ABSTRACT

Title of Dissertation: SINGULAR UPWARD FLOWS
AND CONFORMAL LIMITS
IN THE HITCHIN MODULI SPACE

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The moduli space of Higgs bundles admits a $\mathbb{C}^*$-action, where taking the downward limits contracts the moduli space to the connected components of the fixed-point locus. Through a $\mathbb{C}^*$-fixed point, the locus which contracts to it is referred to as the upward flow through it. In this thesis, we construct a sublocus in the upward flow through a polystable fixed point called the central locus using the local Kuranishi model around the fixed point. We show that its intersection with the stable locus of the moduli space is a complex Lagrangian. When either the polystable $\mathbb{C}^*$-fixed point has a zero field or when its automorphism group is abelian, we prove that this intersection non-empty. Under the same assumption, we prove that the existence of the conformal limit of a stable point along the central locus. Finally, in rank two, we compute and give a concrete description of the central locus and the upward flow through a generic strictly polystable $\mathbb{C}^*$-fixed point. We compute the dimension of the upward flow and demonstrate that it has two components.

SINGULAR UPWARD FLOWS AND CONFORMAL LIMITS IN THE HITCHIN MODULI SPACE

by

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Dedication

To my parents.

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Chapter 1: Introduction

1.1 Background

Let X be a closed Riemann surface of genus $g \geq 2$. A Higgs bundle $(\mathcal{E}, \Phi)$ consists of a holomorphic vector bundle $\mathcal{E}$ over X and a Higgs field $\Phi \in H^0(\text{End}(\mathcal{E}) \otimes K_X)$, where K_X is the canonical line bundle of X. The moduli space of Higgs bundles $\mathcal{M}_H$, whose study was initiated in [Hit87], turns out to have rich geometric structures. On the open locus of stable Higgs bundles, denoted by $\mathcal{M}_H^s$ the moduli space admits a hyperkähler structure, a manifold equipped with three complex structures generating the quaternionic relations and sharing a common Kähler metric. Restricted to the case where the underlying vector bundle has degree zero, the non-abelian Hodge correspondence (NAH) asserts that for each polystable Higgs bundle, defined by a Dolbeault operator $\overline{\partial}_E$ on a smooth vector bundle E and Higgs field $\Phi \in \Omega^{1,0}(\text{End}(E))$ satisfying $\overline{\partial}_E \Phi = 0$, there exists a hermitian metric such that the connection

$$D = \overline{\partial}_E + \partial_E^h + \Phi + \Phi^{*_h}$$

is flat and completely reducible. The connection $\overline{\partial}_E + \partial_E^h$ in the above expression is the Chern connection of $\overline{\partial}_E$ with respect to the hermitian metric h. Now, if we write F_D and $F_{(\overline{\partial}_E, h)}$ as the

curvature of the connection D and $\overline{\partial}_E + \partial_E^h$ respectively, then the equation

$$F_D = F_{(\overline{\partial}_E, h)} + [\Phi, \Phi^*] = 0$$

is known as the Hitchin equation. Conversely, for any completely reducible flat connection D, one can find a hermitian metric h such that when writing $D = d_A + \Psi$ as a sum of an h-unitary connection d_A and h-hermitian endomorphism Ψ, we have

$$d_A^{0,1}\Psi^{1,0} = 0$$

where here, the superscripts in $d_A = d_A^{1,0} + d_A^{0,1}$ and $\Psi = \Psi^{1,0} + \Psi^{0,1}$ refer to the type decomposition. In addition, the pair $(d_A^{0,1}, \Psi^{1,0})$ defines a polystable Higgs bundle. The metric h is called a harmonic metric. This correspondence descends to a homeomorphism between $\mathcal{M}_H$ and $\mathcal{M}_{dR}$, the moduli space of flat connections. Over the stable locus $\mathcal{M}_H^s$, the NAH may be viewed as rotating the complex structure from one to another.

The moduli space $\mathcal{M}_H$ can be constructed by algebraic or analytic means. In rank two, $\mathcal{M}_H^s$ was constructed in [Hit87] by the Kuranishi slice method and was shown to be a complete hyperkähler manifold. The Kuranishi slice technique was enhanced in [Fan22] to provide a local model around polystable Higgs bundles, giving the full $\mathcal{M}_H$ the structure of a (possibly singular) complex analytic space. On the other hand, the moduli space of semistable Higgs bundles $\mathscr{M}_H$ was constructed by algebro-geometric method in [Nit91] and [Sim94], and was shown to be quasi-projective. Viewed as a complex space through analytification, it was shown that the comparison map $\mathcal{M}_H \to \mathscr{M}_H$ is an isomorphism between complex analytic spaces. Therefore,

the analytic and algebraic constructions are equivalent.

The geometric objects of interest in this thesis are certain complex Lagrangians coming from the $\mathbb{C}^*$-action on $\mathcal{M}_H$ given by scaling the Higgs fields by non-zero scalars: for $t \in \mathbb{C}^*$ and $[(\mathcal{E}, \Phi)] \in \mathcal{M}_H$,

$$t \cdot [(\mathcal{E}, \Phi)] = [(\mathcal{E}, t\Phi)].$$

The downward limit $\lim_{t\to 0}[(\mathcal{E}, t\Phi)]$ coincides with $[(\mathcal{E}, 0)]$ only when the underlying vector bundle $\mathcal{E}$ is semistable. The locus in $\mathcal{M}_H$ consisting of $[(\mathcal{E}, \Phi)]$ with $\mathcal{E}$ semistable is referred to as the open stratum. Interestingly, outside the open stratum, it can be shown that the downward limit always exists. The downward limit is necessarily a $\mathbb{C}^*$-fixed point in $\mathcal{M}_H$, whose underlying Higgs bundle can be given the structure of a system of Hodge bundles (SHB). This $\mathbb{C}^*$-limit results in a stratification of $\mathcal{M}_H$, each stratum W_α fibred over a connected component C_α of the $\mathbb{C}^*$-fixed point locus $\mathcal{M}_H^{\mathbb{C}^*}$. For $[(\overline{\partial}_E, \Phi)] \in C_\alpha$, the fibre $W_0 = W_0([(\overline{\partial}_E, \Phi)]) \subset W_\alpha$ defined by

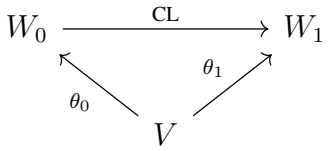
$$W_0([(\overline{\partial}_E, \Phi)]) := \{p \in W_\alpha : \lim_{t\to 0} t \cdot p = [(\overline{\partial}_E, \Phi)]\}$$

is called the *upward flow* through $[(\overline{\partial}_E, \Phi)]$. When $[(\overline{\partial}_E, \Phi)]$ is stable, its upward flow W_0 is a locally closed Lagrangian. Of special interest is the very stable ones, which were recently considered in [HH22] and was related to mirror symmetry in special cases.

Another interest in the upward flows comes from conformal limits, a notion originating from mathematical physics, defined by taking the limit of the family of connections

$$\mathrm{CL}(\overline{\partial}_E, \Phi) := \lim_{R\to 0^-} (\overline{\partial}_E + \partial_E^{h_R} + \Phi + R^2\Phi^{*_{h_R}}).$$

We refer to Section 2.6 for the definition of the expressions appeared in the formula and additional mathematical detail. It was first considered in [Gai14], as the limit of the integral equation coming from a special class of supersymmetric gauge theory. It is not known whether $\mathrm{CL}(\overline{\partial}_E, \Phi)$ exists for any polystable Higgs bundle. It was conjectured [Gai14] that the map CL exists on the Hitchin section, which is sent to the space of opers. The conjecture was proved in [DFK+]. Generalizing the Hitchin section to any upward flow through a stable $\mathbb{C}^*$-fixed point, the method in [DFK+] is extended in [CW19] to prove the existence of the conformal limit along a model space V of the upward flow W_0, which is affine. The space V is a vector space living in the first cohomology $\mathcal{H}^1$ of the deformation complex of $(\overline{\partial}_E, \Phi)$. In this case, the conformal limit can be written down explicity, and the image turns out to be a fibre W_1 of another stratification in $\mathcal{M}_{dR}$ which can also be parametrized by V. Altogether, in [CW19] the following biholomorphic correspondence is shown:



Here, θ_1 and θ_2 are holomorphic bijection.

1.2 Main results

The upward flow through a polystable $\mathbb{C}^*$-fixed point has not been thoroughly studied. In this case, W_0 may incur singularities. The goal of this thesis is to explore the plausibility of extending the picture described above to these singular cases. Since we will consider the problem of the existence of conformal limits, it is necessary for us rely on the analytic construction of $\mathcal{M}_H$. In particular, we will apply the local model developed in [Fan22].

To begin with, let $[(\overline{\partial}_E, \Phi)]$ be a strictly polystable $\mathbb{C}^*$-fixed point. One feature distinct from the stable case is that now $(\overline{\partial}_E, \Phi)$ has a non-trivial automorphism group G, which creates redundant degrees of freedom in our constructions. We then show that there is a particular procedure for constructing an SHB structure on any polystable $\mathbb{C}^*$-fixed point, such that G acts by degree zero with respect to the grading induced by that SHB structure. From the induced grading, the recipe in the stable case for defining the vector space V works, obtaining the following

Theorem 1.2.1 (Proposition 4.3.2)**.** *V is equipped with an G-action. The affine invariant theoretic quotient $V /\!\!/ G$ parametrizes a sublocus of the upward flow W_0 through $[(\overline{\partial}_E, \Phi)]$ which we call the* central locus *through $[(\overline{\partial}_E, \Phi)]$.*

It turns out the the central locus almost always does not capture the full upward flow. However, learning from the stable case and from the virtual dimension count it is expected to be a complex Lagrangian whenever it intersects $\mathcal{M}_H^s$. We show that this is indeed true.

In working with the space V and the local Kuranishi model, it is found that the following technical assumptions are useful:

Assumption 1. $\Phi_0 = 0$.

Assumption 2. *The automorphism group G of $(\mathcal{E}_0, \Phi_0)$ is abelian.*

The advantage of working under Assumption 1 is that in this case the Hodge grading coming from the preferred SHB structure we constructed earlier is trivial. As a result, we can separate out the complication arising from the Hodge grading and focus on dealing with the problems created by the presence of non-trivial G. For a similar reason, Assumption 2 is formulated to make it easier to explore the possible obstacles brought by the Hodge grading. The

abelian assumption on G moderately simplifies the setup to allow some direct computations in some places. Lastly, regarding the effectiveness of the theorems proved under either of these assumptions, we remark that these are quite generic conditions on $\mathbb{C}^*$-fixed point, in the sense that the locus of $\mathbb{C}^*$-fixed points not satisfying Assumption 1 and 2 has lower dimension than the locus which does.

Assuming either one of the two assumption, we can show that

Theorem 1.2.2 (Proposition 4.3.5, Theorem 4.3.6)**.** *The intersection of the central locus S_+ through $[(\overline{\partial}_E, \Phi)]$ with the stable locus $\mathcal{M}_H^s$ is an immersed complex Lagrangian. Assuming Assumption 1 or 2, the central locus S_+ and $\mathcal{M}_H^s$ always intersect.*

Then, we return the problem of conformal limits. This forms the most technical part of the thesis. We will give below a short explanation of the obstacles to overcome and the idea of their solutions.

The goal is to show the existence of conformal limits along the central locus S_+ under either Assumption 1 or 2. For the former, it is understood that the only obstacle is the non-uniqueness of the harmonic metrics of $(\overline{\partial}_E, 0)$, which creates extra degrees of freedom in solving the equations and prohibits the direct application of the implicit function theorem. Given a stable $(\overline{\partial}_E, \varphi) \in S_+$, the idea is that we first solve a minimization problem on a certain functional, whose property of having a unique minimizer will pin down a unique way of removing the degree of freedom coming from G. By the Kempf-Ness theorem, the condition of having a unique minimizer is equivalent to $(\overline{\partial}_E, \varphi)$ being stable.

For the second case, apart from the extra degree of freedom coming from G, there are extra complications coming from the presence of a Hodge grading. The abelian assumption on

G allows us to implement a formally algorithmic iterative procedure that removes the degree of freedom bit by bit.

After we managed to remove the degree of freedom, we can implement the approach in [DFK$^+$] and [CW19], obtaining

Theorem 1.2.3 (Theorem 5.1.5, Corollary 5.2.7)**.** *Under either Assumption 1 or Assumption 2, the conformal limit of a stable Higgs bundle $(\overline{\partial}_u, \Phi_u) \in S_+$ exists and can be written explicitly.*

Finally, we return to the study of the upward flow in a simple rank two example in which the $\mathbb{C}^*$-fixed point has the form $[(\overline{\partial}_E, 0)]$ where $\mathcal{E} = \mathcal{L} \oplus \mathcal{L}^*$ where $\mathcal{L}$ is a degree zero line bundle and $\mathcal{L}^*$ is its dual such that $\mathcal{L}^2 \ncong \mathcal{O}_X$. In this simple case, an explicit description can be given to the central locus through $[(\overline{\partial}_E, 0)]$.

Theorem 1.2.4 (Theorem 6.1.1)**.** *Let $E = L \oplus L^*$ and $\mathcal{L} = (L, \overline{\partial}_L)$ is a degree zero line bundle with $\mathcal{L}^2 \ncong \mathcal{O}_X$. The central locus through $[(\overline{\partial}_E, 0)]$ is of the form*

$$\mathbb{A}^g_{\mathbb{C}} \times Z$$

where $\mathbb{A}^g_{\mathbb{C}}$ is the affine g-space and Z is a determinantal variety.

It is natural to ask if the singular upward flow can be modeled as an affine quotient. In this simple case, we check that this is true. Here, we take the advantage that the underlying vector bundle is semistable. Unlike the case of central locus, we also need to take care of the obstruction condition. Writing down all the conditions explicitly, we check that these conditions form a set of equations preserved by G. Call this zero locus W. We then show that $W /\!\!/ G$ parametrizes the full upward flow.

Theorem 1.2.5. *The upward flow W_0 through $[(\overline{\partial}_E, 0)]$, where $(E, \overline{\partial}_E)$ is as above, has dimension $4g - 5$. It has two irreducible components.*

We also give a description of the model $W /\!\!/ G$ using matrices in Theorem 6.2.5.

The next section contains a brief summary of each subsequent sections,which the readers can refer to from time to time.

1.3 Summary of chapters

Chapter 2 contains preliminary sections needed to understand the setup of this thesis.

Section 2.1: This section discusses the notions in hyperkähler geometry that we will need to formulate the Hitchin-Simpson correspondence. Other chapters have no logical dependence to the results presented here. The materials in this section are only meant to provide a guiding picture.

Section 2.2: Basic properties of Higgs bundles are reviewed. The notion of slope stability is defined and illustrated with a couple of elementary properties.

Section 2.3: We explain the use of Dolbeault operators to specify complex structures on a smooth vector bundle, a perspective that is needed to carry out the gauge theoretic construction of the moduli space of Higgs bundles. We verify in detail the hyperkähler structure on the infinite dimensional configuration space of Higgs pairs, and interpret the holomorphicity condition together with the Hitchin equation as a hyperkähler moment map. We end the section by recalling the Hitchin-Simpson correspondence, and defining the moduli space of semistable Higgs bundles.

Section 2.4: We review the statement of the non-abelian Hodge correspondence, and define conformal limits, followed by an overview of known results in the literature.

Section 2.5: We review the notion of stability in linear representations of a complex reductive group. Here, we recall the Kempf-Ness theorem, which relates stability with minima properties of the Kempf-Ness functional. Lastly, we specialize to toric representation and add a criterion of stability by combinatorial data. This will be used later in Chapter 5.

Section 2.6: We review here the deformation complex at a polystable Higgs bundle and recall the Kähler identities. Then, we proceed to introduce the setup needed in the local model construction in [Fan22], which is an important tool this thesis is based on.

Section 2.7: We review the definition of system of Hodge bundles and their relation to $\mathbb{C}^*$-fixed point of the moduli space.

Chapter 3 is concerned with known facts about the automorphism group of polystable Higgs bundles. We present the results in a way that facilitates the later chapters. One important aspect is that it acts transivitely on the set of harmonic metrics, and that it is covariantly constant with respect to the NAH connection.

Chapter 4 and Chapter 5 is based on the work of the author in [Kwo25].

In Chapter 4, we define and construct the central locus through a polystable $\mathbb{C}^*$-fixed point as the image of an affine GIT quotient. It is a distinguished sublocus of the upward flow. We will show that under Assumption 1 or 2, its intersection with the stable locus of $\mathcal{M}_H$ is a complex Lagrangian.

Chapter 5 gives a proof of the existence of conformal limits along the central locus defined in Chapter 4 under Assumption 1 or 2.

In Chapter 6, we restrict our attention to the concrete computation of a rank two example. We give a description of the central locus as a vector bundle over a certain determinantal variety. We also determine the upward flow in this simple case, showing that it is obtained by iteratively

forming affine bundle over a quadric hypersurface defined by the Kuranishi map. Lastly, we construct an affine GIT quotient model for it.

Chapter 2: Preliminaries

While we will not use the results here directly, they set up the right framework for what to expect in the infinite dimensional setting, whose analysis was carried out in full detail in [Fan22].

2.1 The finite dimensional analogue

2.1.1 Symplectic quotients

We begin with a review of symplectic quotients. Recall that a *symplectic manifold* (M, ω) consists of a smooth manifold M and a non-degenerate closed two form ω. The symplectic form ω induces an isomorphism of bundles $\omega : TM \xrightarrow{\cong} T^*M$ given by

$$\omega_p(v)(w) := \iota_v \omega_p(w) = \omega_p(v, w)$$

for all $p \in M$ and $v, w \in T_pM$. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Suppose (M, ω) is equipped with a smooth G-action which preserves ω. For each p, we will denote the infinitesimal action at p by $\alpha_p : \mathfrak{g} \to TM$. A *moment map* with respect to this G-action is a smooth G-equivariant map $\mu : M \to \mathfrak{g}^*$ with the property that for all $X \in \mathfrak{g}$, we have

$$d\mu_X := \langle d\mu, X \rangle = \iota_{\alpha(X)} \omega.$$

The quotient space

$$\mathcal{M} := \mu^{-1}(0)/G$$

is called the *symplectic quotient* with respect to the moment map μ. Assuming the G-action is proper, it is well-known that $\mathcal{M}$ can be endowed with a symplectic structure if the G-action on μ is free. More generally, the symplectic quotient is a stratified symplectic space (cf. [SL91]).

We are going to look the structure of the quotient at the linear level. Let $p \in \mu^{-1}(0)$. Linearizing the group action and the moment map at p, we obtain

$$C^\bullet : \mathfrak{g} \xrightarrow{\alpha_p} T_pM \xrightarrow{d_p\mu} \mathfrak{g}^*. \tag{2.1}$$

By G-equivariance, (2.1) is a chain complex, i.e. $d_p\mu \circ \alpha_p = 0$. Dualizing, we see that the map $(d_p\mu)^* : \mathfrak{g} \to T_p^*X$ is given by

$$(d_p\mu)^*(X)(v) = \langle d_p\mu(v), X\rangle = \iota_{\alpha_p(X)}\omega_p(v) = \omega_p(\alpha_p(X))(v).$$

Similarly, for $\varphi = \omega(v)$, the map $\alpha_p^* : T_p^*M \to \mathfrak{g}^*$ is given by

$$\alpha_p^*(\varphi)(X) = \omega(v, \alpha_p(X)) = -\langle d_p\mu(\omega^{-1}(\varphi)), X\rangle.$$

Altogether, we obtain an isomorphism between chain complexes:

$$\begin{array}{ccccc} \mathfrak{g} & \xrightarrow{\alpha_p} & T_pM & \xrightarrow{d_p\mu} & \mathfrak{g}^* \\ \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \omega_p} & & \downarrow{\scriptstyle -\mathrm{Id}} \\ \mathfrak{g} & \xrightarrow{(d_p\mu)^*} & T_p^*M & \xrightarrow{\alpha_p^*} & \mathfrak{g}^* \end{array} . \tag{2.2}$$

Therefore, we have

$$\dim(\ker \alpha_p) = \dim(\text{coker } d_p\mu) = \dim(\ker(d_p\mu)^*). \tag{2.3}$$

In particular, p is a regular point of $\mu^{-1}(0)$ iff the action is locally free at p. In general, the space $\ker \alpha_p = H^0(C^\bullet)$ is the Lie algebra of the connected component of the isotropy group of p. If, in addition, $\mathfrak{g}$ can be equipped with an invariant non-degenerate inner product, then we can identify $H^0(C^\bullet)$ and $H^2(C^\bullet)$. Let K be an invariant inner product on $\mathfrak{g}$. Write $K : \mathfrak{g} \xrightarrow{\cong} \mathfrak{g}^*$ the invariant identification between $\mathfrak{g}$ and its dual. The composition

$$\ker \alpha_p = \ker(d_p\mu)^* \hookrightarrow \mathfrak{g} \xrightarrow{K} \mathfrak{g}^* \to \text{coker } d_p\mu \tag{2.4}$$

is injective and hence induces an isomorphism

$$H^0(C^\bullet) \cong H^2(C^\bullet). \tag{2.5}$$

Indeed, suppose $X \in \ker \alpha_p$ such that $K(X) = d_p\mu(v)$ for some $v \in T_pM$. Then we have

$$K(X, Y) = \langle d_p\mu(v), Y\rangle = \omega_p(\alpha_p(Y), v)$$

for all $Y \in \mathfrak{g}$. Putting $Y = X$, we have $K(X, X) = 0$ and hence $X = 0$.

Lastly, we take a look at the local structure around a regular point of $p \in \mu^{-1}(0)$. For

simplicity, assume that the G-action restricted to $\mu^{-1}(0)$ is free. There is a natural identification

$$H^1(C^\bullet) \cong T_{\pi(p)}\mathcal{M}, \tag{2.6}$$

where $\pi : \mu^{-1}(0) \to \mathcal{M}$ is the quotient map. As $\ker d_p\mu$ is coisotropic with $(\ker d_p\mu)^{\perp\omega_p} = \text{im } \alpha_p$, $H^1(C^\bullet)$ is naturally symplectic which, under the identification (2.6), coincides with the symplectic form ω' at $T_{\pi(p)}\mathcal{M}$ obtained from the quotient map $\pi : \mu^{-1}(0) \to \mathcal{M}$. The symplectic form ω' may be locally represented by the restriction of ω to any slice through p. More precisely, let U a submanifold of $\mu^{-1}(0)$ containing p so that the inclusion is a slice. Then, one has

$$(\pi^*\omega')|_U = \omega|_U. \tag{2.7}$$

2.1.2 Kähler quotients and symplectic quotients

We will mostly be interested in the symplectic quotient of a Kähler manifold. Let (M, I, g) be a hermitian manifold consisting of a complex manifold (M, I) with a complex structure I and a hermitian metric g, i.e. g is a riemannian metric on the smooth manifold M that satisfies $I^*g := g(I\cdot, I\cdot) = g$, from which one obtains a non-degenerate two-form $\omega_I = g(I\cdot, \cdot)$. If ω_I is closed, then (M, I, g) is *Kähler*. In this case, we will say that g is a *Kähler metric* and ω_I is the *Kähler form* of the hermitian manifold M.

Suppose G is a Lie group acting holomorphically on M and having a moment map μ (so G also preserves the metric g). If 0 is a regular value of the moment map μ, one can show that the symplectic quotient $\mu^{-1}(0)/G$ in fact inherits a Kähler structure. At a point $p \in \mu^{-1}(0)$, using

the metric g_p, we identify $H^1(C^\bullet)$ with

$$\mathcal{H}^1 := \ker d_p\mu \cap (\text{im } \alpha_p)^{\perp_g}.$$

Notice that for all $X \in \mathfrak{g}$ and $v \in T_pM$, we have

$$-g_p(Iv, \alpha_p(X)) = \langle d_p\mu(v), X\rangle = g_p(I\alpha(X), v), \tag{2.8}$$

from which we see that $I_p(\ker d_p\mu) = (\text{im } \alpha_p)^{\perp_g}$ and $\ker d_p\mu \perp_g I(\text{im } \alpha_p)$. In particular, $\mathcal{H}^1$ is a complex subspace of (T_pM, I). Now, let $g_p : T_pM \to T_p^*M$ be the identification coming from the Kähler metric g, i.e.

$$g_p(v)(w) := g_p(v, w)$$

for all $v, w \in T_pM$. Then, we can write

$$(g_p)^{-1} \circ (d_p\mu)^* = (g_p)^{-1}\omega_p \circ \alpha_p = I_p \circ \alpha_p. \tag{2.9}$$

A dimension count shows that

$$T_pM = \ker d_p\mu \oplus^{\perp_g} I(\text{im } \alpha_p),$$

so complex (2.1) can be 'rolled up' to give a $\mathbb{C}$-linear map

$$\mathfrak{g}_{\mathbb{C}} \xrightarrow{(\alpha_p)_{\mathbb{C}}} (T_pM, I_p), \tag{2.10}$$

where $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of $\mathfrak{g}$ and $(\alpha_p)_{\mathbb{C}}$ is the $\mathbb{C}$-linear extension of α_p. Then, we have

$$\ker((\alpha_p)_{\mathbb{C}}) \cong H^0(C^\bullet) \otimes_{\mathbb{R}} \mathbb{C} \tag{2.11}$$

and

$$\operatorname{coker}((\alpha_p)_{\mathbb{C}}) \cong \mathcal{H}^1 \cong H^1(C^\bullet). \tag{2.12}$$

We note that (2.10) makes sense whether or not the G-action can be extended to a holomorphic action of its complexification $G_{\mathbb{C}}$. If the G-action can be extended to a $G_{\mathbb{C}}$-action, then (2.10) is indeed the linearization of the complex group action at p. The identification (2.12) is a linearized version of the correspondence between the symplectic quotient of M by G and the *Kähler quotient* of M^{ss} by $G_{\mathbb{C}}$, a notion we are going to define.

Assume that G is a compact Lie group with complexfication $G_{\mathbb{C}}$. Suppose the moment map μ is proper. We say a point $p \in M$ is (μ-)*semistable* if the closure of its complex orbit contains zeros of the moment map μ. It was shown in [Kir84] that the *semistable locus*

$$M^{ss} := \{p \in M : \overline{G_{\mathbb{C}} \cdot p} \cap \mu^{-1}(0) \neq \emptyset\}$$

is a $G_{\mathbb{C}}$-invariant open subset of M containing $\mu^{-1}(0)$. Moreover, using an invariant inner product on $\mathfrak{g}$, one can define the function $-||\mu||^2$, whose gradient flow φ_t exists for all time $t \geq 0$ and the limit

$$\varphi_\infty(p) := \lim_{t \to \infty} \varphi_t(p) \tag{2.13}$$

exists for all $p \in M$, defining a map φ_∞ on M. Restricted to M^{ss}, φ_∞ is a continuous retraction onto $\mu^{-1}(0)$, and M^{ss} can be characterized as the set of points whose image under φ_∞ lies in

$\mu^{-1}(0)$. The gradient of $-||\mu||^2$ is tangent to $G_{\mathbb{C}}$-orbits. As a result, for $y \in M^{ss}$, $\varphi_\infty(y)$ lies in $\overline{G_{\mathbb{C}} \cdot y}$.

The following properties are proved in [Kir84] in the case when M is projective with ω the Fubuni-Study form and in [Sja95] for M Kähler. Here, when talking about a $G_{\mathbb{C}}$-orbit being closed or the closure of a $G_{\mathbb{C}}$-orbit, it is the subspace topology of M^{ss} that is referred to.

Proposition 2.1.1. *Under the conditions stated above, the following statements about $G_{\mathbb{C}}$-orbits in M^{ss} are true. Let $p, q \in M^{ss}$.*

(i) The orbit $G_{\mathbb{C}} \cdot y$ is closed iff $G_{\mathbb{C}} \cap \mu^{-1}(0) \neq \emptyset$.

(ii) If $G_{\mathbb{C}} \cdot y$ is closed, then the intersection $G_{\mathbb{C}} \cdot y \cap \mu^{-1}(0) \neq \emptyset$ is precisely one G-orbit.

(iii) The retraction $r(G_{\mathbb{C}} \cdot y) \subset \mu^{-1}(0)$ is precisely one G-orbit.

(iv) If $x, y \in \mu^{-1}(0)$ and they do not lie in the same G-orbit, then there exists disjoint G_C-invariant open neighborhoods U_x and U_y of x and y respectively.

Consequently, there is a one-one correspondence between closed $G_{\mathbb{C}}$-orbits of M^{ss} and G-orbits of $\mu^{-1}(0)$. The closure of any $G_{\mathbb{C}}$-orbit contains precisely one closed $G_{\mathbb{C}}$-orbit. For $x, y \in M^{ss}$, we say x and y are *S-equivalent*, written as $x \sim_s y$ if $\overline{G_{\mathbb{C}} \cdot x} \cap \overline{G_{\mathbb{C}} \cdot y} \neq \emptyset$. This defines an equivalence relation on M^{ss}. By the above, we have a one-one correspondence between closed $G_{\mathbb{C}}$-orbits and S-equivalence classes as well. The quotient space

$$M^{ss} /\!\!/ G_{\mathbb{C}} := M^{ss} / \sim_s \tag{2.14}$$

is called the *Kähler quotient* of M associated with the moment map μ. By the above proposition, $M^{ss} /\!\!/ G_{\mathbb{C}}$ is Hausdorff. Let $M^{ps} \subset M^{ss}$ be the subset of points with closed $G_{\mathbb{C}}$-orbits. Then, the

retraction φ_∞ induces homeomorphisms

$$M^{ps}/G_{\mathbb{C}} \cong M^{ss} /\!\!/ G_{\mathbb{C}} \cong \mu^{-1}(0)/G. \tag{2.15}$$

2.1.3 Hyperkähler quotients and the twistor space

We begin with an overview of basic notions in the differential geometry of hyperkähler manifolds relevant to our discussion. For more details, we refer to [HKLR87] and the nice exposition in [Nei21].

Definition 2.1.2. A *hyperkähler manifold* is a tuple (M, I_1, I_2, I_3, g), where g is a riemannian metric, such that (M, I_j, g) is Kähler for each $j = 1, 2, 3$ and $I_1 I_2 = I_3$.

