dQ^2}$.

As before, resumming the large logarithmic ratios is performed by considering the scale dependence of the matching coefficient, which is controlled by the anomalous dimension of the effective current. By running down the scale from $\mu_H^2 \sim Q^2$ to the intermediate scale μ_I^2 , all the large logarithms exponentiate and give rise to the Sudakov form factor.

5.3 Matching at the Intermediate Scale μ_I

To perform matching at the lower scale $\mu_I^2 \ln^2 \left(\frac{Q^2}{\mu^2} \right) Q^2$ we calculate the cross section in SCET and match it to a product of quark distributions. From the result

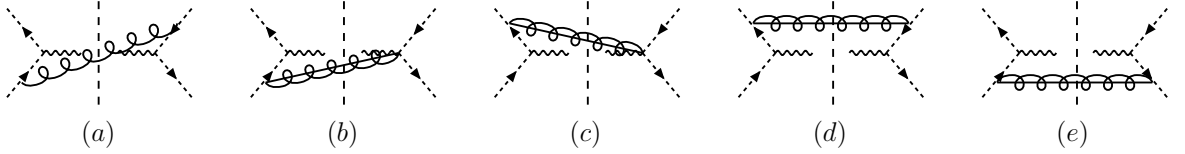


Fig. 5.1: Non-vanishing Feynman diagrams contributing to Drell-Yan production in the soft-collinear-effective theory: (a) for the soft gluon radiation; (b)-(e) for $n = \frac{1}{\sqrt{2}}(1, 0, 0, -1)$ and $\bar{n} = \frac{1}{\sqrt{2}}(1, 0, 0, 1)$ collinear gluon radiations. The mirror diagrams of (a-c) are not shown here but are included in the results.

we can extract the coefficient functions. Below the scale Q^2 , the hard modes have been integrated out, and have been taken into account by the matching condition at Q^2 . Therefore, the calculation of the cross section at μ_f^2 is performed with the SCET diagrams, including both virtual and real contributions. As mentioned earlier the virtual diagrams in SCET are scaleless and vanish in pure DR. As such one can ignore them, but the counterterm for the effective current must be taken into account. This is equivalent to taking the UV-subtracted contribution from the virtual diagrams. The counterterm has been written down in Eq. (3.27),

$$\text{c.t.} = \frac{\alpha_s}{4\pi} C_F \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} + \frac{2}{\epsilon} \ln \frac{Q^2}{\mu^2} \right], \quad (5.7)$$

where μ^2 will be taken as μ_f^2 . The real contribution contains collinear and soft gluon radiation diagrams. Some of these diagrams are identically zero because of $n^2 = 0$ or $\bar{n}^2 = 0$, or because of the equation of motion for the external collinear quark or anti-quark. The non-vanishing diagrams are shown in Fig. (4.1), including soft-gluon interference contribution, n and \bar{n} collinear gluon radiations and their interferences. The contribution from diagram (a) is, including that from the mirror

diagram

$$\left. \frac{d^3\sigma^{(1)}}{dQ^2 d^2Q_T dy} \right|_{(a)} = \frac{\sigma_0 \alpha_s C_F}{\pi \pi^2} \frac{1}{Q_T^2} \left[-\frac{\delta(x_2 - 1)}{(1 - x_1)_+} - \frac{\delta(x_1 - 1)}{(1 - x_2)_+} - \delta(x_1 - 1)\delta(x_2 - 1) \ln \frac{Q^2}{Q_T^2} \right]. \quad (5.8)$$

Diagram (b) represents the interference between the \bar{n} -collinear gluon radiation with collinear expansion of the current operator, and its contribution is

$$\left. \frac{d^3\sigma^{(1)}}{dQ^2 d^2Q_T dy} \right|_{(b)} = \frac{\sigma_0 \alpha_s C_F}{\pi \pi^2} \frac{1}{Q_T^2} \left[\frac{x_1 \delta(x_2 - 1)}{(1 - x_1)_+} + \frac{\delta(x_1 - 1)}{(1 - x_2)_+} + \delta(x_1 - 1)\delta(x_2 - 1) \ln \frac{Q^2}{Q_T^2} \right], \quad (5.9)$$

while diagram(c) for the n -collinear gluon radiation yields

$$\left. \frac{d^3\sigma^{(1)}}{dQ^2 d^2Q_T dy} \right|_{(c)} = \frac{\sigma_0 \alpha_s C_F}{\pi \pi^2} \frac{1}{Q_T^2} \left[\frac{x_2 \delta(x_1 - 1)}{(1 - x_2)_+} + \frac{\delta(x_2 - 1)}{(1 - x_1)_+} + \delta(x_1 - 1)\delta(x_2 - 1) \ln \frac{Q^2}{Q_T^2} \right] \quad (5.10)$$

Diagrams (d) and (e) stand for the n and \bar{n} collinear gluon radiations, respectively, and their sum is

$$\left. \frac{d^3\sigma^{(1)}}{dQ^2 d^2Q_T dy} \right|_{(d+e)} = \frac{\sigma_0 \alpha_s C_F}{\pi 2\pi^2} \frac{1}{Q_T^2} (1 - \epsilon) [(1 - x_1)\delta(x_2 - 1) + (1 - x_2)\delta(x_1 - 1)]. \quad (5.11)$$

The sum of the above contributions, as expected, reproduces the result of the full QCD calculation in the limit of low transverse momentum (see, e.g., [102]).

From the above, the real contribution contains soft divergences (i.e, when $Q_{\perp}^2 \rightarrow 0$), which will be cancelled by the virtual contribution. To see this cancellation explicitly, we Fourier-transform the cross section from the transverse momentum space into the impact parameter b -space. For this we need to calculate the Fourier transform of the functions $1/Q_T^2$ and $\ln(Q_T^2/Q_T^2)$. This calculation is carried out in

the last section of this chapter. The result is $W(b, Q^2, \mu_L^2)$ in SCET, including real and virtual contributions (the counterterm, Eq. (5.7)),

$$\begin{aligned}
& W(b, Q^2, \mu_L^2, x_1, x_2) \\
&= \delta(x_1 - 1)\delta(x_2 - 1) + \frac{\alpha_s C_F}{2\pi} \left[\mathcal{P}_{q/q}(x_1)\delta(x_2 - 1) + (x_1 \leftrightarrow x_2) \right] \times \\
& \left(-\frac{1}{\epsilon} - \gamma_E + \ln \frac{4}{4\pi\mu_L^2 b^2} \right) + \frac{\alpha_s C_F}{2\pi} \left[(1 - x_1)\delta(x_2 - 1) + (1 - x_2)\delta(x_1 - 1) \right] \\
& - \frac{\alpha_s C_F}{2\pi} \delta(x_1 - 1)\delta(x_2 - 1) \left[\ln^2 \left(\frac{\mu_L^2 b^2}{4} e^{2\gamma_E - \frac{3}{2}} \right) + 2 \ln \frac{Q^2}{\mu_L^2} \ln \left(\frac{\mu_L^2 b^2}{4} e^{2\gamma_E} \right) - \frac{9}{4} + \frac{\pi^2}{6} \right], \tag{5.12}
\end{aligned}$$

where $\mathcal{P}_{q/q}$ is the one-loop quark splitting function

$$\mathcal{P}_{q/q}(x) = \left(\frac{1+x^2}{1-x} \right)_+ = \frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x). \tag{5.13}$$

The soft divergences in $W(b, Q^2, \mu_L^2)$ have been cancelled. There is, however, the collinear divergence left which can be absorbed into the quark distribution at one-loop order. The cross section depends on the ultraviolet scale μ_I^2 . It is somewhat surprising that the above result also depends on $\ln Q^2$, but this is expected from the kinematical constraints of the process.

In order to eliminate the large logarithms in the coefficient function, the best choice for the scale μ_I is $\mu_I^2 = C_1^2/b^2$ with $C_1 = 2e^{-\gamma_E}$. In addition, $W(b, Q^2, \mu_I^2)$ no longer depends on Q^2 at this order. In principle, this observation has to be true at higher orders and one should be able to eliminate all the $\ln Q^2$ by making a single choice of μ_I^2 . This, however, needs to be verified explicitly. Considering the quark distribution at one-loop order, we can cast Eq. (5.12) into the form

$$W(b, \mu_I^2 = C_1^2/b^2, x_1, x_2) = C_q(b, x_1, \mu_I^2) \otimes q(x, \mu_I^2) C_{\bar{q}}(b, x_2, \mu_I^2) \otimes \bar{q}(x, \mu_I^2), \tag{5.14}$$

where C_q reads

$$C_q(b, x, \mu_I^2) = \delta(1-x) + \frac{\alpha_s C_F}{2\pi} \left[(1-x) - \delta(x-1) \left(\frac{\pi^2}{12} \right) \right], \quad (5.15)$$

and $C_{\bar{q}} = C_q$. Note that we have set the quark charge e_q to 1.

5.4 Resummation and Comparison with Conventional Approach

Having obtained the matching coefficient at the intermediate scale μ_I , we can now write down the final result for $W(b, Q^2)$ following our discussion in the introduction

$$W(b, Q^2, x_1, x_2) = \sum_{q\bar{q}} C^2(Q^2, \alpha_s(Q^2)) e^{-\mathcal{S}(Q^2, \mu_I^2)} (C_q \otimes f_{q/q})(x_1, b, Q^2, \mu_I^2) \times (C_{\bar{q}} \otimes f_{\bar{q}/\bar{q}})(x_2, b, Q^2, \mu_I^2), \quad (5.16)$$

where the exponential suppression factor can be calculated from the anomalous dimension of the quark form factor γ_1

$$\mathcal{S} = \int_{\mu_I}^Q \frac{d\mu}{\mu} 2\gamma_1(Q^2, \mu^2) = \int_{\mu_I^2}^{Q^2} \frac{d\mu^2}{\mu^2} \left[A \ln \frac{Q^2}{\mu^2} + B \right]. \quad (5.17)$$

Here A and B are the single logarithmic and constant terms in the anomalous dimension γ_1 , respectively. The separation of the anomalous dimension into a sum of these two terms holds to any order in perturbation theory. The Sudakov form factor comes from the running of the matching coefficient $C(Q^2, \mu^2)$ from $\mu^2 = Q^2$ down to the scale of μ_I^2 which can be taken as C_1^2/b^2 .

The above is the final resummation result in the effective theory. To compare with the CSS approach we follow the procedure outlined in [103] by absorbing the

factor $C(Q^2, \alpha_s(Q^2))$ into B and C functions, for example, up to order α_s ,

$$\begin{aligned} C_{q/\text{CSS}}^{(1)}(x) &= C_q^{(1)}(x) + \delta(1-x) \frac{1}{2} \left(C(Q^2, \alpha_s(Q^2)) \right)^2, \\ B_{\text{CSS}}^{(2)} &= B^{(2)} + \beta_0 \frac{\alpha_s}{4\pi} \left(C(Q^2, \alpha_s(Q^2)) \right)^2. \end{aligned} \quad (5.18)$$

At two-loop level and beyond, one has to shuffle the $\ln Q^2$ -dependent part of C_q into B_{CSS} as well. In this way, we get the CSS resummation formula as

$$W(Q^2, b^2) = e^{-\mathcal{S}_{\text{sud}}} \left(C_{q/\text{CSS}} \otimes q \right) (x_1, b, \mu_I^2) \left(C_{\bar{q}/\text{CSS}} \otimes \bar{q} \right) (x_2, b, \mu_I^2), \quad (5.19)$$

with

$$\mathcal{S}_{\text{sud}} = \int_{\mu_L^2}^{Q^2} \frac{d\mu^2}{\mu^2} \left[A_{\text{CSS}} \ln \frac{Q^2}{\mu^2} + B_{\text{CSS}} \right]. \quad (5.20)$$

A_{CSS} will be the same as our A functions, as does $B^{(1)}$. For $C_{q/\text{CSS}}$ (with its usual expansion in $a_s = \alpha_s/4\pi$), we have

$$C_{q/\text{CSS}}^{(1)}(b, x, \mu_L^2) = 2 \left[(1-x) - \delta(x-1) \left(4 - \frac{\pi^2}{2} \right) \right]. \quad (5.21)$$

Comparing with the results in [99], we find that we can reproduce the $A_{\text{CSS}}^{(1)}$, $A_{\text{CSS}}^{(2)}$, $B_{\text{CSS}}^{(1)}$, $C_{\text{CSS}}^{(1)}$, which are all the coefficients and functions needed for resummation at NLL order. $B_{\text{CSS}}^{(2)}$ will be needed to resum NNLL. However, to fully achieve NNLL resummation we need to calculate the matching coefficient at the lower scale up to order α_s^2 [104].

Following the above, the resummation for Semi-Inclusive DIS (where a hadron in the final state is observed) can be performed similarly. As we stated earlier, the anomalous dimension will be the same. The only difference is the process-dependent matching coefficients at Q^2 and μ_I^2 in the resummation formula. For DIS, one has

the one-loop result,

$$C_{\text{DIS}}^{(1)}(Q^2, \alpha_s(Q^2)) = -[8 - \zeta_2] , \quad (5.22)$$

which leads to the result for the $C_{q/\text{CSS}}^{(1)}$ in DIS,

$$C_{q/\text{CSS}}^{(1)} = [2(1-x) - 8\delta(x-1)] . \quad (5.23)$$

These results agree with those from the conventional resummation approach [105].

5.5 Higgs Production

Transverse-momentum dependence of the Standard Model Higgs production can also be studied through resummation of large double logarithms. Higgs production, and as we have already discussed in Chapter 2, for a large range of Higgs mass, can be described by an effective action with a pointlike coupling between the Higgs particle and gluon fields. The effective coupling is of course scale dependent, balancing the renormalization dependence of the composite operator. Let us write the effective lagrangian as

$$\mathcal{L}_{\phi gg} = -\frac{1}{4}C_\phi \phi G^{\mu\nu}G_{\mu\nu} \equiv C_{\text{EW}}(M_t)C_T(M_t, \mu) \phi G^{\mu\nu}G_{\mu\nu} , \quad (5.24)$$

where, as before, ϕ is the scalar Higgs field and $F_{\mu\nu}^a$ is the gluon field strength tensor. C_{EW} represents the electroweak coupling coefficient from the heavy-top-quark loop calculation, while C_T comes from the strong interaction radiative corrections. The coefficient C_T will depend, in general, on the top quark mass and the renormalization scale μ . To our interest, we quote at two-loop [106, 107],

$$C_T(M_t, \mu) = \frac{\alpha_s(\mu)}{4\pi} \left\{ 1 + \frac{\alpha_s(\mu)}{4\pi} (5C_A - 3C_F) \right.$$

$$+ \frac{\alpha_s^2(\mu)}{(4\pi)^2} \left[\ln \frac{\mu^2}{M_t^2} (7C_A^2 - 11C_A C_F + 8N_f T_F C_F) + \dots \right] \} .(5.25)$$

Here we have omitted the constant terms of order α_s^2 because they do not contribute to the renormalization group running at two-loop order.

In SCET, the Higgs production cross section can be calculated from its coupling with the collinear gluon fields,

$$\mathcal{L}_{\phi-SCET} = C(M_H, \mu) \phi \mathcal{G}_{n\mu\nu}^a \mathcal{G}^{a\bar{n}\mu\nu} , \quad (5.26)$$

where $\mathcal{G}_{n,\bar{n}\mu\nu}^a$ represent the n and \bar{n} collinear gluon field strength tensors in SCET [35, 36, 38]. $C(M_H, \mu)$ is the matching coefficient which contains the coupling of Higgs boson to gluons in full QCD,

$$C(M_H, \mu) = C_{EW}(M_t) C_T(M_t, \mu) C_G(M_H, \mu) . \quad (5.27)$$

The last factor comes from matching between the operator $\phi G_{\mu\nu}^a G^{a\mu\nu}$ in full QCD and $\phi \mathcal{G}_{n\mu\nu}^a \mathcal{G}^{a\bar{n}\mu\nu}$ in SCET. Because C_{EW} has no QCD effects, we will not discuss it further. C_T and C_G contain the QCD evolution effects, and thus the anomalous dimension of the SCET operator $\phi \mathcal{G}_{n\mu\nu}^a \mathcal{G}^{a\bar{n}\mu\nu}$ is the sum of the two¹ :

$$\tilde{\gamma}_1 = \gamma_T + \gamma_1 , \quad (5.28)$$

where γ_T and γ_1 are defined as

$$\begin{aligned} \gamma_T &= \frac{d \ln C_T(\mu)}{d \ln \mu} , \\ \gamma_G &= \frac{d \ln C_G(\mu)}{d \ln \mu} . \end{aligned} \quad (5.29)$$

¹ $\tilde{\gamma}_1$, γ_1 and γ_T are the same as ones introduced in Chapter 4 for the Higgs case.

From Eq. (5.25), it is easy to show that

$$\begin{aligned} \gamma_T = & -2\beta_0 \frac{\alpha_s}{4\pi} + \frac{\alpha_s^2}{(4\pi)^2} [-2\beta_1 - 2\beta_0(5C_A - 3C_F) \\ & + 2(7C_A^2 - 11C_A C_F + 8N_f T_F C_F)] = a_s[-2\beta_0] + a_s^2[-4\beta_1], \end{aligned} \quad (5.30)$$

up to two-loop order, and γ_1 for the Higgs case was already given in Chapter 4 in Eq. (4.19).

Including the matching at the intermediate scale μ_I , the result for Higgs production $W(b, M_H^2, x_1, x_2)$ at low transverse momentum can be written as

$$W(M_H^2, b) = C_H^2(M_H^2, \alpha_s(M_H^2)) e^{-S} (C_g \otimes g)(x_1, b, \mu_L^2) (C_g \otimes g)(x_2, b, \mu_L^2), \quad (5.31)$$

where C_{EW} and the leading factor in C_T have been absorbed in the Born cross section, and the remainder C_H is

$$C_H(M_H^2, \alpha_s(M_H^2)) = 1 + \frac{\alpha_s(M_H^2)}{4\pi} [5C_A - 3C_F + C_A(\zeta(2) + \pi^2)], \quad (5.32)$$

up to one-loop order. The Sudakov suppression form factor is the same as the one already introduced in Chapter 4, i.e., Eq. (4.46). With these results we can reproduce the conventional resummation for Higgs-boson production at NLL order [104].

The coefficient C_g to one-loop order is calculated from the matching at the scale of μ_I , with real and virtual contributions. It is needed for resummation at NLL. The result is

$$C_g(b, x, \mu_L^2) = \delta(1-x) + \frac{\alpha_s C_A}{2\pi} \delta(x-1) \left(-\frac{\pi^2}{12} \right), \quad (5.33)$$

where we have chosen $\mu_L = C_1/b$ with $C_1 = 2e^{-\gamma_E}$ to eliminate the large logarithms.

The corresponding coefficient for CSS resummation is

$$C_{g/CSS}(b, x, \mu_L^2) = \delta(1-x) + \frac{\alpha_s C_A}{4\pi} \delta(1-x) [(5 + \pi^2)C_A - 3C_F] \quad (5.34)$$

which is consistent with [104].

5.6 Summary

In this Chapter, we have demonstrated how to perform resummation of large double logarithms in hard processes involving low transverse momentum in the framework of effective field theory. The analysis has been performed up to NLL logarithmic accuracy and along the same lines that were outlined in the previous chapter. The effective field method could also be extended to higher logarithmic accuracies as well, for the case of small transverse momentum distributions, however, it is technically more involved than the case of threshold logarithms.

As a study case, we considered the Drell-Yan resummation in detail, and outlined how to extend the procedure to other processes as well, and reproduced the conventional resummation results to NLL accuracy. We have shown that the SCET calculation for DY cross section reproduces the full QCD results for small transverse momentum distributions. This observation could be very helpful when trying to extend the analysis to higher logarithmic accuracies as we have done in the previous chapter for threshold resummation.