The span $\langle \mathrm{Id}_{TM}, I_1, I_2, I_3 \rangle$ generates the ring of quaternions $\mathbb{H} = \langle 1, i, j, k \rangle$, realizing TM as an $\mathbb{H}$-module. For each $s = (s_1, s_2, s_3) \in S^2 \subset \mathbb{R}^3$, the corresponding unit quaternion

$$I_s := s_1 I_1 + s_2 I_2 + s_3 I_3$$

also defines a complex structure on M so that (M, I_s, g) is Kähler. Hence, the data of a hyperkähler manifold actually gives rise to an S^2-worth of Kähler structures. We will denote the I_s-Kähler form by ω_{I_s}.

In addition, let

$$\Omega_1 := \omega_2 + \sqrt{-1}\omega_3. \tag{2.16}$$

One can verify that Ω_1 is in fact a $(2,0)$-form with respect to the complex structure I_1 non-degenerate on the I_1-holomorphic tangent bundle $T^{1,0}_{I_1} M$. As $d\Omega_1 = 0$, Ω_1 is I_1-holomorphic, so

(M, I_1, Ω_1) is *holomorphic symplectic*.

A hyperkähler manifold (M, I_1, I_2, I_3, g) is associated with a *twistor space* $Z = M \times S^2$ with a fibration $p : Z \to S^2$ given by projection to S^2. We identify S^2 with $\mathbb{C}P^1$ via

$$s = (s_1, s_2, s_3) \mapsto \xi = \frac{s_2 - is_3}{1 + s_1}, \tag{2.17}$$

which is chosen so that I_1, I_2 and $-I_1$ are mapped to 0, 1 and ∞ respectively. Under this identification, the antipodal map of S^2 translates to an anti-holomorphic involution σ on $\mathbb{C}P^1$ given by

$$\sigma(\xi) = -\frac{1}{\bar{\xi}}.$$

Let $I_{\mathbb{C}P^1}$ be the complex structure of the complex projective line. For each point $(p, \xi) \in Z$, we endow with its tangent space the almost complex structure $J_{p,\xi} := I_\xi \oplus I_{\mathbb{C}P^1}$. It can be shown that this defines a complex structure J on Z. The involution σ lifts to an anti-holomorphic involution $\tau : Z \to Z$ covering σ. Each point $p \in M$ defines a holomorphic section $s_p : \mathbb{C}P^1 \to Z$ given by $s_p(\xi) = (p, \xi)$, which is *real* in the sense that the diagram below commutes:

$$\begin{array}{ccc} Z & \xrightarrow{\tau} & Z \\ {\scriptstyle s_p}\uparrow & & \uparrow{\scriptstyle s_p} \\ \mathbb{C}P^1 & \xrightarrow{\sigma} & \mathbb{C}P^1 \end{array} \tag{2.18}$$

In general, such a section is called a *real twistor line*.

We end this section with a discussion about hyperkähler quotient. Suppose we have a G-action on M preserving the hyperkähler structure. A *hyperkähler moment map* with respect to the G-action is a map $\mu = (\mu_1, \mu_2, \mu_3) : M \to (\mathfrak{g}^*)^{\oplus 3}$, such that each μ_i is a moment map

with respect to the symplectic form ω_{I_i}, $i = 1, 2, 3$. In particular, for each $s \in S^2$, G acts I_s-holomorphically and

$$\mu_s := s_1\mu_1 + s_2\mu_2 + s_3\mu_3$$

is a moment map with respect to the symplectic form ω_{I_s}. The quotient space

$$\mathcal{M}_{hk} := \mu^{-1}(0)/G.$$

is called the *hyperkähler quotient* with respect to hyperkähler moment map μ. We denote the quotient map by $\pi : \mu^{-1}(0) \to \mathcal{M}_{hk}$.

Let $p \in \mu^{-1}(0)$. We linearize the G-action and μ at p:

$$C^\bullet_{hk} : \quad \mathfrak{g} \xrightarrow{\alpha_p} T_pM \xrightarrow{d_p\mu} (\mathfrak{g}^*)^{\oplus 3} \ . \tag{2.19}$$

As in the case for symplectic quotients, $C^\bullet_{hk}$ is a chain complex. We compute $(d_p\mu)^*$ next. By (2.9), for $(X_1, X_2, X_3) \in \mathfrak{g}^{\oplus 3}$, we have

$$(g_p)^{-1} \circ (d_p\mu)^*(X_1, X_2, X_3) = I_1\alpha_p(X_1) + I_2\alpha_p(X_2) + I_3\alpha_p(X_3). \tag{2.20}$$

For $I_jI_k = I_\ell$, $j,k,\ell \in \{1,2,3\}$, we have

$$\begin{aligned} g_p(I_j\alpha_p(X), I_k\alpha_p(Y)) &= g(I_\ell\alpha_p(X), \alpha_p(Y)) \\ &= \omega_{I_\ell}(\alpha_p(X), \alpha_p(Y)) \\ &= \langle d_p\mu \circ \alpha_p(X), \alpha_p(Y)\rangle \\ &= 0 \end{aligned}$$

for all $X, Y \in \mathfrak{g}$. It follows that im α_p is totally real. As in the Kähler case, we identify $H^1(C^\bullet_{hk})$ with

$$\mathcal{H}^1_{hk} := \ker d_p\mu \cap (\mathrm{im}\alpha_p)^{\perp_g}.$$

Notice that $\ker d_p\mu = \ker d_p\mu_1 \cap \ker d_p\mu_2 \cap \ker d_p\mu_3$. Moreover, for $j,k,\ell \in \{1,2,3\}$ with $I_jI_k = I_\ell$ and $X \in \mathfrak{g}$,

$$\langle d_p\mu_\ell(v), X\rangle = -g_p(I_\ell v, \alpha_p(X)) = -g_p(I_jI_kv, \alpha_p(X)) = \langle d_p\mu_j(I_kv), X\rangle. \tag{2.21}$$

Together with (2.8), it follows that $\mathcal{H}^1_{hk}$ is an $\mathbb{H}$-module.

The linearization (2.19) and its cohomology H^1 admits several interpretations which, in some suitably nice situations, come from the linearizations of other geometric setup. To better illustrate these situations, it is helpful to assume G is a compact Lie group with complexification $G_\mathbb{C}$. The complexfied Lie algebra $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$ is naturally identified with the Lie algebra of $G_\mathbb{C}$.

First, as in the Kähler case, we have the complexified infinitesimal action $(\alpha_p)_\mathbb{C} : \mathfrak{g}_\mathbb{C} \to$

(T_pM, I_1). For the remaining two components of the moment map (μ_2, μ_3), we consider

$$\mu_1^c := \mu_2 + \sqrt{-1}\mu_3 : M \to \mathfrak{g}_{\mathbb{C}}^*. \tag{2.22}$$

For each $X \in \mathfrak{g}$, we have

$$\langle d\mu_1^c, X\rangle = \iota_{\alpha(X)}\Omega_1. \tag{2.23}$$

As both $\alpha(X)$ and Ω_1 are I_1-holomorphic, we see that μ_1^c is I_1-holomorphic. A similar calculation as (2.21) shows that $d_p\mu_j \circ (\alpha_p)_{\mathbb{C}} = 0$ for $j = 2, 3$. Notice that we have used the full hyperkaähler structure in this verification. As a result, we obtain the following chain complex of complex vector spaces:

$$C_{\mathbb{C}}^{\bullet} : \quad \mathfrak{g}_{\mathbb{C}} \xrightarrow{(\alpha_p)_{\mathbb{C}}} (T_pM, I_1) \xrightarrow{d_p\mu_1^c} (\mathfrak{g}_{\mathbb{C}})^* \tag{2.24}$$

which formally looks like the linearization at a zero of a complex moment map, which often happens to be the case. Without this interpretation, at the linear level one can still apply the exact same reasoning as in the real symplectic case to analyze (2.24). More precisely, we fix an invariant inner product on $\mathfrak{g}$ and extend it to an invariant hermitian inner product on $\mathfrak{g}_{\mathbb{C}}$, which induces a $\mathbb{C}$-linear identification

$$K : \mathfrak{g}_{\mathbb{C}} \xrightarrow{\cong} \overline{(\mathfrak{g}_{\mathbb{C}})^*}.$$

Then, same as (2.1.1) and (2.4), we have a $\mathbb{C}$-conjugate linear identification

$$H^0(C_{\mathbb{C}}^{\bullet}) \cong_{\overline{\mathbb{C}}} H^2(C_{\mathbb{C}}^{\bullet}). \tag{2.25}$$

Moreover, we have $\ker\ d_p\mu_1^c$ coisotropic with $\ker(d_p\mu_1^c)^{\perp_{\Omega_1}} = \mathrm{im}((\alpha_p)_{\mathbb{C}})$, so $H^1(C_{\mathbb{C}}^{\bullet})$ is naturally

equipped with a complex symplectic form.

Formally, the cohomology $H^1(C^\bullet_{\mathbb{C}}) = \text{coker}((\alpha_p)_{\mathbb{C}})$ looks like the 'tangent space' of the quotient $(\mu_1^c)^{-1}(0)/G_{\mathbb{C}}$. Indeed, suppose the G-action can be extended to a $G_{\mathbb{C}}$-action, then one can look for the Kähler quotient of the complex analytic subspace $(\mu_1^c)^{-1}(0)$. On the other hand, the symplectic form ω_1 restricts to a Kähler form on the non-singular locus of $(\mu_1^c)^{-1}(0)$, so one might expect a correspondence between the Kähler quotient and the symplectic quotient. This motivates the following sequence

$$C^\bullet_s : \quad \mathfrak{g} \xrightarrow{\alpha_p} \ker d_p\mu_1^c \xrightarrow{d_p\mu_1} \mathfrak{g}^* \ . \tag{2.26}$$

It is easy to see that $C^\bullet_s$ is indeed a complex and $H^1(C^\bullet_s) = H^1(C^\bullet_{hk})$. To sum up, we have established the isomorphisms

$$\mathcal{H}^1_{hk} \cong H^1(C^\bullet_{hk}) \cong H^1(C^\bullet_s) \cong H^1(C^\bullet_{\mathbb{C}}).$$

The motivation picture of a correspondence between closed $G_{\mathbb{C}}$-orbits of $(\mu_c^{-1})^{-1}(0)$ and closed G-orbits of $\mu^{-1}(0)$ is the finite dimensional analogue of the *Hitchin-Simpson correspondence*.

We end this section by looking at how the quotient space $\mathcal{M}_{hk} = \mu^{-1}(0)/G$ inherits a hyperkähler structure in some simple nice cases where we do not need to worry about singularities. First, notice that (2.20) implies that p is a regular point of μ if and only if the G-action is locally free at p. Hence, it is sufficient to assume that G acts freely on M, implying that 0 is a regular value of μ, and we have

$$\mathcal{H}^1_{hk} \cong H^1(C^\bullet_{hk}) \cong T_{\pi(p)}\mathcal{M}_{hk}. \tag{2.27}$$

By a similar reasoning as above, $d_p\mu_1^c$ is surjective at $p \in (\mu_1^c)^{-1}(0)$. By the implicit function theorem, the G-invariant zero set $(\mu_1^c)^{-1}(0)$ is then an I_1-complex submanifold of (M, I_1), and is thus Kähler. Now, restriction of the moment map μ_1 to $(\mu_1^c)^{-1}(0)$ gives a moment map on $(\mu_1^c)^{-1}(0)$. We then apply the symplectic reduction procedure for Kähler manifolds to conclude that the quotient

$$(\mu_1^{-1}(0) \cap (\mu_1^c)^{-1}(0))/G = \mu^{-1}(0)/G$$

inherits a Kähler structure from I_1. Apply the same reasoning to I_2 and I_3, we obtain the desired hyperkähler structure on the quotient $\mathcal{M}_{hk}$.

2.2 Higgs bundles

In this section, we will collect some basic notions one encounters in studying Higgs bundles. For more details, we recommend [Wen16] and the appendix of [Wel08].

Throughout this thesis, we fix a closed Riemann surface X of genus $g(X) \geq 2$ equipped with a Kähler metric with unit volume. We denote the trivial line bundle and the canonical line bundle of X by $\mathcal{O}_X$ and K_X respectively. For a holomorphic vector bundle $\mathcal{F}$ over X, we denote the rank of $\mathcal{F}$ by $\mathrm{rk}(\mathcal{F})$ and the space of global holomorphic sections of $\mathcal{F}$ by $H^0(\mathcal{F})$.

Definition 2.2.1. A *Higgs bundle* or a *Higgs pair* is a pair $(\mathcal{E}, \Phi)$ where $\mathcal{E}$ is a holomorphic vector bundle over X and $\Phi \in H^0(\mathrm{End}(\mathcal{E}) \otimes K_X)$. The section Φ is called a *Higgs field* of $\mathcal{E}$.

In this thesis, we will mostly be interested in $\mathrm{SL}(r, \mathbb{C})$-Higgs bundles. We remark that it is possible to define holomorphic principal G-Higgs bundle for any reductive Lie group G, real or complex (see, for instance, [GPGiR09]). As we will not need such a wide generality we will introduce $\mathrm{SL}(r, \mathbb{C})$-Higgs bundles using the more classical formulation. First, we recall

that specifying a holomorphic $\mathrm{SL}(r,\mathbb{C})$-principal bundle is equivalent to giving a holomorphic vector bundle $\mathcal{E}$ of rank r and a volume form on $\mathcal{E}$, i.e. a fixed isomorphism $\mathcal{O}_X \xrightarrow{\cong} \det(\mathcal{E})$. The adjoint bundle $\mathfrak{sl}(\mathcal{E})$ may then be identified with the subbundle of $\mathrm{End}(\mathcal{E})$ consisting of traceless endomorphisms.

Definition 2.2.2. An *$SL(r,\mathbb{C})$-Higgs bundle* consists of a pair $(\mathcal{E},\Phi)$, where $\mathcal{E}$ is a vector bundle of rank r with a fixed identification $\det(\mathcal{E}) \xrightarrow{\cong} \mathcal{O}_X$ and $\Phi \in H^0(\mathrm{End}(\mathcal{E}) \otimes K_X)$ satisfying $\mathrm{tr}\,\Phi = 0$.

A homomorphism between two Higgs pairs $(\mathcal{E}_1,\Phi_1)$ and $(\mathcal{E}_2,\Phi_2)$ is homomorphism of vector bundles $f : \mathcal{E}_1 \to \mathcal{E}_2$ such that $\Phi_2 \circ f = f \otimes \mathrm{Id}_{K_X} \circ \Phi_1$. We say f is an *isomorphism* if it is an isomorphism between the two vector bundles.

Given Higgs pairs $\{(\mathcal{E}_i,\Phi_i)\}_{i=1}^k$, we can form their *direct sum*, denoted by $\bigoplus_{i=1}^k(\mathcal{E}_i,\Phi_i)$, obtaining a Higgs pair $(\mathcal{E},\Phi)$ where

$$\mathcal{E} = \bigoplus_{i=1}^k \mathcal{E}_i \quad \text{and} \quad \Phi = \sum_{i=1}^k \Phi_i \in H^0(\mathrm{End}(\mathcal{E}) \otimes K_X).$$

Here, we regard each Φ_i as a K_X-twisted endomorphism on $\mathcal{E}$ by sending components other than $\mathcal{E}_i$ to zero.

Given a proper holomorphic sub-bundle $\mathcal{E}' \subset \mathcal{E}$, by tensoring on both sides by K_X we have the proper sub-bundle $\mathcal{E}' \otimes K_X \subset \mathcal{E} \otimes K_X$. We say $\mathcal{E}'$ is Φ-*invariant* if restricted to $\mathcal{E}'$, the Higgs field Φ factors through $\mathcal{E}' \otimes K_X$. The subbundle $\mathcal{E}'$ then inherits a Higgs field giving a Higgs pair $(\mathcal{E}',\Phi')$. We may sometimes refer to $(\mathcal{E}',\Phi')$ as a sub-Higgs bundle of $(\mathcal{E},\Phi)$, from which we obtain a quotient Higgs bundle $(\mathcal{E}'' = \mathcal{E}/\mathcal{E}', \Phi'')$.

Next, we recall the notion of (slope) stability. For a vector bundle $\mathcal{F}$, let $\deg \mathcal{F}$ by the *degree*

of $\mathcal{F}$, which can be defined to be deg $(\det\mathcal{F}) := c_1(\det\mathcal{F}) \cdot [X]$, where $c_1(\det\mathcal{F}) \in H^2(X, \mathbb{Z})$ is the first Chern class of the line bundle det $\mathcal{F}$ and $[X] \in H_2(X, \mathbb{Z})$ is the fundamental class of X. Through the Chern-Weil theory, one may compute

$$\deg \mathcal{F} = \frac{i}{2\pi} \int_X \text{tr } F_A \tag{2.28}$$

where F_A is the curvature two-form of any connection A on $\mathcal{F}$. Define also

$$\mu(\mathcal{F}) := \frac{\deg \mathcal{F}}{\text{rk } \mathcal{F}},$$

called the *slope* of $\mathcal{F}$. From (2.28), it is easy to see that given a short exact sequence of vector bundles

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

we have

$$\deg \mathcal{F}' + \mathcal{F}'' = \deg \mathcal{F}.$$

Consequently, either one of the following holds:

(i) $\mu(\mathcal{F}') < \mu(\mathcal{F}) < \mu(\mathcal{F}'')$

(ii) $\mu(\mathcal{F}') > \mu(\mathcal{F}) > \mu(\mathcal{F}'')$

(iii) $\mu(\mathcal{F}') = \mu(\mathcal{F}) = \mu(\mathcal{F}'')$.

Although $\deg \mathcal{F}$ only depends on the underlying smooth vector bundle of $\mathcal{F}$, restrictions on the degree of its holomorphic sub-bundles will reflect holomorphic information of $\mathcal{F}$.

Definition 2.2.3. Let $(\mathcal{E}, \Phi)$ be a Higgs bundle. We say $(\mathcal{E}, \Phi)$ is *(semi)stable* if for any Φ-invariant holomorphic subbundle $\mathcal{E}' \subset \mathcal{E}$, we have $\mu(\mathcal{E}')(\leq) < \mu(\mathcal{E})$. If $(\mathcal{E}, \Phi) = \bigoplus_{i=1}^{k}(\mathcal{E}_i, \Phi_i)$ where each $(\mathcal{E}_i, \Phi_i)$ is stable and $\mu(\mathcal{E}_\rangle) = \mu(\mathcal{E})$, we say $(\mathcal{E}, \Phi)$ is *polystable*. A Higgs bundle that is not semistable is said to be *unstable*.

Notice that for an $\mathrm{SL}(r, \mathbb{C})$-Higgs bundle $(\mathcal{E}, \Phi)$, $\deg \mathcal{E} = \deg \mathcal{O}_X = 0$.

Definition 2.2.4. Let $(\mathcal{E}, \Phi)$ be an $\mathrm{SL}(r, \mathbb{C})$-Higgs bundle. We say $(\mathcal{E}, \Phi)$ is (semi)stable if for any Φ-invariant holomorphic subbundle $\mathcal{E}' \subset \mathcal{E}$, we have $\deg \mathcal{E}'(\leq) < 0$. If $(\mathcal{E}, \Phi)$ is a direct sum of stable Higgs bundles of degree zero, we say it is polystable.

One may regard a vector bundle $\mathcal{E}$ as a Higgs bundle $(\mathcal{E}, \Phi)$ with $\Phi = 0$. In particular, the slope-stability notions above reduce to the usual slope-stability notions of vector bundles. There exist Higgs bundles whose underlying vector bundles are unstable. For instance, take a line bundle $\mathcal{L}$ with $0 < 2 \deg \mathcal{L} \leq g(X) - 1$. Let $\mathcal{E} = \mathcal{L} \oplus \mathcal{L}^{-1}$ and

$$\Phi = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$$

where $b \in H^0(\mathcal{L}^{-2} \otimes K_X) - \{0\}$.

We end this section with the introduction of a *Jordan-Hölder filtration* for a semistable Higg bundle.

Proposition 2.2.5. *A semistable Higgs bundle $(\mathcal{E}, \Phi)$ admits Jordan-Hölder filtration, i.e. there exists a filtration*

$$0 = (\mathcal{E}_0, \Phi_0) \subset (\mathcal{E}_1, \Phi_1) \subset \cdots \subset (\mathcal{E}_r, \Phi_r) = (\mathcal{E}, \Phi) \tag{2.29}$$

such that the successive quotients $(\mathcal{E}_i/\mathcal{E}_{i-1}, \Phi_i')$ *are stable of the same slope as* $(\mathcal{E}, \Phi)$. *Consequently, the* associated graded

$$gr\,(\mathcal{E}, \Phi) := \bigoplus_{i=1}^{r} (\mathcal{E}_i/\mathcal{E}_{i-1}, \Phi_i')$$

is polystable. Any two associated graded bundles from two Jordan-Hölder filtrations are isomorphic.

2.3 The moduli space of semistable Higgs bundles as a hyperkähler quotient

2.3.1 Dolbeault operators and unitary connections

To construct the moduli space of semistable Higgs bundles by analytic means, we will switch our point of view about holomorphic structures to integrable Dolbeault operators. We will give a brief overview below. These materials are by now standard and can be found in standard texts, for instance [Kob87].

A holomorphic vector bundle $\mathcal{E}$ determines an operator $\overline{\partial}_{\mathcal{E}}$, so that holomorphic sections of $\mathcal{E}$ are precisely those smooth sections s of its underlying smooth vector bundle such that $\overline{\partial}_{\mathcal{E}} s = 0$. On the other hand, on a smooth vector bundle $E \to X$, let $\Omega^{p,q}(E)$ be the space of smooth sections of E with values in (p, q)-forms. A *Dolbeault operator* is a $\mathbb{C}$-linear operator $\overline{\partial}_E : \Omega^0(E) \to \Omega^{0,1}(E)$ satisfying the $\overline{\partial}$-Leibniz rule

$$\overline{\partial}_E(fs) = \overline{\partial} f \wedge s + f\overline{\partial}_E s$$

for any smooth function f and smooth section $s \in \Omega^0(E)$. For all p and q, it naturally extends to

$\overline{\partial}_E : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ so that for any k-form $\alpha \in \Omega^k$ and $s \in \Omega^{p,q}(E)$, we have

$$\overline{\partial}_E(\alpha \wedge s) = \overline{\partial}\alpha \wedge s + (-1)^k \alpha \wedge \overline{\partial}_E s.$$

We say a Dolbeault operator is *integrable* if $\overline{\partial}_E \circ \overline{\partial}_E = 0$. By a vector bundle version of the Newlander-Nirenberg theorem, an integrable Dolbeault operator uniquely defines a holomorphic vector bundle structure $\mathcal{E}$ on E so that holomorphic sections of $\mathcal{E}$ are precisely those annihilated by $\overline{\partial}_E$. Since our base manifold X is a Riemann surface, the integrability condition $\overline{\partial}_E \circ \overline{\partial}_E = 0$ is automatically satisfied by any Dolbeault operator $\overline{\partial}_E$. From now on, we will drop the adjective 'integrable'. A simple proof of the Newlander-Nirenberg theorem for vector bundles in the case of Riemann surfaces can be found in [AB82]. We also note that a Dolbeault operator $\overline{\partial}_E$ on E induces compatible Dolbeault operators on bundles associated with E such as its dual $E^\vee$ and the space of endomorphisms $\mathrm{End}(E)$, the determinant line bundle $\det E$, and so on.

Now, giving a holomorphic vector bundle $\mathcal{E}$ is equivalent to giving a pair $(E, \overline{\partial}_E)$, where E is a smooth vector bundle and $\overline{\partial}_E$ is a Dolbeault operator. In the following, we will identify $\mathcal{E} = (E, \overline{\partial}_E)$ and use the Dolbeault operator $\overline{\partial}_E$ to encode the holomorphic structure. Over a closed surface, smooth vector bundles are classified by degrees and ranks. Therefore, isomorphism classes of holomorphic vector bundles of fixed degree d and r are in one-one correspondence with isomorphism classes of Dolbeault operators on a *fixed* smooth vector bundle E of degree d and r.

Next, we review the correspondence between unitary connections and holomorphic structures. Fix a smooth hermitian vector bundle (E, h), where h is a smooth hermitian metric on E. Our convention here is that h is $\mathbb{C}$-linear in the first variable. Recall that a connection A on E defines

a covariant derivative $d_A : \Omega^0(E) \to \Omega^1(E)$ satisfying the Leibniz rule

$$d_A(fs) = df \wedge s + f d_A s.$$

We say d_A is h-*unitary* if it is compatible with the hermitian inner product, i.e. $d_A h = 0$ or equivalently, for all sections s and t,

$$dh(s,t) = h(d_A s, t) + h(s, d_A t).$$

Write the $(1,0)$- and $(0,1)$-part of d_A as d'_A and d''_A respectively. Then d''_A defines a Dolbeault operator on E. Conversely, given a Dolbeault operator $\overline{\partial}_E$, one can show that there exists a unique h-unitary connection d_A such that $d''_A = \overline{\partial}_E$, where the uniqueness follows from the condition

$$\overline{\partial} h(s,t) = h(d''_A s, t) + h(s, d'_A t). \tag{2.30}$$

We will write

$$\partial^h_E := d'_A \tag{2.31}$$

and call $\overline{\partial}_E + \partial^h_E$ the *Chern connection* of $\overline{\partial}_E$ (considered h as fixed). To sum up, the construction of Chern connections gives a correspondence between Dolbeaut operators and unitary connections

$$\overline{\partial}_E \;\; \mapsto \;\; \overline{\partial}_E + \partial^h_E \tag{2.32}$$

$$d_A \;\; \mapsto \;\; d''_A \tag{2.33}$$

This correspondence may be formulated as an affine space isomorphism between the spaces of these two structures, as we will now explain.

Denote by $\mathcal{A}(E)$ and $\mathcal{A}(E,h)$ the space of Dolbeault operators and h-unitary connections on E respectively. Let $\mathfrak{u}(E,h) \subset \mathrm{End}(E)$ be the bundle of skew-hermitian endomorphisms of E. Then the two spaces are affine spaces modeled on $\Omega^{0,1}(\mathrm{End}(E))$ and $\Omega^1(\mathfrak{u}(E,h))$ respectively. Let $\overline{\partial}_E$ be a Dolbeault operator with Chern connection $\overline{\partial}_E + \partial_E^h$. With respect to $\overline{\partial}_E$, the correspondence (2.32) and (2.33) can be re-written as

$$\overline{\partial}_E + \beta \;\; \mapsto \;\; \overline{\partial}_E + \partial_E^h + \beta - \beta^{*_h} \tag{2.34}$$

$$\overline{\partial}_E + \partial_E^h + u \;\; \mapsto \;\; \overline{\partial}_E + u^{0,1}, \tag{2.35}$$

where $\beta \in \Omega^{0,1}(\mathrm{End}(E))$ and $u \in \Omega^1(\mathfrak{u}(E,h))$. Here, β^{*_h} is the h-hermitian adjoint of β and $u^{0,1}$ is the $(0,1)$-part of u. From now on, once a hermitian metric h is fixed, we will use $\mathcal{A}(E)$ to refer to both the space of Dolbeault operators and h-unitary connections without further mention, and will implicitly identify $\Omega^{0,1}(\mathrm{End}(E))$ with $\Omega^1(\mathfrak{u}(E,h))$ via the real isomorphisms

$$\begin{cases} \beta \;\; \mapsto \beta - \beta^* \\ a \;\; \mapsto a^{0,1} \end{cases} \tag{2.36}$$

where $\beta \in \Omega^{0,1}(\mathrm{End}(E))$ and $a \in \Omega^1(\mathfrak{u}(E,h))$.

2.3.2 The Hitchin equation and Hitchin-Simpson correspondence

We will introduce a hyperkähler structure on the configuration space of pairs $(\overline{\partial}_E, \Phi)$ with structure group $\mathrm{SL}(r, \mathbb{C})$. Throughout, we fix a smooth vector bundle $E \to X$ of rank r and degree 0, with a fixed identification $\det E \xrightarrow{\cong} \underline{\mathbb{C}}$ where $\underline{\mathbb{C}} = X \times \mathbb{C}$ is the trivial line bundle. The standard $\overline{\partial}$-operator on functions gives a natural identification $(\underline{\mathbb{C}}, \overline{\partial}) \cong \mathcal{O}_X$. In addition, we fix a hermitian metric h on E, so that the induced metric $\det h$ on $\det E$ is compatible with the standard inner product on $\underline{\mathbb{C}}$. The Dolbeault operators of interest are those $\overline{\partial}_E$ whose induced complex structure on $\det E$ coincides with $(\underline{\mathbb{C}}, \overline{\partial})$. In the following, we will simply refer to these operators as Dolbeault operators or complex structures, making the compatibility conditions implicit.

Let $\mathfrak{sl}(E) \subset \mathrm{End}(E)$ be the bundle of traceless endomorphisms, $\mathfrak{su}(E, h)$ be the bundle of traceless skew-hermitian endomorphisms of E and $\mathcal{A} = \mathcal{A}(E)$ the space of complex structures on E. The *configuration space* is defined to be

$$\mathcal{C} = \mathcal{C}(E) := \mathcal{A} \times \Omega^{1,0}(\mathfrak{sl}(E)). \tag{2.37}$$

Note that $\mathcal{A}$ is an affine space modeled on $\Omega^{0,1}(\mathfrak{sl}(E))$, so the tangent space at each point $(\overline{\partial}_E, \Phi) \in \mathcal{C}$ may be identified with $\Omega^{0,1}(\mathfrak{sl}(E)) \oplus \Omega^{1,0}(\mathfrak{sl}(E))$, which is a complex vector space. Thus, $\mathcal{C}$ is naturally equipped with a complex structure, denoted by I_1.

The hermitian metric h defines the hermitian adjoint of each $A \in \Omega^{p,q}(\mathrm{End}(E))$, denoted

by $A^* = A^{*_h}$. Define a riemannian metric on $\mathcal{C}$, given by

$$g((\beta_1, \varphi_1), (\beta_2, \varphi_2)) := -\text{Im} \int_X \text{tr}(\varphi_1 \wedge \varphi_2^* + \beta_2^* \wedge \beta_1). \tag{2.38}$$

It immediately follows that $g(I_1 \cdot, I_1 \cdot) = g(\cdot, \cdot)$. Let $\omega_1 = g(I_1, \cdot, \cdot)$ be the Kähler form with respect to I_1. Explicity, we have

$$\omega_1((\beta_1, \varphi_1), (\beta_2, \varphi_2)) = -\text{Re} \int_X \text{tr}(\varphi_1 \wedge \varphi_2^* + \beta_2^* \wedge \beta_1). \tag{2.39}$$

To define another complex structure I_2, we first observe that the configuration space $\mathcal{C}$ may be identified with the space of $\text{SL}(r, \mathbb{C})$-connections on E, i.e. those complex connections whose induced connection on $\det E$ is compatible with the identification $\det E \cong \underline{\mathbb{C}}$). First, we notice that any connection d_A can be uniquely decomposed into $d_a + \Psi$, where d_a is an h-unitary connection and $\Psi \in \Omega^1(i\mathfrak{su}(E, h))$. Indeed, let d_{a_0} be any unitary connection. Write $d_A = d_{a_0} + u$ where $u \in \Omega^1(\mathfrak{sl}(E))$. Then we can take $d_a = d_{a_0} + \frac{1}{2}(u - u^*)$ and $\Psi = \frac{1}{2}(u + u^*)$. If $d_A = d_{a'} + \Psi'$ is another such decomposition, then $d_{a'} = d_a + \Psi - \Psi'$. But then with $\Psi - \Psi'$ is both skew-hermitian and hermitian, so $\Psi = \Psi'$ and $d_a = d_{a'}$. It follows that the space of $\text{SL}(r, \mathbb{C})$-connections can be identified with $\mathcal{A} \times \Omega^1(i\mathfrak{su}(E, h))$, which may then be identified with $\mathcal{C}$ via

$$\begin{cases} (d_A, \Psi) & \mapsto (d_A'', \Psi^{1,0}) \\ (\overline{\partial}_E, \Phi) & \mapsto (\overline{\partial}_E + \partial_E^h, \Phi + \Phi^*). \end{cases}$$

The natural complex structure on the space of $\text{SL}(r, \mathbb{C})$-connections coming from $\Omega^1(\mathfrak{sl}(E))$ thus induces a complex structure I_2 on $\mathcal{C}$. We then obtain the explicit expression of I_2 on $\mathcal{C}$ by tracing

the identifications made above:

$$(\beta,\varphi)\mapsto\beta-\beta^*+\varphi+\varphi^*\mapsto i(\beta-\beta^*+\varphi+\varphi^*)\mapsto(i(\varphi+\varphi^*),i(\beta-\beta^*))\mapsto(i\varphi^*,-i\beta^*),$$

from which we can check that $I_1I_2=-I_2I_1$. Now, define $I_3=I_1I_2$. We have thus obtained three complex structures on $\mathcal{C}$: for $(\beta,\varphi)\in\Omega^{0,1}(\mathfrak{sl}(E))\oplus\Omega^{1,0}(\mathfrak{sl}(E))$:

$$I_1(\beta,\varphi) = (i\beta,i\varphi) \tag{2.40}$$

$$I_2(\beta,\varphi) = (i\varphi^*,-i\beta^*) \tag{2.41}$$

$$I_3(\beta,\varphi) = (-\varphi^*,\beta^*). \tag{2.42}$$

From (2.38), we can verify that

$$\begin{aligned} g(I_2(\beta_1,\varphi_1),I_2(\beta_2,\varphi_2)) &= -\mathrm{Im}\int_X \mathrm{tr}(\beta_1^*\wedge\beta_2+\varphi_2\wedge\varphi_1^*)\\ &= -\mathrm{Im}\int_X -\mathrm{tr}(\varphi_1\wedge\varphi_2^*+\beta_2^*\wedge\beta_1)^*\\ &= \mathrm{Im}\overline{\int_X \mathrm{tr}(\varphi_1\wedge\varphi_2^*+\beta_2^*\wedge\beta_1)}\\ &= g((\beta_1,\varphi_1),(\beta_2,\varphi_2)). \end{aligned}$$

It follows that I_2 and thus I_3 are orthogonal, with corresponding Kähler forms ω_2 and ω_3 respectively

given by

$$\omega_2((\beta_1,\varphi_1),(\beta_2,\varphi_2)) = \text{Re}\int_X \text{tr}(\varphi_2\wedge\beta_1-\varphi_1\wedge\beta_2) \tag{2.43}$$

$$\omega_3((\beta_1,\varphi_1),(\beta_2,\varphi_2)) = \text{Im}\int_X \text{tr}(\varphi_2\wedge\beta_1-\varphi_1\wedge\beta_2). \tag{2.44}$$

The tuple (g, I_1, I_2, I_3) a hyperkähler structure on $\mathcal{C}$. The I_1-holomorphic symplectic form is given by

$$\Omega_1((\beta_1,\varphi_1),(\beta_2,\varphi_2)) = \int_X \text{tr}(\varphi_2\wedge\beta_1-\varphi_1\wedge\beta_2). \tag{2.45}$$

Let

$$\mathcal{G}^{\mathbb{C}} = \{g\in\Omega^0(\text{End}(E)) : \det g = \text{Id}\} \tag{2.46}$$

be the group of smooth bundle isomorphisms on E that preserve $\det E$, and

$$\mathcal{G} := \{g\in\mathcal{G}^{\mathbb{C}} : g^*h = h\} = \{g\in\mathcal{G}^{\mathbb{C}} : g^{*_h}g = \text{Id}\}. \tag{2.47}$$

be the subgroup of $\mathcal{G}^{\mathbb{C}}$ preserving the metric h. We will refer to $\mathcal{G}^{\mathbb{C}}$ and $\mathcal{G}$ as the complex and unitary gauge group respectively. Formally, $\mathcal{G}^{\mathbb{C}}$ and $\mathcal{G}$ have Lie algebra $\Omega^0(\mathfrak{sl}(E))$ and $\Omega^0(\mathfrak{su}(E,h))$ respectively. Via the trace pairing $\int_X$ tr, we may identify $\Omega^{1,1}(\mathfrak{sl}(E)) \cong \Omega^0(\mathfrak{sl}(E))^*$ and $\Omega^2(\mathfrak{su}(E,h)) \cong \Omega^0(\mathfrak{su}(E,h)^*$.

We introduce a $\mathcal{G}^{\mathbb{C}}$-action on $\mathcal{C}$ as follows: for $g\in\mathcal{G}^{\mathbb{C}}$ and $(\overline{\partial}_E,\Phi)\in\mathcal{C}$,

$$(\overline{\partial}_E,\Phi)\cdot g := (g^{-1}\circ\overline{\partial}_E\circ g, g^{-1}\circ\Phi\circ g) = (\overline{\partial}_E + g^{-1}\overline{\partial}_E g, g^{-1}\Phi g). \tag{2.48}$$

Restricted to $\mathcal{G}$, we have the unitary gauge group action on $\mathcal{C}$. Define $\mu_1^c : \mathcal{G}\to\Omega^{1,1}(\mathfrak{sl}(E))$ given

by

$$\mu_1^c((\overline{\partial}_E, \Phi)) := \overline{\partial}_E \Phi \in \Omega^{1,1}(\mathfrak{sl}(E)). \tag{2.49}$$

We claim that μ_1^c is a complex moment map with respect to the complex structure I_1. First, for $g \in \mathcal{G}^{\mathbb{C}}$ and $a \in \Omega^0(\mathfrak{sl}(E))$, we compute $(\overline{\partial}_E \cdot g)(g^{-1}\Phi g) = g^{-1}(\overline{\partial}_E \Phi)g$ and

$$\begin{aligned} \langle \mu_1^c((\overline{\partial}_E, \Phi) \cdot g), a \rangle &= \int_X \text{tr } a g^{-1} \overline{\partial}_E \Phi g \\ &= \int_X \text{tr } (gag^{-1}) \overline{\partial}_E \Phi \\ &= \langle \mu_1^c((\overline{\partial}_E, \Phi)) \cdot g, a \rangle. \end{aligned}$$

It follows that μ_c^1 is $\mathcal{G}^{\mathbb{C}}$-equivariant. Next, let $p = (\overline{\partial}_E, \Phi)$ and $u = (\beta, \varphi) \in \Omega^{0,1}(\mathfrak{sl}(E)) \oplus \Omega^{1,0}(\mathfrak{sl}(E))$. We compute

$$d_p \mu_1^c(u) = \overline{\partial}_E \varphi + [\beta, \Phi] = D''_{(\overline{\partial}_E, \Phi)} u, \tag{2.50}$$

where for each $k \geq 0$, $D''_{(\overline{\partial}_E, \Phi)} : \Omega^k(\mathfrak{sl}(E)) \to \Omega^{k+1}(\mathfrak{sl}(E))$ is given by $D''_{(\overline{\partial}_E, \Phi)} u = \overline{\partial}_E u + [\Phi, u]$. We remark that here, for $A \in \Omega^k(\text{End}(E))$ and $B \in \Omega^\ell(\text{End}(E))$,

$$[A, B] = A \wedge B - (-1)^{k\ell} B \wedge A.$$

On the other hand,

$$\rho_p(a) = \frac{d}{dt}\bigg|_{t=0} (\overline{\partial}_E, \Phi) \cdot \exp(ta) = \overline{\partial}_E a + [\Phi, a] = D''_{(\overline{\partial}_E, \Phi)} a. \tag{2.51}$$

From (2.50) and (2.51), we have

$$\begin{aligned}
\langle d_p\mu_1^c(\beta,\varphi), a\rangle &= \int_X \text{tr}\ a(\overline{\partial}_E\varphi + [\beta,\Phi]) \\
&= \int_X \text{tr}\left[\overline{\partial}_E(a\varphi) - (\overline{\partial}_E a)\varphi + a\Phi\beta + a\beta\Phi\right] \\
&= \int_X \overline{\partial}_E\text{tr}(au) + \int_X \text{tr}\left[\varphi\wedge(\overline{\partial}_E a) - [\Phi,a]\wedge\beta\right] \\
&= \Omega_1(\rho_p(a),u) \\
&= \iota_{\rho_p(a)}\Omega_1(u).
\end{aligned}$$

Hence μ_1^c is a moment map with respect to Ω_1. The zero set of μ_1^c

$$\mathcal{B} := \{(\overline{\partial}_E,\Phi) : \overline{\partial}_E\Phi = 0\} = (\mu_1^c)^{-1}(0) \subset \mathcal{C} \tag{2.52}$$

is precisely the space of Higgs bundles. Let $\mathcal{B}^{ss}$ and $\mathcal{B}^{ps}$ be space of semistable and polystable Higgs bundles respectively.

Next, we define a moment map for the Kähler form ω_1. Let

$$\mu_1(\overline{\partial}_E,\Phi) := \frac{1}{2}F_{(\overline{\partial}_E,h)} + \frac{1}{2}[\Phi,\Phi^{*_h}] \tag{2.53}$$

and $\mu := (\mu_1,\mu_1^c) : \mathcal{C} \to \Omega^2(\mathfrak{su}(E,h)) \oplus \Omega^{1,1}(\mathfrak{sl}(E))$. Here, $F_{(\overline{\partial}_E,h)}$ denotes the curvature of the Chern connection associated with $\overline{\partial}_E$ and h. The equation

$$\mu(\overline{\partial}_E,\Phi) = 0 \tag{2.54}$$

is known as the *Hitchin equation*. We first show that μ_1 is $\mathcal{G}$-equivariant. First, we note that for any $g \in \mathcal{G}^{\mathbb{C}}$, the Chern connection for $\overline{\partial}_E \cdot g$ is given by

$$g^{-1} \circ \overline{\partial}_E \circ g + g^* \circ \partial_E^h \circ (g^*)^{-1}.$$

It follows that if $g \in \mathcal{G}$, then the Chern connection for $\overline{\partial}_E$ is $g^{-1} \circ (\overline{\partial}_E + \partial_E^h) \circ g$, and

$$F_{(\overline{\partial}_E \cdot g, h)} = g^{-1} F_{(\overline{\partial}, h)} g.$$

For $g \in \mathcal{G}$ and $(\overline{\partial}_E, \Phi) \in \mathcal{C}$, we thus have

$$\mu_1(\overline{\partial}_E \cdot g, \Phi \cdot g) = g^{-1} \mu_1(\overline{\partial}_E, \Phi) g.$$

This shows the $\mathcal{G}$-equivariance of μ_1. Next, for $a \in \Omega^0(\mathfrak{su}(E,h))$, $p = (\overline{\partial}_E, \Phi)$ and $u = (\beta, \varphi)$, we compute

$$\begin{aligned}
2\langle d\mu_1(\beta,\varphi), a\rangle &= \int_X \operatorname{tr} a \Big[(\overline{\partial}_E + \partial_E^h)(\beta - \beta^*) + [\Phi, \varphi^*] + [\varphi, \Phi^*] \Big] \\
&= \int_X \operatorname{tr} \Big(-\beta^* (\overline{\partial}_E a) + \beta(\partial_E^h a) - [\Phi, a]\varphi^* - [\Phi^*, a]\varphi \Big) \\
&= -2\mathrm{Re} \int_X \operatorname{tr}(\beta^* \wedge \overline{\partial}_E a + [\Phi, a] \wedge \varphi^*) \\
&= 2\omega_1(\rho_p(a), (\beta, \varphi)).
\end{aligned}$$

Therefore, μ_1 is a moment map with respect to the Kähler form ω_1.

It turns out those $\mathcal{G}^{\mathbb{C}}$-orbits which intersects $\mu_1^{-1}(0)$ are the precisely the polystable ones. The following theorem is proved in [Hit87] for the rank two case and [Sim88] for the general

case.

Theorem 2.3.1 ([Hit87] and [Sim88]). *Let $(\overline{\partial}_E, \Phi) \in \mathcal{B}$ be an $SL(r,\mathbb{C})$-Higgs bundle. Then $(\overline{\partial}_E, \Phi)$ is polystable if and only there exists $g \in \mathcal{G}^{\mathbb{C}}$ such that*

$$F_{(\overline{\partial}_E \cdot g, h)} + [\Phi \cdot g, (\Phi \cdot g)^{*_h}] = 0. \tag{2.55}$$

If g_1 and g_2 satisfy (2.55), then $g_2 g_1^{-1} \in \mathcal{G}$.

Remark 2.3.2. Let $(\overline{\partial}_E, \Phi)$ be polystable and $g \in \mathcal{G}^{\mathbb{C}}$ be given in Theorem 2.3.1. Let $h' := (g^{-1})^* h$ be the pullback metric. By a direct computation, one can show that the Chern connection of $\overline{\partial}_E$ with respect to the metric h' and the Chern connection of $\overline{\partial}_E \cdot g$ with respect to h, denoted by $d_{A,h'}$ and $d_{A',h}$ respectively, is related by $d_{A,h'} = g \circ d_{A',h} \circ g^{-1}$. Thus, we have

$$F_{(\overline{\partial}_E \cdot g, h)} = g^{-1} F_{(\overline{\partial}_E, h')} g.$$

Moreover, we have $\Phi^{*_{h'}} = g(g^{-1}\Phi g)^* g^{-1}$, from which we obtain

$$[\Phi \cdot g, (\Phi \cdot g)^{*_h}] = g^{-1}[\Phi, \Phi^{*_{h'}}] g.$$

Together with (2.55), we have

$$F_{(\overline{\partial}_E, h')} + [\Phi, \Phi^{*_{h'}}] = 0. \tag{2.56}$$

Definition 2.3.3. Let $(\overline{\partial}_E, \Phi) \in \mathcal{B}^{ps}$. We call a metric h' that satisfies (2.56) a *harmonic metric* of $(\overline{\partial}_E, \Phi)$.

The discussion in the above remark provides an alternative viewpoint on solving the Hitchin

equation. Given $(\overline{\partial}_E, \Phi)$, one might view the Hitchin equation as a PDE for solving the harmonic metrics. On the other hand, keeping h fixed and given a $\mathcal{G}^{\mathbb{C}}$-orbit in $\mathcal{B}$, one may view the Hitchin equation as solving for the $\mathcal{G}$-coset in the $\mathcal{G}^{\mathbb{C}}$-orbit.

The quotient $\mathcal{B}/\mathcal{G}^{\mathbb{C}}$ is badly behaved. For instance, it is not Hausdorff. Instead, we will look at the 'moduli space' of semi-stable Higgs bundles. We define the moduli space of semistable $\mathrm{SL}(r, \mathbb{C})$-Higgs bundles to be

$$\mathcal{M}_H = \mathcal{M}_H(\mathrm{SL}(r, \mathbb{C})) = \mathcal{B}^{ss} /\!\!/ \mathcal{G}^{\mathbb{C}} = \mathcal{B}^{ps}/\mathcal{G}^{\mathbb{C}}. \tag{2.57}$$

Here, $\mathcal{B}^{ss} /\!\!/ \mathcal{G}^{\mathbb{C}}$ refers to the quotient by S-equivalence classes (two semistable Higgs bundles are S-equivalent if their associated graded objects are gauge equivalent). We shall use $[(\overline{\partial}_E, \Phi)]$ to denote the S-equivalence class of a semistable Higgs bundle $(\overline{\partial}_E, \Phi)$. More generally, we will use $\mathcal{M}_H(r, d)$ to denote the moduli space of semistable Higgs bundles of rank r and degree d. The corresponding notions of slope stability are the same, except that we replace 0 by d.

The algebro-geometric construction of $\mathcal{M}_H(r, d)$ was carried at [Nit91] in the case of smooth projective curves and in [Sim94] in the case of smooth projective varieties. In the next section, we will review the Kuranishi slice method to construct a local model around a polystable Higgs bundle, which is carried out in [Fan22]. These local models were used to give a complex analytic structure on $\mathcal{M}_H(r, d)$, which is shown in [Fan22] to be isomorphic to the analytification of the algebraic moduli space constructed in [Nit91]. For the purpose of this thesis, we only need the local theory.

2.4 The non-abelian Hodge correspondence and conformal limits

We have seen that for a polystable Higgs bundle $(\overline{\partial}_E, \Phi)$, there exists a metric such that

$$F_{(\overline{\partial}_E, h)} + [\Phi, \Phi^{*_h}] = 0.$$

In fact, it is the curvature of the connection $\overline{\partial}_E + \partial_E^h + \Phi + \Phi^{*_h}$. Developed in [Hit87] and [Don87], and in [Cor88] and [Sim88] for the general case, the nonabelian Hodge correspondence (NAH for short) asserts that a Higgs bundle $(\overline{\partial}_E, \Phi)$ is polystable (stable) if and only if there exists a metric h that solves the Hitchin equation, in which case the connection

$$\text{NAH}(\overline{\partial}_E, \Phi) = \overline{\partial}_E + \partial_E^h + \Phi + \Phi^{*_h}$$

is flat and completely reducible (irreducible). Such a connection is unique. Conversely, a flat connection D is completely reducible (irreducible) if and only if there exists a metric h such that in the decomposition $d_A + \Psi$ where d_A is k-unitary and Ψ is h-hermitian, we have

$$d_A \Psi^{1,0} = 0,$$

in which case $\overline{\partial}_E = d_A''$, the $(0,1)$-part of d_A, and $\Phi = \Psi^{1,0}$ defines a polystable (stable) Higgs bundle $(\overline{\partial}_E, \Phi)$. In both directions, the metric h is called a *harmonic metric* to $(\overline{\partial}_E, \Phi)$ or D.

Now, for each $R > 0$ and each $\xi \in \mathbb{C} - \{0\}$, the connection

$$\overline{\partial}_E + \partial_E^{h_R} + \xi^{-1} R\Phi + \xi R \Phi^{*_{h_R}}$$

is also flat. Fixing a constant $\hbar \in \mathbb{C} - \{0\}$ and letting $R = \hbar^{-1}\xi$, the limit

$$\lim_{R\to 0}(\overline{\partial}_E + \partial_E^{h_R} + \hbar^{-1}\Phi + \hbar R^2 \Phi^{*_{h_R}}),$$

if it exists, is said to be the *($\hbar$-)conformal limit* of $(\overline{\partial}_E, \Phi)$.

While the conformal limit is associated with a single Higgs bundle, its existence is a gauge-invariant property. Indeed, let $(\overline{\partial}_E, \Phi)$ be a polystable Higgs bundle so that its conformal limit

$$D_0 = \lim_{R\to 0} D_R = \lim_{R\to 0}(\overline{\partial}_E + \partial_E^{h_R} + \hbar^{-1}\Phi + R^2\hbar\Phi^{*_{h_R}})$$

exists. Let $g \in \mathcal{G}^{\mathbb{C}}$ and $(\overline{\partial}', \Phi') = (\overline{\partial}_E, \Phi)\cdot g$. Since $(R\Phi)\cdot g = R(\Phi \cdot g)$, then the harmonic metric for $(\overline{\partial}', R\Phi')$ is $h'_R := g^* h_R$. It follows that

$$\overline{\partial}' + (\partial')^{h'_R} = g^{-1} \circ (\overline{\partial}_E + \partial^{h_R}) \circ g,$$

where $\overline{\partial}' + (\partial')^{h'_R}$ is the Chern connecton of $\overline{\partial}'$ with respect to the metric h'_R. Also,

$$(\Phi \cdot g)^{*_{h'_R}} = \Phi^{*_{h_R}} \cdot g.$$

Therefore,

$$\lim_{R\to 0}(\overline{\partial}' + \partial^{h'_R} + \hbar^{-1}\Phi\cdot g + R^2\hbar(\Phi\cdot g)^{*_{h'_R}}) = \lim_{R\to 0} g^{-1}\circ D_R \circ g = g^{-1}\circ D_0 \circ g.$$

When $(\overline{\partial}_E, \Phi)$ lies on the Hitchin component, it was conjectured by Gaiotto in [Gai14],

that its conformal limit exists and lies in the space of opers. The origin of this conjecture comes from superconformal field theory of class $\mathcal{S}$ which arises from compactification on the product of a Riemann surface and a circle of radius R. The non-punctured case was later proved in [DFK$^+$], whose methodology was then generalized in [CW19] to show the existence of the conformal limit for $(\overline{\partial}_E, \Phi)$ lying on the global parametrization of the upward flow through a stable $\mathbb{C}^*$-fixed point. The case for parabolic Higgs bundles is treated in [CFW24].

2.5 Stability in a linear representations

The local model constructed in [Fan22] relates certain polystable orbits in the cohomology space $\mathcal{H}^1$ of the deformation complex of a polystable Higgs bundle $(\overline{\partial}_E, \Phi)$ (to be introduced in the next section) and polystable orbits in $\mathcal{M}_H$. The former is a representation of the automorphism group of $(\overline{\partial}_E, \Phi)$. In this section, we review the definitions of stability in the linear case and the then recall the Kempf-Ness theorem [KN78], which provides a criterion that relates polystability and and stability of a nonzero vector in the representation space in terms of the existence of minimizers of the associated length squared functional. Then, we specialize to the toric case. The weight space decomposition of the representation space allows us to detect the stability of a vector v by the affine properties of the convex hull of its effective weights (see Theorem 9.2 in [Dol03] for example).

Let G be a connected complex reductive group with a maximal compact subgroup K and V a complex G-representation.

Definition 2.5.1. Let $v \in V - \{0\}$. We say v is *polystable* if its orbit $G \cdot v$ is closed in V. If, in addition, the stabilizer of v is finite, then we say v is *stable*.

Recall that we have the decomposition $G = K\exp(i\mathfrak{k})$, where $\mathfrak{k} = \mathrm{Lie}(K)$. Suppose V is equipped with a hermitian inner product such that K acts unitarily. Now, let $v \in V - \{0\}$. The *Kempf-Ness functional* of v, denoted by $\ell_v : i\mathfrak{k} \to \mathbb{R}$, is given by

$$\ell_v(x) := ||e^x \cdot v||^2. \tag{2.58}$$

Theorem 2.5.2 ([KN78], Kempf-Ness)**.** *A nonzero vector v is polystable if and only if* $\inf \ell_v$ *can be attained, and is stable if and only if the minimizer is unique.*

Next, we will specialize to the case where G is an k-dimensional complex torus, i.e. $G \cong (\mathbb{C}^*)^k$. The representation space V admits an orthogonal decomposition into weight spaces

$$V = \bigoplus_{\lambda \in \Lambda_G} V_\lambda \tag{2.59}$$

where $\Lambda_G \subset X(G) := \mathrm{Hom}(G, \mathbb{C}^*)$ is the set of weights of the G-representation. We identify $X(G) \otimes_{\mathbb{Z}} \mathbb{R}$ with $(i\mathfrak{k})^\vee \cong \mathbb{R}^k$ so that we can regard $X(G) \subset (i\mathfrak{k})^\vee$ as an integral lattice of the dual space $(i\mathfrak{k})^\vee$.

Let $v \in V - \{0\}$. For $\lambda \in \Lambda_G$, we let v_λ be the projection of v to the λ-weight space. Let

$$\Lambda_G(v) := \{\lambda \in \Lambda_G : v_\lambda \neq 0\}. \tag{2.60}$$

We will call the set $\Lambda_G(v)$ the *effective weight* of v with respect to the G-action. Finally, let $C(v) = \mathrm{Conv}(\Lambda_G(v))$ be the closed convex hull of $\Lambda_G(v)$ in $(i\mathfrak{k})^\vee$.

In Chapter 5, we will need the following criterion of the stability of v in terms of properties

of $C(v)$, which is essentially a visual depiction of the Hilbert-Mumford criterion. It can be found, for example, in Theorem 9.2 of [Dol03]. For completeness, we will give the proof of the equivalence between this criterion and the existence of unique minimizer of the Kempf-Ness functional.

Proposition 2.5.3. *The following are equivalent:*

(i) v *is stable*

(ii) ℓ_v *has a unique minimizer.*

(iii) $0 \in C(v)^\circ$*, the interior of* $C(v)$ *(in the euclidean topology of* $\mathbb{R}^k$*).*

Proof. The equivalence between (i) and (ii) is just a special case of the Kempf-Ness theorem. In terms of the weight space decomposition, we can re-write (2.58) as

$$\ell_v(x) = \sum_{\lambda \in \Lambda_G(v)} ||v_\lambda||^2 e^{2\langle \lambda, x \rangle}. \tag{2.61}$$

First, we assume (ii). Note that for any $v' \in G \cdot v$, $\Lambda_G(v') = \Lambda_G(v)$. Hence, we can assume without loss of generality that the unique minimizer takes place at the origin. Since $\Lambda_G(v)$ is finite, $C(v)$ is a convex polytope. If $k = 1$, then $C(v)$ is a closed interval of the form $[-n, m]$ where $n, m \in \mathbb{Z}_{>0}$. Now, assume $k \geq 2$ We first claim that $C(v)$ is k-dimensional. Suppose to the contrary that $\dim C(v) < k$. Then $C(v)$ is contained in some hyperplane $H = \{v \in (i\mathfrak{k})^\vee : \langle v, u \rangle = a\}$ for some $u \in i\mathfrak{k}$ and $a \leq 0$. For $t \geq 0$, $\ell_v(tu) = \sum ||v_\lambda||^2 e^{2ta} \leq \ell_v(0)$. This contradicts the assumption that 0 is the unique minimizer of ℓ_v. A similar argument together with the hyperplane separation theorem will show that $0 \in C(v)^\circ$.

Conversely, assume (iii). First, we claim that $\inf \ell_v$ can be attained. Let S be the unit sphere in $i\mathfrak{k}$ determined by some inner product. Pick a minimizing sequence $\{y_j\}_{j\geq 0}$. Without loss of generality, we may assume the sequence is unbounded. Hence, passing to subsequences if necessary, we may write $y_j = r_j u_j$ where $0 < r_j \nearrow \infty$, $u_j \in S$ and $u_j \to u \in S$. Now, we evaluate

$$\ell_v(y_j) = \sum ||v_\lambda||^2 (e^{2\langle \lambda, u_j \rangle})^{r_j} \to \inf \ell_v.$$

Since $\ell_v(y_j)$ is bounded, we have $\langle \lambda, u \rangle \leq 0$ for all $\lambda \in \Lambda_G(v)$. As $C(v)$ is the convex hull of $\Lambda_G(v)$, it implies $\langle C(v), u \rangle \leq 0$. This contradicts to the assumption $0 \in C(v)^\circ$ since any open ball centered at 0 contains some point y such that $\langle y, u \rangle > 0$. Therefore, no unbounded minimizing sequences exist and thus $\inf \ell_v$ can be attained.

Let x_0 be a minimizer of ℓ_v. For any $w \neq 0$, consider the function

$$t \mapsto \ell_v(x_0 + tw) = \sum ||v_\lambda||^2 e^{2\langle \lambda, x_0 \rangle} e^{2\langle \lambda, w \rangle t},$$

which is convex and has a global minimum at $t = 0$. Moreover, it is either strictly convex and thereby uniquely minimized, or it is constant. In the latter case, we have $w \in D$, where

$$D = \bigcap_{\lambda \in \Lambda'(v)} \ker \lambda.$$

It is also easy to see that for any $w \in D$, $t \mapsto \ell_v(x_0 + tw) \equiv \ell_v(x_0)$. Therefore, the set of minimizers of ℓ_v is $x_0 + D$. But $\langle \Lambda_G(v), D \rangle \equiv 0$ by definition, so $C(v) \subset D^\perp := \{v \in (i\mathfrak{k})^\vee : \langle v, D \rangle \equiv 0\}$. As $0 \in C(v)^\circ$ by assumption, we must have $D^\perp = (i\mathfrak{k})^\vee$. It follows that $D = \{0\}$ and thus x_0 is the unique minimizer. $\square$

2.6 The analytic local model around a polystable Higgs bundle

In this section, we will review the construction of the local model around a polystable Higgs bundle, which will be applied frequently in later chapters. For more details, we refer to [Fan22].

Let $(\overline{\partial}_E, \Phi)$ be a polystable $\mathrm{SL}(r, \mathbb{C})$-Higgs bundle. Denote by G the automorphism group of $(\overline{\partial}_E, \Phi)$, which can be regarded as the stabilizer of $(\overline{\partial}_E, \Phi)$ under the complex gauge group action. The deformation complex of $(\overline{\partial}_E, \Phi)$ is given by

$$C^\bullet = C^\bullet(\overline{\partial}_E, \Phi) : \Omega^0(\mathfrak{sl}(E)) \xrightarrow{D''} \Omega^{0,1}(\mathfrak{sl}(E)) \oplus \Omega^{1,0}(\mathfrak{sl}(E)) \xrightarrow{D''} \Omega^{1,1}(\mathfrak{sl}(E)) \tag{2.62}$$

where $D'' := \overline{\partial}_E + \Phi$. The first operator comes from linearizing the complex gauge group action, while the second one comes from linearizing the complex momement map μ_1^c. It is an elliptic complex.