5.7 d -Dimensional Fourier Transform

The first relation we show is the following

$$I_1 = \int \frac{d^d \vec{p}_\perp}{(2\pi)^d} \frac{e^{i\vec{p}_\perp \cdot \vec{b}}}{(|\vec{p}_\perp|^2)^m} = \frac{1}{2^{2m} \pi^{\frac{d}{2}}} \frac{\Gamma(\frac{d}{2} - m)}{\Gamma(m)} \frac{1}{|\vec{b}|^{d-2m}}, \quad (5.35)$$

where the integration is performed in Euclidian d dimensional space. The calculation goes as follows. The d dimensional integration measure is given by

$$d^d \vec{p}_\perp = |\vec{p}_\perp|^{d-1} d|\vec{p}_\perp| d\Omega_d. \quad (5.36)$$

We also need the expansion of a plane wave in terms of Bessel functions J and Gegenbauer polynomials C [108],

$$e^{i|\vec{p}_\perp| \cdot \vec{b}} = \Gamma(\nu) \left(\frac{\vec{p}_\perp \cdot \vec{b}}{2} \right)^{-\nu} \sum_{k=0}^{\infty} (\nu + k) i^k J_{\nu+k}(\vec{p}_\perp \cdot \vec{b}) C_k^{(\nu)}(\cos \theta), \quad (5.37)$$

where ν is arbitrary and the polynomials C satisfy the orthogonality relation:

$$\int_{-1}^1 dx (1-x^2)^{\alpha-1/2} C_n^{(\alpha)}(x) C_{n'}^{(\alpha)}(x) = \delta_{nn'} \frac{\pi 2^{1-2\alpha} \Gamma(n+2\alpha)}{n!(n+\alpha)(\Gamma(\alpha))^2}, \quad \alpha > -\frac{1}{2}, \quad (5.38)$$

with the normalization, $C_0^{(\alpha)} = 1$. Since the solid angle $d\Omega_d$ is proportional to $\sin^{d-2} \theta$ we have the following integral

$$\begin{aligned} \int_0^\pi d\theta \sin^{d-2} \theta C_k^{(\nu)}(\cos \theta) &= \int_{-1}^1 dx (1-x^2)^{\frac{d-3}{2}} C_k^{(\nu)}(x) \\ &= \int_{-1}^1 dx (1-x^2)^{\frac{d-3}{2}} C_k^{(\nu)}(x) C_0^{(\nu)}, \end{aligned} \quad (5.39)$$

which is 0 unless $k = 0$ and $\frac{d-3}{2} = \nu - 1/2$. Thus the integral I_1 becomes

$$\begin{aligned} I_1 &= \int_0^\infty \frac{d|\vec{p}_\perp| (|\vec{p}_\perp|)^{d-1-2m}}{(2\pi)^d} \Gamma\left(\frac{d}{2} - 1\right) \left(\frac{\vec{p}_\perp \cdot \vec{b}}{2}\right)^{1-\frac{d}{2}} \left(\frac{d}{2} - 1\right) J_{\left(\frac{d}{2}-1\right)}(|\vec{p}_\perp| |\vec{b}|) \times \\ &\int d\Omega_d. \end{aligned} \quad (5.40)$$

Using

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}, \quad (5.41)$$

and defining, $y = |\vec{p}_\perp| |\vec{b}|$, we get,

$$= \frac{|\vec{b}|^{2m-d}}{(2\pi)^{\frac{d}{2}}} \int_0^\infty dy y^{\frac{d}{2}-2m} J_{\left(\frac{d}{2}-1\right)}(|\vec{p}_\perp| |\vec{b}|), \quad (5.42)$$

where we used the identity $z\Gamma(z) = \Gamma(z+1)$. Using the formula [109],

$$\int_0^\infty dy y^{\frac{d}{2}-2m} J_{(\frac{d}{2}-1)}(y) = 2^{\frac{d}{2}-2m} \frac{\Gamma(\frac{d}{2}-m)}{\Gamma(m)}, \quad (5.43)$$

we get

$$I_1 = \frac{1}{2^{2m}\pi^{\frac{d}{2}}} \frac{\Gamma(\frac{d}{2}-m)}{\Gamma(m)} \frac{1}{|\vec{b}|^{d-2m}}. \quad (5.44)$$

We also need to calculate the following integral

$$I_2 = \int \frac{d^d \vec{p}_\perp}{(2\pi)^d} e^{i\vec{p}_\perp \cdot \vec{b}} \frac{\ln(|\vec{p}_\perp|^2)}{(|\vec{p}_\perp|^2)^m}. \quad (5.45)$$

Following similar steps as before, this integral can be brought into the form

$$I_2 = \frac{1}{(2\pi)^{\frac{d}{2}}} (|\vec{b}|)^{2m-d} \int_0^\infty dy y^{\frac{d}{2}-2m} [\ln y^2 - \ln |\vec{b}|^2] J_{(\frac{d}{2}-1)}(y). \quad (5.46)$$

The integral proportional to $\ln |\vec{b}|^2$ is calculated using Eq. (5.43). For the other one

we need the following relation

$$\int_0^\infty dx x^{\mu+\frac{1}{2}} \ln x J_\nu(x) = \frac{2^{\mu-\frac{1}{2}} \Gamma(\frac{\mu+\nu}{2} + \frac{3}{4})}{\Gamma(\frac{\nu-\mu}{2} + \frac{1}{4})} \times \left[\psi\left(\frac{\mu+\nu}{2} + \frac{3}{4}\right) + \psi\left(\frac{\nu-\mu}{2} + \frac{1}{2}\right) + \ln 4 \right], \quad (5.47)$$

where

$$\psi(z) \equiv \frac{d\Gamma(z)}{\Gamma(z)dz}, \quad (5.48)$$

with the expansion

$$\psi(\varepsilon) = -\frac{1}{\varepsilon} - \gamma_E + \zeta_2 \varepsilon + \mathcal{O}(\varepsilon^2), \quad (5.49)$$

and $\psi(1) = -\gamma_E$. With this, the integral I becomes

$$I_2 = \frac{1}{2^{2m}\pi^{\frac{d}{2}}} \frac{\Gamma(\frac{d}{2}-m)}{\Gamma(m)} \frac{1}{|\vec{b}|^{d-2m}} \left[\psi\left(\frac{d}{2}-m\right) + \psi(m) - \ln \frac{|\vec{b}|^2}{4} \right]. \quad (5.50)$$

It should be remembered that in the above expressions for I_1 and I_2 we have $d =$

$2 - 2\varepsilon$ and each integral has to be multiplied by the renormalization scale μ_0^2 .

6. WAVE DISTRIBUTION AMPLITUDES FOR THE Δ RESONANCE

6.1 *Introduction*

In the previous chapters we have been mainly concerned with inclusive processes where hadrons in the final state are “summed over” and are not observed experimentally. However, exclusive processes like the elastic electron-proton scattering

$$e + P \rightarrow e + P , \tag{6.1}$$

or

$$e + P \rightarrow e + R , \tag{6.2}$$

where R is one of the proton’s resonances, are no less important for studying hadronic structure within the framework of quantum chromodynamics (QCD).

The first reaction, for example, allowed Hofstadter and collaborators [110] half a century ago to demonstrate that nucleons have finite size of order one fermi. Since then, the various nucleon’s form factors were analyzed and studied extensively both theoretically and experimentally for large as well as small values of Q^2 , where Q^2 is the virtuality of the exchanged photon (between the electron and the proton). The

second reaction serves to study the proton and its resonances. Among the most important excited states of the proton is the spin-isospin $3/2$ $\Delta(1232)$ which is the lowest mass member of the baryon flavor decuplet. The $N - \Delta$ transition will be discussed, within perturbative QCD (pQCD), in the next chapter.

For large values of Q^2 exclusive processes are generically referred to as “hard” ones and they admit a pQCD treatment. The theoretical foundation for analyzing such processes was laid down more than twenty years ago [111, 112]. The fundamental point in that approach is that one can treat any amplitude related to a hard exclusive process through the concept of scale separation. This principle is in turn manifested through a factorization formula which incorporates the two (low and high) energy scales through fundamental quantities: a non-perturbative light-cone wave distribution amplitude and a hard scattering amplitude computed within perturbative quantum chromodynamics (pQCD). In order to construct such a factorization formula for an exclusive process with high momentum transfer $-q^2 = Q^2$, one needs to write down light-cone wave functions for the hadrons involved in the process. The notion of a wave distribution amplitude is essential for introducing light-cone wave functions.

Until a few years ago, hard scattering amplitudes were mainly computed with quark transverse momenta $k_{i\perp}$ being neglected. As such, the only relevant distribution amplitudes are “leading twist” ones, in the sense that effects which are suppressed by inverse powers of Q^2 are totally neglected. The leading twist distribution amplitudes describing three-quark Fock state of the proton has been studied extensively in the literature [113, 114, 115, 116, 117]. However, it is becoming more

and more evident, at least in cases where the initial and final baryon helicities are different, that instead of neglecting the quark transverse degrees of freedom, if one treats them perturbatively, a better phenomenological results may be obtained.

One such case is a pQCD analysis of the proton's Pauli form factor [118]. In that work it was shown that if the hard scattering amplitude is computed as a power expansion in the ratio $|k_{i\perp}|/Q$, one then gets a better agreement with experimental data. To perform such an analysis one needs to write down light-cone wave functions of baryons with explicit $k_{i\perp}$ -dependence or, equivalently, with a non-zero orbital angular momentum carried by the constituent quarks of the baryons. Such wave functions were systematically considered in Refs. [119, 120]. It is quite obvious, at least from dimensional analysis arguments, that such light-cone wave functions will depend on higher twist wave distribution amplitudes. Thus one gets to the conclusion that higher twist wave amplitudes are indispensable in order to carry out phenomenological analysis for exclusive processes when quark transverse momenta are taken into account.

Here we will discuss the asymptotic form of the higher twist wave amplitudes of the singly-charged Δ^+ and will extract the number of independent amplitudes for each twist. The results we obtain will be used in the next chapter.

In order to extract the wave distribution amplitudes of a baryon with uud quark structure one needs to consider the baryon-to-vacuum matrix element of trilo-

cal quark fields which live on the light cone

$$\langle 0 | \varepsilon^{ijk} u_{\alpha}^{i'}(a_1 n) [a_1 n, a_0 n]_{i',i} u_{\beta}^{j'}(a_2 n) [a_2 n, a_1 n]_{j',j} d_{\gamma}^{k'}(a_3 n) [a_3 n, a_2 n]_{k',k} | B(P) \rangle , \quad (6.3)$$

where u, d are quark-field operators, and the Greek letters α, β, γ stand for the spinor indices. The Latin letters i, j, k are color indices. n^{μ} is a light-like vector and the a_i are real numbers. The $|B(P)\rangle$ is the baryon Fock state with momentum P . The gauge link $[x, y]$ is defined through path ordering P and it is given by:

$$[x, y] = P \exp \left[ig \int_0^1 dt (x - y)_{\mu} A^{\mu}(tx + (1 - t)y) \right] , \quad (6.4)$$

where $A^{\mu} = A_a^{\mu} t^a$ and t^a are the generators of $SU_C(3)$. With the gauge link $[x, y]$ the matrix element becomes gauge invariant.

Full knowledge of the hadronic wave distribution amplitudes from first principles is beyond the reach of current research due to its nonperturbative nature. Nevertheless one can extract the asymptotic form ($\mu^2 \sim Q^2 \rightarrow \infty$ where μ is some renormalization scale) by making use of conformal symmetry arguments [121] (see also Appendix D). Other pieces of information can also be obtained from QCD sum rules [122, 123, 124, 125]. For the proton, the first systematic higher twist analysis was performed in [126]. That analysis was carried on in the infinite momentum frame and the twist contribution of the various amplitudes, which appear once the matrix element has been covariantly decomposed, was determined according to their asymptotic momentum scaling: the lower the momentum powers, the higher their twist contribution. For example the leading twist-3 amplitudes scale like $p^{+3/2}$, the twist-4 scale like $p^{+1/2}$, and so on. p^{μ} is defined as: $p^{\mu} = P^{\mu} - \frac{M^2}{2p \cdot n} n^{\mu}$. P^{μ} and n^{μ}

have only “plus” and “minus” light-cone components, respectively. $p^\pm \equiv \frac{1}{\sqrt{2}}(p^0 \pm p^3)$, $p^2 = n^2 = 0$, and M is the nucleon’s mass.

Another approach that could be pursued for analyzing the matrix element given in Eq. (6.3) is to decompose it while the quark fields have definite “good” or “bad” components, where for any spinor χ_α these are defined respectively through the action of the projection operators: $\Lambda_\pm \equiv \gamma^\mp \gamma^\pm / 2$,

$$\chi_+ = \Lambda_+ \chi, \quad \chi_- = \Lambda_- \chi. \quad (6.5)$$

We will pursue this approach below. For the Δ particle only the leading twist-3 distribution amplitudes were considered in the literature [127, 128].

6.2 Covariant Decomposition

As a matter of notation, Δ will denote the singly-charged delta particle, Δ^\pm are the “plus” and “minus” light-cone coordinates of the four vector Δ^μ and Δ_\pm will denote the “good” and “bad” components of the spinor Δ_α . The analysis will be carried out in a frame where the momentum of Δ , P^μ , and the light-like vector n^μ have the following light-cone coordinates:

$$P^\mu = (P^+, \frac{M_\Delta^2}{2P^+}, \vec{0}_\perp), \quad n^\mu = (0, 1, \vec{0}_\perp). \quad (6.6)$$

Thus in the limit $P^+ \rightarrow \infty$ we can ignore M_Δ . As it is well known the Δ is a spin-isospin 3/2 particle described by Rarita-Schwinger equations [129, 130]:

$$\begin{aligned} (\gamma \cdot P - M_\Delta) \Delta^\mu(P) &= 0, & \gamma_\mu \Delta^\mu(P) &= 0, \\ P_\mu \Delta^\mu(P) &= 0. \end{aligned} \quad (6.7)$$

In the matrix element given in Eq. (6.3) and following a common practice [126] we will not show explicitly the gauge factors but assume that they are there. With this, one needs to decompose the matrix element,

$$\langle 0 | \varepsilon_{ijk} u_\alpha^i(a_1 n)_\beta^j(a_2 n) d_\gamma^k(a_3 n) | \Delta(P) \rangle, \quad (6.8)$$

into the relevant Lorentz structures taking into account the spin and parity of the Δ .

In principle this decomposition is similar to the one given for the proton [126]. Technically, however, it is more complicated due to the vector index μ carried by the spinor Δ_α . The decomposition reads

$$\begin{aligned} & 4 \langle 0 | \varepsilon_{ijk} u_\alpha^i(a_1 n)_\beta^j(a_2 n) d_\gamma^k(a_3 n) | \Delta(P) \rangle \\ &= C_{\alpha\beta} [\gamma_5 n \cdot \Delta \mathcal{S}_1 + \gamma \cdot n \gamma_5 n \cdot \Delta \mathcal{S}_2 + \varepsilon_{\mu\nu\sigma\rho} \gamma^\nu P^\sigma n^\rho \Delta^\mu \mathcal{S}_3 + \varepsilon_{\mu\nu\sigma\rho} \sigma^{\nu\alpha} n_\alpha P^\sigma n^\rho \Delta^\mu \mathcal{S}_4]_\gamma \\ &+ (\gamma_5 C)_{\alpha\beta} [n \cdot \Delta \mathcal{P}_1 + \gamma \cdot n n \cdot \Delta \mathcal{P}_2 + \varepsilon_{\mu\nu\sigma\rho} \gamma^\nu P^\sigma n^\rho \gamma_5 \Delta^\mu \mathcal{P}_3 + \varepsilon_{\mu\nu\sigma\rho} \sigma^{\nu\alpha} n_\alpha P^\sigma n^\rho \gamma_5 \Delta^\mu \mathcal{P}_4]_\gamma \\ &+ (\gamma_\mu C)_{\alpha\beta} [(\gamma_5 \mathcal{V}_1 + \gamma \cdot n \gamma_5 \mathcal{V}_2) \Delta^\mu + P^\mu (\gamma_5 n \cdot \Delta \mathcal{V}_3 + \gamma \cdot n \gamma_5 n \cdot \Delta \mathcal{V}_4 + \varepsilon_{\alpha\nu\sigma\rho} \gamma^\nu P^\sigma n^\rho \Delta^\alpha \mathcal{V}_5 \\ &+ \varepsilon_{\alpha\nu\sigma\rho} \sigma^{\nu\beta} n_\beta P^\sigma n^\rho \Delta^\alpha \mathcal{V}_6) + n^\mu (\gamma_5 n \cdot \Delta \mathcal{V}_7 + \gamma \cdot n \gamma_5 n \cdot \Delta \mathcal{V}_8 + \varepsilon_{\alpha\nu\sigma\rho} \gamma^\nu P^\sigma n^\rho \Delta^\alpha \mathcal{V}_9 \\ &+ \varepsilon_{\alpha\nu\sigma\rho} \sigma^{\nu\beta} n_\beta P^\sigma n^\rho \Delta^\alpha \mathcal{V}_{10}) + \gamma^\mu \gamma_5 n \cdot \Delta \mathcal{V}_{11} + i \sigma^{\mu\nu} (n_\nu \gamma_5 n \cdot \Delta \mathcal{V}_{12} + \varepsilon_{\nu\alpha\beta\rho} P^\alpha n^\beta \Delta^\rho \mathcal{V}_{13}) \\ &+ \varepsilon^{\mu\nu\sigma\rho} \gamma_\nu P_\sigma n_\rho n \cdot \Delta \mathcal{V}_{14} + \varepsilon^{\mu\nu\sigma\rho} P_\sigma n_\rho \sigma_{\nu\alpha} n^\alpha n \cdot \Delta \mathcal{V}_{15} + \varepsilon^{\mu\nu\sigma\rho} P_\sigma n_\rho \Delta_\nu \mathcal{V}_{16} \\ &+ \varepsilon^{\mu\nu\sigma\rho} P_\sigma n_\rho n \cdot \gamma \Delta_\nu \mathcal{V}_{17} + \varepsilon^{\mu\nu\sigma\rho} \gamma_\nu \Delta_\sigma n_\rho \mathcal{V}_{18} + \varepsilon^{\mu\nu\sigma\rho} \gamma_\nu P_\sigma \Delta_\rho \mathcal{V}_{19} + \varepsilon^{\mu\nu\sigma\rho} \sigma_{\nu\alpha} n^\alpha \Delta_\sigma P_\rho \mathcal{V}_{20} \\ &+ \varepsilon^{\mu\nu\sigma\rho} \sigma_{\nu\alpha} n^\alpha \Delta_\sigma n_\rho \mathcal{V}_{21}]_\gamma + (\gamma_\mu \gamma_5 C)_{\alpha\beta} [\mathcal{V}_i \rightarrow \mathcal{A}_i, \Delta^\mu \rightarrow \gamma_5 \Delta^\mu]_\gamma \\ &+ (\sigma_{\mu\nu} C)_{\alpha\beta} [P^\mu \gamma_5 \Delta^\nu \mathcal{T}_1 + P^\mu \gamma \cdot n \gamma_5 \Delta^\nu \mathcal{T}_2 + n^\mu \gamma_5 \Delta^\nu \mathcal{T}_3 + n^\mu \gamma \cdot n \gamma_5 \Delta^\nu \mathcal{T}_4 + \gamma^\mu \gamma_5 \Delta^\nu \mathcal{T}_5 \\ &+ \varepsilon^{\mu\beta\sigma\rho} \gamma_\beta P_\sigma n_\rho \Delta^\nu \mathcal{T}_6 + \varepsilon^{\mu\alpha\sigma\rho} \sigma_{\alpha\beta} n^\beta P_\sigma n_\rho \Delta^\nu \mathcal{T}_7 + \sigma^{\mu\alpha} n_\alpha \gamma_5 \Delta^\nu \mathcal{T}_8 + P^\mu n^\nu (\gamma_5 n \cdot \Delta \mathcal{T}_9 \\ &+ \gamma \cdot n \gamma_5 n \cdot \Delta \mathcal{T}_{10} + \varepsilon^{\alpha\beta\sigma\rho} \gamma_\beta P_\sigma n_\rho \Delta_\alpha \mathcal{T}_{11} + \varepsilon^{\alpha\beta\sigma\rho} \sigma_{\beta\gamma} n^\gamma P_\sigma n_\rho \Delta_\alpha \mathcal{T}_{12}) + \sigma^{\mu\nu} \gamma_5 n \cdot \Delta \mathcal{T}_{13} \end{aligned}$$

$$\begin{aligned}
& +P^\mu\sigma^{\nu\rho}n_\rho\gamma_5n\cdot\Delta\mathcal{T}_{14}+n^\mu\sigma^{\nu\rho}n_\rho\gamma_5n\cdot\Delta\mathcal{T}_{15}+P^\mu\gamma^\nu\gamma_5n\cdot\Delta\mathcal{T}_{16}+n^\mu\gamma^\nu\gamma_5n\cdot\Delta\mathcal{T}_{17} \\
& +P^\mu(\varepsilon^{\nu\alpha\beta\rho}\gamma_\alpha P_\beta n_\rho n\cdot\Delta\mathcal{T}_{18}+\varepsilon^{\nu\alpha\beta\delta}\sigma_{\alpha\rho}n^\rho P_\beta n_\delta n\cdot\Delta\mathcal{T}_{19}+\varepsilon^{\nu\alpha\beta\rho}P_\alpha n_\beta\Delta_\rho\mathcal{T}_{20} \\
& +\varepsilon^{\nu\alpha\beta\rho}P_\alpha n_\beta\gamma\cdot n\Delta_\rho\mathcal{T}_{21}+\varepsilon^{\nu\alpha\beta\rho}\gamma_\alpha n_\beta\Delta_\rho\mathcal{T}_{22}+\varepsilon^{\nu\alpha\beta\rho}P_\alpha\gamma_\beta\Delta_\rho\mathcal{T}_{23} \\
& +\varepsilon^{\nu\alpha\beta\delta}\sigma_{\alpha\rho}n^\rho n_\beta\Delta_\delta\mathcal{T}_{24}+\varepsilon^{\nu\alpha\beta\delta}\sigma_{\alpha\rho}n^\rho P_\beta\Delta_\delta\mathcal{T}_{25})+n^\mu(\varepsilon^{\nu\alpha\beta\rho}\gamma_\alpha P_\beta n_\rho n\cdot\Delta\mathcal{T}_{26} \\
& +\varepsilon^{\nu\alpha\beta\delta}\sigma_{\alpha\rho}n^\rho P_\beta n_\delta n\cdot\Delta\mathcal{T}_{27}+\varepsilon^{\nu\alpha\beta\rho}P_\alpha n_\beta\Delta_\rho\mathcal{T}_{28}+\varepsilon^{\nu\alpha\beta\rho}P_\alpha n_\beta\gamma\cdot n\Delta_\rho\mathcal{T}_{29} \\
& +\varepsilon^{\nu\alpha\beta\rho}\gamma_\alpha n_\beta\Delta_\rho\mathcal{T}_{30}+\varepsilon^{\nu\alpha\beta\rho}P_\alpha\gamma_\beta\Delta_\rho\mathcal{T}_{31}+\varepsilon^{\nu\alpha\beta\rho}\sigma_{\alpha\gamma}n^\gamma n_\beta\Delta_\rho\mathcal{T}_{32} \\
& +\varepsilon^{\nu\alpha\beta\rho}\sigma_{\alpha\rho}n^\rho P_\beta\Delta_\rho\mathcal{T}_{33})+\gamma^\mu\varepsilon^{\nu\alpha\beta\rho}P_\alpha n_\beta\Delta_\rho\mathcal{T}_{34}+\sigma^{\mu\rho}n_\rho\varepsilon^{\nu\alpha\beta\rho}P_\alpha n_\beta\Delta_\rho\mathcal{T}_{35} \\
& +\varepsilon^{\mu\nu\sigma\rho}(P_\sigma n_\rho n\cdot\Delta\mathcal{T}_{36}+P_\sigma n_\rho\gamma\cdot nn\cdot\Delta\mathcal{T}_{37}+\gamma_\sigma n_\rho n\cdot\Delta\mathcal{T}_{38}+\sigma_{\sigma\alpha}n^\alpha n_\rho n\cdot\Delta\mathcal{T}_{39} \\
& +P_\sigma\gamma_\rho n\cdot\Delta\mathcal{T}_{40}+P_\sigma\sigma_{\rho\alpha}n^\alpha n\cdot\Delta\mathcal{T}_{41}+P_\sigma\Delta_\rho\mathcal{T}_{42}+P_\sigma\gamma\cdot n\Delta_\rho\mathcal{T}_{43}+n_\sigma\Delta_\rho\mathcal{T}_{44} \\
& +n_\sigma\gamma\cdot n\Delta_\rho\mathcal{T}_{45}+\gamma_\sigma\Delta_\rho\mathcal{T}_{46}+\sigma_{\sigma\alpha}n^\alpha\Delta_\rho\mathcal{T}_{47})-(\mu\leftrightarrow\nu)]_\gamma. \tag{6.9}
\end{aligned}$$