In the following, for the degree one piece of the complex $C^1 = \Omega^{0,1}(\mathfrak{sl}(E)) \oplus \Omega^{1,0}(\mathfrak{sl}(E))$, we usually use a tuple $u = (\beta, \varphi)$ to denote its elements. We will also use u to represent $\beta + \varphi \in \Omega^1(\mathfrak{sl}(E))$. For any $v \in \Omega^1(\mathfrak{sl}(E))$, we write $v = v^{0,1} + v^{1,0}$ to denote its type decomposition.

Now, fix a harmonic metric h of $(\overline{\partial}_E, \Phi)$. The unitary gauge group $\mathcal{G}$ will be defined with respect to the metric h. We define $D' := \partial^h + \Phi^*$ where $\overline{\partial}_E + \partial^h$ is the Chern connection with respect to the metric h, Φ^* the h-hermitian adjoint of Φ, and $D := D' + D''$ is the flat connection under NAH. We remark that D' is independent of the choice of harmonic metrics of $(\overline{\partial}_E, \Phi)$ (for instance, see Chapter 3). These operators satisfy the following Kähler identities (for example, see [Sim92]):

$$(D'')^* = -i[\Lambda, D'] \quad \text{and} \quad (D')^* = i[\Lambda, D'']. \tag{2.63}$$

We denote the space of harmonic forms of the deformation complex by

$$\mathcal{H}^i = \mathcal{H}^i(C^\bullet) := \{x \in C^i : D''x = 0, (D'')^*x = 0\}. \tag{2.64}$$

The zero-th cohomology space $\mathcal{H}^0 = \ker D''$ can be identified with the Lie algebra of the automorphism group of $(\overline{\partial}_E, \Phi)$. Moreover, via Serre duality we have $\mathcal{H}^2 \cong (\mathcal{H}^0)^\vee$. The stabilizer group G acts on $\mathcal{H}^i$, $i = 0, 1, 2$, by conjugation. The group G has a maximal compact subgroup $K = K(h)$ which consists of elements in G fixing the metric h. In Chapter 3, we will show that all other harmonic metrics of $(\overline{\partial}_E, \Phi)$ can be obtained from pulling back by elements in G. In general, for a gauge transformation $g \in \mathcal{G}^{\mathbb{C}}$, we shall use g^*h to refer to the pullback metric defined by $g^*h(v, w) = h(gv, gw)$. Finally, we note that when $(\overline{\partial}_E, \Phi)$ is strictly polystable, $\mathcal{H}^0 \neq \{0\}$.

Next, we recall the local model around $[(\overline{\partial}_E, \Phi)] \in \mathcal{M}_H$ constructed in Section 3 of [Fan22]. Define the *Kuranishi map* $\kappa : \Omega^{0,1}(\mathfrak{sl}(E)) \oplus \Omega^{1,0}(\mathfrak{sl}(E)) \to \Omega^{0,1}(\mathfrak{sl}(E)) \oplus \Omega^{1,0}(\mathfrak{sl}(E))$ associated with $(\overline{\partial}_E, \Phi)$ by

$$\kappa(\beta, \varphi) = (\beta, \varphi) + (D'')^*\Gamma([\beta, \varphi]), \tag{2.65}$$

where Γ is the Green's operator of the Laplacian $D''(D'')^*$ on $\Omega^{1,1}(\mathfrak{sl}(E))$. The automorphism group G acts on both sides by conjugation, and κ is G-equivariant. We remark that this G-action is compatible with the action of the gauge group: for $g \in G$, $g \cdot (\overline{\partial}_E + \beta, \Phi + \varphi) =$

$(\overline{\partial}_E + g\beta g^{-1}, \Phi + g\varphi g^{-1})$. Let

$$\tilde{Z} := \{(\beta, \varphi) \in \ker D' : D''(\beta, \varphi) + [\beta, \varphi] \perp \mathcal{H}^2\}.$$

Restricted to $\tilde{Z}$, κ takes values in $\mathcal{H}^1$. Moreover, κ is locally invertible at the origin. There exists an open subset $\tilde{U}$ of $\tilde{Z}$ containing the origin, such that $\kappa|_{\tilde{U}}$ is a local biholomorphism onto an open set $U = \kappa(\tilde{U})$ of $\mathcal{H}^1$ containing 0. By shrinking $\tilde{U}$ if needed, we may take U to be an open ball centered at 0. Then, the restriction $\kappa|_{\tilde{U}}$ is K-equivariant. Let $\kappa^{-1} : U \to \Omega^{0,1}(\mathfrak{sl}(E)) \oplus \Omega^{1,0}(\mathfrak{sl}(E))$ be the inverse of κ.

Let

$$Z := \{(\beta, \varphi) \in \ker D' : D''(\beta, \varphi) + [\beta, \varphi] = 0\} \subset \tilde{Z}. \tag{2.66}$$

Note that $(\overline{\partial}_E, \Phi) + Z \subset \mathcal{B}$. For $u = (\beta, \varphi) \in Z$, we will use interchangeably

$$(\overline{\partial}_u, \Phi_u) = (\overline{\partial}_E, \Phi) + u = (\overline{\partial}_E + \beta, \Phi + \varphi)$$

to denote the Higgs bundle translated from $(\overline{\partial}_E, \Phi)$ by u.

Define $k : \mathcal{H}^1 \to \mathcal{H}^2$ by

$$k(x) := \frac{1}{2} P[x, x], \tag{2.67}$$

where P is the projection to $\mathcal{H}^2$. Let $\mathcal{Z} := k^{-1}(0) \subset \mathcal{H}^1$. We remark that (cf. Theorem 5.1, [Fan22]) for $(\beta, \varphi) \in \tilde{Z}$,

$$D''(\beta, \varphi) + [\beta, \varphi] = k(\kappa(\beta, \varphi)),$$

from which it follows that $\kappa(Z) \subset \mathcal{Z}$. By local biholomorphicity, $\kappa(Z \cap \tilde{U}) = \mathcal{Z} \cap U$ for U sufficiently small. We will refer to either one of Z, $\mathcal{Z}$, $Z \cap \tilde{U}$, $\kappa(Z) \cap U$, or their restrictions to smaller open neighborhoods as the *Kuranishi space*. For U sufficiently small, we have the map $\theta := \kappa^{-1} : \kappa(Z) \cap U \to \mathcal{B}^{ss}$, defined by

$$\theta(x) = (\overline{\partial}_E + (\kappa^{-1}(x))^{0,1}, \Phi + (\kappa^{-1}(x))^{1,0}). \tag{2.68}$$

Remark 2.6.1. When $(\overline{\partial}_E, \Phi)$ is stable, $\mathcal{H}^0 = \{0\}$, the slice method in [Par82] applies to the case in [Hit87], providing local charts for the moduli space of stable Higgs bundles. In the case when H^0 is non-trivial, more work is needed, as is done in [Fan22].

Remark 2.6.2. Strictly speaking, in order to apply the implicit function theorem for various constructions in the above, we need to pass to the Sobolev completion of the gauge groups and the configuration space, which is by now a standard practice in gauge theory. We refer to [AB82] for the case of complex structures of vector bundles over a Riemann surface, and [Pal68] for the foundation of the analytic theory.

As $\mathcal{H}^1$ is a G-representation, there is a corresponding notion of stability for vectors in $\mathcal{H}^1$. One key step in the construction in [Fan22] is to relate polystability in $\mathcal{H}^1$ with polystability of the corresponding Higgs bundles, for vectors in $\kappa(Z) \cap U$ with U sufficiently small.

We are now ready to describe the construction of the local model around $[(\overline{\partial}_E, \Phi)]$. For a sufficiently small open ball U around the origin, $G \cdot (\mathcal{Z} \cap U)$ is an open G-invariant subset of $\mathcal{Z}$. We have the analytic Hilbert quotient

$$p : \mathcal{Z} \to \mathcal{Z} /\!\!/ G.$$

For the details of the analytic Hilbert quotient, we refer to [HMP98]. Important properties of an analytic Hilbert quotient includes the following, which we follow the formulation in [Fan22]:

1. for a p-saturated open set W of $\mathcal{Z}$, the restriction $p : W \to p(W)$ is an analytic Hilbert quotient.

2. If $Y \subset \mathcal{Z}$ is a G-invariant closed complex subspace, then $p(Y)$ is closed and the restriction is an analytic Hilbert quotient.

3. Every fibre of p contains a unique closed G-orbit.

We claim that $G \cdot (\mathcal{Z} \cap U)$ is a p-saturated open set of $\mathcal{Z}$. Let r be the radius of U and $x \in \mathcal{Z}$ such that $p(x) = p(z)$ for some $z \in G \cdot (\mathcal{Z} \cap U)$. Without loss of generality, we assume $z \in U$. Let $y \in \overline{G \cdot x}$ be a point that has the smallest norm among the points in $\overline{G \cdot x}$. Then $||y|| \leq ||z|| < r$. On the other hand, there exists a sequence $\{x_i\} \subset G \cdot x$ with $x_i \to y$. Hence, for a sufficiently large i, $||x_i|| < r$, so $x \in G \cdot (\mathcal{Z} \cap U)$.

It follows that we have an analytic Hilbert quotient $p : G \cdot (\mathcal{Z} \cap U) \to G \cdot (\mathcal{Z} \cap U) /\!\!/ G$. We have the following local parameterization of $\mathcal{M}_H$ around $[(\overline{\partial}_E, \Phi)]$.

Proposition 2.6.3 ([Fan22], Section 3)**.** *For a sufficiently small ball U in $\mathcal{H}^1$ centered at 0, the continuous map $\mathcal{Z} \cap U \xrightarrow{\theta} \mathcal{B}^{ss} \longrightarrow \mathcal{M}_H$ induces an open embedding*

$$G \cdot (\mathcal{Z} \cap U) /\!\!/ G \xrightarrow{\pi} \mathcal{M}_H.$$

In fact, if $x \in \mathcal{Z} \cap U$ is polystable, then $\theta(x)$ is a polystable Higgs bundle.

2.7 System of Hodge bundles (SHB)

The moduli space $\mathcal{M}_H$ admits a $\mathbb{C}^*$-action $t \cdot [(\overline{\partial}_E, \Phi)] \mapsto [(\overline{\partial}_E, t\Phi)]$. For $[(\overline{\partial}_E, \Phi)] \in \mathcal{M}_H$, as a consequence of the properness of the Hitchin fibration, the limit

$$\lim_{t \to 0} [(\overline{\partial}_E, t\Phi)]$$

always exists, and we shall call it the *downward limit* of $[(\overline{\partial}_E, \Phi)]$. Alternatively, the existence of downward limits can be deduced from the constructions in [Sim10].

By a $\mathbb{C}^*$-fixed point, we mean a point in $\mathcal{M}_H$ fixed by the $\mathbb{C}^*$-action. Downward limits are always $\mathbb{C}^*$-fixed points. A polystable representative of a $\mathbb{C}^*$-fixed point can be written as a *system of Hodge bundles*.

We follow the convention in [CW19] in stating the following

Definition 2.7.1. A system of Hodge bundles (SHB for short) is a direct sum of holomorphic vector bundles

$$(E, \overline{\partial}_E) = \bigoplus_{a=1}^{k} (E_a, \overline{\partial}_a)$$

together with a Higgs field

$$\Phi = \sum_{a=1}^{k-1} \Phi_a,$$

where $\Phi_a \in \Omega^{1,0}(\mathrm{Hom}(E_a, E_{a+1})$ and $\overline{\partial}_E \Phi_a = 0$.

Suppose we have a Higgs bundle $(\overline{\partial}_E, \Phi)$, we say it admits an SHB structure if there exists a direct sum decomposition as in the definition above. When an SHB structure on $(\overline{\partial}_E, \Phi)$ is fixed, we will sometimes refer to the underlying Higgs bundle as an SHB.

If $(\overline{\partial}_E, \Phi)$ is an SHB, then it is isomorphic to $(\overline{\partial}_E, t\Phi)$ for any $t \in \mathbb{C}^*$. An isomorphism can be constructed as follows. Let

$$g_t = \begin{pmatrix} t^{n_1}\mathrm{Id}_{E_1} & & \\ & \ddots & \\ & & t^{n_k}\mathrm{Id}_{E_k} \end{pmatrix} \tag{2.69}$$

where $\sum_{a=1}^{k} n_k \mathrm{rk}(E_a) = 0$ and $n_a - n_{a+1} = 1$ for $1 \leq a \leq k-1$. Then $\overline{\partial}_E g_t = 0$ and $g_t^{-1}\Phi g_t = t\Phi$, so g_t is an isomorphism from $(\overline{\partial}_E, t\Phi)$ to $(\overline{\partial}_E, \Phi)$.

From Lemma 4.1 of [Sim92], the converse is also true. For completeness, we record the proof below.

Proposition 2.7.2 ([Sim92], Lemma 4.1)**.** *Suppose $(\overline{\partial}_E, \Phi)$ and $(\overline{\partial}_E, t\Phi)$ are gauge equivalent for some $t \in \mathbb{C}^*$ which is not a root of unity, then $(\overline{\partial}_E, \Phi)$ admits an SHB structure. In particular, if $(\overline{\partial}_E, \Phi)$ is stable, the SHB structure is unique.*

Proof. Since $(\overline{\partial}_E, \Phi)$ and $(\overline{\partial}_E, t\Phi)$ are isomorophic, there exists $g \in \mathcal{G}^{\mathbb{C}}$ such that $\overline{\partial}_E g = 0$ and

$$g\Phi = t\Phi g. \tag{2.70}$$

Hence, the characteristic polynomial of g has constant coefficients over X. In particular, the eigenvalues of g are constant. Let Λ be the set of eigenvalues. Decompose E into generalized eigenspaces

$$E = \bigoplus_{\lambda \in \Lambda} E_\lambda$$

where each E_λ is the kernel of $(g - \lambda \mathrm{Id})^r$ and hence is holomorphic. As g is invertible, $0 \notin \Lambda$.

From (2.70), a direct computation shows that

$$(g - t\lambda \mathrm{Id})^r \Phi = t^r \Phi (g - \lambda \mathrm{Id})^r,$$

so Φ sends E_λ to $E_{t\lambda}$. Iterate in both directions, we obtain a sequence of subbundles $\{E_{t^n\lambda}\}_{n\in\mathbb{Z}}$, where $E_{t^n\lambda} = 0$ for $|n|$ sufficiently large. Since t is not a root of unity, the non-trivial factors are pairwise distinct. The direct sum

$$E' := \bigoplus_n E_{t^n\lambda}$$

then has an SHB structure. If $E \neq E'$, we repeat this process for the remaining eigenvalues. Combining these chains, we obtain an SHB structure on $(\overline{\partial}_E, \Phi)$.

The last statement follows from the uniqueness of g. □

Chapter 3: The automorphism group of a polystable Higgs bundle

In this chapter, we collect some facts about the automorphism group of a polystable Higgs bundle and its harmonic metrics, some of which can be found in [GPGiR09]. While these are well-known, we will develop everything from the ground up here.

Let $(\overline{\partial}_E, \Phi)$ be a polystable $\mathrm{SL}(r, \mathbb{C})$-Higgs bundle. By the non-abelian Hodge correspondence, there exists a harmonic metric h, such that the connection

$$D := \overline{\partial}_E + \partial_E^h + \Phi + \Phi^{*_h}$$

is flat. In the following, we will omit the subscript h in the hermitian adjoint of any endomorphism. Recall that we have defined the operators $D'' = \overline{\partial}_E + \Phi$ and $D' = \partial^h + \Phi^*$ in the deformation complex

$$C^\bullet : \ \Omega^0(\mathfrak{sl}(E)) \xrightarrow{D''} \Omega^{0,1}(\mathfrak{sl}(E)) \oplus \Omega^{1,0}(\mathfrak{sl}(E)) \xrightarrow{D''} \Omega^{1,1}(\mathfrak{sl}(E)) \ .$$

Let G be the automorphism group of $(\overline{\partial}_E, \Phi)$ and $K = K(h) \subset G$ be the subgroup that preserves h. By definition, for $g \in G$, $\overline{\partial}_E g = 0$ and $[\Phi, g] = 0$. Hence, G sits inside the finite dimensional vector space

$$V := \{s \in \Omega^0(\mathrm{End}(E)) : D''s = 0\}. \tag{3.1}$$

In particular, elements in V are holomorphic, so $\mathrm{tr}\, s \in H^0(\mathcal{O}_X)$ and $\det s \in H^0(\mathcal{O}_X)$ are constant for all $s \in V$. For any $p \in X$, we have the evaluation maps

$$\begin{aligned} \mathrm{tr}_p\, s &:= \mathrm{tr}\, s_p \\ \det{}_p s &:= \det s_p \end{aligned}$$

which are independent of p. Hence, we will drop the subscript p. The automorphism group G may be alternatively characterized by $G = \det^{-1}(1)$. The map tr defines a $\mathbb{C}$-linear map with kernel $\ker \mathrm{tr} \subset \Omega^0(\mathfrak{sl}(E))$. As the identity $\mathrm{Id} \in V$ and $\mathrm{trId} = r$, we have

$$V = \ker D'' \oplus \langle \mathrm{Id} \rangle = H^0(C^\bullet) \oplus \langle \mathrm{Id} \rangle. \tag{3.2}$$

Lemma 3.0.1. *Let $g \in \Omega^0(\mathfrak{sl}(E))$. Then $D''g = 0$ if and only if $Dg = 0$. In this case, we have $D''g = D'g = 0$.*

Proof. First, note that $D \circ D = 0$ implies $D'D'' + D''D' = 0$. Recall the Kähler identities $(D'')^* = -i[\Lambda, D'']$ and $(D')^* = i[\Lambda, D'']$. We will use $\langle \cdot, \cdot \rangle$ to denote the inner product induced by h.

Suppose $D''g = 0$. Then

$$\langle D'g, D'g \rangle = \langle g, (D')^* D'g \rangle = \langle g, i\Lambda D''D'g - iD''\Lambda D'g \rangle = 0.$$

Here, we used $D''D' = -D'D''$ and $\Lambda D'g = 0$. Consequently, $D'g = 0$ and hence $Dg = 0$.

Conversely, suppose $Dg = 0$. The same argument shows that $||Dg||^2 = ||D'g||^2 + ||D''g||^2$. Therefore, $D''g = D'g = 0$. □

In particular, Lemma 3.0.1 says that elements in $\ker D''$ are D-covariantly constant. As Id is obviously covariantly constant, elements in V are covariantly constant. Consequently, for any point $p \in X$, the evaluation map $\mathrm{ev}_p : V \to \mathrm{End}(E_p)$ is injective, and maps tr_p and $\det_p$ factor through the corresponding maps on $\mathrm{End}(E_p)$ respectively. This realizes G as a closed linear subgroup of $\mathrm{SL}(r, \mathbb{C})$.

Proposition 3.0.2. *V and G are closed under taking the h-hermitian adjoint: if $g \in V$ or $g \in G$, then so is g^*. It follows that $K(h)$ is precisely the fix point set of the involution $g \mapsto (g^*)^{-1}$ in G.*

Proof. We first prove the case for V. Without loss of generality, assume $g \in \ker D''$. As $\mathrm{tr}\, g^* = \overline{\mathrm{tr}\, g} = 0$, $g^* \in \Omega^0(\mathfrak{sl}(E))$. We compute

$$(D''g^*)^* = (\overline{\partial}_E g^* + \Phi g^* - g^*\Phi)^* = (\partial_E^h - \Phi^*)g.$$

By Lemma 3.0.1, $D''g = D'g = 0$, which implies $\partial_E^h g = 0$ and $[\Phi^*, g] = 0$. It follows that $D''g^* = 0$.

Now, suppose $g \in G$. Then $g^* \in V$. As $\det g^* = \overline{\det g} = 1$, $g^* \in G$. □

Proposition 3.0.3. *1. $Lie(G) = \ker D''$.*

2. $Lie(K) = \ker D'' \cap \Omega^0(\mathfrak{su}(E, h))$, and $Lie(K)_{\mathbb{C}} = Lie(G)$.

Proof. Let $G' := \det^{-1}(\mathbb{C}^*) \subset V$. It is an open subset of V, and a complex Lie group with Lie

algebra V. We have the diagram

$$\begin{array}{ccccc} \ker D'' & \longrightarrow & V & \xrightarrow{\text{tr}} & \mathbb{C} \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\ \ker(\det) & \longrightarrow & G' & \xrightarrow{\det} & \mathbb{C}^* \end{array}$$

Note that $\ker(\det) = G$. So $\text{Lie}(G) = \ker D''$.

The hermitian adjoint $*$ defines an involution on $\ker D''$ and splits $\ker D''$ into $+1$- and -1-eigenspaces. Let $\mathfrak{k}$ be the -1-eigenspace. Multiplication by i exchanges the two eigenspaces. These imply

$$\mathfrak{k} \oplus i\mathfrak{k} = \ker D''.$$

Note that taking hermitian adjoint is a fibrewise operation, so it is compatible with the embedding $V \hookrightarrow \text{End}(E_p)$ for any $p \in X$. This means we have $\mathfrak{k} = \ker D'' \cap \mathfrak{u}(E_p, h_p)$, and thus $\mathfrak{k} = \text{Lie}(K)$. □

The following consequences are immediate.

Corollary 3.0.4. *The complexification $(K^\circ)_\mathbb{C} = G^\circ_\mathbb{C}$. The connected component $G^\circ_\mathbb{C}$ has a maximal compact K°.*

Next, we turn our attention to harmonic metrics of $(\overline{\partial}_E, \Phi)$. Let $\mathcal{HM} = \mathcal{HM}(\overline{\partial}_E, \Phi)$ be the set of harmonic metrics of $(\overline{\partial}_E, \Phi)$. By definition, we have $h \in \mathcal{HM}$. The first thing to notice is that G acts on $\mathcal{HM}$ via pulling back.

Lemma 3.0.5. *Let $h' \in \mathcal{HM}$ and $g \in G$. Then $g^*h' := h'(g\cdot, g\cdot) \in \mathcal{HM}$.*

Proof. Let $h'_2 = g^*h'$. From $\overline{\partial}_E g = 0$ and the relation $h'_2(s, \partial_E^{h'_2} t) = \overline{\partial}_E h'_2(s,t) - h'_2(\overline{\partial}_E s, t)$, a

direct computation leads to the relation

$$\partial^{h_2'} = g^{-1} \circ \partial^{h'} \circ g.$$

In the following, we write $*_{h'} = *$ for convenience. We have

$$\Phi^{*_{h_2'}} = g^{-1}(g^{-1})^* \Phi^* g^* g = g^{-1}(g\Phi g^{-1})^* g = g^{-1}\Phi^* g$$

since $g\Phi g^{-1} = \Phi$ for all $g \in G$. Altogether, we have

$$\begin{aligned} F_{(\overline{\partial}_E, h_2')} + [\Phi, \Phi^{*_{h_2'}}] &= g^{-1} F_{(\overline{\partial}_E, h')} g + [\Phi, g^{-1}\Phi^* g] \\ &= g^{-1} F_{(\overline{\partial}_E, h')} g + [g^{-1}\Phi g, g^{-1}\Phi^* g] \\ &= g(F_{(\overline{\partial}_E, h')} + [\Phi, \Phi^*]) g^{-1} \\ &= 0. \end{aligned}$$

□

It follows that K is the stabilizer of h under the G-action on $\mathcal{HM}$.

Theorem 3.0.6. *The G-action on $\mathcal{HM}$ is transitive.*

Proof. For each $h_1, h_2 \in \mathcal{HM}$, let

$$(E, \overline{\partial}_E) = \bigoplus_{i=1}^{k} (E_i^{h_j}, \overline{\partial}_{h_j, i}) \quad \text{and} \quad \Phi = \sum_{i=1}^{k} \Phi_{h_j, i},$$

where $\Phi_{h_j,i} \in H^0(\text{End}(E_i^{h_j}) \otimes K_X)$, be the h_j-orthogonal direct sum decomposition of $(\overline{\partial}_E, \Phi)$

with stable factors $(\overline{\partial}_{h_j,i}, \Phi_{h_j,i})$, each with degree zero. Upon rearrangement if needed, there exists a holomorphic isomorphism g sending $(E_i^{h_1}, \Phi_{h_1,i})$ to $(E_i^{h_2}, \Phi_{h_2,i})$. Since harmonic metrics on stable $\mathrm{GL}(n,\mathbb{C})$-Higgs bundles are unique up to positive scalar multiples, we have

$$g^*(h_2|_{E_i^{h_2}}) = c_i h_1|_{E_i^{h_1}}$$

from some $c_i > 0$ that satisfy $\prod_{i=1}^k c_i^{\mathrm{rk}(E_i^{h_1})} = 1$.

Finally, let $g' \in G$ by $g'|_{E_i^{h_2}} = c_i^{-1/2}\mathrm{Id}$ for each i. Then $g'g \in G$ and $(g'g)^* h_2 = h_1$. □

Therefore, the set of harmonic metrics $\mathcal{HM}$ has the structure of the quotient space G/K.

Corollary 3.0.7. *The unitary connection $\overline{\partial}_E + \partial^h$ and Φ^* are independent of the choice of harmonic metric. Hence, D' and D are independent of the harmonic metric.*

Proof. If k is any other metric, then there exists a unique inveritble h-hermitian endomorphism A such that $k = h(A\cdot, \cdot)$ (in fact, $A = h^{-1} \circ k$). A similar computation as the above shows that

$$\partial_E^k = \partial_E^h - (\partial_E^h A^{-1})A \tag{3.3}$$

If $k \in \mathcal{HM}$, by Theorem 3.0.6 there exists $g \in G$ such that $k = g^*h$. In this case, we have $A = g^*g$, and

$$A^{-1}\partial^h A = g^{-1}(g^*)^{-1}\partial^h(g^*g) = 0.$$

since $g^*g \in G$ by Proposition 3.0.2. From (3.3), it follows that $\partial_E^k = \partial_E^h$.

The result now follows from $\Phi^{*_k} = A^{-1}\Phi^* A = \Phi^*$, using $g^* \in \mathrm{Aut}(\overline{\partial}_E, \Phi)$. □

Remark 3.0.8. Since $\overline{\partial}_E + \partial_E^h = \overline{\partial}_E + \partial_E^k$ for any harmonic metric $k \in \mathcal{HM}$, we see that the

Chern connection $\overline{\partial}_E + \partial^h$ is simultaneously unitary with respect to all harmonic metrics.

We can give an explicit description of G. With respect to h, as in the proof of Theorem 3.0.6, write

$$(E, \overline{\partial}_E) = \bigoplus_{i=1}^{k} (E_i, \overline{\partial}_i) \quad \text{and} \quad \Phi = \sum_{i=1}^{k} \Phi_i,$$

where $\Phi_i \in H^0(\mathrm{End}(E_i) \otimes K_X)$, be the h-orthogonal direct sum decomposition of $(\overline{\partial}_E, \Phi)$ with stable factors $(\overline{\partial}_i, \Phi_i)$, each with degree zero. Let $f \in \ker D''$. We can decompose f into

$$f = \sum_{i,j} f_{ij}$$

where $f_{ij} \in \Omega^0(\mathrm{Hom}(E_i, E_j))$ for each i and j. By assumption, each f_{ij} is holomorphic and $\Phi_j f_{ij} = f_{ij} \Phi_i$. Hence, when $f_{ij} \neq 0$, $(\overline{\partial}_i, \Phi_i)$ and $(\overline{\partial}_j, \Phi_j)$ are isomorphic and f_{ij} is an isomorphism. In this case, $\mathrm{Hom}((\overline{\partial}_i, \Phi_i), (\overline{\partial}_j, \Phi_j))$ is one-dimensional. These show that

Theorem 3.0.9. *The automorphic group of $(\overline{\partial}_E, \Phi)$ is isomorphic to $S(GL_{s_1} \times \cdots \times GL_{s_m})$, where $\sum_j s_j = k$.*

The following corollary is an immediate consequence.

Corollary 3.0.10. *$(\overline{\partial}_i, \Phi_i)$ and $(\overline{\partial}_j, \Phi_j)$ are pairwise non-isomorphic if and only if G is abelian.*

Chapter 4: Model of the central locus

This chapter is based on the author's paper [Kwo25].

Let $[(\overline{\partial}_E, \Phi)] \in \mathcal{M}_H$ be a $\mathbb{C}^*$-fixed point. If $(\overline{\partial}_E, \Phi)$ is stable, we have seen that $(\overline{\partial}_E, \Phi)$ admits an SHB structure.

We will define and construct a model for the central locus of the upward flow through $[(\overline{\partial}_E, \Phi)]$. In the case where $(\overline{\partial}_E, \Phi)$ is stable, this construction is the same as the global parametrization of the upward flow through $[(\overline{\partial}_E, \Phi)]$ obtained in [CW19]. When $(\overline{\partial}_E, \Phi)$ is strictly polystable, however, we need to take into account the non-trivial automorphism group G of $(\overline{\partial}_E, \Phi)$. Another difference compared with the stable case is that there is an ambiguity in decomposing $(\overline{\partial}_E, \Phi)$ into a system of Hodge bundles (to be recalled below), a step that is needed to decompose the deformation complex of $(\overline{\partial}_E, \Phi)$ into graded pieces.

4.1 Decomposition of a polystable $\mathbb{C}^*$-fixed point

Recall that a system of Hodge bundles (SHB) consists of

$$(E, \overline{\partial}_E) = \bigoplus_{a=1}^{k} (E_a, \overline{\partial}_a) \quad \text{and} \quad \Phi = \sum_{a=1}^{k-1} \Phi_a$$

where $\Phi_a \in \Omega^{1,0}(\mathrm{Hom}(E_a, E_{a+1})$ and $\overline{\partial}_E \Phi_a = 0$ for each a. We will call these subbundles E_a the *Hodge summands* of the SHB structure. Conversely, if $(\overline{\partial}_E, \Phi)$ is polystable and $[(\overline{\partial}_E, \Phi)]$ is a $\mathbb{C}^*$-fixed point, then $(\overline{\partial}_E, \Phi)$ can be endowed with an SHB structure, which is unique when $(\overline{\partial}_E, \Phi)$ is stable. The summands E_a are mutually orthogonal with respect to the harmonic metric of $(\overline{\partial}_E, \Phi)$.