In the above decomposition we used the 16 Dirac matrices,

$$\{\Gamma C\}\equiv\{C,\gamma_5 C,\gamma^\mu C,\gamma^\mu\gamma_5 C,\sigma^{\mu\nu}C\}, \tag{6.10}$$

as a basis for the α, β spinor structure. C is the charge conjugation matrix which in Dirac representation reads as $C = i\gamma^2\gamma^0$, and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. We have also suppressed the mass M_Δ for convenience. The calligraphic functions depend on the scalar product $P \cdot n$. Using the fact that $\{\gamma^\mu C, \sigma^{\mu\nu} C\}$ are symmetric and $\{C, \gamma_5 C, \gamma^\mu\gamma_5 C\}$ are anti-symmetric matrices then the identity of the two up quarks delivers the following:

$$\begin{aligned}
\mathcal{S}_i(1, 2, 3) &= \mathcal{S}_i(2, 1, 3), & \mathcal{P}_i(1, 2, 3) &= -\mathcal{P}_i(2, 1, 3), & \mathcal{V}_i(1, 2, 3) &= \mathcal{V}_i(2, 1, 3), \\
\mathcal{A}_i(1, 2, 3) &= -\mathcal{A}_i(2, 1, 3), & \mathcal{T}_i(1, 2, 3) &= \mathcal{T}_i(2, 1, 3), & &
\end{aligned} \tag{6.11}$$

where the arguments $i = 1, 2, 3$ refer to the three quarks appearing in Eq. (6.3) who are differentiated by their light-cone positions.

Since the Δ is isospin-3/2, the matrix element is restricted by the requirement:

$$\left(T^2 - \frac{3}{2} \left(\frac{3}{2} + 1\right)\right) \langle 0 | \varepsilon_{ijk} u_\alpha^i(1) u_\beta^j(2) d_\gamma^k(3) | \Delta \rangle = 0, \quad (6.12)$$

where

$$T^2 = \frac{1}{2}(T_+ T_- + T_- T_+) + T_3^2, \quad (6.13)$$

and T_\pm are the usual $SU(2)$ raising and lowering operators. From the isospin constraint, Eq. (6.12), one gets the following relation,

$$2\langle 0 | \varepsilon_{ijk} u_\alpha^i(1) u_\beta^j(2) d_\gamma^k(3) | \Delta \rangle - \langle 0 | \varepsilon_{ijk} d_\alpha^i(1) u_\beta^j(2) u_\gamma^k(3) | \Delta \rangle - \langle 0 | \varepsilon_{ijk} u_\alpha^i(1) d_\beta^j(2) u_\gamma^k(3) | \Delta \rangle = 0. \quad (6.14)$$

Or, equivalently, one can write

$$2\langle 0 | \varepsilon_{ijk} u_\alpha^i(1) u_\beta^j(2) d_\gamma^k(3) | \Delta \rangle - \langle 0 | \varepsilon_{ijk} u_\alpha^i(1) u_\gamma^j(3) d_\beta^k(2) | \Delta \rangle - \langle 0 | \varepsilon_{ijk} u_\gamma^i(3) d_\beta^j(2) d_\alpha^k(1) | \Delta \rangle = 0. \quad (6.15)$$

6.3 Twist-3

As mentioned earlier, the twist classification of the invariant amplitudes appearing in Eq. (6.9) will be performed according to their contribution to the matrix element while the quark fields have definite “good” or “bad” components. For the leading twist we consider the matrix element with all quark fields having “good” components:

$$\langle 0 | \varepsilon_{ijk} u_+^i(a_1 n) u_+^j(a_2 n) d_+^k(a_3 n) | \Delta(P) \rangle. \quad (6.16)$$

In order to determine what are the Dirac matrices, in the $\{\Gamma C\}$ basis that contribute in this case, one imposes the following conditions (which are not all independent) on the α, β indices,

$$\Lambda_- \Gamma C \Lambda_+ = 0, \quad \Lambda_+ \Gamma C \Lambda_- = 0, \quad \Lambda_- \Gamma C \Lambda_- = 0, \quad \Lambda_+ \Gamma C \Lambda_+ \neq 0. \quad (6.17)$$

From this, it can be easily shown that the Dirac matrices satisfying the above relations are

$$\{\Gamma\} = \{\gamma^-, \gamma_5 \gamma^-, \sigma^{-i}\}, \quad (6.18)$$

which means that only amplitudes from the vector, axial-vector and tensor parts will contribute to twist-3 case. Once the Lorentz indices on the Dirac matrices are fixed, one can go further and determine the relevant indices of the terms associated with Δ^μ . To do that one has to recall that P^μ and n^μ have only plus and minus light-cone coordinates (in the limit $P^+ \rightarrow \infty$), respectively, and the fact that the vector-spinor Δ_α^μ satisfies Rarita-Schwinger relations, Eq. (6.7). After all the Lorentz indices have been fixed we then impose the condition $(\Lambda_- \Gamma' \Delta^\mu)_\gamma = 0$, where Γ' , like Γ , could be any one of the Dirac matrices.

One then finds that for twist-3 only Δ_+^μ contributes. In the last step we made use of the following useful identities:

$$\gamma^1 \gamma^2 = -i \gamma_5 (-\Lambda_- + \Lambda_+), \quad \gamma^1 = -i \gamma^2 \gamma_5 (-\Lambda_- + \Lambda_+), \quad \gamma^2 = i \gamma^1 \gamma_5 (-\Lambda_- + \Lambda_+). \quad (6.19)$$

Thus the covariant decomposition, Eq. (6.9), gives the following relation of the twist-3 matrix element

$$4 \langle 0 | \varepsilon_{ijk} (u_+^i)_\alpha (1) (u_+^j)_\beta (2) (d_+^k)_\gamma (3) | \Delta(P) \rangle$$

$$\begin{aligned}
&= V_1^{(uud)}(\gamma^- C)_{\alpha\beta}(\gamma_5 \Delta_+^+)_{\gamma} + V_2^{(uud)}(\gamma^- C)_{\alpha\beta}(\gamma_5 \gamma^i \Delta_{+i})_{\gamma} + A_1^{(uud)}(\gamma^- \gamma_5 C)_{\alpha\beta}(\Delta_+^+)_{\gamma} \\
&+ A_2^{(uud)}(\gamma^- \gamma_5 C)_{\alpha\beta}(\gamma^i \Delta_{+i})_{\gamma} + T_1^{(uud)}(i\sigma^{i-} C)_{\alpha\beta}(\gamma_5 \Delta_{+i})_{\gamma} \\
&+ T_2^{(uud)}(i\sigma^{i-} C)_{\alpha\beta}(\gamma_i \gamma_5 \Delta_+^+)_{\gamma} + T_3^{(uud)}(\sigma^{i-} C)_{\alpha\beta} \varepsilon_{ij}(\Delta_+^j)_{\gamma} ,
\end{aligned} \tag{6.20}$$

where the superscript (uud) designates the quarks with ‘‘good’’ components. We will use this notation also for the higher-twist cases.

The symmetry of the the two up quarks gives

$$\begin{aligned}
V_i^{(uud)}(1, 2, 3) &= V_i^{(uud)}(2, 1, 3) , & A_i^{(uud)}(1, 2, 3) &= -A_i^{(uud)}(2, 1, 3) \text{ for } i = 1, 2 , \\
T_i^{(uud)}(1, 2, 3) &= T_i^{(uud)}(2, 1, 3) , \text{ for } i = 1, 2, 3.
\end{aligned} \tag{6.21}$$

The four functions $V_1^{(uud)}$, $V_2^{(uud)}$, $A_1^{(uud)}$ and $A_2^{(uud)}$ can be combined into two functions:

$$\psi_i^{(3)}(1, 2, 3) = V_i^{(uud)}(1, 2, 3) - A_i^{(uud)}(1, 2, 3) \text{ for } i = 1, 2. \tag{6.22}$$

To implement the isospin constraint, Eq. (6.12), we make use of Fierz transformations, discussed in Section 6.6, to shuffle the spinor indices of the different Dirac structures. This gives the following

$$\begin{aligned}
T_3^{(uud)}(1, 2, 3) &= 0 , \\
\psi_i^{(3)}(1, 2, 3) &= \psi_i^{(3)}(3, 2, 1) , \\
T_1^{(uud)}(1, 2, 3) &= -\psi_1^{(3)}(1, 3, 2) + \frac{1}{2}\psi_1^{(3)}(2, 3, 1) = -\frac{1}{2}\psi_1^{(3)}(1, 3, 2) , \\
T_2^{(uud)}(1, 2, 3) &= -2\psi_2^{(3)}(1, 3, 2) + \psi_2^{(3)}(2, 3, 1) = -\psi_2^{(3)}(1, 3, 2).
\end{aligned} \tag{6.23}$$

Thus at twist-3 there are only two independent amplitudes, $\psi_i^{(3)}(1, 2, 3)$, $i = 1, 2$. This is consistent with earlier results given in [127]. To gain more physical

insight of the meaning of ψ_i , let us define them in terms of left and right chiral fields. From Eq. (6.20) we have

$$\begin{aligned}\langle 0|\varepsilon_{ijk}u_+^{iT}(1)C\gamma^+u_+^j(2)d_{+\gamma}^k(3)|\Delta\rangle &= -\left(V_1^{(uud)}\gamma_5\Delta_+^+ + V_2^{(uud)}\gamma_5\gamma^i\Delta_{+i}\right)_\gamma, \\ \langle 0|\varepsilon_{ijk}u_+^{iT}(1)C\gamma^+\gamma_5u_+^j(2)d_{+\gamma}^k(3)|\Delta\rangle &= -\left(A_1^{(uud)}\Delta_+^+ + A_2^{(uud)}\gamma^i\Delta_{+i}\right)_\gamma,\end{aligned}\quad (6.24)$$

where T stands for the transpose of a matrix. Introduce now the chiral fields $\chi_{L,R} \equiv P_{L,R}\chi$ where $P_{L,R} = \frac{1}{2}(1 \mp \gamma_5)$ are the chiral projection operators. Combining the two equations in Eq. (6.24) and applying P_R on the spinors with the γ -index, one obtains

$$\langle 0|\varepsilon_{ijk}u_{+R}^{iT}(i)C\gamma^+u_{+L}^j(2)d_{+R}^k(3)|\Delta\rangle = -\frac{1}{2}[\psi_1^{(3)}(1,2,3)P_R\Delta_+^+ + \psi_2^{(3)}(1,2,3)P_R\gamma^i\Delta_{+i}].\quad (6.25)$$

From the last equation it is clear that the two up quarks are combined to give a helicity zero. This could be considered as the definition of $\psi_i^{(3)}$, $i = 1, 2$. Denote the Fourier transform to the momentum space of $\psi_i^{(3)}(a_in)$ by $\tilde{\psi}_i^{(3)}(x_1, x_2, x_3)$ where $\psi_i^{(3)}(a_1n, a_2n, a_3n) = \int_0^1 dx_1 dx_2 dx_3 \delta(1 - x_1 - x_2 - x_3) e^{-iP^+n^-(\sum a_i x_i)} \tilde{\psi}_i^{(3)}(x_1, x_2, x_3)$,

for $i = 1, 2$ where x_j are the momenta fractions. Based on conformal symmetry arguments [131, 132] (see also the discussion in Appendix D and Eq. (D.27)) the asymptotic form of $\tilde{\psi}_i^{(3)}(x_1, x_2, x_3)$ reads

$$\tilde{\psi}_i^{(3)}(x_1, x_2, x_3) \sim x_1 x_2 x_3.\quad (6.27)$$

where we have used

$$\Sigma_{+-}u_\pm \equiv su_\pm = \pm(1/2)u_\pm,\quad (6.28)$$

and s stands for the projection of the particle's spin along the light-cone. Since the canonical dimension of a quark field is $3/2$ then the conformal spin $j = (1/2)(l + s)$ for u_+ and d_- is 1 and $1/2$, respectively.

6.4 Twist-4

At twist-4 level there are three different matrix elements each with two quarks having “good” components and one quark with a “bad” component. However, due to the symmetry of the up quarks, we need to consider only two of them: $u_+u_+d_-$ and $u_+u_-d_+$. For the first case, the decomposition of Eq. (6.9) will deliver a relation similar to the twist-3 case but the Δ will have a “bad” component. Thus we write

$$\begin{aligned}
& 4\langle 0 | \varepsilon_{ijk} (u_+^i)_\alpha (1) (u_+^j)_\beta (2) (d_-^k)_\gamma (3) | \Delta(P) \rangle \\
&= V_1^{(uu)} (\gamma^- C)_{\alpha\beta} (\gamma_5 \Delta_-^+)_\gamma + V_2^{(uu)} (\gamma^- C)_{\alpha\beta} (\gamma_5 \gamma_i \Delta_-^i)_\gamma + A_1^{(uu)} (\gamma^- \gamma_5 C)_{\alpha\beta} (\Delta_-^+)_\gamma \\
&+ A_2^{(uu)} (\gamma^- \gamma_5 C)_{\alpha\beta} (\gamma_i \Delta_-^i)_\gamma + T_1^{(uu)} (i\sigma^{i-} C)_{\alpha\beta} (\gamma_5 \Delta_{-i})_\gamma \\
&+ T_2^{(uu)} (i\sigma^{i-} C)_{\alpha\beta} (\gamma_i \gamma_5 \Delta_-^+)_\gamma + T_3^{(uu)} (\sigma^{i-} C)_{\alpha\beta} \varepsilon_{ij} (\Delta_-^j)_\gamma.
\end{aligned} \tag{6.29}$$

We also have

$$\begin{aligned}
V_i^{(uu)}(1, 2, 3) &= V_i^{(uu)}(2, 1, 3), \quad A_i^{(uu)}(1, 2, 3) = -A_i^{(uu)}(2, 1, 3) \text{ for } i = 1, 2 \\
T_i^{(uu)}(1, 2, 3) &= T_i^{(uu)}(2, 1, 3) \text{ for } i = 1, 2, 3.
\end{aligned} \tag{6.30}$$

For the second matrix element we impose the following conditions on the α, β structure:

$$\Lambda_- \Gamma C \Lambda_- = 0, \quad \Lambda_+ \Gamma C \Lambda_+ = 0, \quad \Lambda_- \Gamma C \Lambda_+ = 0, \quad \Lambda_+ \Gamma C \Lambda_- \neq 0. \tag{6.31}$$

From this one finds that only

$$\{\Gamma\} = \{1, \gamma_5, \gamma^i, \gamma^i \gamma_5, \sigma^{ij}\} \quad (6.32)$$

contributes where $i = 1, 2$. We now again fix all the Lorentz indices on the right-hand side of Eq. (6.9) and impose the condition $(\Lambda_+ \Gamma' C)_\gamma = 0$ where Γ' , like Γ , could any one of the 16 Dirac matrices. We then find that the Δ will have a “bad” component and get the following decomposition:

$$\begin{aligned} & 4\langle 0 | \varepsilon_{ijk} (u_+^i)_\alpha (1) (u_-^j)_\beta (2) (d_+^k)_\gamma (3) | \Delta \rangle \\ &= S_1^{(ud)} C_{\alpha\beta} (\gamma^- \gamma_5 \Delta_-^+)_\gamma + S_2^{(ud)} C_{\alpha\beta} (\gamma^- \gamma_i \gamma_5 \Delta_-^i)_\gamma + P_1^{(ud)} (\gamma_5 C)_{\alpha\beta} (\gamma^- \Delta_-^+)_\gamma \\ &+ P_2^{(ud)} (\gamma_5 C)_{\alpha\beta} (\gamma^- \gamma_i \Delta_-^i)_\gamma + V_1^{(ud)} (\gamma^i C)_{\alpha\beta} (\gamma^- \gamma_5 \Delta_-^i)_\gamma + V_2^{(ud)} (\gamma^i C)_{\alpha\beta} (\gamma^- \gamma_i \gamma_5 \Delta_-^+)_\gamma \\ &+ iV_3^{(ud)} \varepsilon_{ij} (\gamma^i C)_{\alpha\beta} (\gamma^- \Delta_-^j)_\gamma + A_1^{(ud)} (\gamma^i \gamma_5 C)_{\alpha\beta} (\gamma^- \Delta_-^i)_\gamma + A_2^{(ud)} (\gamma^i \gamma_5 C)_{\alpha\beta} (\gamma^- \gamma_i \Delta_-^+)_\gamma \\ &+ iA_3^{(ud)} \varepsilon_{ij} (\gamma^i \gamma_5 C)_{\alpha\beta} (\gamma^- \gamma_5 \Delta_-^j)_\gamma + (\sigma^{ij} C)_{\alpha\beta} [(T_2^{(ud)} (\gamma^- \gamma_5 \gamma_i \Delta_-^j) - (i/2) T_1^{(ud)} \varepsilon_{ij} (\gamma^- \Delta_-^+)_\gamma) \\ &- (i \leftrightarrow j)]. \end{aligned} \quad (6.33)$$

The isospin constraint given in Eq. (6.12) and (6.34) can also be written in the form:

$$\langle 0 | \varepsilon_{ijk} (u_+^i(1))_\alpha (u_+^j(3))_\gamma (d_-^k(2))_\beta | \Delta \rangle = \langle 0 | \varepsilon_{ijk} (u_+^i(1))_\alpha (u_-^j(2))_\beta (d_+^k(3))_\gamma | \Delta \rangle. \quad (6.34)$$

This form allows us to relate the two matrix elements with $u_+ u_+ d_-$ and $u_+ u_- d_+$ by performing, again, a Fierz transformation on the $u_+ u_+ d_-$ matrix element from $(\alpha\beta, \gamma)$ to $(\alpha\gamma, \beta)$ and switching the 2 and 3 arguments. From this we get the $u_+ u_- d_+$ matrix element. Thus we find that all the functions appearing in Eq. (6.33) can be expressed by those of Eq. (6.29):

$$S_1^{(ud)}(1, 2, 3) = \frac{1}{4} \left(-V_1^{(uu)}(1, 3, 2) + A_1^{(uu)}(1, 3, 2) - 2T_2^{(uu)}(1, 3, 2) \right),$$

$$\begin{aligned}
S_2^{(ud)}(1, 2, 3) &= \frac{1}{4} \left(V_2^{(uu)}(1, 3, 2) - A_2^{(uu)}(1, 3, 2) - T_1^{(uu)}(1, 3, 2) + T_3^{(uu)}(1, 3, 2) \right) , \\
P_1^{(ud)}(1, 2, 3) &= \frac{1}{4} \left(-V_1^{(uu)}(1, 3, 2) + A_1^{(uu)}(1, 3, 2) + 2T_2^{(uu)}(1, 3, 2) \right) , \\
P_2^{(ud)}(1, 2, 3) &= \frac{1}{4} \left(-V_2^{(uu)}(1, 3, 2) + A_2^{(uu)}(1, 3, 2) - T_1^{(uu)}(1, 3, 2) + T_3^{(uu)}(1, 3, 2) \right) , \\
V_1^{(ud)}(1, 2, 3) &= \frac{1}{4} \left(-V_2^{(uu)}(1, 3, 2) - A_2^{(uu)}(1, 3, 2) + T_1^{(uu)}(1, 3, 2) + T_3^{(uu)}(1, 3, 2) \right) , \\
V_2^{(ud)}(1, 2, 3) &= \frac{1}{4} \left(V_1^{(uu)}(1, 3, 2) + A_1^{(uu)}(1, 3, 2) \right) , \\
V_3^{(ud)}(1, 2, 3) &= \frac{1}{4} \left(-V_2^{(uu)}(1, 3, 2) - A_2^{(uu)}(1, 3, 2) - T_1^{(uu)}(1, 3, 2) - T_3^{(uu)}(1, 3, 2) \right) , \\
A_1^{(ud)}(1, 2, 3) &= \frac{1}{4} \left(V_2^{(uu)}(1, 3, 2) + A_2^{(uu)}(1, 3, 2) + T_1^{(uu)}(1, 3, 2) + T_3^{(uu)}(1, 3, 2) \right) , \\
A_2^{(ud)}(1, 2, 3) &= \frac{1}{4} \left(V_1^{(uu)}(1, 3, 2) + A_1^{(uu)}(1, 3, 2) \right) , \\
A_3^{(ud)}(1, 2, 3) &= \frac{1}{4} \left(V_2^{(uu)}(1, 3, 2) + A_2^{(uu)}(1, 3, 2) - T_1^{(uu)}(1, 3, 2) - T_3^{(uu)}(1, 3, 2) \right) .
\end{aligned} \tag{6.35}$$

By using the identities in Eq. (6.19) one finds

$$T_1^{(ud)} = \frac{i}{2} P_i^{(ud)} , \quad T_2^{(ud)} = \frac{-i}{2} P_2^{(ud)} . \tag{6.36}$$

Hence there are five independent amplitudes at twist-4 which we denote by $\psi_i^{(4)}$ for $i = 1, 2, \dots, 5$. For $i = 1, 2$

$$\psi_i^{(4)}(1, 2, 3) = V_i^{(uu)}(1, 2, 3) - A_i^{(uu)}(1, 2, 3) , \tag{6.37}$$

For $i = 1, 2, 3$,

$$\psi_{i+2}^{(4)}(1, 2, 3) = T_i^{(uu)}(1, 2, 3) . \tag{6.38}$$

These five functions are symmetric under the interchange $1 \rightleftharpoons 2$. Similar to the case of twist-3 we can write

$$\langle 0 | \varepsilon_{ijk} u_{+R}^{iT}(1) C \gamma^+ u_{+L}^j(2) d_{-R}^k(3) | \Delta \rangle = -\frac{1}{2} [\psi_1^{(4)}(1, 2, 3) P_R \Delta_-^+ + \psi_2^{(4)}(1, 2, 3) P_R \gamma^i \Delta_-^i] .
\tag{6.39}$$

The Fourier transform of $\psi_i^{(4)}$ for $i = 1, \dots, 5$ is defined analogously to the twist-3 case. Again from conformal symmetry arguments the asymptotic form reads

$$\tilde{\psi}_i^{(4)}(x_1, x_2, x_3) \sim x_1 x_2. \quad (6.40)$$

6.5 Twist-5 and 6

At twist-5 we need to consider the two matrix elements with $u_- u_- d_+$ and $u_- u_+ d_-$. The decomposition can be read off directly from that of the twist-4 case with suitable modifications:

$$\begin{aligned} & 4\langle 0 | \varepsilon_{ijk} (u_-^i)_\alpha (1) (u_-^j)_\beta (2) (d_+^k)_\gamma (3) | \Delta(P) \rangle \\ &= V_1^{(d)} (\gamma^+ C)_{\alpha\beta} (\gamma_5 \Delta_-^-)_\gamma + V_2^{(d)} (\gamma^+ C)_{\alpha\beta} (\gamma_5 \gamma_i \Delta_+^i)_\gamma + A_1^{(d)} (\gamma^+ \gamma_5 C)_{\alpha\beta} (\Delta_-^-)_\gamma \\ &+ A_2^{(d)} (\gamma^+ \gamma_5 C)_{\alpha\beta} (\gamma_i \Delta_+^i)_\gamma + T_1^{(d)} (i\sigma^{i+} C)_{\alpha\beta} (\gamma_5 \Delta_{+i})_\gamma + T_2^{(d)} (i\sigma^{i+} C)_{\alpha\beta} (\gamma_i \gamma_5 \Delta_+^-)_\gamma \\ &+ T_3^{(d)} (\sigma^{i+} C)_{\alpha\beta} \varepsilon_{ij} (\Delta_+^j)_\gamma. \end{aligned} \quad (6.41)$$

The symmetry of the two up quarks again delivers

$$\begin{aligned} V_i^{(d)}(1, 2, 3) &= V_i^{(d)}(2, 1, 3), \quad A_i^{(d)}(1, 2, 3) = -A_i^{(2,1,3)} \text{ for } i = 1, 2 \\ T_i^{(d)}(1, 2, 3) &= T_i^{(d)}(2, 1, 3) \text{ for } i = 1, 2, 3. \end{aligned} \quad (6.42)$$

The second matrix element decomposes into

$$\begin{aligned} & 4\langle 0 | \varepsilon_{ijk} (u_-^i)_\alpha (1) (u_+^j)_\beta (2) (d_-^k)_\gamma (3) | \Delta \rangle \\ &= S_1^{(u)} C_{\alpha\beta} (\gamma^+ \gamma_5 \Delta_+^-)_\gamma + S_2^{(u)} C_{\alpha\beta} (\gamma^+ \gamma_i \gamma_5 \Delta_+^i)_\gamma + P_1^{(u)} (\gamma_5 C)_{\alpha\beta} (\gamma^+ \Delta_+^-)_\gamma \\ &+ P_2^{(u)} (\gamma_5 C)_{\alpha\beta} (\gamma^+ \gamma_i \Delta_+^i)_\gamma + V_1^{(u)} (\gamma^i C)_{\alpha\beta} (\gamma^+ \gamma_5 \Delta_{+i})_\gamma + V_2^{(d)} (\gamma^i C)_{\alpha\beta} (\gamma^+ \gamma_i \gamma_5 \Delta_+^-)_\gamma \\ &+ iV_3^{(u)} \varepsilon_{ij} (\gamma^i C)_{\alpha\beta} (\gamma^+ \Delta_+^j)_\gamma + A_1^{(u)} (\gamma^i \gamma_5 C)_{\alpha\beta} (\gamma^+ \Delta_{+i})_\gamma + A_2^{(u)} (\gamma^i \gamma_5 C)_{\alpha\beta} (\gamma^+ \gamma_i \Delta_+^-)_\gamma \end{aligned}$$

$$\begin{aligned}
& +iA_3^{(u)}\varepsilon_{ij}(\gamma^i\gamma_5 C)_{\alpha\beta}(\gamma^+\gamma_5\Delta_+^j)_\gamma + (\sigma^{ij}C)_{\alpha\beta}[(T_2^{(u)}(\gamma^+\gamma_5\gamma_i\Delta_{+j} - \frac{i}{2}T_1^{(u)}\varepsilon_{ij}(\gamma^+\Delta_+^-)_\gamma) \\
& -(i \leftrightarrow j)]. \tag{6.43}
\end{aligned}$$

The isospin constraint of Eq. (6.34) takes the form

$$\langle 0|\varepsilon_{ijk}(u_-^i(1))_\alpha(u_-^j(3))_\gamma(d_+^k(2))_\beta|\Delta\rangle = \langle 0|\varepsilon_{ijk}(u_-^i(1))_\alpha(u_+^j(2))_\beta(d_-^k(3))_\gamma|\Delta\rangle. \tag{6.44}$$

Thus we see that all functions of Eq. (6.43) can be expressed with those of Eq. (6.41). This gives five independent amplitudes for the twist-5 case which we denote by $\psi_i^{(5)}$ for $i = 1, 2, \dots, 5$. For $i = 1, 2$,

$$\psi_i^{(5)}(1, 2, 3) = V_i^{(d)}(1, 2, 3) - A_i^{(d)}(1, 2, 3). \tag{6.45}$$

And the other three ($i = 1, 2, 3$) are given by

$$\psi_{i+2}^{(4)}(1, 2, 3) = T_i^{(d)}(1, 2, 3). \tag{6.46}$$

In terms of the chiral fields we get

$$\langle 0|\varepsilon_{ijk}u_{-R}^{iT}(1)C\gamma^-u_{-L}^j(2)d_{+R}^k(3)|\Delta\rangle = -\frac{1}{2}[\psi_1^{(5)}(1, 2, 3)P_R\Delta_+^- + \psi_2^{(5)}(1, 2, 3)P_R\gamma^i\Delta_+^i]. \tag{6.47}$$

The Fourier transform of $\psi_i^{(5)}$ for $i = 1, \dots, 5$ is defined in the usual manner.

Again from conformal symmetry arguments the asymptotic form reads

$$\tilde{\psi}_i^{(5)}(x_1, x_2, x_3) \sim x_3 \text{ for } i = 1 \dots 5. \tag{6.48}$$

For twist-6 we need to consider one matrix element with $u_-u_-d_-$ structure. The decomposition of this matrix element can be read off from that of the twist-3 given

by Eq. (6.20) with obvious modifications:

$$\begin{aligned}
& 4\langle 0 | \varepsilon_{ijk} (u_-^i)_\alpha (1) (u_-^j)_\beta (2) (d_-^k)_\gamma (3) | \Delta(P) \rangle \\
& = V_1(\gamma^+ C)_{\alpha\beta} (\gamma_5 \Delta_-)_\gamma + V_2(\gamma^+ C)_{\alpha\beta} (\gamma_5 \gamma^i \Delta_{-i})_\gamma + A_1(\gamma^+ \gamma_5 C)_{\alpha\beta} (\Delta_-)_\gamma \\
& + A_2(\gamma^+ \gamma_5 C)_{\alpha\beta} (\gamma^i \Delta_{-i})_\gamma + T_1(i\sigma^{i+} C)_{\alpha\beta} (\gamma_5 \Delta_{-i})_\gamma + T_2(i\sigma^{i+} C)_{\alpha\beta} (\gamma_i \gamma_5 \Delta_-)_\gamma \\
& + T_3(\sigma^{i+} C)_{\alpha\beta} \varepsilon_{ij} (\Delta_-^j)_\gamma. \tag{6.49}
\end{aligned}$$

The symmetry between the up quarks gives

$$\begin{aligned}
V_i(1, 2, 3) & = V_i(2, 1, 3), \quad A_i(1, 2, 3) = -A_i(2, 1, 3) \text{ for } i = 1, 2, \\
T_i(1, 2, 3) & = T_i(2, 1, 3) \text{ for } i = 1, 2, 3. \tag{6.50}
\end{aligned}$$

The four functions V_1, V_2, A_1 and A_2 can be combined into two functions;

$$\psi_i^{(6)}(1, 2, 3) = V_i(1, 2, 3) - A_i(1, 2, 3) \text{ for } i = 1, 2. \tag{6.51}$$

The isospin constraint, Eq. (6.12), gives relations among the five functions similar to those given in Eq. (6.23) with the replacement

$$\begin{aligned}
V_i^{(uud)}(1, 2, 3) & \rightarrow V_i(1, 2, 3), \quad A_i^{(uud)} \rightarrow A_i(1, 2, 3), \quad T_i^{(uud)} \rightarrow T_i(1, 2, 3), \quad \text{for } i = 1, 2, \\
T_3^{(uud)} & \rightarrow T_3(1, 2, 3). \tag{6.52}
\end{aligned}$$

Thus at twist-6 there are only two independent wave amplitudes. We denote them by $\psi_i^{(6)}$. In terms of chiral fields we get

$$\langle 0 | \varepsilon_{ijk} u_{-R}^{iT} (1) C \gamma^- u_{-L}^j (2) d_{-R}^k (3) | \Delta \rangle = -\frac{1}{2} [\psi_1^{(6)}(1, 2, 3) P_R \Delta_- + \psi_2^{(6)}(1, 2, 3) P_R \gamma^i \Delta_-^i], \tag{6.53}$$

and the asymptotic form is,

$$\tilde{\psi}_i^{(6)}(x_1, x_2, x_3) \sim 1, \quad \text{for } i = 1, 2. \tag{6.54}$$

6.6 Fierz Transformations

In this section we show how to carry out the Fierz transformations. Specifically it is applied to twist-4 Lorentz structures. The general formula to be used is Eq. (A.2) in Ref. [126] with the following modifications:

$$\not{p} \rightarrow \gamma^-, \quad \not{z} \rightarrow \gamma^+, \quad p \cdot z = 1, \quad (6.55)$$

we then get

$$\begin{aligned} & (\Gamma C)_{\alpha\beta}(\Gamma' \Delta)_\gamma \\ &= \frac{1}{4} [C_{\gamma\beta}(\Gamma\Gamma' \Delta)_\alpha + (\gamma_5 C)_{\gamma\beta}(\Gamma\gamma_5\Gamma' \Delta)_\alpha + (\gamma^- C)_{\gamma\beta}(\Gamma\gamma^+\Gamma' \Delta)_\alpha + (\gamma^+ C)_{\gamma\beta}(\Gamma\gamma^-\Gamma' \Delta)_\alpha \\ &+ (\gamma^i C)_{\gamma\beta}(\Gamma\gamma_i\Gamma' \Delta)_\alpha - (\gamma^- \gamma_5 C)_{\gamma\beta}(\Gamma\gamma^+\gamma_5\Gamma' \Delta)_\alpha - (\gamma^+ \gamma_5 C)_{\gamma\beta}(\Gamma\gamma^-\gamma_5\Gamma' \Delta)_\alpha \\ &- (\gamma^i \gamma_5 C)_{\gamma\beta}(\Gamma\gamma_i\gamma_5\Gamma' \Delta)_\alpha + (\sigma^{-+} C)_{\gamma\beta}(\Gamma\sigma_{+-}\Gamma' \Delta)_\alpha + \frac{1}{2}(\sigma^{ij} C)_{\gamma\beta}(\Gamma\sigma_{ij}\Gamma' \Delta)_\alpha \\ &+ (i\sigma^{i-} C)_{\gamma\beta}(\Gamma\gamma_i\gamma^+\Gamma' \Delta)_\alpha + (i\sigma^{i+} C)_{\gamma\beta}(\Gamma\gamma_i\gamma^-\Gamma' \Delta)_\alpha] \end{aligned} \quad (6.56)$$

The above formula can be used for any μ -index carried by the Δ . Moreover the Δ will carry either a “good” or “bad” component according to the specific twist considered. In any case, Eq. (6.56) is applicable in both cases. We will apply it now to the twist-4 matrix elements.

Let us consider the first one with $u_+ u_+ d_-$ which has decomposition as given in Eq. (6.29). The second matrix element given in Eq. (6.33) can be found from the first one by making a Fierz transformation $(\alpha\beta, \gamma) \rightarrow (\alpha\gamma, \beta)$, on the first matrix element. As it is clear from Eq. (6.29), there are seven different Lorentz structures that contribute to the first matrix element. We will carry out the Fierz transformation for each one of them using Eq. (6.56) above.

We notice that since the Δ has a “bad” component for twist-4 then many terms will vanish. Moreover, we notice that there are two distinct “ γ ” structures (where by γ we mean the spinor index) that do not mix when performing the Fierz transformation. For convenience we introduce the following notation:

$$\begin{aligned}
(v_1)_{\alpha\beta,\gamma}^{(uu)} &\equiv (\gamma^- C)_{\alpha\beta}(\gamma_5 \Delta_-^+)_{\gamma}, & (v_2)_{\alpha\beta,\gamma}^{(uu)} &\equiv (\gamma^- C)_{\alpha\beta}(\gamma_5 \gamma_i \Delta_-^i)_{\gamma}, \\
(a_1)_{\alpha\beta,\gamma}^{(uu)} &\equiv (\gamma^- \gamma_5 C)_{\alpha\beta}(\Delta_-^+)_{\gamma}, & (a_2)_{\alpha\beta,\gamma}^{(uu)} &\equiv (\gamma^- \gamma_5 C)_{\alpha\beta}(\gamma_i \Delta_-^i)_{\gamma}, \\
(t_1)_{\alpha\beta,\gamma}^{(uu)} &\equiv (i\sigma^{i-} C)_{\alpha\beta}(\gamma_5 \Delta_{-i})_{\gamma}, & (t_2)_{\alpha\beta,\gamma}^{(uu)} &\equiv (i\sigma^{i-} C)_{\alpha\beta}(\gamma_i \gamma_5 \Delta_-^+)_{\gamma}, \\
(t_3)_{\alpha\beta,\gamma}^{(uu)} &\equiv (\sigma^{i-} C)_{\alpha\beta} \varepsilon_{ij} (\Delta_-^j)_{\gamma}, & & (6.57)
\end{aligned}$$

and

$$\begin{aligned}
(s_1)_{\alpha\beta,\gamma}^{(ud)} &\equiv C_{\alpha\beta}(\gamma^- \gamma_5 \Delta_-^+)_{\gamma}, & (s_2)_{\alpha\beta,\gamma}^{(ud)} &\equiv C_{\alpha\beta}(\gamma^- \gamma_i \gamma_5 \Delta_-^i)_{\gamma}, \\
(p_1)_{\alpha\beta,\gamma}^{(ud)} &\equiv (\gamma_5 C)_{\alpha\beta,\gamma}(\gamma^- \Delta_-^+)_{\gamma}, & (p_2)_{\alpha\beta,\gamma}^{(ud)} &\equiv (\gamma_5 C)_{\alpha\beta}(\gamma^- \gamma_i \Delta_-^i)_{\gamma}, \\
(v_1)_{\alpha\beta,\gamma}^{(ud)} &\equiv (\gamma^i C)_{\alpha\beta}(\gamma^- \gamma_5 \Delta_{-i})_{\gamma}, & (v_2)_{\alpha\beta,\gamma}^{(ud)} &\equiv (\gamma^i C)_{\alpha\beta}(\gamma^- \gamma_i \gamma_5 \Delta_-^+)_{\gamma}, \\
(v_3)_{\alpha\beta,\gamma}^{(ud)} &\equiv (\gamma^i C)_{\alpha\beta} i \varepsilon_{ij} (\gamma^- \Delta_-^j)_{\gamma}, & & \\
(a_1)_{\alpha\beta,\gamma}^{(ud)} &\equiv (\gamma^i \gamma_5 C)_{\alpha\beta}(\gamma^- \Delta_{-i})_{\gamma}, & (a_2)_{\alpha\beta,\gamma}^{(ud)} &\equiv (\gamma^i \gamma_5 C)_{\alpha\beta}(\gamma^- \gamma_i \Delta_-^+)_{\gamma}, \\
(a_3)_{\alpha\beta,\gamma}^{(ud)} &\equiv (\gamma^i \gamma_5 C)_{\alpha\beta} i \varepsilon_{ij} (\gamma^- \gamma_5 \Delta_-^j)_{\gamma}, & & \\
(t_1)_{\alpha\beta,\gamma}^{(ud)} &\equiv (\sigma^{ij} C)_{\alpha\beta} i \varepsilon_{ij} (\Delta_-^+)_{\gamma}, & (t_2)_{\alpha\beta,\gamma}^{(ud)} &\equiv (\sigma^{ij} C)_{\alpha\beta}(\gamma^- \gamma_i \gamma_5 \Delta_{-j})_{\gamma}. \quad (6.58)
\end{aligned}$$

Let us apply now Eq. (6.56) for the Lorentz structures of the first matrix elements. We start with $v_1^{(uu)}$ and we get the following:

$$(v_1)_{\alpha\beta,\gamma}^{(uu)} = \frac{1}{4} [(s_1) + (p_1) + (v_2) - (a_2)]_{\gamma\beta,\alpha}^{(ud)}, \quad (6.59)$$

which gives

$$\begin{aligned}
(v_1)_{\gamma\beta,\alpha}^{(uu)} &= \frac{1}{4}[(s_1) + (p_1) + (v_2) - (a_2)]_{\alpha\beta,\gamma}^{(ud)} , \\
(v_1)_{\gamma\alpha,\beta} &= \frac{1}{4}[(s_1) + (p_1) + (v_1) - (a_2)]_{\beta\alpha,\gamma}^{(ud)} , \\
(v_1)_{\alpha\gamma,\beta}^{(uu)} = (v_1)_{\gamma\alpha,\beta}^{(uu)} &= \frac{1}{4}[-(s_1) - (p_1) + (v_2) + (a_2)]_{\alpha\beta,\gamma}^{(ud)} ,
\end{aligned} \tag{6.60}$$

where we used the fact that C and $\gamma_5 C$ and $\gamma^i \gamma_5 C$ are anti-symmetric and $\gamma^i C$ is symmetric. Similarly we get

$$\begin{aligned}
(a_1)_{\alpha\beta,\gamma}^{(uu)} &= \frac{1}{4}[(s_1) + (p_1) - (v_2) + (a_2)]_{\gamma\beta,\alpha}^{(ud)} , \\
(a_1)_{\alpha\gamma,\beta}^{(uu)} &= \frac{1}{4}[(s_1) + (p_1) + (v_2) + (a_2)]_{\alpha\beta,\gamma}^{(ud)} , \\
(t_2)_{\alpha\beta,\gamma}^{(uu)} &= \frac{1}{2}[(s_1) - (p_1)]_{\gamma\beta,\alpha}^{(ud)} , \\
(t_2)_{\alpha\gamma,\beta}^{(uu)} &= \frac{1}{2}[-(s_1) + (p_1)]_{\alpha\beta,\gamma}^{(ud)} .
\end{aligned} \tag{6.61}$$

From Eqs. (6.60) and (6.61) we get the following:

$$S_1^{(ud)} = \frac{1}{4}[-V_1 + A_1 - 2T_2]^{(uu)} , \quad P_1^{(ud)} = \frac{1}{4}[-V_1 + A_1 + 2T_2]^{(uu)} , \tag{6.62}$$

$$V_2^{(ud)} = \frac{1}{4}[V_1 + A_1]^{(uu)} , \quad A_2^{(ud)} = \frac{1}{4}[V_1 + A_1]^{(uu)} . \tag{6.63}$$

Similarly we can obtain all of the relations given in Eq. (6.35).