In this article, there is one SHB structure for a polystable $(\overline{\partial}_E, \Phi)$ that is convenient for our purposes. Its construction will be carried out in the next section. In preparation for that, we first establish the following

Lemma 4.1.1. *Suppose $(\overline{\partial}_E, \Phi)$ is polystable and $[(\overline{\partial}_E, \Phi)] \in \mathcal{M}_H$ is a $\mathbb{C}^*$-fixed point. Then $(\overline{\partial}_E, \Phi)$ can be written as a direct sum of stable Higgs bundles which represent $\mathbb{C}^*$-fixed points.*

Proof. First, we claim that all $t \in \mathbb{C}^*$, $(\overline{\partial}_E, t\Phi)$ lie in the same gauge equivalence class. This is true for $t \in S^1$ since $[(\overline{\partial}_E, \Phi)] = [(\overline{\partial}_E, t\Phi)]$ and the same harmonic metrics solve the Hitchin equation for each $(\overline{\partial}_E, t\Phi)$. Pick a t, $|t| = 1$, which is not a root of unity. Then there exists an isomorphism f_t from $(\overline{\partial}_E, \Phi)$ to $(\overline{\partial}_E, t\Phi)$. In particular, f_t is holomorphic with respect to the holomorphic structure defined by $\overline{\partial}_E$. Proved in Lemma 4.1 of [Sim92], the generalized eigenspaces of f_t give $(\overline{\partial}_E, \Phi)$ a system of Hodge bundles structure, with which we can construct isomorphisms to all $(\overline{\partial}_E, t\Phi)$, $t \in \mathbb{C}^*$.

Next, we decompose $(\overline{\partial}_E, \Phi)$ into a direct sum of stable summands $\bigoplus_{i=1}^k (\overline{\partial}_{E^i}, \Phi^i)$. We claim each $(\overline{\partial}_{E^i}, \Phi^i)$ is a $\mathbb{C}^*$-fixed point in the moduli space of degree zero polystable Higgs bundles of the corresponding ranks. Pick a non-zero scalar $t \in \mathbb{C}^*$ which is not a root of unity. Each $(\overline{\partial}_{E^i}, t\Phi^i)$ is stable and we have $(\overline{\partial}_E, \Phi) \cong (\overline{\partial}_E, t\Phi) = \bigoplus_i (\overline{\partial}_{E^i}, t\Phi^i)$. It follows that $(\overline{\partial}_{E^i}, t\Phi^i) \cong (\overline{\partial}_{E^{\sigma(i)}}, \Phi^{\sigma(i)})$ for some permutation σ of $\{1, \cdots, k\}$. Composing these isomorphisms,

we have $(\overline{\partial}_{E^i}, t^j\Phi^i) \cong (\overline{\partial}_{E^{\sigma^j(i)}}, \Phi^{\sigma^j(i)})$. Now, setting $j = \mathrm{ord}(\sigma)$, we have $(\overline{\partial}_{E^i}, \Phi^i) \cong (\overline{\partial}_{E^i}, t^j\Phi^i)$. Applying the procedure in Lemma 4.1 of [Sim92] again, we see that each $[(\overline{\partial}_{E^i}, \Phi^i)]$ is a $\mathbb{C}^*$-fixed point. □

We will also need the following lemma, which roughly speaking says that taking downward limits commutes with direct sums.

Lemma 4.1.2. *Let $(\mathcal{E}, \overline{\partial}_E, \Phi)$ be a polystable Higgs bundle. Suppose $(\mathcal{E}, \overline{\partial}_E, \Phi) = \bigoplus_{i=1}^k (\mathcal{E}^i, \overline{\partial}^i, \Phi^i)$, where each summand $(\mathcal{E}^i, \Phi^i)$ is a stable Higgs bundle with downward limit $\lim_{t\to 0}[(\mathcal{E}^i, \overline{\partial}^i, t\Phi^i)] = [(\mathcal{E}^i_0, \overline{\partial}^i_0, \Phi^i_0)]$ represented by $(\overline{\partial}^i_0, \Phi^i_0)$ which is polystable. Then*

$$\lim_{t\to 0}[(\mathcal{E}, \overline{\partial}_E, t\Phi)] = [\bigoplus_{i=1}^k (\mathcal{E}^i_0, \overline{\partial}^i_0, \Phi^i_0)].$$

Proof. For each i, using Lemma 4.1.1, we put an SHB structure on $(\overline{\partial}^i_0, \Phi^i_0)$ in the form (4.3) and fix a harmonic metric h_i. Let G_i be the automorphism group of $(\overline{\partial}^i_0, \Phi^i_0)$, U_i a small ball around 0 in $\mathcal{H}^1$ of the deformation complex of $(\overline{\partial}^i_0, \Phi^i_0)$ as in Proposition 2.6.3, $\mathcal{Z}_i$ the Kuranishi space and let

$$\pi_i : G_i \cdot (\mathcal{Z}_i \cap U_i) /\!\!/ G_i \to \mathcal{M}_H(r_i, 0)$$

be the local Kuranishi model into the moduli space of semistable Higgs bundles of rank r_i and degree zero, where $r_i = \mathrm{rk}(\mathcal{E}^i)$. For small $|t|$, every $[(\overline{\partial}^i, t\Phi^i)]$ has representatives x^t_i in U_i, i.e. $\pi_i([x^t_i]) = [(\overline{\partial}^i, t\Phi^i)]$. Shrinking the radius of each U_i to zero, we see that there exists a sequence $t_n \to 0$ such that $x^{t_n}_i \to 0$ for all i. For each $x^{t_n}_i$, choose $y^{t_n}_i \in \overline{G_i \cdot x^{t_n}_i}$ which minimizes the h_i-norm along $\overline{G_i \cdot x^{t_n}_i}$. Then $y^{t_n}_i \in U_i$, $y^{t_n}_i \to 0$ as $n \to 0$ and the G_i-orbit of each $y^{t_n}_i$ is closed. By Proposition 2.6.3, $\theta(y^{t_n}_i)$ is gauge-equivalent to $(\overline{\partial}^i, t_n\Phi^i)$ (for convenience, here

we use the same notation θ to denote the inverses of the Kuranishi maps with respect to all $(\overline{\partial}_0^i, \Phi_0^i)$). It follows that $(\overline{\partial}_E, t_n\Phi)$ is gauge equivalent to $\bigoplus_{i=1}^k \theta(y_i^{t_n})$. For each n, we may take the gauge transformations to lie in SL. Hence, $\lim_{n\to 0}[(\overline{\partial}_E, t_n\Phi)] = [\bigoplus_{i=1}^k (\overline{\partial}_0^i, \Phi_0^i)]$. Since $\lim_{t\to 0}[(\overline{\partial}_E, t\Phi)]$ exists, the result follows. □

4.2 The deformation complex at a polystable SHB

Now, suppose $[(\overline{\partial}_E, \Phi)]$ is a $\mathbb{C}^*$-fixed point, with $(\overline{\partial}_E, \Phi)$ polystable. Let G be its automorphism group. We will specify an SHB structure on $(\overline{\partial}_E, \Phi)$ in the following manner. By Lemma 4.1.1, there exists a holomorphic decomposition into k stable summands

$$(E, \overline{\partial}_E) = \bigoplus_{i=1}^k (E^i, \overline{\partial}^i) \quad \text{and} \quad \Phi = \sum_{i=1}^k \Phi^i, \tag{4.1}$$

where $\Phi^i \in H^0(\mathrm{End}(E^i) \otimes K_X)$, such that each $(\overline{\partial}^i, \Phi^i)$ is a stable Higgs bundle of degree zero representing a $\mathbb{C}^*$-fixed point. Each $(\overline{\partial}^i, \Phi^i)$ admits a unique SHB structure given by

$$(E^i, \overline{\partial}^i) = \bigoplus_{a=1}^{\ell_i} (E_a^i, \overline{\partial}_a^i) \quad \text{and} \quad \Phi^i = \sum_{a=1}^{\ell_i - 1} \Phi_a^i \tag{4.2}$$

where $\Phi_a^i \in H^0(\mathrm{Hom}(E_a^i, E_{a+1}^i) \otimes K_X)$. Let

$$(E_a, \overline{\partial}_a) := \bigoplus_i (E_a^i, \overline{\partial}_a^i) \quad \text{and} \quad \Phi_a := \sum_i \Phi_a^i \tag{4.3}$$

so that $\Phi_a \in H^0(\mathrm{Hom}(E_a, E_{a+1}) \otimes K_X)$. The direct sum $(E, \overline{\partial}_E) = \bigoplus_{a=1}^{\ell} (E_a, \overline{\partial}_a)$, where $\ell = \max_i\{\ell_i\}$. (4.3) then defines an SHB structure on $(\overline{\partial}_E, \Phi)$. We can take a harmonic metric h

of $(\overline{\partial}_E, \Phi)$ such that the summands E^i and E^j are mutually orthogonal with respect to h. Thus, E_a and E_b are mutually orthogonal. Henceforth, this is be the SHB structure we will refer to. We remark that the resulting Hodge summands (4.3) are independent of the choice of the stable summand decomposition (4.1).

Next, we recall the deformation complex (2.62). As in [CW19], we introduce a grading on $\mathrm{End}(E)$, whose ℓ-th component is given by

$$\mathrm{End}_\ell(E) := \bigoplus_{i-j=\ell} \mathrm{Hom}(E_i, E_j). \tag{4.4}$$

There is a $\mathbb{C}^*$-action on $C^\bullet$ defined in the following way. Let $w \in \Omega^0(\mathfrak{sl}(E))$, $(\beta, \varphi) \in \Omega^{0,1}(\mathfrak{sl}(E)) \oplus \Omega^{1,0}(\mathfrak{sl}(E))$ and $v \in \Omega^{1,1}(\mathfrak{sl}(E))$. We will add a subscript j to denote the j-th graded component with respect to the grading on $\mathrm{End}(E)$. For $t \in \mathbb{C}^*$, the action is given by

$$t \cdot w := \sum_j t^j w_j \tag{4.5}$$

$$t \cdot (\beta, \varphi) := \left(\sum_j t^j \beta_j, \sum_j t^{j+1} \varphi_j \right) \tag{4.6}$$

$$t \cdot v := \sum_j t^{j+1} v_j. \tag{4.7}$$

It is a direct computation to verify that D'' and $(D'')^*$ are $\mathbb{C}^*$-equivariant. The deformation complex is then decomposed into subcomplexes

$$(C^\bullet)_j : \Omega^0(\mathfrak{sl}(E)_j) \xrightarrow{D''} \Omega^{0,1}(\mathfrak{sl}(E)_j) \oplus \Omega^{1,0}(\mathfrak{sl}(E)_{j-1}) \xrightarrow{D''} \Omega^{1,1}(\mathfrak{sl}(E)_{j-1}). \tag{4.8}$$

The space of harmonic forms are decomposed accordingly: $\mathcal{H}^i = \bigoplus_j (\mathcal{H}^i)_j$ where $(\mathcal{H}^i)_j =$

$\mathcal{H}^i((C^\bullet)_j)$.

Lemma 4.2.1. *(i)* $\mathcal{H}^0 = (\mathcal{H}^0)_0$ *and* $\mathcal{H}^2 = (\mathcal{H}^2)_1$.

(ii) The automorphism group G acts by graded degree zero.

Proof. Elements of $\mathcal{H}^0$ are endomorphisms of $(\overline{\partial}_E, \Phi)$. Decompose them into $\mathrm{Hom}((\overline{\partial}^i, \Phi^i), (\overline{\partial}^j, \Phi^j))$. Notice that $\mathrm{Hom}((\overline{\partial}^i, \Phi^i), (\overline{\partial}^j, \Phi^j))$ is trivial when $(\overline{\partial}^i, \Phi^i)$ and $(\overline{\partial}^j, \Phi^j)$ are not equivalent and is one-dimensional otherwise. In the latter case, the non-zero elements preserve the SHB structures. Putting them back together, we see that elements in $\mathcal{H}^0$ preserve each E_i. This proves (ii) and the case for $\mathcal{H}^0$ in (i). The case for $\mathcal{H}^2$ follows from duality. □

Now, let $C^\bullet_+ := \bigoplus_{j>0}(C^\bullet)_j$ and $\mathcal{H}^1_+ := \mathcal{H}^1(C^\bullet_+) = \bigoplus_{j>0}(\mathcal{H}^1)_j$. The computation in Lemma 3.6 of [CW19] together with (i) of Lemma 4.2.1 prove the following

Proposition 4.2.2. $\chi(C^\bullet_+) = \dim \mathcal{H}^2 - \dim \mathcal{H}^1_+ = -(r^2-1)(g-1) = -\frac{1}{2}\dim \mathcal{M}_H$.

4.3 Model for the central locus

Associated with the polystable SHB $(\overline{\partial}_E, \Phi)$, we recall the Kuranishi map (2.65)

$$\kappa(\beta, \varphi) = (\beta, \varphi) + (D'')^*\Gamma([\beta, \varphi]).$$

In addition to the G-equivariance, κ is also $\mathbb{C}^*$-equivariant with respect to the $\mathbb{C}^*$-action on $\Omega^{0,1}(\mathfrak{sl}(E)) \oplus \Omega^{1,0}(\mathfrak{sl}(E))$ defined in the previous section. Moreover, the two actions commute with each other.

Define the space

$$S_+ = S_+(\overline{\partial}_E, \Phi) := C^1_+ \cap Z = C^1_+ \cap \{(\beta, \varphi) : D''(\beta, \varphi) + [\beta, \varphi] = D'(\beta, \varphi) = 0\}, \tag{4.9}$$

which is a closed complex subspace of Z. By examining the graded pieces, we see that the restriction of κ to S_+ takes values in $\mathcal{H}^1_+$. Also, note that both S_+ and $\mathcal{H}^1_+$ are preserved by $G \times \mathbb{C}^*$. In [CW19], where $(\overline{\partial}_E, \Phi)$ is assumed to be stable, S_+ is referred to as the BB-*slice*. In that case, the image of S_+ in $\mathcal{M}_H$ is the upward flow through $[(\overline{\partial}_E, \Phi)]$. However, when $(\overline{\partial}_E, \Phi)$ is only assumed to be polystable, S_+ almost never captures the full upward flow. In any case, κ gives a biholomorphism between the S_+ and $\mathcal{H}^1_+$.

Lemma 4.3.1. *The restriction $\kappa : S_+ \to \mathcal{H}^1_+$ is a G- and $\mathbb{C}^*$-equivariant biholomorphism.*

Proof. We claim that for $x = (\alpha, \upsilon) \in \mathcal{H}^1_+$, there exists a unique $(\beta, \varphi) \in S_+$ with $\kappa(\beta, \varphi) = x$. It amounts to solving

$$(\alpha, \upsilon) = (\sum_{j>0} \beta_j, \sum_{\ell \geq 0} \varphi_\ell) + iD'\Lambda\Gamma\left(\sum_{j,\ell} [\beta_j, \varphi_\ell]\right). \tag{4.10}$$

As each $[\beta_j, \varphi_\ell]$ has graded degree $j + \ell$, we see that in particular $(\beta_1, \varphi_0) = (\alpha_1, \upsilon_0)$. Next, we solve inductively,

$$(\beta_{j+1}, \varphi_j) = (\alpha_{j+1}, \upsilon_j) - (D'')^*\Gamma\left(\sum_{a+b=j} [\beta_a, \varphi_b]\right). \tag{4.11}$$

Putting these graded pieces together determine the solution (β, φ) uniquely.

Lastly, we check that $(\beta, \varphi) \in S_+$. Since $D'x = 0$, from (4.10) we have $D'(\beta, \varphi) = 0$. Similarly, we have $0 = D''x = D''(\beta, \varphi) + (1 - P)([\beta, \varphi]) = D''(\beta, \varphi) + [\beta, \varphi] - P([\beta, \varphi])$.

Since $[\beta, \varphi]$ has no non-negative graded degree pieces, $P([\beta, \varphi]) = 0$ by Lemma 4.2.1. Therefore, $(\beta, \varphi) \in S_+$. From (4.11), we see that the inverse map is holomorphic. $\square$

Hence, κ induces a $\mathbb{C}^*$-equivariant analytic isomorphism

$$\overline{\kappa} : S_+ /\!\!/ G \xrightarrow{\cong} \mathcal{H}^1_+ /\!\!/ G. \tag{4.12}$$

Note that $\mathcal{H}^1_+ /\!\!/ G$ here is an affine GIT quotient. At the pre-quotient level, the resulting map of the following composition

$$S_+ \to (\overline{\partial}_E, \Phi) + S_+ \hookrightarrow \mathcal{B} \to \mathcal{B}/\mathcal{G}^{\mathbb{C}}$$

is $\mathbb{C}^*$-equivariant. As $(\overline{\partial}_E, \Phi) + S_+ \cap \tilde{U}$ consists of semistable Higgs bundles for small open $\tilde{U}$ containing 0, the $\mathbb{C}^*$-equivariance implies that we have the global map

$$S_+ \longrightarrow \mathcal{M}_H.$$

Composing with κ^{-1}, we also have the map $\mathcal{H}^1_+ \to \mathcal{M}_H$. When $(\overline{\partial}_E, \Phi)$ is stable, it is shown in [CW19] that this gives a parametrization of the upward flow through $[(\overline{\partial}_E, \Phi)]$. This is not the case if $(\overline{\partial}_E, \Phi)$ is strictly polystable. Nonetheless, using $\mathbb{C}^*$-equivariance again and the local complex analyticity of the map $(S_+ \cap G \cdot \tilde{U}) /\!\!/ G \hookrightarrow \mathcal{M}_H$, we obtain a global injective $\mathbb{C}^*$-equivariant complex analytic map $L : S_+ /\!\!/ G \to \mathcal{M}_H$ given by

$$L([(\beta, \varphi)]) = [(\overline{\partial}_u, \Phi_u)]. \tag{4.13}$$

We call the image of L the *central locus* of the upward flow through $[(\overline{\partial}_E, \Phi)]$. Composing with $\overline{\kappa}^{-1}$, we then obtain

Proposition 4.3.2. *The map $L \circ \overline{\kappa}$ gives a parametrization of the central locus of the upward flow through the polystable $\mathbb{C}^*$-fixed point $[(\overline{\partial}_E, \Phi)]$ by the affine quotient $\mathcal{H}^1_+ /\!\!/ G$.*

For further discussion about the central locus, it will be of interest to us to consider certain subloci in S_+. To define them, we first set up some notation and terminology. Let $\{(\mathcal{E}^\alpha, \phi^\alpha)\}$ be the maximal set of pairwise non-equivalent stable Higgs bundles, each gauge equivalent to some $(\overline{\partial}^i, \Phi^i)$ in (4.1). Let C be the collection containing n_α copies of $(\mathcal{E}^\alpha, \phi^\alpha)$, where n_α is the number of copies of $(\mathcal{E}^\alpha, \phi^\alpha)$ in $(\overline{\partial}_E, \Phi)$. We do not distinguish members in C which are isomorphic to each other.

Define a *partition* (of C) to be a collection $P = \{P_\lambda\}$ of subcollections of C which partitions C. Let $P^C = \{C\}$ be the trivial partition. We say P is *proper* if $P \neq P^C$. Each $P_\lambda \in P$ defines a polystable Higgs bundle $(\overline{\partial}(P_\lambda), \Phi(P_\lambda))$ obtained from taking the direct sum of members in P_λ. Note that $(\overline{\partial}(P_\lambda), \Phi(P_\lambda))$ represents a $\mathbb{C}^*$-fixed point. Now, for each partition P, we fix an isomorphism between each factor in each P_λ and a stable factor of $(\overline{\partial}_E, \Phi)$ so that the resulting holomorphism $\bigoplus_\lambda (\overline{\partial}(P_\lambda), \Phi(P_\lambda)) \to (\overline{\partial}_E, \Phi)$ is an isomorphism of Higgs bundles. Endow each $(\overline{\partial}(P_\lambda), \Phi(P_\lambda))$ with the SHB structure laid out in (4.1), and fix a harmonic metric that is compatible with that of $(\overline{\partial}_E, \Phi)$. For each $(\overline{\partial}(P_\lambda), \Phi(P_\lambda))$, the induced $\mathbb{C}^*$-action allows one to define the model spaces, denoted by $S_+(P_\lambda)$ and $\mathcal{H}^1_+(P_\lambda) \subset \mathcal{H}^1(P_\lambda)$ respectively, as well as the biholomorphism coming from the restricted Kuranishi map

$$\kappa(P_\lambda) : S_+(P_\lambda) \to \mathcal{H}^1_+(P_\lambda).$$

Let $S_+(P) := \prod_\lambda S_+(P_\lambda)$ and $\mathcal{H}^1_+(P) := \prod_\lambda \mathcal{H}^1_+(P_\lambda)$. Notice that $S_+(P^C) \cong S_+$ and $\mathcal{H}^1_+(P^C) \cong \mathcal{H}^1_+$.

We introduce a partial ordering on partitions as follows. For two distinct partitions $P = \{P_\lambda\}$ and $P' = \{P'_{\lambda'}\}$, we write $P > P'$ if we can group members in P' into a disjoint union $P' = \coprod_\lambda Q_\lambda$ such that for each λ, the union of members in Q_λ is P_λ. Given a pair $P > P'$ and a grouping, we have the following $\mathbb{C}^*$-equivariant commutative diagram

$$\begin{array}{ccccc} S_+(P') & \xrightarrow{\prod_{\lambda'} \kappa(P'_{\lambda'})} & \mathcal{H}^1_+(P') & \longrightarrow & \mathcal{H}^1(P') \\ \downarrow & & \downarrow & & \downarrow \\ S_+(P) & \xrightarrow{\prod_\lambda \kappa(P_\lambda)} & \mathcal{H}^1_+(P) & \longrightarrow & \mathcal{H}^1(P) \end{array} \tag{4.14}$$

where the vertical arrows are closed embeddings.

Remark 4.3.3. In particular, since $P^C > P$ for any proper partition P, we may regard $S_+(P)$, $\mathcal{H}^1_+(P)$ and $\mathcal{H}^1(P)$ as closed subspaces of S_+, $\mathcal{H}^1_+$ and $\mathcal{H}^1$ respectively. In general, as subspaces of S_+, $\mathcal{H}^1_+$ and $\mathcal{H}^1$ respectively, $S_+(P)$, $\mathcal{H}^1_+(P)$ and $\mathcal{H}^1(P)$ are not preserved by G unless G is abelian. When G is abelian and $P = P^C$, (4.14) is G-equivariant in every arrow.

It is reasonable to expect that the (inverse) Kuranishi map θ (2.68) should give a correspondence between G-(poly)stability in $\mathcal{H}^1$ and (poly)stability of the image Higgs bundles. Indeed, it has been proved to be the case for holomorphic vector bundles in [BS20]. While this does not appear to follow immediately from the construction in [Fan22], for our purposes it suffices to have a partial correspondence along $\mathcal{H}^1_+$ as well as imposing the abelian assumption on G. The precise formulation is given in the following

Proposition 4.3.4. *Let $u = (\beta, \varphi) \in S_+(\overline{\partial}_E, \Phi) - \{0\}$, and assume $(\overline{\partial}_u, \Phi_u)$ is not gauge equivalent to $(\overline{\partial}_E, \Phi)$.*

(i) Suppose $(\overline{\partial}_u, \Phi_u)$ is stable. Then $\kappa(u)$ is stable in $\mathcal{H}^1_+$.

(ii) Suppose $(\overline{\partial}_u, \Phi_u)$ is strictly polystable and the automorphism group G of $(\overline{\partial}_E, \Phi)$ is abelian. Then $\overline{G \cdot \kappa(u)}$ contains a polystable vector y which lies in $\mathcal{H}^1_+(P)$ for some proper partition P of C.

Proof. Assume that $(\overline{\partial}_u, \Phi_u)$ is stable. Using the $\mathbb{C}^*$-action, we may assume that $x = \kappa(u) \in U$ where U is a sufficiently small ball for which Proposition 2.6.3 applies. Take a point $y \in \overline{G \cdot x}$ that is closest to the origin among points in $\overline{G \cdot x}$. In particular, $y \in U$ and $G \cdot y$ is closed. It follows from Proposition 2.6.3 that $\theta(y)$ is polystable and lies in the same gauge equivalence class as $(\overline{\partial}_u, \Phi_u)$. Hence, $\theta(y)$ is also stable. It implies that its isotropy group is trivial, and so is its restriction to G. As the Kuranishi map κ is a G-equivariant biholomorphism, y is stable. Since stability is an open condition, x is also stable and $y \in G \cdot x$. This proves (i).

Next, assume $(\overline{\partial}_u, \Phi_u)$ is strictly polystable and G is abelian. It follows that $\kappa(u)$ is semistable and that the stable factors in (4.1) are pairwise non-equivalent. Let $\bigoplus_{i=1}^k (\mathcal{E}^i_u, \overline{\partial}^i_u, \Phi^i_u) = (\overline{\partial}_u, \Phi_u)$ be its stable summand decomposition. From Lemma 4.1.2, we see that the downward limits of $(\mathcal{E}^i_u, \overline{\partial}^i_u, \Phi^i_u)$ define a partition

$$P = \{P_i : i = 1, \cdots, k\},$$

where each P_i contains stable summands of the downward limit of $[(\mathcal{E}^i_u, \overline{\partial}^i_u, \Phi^i_u)]$. Let U_i and U be sufficiently small open balls around 0 in $\mathcal{H}^1(P_i)$ and $\mathcal{H}^1$ respectively in the construction of the local Kuranishi model in Proposition 2.6.3, such that $\prod_i U_i \subset U \subset \mathcal{H}^1$. Here, the inclusion comes from the remark after (4.14). Without loss of generality, we can assume $[(\mathcal{E}^i_u, \overline{\partial}^i_u, \Phi^i_u)]$

has a representative $x_i \in U_i$. Writing $x_P = \sum_i x_i$ and $u_P = \kappa^{-1}(x_P)$, we have $[(\overline{\partial}_{u_P}, \Phi_{u_P})] = [(\overline{\partial}_u, \Phi_u)]$. Thus, $\overline{G \cdot \kappa(u)} \cap \overline{G \cdot x_P} \neq \emptyset$. Pick a point y that lies in the unique closed G-orbit of this intersection. As both $\mathcal{H}^1(P)$ and $\mathcal{H}^1_+$ are closed and G-invariant, $y \in \mathcal{H}^1_+(P)$. □

It was shown in [CW19] that the upward flow through a stable $\mathbb{C}^*$-fixed point is a complex Lagrangian with respect to the holomorphic structure I of $\mathcal{M}^s_H$. The following proposition, assuming that the central locus intersects with the stable locus $\mathcal{M}^s_H$, can be viewed as an analogue to this statement.

Proposition 4.3.5. *Let $u = (\beta, \varphi) \in S_+$ such that $(\overline{\partial}_u, \Phi_u)$ is stable. Then, L is locally an immersion in a neighborhood of $[(\beta, \varphi)] \in S_+ /\!\!/ G$ and is Lagrangian.*

Proof. Without loss of generality we assume that $\kappa(u) \in \mathcal{H}^1_+ \cap U$, where U is a sufficeintly small ball in the construction of the local Kuranishi model in Proposition 2.6.3. First, we show that L is an immersion at $[(\beta, \varphi)]$. Let $G \cdot (\mathcal{Z} \cap U)^s$ be the stable locus of $G \cdot (\mathcal{Z} \cap U)$. As $(\overline{\partial}_u, \Phi_u)$ is stable and $\mathcal{M}^s_H$ is open in $\mathcal{M}_H$, by the proof of (i) in Proposition 4.3.4, there exists an open neighborhood $V \subset U$ of $\kappa(u)$ where $\theta(x)$ (cf. (2.68) is stable for $x \in \mathcal{Z} \cap V$. Let $G \cdot (\mathcal{Z} \cap U)^s$ be the stable locus of $G \cdot (\mathcal{Z} \cap U)$. We have the G-invariant open set $G \cdot (\mathcal{Z} \cap V) \subset G \cdot (\mathcal{Z} \cap U)^s$ where G acts freely and properly. Restricted to $\mathcal{H}^1_+$, we have the closed embedding $\mathcal{H}^1_+ \cap G \cdot V \hookrightarrow G(\mathcal{Z} \cap V)$. At $\kappa(u)$, this gives the inclusions

$$\mathrm{Lie}(G) \hookrightarrow T_{\kappa(u)}\mathcal{H}^1_+ \hookrightarrow T_{\kappa(u)}\mathcal{Z}.$$

Quotient by $\mathrm{Lie}(G)$, we have $T_{[(\beta,\varphi)]}S_+ /\!\!/ G \hookrightarrow T_{[(\overline{\partial}_u, \Phi_u)]}\mathcal{M}^s_H$. This shows L is locally an immersion around $[(\beta, \varphi)]$.

Let $\overline{V} := (\mathcal{H}^1_+ \cap G \cdot V)/G$. Now, take a slice s_u through $\kappa(u)$. Through $\kappa^{-1}|_{s_u}$, tangent vectors of the image of $L \circ \overline{\kappa}(\overline{V})$ are represented by elements in $C^1_+ \subset \Omega^{0,1}(\mathfrak{sl}(E)) \oplus \Omega^{1,0}(\mathfrak{sl}(E))$. It follows that $\Omega|_{L\circ\overline{\kappa}(\overline{V})} \equiv 0$. As $\dim \overline{V} = \dim \mathcal{H}^1_+ - \dim G = \frac{1}{2}\mathcal{M}_H$ by Proposition 4.2.2, we see that L is Lagrangian around $\overline{\kappa}^{-1}(\overline{V})$. □

Finally, we show that under Assumption 1 or Assumption 2, the central locus always intersect with $\mathcal{M}^s_H$.

Theorem 4.3.6. *Suppose $\Phi = 0$, or the automorphism group of $(\overline{\partial}_E, \Phi)$ is abelian. Then $(\overline{\partial}_E, \Phi) + S_+$ contains a stable Higgs bundle.*

Proof. For the first case $\Phi = 0$, the underlying holomorphic vector bundle is polystable. Fix a harmonic metric h. Let $(\mathcal{H}^1)^{0,1} := \{\beta \in \Omega^{0,1}(\mathfrak{sl}(E)) : \partial\beta = 0\}$. We have the decomposition $\mathcal{H}^1 = (\mathcal{H}^1)^{0,1} \oplus \mathcal{H}^1_+$, where each factor is G-invariant. Taking the h-hermitian adjoint gives the norm-preserving identification $(\mathcal{H}^1)^{0,1} \xrightarrow{\cong_{\mathbb{R}}} \mathcal{H}^1_+$. Since $G^{*_h} = G$, this identification maps G-orbits onto G-orbits. Now, there exists $\beta \in (\mathcal{H}^1)^{0,1}$ with small h-norm such that $\theta(\beta) = (\overline{\partial}_E + \beta, 0)$ is a stable vector bundle. By the proof of (i) of Proposition 4.3.4, β is G-stable. The Kempf-Ness theorem then implies that $\varphi := \beta^*$ is also G-stable. In particular, its isotropy group has dimension zero. By Proposition 2.6.3, the resulting Higgs bundle $(\overline{\partial}_E, \varphi)$ is polystable. But the automorphism group of $(\overline{\partial}_E, \varphi)$ is a subgroup of G, and hence coincides with the isotropy group of $\varphi \in \mathcal{H}^1_+$. Therefore, $(\overline{\partial}_E, \varphi)$ is a stable Higgs bundle.

Next, suppose the automorphism group G of $(\overline{\partial}_E, \Phi)$ is abelian. We use the notations in (4.1) and (4.3) respectively to denote the stable summands and Hodge summands respectively. First, we claim that there exists non-trivial $\varphi_i \in H^0(\mathrm{Hom}(E^i_{\ell_i}, E^{i+1}_1) \otimes K_X)$ for each i where

here we let $E^{k+1} = E^1$. Using Riemann-Roch, we compute

$$h^1((E_1^{i+1})^\vee E^i_{\ell_i}) = h^0((E_1^{i+1})^\vee E^i_{\ell_i}) - \deg((E_1^{i+1})^\vee E^i_{\ell_i}) + \mathrm{rk}(E_1^{i+1})\mathrm{rk}(E^i_{\ell_i})(g-1). \tag{4.15}$$

Recall that each $(\overline{\partial}^i, \Phi^i)$ is a stable SHB, so $\deg E^i_{\ell_i} \leq 0$ while $\deg E^i_1 \geq 0$, where the equalities hold if and only if $\ell_i = 1$. Hence, $\deg((E_1^{i+1})^\vee E^i_{\ell_i}) \leq 0$ and therefore $h^1((E_1^{i+1})^\vee E^i_{\ell_i}) > 0$, which proves the claim. Now, we take φ to be $\sum_{i=1}^k \varphi_i$. Write

$$G = \{\xi = (\xi_1, \cdots, \xi_{k-1}) : \prod_{i=1}^{k-1} \xi_i^{r_i} = 1\}.$$

The G-action on φ (writing $\xi_{k+1} = \xi_1$) can be written as

$$\xi \cdot \varphi = \sum_{i=1}^{k} \xi_{i+1}\xi_i^{-1}\varphi_i.$$

Let $t = (t^{c_1}, \cdots, t^{c_k})$ be a one-parameter subgroup of G, so $\sum_{i=1}^k r_i c_i = 0$. The action of t on φ is given by $t \cdot \varphi = \sum_{i=1}^k t^{c_{i+1}-c_i}\varphi_i$. Notice that $c_{i+1} = c_i$ for all i if and only if $c_i \equiv 0$. Moreover, $\sum_{i=1}^k (c_{i+1} - c_i) = 0$. It follows that either t is trivial or there exists $i \neq j$ such that $c_{i+1} - c_i > 0$ and $c_{j+1} - c_j < 0$. By the Hilbert-Mumford criterion, φ is G-stable, so $\mathcal{H}^1_+$ contains some G-stable point. In particular, together with Proposition 4.2.2, we have

$$\dim(\mathcal{H}^1_+ /\!\!/ G) = \dim \mathcal{H}^1_+ - \dim G = \frac{1}{2}\dim \mathcal{M}_H = (r^2-1)(g-1).$$

Now, let $(\overline{\partial}_u, \Phi_u) \in (\overline{\partial}_E, \Phi) + S_+$. Replacing u with some point in $\overline{G \cdot u}$ we may assume $\kappa(u)$ is G-polystable and $(\overline{\partial}_u, \Phi_u)$ is polystable. Also, using the $\mathbb{C}^*$-action, we may move $\kappa(u)$

into an open ball U used in the construction of the local Kuranishi model in Proposition 2.6.3. If $u \neq 0$ and $(\overline{\partial}_u, \Phi_u)$ is not stable, then by (ii) of Proposition 4.3.4 and upon replacement if needed, $(\overline{\partial}_u, \Phi_u) \in (\overline{\partial}_E, \Phi) + S_+(P)$ for some proper partition P. Hence, if $(\overline{\partial}_E, \Phi) + S_+$ contains no stable Higgs bundles, then we can write $\mathcal{H}^1_+ /\!\!/ G = \bigcup_P \mathcal{H}^1_+(P) /\!\!/ G$, a finite union of closed subsets ranging over proper partitions P.

Lastly, we proceed by induction on $\dim G$. First, we look at the case $\dim G = 1$, in whise case $P = \{P_1, P_2\}$ is the unique proper partition and G acts trivially on $\mathcal{H}^1_+(P)$. Let r_1 and r_2 be the rank of the underlying vector bundles of the two stable factors so that $r = r_1 + r_2$. We have

$$\dim(\mathcal{H}^1_+(P) /\!\!/ G) = \dim \mathcal{H}^1_+(P_1) + \dim \mathcal{H}^1_+(P_2) = (r_1^2 + r_2^2 - 2)(g-1) < \dim \mathcal{H}^1_+ /\!\!/ G.$$

Therefore, there must exist stable Higgs bundles on $(\overline{\partial}_E, \Phi) + S_+$ since otherwise one would have $\mathcal{H}^1_+ /\!\!/ G = \mathcal{H}^1_+(P) /\!\!/ G \cong \mathcal{H}^1_+(P)$.

Now, let $r > 1$. We suppose it is true that the $(\overline{\partial}_E, \Phi) + S_+$ contains a stable Higgs bundle whenever $\dim G < r$. Assume now that $\dim G = r \geq 2$. We claim that for each proper P, there exists polystable $w \in \mathcal{H}^1_+(P)$ such that $w \notin \mathcal{H}^1_+(P')$ for all $P' \not> P$. It suffices to show that for each subcollection of stable factors $P_\lambda \in P$, the locus $(\overline{\partial}(P_\lambda), \Phi(P_\lambda)) + S_+(P_\lambda)$ contains a stable Higgs bundle. If $|P_\lambda| = 1$, $(\overline{\partial}(P_\lambda), \Phi(P_\lambda))$ is stable, in which case it is shown in [CW19] that $(\overline{\partial}(P_\lambda), \Phi(P_\lambda)) + S_+(P_\lambda)$ consists of stable Higgs bundles. For $|P_\lambda| = r_\lambda + 1 > 1$, we have $0 < r_\lambda < r$ so this case follows from the induction hypothesis. Hence, if $(\overline{\partial}_E + \Phi) + S_+$ contains no stable Higgs bundles, then $\mathcal{H}^1_+ /\!\!/ G$ is a finite union of proper closed subsets, which contradicts to the fact that $\mathcal{H}^1_+$ is irreducible. Therefore, $(\overline{\partial}_E, \Phi) + S_+$ must contain a stable Higgs bundle. $\square$

Chapter 5: Conformal limits

This chapter is based on the author's paper [Kwo25].

Let $(\overline{\partial}_E, \Phi)$ be a strictly polystable Higgs bundle such that $[(\overline{\partial}_E, \Phi)]$ is a $\mathbb{C}^*$-fixed point. We will use the SHB structure introduced in Section 4.2 and follow the notation therein. From the SHB structure, we obtain a family of isomorphisms from $(\overline{\partial}_E, R\Phi)$ to $(\overline{\partial}_E, \Phi)$ for each $R > 0$. For a harmonic metric h of $(\overline{\partial}_E, \Phi)$, the summands E_a and E_b are mutually orthogonal. By pulling back h, we obtain harmonic metrics h_R for $(\overline{\partial}_E, R\Phi)$.

Let $(\overline{\partial}_u, \Phi_u) \in S_+$ be stable. The idea in [DFK$^+$] of establishing the existence of the conformal limit of $(\overline{\partial}_u, \Phi_u)$ is the following. Using h_R, one first obtain a family of connections

$$\overline{\partial}_u + \partial_u^{h_R} + \hbar^{-1}\Phi_u + \hbar R^2 \Phi_u^{*_{h_R}}$$

which converges to the flat connection

$$D_0 := \overline{\partial}_u + \partial_E^h + h^{-1}\Phi_u + \hbar\Phi^{*_h}. \tag{5.1}$$

Then, one attempt to correct the metrics to $(e^{f_R})^* h_R$, where f_R is h_R-hermitian, so that the

resulting connections are flat. This amounts to solving

$$N_{u,R}(f_R) := \overline{\partial}_u(e^{-2f_R}\partial_u^{h_R}(e^{2f_R})) + F_{(\overline{\partial}_u,h_R)} + [\Phi_u, e^{-2f_R}(R^2\Phi_u^{*_{h_R}})e^{2f_R}] = 0. \tag{5.2}$$

Recall that the SHB structure induces a grading on $\Omega^{p,q}(\mathrm{End}(E))$. Using the notation in Section 4.2, for $d > 0$, we have

$$(f_R)_{-d} = R^{2d}(f_R)_d^{*_h}.$$

This fact is used in [CW19] to parametrize f_R by non-negatively graded pieces so that the operator $M_{u,R}$ is extended to all R. When the SHB $(\overline{\partial}_E, \Phi)$ is stable, using the implicit function theorem, it is shown that for small R there exists f_R with $f_0 = 0$ such that $M_{u,R}(f_R) \equiv 0$, which confirms that the conformal limit of $(\overline{\partial}_u, \Phi_u)$ is D_0.

In the present context, the last perturbation step in the method outlined above breaks down due to the non-uniqueness of the harmonic metrics of $(\overline{\partial}_E, \Phi)$. In the following, we will see that the issues that arise from this can be overcome under Assumption 1 or Assumption 2. In both cases, we will determine a harmonic metric h which depends on u, from which we determine a family of metrics h'_R modified from h_R. The first case is technically simpler due to the absence of non-trivial SHB structures and will be addressed in the next section. On the other hand, more preparations are needed to treat the second case. Combining Theorem 5.1.5 and Corollary 5.2.7, we then obtain Theorem 1.2.3.

5.1 On the open stratum

Suppose $\Phi = 0$, so that $(E, \overline{\partial}_E)$ is a polystable vector bundle. In this case,

$$S_+ = \mathcal{H}^1_+ = H^0(\mathfrak{sl}(E) \otimes K_X)$$

and the Kuranishi map κ restricted to S_+ is the identity (cf. (4.9)). For a harmonic metric h of $(\overline{\partial}_E, 0)$, we denote the subgroup of the automorphism group G of $(\overline{\partial}_E, 0)$ that fixes h by $K(h)$ or simply by K whenever no confusions arise. It is a maximal compact subgroup of G whose Lie algebra is given by $iH_0(h)$, where

$$H_0(h) := \{v \in \mathcal{H}^0 : v^{*_h} = v\}. \tag{5.3}$$

As G acts transitively by pulling back on the set of harmonic metrics of $(\overline{\partial}_E, 0)$, by Cartan decomposition we may parametrize the set of harmonic metrics by $H_0(h)$. Under Assumption 1, we have $(\overline{\partial}_u, \Phi_u) = (\overline{\partial}_E, \varphi)$.

Start with an arbitrary harmonic metric h_0 of $(\overline{\partial}_E, 0)$. Consider the G-representation $V = \mathcal{H}^1_+$, equipped with an inner product induced from h_0 so that $K(h_0)$ acts unitarily. Recall that we assume $(\overline{\partial}_u, \Phi_u)$ is stable. By Proposition 4.3.4, φ is a stable vector in V. Hence, there exists $g \in G$ such that the norm of $g \cdot \varphi$ is minimized along the G-orbit of φ. Let $h = g^* h_0$, i.e. $h(v, w) = h_0(gv, gw)$. To make the notations more concise, below we shall adopt the abbreviations $H_0 = H_0(h)$ and we denote the h-hermitian adjoint simply by $\varphi^* = \varphi^{*_h}$.

Define the Kempf-Ness functional $\ell_\varphi : H_0 \to \mathbb{R}$ by

$$\ell_\varphi(g) := ||e^g \varphi e^{-g}||_h^2 = i \int_X \mathrm{tr}\big[e^g \varphi e^{-g} (e^g \varphi e^{-g})^*\big] = i \int_X \mathrm{tr}\big[\varphi (e^{-2g} \varphi^* e^{2g})\big]. \tag{5.4}$$

By construction and the Kempf-Ness theorem ([KN78]), ℓ_φ attains its unique minimum at 0. Hence, its Jacobian $D\ell_\varphi$ vanishes and the Hessian $H(\ell_\varphi)$ is non-degenerate at the origin. For $g, v \in H_0$, we write

$$v_g := D_g \exp^2(v) = \frac{d}{dt}\Big|_{t=0} e^{2(g+tv)} \in \mathcal{H}^0. \tag{5.5}$$

The Jacobian of ℓ_φ is computed as follow:

$$\begin{aligned} -iD_g\ell_\varphi(v) &:= \frac{d}{dt}\Big|_{t=0} \int_X \mathrm{tr}\big[\varphi (e^{2(g+tv)})^{-1} \varphi^* e^{2(g+tv)}\big] \qquad (5.6)\\ &= \int_X \mathrm{tr}\big[\varphi(-e^{-2g} v_g e^{-2g}) \varphi^* e^{2g} + \varphi e^{-2g} \varphi^* v_g\big] \\ &= \int_X \mathrm{tr}\big(e^{-2g} \varphi^* e^{2g} \varphi + \varphi e^{-2g} \varphi^* e^{2g}\big) e^{-2g} v_g \\ &= \int_X \mathrm{tr}[\varphi, e^{-2g} \varphi^* e^{2g}] e^{-2g} v_g. \qquad (5.7) \end{aligned}$$

In particular, letting $g = 0$, we have

$$-iD_0\ell_\varphi(v) = 2 \int_X \mathrm{tr}[\varphi, \varphi^*] v = 0 \tag{5.8}$$

for any $v \in H_0$.

The second variation can be computed as follow.

$$-i\frac{\partial^2}{\partial w\partial v}\ell_\varphi(g) := \frac{d}{dt}\Big|_{t=0}\int_X \mathrm{tr}\big[\varphi, (e^{2(g+tw)})^{-1}\varphi^* e^{2(g+tw)}\big](e^{2(g+tw)})^{-1}v_{g+tw} \tag{5.9}$$

$$= \int_X \mathrm{tr}\Big\{\big[\varphi, [e^{-2g}\varphi^* e^{2g}, e^{-2g}w_g]\big]e^{-2g}v_g - \big[\varphi, e^{-2g}\varphi^* e^{2g}\big]\frac{d}{dt}\Big|_{t=0}(e^{2(g+tw)})^{-1}v_{g+tw}\Big\}. \tag{5.10}$$

Specializing at 0, the second term vanishes because of (5.8). Therefore, we obtain

Lemma 5.1.1. *Let $(\overline{\partial}_E, \varphi)$ be a stable Higgs bundle whose underlying holomorphic vector bundle is polystable. Then there exists a unique harmonic metric h of $(\overline{\partial}_E, 0)$ such that*

1. *$\int_X tr[\varphi, \varphi^{*_h}]v = 0$ for every $v \in H_0$, and*
2. *for $w \in H_0$, if $\int_X tr[\varphi, [\varphi^{*_h}, w]]v \equiv 0$ for all $v \in H_0$, then $w = 0$.*

Henceforth, we fix the harmonic metric h to be the one obtained in Lemma 5.1.1. Before we proceed to the proof the existence of the conformal limit of $(\overline{\partial}_E, \varphi)$, we first fix some notation and setting. Let

$$p : \mathfrak{sl}(E) \to i\mathfrak{su}(E, h) \tag{5.11}$$

be the bundle morphism which fibrewise projects a traceless endomorphism to its hermitian part. Let

$$H_1 := H_0^\perp \subset L_k^2(i\mathfrak{su}(E, h))$$

be the L^2 orthogonal complement in L_k^2-completion of $\Omega^0(i\mathfrak{su}(E, h))$ for some sufficiently large k. The two spaces of $\Omega^0(i\mathfrak{su}(E, h))$ and $\Omega^2(i\mathfrak{su}(E, h))$ are identified via the Hodge star. This identification extends to their L_k^2-completions and preserves the harmonic forms.

As $\Phi = 0$, we will set $h_R = h$ for all $R > 0$. We then obtain the following family of connections

$$\overline{\partial}_E + \partial_E^h + \hbar^{-1}\varphi + \hbar R^2 \varphi^*. \tag{5.12}$$

The operator (5.2) is reduced to

$$N(f, R) := \overline{\partial}_E(e^{-2f}(\partial_E^h e^{2f})) + R^2[\varphi, e^{-2f}\varphi^* e^{2f}]. \tag{5.13}$$

First, we notice the following

Lemma 5.1.2. *For all $v \in H_0$, $\int_X$ tr $p(i\overline{\partial}_E(e^{-2f}\partial_E^h e^{2f}))v = 0$.*

Proof. For any $A \in \Omega^1(\mathrm{End}(E))$, we notice that

$$\begin{aligned} 2\int_X \text{tr } p(\overline{\partial}_E A)v &= \int_X \text{tr } (\overline{\partial}_E A + (\overline{\partial}_E A)^*)v \\ &= \int_X \text{tr } [\overline{\partial}_E(Av) + (\overline{\partial}_E(vA))^*] \\ &= \int_X \overline{\partial}_E \text{tr } Av + \overline{\int_X \overline{\partial}_E \text{tr } vA} \\ &= 0. \end{aligned}$$

□

We now arrive at the most important outcome of the above setup.

Proposition 5.1.3. *There exists an open neighborhood U of 0 in H_1 and a smooth map $g : U \to$*

H_0 *such that* $g(0) = 0$ *and for any* $v \in H_0$,

$$\int_X tr\, p\left(i[\varphi, e^{-2(f+g(f))}\varphi^* e^{2(f+g(f))}]\right)v = 0.$$

Proof. Using the trace pairing, we regard the term $[\varphi, e^{-2f}\varphi^* e^{2f}]$ as a dual element of H_0. More precisely, define a map $n : H_1 \oplus H_0 \to H_0^\vee$ as follows: for $(f, g) \in H_1 \oplus H_0$ and $v \in H_0$,

$$n(f, g)(v) := \int_X \text{tr}\, p\left(i[\varphi, e^{-2(f+g)}\varphi^* e^{2(f+g)}]\right)v. \tag{5.14}$$

Notice that $n(0, 0) = 0$ by Lemma 5.1.1. Next, we compute the differential at the origin and along the H_0-direction: for $w \in H_0$,

$$\begin{aligned} D_{(0,0)}n(w)(v) &= 2\int_X \text{tr}\, p(i[\varphi, [\varphi^*, w]])v \\ &= -2\text{Im}\int_X \text{tr}\, [\varphi, [\varphi^*, w]]v \\ &= -2\int_X \text{tr}\, [\varphi, [\varphi^*, w]]v. \end{aligned}$$

By the second part of Lemma 5.1.1, $D_{(0,0)}n$ is non-singular. Hence, by the implicit function theorem, we obtain a map $g : U \to H_0$, for some open neighborhood U in H_1 containing 0, such that $g(0) = 0$ and $n(f, g(f)) \equiv 0$. □

Define the map

$$N'(f, R) := p(iN(f + g(f), R)),$$

where $f \in H_1$ and $g : U \to H_0$ is the map obtained in the above proposition. The image of

N' lies in the Hodge dual of H_1. Next, we compute the differential of N' at $(0,0)$ along the H_1-direction:

$$dN'_{(0,0)}(f) = 2p\big(i\overline{\partial}_E\partial_E^h(f + D_0g(f))\big) = 2i\overline{\partial}_E\partial_E^h f$$

since $D_0g(f) \in \mathcal{H}^0$ and $i\overline{\partial}_E\partial_E^h f$ is h-hermitian. As the Laplacian $i\overline{\partial}_E\partial_E^h$ has index zero with trivial kernel on H_1, by the implicit function theorem, we have a family $f_R \in H_1$ with $f_0 = 0$ for small R, such that

$$p(iN(f_R + g(f_R), R)) \equiv 0.$$

We notice that $iN(f_R + g(f_R), R)$ is $(e^{f_R+g(f_R)})^*h$-hermitian. To see that $N(f_R + g(f_R), R) \equiv 0$ for small R, we use the following

Lemma 5.1.4. *Let h_1 and h_2 be two hermitian inner products on a finite dimensional vector space V that induce the same volume form. Let A be the h_1-hermitian endomorphism such that $h_2(\cdot,\cdot) = h_1(A\cdot,\cdot)$. Write Herm$_0(V, h_i)$ to be the space of traceless h_i-hermitian endomorphisms, $i = 1, 2$, and*

$$p_A : \textit{Herm}_0(V, h_2) \to \textit{Herm}_0(V, h_1)$$

the projection map with respect to the inner product on End(V) induced from h_1. If $||A||_{h_1}$ is sufficiently small, then p_A is an $\mathbb{R}$-linear isomorphism.

Proof. Let U be a neighborhood around 0 of $\text{Herm}_0(V, h_1)$ such that if $B \in U$ then $h_B := h_1(B\cdot,\cdot)$ is a hermitian inner product. Let $*_{h_B}$ be the conjugation with respect to the metric h_B. It follows that $\text{Herm}_0(V, h_B)$ is the fixed point set of $*_{h_B}$ and $\text{Herm}_0 := \{(f, B) \in \text{End}_0(V) \times U : f \in \text{Herm}_0(V, h_B)\}$ is a vector bundle over U.

Consider the bundle morphism $p_\cdot : \text{Herm}_0 \to \text{Herm}_0(V, h_1)$. The result then follows from

continuity and that at $B = 0$, p is the identity. $\square$

Now, if k is sufficiently large, then $N(f_R + g(f_R), R) \in L^2_{k-2}(\mathfrak{su}(E,h) \otimes \bigwedge^2 T^*X) \subset C^0(\mathfrak{su}(E,h) \otimes \bigwedge^2 T^*X)$. By compactness of X and Lemma 5.1.4, $N(f_R + g(f_R), R) \equiv 0$ for sufficiently small R. This proves

Theorem 5.1.5. *For a stable Higgs bundle $(\overline{\partial}_E, \varphi)$ whose underlying holomorphic vector bundle $(E, \overline{\partial}_E)$ is polystable, its $\hbar$-conformal limit is $\overline{\partial}_E + \partial_E^h + \hbar^{-1}\varphi$.*

5.2 On polystable SHBs with pairwise distinct stable summands

In this section, we assume the stabilizer of $(\overline{\partial}_E, \Phi)$ is abelian, i.e. the stable summands $(\overline{\partial}^i, \Phi^i)$ in (4.1) are pairwise non-isomorphic. For $v \in \Omega^{p,q}(\mathrm{End}(E))$, we write $v = \sum_{i,j,a,b} v_{jb|ia}$, where $v_{jb|ia} \in \Omega^{p,q}(\mathrm{Hom}(E_a^i, E_b^j))$. To write down an explicit description of the G-action on S_+, we first embed G into $(\mathbb{C}^*)^k$. Let χ_i be the standard character of $(\mathbb{C}^*)^k$ obtained from projection to the i-th factor. Introduced in Section 4.2, $G \times \mathbb{C}^*$ acts on (β, φ) in the following way:

$$(t,s) \cdot (\beta, \varphi) = \left(\sum s^{b-a} t_i (t_j)^{-1} \beta_{jb|ia}, \sum s^{b-a+1} t_i (t_j)^{-1} \varphi_{jb|ia} \right). \tag{5.15}$$

Consider the finite dimensional vector space V spanned by the components $\beta_{jb|ia}$ and $\varphi_{jb|ia}$. Note that $G \times \mathbb{C}^*$ acts on the whole V linearly, and V inherits a hermitian inner product from a harmonic metric h of $(\overline{\partial}_E, \Phi)$, such that $K(h) \times S^1 \subset G \times \mathbb{C}^*$ acts unitarily. Since the G-orbit of u is closed in S_+, it is also closed in V. Hence, u is a G-stable vector in V. In the following, we will use the notation introduced in Section 2.5.

Next, we are going to introduce the following procedure:

Start with (i) $u_0 = u = (\beta, \varphi)$, (ii) $G_0 = G$, (iii) $c_{-1} = 0 \in \mathbb{Q}$, (iv) $x_{-1} = 0 \in X(G)^\vee \otimes_{\mathbb{Z}} \mathbb{Q} \subset i\mathfrak{k}$ and (v) $S_{-1} = \emptyset$.

Suppose we have determined u_{i+1}, G_{i+1}, c_i , x_i and S_i for $-1 \leq i < n$. In the n-th iteration (starting with $n = 0$):

Let ν_n be the sum of components of u_n fixed by G_n. Let

$$\phi_n := \sum_{i=0}^{n} \nu_i + \sum_{i=-1}^{n-1} P_{S_i}(u). \tag{5.16}$$

Here, $P_{S_i}(u)$ refers to the projection of u to $\bigoplus_{\lambda \in S_i} V_\lambda$.

Consider $\Lambda_{G\times\mathbb{C}^*}(u - \phi_n) \subset (i\mathfrak{k}_0)^\vee \oplus \mathbb{R}$. The inclusion $G_n \to G$ induces a projection

$$\pi_n : (i\mathfrak{k})^\vee \oplus \mathbb{R} \to (i\mathfrak{k}_n)^\vee \oplus \mathbb{R},$$

where $\mathfrak{k}_n$ is the Lie algebra of the maximal compact of G_n. Define the transformation the following transformation on $i\mathfrak{k} \oplus \mathbb{R}$

$$s_n(\ell, \rho) := (\ell, \rho + \langle \ell, \sum_{i=-1}^{n-1} x_i \rangle). \tag{5.17}$$

Let C_n be the convex hull of $\pi_n \circ s_n(\Lambda_{G\times\mathbb{C}^*}(u - \phi_n))$. Let

$$\{0\} \times [c_n, c_n'] := C_n \cap \{0\} \times \{\rho : \rho > 0\} \tag{5.18}$$

the intersection of the positive ρ-ray with C_n. Then, verify that $c_{n-1} < c_n \in \mathbb{Q}$.

Let F_n be the face of C_n that contains the point $(0, c_n)$ in its interior (as a CW-complex),

and

$$S_n := \Lambda_{G\times\mathbb{C}^*}(u) \cap s_n^{-1}(F_n). \tag{5.19}$$

Verify that $P_{S_n}(u)$ is G_n-polystable. Replace the metric h by some metric in $(G_n)^*h$, with respect to which $P_{S_n}(u)$ attains the minimum norm along its G_n-orbit, which exists by the Kempf-Ness theorem.

Lastly, we find some $x_n \in X(G_n)^\vee \otimes_{\mathbb{Z}} \mathbb{Q} \subset i\mathfrak{k}$ such that the following hold: for all $(\ell, \rho) \in S_n$ and $(\ell', \rho') \in \Lambda_{G\times\mathbb{C}^*}(u - \phi_n) - S_n$, we have

$$\rho + \langle \ell, \sum_{i=-1}^{n} x_i \rangle = c_n \tag{5.20}$$

and

$$\rho' + \langle \ell', \sum_{i=-1}^{n} x_i \rangle > c_n. \tag{5.21}$$

Now, let G_{n+1} be the stablizer of $P_{S_n}(u)$ and $u_{n+1} := u - \phi_n - P_{S_n}(u)$. If $\dim G_{n+1} = 0$, we stop. Otherwise, we proceed to the $(n+1)$-iteration.

The following is the first main technical input for the main result of this section.

Proposition 5.2.1. *Suppose $u = (\beta, \varphi)$ is stable. Then the iterative procedure described above is well-defined and terminates at the $(k-1)$-th iteration for some $k \geq 1$. Along the way, the following data is obtained:*

(i) a harmonic metric h of $(\overline{\partial}_E, \Phi)$, which induces a hermitian inner product on V

(ii) $(x, \sigma) \in X(G \times \mathbb{C}^*)^\vee$, where $\sigma \in \mathbb{Z}_{>0}$ determines a one-parameter subgroup of $\mathbb{C}^*$ of weight σ

(iii) a sequence of tori $G = G_0 \supset G_1 \supset \cdots \supset G_k = \{1\}$

(iv) $\nu_0, \cdots, \nu_{k-1} \in V$ *with* ν_i *fixed by* G_i *for each* $i = 0, \cdots k-1$

(v) $S_0, \cdots, S_{k-1} \subset \Lambda_{G\times\mathbb{C}^*}(u)$*, where each* $P_{S_i}(u)$ *is* G_i*-polystable and fixed by* G_{i+1}*, for* $i = 0, \cdots, k-1$*, each having minimal* h*-norm along its* G_i*-orbit, and*

(vi) a sequence of integers $0 < d_0 < \cdots < d_{k-1} \in \mathbb{Z}_{>0}$*, such that under the action of the one-parameter subgroup* (x, σ)*, we have*

$$(x(t), \sigma(t)) \cdot u = \sum_{i=0}^{k-1} (x(t), \sigma(t)) \cdot \nu_i + \sum_{i=0}^{k-1} t^{d_i} P_{S_i}(u) + (x(t), \sigma(t)) \cdot u_k. \tag{5.22}$$

Each $(x(t), \sigma(t)) \cdot \nu_i$ *can be further decomposed into*

$$(x(t), \sigma(t)) \cdot \nu_i = \sum_j t^{d_{ij}} \nu_{ij}, \tag{5.23}$$

where all d_{ij}*'s are positive integers and* $\min_j d_{ij} > d_{i-1}$*. Likewise, the residual term* $(x(t), \sigma(t)) \cdot u_k$ *can be decomposed into*

$$(x(t), \sigma(t)) \cdot u_k = \sum_j t^{e_j} (u_k)_j, \tag{5.24}$$

where e_j*'s are positive integers and* $\min_j e_j > d_{k-1}$.

Proof. Assume we have performed the procedure described above successfully from the 0-th up to the $(n-1)$-th iteration. We are going to show that the n-th iteration can be carried out, giving (i) ν_n, (ii) c_n, (iii) an updated metric h, (iv) a subset of weights S_n, and (v) a solution x_n to (5.20) and (5.21), all of which satisfy the required properties.