6.7 Summary

The motivation behind this work was the need to obtain in a systematic manner the higher light-cone distribution amplitudes of the singly-charged Δ^+ particle thereby extending an already existing similar work for the nucleon. In this work we considered only the asymptotic form of these amplitudes using well-known conformal

symmetry results. Such higher-twist amplitudes (at least their asymptotic form) of the Δ^+ are to be utilized whenever the Δ^+ is observed as a resonance in exclusive process with high momentum transfer; for example, in the case of nucleon to Δ^+ transition when the baryon helicities in the initial and final states are different.

We analyzed the matrix element of delta-to-vacuum trilocal quark field operators and decomposed it in a covariant manner, taking into account the spin, isospin and parity of the Δ^+ . We have ignored contributions coming from Fock components with a number of partons other than three. The twist classification was performed according to the number of the “good” components that the quark fields have. The higher this number the lower the twist. The isospin constraint limits the number of the independent amplitudes. For twist-3 and 6 we found two independent amplitudes for each twist and for twist-4 and 5 there were five of them in each case. It is easy to see that for the negatively-charged Δ^- one will also get the same results by simply interchanging one of two up quarks with a down quark and repeating identical analysis as the one already performed. For the Δ^{++} and Δ^{--} similar analysis is also straightforward.

7. NUCLEON TO DELTA TRANSITION

7.1 Introduction

The nucleon transition to one of its resonances, mainly the $\Delta(1232)$, has been studied for decades within many of the theoretical frameworks of QCD. Phenomenologically, this transition is important one since the Δ , among all the nucleon resonances, admits the highest production cross section. Moreover, the Δ resonance has the same quark structure like the proton, hence one may expect that they have some common features. Theoretically, it turns out, that “naive” treatments of this transition, yield predictions that are inconsistent with experimental data. Thus, this transition, and its electromagnetic form factors, has attracted much theoretical attention in recent years and studies ranging from perturbative QCD (pQCD), lattice calculations, large N_c , chiral perturbation theory and model dependent calculations where performed in order to gain a better understanding of this process¹.

Some of the earlier interests was due to the work of Becchi and Morpurgo [135]. In that work they showed that in the context of the symmetric, non-relativistic, spin-flavor $SU(6)$ quark model, the transition $N \rightarrow \Delta$ is a pure magnetic dipole $M1$ and the contribution from the electric quadrupole $E2$ is zero. We first note, that based

¹ for very recent treatments see Refs. [133, 134, 125].

on parity invariance and angular momentum conservation, only the above mentioned electromagnetic multipoles can contribute. One crucial assumption in the derivation of this ‘selection rule’ is that the quarks in both the nucleon and the delta are in the zero orbital angular momentum states.

These predictions were made prior to QCD and they were considered to be as a check to the validity of the quark model, which was still questionable in those days. Later experimental measurements [136] showed that, indeed, the bulk of the contribution to the $N \rightarrow \Delta$ transition is due to the magnetic dipole $M1$, while the contribution from $E2$ is small, however, non-vanishing.

The non-vanishing value of $E2$, and also the Coulomb quadrupole $C2$ in the case of a virtual photon, has generated much theoretical interest. One way to account for $E2 \neq 0$ is through the D -wave mixtures in the N and Δ wave functions [137]. Another way is through the two-body electromagnetic currents from one-gluon and/or one-pion exchange between constituent quarks [138]. In the latter case, it is argued that $E2$ transition is, mainly, due to a two-quark spin-flip operator. On the other hand, in the large N_c limit of QCD it has been shown [139] that $R_{EM} \equiv E2/M1$ is of order $1/N_c^2$. To derive this result, no assumption about orbital angular momentum of the quarks was necessary. More work in this direction can be found in Refs. [140, 141, 142, 143].

Another major issue related to the $N \rightarrow \Delta$ transition, and in general to any hadronic exclusive process, is the applicability of pQCD at the range of values of momentum transfer Q^2 accessible in the current generation of experiments. In terms of the ratios R_{EM} and $R_{SM} \equiv C2/M1$, pQCD power counting predicts that in the

limit $Q^2 \rightarrow \infty$, $R_{EM} \rightarrow 1$ and $R_{SM} \rightarrow \text{const.}$, up to logarithmic corrections to be discussed below [144]. The former prediction has not yet been observed experimentally. In fact, up to $Q^2 = 4 \text{ GeV}^2$, R_{EM} stays negative and very close to zero [145, 146, 147, 148, 149]. The comparison between data and pQCD prediction for R_{SM} is the main topic of this chapter.

To make the pQCD predictions more relevant at finite Q^2 where data have been and will be taken, one has to go beyond the asymptotic power counting, make detailed pQCD calculations of hard scattering amplitudes and derive the factorization formula for the experimental observable. In the present case, the relevant quantities are the three independent helicity matrix elements [144]:

$$G_\lambda = \frac{1}{2M_N} \langle \Delta, \lambda_\Delta | \varepsilon(\lambda) \cdot J | P, \lambda_P = 1/2 \rangle, \quad (7.1)$$

where M_N is the mass of the nucleon, $\lambda = 0, \pm 1$, and $\lambda_\Delta + 1/2 = \lambda$. A pQCD factorization formula of the helicity-conserving matrix element G_+ has been known for many years [112, 113, 144]. However, a pQCD calculation of the helicity non-conserving amplitudes has not been explored in the literature until recently, because *it necessarily involves the orbital motion of the quarks* [118, 150].

Below, we derive a pQCD factorization formula for the helicity-flip matrix element G_0 . The technical details are basically similar to those in the calculation of the Pauli form factor $F_2(Q^2)$ of the proton performed in [118]. In this calculation, the quarks in the nucleon and the Δ have one unit of relative orbital angular momentum along the direction of motion (taken as the z axis) *i.e.*, $|\Delta L_z| = 1$. With this we calculate the “hard amplitudes” which, in turn, are to be convoluted with the soft

light-cone distribution amplitudes. The light-cone wave functions of the nucleon and delta have been classified according to their orbital angular momentum dependence in [119, 120]. Note that we have to add the gluon contribution to maintain color gauge invariance. We neglect the dynamical gluon effects proportional to the gluon field strength $F^{\alpha\beta}$ which seems numerically suppressed in general [151, 152].

We remark that a detailed calculation of the hard amplitudes for G_- will involve two units of quark relative orbital angular momentum. Due to computational complexity it will not be pursued here. Nevertheless, it can be easily shown that G_- is suppressed, in the high Q^2 limit by $O(1/Q)$ relative to G_0 [120, 144].

Assuming our calculation for G_0 is relevant at Q^2 that is currently explored at Jefferson Lab, our result and a recent experimentally measured values of R_{SM} enable us to make a phenomenological prediction of this ratio for higher values of Q^2 . We mention in this regard that the double logarithmic term, $\log^2(Q^2/\Lambda^2)$, will play an important role in our analysis. The importance of $\log^2(Q^2/\Lambda^2)$ has also been demonstrated recently in the Jefferson Lab data on the nucleon elastic form factor $Q^2 F_2/F_1$ [118].

7.2 Kinematics and Notation

Although in experiments the delta appears only as a resonance (in certain scattering cross sections) which decays back, mainly, to nucleon and a pion, however, at the resonance peak, where the contribution from the background vanishes by definition, we treat the the bare delta as if it were a bound state of QCD. Thus,

our analysis is relevant only at the resonance peak. Moreover, we will not consider the so-called the dressing of the resonance due to the pion cloud which might be important at small Q^2 [148, 149].

The production of the delta proceeds through the following scattering:

$$P(P_N) + \gamma^*(q) \rightarrow \Delta(P_\Delta). \quad (7.2)$$

which appears as a sub-process of the electro-production of pions. The quantity of interest is the G_0 matrix element, in which case the virtual photon is longitudinally-polarized and with a large virtuality $Q^2 = -q^2$. Our analysis will be carried out in a frame in which the virtual photon γ^* , the incoming proton and the outgoing Δ are collinear. The three momenta of the proton and the Δ are in opposite directions and the γ^* is moving in the $-z$ direction. The photon polarization vector is given by

$$\varepsilon(\lambda = 0) = \frac{-1}{\sqrt{Q^2}}(q^3, 0, 0, q^0), \quad (7.3)$$

with $q^\mu = (q^0, 0, 0, q^3)$.

We will also need the light-cone wave functions of the proton and the Δ^+ (with the same charge as the proton). These wave functions are given by [119, 120]

$$\begin{aligned} |P, \lambda = 1/2\rangle &= \frac{1}{12} \int d[1][2][3] \varepsilon^{ijk} \left\{ \psi_P^{(1)}(1, 2, 3) u_{i\uparrow}^\dagger(1) \right. \\ &\quad \times \left[u_{j\downarrow}^\dagger(2) d_{k\uparrow}^\dagger(3) - d_{j\downarrow}^\dagger(2) u_{k\uparrow}^\dagger(3) \right] |0\rangle \\ &\quad + \left(k_{1\perp}^+ \psi_P^{(3)}(1, 2, 3) + k_{2\perp}^+ \psi_P^{(4)}(1, 2, 3) \right) \\ &\quad \left. \times \left[u_{i\uparrow}^\dagger(1) u_{j\downarrow}^\dagger(2) d_{k\downarrow}^\dagger(3) - d_{i\uparrow}^\dagger(1) u_{j\downarrow}^\dagger(2) u_{k\downarrow}^\dagger(3) \right] |0\rangle \right\} + \dots, \quad (7.4) \end{aligned}$$

$$|\Delta^+, \lambda = -1/2\rangle = -\frac{1}{12} \int d[1'][2'][3'] \varepsilon^{ijk} \left\{ \psi_\Delta^{(1)}(1', 2', 3') \right.$$

$$\begin{aligned}
& \times \left[u_{i\downarrow}^\dagger(1') u_{j\downarrow}^\dagger(2') d_{k\uparrow}^\dagger(3') + u_{i\downarrow}^\dagger(1') d_{j\downarrow}^\dagger(2') u_{k\uparrow}^\dagger(3') \right. \\
& \left. + d_{i\downarrow}^\dagger(1') u_{j\downarrow}^\dagger(2') u_{k\uparrow}^\dagger(3') \right] |0\rangle + k_{2\perp}^- \psi_\Delta^{(3)}(1', 2', 3') \\
& \times \left[u_{i\downarrow}^\dagger(1') u_{j\uparrow}^\dagger(2') d_{k\uparrow}^\dagger(3') + u_{i\downarrow}^\dagger(1') d_{j\uparrow}^\dagger(2') u_{k\uparrow}^\dagger(3') \right. \\
& \left. + d_{i\downarrow}^\dagger(1') u_{j\uparrow}^\dagger(2') u_{k\uparrow}^\dagger(3') \right] |0\rangle \} + \dots, \tag{7.5}
\end{aligned}$$

where $k_\perp^\pm = k_x \pm ik_y$, the ellipses denote other components of the wave functions which do not enter the following calculation, and \uparrow (\downarrow) on the quark creation operators denotes the positive (negative) helicity of the quarks. The amplitudes $\psi_{P,\Delta}^{(1)}$ ($\psi_{P,\Delta}^{(3)}$, $\psi_P^{(4)}$) have zero (one) unit of the orbital angular momentum projection. $\psi_\Delta^{(1)}(1, 2, 3)$ is symmetric in 1 and 2.

The integration measure is given by

$$\begin{aligned}
d[1]d[2]d[3] &= \frac{dx_1 dx_2 dx_3}{\sqrt{x_1 x_2 x_3}} \frac{d^2 k_{1\perp} d^2 k_{2\perp} d^2 k_{3\perp}}{(2\pi)^9} \\
&\times 2\pi \delta(1 - x_1 - x_2 - x_3) (2\pi)^2 \delta^{(2)}(\mathbf{k}_{1\perp} + \mathbf{k}_{2\perp} + \mathbf{k}_{3\perp}), \tag{7.6}
\end{aligned}$$

and the x_i are the fractions of the proton linear momentum P_N carried by the quarks, and $k_{i\perp}$ are the corresponding transverse momenta. For the Δ we use y_i instead of x_i and $k'_{i\perp}$ instead of $k_{i\perp}$. It should be noted that since the Δ moves in $-z$ direction then the orbital angular wave function must be $k_{2\perp}^-$ as shown in Eq. (7.5). The matrix element G_0 has two contributions: one in which the proton has $l_z = 0$ and the Δ carries $l_z = -1$ and in the second one the proton has $l_z = 1$ and the Δ carries $l_z = 0$. For the first case, we introduce the hard amplitudes $T_i(1, 2, 3, 1', 2', 3')$, which represent three quark scattering off the external photon with two-gluon exchange. The index $i = 1, 2, 3$ indicates that the photon is attached to i -th quark. They are

given by

$$\begin{aligned}
T_1(1, 2, 3, 1', 2', 3') &= \varepsilon^{ijk} \varepsilon^{i'j'k'} \langle c_{k'\uparrow}(3') b_{j'\downarrow}(2') a_{i'\uparrow}(1') | \bar{a} \varepsilon_L \cdot \gamma a | a_{i\uparrow}(1) b_{j\downarrow}(2) c_{k\uparrow}(3) \rangle, \\
T_2(1, 2, 3, 1', 2', 3') &= \varepsilon^{ijk} \varepsilon^{i'j'k'} \langle c_{k'\uparrow}(3') b_{j'\downarrow}(2') a_{i'\uparrow}(1') | \bar{b} \varepsilon_L \cdot \gamma b | a_{i\uparrow}(1) b_{j\downarrow}(2) c_{k\uparrow}(3) \rangle, \\
T_3(1, 2, 3, 1', 2', 3') &= \varepsilon^{ijk} \varepsilon^{i'j'k'} \langle c_{k'\uparrow}(3') b_{j'\downarrow}(2') a_{i'\uparrow}(1') | \bar{c} \varepsilon_L \cdot \gamma c | a_{i\uparrow}(1) b_{j\downarrow}(2) c_{k\uparrow}(3) \rangle, \tag{7.7}
\end{aligned}$$

where a, b, c are labels of the different quarks and $\varepsilon_L \equiv \varepsilon(\lambda = 0)$ is the polarization vector of a longitudinal photon. For the second case we label the hard amplitude as $T'_i(1, 2, 3, 1', 2', 3')$, in which all quark helicities are reversed. We will explain how to compute these hard amplitudes in the next section.

7.3 Leading-Order pQCD Factorization Formula for G_0

In this section, we derive a leading-order pQCD factorization formula for the helicity-flip $N\gamma^* \rightarrow \Delta$ matrix element G_0 . As shown in Eq. (7.1), this is generated by a virtual photon with longitudinal polarization, corresponding to the Coulomb quadrupole amplitude.

The first step in calculating G_0 is to obtain the hard scattering amplitudes of partons. For this we follow the guidelines of Ref. [118]. In principle, one has to consider both three-quark scattering with one unit of orbital angular momentum or three-quark-one-gluon scattering without orbital motion. Here we consider explicitly only the former and take into account the latter through the requirement of color gauge invariance, leaving out the contribution with the gluon field strength tensor. This practice of leaving out dynamical gluon contribution corresponds to the so-called Wandzura-Wilczek approximation in the literature [152]. A key point in

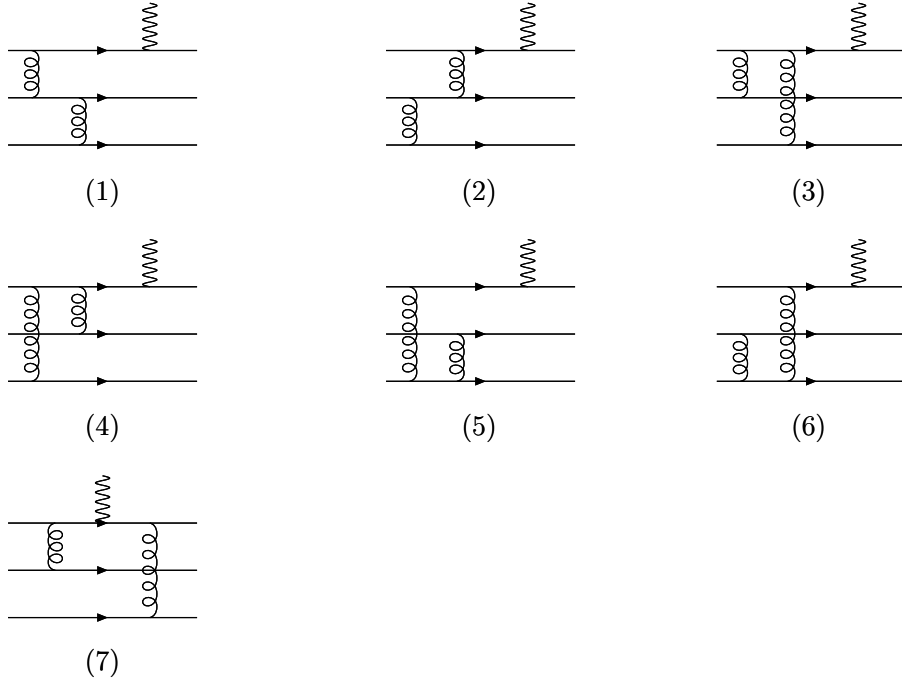


Fig. 7.1: The leading pQCD diagrams contributing to the nucleon to Δ transition amplitudes. The mirror diagrams should be added.

the calculation is that since the quark masses are negligible, the quark helicity is conserved. This is due to the vectorial nature of the electromagnetic coupling between the struck quark and the virtual photon and of QCD quark-gluon coupling [113]. When a Fock component of the light-cone wave function of the proton contains two quarks of the same flavor and helicity then, clearly, it contributes only when there are two quarks of the same flavor and helicity in the Δ wave function. The exchange contribution that results due to the presence of the same two quarks has to be taken into account properly through the second quantized calculation.

Using the light-cone wave functions of the proton and the Δ in Eqs. (7.4) and (7.5) and keeping terms that are linear in the quark transverse momenta, one

obtains

$$\begin{aligned}
G_0 = & -\frac{e_u - e_d}{24(2M_N)} \int d[1][2][3]d[1'][2'][3'] \{ k_{1\perp}' \psi_{\Delta}^{(3)*}(2', 1', 3') \psi_P^{(1)}(1, 2, 3) \\
& \times [T_2(1, 2, 3, 1', 2', 3') + T_2(1, 2, 3, 3', 2', 1') - T_3(1, 2, 3, 1', 2', 3') \\
& - T_3(1, 2, 3, 3', 2', 1')] + \psi_{\Delta}^{(1)*}(1', 3', 2') (k_{2\perp}' \psi_P^{(3)} + k_{1\perp}' \psi_P^{(4)})(2, 1, 3) \\
& \times [T_2'(1, 2, 3, 1', 2', 3') + T_2'(3, 2, 1, 1', 2', 3') - T_3'(1, 2, 3, 1', 2', 3') \\
& - T_3'(3, 2, 1, 1', 2', 3')] \} , \} \tag{7.8}
\end{aligned}$$

where T_i and T_i' are the three-quark scattering amplitudes introduced in the previous section. The color wave function of the three quarks is normalized to unity.

The perturbative Feynman diagrams for T_i are given in Fig. 7.1. There are fourteen such diagrams, taking into account the relative position of the photon vertex and the gluon interactions. In each diagram we have three collinear incoming and outgoing quarks exchanging two gluons. The photon could interact with any one of the quarks. With the spin-isospin structure of the proton and the Δ state functions, we assume that the first and third quarks have positive helicities while the second quark has negative one.

To find the amplitudes T_i , we let the outgoing quarks carry orbital angular momenta $k_{i\perp}'$ and the incoming quarks have zero orbital angular momenta, *i.e.*, $k_{i\perp} = 0$. From each one of the Feynman diagrams we write the amplitude following the usual QED and QCD Feynman rules. Since the calculation is performed in the collinear frame in which the particles are highly relativistic, we can set the masses of the proton and the Δ to zero. We expand the quark spinors in the final state and the quark propagators to first order in the transverse momenta. Then we collect

all such contributions from the given diagram and sum up the results of all the Feynman diagrams. This yields $T_i(k'_{i\perp}, x_i, y_i, Q^2)$. To find $T'_i(k_{i\perp}, x_i, y_i, Q^2)$ we set $k'_{i\perp} = 0$ and $k_{i\perp} \neq 0$ and follow similar steps.