First, note that ϕ_n is fixed by G_n. As u is G-stable, $u - \phi_n$ is G_n-stable by the Hilbert-Mumford criterion. Let

$$q_n : (i\mathfrak{k}_n)^\vee \oplus \mathbb{R} \to (i\mathfrak{k}_n)^\vee$$

be the projection map. By Proposition 2.5.3, the convex hull of $q_n \circ \pi_n \circ s_n(\Lambda_{G\times\mathbb{C}^*}(u - \phi_n))$, which is $q_n(C_n)$, contains 0 in its interior. Hence, $C_n \cap \{0\} \times \{\rho : \rho > 0\}$ is non-empty. As the ρ-coordinates of all points in $s_n(\Lambda_{G\times\mathbb{C}^*}(u-\phi_n))$ are greater than c_{n-1}, we have $c_{n-1} < c_n \in \mathbb{Q}_{>0}$ ($(0, c_n)$ is a convex combination of these points).

Now, following the description of iterative procedure, the face F_n contains $(0, c_n)$ in its interior. Note that F_n is not a vertex, since otherwise $F_n = \{(0, c_n)\}$ but we have already removed ν_n from $u - \phi_n$. Next, we claim that $P_{S_n}(u)$ is G_n-polystable. Take a complementary torus of G_{n+1} inside G_n and let $\mathfrak{k}'_n$ be the Lie algebra of its maximal compact. So we have $(i\mathfrak{k}_n)^\vee = (i\mathfrak{k}_{n+1})^\vee \oplus (i\mathfrak{k}'_n)^\vee$. Let $\pi'_n : (i\mathfrak{k}_n)^\vee \to (i\mathfrak{k}'_n)^\vee$ be the projection map. It follows that the convex hull of $\pi'_n(\Lambda_{G_n}(P_{S_n}(u)))$ contains 0 in its interior. By Proposition 2.5.3, $P_{S_n}(u)$ is then G'_n-stable. In particular, its G'_n-orbit is closed. Since the G_n-orbit of $P_{S_n}(u)$ coincides with the G'_n-orbit, it follows that $P_{S_n}(u)$ is G_n-polystable. We replace the metric h by pullback from some element $g \in G_n$ such that with respect to this metric g^*h, $P_{S_n}(u)$ attains the minimal norm along its G_n-orbit. By abuse of notation, we still denote the updated metric by h.

Next, we show that solutions to (5.20) and (5.21) exists. Recall that $\{0\} \times [c_n, c'_n] := C_n \cap \{0\} \times \{\rho : \rho > 0\}$. We note that $\dim q_n(C_n) \leq \dim C_n \leq \dim q_n(C_n) + 1$. Consider the possibilty $\dim q_n(C_n) = \dim C_n$ first. In this case, $C_n = F_n$ is contained in a hyperplane defined by $\rho t + \langle \ell, x\rangle = c_n t$ for some $(x, t) \in X(G_n \times \mathbb{C}^*)^\vee \otimes_{\mathbb{Z}} \mathbb{Q} - \{0\}$. Observe that $t \neq 0$, since otherwise, we have $\langle \ell, x\rangle = 0$ for all $\ell \in C_n$, which contradicts G_n-stability of $u - \phi_n$. As $t \neq 0$,

we obtain the unique normalized solution $(x_n, 1)$, where this x_n solves (5.20) and (5.21). By stability, $\dim G_{n+1} = 0$, so we stop the iteration here. In this case, $u_{n+1} = u - \phi_n - P_{S_n}(u) = 0$.

Suppose $\dim C_n = \dim q_n(C_n) + 1$. We first consider the 'homogenized' system of inequalities

$$\left(\rho + \langle \ell, \sum_{i=-1}^{n-1} x_i \rangle\right) t + \langle \ell, x_n \rangle = c_n t \tag{5.25}$$

and

$$\left(\rho' + \langle \ell', \sum_{i=-1}^{n-1} x_i \rangle\right) t + \langle \ell', x_n \rangle > c_n t. \tag{5.26}$$

for $(\ell, \rho) \in F_n$ and $(\ell, \rho') \in C_n - F_n$. By the condition that F_n is a face and all vertices of C_n have rational coordinates, this system always has rational solutions (x_n, t). For all such (x_n, t), $t \neq 0$. Otherwise, we have $\langle \ell'', x_n \rangle \geq 0$ for all $\ell'' \in q_n \circ \pi_n(C_n)$, but this contradicts G_n-stability of $u - \phi_n$. Moreover, as $(0, c_n')$ also satisfies (5.26), we have $c_n' t > c_n t$. Hence, we can find a normalized solution $(x_n, 1)$, where x_n solves (5.20) and (5.21), once we obtain the following

Lemma 5.2.2. *Suppose* $\dim C_n = \dim q_n(C_n) + 1$, *then* $c_n < c_n'$.

Proof. Note that if (ℓ, ρ) is an interior point of C_n, then the small variations $(\ell, \rho + \epsilon)$ stay within the interior of C_n when $|\epsilon|$ is sufficiently small. For $\dim C_n \geq 3$, take an interior point (ℓ, ρ) of C_n, find a hyperplane containing ℓ and 0 in $(i\mathfrak{k}_n)^*$, say defined by $\ker x$ where $x \in i\mathfrak{k}_n - \{0\}$. It suffices to prove the statement on the restriction $\{(\ell, \rho) : \langle \ell, x \rangle = 0\} \cap C_n$. Inductively, we can reduce to the case where $\dim C_n = 2$ and $\dim q_n(C_n) = 1$. In this case, the vertical line $\{(\ell, \rho) : \ell = 0\}$ separates C_n into two portions. Both of them must contain interior points of C_n, since otherwise the closure of one of them is the whole C_n, leaving the other portion empty. Now, pick some interior points (ℓ_1, ρ_1) and (ℓ_2, ρ_2) from each portion. There exists $\epsilon > 0$ such

that the slight variation $(\ell_1, \rho_1 + t)$ and $(\ell_2, \rho_2 + t)$ remain in the same portion for $t \in (-\epsilon, \epsilon)$. Now, for $0 \leq \lambda \leq 1$, the segments $(\lambda\ell_1 + (1-\lambda)\ell_2, \lambda\rho_1 + (1-\lambda)\rho_2 + t)$ must cross the vertical line. Above the spot where $\lambda\ell_1 + (1-\lambda)\ell_2 = 0$, we see a variation $\lambda\rho_1 + (1-\lambda)\rho_2 + t$ with $|t| < \epsilon$, so we must have $c_n < c'_n$. □

Therefore, we can find such x_n. The n-th iteration is thus completed. For the stabilizer G_{n+1} of $P_{S_n}(u)$, if $\dim G_{n+1} \neq 0$, we let $u_{n+1} = u - \phi_n - P_{S_n}(u)$ and proceed to the $(n+1)$-iteration. Else, we stop. Since $P_{S_n}(u)$ is not fixed by G_n, $\dim G_{n+1} < \dim G_n$, so the process will stop at the $(k-1)$-iteration for some $k \geq 1$.

By the above construction, we have

$$u = \sum_{i=0}^{k-1} \nu_i + \sum_{i=0}^{k-1} P_{S_i}(u) + u_k.$$

We let $x = \sigma \sum_{i=0}^{k-1} x_i$ for some $\sigma \in \mathbb{Z}_{>0}$ so that $X(G)^\vee$. Regarding (x, σ) as a one-parameter subgroup, its action on u gives

$$(x(t), \sigma(t)) \cdot u = \sum_{i=0}^{k-1} (x(t), \sigma(t))\nu_i + \sum_{i=0}^{k-1} t^{c_i\sigma} P_{S_i}(u) + (x, t) \cdot u_k.$$

As in the statement of the lemma, write $(x(t), \sigma(t)) \cdot \nu_i = \sum_j t^{d_{ij}\sigma}\nu_{ij}$ and $(x(t), \sigma(t)) \cdot = \sum_j t^{e_j\sigma}(u_k)_j$. By construction, we have the bounds $\min_j d_{ij} > c_{i-1}$ and $\min_j e_j > c_{k-1}$. Now, pick σ so that all $c_i\sigma$, $d_{ij}\sigma$ and $e_j\sigma$ are integers. Finally, we let d_i to be $c_i\sigma$. □

With (x, t) and h determined in Proposition 5.2.1, as a one-parameter subgroup of gauge

transformations, we obtain a family of metrics h'_R parametrized by $R > 0$,

$$h'_R := (x(R), \sigma(R))^* h. \tag{5.27}$$

For $v \in \Omega^{p,q}(\mathrm{End}(E))$, we have

$$v^{*_{h'_R}} = \sum_{i,a,j,b} R^{2\sigma(b-a)+2\chi_i(x)-2\chi_j(x)} v^{*_h}_{jb|ia}. \tag{5.28}$$

Let $I(d)$ be the set of indices (i, a, j, b) such that $2\sigma(b-a) + 2\chi_i(x) - 2\chi_j(x) = d$. Define the bundle

$$Q_d := \bigoplus_{(i,a,j,b)\in I(d)} \mathrm{Hom}(E^i_a, E^j_b). \tag{5.29}$$

Sections of $Q_d \otimes \bigwedge^{p,q} T^*X$ are said to have *degree* d. Note that if for a section v of degree d, we have $v^{*_{h'_R}} = R^d v^{*_h}$. Next, we let

$$\mathcal{F}^{p,q}_d = \text{ the set of sections of } \bigoplus_{d'\geq d} Q_d \otimes \bigwedge^{p,q} T^*X. \tag{5.30}$$

When we do not distinguish the type of forms, we will omit the superscripts. The following are a few simple but important observations.

Lemma 5.2.3. *(i)* $\mathcal{F}^{p,q}_d \cdot \mathcal{F}^{p',q'}_{d'} \subset \mathcal{F}^{p+p',q+q'}_{d+d'}$.

(ii) If $v \in \mathcal{F}^{p,q}_d$ *and* $d > 0$*, then tr* $v = 0$.

(iii) Elements of $\mathcal{H}_0$ *have degree zero.*

(iv) $(u - \phi_j)^{1,0} \in \mathcal{F}_{2d_j-2\sigma}$ *for* $j = 1, \cdots, k-1$.

(v) $(u-\phi_j)^{0,1} \in \mathcal{F}_{2d_j}$ *for* $j = 1, \cdots, k-1$.

Proof. For $v \in \mathcal{F}_d^{p,q}$ and $v' \in \mathcal{F}_{d'}^{p',q'}$, we have

$$(vv')_{jb|ia} = \sum_{j',b'} v_{jb|j'b'} v'_{j'b'|ia}.$$

(i) then follows from (5.28). (ii) and (iii) are true because $v_{ia|ia}$ has degree zero for all i and a. For (iv) and (v), we note that $(P_{S_j}(u)^{1,0})^{*_{h'_R}} = R^{2d_j-2\sigma}(P_{S_j}(u)^{1,0})^*$ and $(P_{S_j}(u)^{0,1})^{*_{h'_R}} = R^{2d_j}(P_{S_j}(u)^{1,0})^*$. □

At degree zero, we define $Q_0(h)$ be the subbundle of h-hermitian endomorphisms of Q_0. Observe that $Q_0(h) = Q_0(h'_R)$ for all $R > 0$. We define

$$Q := Q_0(h) \oplus \bigoplus_{i>0} Q_i. \tag{5.31}$$

Let

$$p : \mathfrak{sl}(E) \to Q \tag{5.32}$$

be the bundle morhpism given by projection. We let $P(E,h)$ be the space of L_k^2-sections of Q where k is sufficiently large. For $f \in P(E,h)$, we write $f = \sum f_d$, where f_d is the degree d component of f. There is an isomorphism from $P(E,h)$ to the space of L_k^2-sections of h'_R-hermitian endomorphism, given by

$$f \mapsto f_R := f_0 + \sum_{d>0} f_d + \sum_{d>0} R^d f_d^*. \tag{5.33}$$

Note that the space

$$H_0 = \{v \in \mathcal{H}^0 : v^* = v\}$$

is contained in $P(E,h)$, by (iii) of Lemma 5.2.3. Lastly, we let H_1 to be the L^2-orthogonal complement of H_0 in $P(E,h)$.

Now, we return to the problem of the existence of the conformal limit of $(\overline{\partial}_u, \Phi_u)$. For $R > 0$, define the connections

$$D_R := \overline{\partial}_u + \partial_u^{h'_R} + \hbar^{-1}\Phi_u + \hbar(R^{2\sigma}\Phi_u)^{*_{h'_R}}, \tag{5.34}$$

which has the limit

$$D_0 := \lim_{R\to 0} D_R = \overline{\partial}_E + \partial_E^h + \Phi_u + \Phi^* \tag{5.35}$$

since by (5.22), (5.23) and (5.24) we have

$$\partial_u^{h'_R} = \partial^h - \beta^{*_{h'_R}} \xrightarrow{R\to 0} \partial_E^h$$

and

$$(R^{2\sigma}\Phi_u)^{*h'_R} = \Phi^* + R^{2\sigma}\varphi^{*_{h'_R}} \xrightarrow{R\to 0} \Phi^*.$$

More genearlly, for $f \in P(E,h)$, we have the expressions

$$\begin{aligned} D_R(f) &:= \overline{\partial}_u + \partial_u^{(e^{f_R})^* h'_R} + \hbar^{-1}\Phi_u + \hbar(R^{2\sigma}\Phi_u)^{*_{(e^{f_R})^* h'_R}} \\ &= \overline{\partial}_u + \partial_u^{h'_R} + e^{-2f_R}(\partial_u^{h'_R} e^{2f_R}) + \hbar^{-1}\Phi_u + \hbar e^{-2f_R}(R^{2\sigma}\Phi_u)^{*_{h'_R}} e^{2f_R}. \end{aligned} \tag{5.36}$$

For $f \in P(E,h)$, consider the following operator as in [DFK+] and [CW19]:

$$N(f,R) := F_{(\overline{\partial}_u,(e^{f_R})^*h'_R)} + [\Phi_u, (R^{2\sigma}\Phi_u)^{*_{(e^{f_R})^*h'_R}}]. \tag{5.37}$$

Recall that $F_{(\overline{\partial}_u,(e^{f_R})^*h'_R)}$ refers to the curvature of the Chern connection associated with the complex structure $\overline{\partial}_u$ and metric $(e^{f_R}))^*h'_R$, where f_R is defined in (5.33). Similar to [CW19], one compute that

$$\begin{aligned}
[\Phi_u, (R^{2\sigma}\Phi_u)^{*_{(e^{f_R})^*h'_R}}] &= [\Phi_u, e^{-2f_R}(\Phi^* + R^{2\sigma}\varphi^{*_{h'_R}})e^{2f_R}] \\
\overline{\partial}_u(e^{-2f_R}(\partial_u^{h'_R}e^{2f_R})) &= [\beta, e^{-2f_R}(\partial_E^h e^{2f_R}) - e^{2f_R}\beta^{*_{h'_R}}e^{2f_R}] + [\beta, \beta^{*_{h'_R}}] + \overline{\partial}_E(e^{-2f_R}\partial_u^{h'_R}e^{2f_R}) \\
F_{(\overline{\partial}_u,h'_R)} &= F_{(\overline{\partial}_E,h)} + \partial_E^h\beta - \overline{\partial}_E\beta^{*_{h'_R}} - [\beta, \beta^{*_{h'_R}}].
\end{aligned}$$

So, we have

$$\begin{aligned}
F_{(\overline{\partial}_u,(e^{f_R})^*h'_R)} &= \overline{\partial}_u(e^{-2f_R}((\partial_E^h - \beta^{*_{h'_R}})e^{2f_R})) + F_{(\overline{\partial}_u,h'_R)} \\
&= [\beta, e^{-2f_R}(\partial_E^h e^{2f_R}) - e^{-2f_R}\beta^{*_{h'_R}}e^{2f_R}] + F_{(\overline{\partial}_E,h)} + \partial_E^h\beta - \overline{\partial}_E\beta^{*_{h'_R}} + \overline{\partial}_E(e^{-2f_R}\partial_u^{h'_R}e^{2f_R}).
\end{aligned} \tag{5.38}$$

While h'_R is only defined for $R > 0$, the right side of (5.36) and (5.38), and thus $D_R(f)$ and $N(f,R)$, are defined for all f and R.

Lastly, for any $f \in H_0$, we compute that

$$N(f,0) = F_{(\overline{\partial}_E,h)} + \partial^h\beta + [\Phi_u, \Phi^*] = F_{(\overline{\partial}_E,h)} + [\Phi, \Phi^*] + D'u = 0.$$

As explained earlier, one would have shown that the $\hbar$-conformal limit of $(\overline{\partial}_u, \Phi_u)$ is D_0 if there exists, locally around $R = 0$, a family of solutions $R \mapsto f(R)$ with $f(0) = 0$ such that $N(f(R), R) \equiv 0$. Unfortunately, the presence of the non-trivial H_0 obstructs the construction of such $f(R)$ by means of a direct application of the implicit function theorem. This issue will be addressed with the help of Proposition 5.2.1.

First, we note that similar to Lemma 5.1.2, the last few terms at the right side of (5.38) are in some sense 'orthogonal' to H_0.

Lemma 5.2.4. *For all* $v \in H_0$, $\int_X tr\, p\Big(i(F_{(\overline{\partial}_E,h)} + \partial^h\beta - \overline{\partial}_E\beta^{*_{h'_R}} + \overline{\partial}_E(e^{-2f_R}\partial_u^{h'_R}e^{2f_R}))\Big)v = 0.$

Proof. Recall that $D''v = 0$ and $D'v = 0$. For any $A \in \Omega^1(\mathrm{End}(E))$, we have $\int_X \mathrm{tr}\, p(\overline{\partial}_E A)v = \int_X \mathrm{tr}\, p(\overline{\partial}_E A_0)v$, where A_0 is the degree zero component of A. Thus, similar to the proof of Lemma 5.1.2, we have $\int_X \mathrm{tr}\, p(\overline{\partial}_E A)v = 0$. The case for $\int_X \mathrm{tr}\, p(\partial_E^h A)v$ is similar. Hence, the last three terms vanish. For the first term, recall that $iF_{(\overline{\partial}_E,h)} = -i[\Phi, \Phi^*]$ and is h-hermitian. So $\int_X \mathrm{tr}\, p(-i[\Phi, \Phi^*])v = -i\int_X \mathrm{tr}\,[\Phi, \Phi^*]v = -i\int_X \mathrm{tr}\,\Phi^*[\Phi, v] = 0$. □

Next, we will consider the functional $n : H_1 \times H_0 \times \mathbb{R} \to H_0^\vee$, defined by

$$n(f, g, R)(v) := \int_X \mathrm{tr}\, p\big(iN(f, g, R)\big)v \tag{5.39}$$

$$= \int_X \mathrm{tr}\,\Big\{p\Big(i[\beta + \Phi_u, e^{-2f_R}(R^{2\sigma}\Phi_u - \beta)^{*_{h'_R}})e^{2f_R} + e^{-2f_R}(\partial_E^h e^{2f_R})]\Big)v\Big\} \tag{5.40}$$

$$= \mathrm{Re}\int_X \mathrm{tr}\, i[\beta + \Phi_u, e^{-2f_R}(R^{2\sigma}\Phi_u - \beta)^{*_{h'_R}})e^{2f_R} + e^{-2f_R}(\partial_E^h e^{2f_R})]v. \tag{5.41}$$

Here, we have adopted the following conventions on notation. First, we write $N(f, g, R) = N(f + g, R)$ to emphasize that we treat f and g as two independent variables. However, to save spaces, we also denote $(f + g)_R = f_R + g_R = f_R + g$ simply by f_R.

Lemma 5.2.5. *For $n \geq 0$, we have*

(i) $\left.\dfrac{d^n}{dR^n}\right|_{R=0} 2f_R \in \mathcal{F}^{0,0}_{-n}$

(ii) $\left.\dfrac{d^n}{dR^n}\right|_{R=0} e^{2f_R} \in \mathcal{F}^{0,0}_{-n}$

(iii) $\left.\dfrac{d^n}{dR^n}\right|_{R=0} \partial^h_E e^{2f_R} \in \mathcal{F}^{1,0}_{-n}$

(iv) $\left.\dfrac{d^n}{dR^n}\right|_{R=0} \beta^{*_{h'_R}} \in \mathcal{F}^{1,0}_{-n}$

(v) $\left.\dfrac{d^n}{dR^n}\right|_{R=0} R^{2\sigma}\Phi_u^{*_{h'_R}} \in \mathcal{F}^{0,1}_{2\sigma-n}$.

Proof. (i), (iv), and (v) follow from $f^*_d \in \mathcal{F}^0_{-d}$. For (ii), first we have

$$\left.\frac{d^n}{dR^n}\right|_{R=0} \frac{(2f_R)^j}{j!} = \sum_{(j_k)\in\mathcal{P}(n)} \prod_k \frac{1}{j_k!} \left.\frac{d^{j_k}}{dR^{j_k}}\right|_{R=0} 2f_R.$$

Here, $\mathcal{P}(n)$ denotes the set of tuples $(j_k)_{k=1}^j$ of integers such that $j_k \geq 0$ and $\sum_{k=1}^j j_k = n$. Then, apply (i) and Lemma 5.2.3. Finally, for (iii), just note that ∂^h_E preserves degrees. □

We finally arrive at the second important technical input for the main result of this section:

Theorem 5.2.6. *There exists a smooth map $g : U \times (-\epsilon, \epsilon) \to H_0$, where U is an open neighborhood of 0 in H_1 and $\epsilon > 0$, such that $g(0,0) = 0$ and $n(f, g(f,R), R) \equiv 0$.*

Proof. Note that $[\Phi, v] = 0$. For any $A \in \Omega^1(\mathrm{End}(E))$, tr $[\Phi, A]v = $ tr $A[\Phi, v] = 0$. Hence, we can drop the Higgs field Φ in the term $[\beta + \Phi + \varphi, e^{-2f_R}(R^{2\sigma}\Phi_u - \beta)^{*_{h'_R}})e^{2f_R} + e^{-2f_R}(\partial^h e^{2f_R})]$.

Recall that from Proposition 5.2.1, we have determined a decreasing chain of tori $G = G_0 \supset G_1 \supset \cdots \supset G_k = \{1\}$, with Lie algebras denoted by $\mathrm{Lie}(G_i) = \mathfrak{k}_i \otimes_{\mathbb{R}} \mathbb{C}$ for $i = 0, \cdots, k$.

Let $\mathfrak{k}_i' \otimes_{\mathbb{R}} \mathbb{C}$ be the Lie algebra of a complementary torus of G_i in G_{i-1}. Then H_0 admits the following decomposition

$$H_0 = \bigoplus_{j=1}^{k} \sqrt{-1}\mathfrak{l}_j'.$$

For each summand W of H_0, let n_W be the restriction of n to W and $W^\vee$. For $1 \leq j \leq k$ and $v \in i\mathfrak{k}_j'$, $[\phi_{j-1}, v] = 0$. Hence, by the same argument for Φ above, we have, for $g, v \in i\mathfrak{k}_j'$,

$$n_{i\mathfrak{k}_j'}(f, g, R)(v) = \mathrm{Re} \int_X \mathrm{tr}\, i[u - \phi_{j-1}, e^{-2f_R}(R^{2\sigma}\Phi_u - \beta)^{*_{h_R'}})e^{2f_R} + e^{-2f_R}(\partial^h e^{2f_R})]v. \quad (5.42)$$

From Lemma 5.2.3, $(u - \phi_{j-1})^{1,0} \in \mathcal{F}_{2d_{j-1}-2\sigma}$ and $(u - \phi_{j-1})^{0,1} \in \mathcal{F}_{2d_{j-1}}$. By Lemma 5.2.5, we then have

$$\left.\frac{d^n}{dR^n}\right|_{R=0} [u - \phi_{j-1}, e^{-2f_R}(R^{2\sigma}\Phi_u - \beta)^{*_{h_R'}})e^{2f_R} + e^{-2f_R}(\partial^h e^{2f_R})]v \in \mathcal{F}^{1,1}_{2d_{j-1}-n}. \quad (5.43)$$

Using (ii) of Lemma 5.2.3, it follows that the functional

$$\tilde{n}_j := R^{-2d_{j-1}} n_{i\mathfrak{k}_j'} \quad (5.44)$$

is well-defined everywhere. We compute

$$\begin{aligned} n_{i\mathfrak{k}_j'}(0, g, R)(v) &= \mathrm{Re} \int_X \mathrm{tr}\, i[u - \phi_{j-1}, e^{-2g}(R^{2\sigma}\Phi_u - \beta)^{*_{h_R'}})e^{2g}]v \\ &= \mathrm{Re} \int_X \mathrm{tr}\, i[u - \phi_{j-1}, \Phi^* + e^{-2g}(R^{2\sigma}\varphi - \beta)^{*_{h_R'}})e^{2g}]v \\ &= \mathrm{Re} \int_X \mathrm{tr}\, i[u - \phi_{j-1}, e^{-2g}(R^{2\sigma}\varphi - \beta)^{*_{h_R'}})e^{2g}]v. \end{aligned}$$

Notice that the dependence on R only occurs in the term $(R^{2\sigma}\varphi - \beta)^{*_{h'_R}}$. Therefore, from (5.44), we obtain

$$n_{i\mathfrak{k}'_j}(0,g,R)(v) = \mathrm{Re}\int_X \mathrm{tr}\, i[u-\phi_{j-1}, e^{-2g}\big(R^{2\sigma}(u-\phi_{j-1})^{1,0} - (u-\phi_{j-1})^{0,1}\big)^{*_{h'_R}}e^{2g}]v, \quad (5.45)$$

from which we have

$$\tilde{n}_j(0,g,0)(v) = \mathrm{Re}\int_X \mathrm{tr}\, i[u-\phi_{j-1}, e^{-2g}\bigg(\big(P_{S_{j-1}}(u)\big)^{1,0}\big)^* - \big((P_{S_{j-1}}(u))^{0,1}\big)^*\bigg)e^{2g}]v. \quad (5.46)$$

In particular, putting $g=0$ and $R=0$, we have

$$\begin{aligned}\tilde{n}_j(0,0,0)(v) &= \mathrm{Re}\int_X \mathrm{tr}\, i[u-\phi_{j-1}, \big(\big(P_{S_{j-1}}(u)\big)^{1,0}\big)^* - \big((P_{S_{j-1}}(u))^{0,1}\big)^*]v\\ &= \mathrm{Re}\int_X \mathrm{tr}\, i[P_{S_{j-1}}(u), \big(\big(P_{S_{j-1}}(u)\big)^{1,0}\big)^* - \big((P_{S_{j-1}}(u))^{0,1}\big)^*]v\\ &= \frac{1}{2}D_0\ell_{P_{S_{j-1}}(u)}(v),\end{aligned}$$

where $\ell_{P_{S_{j-1}}(u)} : i\mathfrak{k}'_j \to \mathbb{R}$ is the Kempf-Ness functional (cf. (5.4)) of $P_{S_{j-1}}(u)$ given by

$$\ell_{P_{S_{j-1}}(u)}(g) = ||e^g P_{S_{j-1}}(u)e^{-g}||^2.$$

By the choice of the harmonic metric h stipulated in Proposition 5.2.1, and applying the Kempf-Ness theorem as in the proof of Lemma 5.1.1, we have (i) $\tilde{n}_j(0,0,0)=0$, and (ii) the differential at 0 along the $i\mathfrak{k}'_j$-direction is non-degenerate. Hence, by the implicit function theorem, there

exists an open neighborhood U_j of 0 in $H_1 \oplus \bigoplus_{j' \neq j} i\mathfrak{k}'_{j'} \oplus \mathbb{R}$, such that there is a unique map

$$\tau_j : U_j \to i\mathfrak{k}'_j, \tag{5.47}$$

that satisfies

$$\tau_j(0,0,0) = 0, \text{ and}$$

$$\tilde{n}_j(f, \hat{g}_j + \tau_j(f, \hat{g}_j, R), R) \equiv 0,$$

where we have made the following convention on the notation used: for $g \in H_0$, we write $g = \sum_{j'=1}^{k} g_{j'}$ where $g_{j'} \in i\mathfrak{k}'_{j'}$ for each $j' = 1, \cdots, k$, and $\hat{g}_j := g - g_j$. Putting the factor $R^{2d_{j-1}}$ back, we see that the same holds for $n_{i\mathfrak{k}'_j}$.

If $k = 1$, taking $g(f, R)$ to be τ_1 will finish the proof. In what follows, we assume that $k \geq 2$. Recall that $P_{S_{j-1}}(u)$ is fixed by G_j. For each $j < k$ and $v \in i\mathfrak{k}'_j$, from (5.46) it follows that for all v,

$$\tilde{n}_j(0, \sum_{j'>j} g_{j'}, 0)(v) \equiv 0.$$

By uniqueness of τ_j, we have

$$D_{(0,0,0)}\tau_j(\sum_{j'>j} w_{j'}) \equiv 0$$

for $w_{j'} \in i\mathfrak{k}'_{j'}$, $j' = j+1, \cdots, k$.

Finally, consider the map

$$F = (\tau_1, \cdots, \tau_k) - \text{pr}_{H_0} : \bigcap_{j=1}^{k} (U_j \times i\mathfrak{k}'_j) \to H_0,$$

where $\mathrm{pr}(f, g, R) := g$ is the projection map. From the above, it follows that $D_{(0,0,0)}F\big|_{H_0}$ is block lower-triangular with $-\mathrm{id}$ on the diagonal. Restricting to a product neighborhood and applying the implicit function theorem, there exists a map

$$g : U \times (-\epsilon, \epsilon) \to H_0$$

from an open neighborhood of 0 of $H_1 \times \mathbb{R}$ to H_0 such that $g(0,0) = 0$ and $F(f, g(f,R), R) \equiv 0$. In other words, $\tau_j(f, \hat{g}_j(f,R), R) \equiv g_j(f,R)$ for each j. For each j and $v \in i\mathfrak{k}'_j$,

$$n(f, g(f,R), R)(v) = n_{i\mathfrak{k}'_j}(f, \hat{g}_j + \tau_j, R)(v) = 0.$$

Therefore, $n(f, g(f,R), R) \equiv 0$. □

Corollary 5.2.7. *The $\hbar$-conformal limit of $(\overline{\partial}_u, \Phi_u) \in S_+$ is $D_0 = \overline{\partial}_E + \partial_E^h + \hbar^{-1}\Phi_u + \hbar\Phi^*$.*

Proof. Consider

$$N'(f,R) := p\big(iN(f, g(R), R)\big).$$

By Theorem 5.2.6, $\mathrm{im}N' \perp H_0$. Differentiating at $(0,0)$ along the H_1-direction, we obtain

$$D_{(0,0)}N'(f) = 2p\Big(iD''_u D'(f + D_{(0,0)}g(f))\Big) = 2p\Big(iD''_u D' f\Big)$$

since $D_{(0,0)}g(f) \in H_0 \subset \ker D'$.