The results for $T_1(1, 2, 3, 1', 2', 3')$ at order of $\mathcal{O}(\mathbf{k}_\perp)$ are as follows:

$$T_1(1, 2, 3, 1', 2', 3') = -g_s^4 \frac{8C_B^2}{Q^4} \left[k'_{1\perp}{}^+ w_1(x_1, x_2, x_3, y_1, y_2, y_3) + k'_{3\perp}{}^+ w_2(x_1, x_2, x_3, y_1, y_2, y_3) \right], \quad (7.9)$$

where $C_B = 2/3$ and the functions w_1 and w_2 are

$$\begin{aligned} w_1(x_1, x_2, x_3, y_1, y_2, y_3) &= \frac{1}{\bar{x}_1 \bar{y}_1^2 x_3^2 y_3} + \frac{1}{\bar{x}_1^2 \bar{y}_1^2 x_3 y_3} - \frac{1}{\bar{y}_1 \bar{x}_3 x_2 y_2 x_3 y_3} + \frac{1}{\bar{y}_1^2 \bar{x}_1^2 x_2 y_2} \\ &\quad - \frac{1}{\bar{x}_3 y_1 y_2 y_3 x_2 x_3^2}, \\ w_2(x_1, x_2, x_3, y_1, y_2, y_3) &= \frac{1}{\bar{x}_1 \bar{y}_1 x_3^2 y_3^2} + \frac{1}{\bar{x}_1^2 \bar{y}_1 x_3 y_3^2} - \frac{1}{\bar{x}_3 y_2 x_2 x_3 y_3^2} \\ &\quad + \frac{1}{\bar{x}_1^2 \bar{y}_1 y_3 x_2 y_2} + \frac{1}{\bar{x}_3 x_2 y_2 x_3^2 y_3^2}. \end{aligned} \quad (7.10)$$

Similarly for T_2 we have

$$T_2(1, 2, 3, 1', 2', 3') = -g_s^4 \frac{8C_B^2}{Q^4} \left[k'_{1\perp}{}^+ w_0(y_1, y_2, y_3, x_1, x_2, x_3) + k'_{3\perp}{}^+ w_0(y_3, y_2, y_1, x_3, x_2, x_1) \right], \quad (7.11)$$

the function w_0 is given by

$$\begin{aligned} w_0(x_1, x_2, x_3, y_1, y_2, y_3) &= -\frac{1}{\bar{x}_2 \bar{y}_2 x_1 x_3 y_3^2} + \frac{1}{x_1^2 y_1^2 x_3 y_3} + \frac{1}{\bar{x}_1 x_1 y_1 x_3 y_3^2} \\ &\quad + \frac{1}{\bar{x}_2 \bar{y}_2 x_1^2 y_1^2} + \frac{1}{\bar{x}_1 \bar{y}_3 y_1 x_1^2 x_3 y_3}. \end{aligned} \quad (7.12)$$

To obtain $T_3(1, 2, 3, 1', 2', 3')$ we interchange $x_1, y_1, k'_{1\perp}{}^+$ and $x_3, y_3, k'_{3\perp}{}^+$ and to obtain $T'_i(1, 2, 3, 1', 2', 3')$ we replace $k'_{i\perp}{}^+$ with $k_{i\perp}^+$ and interchange x_i and y_i . The above results are consistent with those derived in Ref. [118].

To further simplify the above expressions, let us define k_{\perp} -integrated wave functions:

$$\begin{aligned}
\phi_P^{(3)}(x_1, x_2, x_3) &= 2 \int [d^2 \mathbf{k}_{\perp}] \psi_P^{(1)}(k_1, k_2, k_3) , \\
\phi_P^{(4)}(x_1, x_2, x_3) &= \frac{2}{x_3} \int [d^2 \mathbf{k}_{\perp}] \mathbf{k}_{3\perp} \cdot (\mathbf{k}_{2\perp} \psi_P^{(3)} + \mathbf{k}_{1\perp} \psi_P^{(4)})(k_2, k_1, k_3) , \\
\psi_P^{(4)}(x_1, x_2, x_3) &= \frac{2}{x_1} \int [d^2 \mathbf{k}_{\perp}] \mathbf{k}_{1\perp} \cdot (\mathbf{k}_{2\perp} \psi_P^{(3)} + \mathbf{k}_{1\perp} \psi_P^{(4)})(k_2, k_1, k_3) , \\
\phi_{\Delta}^{(3)}(x_1, x_2, x_3) &= 2 \int [d^2 \mathbf{k}_{\perp}] \psi_{\Delta}^{(1)}(k_1, k_2, k_3) , \\
\phi_{\Delta}^{(4)}(x_1, x_2, x_3) &= \frac{2}{x_3} \int [d^2 \mathbf{k}_{\perp}] \mathbf{k}_{3\perp} \cdot \mathbf{k}_{1\perp} \psi_{\Delta}^{(3)}(k_2, k_1, k_3) , \\
\psi_{\Delta}^{(4)}(x_1, x_2, x_3) &= \frac{2}{x_1} \int [d^2 \mathbf{k}_{\perp}] \mathbf{k}_{1\perp} \cdot \mathbf{k}_{1\perp} \psi_{\Delta}^{(3)}(k_2, k_1, k_3) , \tag{7.13}
\end{aligned}$$

where the integration measure is defined as

$$[d^2 \mathbf{k}_{\perp}] = \frac{d^2 k_{1\perp} d^2 k_{2\perp} d^2 k_{3\perp}}{(2\pi)^6} \delta^{(2)}(\mathbf{k}_{1\perp} + \mathbf{k}_{2\perp} + \mathbf{k}_{3\perp}).$$

In terms of these wave functions and by using the obtained expressions for the hard amplitudes we get the following expression for G_0 :

$$\begin{aligned}
G_0 &= (4\pi\alpha_s)^2 \frac{C_B^2}{(2M_N)4Q^4} \frac{e_u - e_d}{3} \int [dx][dy] \\
&\left\{ y_3 \phi_{\Delta}^{*(4)}(y_1, y_2, y_3) \phi_P^{(3)}(x_1, x_2, x_3) \left[w_0(y_1, y_2, y_3, x_1, x_2, x_3) + w_0(y_1, y_2, y_3, x_3, x_2, x_1) \right. \right. \\
&\quad \left. \left. - w_1(x_3, x_2, x_1, y_1, y_2, y_3) - w_2(x_3, x_2, x_1, y_3, y_2, y_1) \right] \right. \\
&\quad \left. + y_1 \psi_{\Delta}^{*(4)}(y_1, y_2, y_3) \phi_P^{(3)}(x_1, x_2, x_3) \left[w_0(y_3, y_2, y_1, x_3, x_2, x_1) + w_0(y_3, y_2, y_1, x_1, x_2, x_3) \right. \right. \\
&\quad \left. \left. - w_1(x_3, x_2, x_1, y_3, y_2, y_1) - w_2(x_3, x_2, x_1, y_1, y_2, y_3) \right] \right. \\
&\quad \left. + x_3 \phi_P^{(4)}(x_1, x_2, x_3) \phi_{\Delta}^{*(3)}(y_1, y_3, y_2) \left[w_0(x_3, x_2, x_1, y_3, y_2, y_1) + w_0(x_3, x_2, x_1, y_1, y_2, y_3) \right. \right. \\
&\quad \left. \left. - w_1(y_3, y_2, y_1, x_3, x_2, x_1) - w_2(y_3, y_2, y_1, x_1, x_2, x_3) \right] \right. \\
&\quad \left. + x_1 \psi_P^{(4)}(x_1, x_2, x_3) \phi_{\Delta}^{*(3)}(y_1, y_3, y_2) \left[w_0(x_1, x_2, x_3, y_1, y_2, y_3) + w_0(x_1, x_2, x_3, y_3, y_2, y_1) \right] \right\}
\end{aligned}$$

$$-w_1(y_3, y_2, y_1, x_1, x_2, x_3) - w_2(y_3, y_2, y_1, x_3, x_2, x_1)]\}, \quad (7.14)$$

where $[dx] = dx_1 dx_2 dx_3 \delta(1 - x_1 - x_2 - x_3)$ and $\alpha_s = g_s^2/4\pi$. We see from Eq. (7.14) that G_0 scales like $1/Q^4$ in the high Q^2 limit, consistent with the general power counting. To find the normalization of G_0 we need to know the light-cone distribution amplitudes defined in Eq. (7.13). For the proton, a set of such functions have been given in Ref. [126] based on conformal expansion, QCD sum rules and Lorentz symmetry. For the Δ no such work has been done explicitly.

However, as we have shown in the previous chapter, if the Δ is treated as a three-quark state, at the asymptotically large Q^2 , the light-cone distribution amplitudes of the Δ have a y_i -dependence which is identical to the x_i -dependence of the light-cone distribution amplitudes of the proton. For the proton we have, for example, the asymptotic form of $\phi_p^{(3)}$ is $x_1 x_2 x_3$ and $\phi_p^{(4)}$ is $x_1 x_2$ [112]. If we perform the x_i and y_i integration in Eq. (7.14) with such functions we get, due to end-point singularities, a double logarithmic divergence results. This divergence indicates that quarks with small Feynman x contribute significantly to the hard scattering. However, for very small x for which the parton longitudinal momentum is on the order of Λ_{QCD} , the hard scattering picture breaks down, and the above calculation is invalid. Then one has to add the so-called Feynman contribution which is relatively unimportant in the limit of large Q^2 .

Therefore, these divergent integrals are regulated physically by a cut-off from below of order Λ^2/Q^2 where Λ is some parameter that represents the soft energy scale (order of few hundreds of MeV). With this cut-off, the result of the momentum

fraction integration will be a Q^2 -dependent term of the form, $\log^2(Q^2/\Lambda^2)$. Thus we can write

$$G_0 = c' \log^2(Q^2/\Lambda^2)/Q^4 \quad (7.15)$$

where c' is a numerical factor that depends on the explicit expressions of the light-cone distribution amplitudes. With this form of G_0 and with the fact that G_- is of order $1/Q^2$ relative to G_+ we will give in the next section a phenomenological prediction of the ratio R_{SM} in the high Q^2 limit.

7.4 Phenomenology

In the electro-production of the Δ resonance, the multipoles $M1$, $E2$ and $C2$ of the exchanged virtual photon are the only ones that contribute [153, 154, 155, 156]. One of the experimentally extracted quantities related to this process is the ratio $R_{SM} \equiv -C2/M1$. In order to express this ratio in terms of the helicity matrix elements G_λ we introduce the resonance ‘‘helicity amplitudes’’ $A_{1/2}$, $A_{3/2}$ and $S_{1/2}$ [157]. These amplitudes are computed in the rest frame of the Δ and the sub-indices refer to the helicity of the Δ . The scalar amplitude $S_{1/2}$ is relevant only for virtual photons. These amplitudes are related to the helicity matrix elements through the relations [158]:

$$A_{1/2} = \eta G_+, \quad A_{3/2} = \eta G_-, \quad S_{1/2} = \eta \frac{|\vec{q}|}{Q} G_0, \quad (7.16)$$

where η is some kinematical factor and $|\vec{q}|$ is the photon three-momentum in the rest frame of the Δ given by : $|\vec{q}| = \sqrt{\left(\frac{M_\Delta^2 + M_N^2 + Q^2}{2M_\Delta}\right)^2 - M_N^2}$. The multipole $C2$ equals

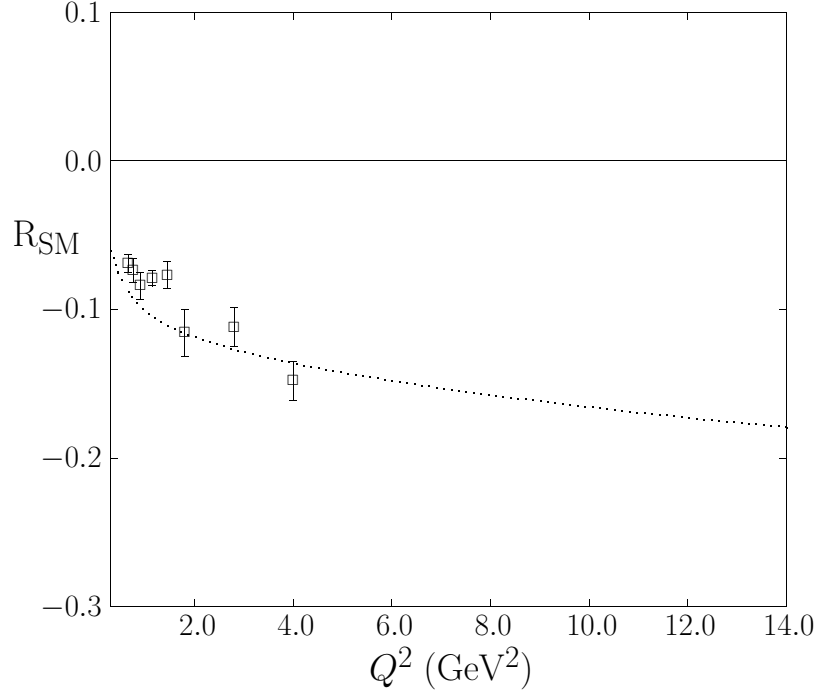


Fig. 7.2: A phenomenological prediction for the ratio R_{SM} .

$\frac{1}{\sqrt{2}}S_{1/2}$ and we also have [159]

$$M1 = -\frac{1}{2}A_{1/2} - \frac{\sqrt{3}}{2}A_{3/2}, \quad (7.17)$$

From the results of the previous section we see that $A_{3/2}$ is of order $1/Q^2$ relative to $A_{1/2}$ in the high Q^2 limit and thus it will be neglected. With this approximation the ratio R_{SM} becomes

$$R_{SM} = \frac{|\vec{q}|}{Q} \frac{G_0}{G_+}, \quad (7.18)$$

Substituting in the last equation the expression for $|\vec{q}|$ and using Eq. (7.15) we get,

$$R_{SM} = c \sqrt{\left(\frac{2.4 + Q^2}{2.464}\right)^2 - 0.88} \frac{\log^2(Q^2/\Lambda^2)}{Q^2}, \quad (7.19)$$

where the pre-factor c is determined by light-cone wave functions.

Now we *assume* that the above expression works in the region probed by

current experiments. There are many theoretical arguments against this practice. For example, the virtuality of the quarks and gluons in the hard part might not be large enough for pQCD to be justified [160], the Feynman contribution may not be small, hadron mass effects could also be large, etc. Moreover, the G_+ form factor calculated using the asymptotic wave function falls far below the experimental data in the region of interest. However here we are considering the *ratio* of form factors. So it might be possible that for not-yet-understood reason, the asymptotic prediction for the ratio might work in the region where it has no right to. This phenomenon of precocious scaling has been observed in various hadronic observables. Thus we use the above asymptotic result to make a phenomenological analysis.

We determine the the pre-factor c by taking $\Lambda = 0.25$ GeV and fitting with the results of R_{SM} [147]:

$$R_{SM} = -.112 \pm .013 \text{ at } Q^2 = 2.8 \text{ GeV}^2 ,$$

$$R_{SM} = -.148 \pm .013 \text{ at } Q^2 = 4.0 \text{ GeV}^2 .$$

The result is

$$R_{SM} = -\frac{.013 \sqrt{-.88 + (2.4 + Q^2)^2} \log^2 (16Q^2)}{Q^2} . \quad (7.20)$$

R_{SM} is displayed in Fig. 7.2. We have added also a more measurements of R_{SM} with Q^2 below 2 GeV² [161]. From that figure it is clear that R_{SM} has a slow variation at the presently-accessible momentum transfer Q^2 at JLab which is about 6 \sim 7 GeV². The pQCD predictions for R_{SM} have been found recently to be in good qualitative agreement with the results of Ref. [133]. We remark here that for the E2-to-M1 ratio R_{EM} , two units of orbital angular momentum are need to induce

the transition, and higher powers of the logarithmic correction may arise when one tries to calculate the matrix element G_- .

7.5 Summary

A leading-order perturbative QCD factorization for the helicity flip matrix element of the electromagnetic $N \rightarrow \Delta$ transition has been given. We showed that the transverse momenta of the quarks in the proton and Δ are essential to obtain a finite result. An explicit calculation of the hard amplitude $T_i(x_i, y_i, k'_{i\perp}, Q^2)$ and $T'_i(x_i, y_i, k_{i\perp}, Q^2)$ has been given and the techniques of the calculation were outlined in some detail. The essential steps of the calculation are to draw the relevant perturbative Feynman diagrams and then to compute the contribution from each such diagram by expanding the quark spinors and propagators to first power of the transverse momenta. The hard amplitudes are then to be convoluted with light-cone amplitudes. Our pQCD results for the scaling behavior of G_+ , G_0 and G_- confirm an earlier scaling prediction [144]. In addition we outlined how one obtains the double logarithmic correction. Based on our result for G_λ we gave a phenomenological prediction for the ratio R_{SM} . Roughly speaking, $|R_{SM}|$ will be of order of 20% at Q^2 of order 10 GeV².

APPENDIX

A. APPENDIX A

A.1 Mellin Moments

The convolution of functions $f_i(x)$, $1 = 1, 2, \dots, n$ denoted by the symbol \otimes is defined as

$$f(x) \equiv (f_1 \otimes f_2 \otimes \dots \otimes f_n)(x) = \int_0^1 dx_1 \int_0^1 dx_2 \dots \int_0^1 dx_n \delta\left(x - \prod_{i=1}^n x_i\right) \times f_1(x_1) f_2(x_2) \dots f_n(x_n). \quad (\text{A.1})$$

It is easy to see that $f(x)$ can also be expressed as

$$f(x) = \int_x^1 \frac{dx_1}{x_1} \int_{x/x_1}^1 \frac{dx_2}{x_2} \dots \int_{x/x_1 x_2 \dots x_n}^1 \frac{dx_n}{x_n} f_1(x_1) f_2(x_2) \dots f_n\left(\frac{x}{\prod_{i=1}^n x_i}\right). \quad (\text{A.2})$$

The Mellin transform of a function $f(x)$ to a conjugate variable N is defined as

$$f_N = \int_0^1 dx x^{N-1} f(x), \quad (\text{A.3})$$

where f_N is the ‘‘moment’’ or order N of $f(x)$. Under the action of the Mellin transform, the convolution defined in Eq. (A.1) becomes a simple product

$$\begin{aligned} f_N &= \int_0^1 dx_1 \int_0^1 dx_2 \dots \int_0^1 dx_n \int_0^1 dx x^{N-1} \delta\left(x - \prod_{i=1}^n x_i\right) f_1(x_1) f_2(x_2) \dots f_n(x_n), \\ &= \int_0^1 dx_1 \int_0^1 dx_2 \dots \int_0^1 dx_n \int_0^1 \left(\prod_{i=1}^n x_i\right)^{N-1} f_1(x_1) f_2(x_2) \dots f_n(x_n), \\ &= \prod_i^n \int_0^1 dx_i x_i^{N-1} f_i(x_i) = f_{1N} \times f_{2N} \dots \times f_{nN}. \end{aligned} \quad (\text{A.4})$$

Thus the integro-differential equations, like the evolution equations of parton distribution functions encountered in Chapter 1, are easier to solve in moment space due to Eq. (A.4).

The variable N is the “conjugate” one of x . The constant $\gamma_E = 0.5772\dots$, known as Euler-Mascheroni constant, and may be defined through the following algebraic representation

$$\gamma_E = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \ln n \right). \quad (\text{A.5})$$

By defining: $\bar{N} \equiv Ne^{\gamma_E}$, it turns out that many expressions get simplified. This can be seen more clearly from the discussion of Chapter 4.

A.2 The Plus Distributions

Let us assume that a given function $g(x)$ approaches infinity at $x \rightarrow 1^-$, then we define its “plus” distribution, denoted by $(g(x))_+$ such that its integral against any smooth function $f(x)$ which is sufficiently vanishing as $x \rightarrow 1^-$ is convergent:

$$\int_0^1 dx f(x)g(x)_+ = \int_0^1 dx \frac{f(x) - f(1)}{g(x)}. \quad (\text{A.6})$$

The “plus” distributions encountered in perturbative calculations of Feynman diagrams with real gluon emission are of the type

$$\mathcal{D}_i(x) = \left[\frac{\ln^i(1-x)}{(1-x)} \right]_+, \quad (\text{A.7})$$

The Mellin moments of these distributions are given by

$$\mathcal{D}_i(\bar{N}) = \int_0^1 dx x^{N-1} \left[\frac{\ln^i(1-x)}{(1-x)} \right]_+, \quad (\text{A.8})$$

are

$$\begin{aligned}
\mathcal{D}_0(\bar{N}) &= -\ln \bar{N} + \mathcal{O}\left(\frac{\ln \bar{N}}{\bar{N}}\right), \\
\mathcal{D}_1(\bar{N}) &= \frac{1}{2} \ln^2 \bar{N} + \frac{1}{2} \zeta_2 + \mathcal{O}\left(\frac{\ln \bar{N}}{\bar{N}}\right), \\
\mathcal{D}_2(\bar{N}) &= -\frac{1}{3} \ln^3 \bar{N} - \zeta_2 \ln \bar{N} - \frac{2}{3} \zeta_3 + \mathcal{O}\left(\frac{\ln \bar{N}}{\bar{N}}\right), \\
\mathcal{D}_3(\bar{N}) &= \frac{1}{4} \ln^4 \bar{N} + \frac{3}{2} \zeta_2 \ln^2 \bar{N} + 2\zeta_3 \ln \bar{N} + \frac{27}{20} \zeta_2^2 + \mathcal{O}\left(\frac{\ln \bar{N}}{\bar{N}}\right). \quad (\text{A.9})
\end{aligned}$$

The Riemann $\zeta(s)$ -function,

$$\zeta_s \equiv \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad (\text{A.10})$$

has the values $\zeta(2) = \pi^2/6$, $\zeta(3) = 1.202057$ and $\zeta(4) = \pi^4/90$.

A.3 Gamma Function

The Euler Γ -function is defined through the integral representation

$$\Gamma(z) = \int_0^{\infty} dt t^{z-1} e^{-t}, \quad \text{where } \text{Re}(z) > 0. \quad (\text{A.11})$$

Integration by parts gives

$$\Gamma(z+1) = z\Gamma(z). \quad (\text{A.12})$$

For $z = 1$ it is easy to get from the definition that: $\Gamma(1) = 1$ which implies that for integer values of z one gets, $\Gamma(z) = (z-1)!$. It also has the expansion

$$\Gamma(1+\varepsilon) = 1 - \gamma_E \varepsilon + \left(\frac{\pi^2}{12} + \frac{1}{2} \gamma_E^2\right) \varepsilon^2 + \mathcal{O}(\varepsilon^3), \quad (\text{A.13})$$

which shows that $\Gamma(\varepsilon)$ diverges as $\varepsilon \rightarrow 0$. It is from the last equation (and similar ones) where the γ_E appears in loop integrals. The following relation is a very useful

one when carrying out phase space integrals for Feynman diagrams with real gluon emission.

$$\int_0^1 dx x^m (1-x)^n = \frac{\Gamma(1+m)\Gamma(1+n)}{2+m+n}, \quad \text{Re}\{m, n\} > -1. \quad (\text{A.14})$$

B. FULL QCD CALCULATION FOR DRELL-YAN PROCESS

In this appendix we show in some detail, some technical calculations related to Drell-Yan (DY) process up to first order in α_s . The aim is to show explicitly how the “plus” distributions appear in perturbative calculations of Feynman diagrams with real gluon emission. Then we consider the “soft limit” in which we keep only the most singular terms in the final result, which are the “plus” distributions and the Dirac delta function.

The starting point is the phase space for production of a massive particle (the photon) with mass squared Q^2 and another massless gluon, $k^2 = 0$. In $d = 4 - 2\varepsilon$, we have

$$d\Phi_2 = \int \frac{d^d q}{(2\pi)^{d-1}} \int \frac{d^d k}{(2\pi)^{d-1}} (2\pi)^d \delta^{(d)}[p_1 + p_2 - q - k] \delta^+(q^2 - Q^2) \delta^+(k^2), \quad (\text{B.1})$$

where p_i are the momenta of the incoming partons. Performing the q integral one simply gets

$$d\Phi_2 = 2\pi \int \frac{d^d k}{(2\pi)^{d-1}} \delta^+((p_1 + p_2 - k)^2 - Q^2) \delta^+(k^2). \quad (\text{B.2})$$

In the center of mass frame of the incoming partons we let

$$k^\mu = (|k|, 0, \dots, |k| \cos \theta). \quad (\text{B.3})$$

By simple kinematics we also have

$$(p_1 + p_2 - k)^2 = s - 2|k|\sqrt{s} \quad , \quad s \equiv (p_1 + p_2)^2. \quad (\text{B.4})$$

We now perform all the angular integrations in $d - 2$ (Euclidian) space and use the relation

$$\Omega_{d-2} = \frac{2\pi^{\frac{d-2}{2}}}{\Gamma(\frac{d-2}{2})} \quad , \quad (\text{B.5})$$

where Ω is the solid angle. Thus we are left with

$$d\Phi_2 = \frac{(4\pi)^\varepsilon}{2\pi} \frac{1}{\Gamma(1-\varepsilon)} \int dk d\theta |k|^{d-2} (\sin \theta)^{d-3} \delta(s - Q^2 - 2|k|\sqrt{s}) \delta^+(k^2). \quad (\text{B.6})$$

Using: $\delta^+(k^2) = \frac{1}{2|k|} \delta(|k| - |\vec{k}|)$, we get

$$d\Phi_2 = \frac{1}{4\pi} \frac{(4\pi)^\varepsilon}{\Gamma(1-\varepsilon)} \int_0^\infty d|k| |k|^{1-2\varepsilon} \int_{-1}^1 d(\cos \theta) [1 - \cos^2 \theta]^{-\varepsilon} \delta(s - Q^2 - 2|k|\sqrt{s}). \quad (\text{B.7})$$

Defining $y = \frac{1}{2}(1 + \cos \theta)$ and performing the $|k|$ integral using the delta function we get

$$d\Phi_2 = \frac{1}{8\pi} \left(\frac{4\pi}{Q^2} \right)^\varepsilon \frac{1}{\Gamma(1-\varepsilon)} z^\varepsilon (1-z)^{1-2\varepsilon} \int_0^1 dy [y(1-y)]^{-\varepsilon} \quad , \quad (\text{B.8})$$

where $z = \frac{Q^2}{s}$.

In full QCD, the amplitude squared for the process $q\bar{q} \rightarrow \gamma^* g$ is given by:

$$|M|^2 = 4\alpha_s C_F (\mu_0^2)^\varepsilon \left[(1-\varepsilon) \left(\frac{u}{t} + \frac{t}{u} \right) + \frac{2Q^2 s}{ut} - 2\varepsilon \right] \quad , \quad (\text{B.9})$$

where the Mandelstam variables, s , u and t are given in the c.o.m frame by:

$$\begin{aligned} s &\equiv (p_1 + p_2)^2 = \frac{Q^2}{z} \quad , \\ t &\equiv (p_2 - k)^2 = -\frac{Q^2}{z} (1-z)(1-y) \quad , \\ u &\equiv (p_1 - k)^2 = -\frac{Q^2}{z} (1-z)y. \end{aligned} \quad (\text{B.10})$$

Substituting the above relations in Eq. (B.9) and performing the phase space integral using Eqs. (A.12) and (A.14) we get

$$\frac{d\sigma(z, Q^2)}{dQ^2} \Big|_{\text{real}} = \frac{\alpha_s}{4\pi} C_F \left(\frac{\mu^2 e^{\gamma_E}}{Q^2} \right)^\varepsilon \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \times 2 \left\{ -\frac{2}{\varepsilon} \left[(1-z)^{1-2\varepsilon} z^\varepsilon + 2z^{1+\varepsilon} (1-z)^{-1-2\varepsilon} \right] \right\}, \quad (\text{B.11})$$

where we defined, in $\overline{\text{MS}}$ renormalization scheme, $\mu^2 = 4\pi\mu_0^2 e^{-\gamma_E}$. We first notice the appearance of the pole $1/\varepsilon$. This arises from the y integration due to integrals of the form

$$\int_0^1 dy y^{-1-\varepsilon} (1-y)^{1-\varepsilon} = \int_0^1 dy y^{1-\varepsilon} (1-y)^{(-1-\varepsilon)} = \frac{\Gamma(-\varepsilon)\Gamma(2-\varepsilon)}{\Gamma(2-2\varepsilon)} = \frac{-1}{\varepsilon} + \text{finite}, \quad (\text{B.12})$$

and

$$\int_0^1 dy y^{-1-\varepsilon} (1-y)^{-1-\varepsilon} = \frac{\Gamma^2(-\varepsilon)}{\Gamma(-2\varepsilon)} = \frac{-2}{\varepsilon} + \text{finite}. \quad (\text{B.13})$$

Thus we see that the divergence at $y = 0$ or $y = 1$ is regularized by ε . Moreover we see that this divergence emerges from collinear emission of gluons since $y = 0$ ($y = 1$) corresponds to $\theta = \pi$ ($\theta = 0$). We also notice that this holds, to all values of z for $0 < z < 1$.

Equation (B.11) has another pole that is ‘‘hidden’’ in the term $(1-z)^{-1-2\varepsilon}$. It is clear that power of -2ε will somehow regularize any possible divergence in the region $z \rightarrow 1$. To see how this works, we need to consider the term $(1-z)^{-1-2\varepsilon}$ as a distribution function to be integrated against some other smooth function $f(z)$. To isolate the divergence as $z \rightarrow 1$, we write

$$\int_0^1 dz \frac{f(z)}{(1-z)^{1+2\varepsilon}} = \int_0^1 dz \frac{f(z) - f(1) + f(1)}{(1-z)^{1+2\varepsilon}} = \int_0^1 dz \frac{f(z) - f(1)}{(1-z)^{1+2\varepsilon}}$$

$$-\frac{1}{2\varepsilon}f(1) + \text{finite} , \quad (\text{B.14})$$

where to get the last equality we used

$$\int dz \frac{1}{(1-z)^{1+2\varepsilon}} = \frac{\Gamma(1)\Gamma(-2\varepsilon)}{\Gamma(1-2\varepsilon)} = -\frac{1}{2\varepsilon} + \text{finite}. \quad (\text{B.15})$$

Since the term $(1-z)^{-1-2\varepsilon}$ is multiplied by the collinear pole $1/\varepsilon$ in Eq. (B.11), we need to expand it up to $\mathcal{O}(\varepsilon)$ to get all the finite contributions. Thus we have,

$$(1-z)^{-1-2\varepsilon} = \frac{1}{(1-z)}[1 - 2\varepsilon \ln(1-z) + \mathcal{O}(\varepsilon^2)] \quad (\text{B.16})$$

Substituting in Eq. (B.14), we get,

$$\begin{aligned} \int_0^1 dz \frac{f(z)}{(1-z)^{1+2\varepsilon}} &= -\frac{1}{2\varepsilon} \int_0^1 dz f(z) \delta(1-z) + \int_0^1 dz \frac{f(z) - f(1)}{(1-z)} \\ &\quad - 2\varepsilon \int dz [f(z) - f(1)] \frac{\ln(1-z)}{(1-z)}. \end{aligned} \quad (\text{B.17})$$

The last two terms are exactly the convolution of $f(z)$ with $\mathcal{D}_0(z)$ and $\mathcal{D}_1(z)$ introduced in Appendix A. Thus in distribution sense we have,

$$\frac{1}{(1-z)^{1+2\varepsilon}} = -\frac{1}{2\varepsilon} \delta(1-z) + \mathcal{D}_0(z) - 2\varepsilon \mathcal{D}_1(z) + \mathcal{O}(\varepsilon^2). \quad (\text{B.18})$$

The last result is a special case of following one

$$f_{k,\varepsilon}(z) \equiv \varepsilon[(1-z)^{-1-k\varepsilon}] = -\frac{1}{k} \delta(1-z) + \sum_{i=0} \frac{(-k\varepsilon)^i}{i!} \varepsilon \mathcal{D}_i(z). \quad (\text{B.19})$$

Concentrating on the z dependent term in Eq. (B.11);

$$-\frac{2}{\varepsilon} \left[(1-z)^{1-2\varepsilon} z^\varepsilon + 2z^{1+\varepsilon} (1-z)^{-1-2\varepsilon} \right] , \quad (\text{B.20})$$

and using Eq. (B.18), we obtain

$$\begin{aligned} &= \frac{2}{\varepsilon^2} \delta(1-z) - \frac{1}{\varepsilon} \left[2(1-z) + 4z \mathcal{D}_0(z) \right] \\ &\quad + 8z \mathcal{D}_1(z) - 4z \ln z \mathcal{D}_0(z) + 4(1-z) \ln(1-z) - 2(1-z) \ln z. \end{aligned} \quad (\text{B.21})$$

The double pole in ε has now a clear meaning: it results from a single collinear pole and a pole from the $z \rightarrow 1$ region, i.e., the soft region, and it gets cancelled from a similar term, when the contribution from virtual diagrams to the cross section, is taken into account.

To simplify the algebra in the last expression, we note, that in distributional sense

$$\begin{aligned}
\int_0^1 dz f(z)[(1-z) + 2z\mathcal{D}_0(z)] &= \int_0^1 dz \frac{f(z)(1-z)^2}{(1-z)} + \int_0^1 dz \frac{2zf(z) - 2f(1)}{(1-z)} \\
&= \int_0^1 dz \frac{f(z)(1+z^2) - 2f(1)}{(1-z)} \\
&= \int_0^1 dz (1+z^2)f(z)\mathcal{D}_0(z). \tag{B.22}
\end{aligned}$$

Similarly we can show that the term

$$2z \ln z \mathcal{D}_0(z) + (1-z) \ln z, \tag{B.23}$$

is equivalent to

$$\frac{1+z^2}{1-z} \ln z. \tag{B.24}$$

And

$$8z\mathcal{D}_1(z) + 4(1-z) \ln(1-z), \tag{B.25}$$

gives

$$4(1+z^2)\mathcal{D}_1(z). \tag{B.26}$$

Putting all of this together we get

$$= \frac{2}{\varepsilon^2} \delta(1-z) - \frac{2}{\varepsilon} (1+z^2)\mathcal{D}_0(z) + 4(1+z^2)\mathcal{D}_1(z) - 2\frac{1+z^2}{1-z} \ln z. \tag{B.27}$$

The last result is an exact one and holds to all values of z . Going back to Eq. (B.11), throwing away the regular contribution $z^\varepsilon(1-z)^{1-2\varepsilon}$, we get

$$= \frac{2}{\varepsilon^2}\delta(1-z) - \frac{4}{\varepsilon}\mathcal{D}_0(z) + 8\mathcal{D}_1(z) \quad (\text{B.28})$$

which only contains singular contributions in the limit $z \rightarrow 1$. The last result is also the one obtained using the SCET Feynman rules.

C. SECOND ORDER RESULTS IN THE SOFT LIMIT

Here we collect the coefficient functions for deep-inelastic scattering, drell-Yan and the Higgs production (within the large top quark mass limit effective theory) to next-to-next-to-leading order in α_s and in the soft+virtual limit. For DIS

$$\begin{aligned}
& G_{\text{DIS}}^{(2),s+v}(x) \\
&= C_F^2 \left\{ \left[16\mathcal{D}_1(x) + 12\mathcal{D}_0(x) + \delta(1-x) \left(\frac{9}{2} - 8\zeta_2 \right) \right] \ln^2 \left(\frac{Q^2}{\mu^2} \right) \right. \\
&+ \left[24\mathcal{D}_2(x) - 12\mathcal{D}_1(x) - (45 + 32\zeta_2)\mathcal{D}_0(x) \right. \\
&+ \left. \left. \delta(1-x) \left(-\frac{51}{2} - 12\zeta_2 + 40\zeta_3 \right) \right] \ln \left(\frac{Q^2}{\mu^2} \right) \right. \\
&+ 8\mathcal{D}_3(x) - 18\mathcal{D}_2(x) - (27 + 32\zeta_2)\mathcal{D}_1(x) + 2 \times 48 \times \left(-\frac{3}{4} \right) \zeta_3 \mathcal{D}_0(x) \\
&+ \left. \left. \left(\frac{51}{2} + 36\zeta_2 + 64\zeta_3 \right) \mathcal{D}_0(x) + \delta(1-x) \left(\frac{331}{8} + 69\zeta_2 - 78\zeta_3 + 6\zeta_2^2 \right) \right\} \right. \\
&+ C_F N_F \left\{ \left[\frac{4}{3}\mathcal{D}_0(x) + \delta(1-x) \right] \ln^2 \left(\frac{Q^2}{\mu^2} \right) + \left[\frac{8}{3}\mathcal{D}_1(x) - \frac{58}{9}\mathcal{D}_0(x) \right. \right. \\
&- \left. \left. \delta(1-x) \left(\frac{19}{3} + \frac{16}{3}\zeta_2 \right) \right] \ln \left(\frac{Q^2}{\mu^2} \right) + \frac{4}{3}\mathcal{D}_2(x) - \frac{58}{9}\mathcal{D}_1(x) + \left(\frac{247}{27} - \frac{8}{3}\zeta_2 \right) \mathcal{D}_0(x) \right. \\
&+ \left. \left. \delta(1-x) \left(\frac{457}{36} + \frac{38}{3}\zeta_2 + \frac{4}{3}\zeta_3 \right) \right\} \right. \\
&+ C_A C_F \left\{ \left[-\frac{22}{3}\mathcal{D}_0(x) - \frac{11}{2}\delta(1-x) \right] \ln^2 \left(\frac{Q^2}{\mu^2} \right) \right. \\
&+ \left[-\frac{44}{3}\mathcal{D}_1(x) + \left(\frac{367}{9} - 8\zeta_2 \right) \mathcal{D}_0(x) + \left(\frac{215}{6} + \frac{88}{3}\zeta_2 - 12\zeta_3 \right) \delta(1-x) \right] \ln \left(\frac{Q^2}{\mu^2} \right) \\
&- \frac{22}{3}\mathcal{D}_2(x) + \left(\frac{367}{9} - 8\zeta_2 \right) \mathcal{D}_1(x) + 36\zeta_3 \mathcal{D}_0(x) + \left(-\frac{3155}{54} + \frac{44}{3}\zeta_2 + 4\zeta_3 \right) \mathcal{D}_0(x) \\
&+ \left. \left. \delta(1-x) \left(-\frac{5465}{72} - \frac{251}{3}\zeta_2 + \frac{140}{3}\zeta_3 + \frac{71}{5}\zeta_2^2 \right) \right\}. \tag{C.1}
\end{aligned}$$

For Drell-Yan we have

$$\begin{aligned}
& G_q^{(2),s+v}(z) \\
&= C_F^2 \left\{ [64\mathcal{D}_1(z) + 48\mathcal{D}_0(z) + \delta(1-z)(18 - 32\zeta_2)] \ln^2 \left(\frac{Q^2}{\mu^2} \right) \right. \\
&+ [192\mathcal{D}_2(z) + 96\mathcal{D}_1(z) - (128 + 64\zeta_2)\mathcal{D}_0(z) \\
&+ \delta(1-z)(-93 + 24\zeta_2 + 176\zeta_3)] \ln \left(\frac{Q^2}{\mu^2} \right) \\
&+ 128\mathcal{D}_3(z) - (256 + 128\zeta_2)\mathcal{D}_1(z) + 256\zeta_3\mathcal{D}_0(z) \\
&+ \delta(1-z) \left(\frac{511}{4} - 70\zeta_2 - 60\zeta_3 + \frac{8}{5}\zeta_2^2 \right) \left. \right\} \\
&+ C_F N_F \left\{ \left[\frac{8}{3}\mathcal{D}_0(z) + 2\delta(1-z) \right] \ln^2 \left(\frac{Q^2}{\mu^2} \right) \right. \\
&+ \left[\frac{32}{3}\mathcal{D}_1(z) - \frac{80}{9}\mathcal{D}_0(z) - \frac{34}{3}\delta(1-z) \right] \ln \left(\frac{Q^2}{\mu^2} \right) \\
&+ \frac{32}{3}\mathcal{D}_2(z) - \frac{160}{9}\mathcal{D}_1(z) + \left(\frac{224}{27} - \frac{32}{3}\zeta_2 \right) \mathcal{D}_0(z) + \delta(1-z) \left(\frac{127}{6} - \frac{112}{9}\zeta_2 + 8\zeta_3 \right) \left. \right\} \\
&+ C_A C_F \left\{ \left(-\frac{44}{3}\mathcal{D}_0(z) - 11\delta(1-z) \right) \ln^2 \left(\frac{Q^2}{\mu^2} \right) \right. \\
&+ \left[-\frac{176}{3}\mathcal{D}_1(z) + \left(\frac{536}{9} - 16\zeta_2 \right) \mathcal{D}_0(z) + \left(\frac{193}{3} - 24\zeta_3 \right) \delta(1-z) \right] \ln \left(\frac{Q^2}{\mu^2} \right) \\
&- \frac{176}{3}\mathcal{D}_2(z) + \left(\frac{1072}{9} - 32\zeta_2 \right) \mathcal{D}_1(z) + \left(-\frac{1616}{27} + \frac{176}{3}\zeta_2 + 56\zeta_3 \right) \mathcal{D}_0(z) \\
&+ \delta(1-z) \left(-\frac{1535}{12} + \frac{592}{9}\zeta_2 + 28\zeta_3 - \frac{12}{5}\zeta_2^2 \right) \left. \right\}. \tag{C.2}
\end{aligned}$$