We claim that $\ker D_{(0,0)}N' = \{0\}$. Suppose $f \in \ker D_{(0,0)}N'$. Then

$$\big(p(iD''_u D' f)\big)_0 = p(iD'' D' f_0) = 0.$$

Now,

$$(D''D'f_0)^* = (\overline{\partial}_E \partial^h f_0)^* + ([\Phi, [\Phi^*, f_0]])^* = \partial^h \overline{\partial}_E f_0^* + [\Phi^*, [\Phi, f_0^*]] = D'D''f_0, \text{ and}$$

$$0 = (D'' + D')^2 f_0 = D''D'f_0 + D'D''f_0.$$

It follows that $0 = p(iD''D'f_0) = iD''D'f_0$. But this implies $f_0 \in H_0$, so $f_0 = 0$. Then, using (iii) in Lemma 5.2.3, one can argue inductively similar to [CW19], showing that $f_d = 0$ if $f_{d'} = 0$ for all $d' < d$. Hence, $f = 0$ and the claim follows.

The index of differential operator $p \circ iD''_u D'$ is the same as that of $p \circ iD''D' = iD''D'|_{P(E,h)}$, the latter being zero. Hence, we can apply the implicit function theorem, obtaining the solution $f(R)$ for small R with $f(0) = 0$, so that $N'(f(R), R)) \equiv 0$.

To see that $N(f(R), R) \equiv 0$, analogous to Lemma 5.1.4, it suffices to prove the following

Lemma 5.2.8. *There exists $\delta > 0$ such that for continuous f with sup-norm $||f||_h < \delta$, the projection map p gives an isomorphism between continuous sections of $(e^{f_R})^* h'_R$-hermitian endomorphisms and continuous sections of Q.*

Proof. Let $x \in X$. At a fibre, it suffices to show that the restriction of p to $i\mathfrak{u}(E_x, (e^{f_R})^* h'_R)$, the space of $(e^{f_R})^* h'_R$-hermitian endomorphisms of E_x, is injective.

Let

$$Q' = iQ_0(h) \oplus \bigoplus_{d<0} Q_d$$

be the complementary bundle of Q. In other words, Q' is the kernel of p. For $g \in Q'_x$, define

$$g(R) := \sum_{d \leq 0} R^{-d} g_d.$$

Consider the map $\beta : Q_x \times Q'_x \times \mathbb{R} \to \mathrm{End}(E)_x$ defined by

$$\beta(f, g, R) = g(R) - e^{-2f_R} g^* e^{2f_R}.$$

Note that for fixed f and R, β is linear in g. For $f = 0$ and $R = 0$, we have $\beta(0, g, 0) = g_0 - g^*$, which vanishes if and only if $g = 0$. Hence, $\beta(0, \cdot, 0)$ is injective. Then, for small R and f with small h-norm, $\beta(f, \cdot, R)$ is injective. For such f and positive R, $\beta(f, g, R) = g(R) - g(R)^{*_{(e^{f_R})^* h'_R}}$, so $g(R)$ is $(e^{f_R})^* h'_R$-hermitian if and only if $g = 0$. Finally, suppose $n \in i\mathfrak{u}(E_x, (e^{f_R})^* h'_R)$ and $p(n) = 0$. Then $n \in Q'_x$. Since $n = n'(R)$ for some $n' \in Q'_x$, $\beta(f, n', R) = 0$. Thus, $n' = n = 0$.

Choosing local unitary frames and using compactness of X, the result then follows. □

Since k is sufficiently large, $N(f(R), R) \in C^0(\Omega^{1,1}(\mathrm{End}(E))$. Therefore, we can conculde the proof by the above lemma. □

Chapter 6: A case study in rank two

6.1 Computation of the central locus

In this section, we look at an explicit example in the $\mathrm{SL}(2,\mathbb{C})$ case. Let $\mathcal{L}$ be a degree zero holomorphic line bundle whose square $\mathcal{L}^2$ is not trivial. In terms of Dolbeault operators, we fix $\underline{\mathbb{C}}$ as the underlying smooth line bundle of $\mathcal{L}$, so $\mathcal{L} = (\underline{\mathbb{C}}, \overline{\partial}_L)$, where $\overline{\partial}_L = \overline{\partial} + \beta_L$ for some smooth $(0,1)$-form $\beta_L \in \Omega^{0,1}$. The induced Dolbeault operator on $\mathcal{L}^k$ can then be written as $\overline{\partial}_{L^k} = \overline{\partial} + k\beta_L$. So, we have

$$\mathcal{L}^* = (\underline{\mathbb{C}}, \overline{\partial}_{L^*}) = (\underline{\mathbb{C}}, \overline{\partial} - \beta_L).$$

Now, let

$$E = \underline{\mathbb{C}^2} \quad \text{and} \quad \overline{\partial}_E = \begin{pmatrix} \overline{\partial}_L & 0 \\ 0 & \overline{\partial}_{L^*} \end{pmatrix}. \tag{6.1}$$

It follows that $(E, \overline{\partial}_E)$ is a polystable $\mathrm{SL}(2,\mathbb{C})$-bundle and $[(\overline{\partial}_E, 0)]$ is a polystable $\mathbb{C}^*$-fixed point. Since $\mathcal{L}$ and its dual $\mathcal{L}^*$ are not isomorphic, the automorphism group G is one-diemensional.

Fix a harmonic metric h of $(\overline{\partial}_E, 0)$. In the following, we will explicitly compute $\mathcal{H}^1_+$ of $(\overline{\partial}_E, 0)$. In this simple case,

$$D'' = \overline{\partial}_E \quad \text{and} \quad D' = \partial^h_E.$$

The complex $C_+^\bullet$ is reduced to

$$C_+^\bullet : \ 0 \longrightarrow \Omega^{1,0}(\mathfrak{sl}(E)) \xrightarrow{\overline{\partial}_E} \Omega^{1,1}(\mathfrak{sl}(E)) \ . \tag{6.2}$$

The Kuranishi map κ restricted to the central locus S_+ is simply the identity. Explicity,

$$\begin{aligned} S_+ &= \mathcal{H}^1_+ \\ &= H^0(\mathfrak{sl}(E) \otimes K_X) \\ &= \left\{ \varphi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a \in H^0(K_X), b \in H^0(L^2 K_X), c \in H^0(L^{-2} K_X) \right\} \end{aligned} \tag{6.3}$$

In this simple case, we can compute $h^1_+ = \dim \mathcal{H}^1_+$ by hand using Riemann-Roch and Serre duality:

$$\begin{aligned} h^0(K_X) &= g \\ h^0(L^2 K_X) &= h^1(L^2 K_X) + 2g - 2 + 1 - g = h^0(L^{-2}) + g - 1 = g - 1 \\ h^0(L^{-2} K_X) &= g - 1. \end{aligned}$$

Hence, $h^1_+ = 3g - 2$.

By the Kähler identities, $\mathcal{H}^2(C_+^\bullet) = \Lambda \ker \partial_E^h$. Since $(\partial_E^h a)^* = \overline{\partial}_E a^*$ and $\ker \overline{\partial}_E$ is invariant under taking the hermitian adjoint, by Riemann Roch, we have

$$\mathcal{H}^2(C_+^\bullet) = \Lambda \ker \overline{\partial}_E = \{ \Lambda \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} : t \in H^0(\mathcal{O}_X) = \mathbb{C} \}. \tag{6.4}$$

In particular, $h^2_+ := \dim \mathcal{H}^2(C^\bullet_+) = 1$, so $-\chi(C^\bullet_+) = 3g - 2 - 1 = 3g - 3$, which matches the general statement in Proposition 4.2.2.

From $G \subset \ker D'' \oplus \langle \mathrm{Id}_E \rangle$, we write

$$G = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{C}^* \right\}. \tag{6.5}$$

We will parametrize G by the non-zero scalar $t \in \mathbb{C}^*$. The G-action on $\mathcal{H}^1_+$ may then be written as

$$t \cdot \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} a & t^2 b \\ t^{-2} c & -a \end{pmatrix}. \tag{6.6}$$

Let $\{a_i\}_{i=1}^g, \{b_i\}_{i=1}^{g-1}$ and $\{c_i\}_{i=1}^{g-1}$ dual bases of $H^0(K)$, $H^0(L^2K)$ and $H^0(L^{-2}K)$ respectively. Then as an affine variety, $\mathcal{H}^1_+ = \mathrm{Spec}\mathbb{C}[a_i, b_j, c_k]$. We aim to give a descriptiong of the affine quotient $\mathcal{H}^1_+ /\!\!/ G$.

Theorem 6.1.1. *The affine quotient $\mathcal{H}^1_+//G$ can be identified with $\mathbb{A}^g_{\mathbb{C}} \times V$, where $\mathbb{A}^g_{\mathbb{C}}$ is the affine space of dimension g and $V \subset Mat_{(g-1)\times(g-1)}$ is the subspace of $(g-1) \times (g-1)$ matrices of rank at most one.*

Proof. Let $P = \sum_{i,j,k} \lambda_{ijk} a_i^{n_i} b_j^{n_j} c_k^{n_k} \in \mathbb{C}[a_i, b_j, c_k]$, where each $\lambda_{ijk} \neq 0$. We have

$$\xi \cdot P = \sum_{ijk} \lambda_{ijk} \xi^{-2n_j + 2n_k} a_i^{n_i} b_j^{n_j} c_k^{n_k}. \tag{6.7}$$

It follows that $P \in \mathbb{C}[a_i, b_j, c_k]^G$ if and only if $n_j = n_k$ for all monomial $a_i^{n_i} b_j^{nj} c_k^{n_k}$ in P. Let

$$x_{ij} := b_i c_j.$$

We can then write $\mathbb{C}[a_i, b_j, c_k]^G = \mathbb{C}[a_i, x_{ij}]$. The variables x_{ij} are subject to the relations

$$x_{ij}x_{kl} - x_{il}x_{kj},$$

so we have

$$\mathcal{H}^1_+ /\!\!/ G = \text{Spec}\mathbb{C}[a_i, x_{ij}]/\langle x_{ij}x_{kl} - x_{il}x_{kj}\rangle. \tag{6.8}$$

Now, define the matrix

$$x = (x_{ij})_{i,j=1}^{g-1} = bc^t \tag{6.9}$$

where $b = (b_1, \cdots, b_{g-1})^t$ and $c = (c_1, \cdots, c_{g-1})^t$. The map

$$\begin{aligned} a_i &\mapsto a_i \\ x_{ij} &\mapsto b_i c_j \end{aligned}$$

defines a morphism from $\mathcal{H}^1_+$ to $\mathbb{A}^g_{\mathbb{C}} \times \text{Mat}_{(g-1)\times(g-1)}$, the space of $(g-1) \times (g-1)$ matrices. The image is $\mathbb{A}^g_{\mathbb{C}} \times V$, where V is the space of matrices of rank at most one, defined by the set of equations

$$x_{ij}x_{kl} - x_{il}x_{kj} = 0.$$

This gives the isomorphism $\mathcal{H}^1_+ /\!\!/ G \cong \mathbb{A}^g_{\mathbb{C}} \times V$. □

Remark 6.1.2. (i) Since $\mathcal{H}^1_+ /\!\!/ G$ is irreducible, the isomorphism in Theorem 6.1.1 provides a proof that V is irreducible.

(ii) There is a stratification in V by ranks, which induces a stratification on the central locus

via $\mathcal{H}^1 /\!\!/ G$. In terms of the Higgs fields, the open stratum, which corresponds to rank 1 matrices, consists of Higgs bundles where both b and c are non-zero. In this case, neither $\mathcal{L}$ nor $\mathcal{L}^*$ are preserved. Hence, these correspond to stable Higgs bundles on the central locus S_+. The closed stratum consists of Higgs fields of the form

$$\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}.$$

This is the locus of S_+ which intersects the polystable locus of $\mathcal{M}_H$.

Let us write down the explicit form of the variety when $g = 2$ and $g = 3$.

Before we proceed to the upward flow, let us look at the case of low genus. When $g = 2$, writing $b = b_1$ and $c = c_1$, we simply have

$$\mathcal{H}^1_+ /\!\!/ G = \text{Spec}\mathbb{C}[a_1, a_2, x = bc] = \mathbb{A}^3_{\mathbb{C}}.$$

In particular, it is smooth of dimension 3. This is an exceptional case.

When $g = 3$, writing $b_i c_j = x_{ij}$, we have

$$\mathcal{H}^1_+ /\!\!/ G = \text{Spec}\mathbb{C}[a_1, a_2, a_3, x_{11}, x_{12}, x_{21}, x_{22}]/\langle x_{11}x_{22} - x_{12}x_{21}\rangle$$

which is the product of $\mathbb{A}^3_{\mathbb{C}}$ and the space of 2×2 marices with zero determinant.

6.2 The upward flow

The goal of this section is to describe the upward flow W_0 through $(\mathcal{L} \oplus \mathcal{L}^*, 0)$. For this, we need to return to the full $\mathcal{H}^1$ of the complex

$$C^\bullet : \ \Omega^0(\mathfrak{sl}(E)) \xrightarrow{\overline{\partial}_E} \Omega^{0,1}(\mathfrak{sl}(E)) \oplus \Omega^{1,0}(\mathfrak{sl}(E)) \xrightarrow{\overline{\partial}_E} \Omega^{1,1}(\mathfrak{sl}(E)) \ .$$

First, we characterize Higgs bundles whose downward limit is $[(\overline{\partial}_E, 0)]$.

Lemma 6.2.1. *A polystable Higgs bundle whose downward limit is* $[(\overline{\partial}_E, 0)]$ *can be gauged into the form* $(\overline{\partial}_u, \Phi_u)$ *where*

$$\overline{\partial}_u = \begin{pmatrix} \overline{\partial}_L & q \\ 0 & \overline{\partial}_{L^*} \end{pmatrix} \quad \textit{and} \quad \Phi_u = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \tag{6.10}$$

or

$$\overline{\partial}_u = \begin{pmatrix} \overline{\partial}_L & 0 \\ r & \overline{\partial}_{L^*} \end{pmatrix} \quad \textit{and} \quad \Phi_u = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}. \tag{6.11}$$

Proof. Let $(\mathcal{E}, \varphi)$ be a polystable Higgs bundle with $\lim_{t\to 0}[(\mathcal{E}, t\varphi)] = [(\overline{\partial}_E, 0)]$. For small t, we there exists gauge transformation g_t such that $(\mathcal{E}, t\varphi) \cdot g_t$ lies in the local Kuranishi model around $(\overline{\partial}_E, 0)$. Let $\overline{\partial}_t$ be the family of Dolbeault operators that defines the complex structures. Project the Higgs pairs to the configuration space of holomorphic vector bundles. As the semistable locus is open and $(E, \overline{\partial}_t)$ are gauge-equivalent with limit $(E, \overline{\partial}_E)$, we conclude that $\mathcal{E}$ is semistable with associated graded $\mathcal{L} \oplus \mathcal{L}^*$.

Let $0 \subset \mathcal{S} \subset \mathcal{E}$ be a Jordan-Hölder filtration of $\mathcal{E}$. Then $\mathcal{S} \oplus \mathcal{E}/\mathcal{S} \cong \mathcal{L} \oplus \mathcal{L}^*$. Therefore,

we either have the extension

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}^* \longrightarrow 0$$

or

$$0 \longrightarrow \mathcal{L}^* \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0 \ .$$

With respect to the Dolbeault operator $\overline{\partial}_E$, the complex structure for $\mathcal{E}$ with have the form in (6.10) or (6.11) respectively. □

Let $\tilde{W}_U$ and $\tilde{W}_L$ be the locus in $\mathcal{B} \cap ((\overline{\partial}_E, 0) + \ker D')$ corresponding to (6.10) and (6.11) respectively. For any q, one can find a gauge transformation g of the form

$$g = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

so that

$$g^{-1}\overline{\partial}_u \circ g = \begin{pmatrix} \overline{\partial}_L & q + \overline{\partial}_E s \\ 0 & \overline{\partial}_{L^*} \end{pmatrix}$$

where $\partial_E^h(q + \overline{\partial}_E s) = 0$. The same applies for r, replacing g with a lower triangular one. Hence, $\tilde{W}_U \cup \tilde{W}_L$ will capture the full upward flow.

Our next goal is to describe these two sets. We first deal with $\tilde{W}_U$. The case for $\tilde{W}_L$ can be treated in parallel.

Let $u = (\beta, \varphi) \in \tilde{W}_U$. Write

$$\beta = \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \varphi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \tag{6.12}$$

The condition $D'(\beta, \varphi) = \partial^h \beta = 0$ amounts to

$$\partial_E^h q = 0. \tag{6.13}$$

Hence q is the harmonic representative of the extension class $[q] \in H^1(L^2)$. The condition $D''(\beta, \varphi) + [\beta, \varphi] = 0$ is equivalent to

$$\overline{\partial}_E c = 0 \tag{6.14}$$

$$\overline{\partial}_E a + qc = 0 \tag{6.15}$$

$$\overline{\partial}_E b + 2aq = 0. \tag{6.16}$$

From this set of equations, already we can readily describe $\tilde{W}_U$. First, from (6.14) we have

$$c \in H^0(L^{-2} \otimes K_X).$$

The equation (6.15) requires the $(1,1)$-form qc to be $\overline{\partial}$-exact, which amounts to the condition

$$[q] \cdot [c] = 0 \tag{6.17}$$

under the non-degenerate pairing

$$H^1(L^2) \times H^0(L^{-2} \otimes K_X) \to \mathbb{C}.$$

In terms of integral, we require

$$\int_X q \wedge c = 0. \tag{6.18}$$

In particular, Λqc is orthogonal to 1, so there exists a_0 such that

$$\overline{\partial} a_0 + qc = 0.$$

Hence, we have

$$a \in a_0 + H^0(K_X). \tag{6.19}$$

Finally, as the operator $\overline{\partial}_E : \Omega^{1,0}(L^2) \to \Omega^{1,1}(L^2)$ obtained from the restriction $\overline{\partial}_E$ is surjective, given q and a there exists b_0 such that

$$\overline{\partial}_E b_0 + 2aq = 0. \tag{6.20}$$

Hence, given a and q

$$b \in b_0 + H^0(L^2 \otimes K_X). \tag{6.21}$$

Let $Z_{qc} \subset H^1(L^2 \times H^0(L^{-2} \otimes K_X)$ be the zero locus defined by $[q] \cdot [c] = 0$. The solution for (6.13), (6.14) and (6.15) is an affine bundle over Z_{qc} whose fibres are $H^0(K_X)$-torsor. Denote this space by Z_a^U. Then $\tilde{W}_U$ is an affine bundle over Z_a with fibres being $H^0(L^2 \otimes K_X)$-torsors.

Notice that for the case $\tilde{W}_L$, we can swap the role between $\mathcal{L}$ and $\mathcal{L}^*$ and solve the analogous system to define Z_{rb} and Z_a^L. We conclude this computation in the form of the following

Proposition 6.2.2. *$\tilde{W}_U$ and $\tilde{W}_L$ can be obtained from forming iterated affine bundle over a quadric hypersurface Z_{qc} and Z_{rb}, from which we calculate* $\dim \tilde{W}_U = \dim \tilde{W}_L = 4g - 4$.

Proof. It remains to compute the dimension. We treat the case $\tilde{W}_U$. First, we have

$$h^0(L^{-2} \otimes K_X) = h^1(L^{-2} \otimes K_X) + 2g - 2 + 1 - g = g - 1.$$

Hence, $\dim\ Z_{qc} = 2g - 2 - 1 = 2g - 3$. Similarly, $h^0(L^2 \otimes K_X) = g - 1$. So,

$$\dim \tilde{W}_U = 2g - 3 + g + g - 1 = 4g - 4.$$

□

Next, we aim to characterize the image $\kappa(\tilde{W}_U)$. By the construction of κ, we know $\kappa(\tilde{W}_U) \subset \mathcal{H}^1$. Let $u = (\beta, \varphi) \in \mathcal{H}^1$ with the form as in (6.10). This amounts to $\partial^h \beta = 0$ and $\overline{\partial}_E \varphi = 0$. So the two components β and φ seem to be decoupled. However, we need to also consider the Kuranishi space condition $P[\beta, \varphi] = 0$:

$$P \begin{pmatrix} qc & 2aq \\ 0 & cq \end{pmatrix} = 0.$$

As $\mathcal{H}^2 = \Lambda\mathcal{H}^1$ and $\mathcal{H}^1 = \{\mathrm{diag}(a, -a) : a \in \mathbb{C}\}$, this condition reduces to

$$[q] \cdot [c] = 0. \tag{6.22}$$

Let $W_U \subset \mathcal{H}^1$ consists of elements of the form as in (6.10) and satisfying (6.22).

Proposition 6.2.3. *The restricted Kuranishi map $\kappa|_{\tilde{W}_U}$ gives a biholomorphism between $\tilde{W}_U$ and W_U. The same statement is true for $\tilde{W}_L$ and W_L.*

Proof. We first claim that the image of $\tilde{W}_U$ lies in W_U. It is easy to see that the image has the form as in (6.10). Moreover, as $\tilde{W}_U$ consists of holomorphic Higgs bundles, the condition $[q] \cdot [c] = 0$ is satisfied by its image. Hence, the claim is proved.

Next, we compute

$$(\beta, \varphi) + (D'')^*\Gamma([\beta, \varphi]) = \begin{pmatrix} a + i\partial_E^h \Lambda\Gamma(qc) & q + b + i\partial_E^h \Lambda\Gamma(2aq) \\ c & -a + i\partial_E^h \Lambda\Gamma(cq) \end{pmatrix}.$$

Suppose $(\beta', \varphi') \in W_U$ be given. We will use the same set of alphabets as in (β, φ) to denote the components of (β', φ'). We wish to solve

$$\kappa(\beta, \varphi) = (\beta', \varphi').$$

First, we have

$$c = c'. \tag{6.23}$$

By comparing the $(0,1)$-forms on the top-right entry, we have

$$q = q'. \tag{6.24}$$

This also uniquely solves

$$a = a' - i\partial_E^h \Lambda\Gamma(q'c') \tag{6.25}$$

$$b = b' - i\partial_E^h \Lambda\Gamma(2aq'). \tag{6.26}$$

Notice that the formula above for are quadratic, this defines a holomorphic inverse $\kappa^{-1} : W_U \to \Omega^{0,1}(\mathfrak{sl}(E)) \oplus \Omega^{1,0}(\mathfrak{sl}(E))$. It thus remains to so that the image of κ^{-1} lies in $\tilde{W}_U$. We rewrite

$$\kappa^{-1}(\beta', \varphi') = \left(\begin{pmatrix} 0 & q' \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a' - i\partial_E^h \Lambda\Gamma(q'c') & b' - i\partial_E^h \Lambda\Gamma(2aq') \\ c' & -a' \end{pmatrix} \right). \tag{6.27}$$

The quadric condition $[q] \cdot [c] = 0$, $\partial_E^h q = 0$ and $\overline{\partial}_E c = 0$ are automatically satisfied. Next,

$$\overline{\partial}_E a = \overline{\partial}_E a' - i\overline{\partial}_E \partial_E^h \Lambda\Gamma(qc) = \overline{\partial}_E a' - (1 - P)(qc) = -qc.$$

Similary,

$$\overline{\partial}_E b = \overline{\partial}_E b' - (1 - P)(2aq) = -2aq$$

since $2aq$ is off-diagonal. □

Hence, we may consider the G-action on W_U and W_L instead. Recall the G-action acts by

$$t \cdot (\beta, \varphi) = \left(\begin{pmatrix} 0 & t^2 q \\ t^{-2} r & 0 \end{pmatrix}, \begin{pmatrix} a & t^2 b \\ t^{-2} c & -a \end{pmatrix} \right). \tag{6.28}$$

Here, we assume $q = 0$ or $r = 0$, which is a condition preserved by G. Notice also that the Kuranishi condition $[q] \cdot [c] = 0$ is preserved by G. To parametrize the upward flow, we need to combine both W_U and W_L. Now, let

$$W := W_U \cup W_L.$$

As both W_U and W_L are closed in W, we see that W is reducible.

To use W to model the upward flow, we need to know that for $p \in W$ and small R, there exists $g_R \in G$ such that $g_R \cdot R \cdot p$ is near the origin. We demonstrate it on W_U, so let $(\beta, \varphi) \in W_U$ with $r = 0$. Let $t \in \mathbb{C}^*$ and $R > 0$. If we impose $t = R^{1/4}$, we have

$$\begin{aligned} t \cdot R(\beta, \varphi) &= \left(\begin{pmatrix} 0 & t^2 q \\ 0 & 0 \end{pmatrix}, R \begin{pmatrix} a & t^2 b \\ t^{-2} c & -a \end{pmatrix} \right) \\ &= \left(\begin{pmatrix} 0 & R^{1/2} q \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} Ra & R^{3/2} b \\ R^{1/2} c & -Ra \end{pmatrix} \right) \\ &\xrightarrow{R \to 0} 0. \end{aligned}$$

All of the computations above culminate into the following

Theorem 6.2.4. *$W \mathbin{/\!\!/} G$ parametrizes the upward flow W_0.*

Finally, we give a description of the model space $W /\!\!/ G$. We begin by giving an algebraic description of W. Let $\{q_i\}_{i=1}^{g-1}$ and $\{c_i\}_{g-1}$ be dual basis of $H^1(L^2)$ and $H^0(L^{-2}\otimes K_X)$ respectively, such that with respect to the induced bilinear pairing, $q_i \cdot c_j = \delta_{ij}$. Let $\{r_i\}_{i=1}^{g-1}$ be dual basis for $H^1(L^{-2})$. Similarly, we have $\{a_i\}_{i=1}^{g}$, $\{b_i\}_{i=1}^{g-1}$ and $\{c_i\}_{i=1}^{i-1}$. The combination of the Kuranishi space condition and the condition that either $q = 0$ or $r = 0$ can be written as

$$\sum_{i=1}^{g-1} q_i c_i + \sum_{i=1}^{g-1} r_i b_i = 0 \tag{6.29}$$

$$q_i r_j = 0 \tag{6.30}$$

for $i, j = 1, \cdots, g-1$. Let I be the ideal generated by the expressions above, so

$$W = \text{Spec}\mathbb{C}[a_i, b_j, c_k, r_\alpha, q_\gamma]/I.$$

The invariant ring of $\mathbb{C}[a_i, b_j, c_k, r_\alpha, q_\gamma]$ is generated by

$$a_i \quad x_{ij} := b_i c_j \quad y_{ij} := q_i r_j \quad z_{ij} := r_i b_j \quad \text{and} \quad v_{ij} := q_i c_j.$$

We use X, Y, Z and V to denote the matrices with the corresponding entries. The defining equations for the model space W can then be written

$$\text{tr}V + \text{tr}Z = 0 \tag{6.31}$$

$$Y = 0. \tag{6.32}$$

In terms of outer products of the original coordinates, we can write the matrices as

$$X = cb^t \quad Y = rq^t \quad Z = rb^t \quad V = cq^t.$$

An inspection of the matric products above suggests that we should look at the augmented vectors (c, r) and (q, b) so now the outer product $(c, r)(q, b)^t$ gives $\begin{pmatrix} V & X \\ Y & Z \end{pmatrix}$. Under this matrix representation, we obtain

Theorem 6.2.5. *$W /\!\!/ G$ is the product $\mathbb{A}^g_{\mathbb{C}} \times W$, where $W \subset Mat_{(2g-2)\times(2g-2)}$ is the subvariety whose points are $(2g-2) \times (2g-2)$ which, when written in block form, each block of dimension $(g-1) \times (g-1)$, have the form*

$$\begin{pmatrix} V & X \\ 0 & Z \end{pmatrix}$$

and such that

(i) their rank is at most 1

(ii) they are nilpotent, i.e. $trW + trZ = 0$ (or equivalently $trW = trZ = 0$).

Notice that when $g = 2$, W, X and Z are scalars. The rank one condition forces $W = Z = 0$. Hence, in this case $W /\!\!/ G$ coincides with the central locus. This is not surprising: say when $q \neq 0$ and $r = 0$, the pairing condition forces $c = 0$. But in that case, the Higgs bundle obtained leaves $\mathcal{L}$ invariant, so it is strictly semi-stable. Its associated graded is the Higgs bundle which will kill off q and b. So it is a point in the central leaf.

Things start to look different when $g \geq 3$. We claim that when $g \geq 3$, $W /\!\!/ G$ is *not* irreducible. Here is one way to see it. The rank condition forces either V or Z to be zero. This

expresses $W /\!\!/ G$ as a union of two closed subset. When $V = 0$, it is easy to exhibit a point in $W /\!\!/ G$ with $Z \neq 0$. The same applies to the case $Z = 0$. Hence, $W /\!\!/ G$ is reducible.

Next, we count the dimension of the components. It suffices to count the dimension for the case $\begin{pmatrix} X \\ \\ Z \end{pmatrix}$ since the case $\begin{pmatrix} V & X \end{pmatrix}$ can be obtained by symmetry (tr$W$ = trW^T and ranks are preserved by taking transpose). On the closed subset $V = 0$, we exhibit a smooth point whose tangent space intersects transversely with the hyperplane defined by tr$Z = 0$. It follows that the dimension of this closed subset is $(2g-2)(g-1) - (2g-3)(g-2) - 1 = 3g - 5$.

Therefore, the dimension of $N//G$ is $g + (3g - 5) = 4g - 5$, which agrees with the calculation for $\tilde{W}_U /\!\!/ G$. Note that when $g = 2$, $4g - 5 = 3 = 3g - 3$.

The $\mathbb{C}^*$-action descends to $W /\!\!/ G$, givenby

$$R \cdot \begin{pmatrix} V & X \\ \\ 0 & Z \end{pmatrix} \mapsto \begin{pmatrix} RV & R^2X \\ \\ 0 & RZ \end{pmatrix}.$$

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