For the Higgs production

$$\begin{aligned}
& G_g^{(2),s+v}(z) \\
&= C_A^2 \left\{ \left[64\mathcal{D}_1(z) - \frac{44}{3}\mathcal{D}_0(z) - 32\zeta_2\delta(1-z) \right] \ln^2 \left(\frac{Q^2}{\mu^2} \right) \right. \\
&+ \left[192\mathcal{D}_2(z) - \frac{176}{3}\mathcal{D}_1(z) + \left(\frac{536}{9} - 80\zeta_2 \right) \mathcal{D}_0(z) \right. \\
&+ \delta(1-z) \left(-24 - \frac{88}{3}\zeta_2 + 152\zeta_3 \right) \left. \right] \ln \left(\frac{Q^2}{\mu^2} \right) \\
&+ 128\mathcal{D}_3(z) - \frac{176}{3}\mathcal{D}_2(z) + \left(\frac{1072}{9} - 160\zeta_2 \right) \mathcal{D}_1(z)
\end{aligned}$$

$$\begin{aligned}
& + \left(-\frac{1616}{27} + \frac{176}{3}\zeta_2 + 312\zeta_3 \right) \mathcal{D}_0(z) \\
& + \delta(1-z) \left(93 + \frac{536}{9}\zeta_2 - \frac{220}{3}\zeta_3 - \frac{4}{5}\zeta_2^2 \right) \Big\} \\
& + C_F N_F \delta(1-z) \left(4 \ln \left(\frac{Q^2}{\mu^2} \right) - \frac{67}{3} + 16\zeta_3 \right) \\
& + C_A N_F \left\{ \left(\frac{8}{3} \mathcal{D}_0(z) \right) \ln^2 \left(\frac{Q^2}{\mu^2} \right) \right. \\
& + \left[\frac{32}{3} \mathcal{D}_1(z) - \frac{80}{9} \mathcal{D}_0(z) + \delta(1-z) \left(8 + \frac{16}{3}\zeta_2 \right) \right] \ln \left(\frac{Q^2}{\mu^2} \right) \\
& + \frac{32}{3} \mathcal{D}_2(z) - \frac{160}{9} \mathcal{D}_1(z) + \left(\frac{224}{27} - \frac{32}{3}\zeta_2 \right) \mathcal{D}_0(z) \\
& \left. + \delta(1-z) \left(-\frac{80}{3} - \frac{80}{9}\zeta_2 - \frac{8}{3}\zeta_3 \right) \right\}. \tag{C.3}
\end{aligned}$$

The Mellin transform of the above functions with respect to their arguments in the large \bar{N} limit are,

$$\begin{aligned}
& G_{N,\text{DIS}}^{(2),s+v} \\
& = C_F^2 \left\{ \left[8 \ln^2 \bar{N} - 12 \ln \bar{N} + \frac{9}{2} \right] \ln^2 \left(\frac{Q^2}{\mu^2} \right) \right. \\
& + \left[-8 \ln^3 \bar{N} - 6 \ln^2 \bar{N} + (45 + 8\zeta_2) \ln \bar{N} - \frac{51}{2} - 18\zeta_2 + 24\zeta_3 \right] \ln \left(\frac{Q^2}{\mu^2} \right) \\
& + 2 \ln^4 \bar{N} + 6 \ln^3 \bar{N} - \left(\frac{27}{2} + 4\zeta_2 \right) \ln^2 \bar{N} \\
& + \left(-\frac{51}{2} - 18\zeta_2 + 24\zeta_3 \right) \ln \bar{N} + \frac{331}{8} + \frac{111}{2}\zeta_2 - 66\zeta_3 + \frac{4}{5}\zeta_2^2 \Big\} \\
& + C_F N_F \left\{ \left[-\frac{4}{3} \ln \bar{N} + 1 \right] \ln^2 \left(\frac{Q^2}{\mu^2} \right) + \left(\frac{4}{3} \ln^2 \bar{N} + \frac{58}{9} \ln \bar{N} - \frac{19}{3} - 4\zeta_2 \right) \ln \left(\frac{Q^2}{\mu^2} \right) \right. \\
& - \frac{4}{9} \ln^3 \bar{N} - \frac{29}{2} \ln^2 \bar{N} + \left(-\frac{247}{27} + \frac{4}{3}\zeta_2 \right) \ln \bar{N} \\
& + \frac{457}{36} + \frac{85}{9}\zeta_2 + \frac{4}{9}\zeta_3 \Big\} + C_A C_F \left\{ \left[\frac{22}{3} \ln \bar{N} - \frac{11}{2} \right] \ln^2 \left(\frac{Q^2}{\mu^2} \right) \right. \\
& + \left(-\frac{22}{3} \ln^2 \bar{N} - \left(\frac{367}{9} - 8\zeta_2 \right) \ln \bar{N} + \frac{215}{6} + 22\zeta_2 - 12\zeta_3 \right) \ln \left(\frac{Q^2}{\mu^2} \right) \\
& + \frac{22}{9} \ln^3 \bar{N} + \left(\frac{367}{18} - 4\zeta_2 \right) \ln^2 \bar{N} + \left(\frac{3155}{54} - \frac{22}{3}\zeta_2 - 40\zeta_3 \right) \ln \bar{N} \\
& \left. - \frac{5465}{72} - \frac{1139}{18}\zeta_2 + \frac{140}{3}\zeta_3 + \frac{51}{5}\zeta_2^2 \right\}. \tag{C.4}
\end{aligned}$$

$$\begin{aligned}
& G_{N,q}^{(2),s+v} \\
&= C_F^2 \left\{ \left[32 \ln^2 \bar{N} - 48 \ln \bar{N} + 18 \right] \ln^2 \left(\frac{Q^2}{\mu^2} \right) \right. \\
&+ \left[-64 \ln^3 \bar{N} + 48 \ln^2 \bar{N} + (128 - 128\zeta_2) \ln \bar{N} - 93 + 72\zeta_2 + 48\zeta_3 \right] \ln \left(\frac{Q^2}{\mu^2} \right) \\
&+ \left. 32 \ln^4 \bar{N} - (128 - 128\zeta_2) \ln^2 \bar{N} + \frac{511}{4} - 198\zeta_2 - 60\zeta_3 + \frac{552}{5} \zeta_2^2 \right\} \\
&+ C_F N_F \left\{ \left[-\frac{8}{3} \ln \bar{N} + 2 \right] \ln^2 \left(\frac{Q^2}{\mu^2} \right) + \left[\frac{16}{3} \ln^2 \bar{N} + \frac{80}{9} \ln \bar{N} - \frac{34}{3} + \frac{16}{3} \zeta_2 \right] \ln \left(\frac{Q^2}{\mu^2} \right) \right. \\
&- \left. \frac{32}{9} \ln^3 \bar{N} - \frac{80}{9} \ln^2 \bar{N} - \frac{224}{27} \ln \bar{N} + \frac{127}{6} - \frac{192}{9} \zeta_2 + \frac{8}{9} \zeta_3 \right\} \\
&+ C_F C_A \left\{ \left[\frac{44}{3} \ln \bar{N} - 11 \right] \ln^2 \left(\frac{Q^2}{\mu^2} \right) \right. \\
&+ \left[-\frac{88}{3} \ln^2 \bar{N} - \left(\frac{536}{9} - 16\zeta_2 \right) \ln \bar{N} + \frac{193}{3} - \frac{88}{3} \zeta_2 - 24\zeta_3 \right] \ln \left(\frac{Q^2}{\mu^2} \right) \\
&+ \frac{176}{9} \ln^3 \bar{N} + \left(\frac{536}{9} - 16\zeta_2 \right) \ln^2 \bar{N} + \left(\frac{1616}{27} - 56\zeta_3 \right) \ln \bar{N} \\
&- \left. \frac{1535}{12} + \frac{1128}{9} \zeta_2 + \frac{504}{9} \zeta_3 - \frac{92}{5} \zeta_2^2 \right\}. \tag{C.5}
\end{aligned}$$

$$\begin{aligned}
& G_{N,g}^{(2),s+v} \\
&= C_A^2 \left\{ \left[32 \ln^2 \bar{N} + \frac{44}{3} \ln \bar{N} \right] \ln^2 \left(\frac{Q^2}{\mu^2} \right) \right. \\
&+ \left[-64 \ln^3 \bar{N} - \frac{176}{6} \ln^2 \bar{N} - \left(\frac{536}{9} + 112\zeta_2 \right) \ln \bar{N} - 24 - \frac{176}{3} \zeta_2 + 24\zeta_3 \right] \ln \left(\frac{Q^2}{\mu^2} \right) \\
&+ 32 \ln^4 \bar{N} + \frac{176}{9} \ln^3 \bar{N} + \left(\frac{536}{9} + 112\zeta_2 \right) \ln^2 \bar{N} \\
&+ \left(\frac{1616}{27} - 56\zeta_3 \right) \ln \bar{N} + 93 + \frac{1072}{9} \zeta_2 - \frac{308}{9} \zeta_3 + 172\zeta_2^2 \\
&+ C_A N_F \left\{ \left[-\frac{8}{3} \ln \bar{N} \right] \ln^2 \left(\frac{Q^2}{\mu^2} \right) + \left[\frac{16}{3} \ln^2 \bar{N} + \frac{80}{9} \ln \bar{N} + 8 + \frac{32}{3} \zeta_2 \right] \ln \left(\frac{Q^2}{\mu^2} \right) \right. \\
&- \left. \frac{32}{9} \ln^3 \bar{N} - \frac{80}{9} \ln^2 \bar{N} - \frac{224}{27} \ln \bar{N} - \frac{80}{3} - \frac{160}{9} \zeta_2 - \frac{88}{9} \zeta_3 \right\} \\
&+ C_F N_F \left\{ 4 \ln \left(\frac{Q^2}{\mu^2} \right) - \frac{67}{3} + 16\zeta_3 \right\} \tag{C.6}
\end{aligned}$$

D. CONFORMAL GROUP

We all recall that Lorentz group $SO(3, 1)$ is the group of all space-time transformations that leave invariant the inner product $(x, y) = g_{\mu\nu}x^\mu y^\nu$. Equivalently, under Lorentz transformation the metric tensor $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is invariant. Accordingly, all Lorentz transformations are linear in space-time coordinates. The Lorentz group has six generators which are usually denoted by $M_{\mu\nu}$ with property $M_{\mu\nu} = -M_{\nu\mu}$. Combined with four generators of space-time translations, P_μ , one obtains the well-known Poincaré group with its ten generators.

The Lorentz group can be extended in the following manner. Let us perform infinitesimal coordinate transformations $x^\mu \rightarrow x'^\mu$ and allow for the metric tensor to become space-time dependent $g_{\mu\nu}(x)$. The infinitesimal distance ds^2 becomes

$$ds'^2 = g'_{\alpha\beta}(x') dx'^\alpha dx'^\beta = g'_{\alpha\beta}(x') \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} dx^\nu dx^\mu. \quad (\text{D.1})$$

If we restrict ourselves to those transformations such that

$$g'_{\alpha\beta}(x') \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} = \omega(x) g_{\mu\nu}(x), \quad (\text{D.2})$$

then these transformations constitute the *conformal group*, and one gets

$$ds'^2 = \omega(x) g_{\mu\nu}(x) dx^\mu dx^\nu = \omega(x) ds^2. \quad (\text{D.3})$$

The essence of this equation is that up to a space-time dependent scale factor, ds^2

is invariant. Clearly, for those transformations such that $\omega(x) = 1$ we restore the original Poincaré transformations. Thus the Poincaré group is a sub-group of the conformal group. In d -dimensional space-time $x^\mu = (x^0, x^1, \dots, x^{d-1})$ it is isomorphic to the orthogonal group of $(d+2) \times (d+2)$ matrices and is denoted by $SO(2, d)$.

It is straightforward to show that under conformal transformations the following relation holds

$$\frac{dx \cdot dy}{|dx||dy|} = \frac{dx' \cdot dy'}{|dx'||dy'|}, \quad (\text{D.4})$$

which means that the angles between dx and dy are preserved. This is the origin of the word “conformal”.

The generators of the conformal group can be found in the following way. Let $x^\mu \rightarrow x^\mu + \epsilon^\mu$. The infinitesimal distance ds^2 becomes

$$\begin{aligned} ds'^2 &= ds^2 + (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) dx^\mu dx^\nu \\ &= (g_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) dx^\mu dx^\nu. \end{aligned} \quad (\text{D.5})$$

Comparing with Eq.(D.3) and writing $\omega(x) \equiv 1 + h(x)$, we find

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = h(x) g_{\mu\nu}. \quad (\text{D.6})$$

Multiplying both sides of last Equation with $g_{\mu\nu}$ and using $d_{\mu\nu} g^{\mu\nu} = d$, we get

$$\omega(x) = 1 + \frac{2}{d} (\partial \cdot \epsilon), \quad (\text{D.7})$$

thus

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} (\partial \cdot \epsilon) g_{\mu\nu}. \quad (\text{D.8})$$

From this we obtain

$$(d-2) \partial_\nu \partial_\mu (\partial \cdot \epsilon) = -d \partial_\nu \partial_\alpha \partial^\alpha \epsilon_\mu. \quad (\text{D.9})$$

For $\mu \neq \nu$ we have $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = 0$, from which we see that the right hand side of the last equation is antisymmetric under the exchange of μ and ν while, clearly, the left hand side is symmetric. Thus both sides have to vanish for $\mu \neq \nu$. Since this is true to arbitrary $\epsilon(x)$ then the quantity $d\partial_\nu \partial_\alpha \partial^\alpha \epsilon_\mu$ has to be proportional to $g_{\mu\nu} \partial_\alpha \partial^\alpha (\partial \cdot \epsilon)$. By taking the trace we get the proportionality constant to be 1. Thus we have

$$[g_{\mu\nu} \partial_\alpha \partial^\alpha + (d-2) \partial_\mu \partial_\nu] \partial \cdot \epsilon = 0 , \quad (\text{D.10})$$

from which we see that $\epsilon(x)$ could be at most quadratic in x^μ . The last equation is satisfied for the following four-vectors

$$\begin{aligned} \epsilon(x) &= a^\mu , \\ \epsilon(x) &= \omega_\nu^\mu x^\nu , \\ \epsilon(x) &= \lambda x^\mu , \\ \epsilon(x) &= b^\mu x^2 - 2x^\mu b_\nu x^\nu . \end{aligned} \quad (\text{D.11})$$

The Poincaré group corresponds to the first two transformations where the first one contains the four translational parameters. In the second line we have the six parameters of the six lorentz generators $M_{\mu\nu}$. The third gives the scale transformation with generator (D) and the last four stand for the “special conformal transformations” with generators K_μ . Thus in d -dimensional space-time the number in generators is: $d + \frac{d(d-1)}{2} + 1 + d = \frac{1}{2}[d+1][d+2]$, and for $d = 4$ we have 15 generators.

The commutation relations of the generators of the conformal group are the following

$$i[P_\mu, P_\nu] = 0 , \quad i[M_{\alpha\beta}, P_\mu] = g_{\alpha\mu} P_\beta - g_{\beta\mu} P_\alpha ,$$

$$\begin{aligned}
i[M_{\alpha\beta}, M_{\mu\nu}] &= g_{\alpha\mu}M_{\beta\nu} - g_{\beta\mu}M_{\alpha\nu} - g_{\alpha\nu}M_{\beta\mu} + g_{\beta\nu}M_{\alpha\mu} , \\
i[D, P_\mu] &= P_\mu , \quad i[D, K_\mu] = -K_\mu , \\
i[M_{\alpha\beta}, K_\mu] &= g_{\alpha\beta}K_\mu - g_{\beta\mu}K_\alpha , \quad i[P_\mu, K_\nu] = -2g_{\mu\nu}D + 2M_{\mu\nu} , \\
i[D, M_{\mu\nu}] &= i[K_\mu, K_\nu] = 0.
\end{aligned} \tag{D.12}$$

Acting on a generic field $\Phi(x)$, the infinitesimal transformations yield the following

$$\begin{aligned}
\delta_P^\mu \Phi(x) &\equiv i[P^\mu, \Phi(x)] = \partial^\mu \Phi(x) , \\
\delta_M^{\mu\nu} \Phi(x) &\equiv i[M^{\mu\nu}, \Phi(x)] = (x^\mu \partial^\nu - x^\nu \partial^\mu - \Sigma^{\mu\nu})\Phi(x) , \\
\delta_D \Phi(x) &\equiv i[D, \Phi(x)] = (x \cdot \partial + l)\Phi(x) , \\
\delta_K^\mu \Phi(x) &\equiv i[K^\mu, \Phi(x)] = (2x^\mu x \cdot \partial - x^2 \partial^\mu + 2lx^\mu - 2x_\nu \Sigma^{\mu\nu})\Phi(x) ,
\end{aligned} \tag{D.13}$$

where $\Sigma_{\mu\nu}$ is the generator for spin rotations. For scalar, Dirac and vector fields we have

$$\Sigma^{\mu\nu} \phi(x) = 0 , \quad \Sigma^{\mu\nu} \psi(x) = \frac{i}{2} \sigma^{\mu\nu} \psi(x) , \quad \Sigma^{\mu\nu} A^\alpha = g^{\nu\alpha} A^\mu - g^{\mu\alpha} A^\nu. \tag{D.14}$$

respectively. The quantity l is the scaling dimension of the field Φ . At the classical level (i.e., before renormalization effects are taken into account) we have $l = l^{\text{can}}$ where l^{can} is the canonical dimension of the operator Φ (for example, $l = 3/2$ for a spin 1/2 particle and 1 for a spin 1). Taking into account quantum effects, l becomes $\gamma = l - l^{\text{can}}$ where γ is the anomalous dimension of the quantum field Φ that controls the dependence of Φ on the renormalization scale μ . The relations in Eq. (D.13) are proved in Ref. [162].

As the discussion of chapter 4 shows, we are mainly interested in particles (quarks) that move along the light-cone in a specific direction. Let the four-

momentum p^μ have only a plus component p^+ . The conjugate coordinate is $x^\mu = \alpha n^\mu$ where α is real parameter and n^μ has only a minus component n^- and $\Phi(x) = \Phi(\alpha n)$. In this case we can confine ourselves to a sub-group of the conformal group, known as the “collinear group” which maps a light-ray (say in the minus direction) into itself. The generators of this sub-group are P_+ , M_{-+} , D and K_- . Defining:

$$\begin{aligned} L_+ &= L_1 + iL_2 = -iP_+ , & L_- &= L_1 - iL_2 = \frac{i}{2}K_- , \\ L_3 &= \frac{i}{2}(D_+ + M_{-+}) , & E &= \frac{i}{2}(D_- - M_{-+}) , \end{aligned} \quad (\text{D.15})$$

we can easily see that these operators close the algebra of the collinear group

$$[L_3, L_\mp] = \mp L_\mp , \quad [L_-, L_+] = -2L_3 , \quad [E, L_i] = 0. \quad (\text{D.16})$$

Assuming that Φ is an eigenstate of the light-cone spin projection Σ_{+-} : $\Sigma_{+-}\Phi = s\Phi$, one can show that

$$\begin{aligned} [L_+, \Phi(\alpha)] &= -\partial_\alpha \Phi(\alpha) \equiv L_+ \Phi(\alpha) , \\ [L_-, \Phi(\alpha)] &= (\alpha^2 \partial_\alpha + 2j\alpha) \Phi(\alpha) \equiv L_- \Phi(\alpha) , \\ [L_3, \Phi(\alpha)] &= (\alpha \partial_\alpha + j) \Phi(\alpha) \equiv L_3 \Phi(\alpha) , \\ [E, \Phi(\alpha)] &= \frac{1}{2}(l - s) \Phi(\alpha) , \end{aligned} \quad (\text{D.17})$$

where $l - s$ is known as the “geometric twist”. The Casimir operator given by:

$$L^2 = L_3^2 + L_1^2 + L_2^2 = L_3^2 - L_3 - L_- L_+ , \quad (\text{D.18})$$

satisfies

$$[L^2, L_i] = 0 , \quad (\text{D.19})$$

and one can show that

$$\sum_{i=0,1,2} [L_i, [L_i, \Phi(\alpha)]] = j(j-1)\Phi(\alpha) = L^2\Phi(\alpha) , \quad (\text{D.20})$$

where $j = (l + s)/2$ known as the ‘‘conformal spin’’.

Let us define $\Phi(y)$ as the Fourier transform

$$\Phi(y) = \int d\alpha e^{iy\alpha} \Phi(\alpha). \quad (\text{D.21})$$

where y is the fraction of longitudinal momentum of the hadron carried by one of its quark constituents. The states given by:

$$|j, n\rangle = \frac{1}{\Gamma(j+n)} (iy)^{(j+n-1)} , \quad (\text{D.22})$$

form a basis for the irreducible representations of the collinear group

$$L_{\pm}|j, n\rangle = (n \pm j)|j, n \pm 1\rangle , \quad L_3|j, n\rangle = -n|j, n\rangle , \quad (\text{D.23})$$

and the operators L_{\pm} are just a ladder operators for the third component n of the conformal spin j . The lowest state is the one with $-n = -j$ which is annihilated by L_- . This is completely analogous to the $SU(2)$ group. Thus the action of the collinear group generators on a state of k quarks

$$\Phi = \Phi(\alpha_1) \cdots \Phi(\alpha_k) , \quad (\text{D.24})$$

corresponds to addition of k conformal spins and the Clebsch-Gordan coefficients of the conformal group are needed to perform this addition. The task gets simplified by the important observation that the collinear group is isomorphic to Lorentz group in $2 + 1$ dimensional space-time. Owing to the fact that $-j_3 \geq j$ one then can think

of the conformal spin j as the mass of a particle and n stands for its energy. The lowest possible value of the energy of k particles equals the sum of their masses and the lowest energy eigenstate is the product of states with each one of them being at rest, i.e., $-n = -j$,

$$|j = j_{\min}, n\rangle \sim |j_1, j_1\rangle \cdots + |j_k, j_k\rangle, \quad (\text{D.25})$$

and we have

$$j_{\min} = j_1 + \cdots + j_k. \quad (\text{D.26})$$

From Eq. (D.22) we can write

$$|j = j_{\min}, n\rangle \sim y_1^{2j_1-1} \cdots y_k^{2j_k-1} \quad (\text{D.27})$$

The state with lowest value of the total conformal spin is, by definition, the *asymptotic* wave function.

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