ABSTRACT

Title of Dissertation: AVERAGING PRINCIPLE FOR STOCHASTIC DYNAMICAL SYSTEMS

Yichun Zhu
Doctor of Philosophy, 2023

Dissertation Directed by: Professor Sandra Cerrai
Department of Mathematics

The theory of averaging, originated by Laplace and Lagrange, has been applied in its long history in many fields as, for example, celestial mechanics, oscillation theory and radio physics, and for a long period it has been used without a rigorous mathematical justification. A further development in the theory of averaging, which is of great interest in applications, concerns the case of random perturbations of dynamical systems.

This dissertation, discussing some applications of the averaging principle, includes two parts. In Chapter 1 and Chpater 2, we first study the Smoluchowski-Kramers approximation of a system with a finite number of

degrees of freedom, in the presence of a varying magnetic field, where the planar motion of a charged particle of small mass μ is governed by Newton's law. A small friction is added to the magnetic field to regularize the problem so that the small mass approximation can be studied. When $\mu \to 0$, a noise-drift term arises in the limit process due to the dependence of the strength of the magnetic field on the state of the particle and the system can be viewed as a small random perturbation of a Hamiltonian system in a large time scale and the Hamiltonian turns out to be the strength of the magnetic field. Therefore, by generalzing the Freidlin-Wentzell theory of averaging for Hamiltonian systems, we obtain a description of the slow motion of the charged particle when the mass is small.

In the second part(Chapter 3), we study the validity of an averaging principle for a class of slow-fast systems of stochastic partial differential equations. These systems are characterized by having weak regularity assumptions for the nonlinearity in the slow equation. Due to the weakness of these conditions, we are only able to prove the existence of martingale solutions and characterize the distributions of limiting points for the slow motions as solutions of a suitable averaged equation. After the publication of [33] and [31], there have been many papers in recent years that have stud-

ied the validity of averaging principles for various types of slow-fast systems of SPDEs, but these previous papers have all assumed well-posedness in appropriate functional spaces, whereas the systems considered in this paper are so irregular that it is not possible to have the existence and uniqueness of solutions, not even in the martingale sense, for either the slow-fast system or the limiting averaged equation.

In Chapter 4, we proved the existence of martingale solution for the stochastic PDE with rough nonlinear coefficients. We start from the condition introduced in [28] and improve the previous results from the existence of measure solution of the corresponding Fokker-Planck equation to the existence of martingale solution. This result can be applied to prove the existence of martingale solution of the slow-fast system in Chapter 3.

AVERAGING PRINCIPLE FOR STOCHASTIC DYNAMICAL SYSTEMS

by

Yichun Zhu

Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2023

Advisory Committee:
Prof. Sandra Cerrai, Chair/Advisor
Prof. Ilya Ryzhov , Dean's Representative
Prof. Leonid Koralov
Prof. Mark Freidlin
Prof. Yu Gu

Table of Contents

Chapter 1: An averaging approach to the Smoluchowski-Kramers approximation in the presence of a varying magnetic field

1.1 Introduction

We are dealing with the planar motion of a charged particle of a small mass μ in a force field that has a deterministic as well as a stochastic component combined with a magnetic field

$$\begin{cases} \mu\, \ddot{q}_\mu(t) = b(q_\mu(t)) - \lambda(q_\mu(t)) A \dot{q}_\mu(t) + \sigma(q_\mu(t))\, \dot{w}_t, \\ \\ q_\mu(0) = q \in \mathbb{R}^2, \qquad \dot{q}_\mu(0) = p \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

Here b is a vector field in $\mathbb{R}^2$, σ is 2×2-matrix valued mapping defined on $\mathbb{R}^2$ and $w(t)$ is a standard two-dimensional Brownian motion. Moreover,

$\lambda : \mathbb{R}^2 \to \mathbb{R}$ is some mapping, such that $\lambda(x) \geq \lambda_0 > 0$, for every $x \in \mathbb{R}^2$, and

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We are here interested in understanding the limiting behavior of the solution q_μ to equation (1.1), as the mass μ vanishes. This is the so called Smoluchowski-Kramers approximation.

It is well known (see [9] for all details) that when the variable magnetic field considered in the present paper is replaced by a constant friction (that is λ is constant and the matrix A coincides with the identity matrix), then $q_\mu(t)$ can be approximated with the solution of the first order equation

$$dq(t) = b(q(t))\, dt + \sigma(q(t))\, dw(t), \qquad q(0) = q. \tag{1.2}$$

More precisely, for every fixed $T > 0$

$$\lim_{\mu \to 0} \mathbb{E} \max_{t \in [0,T]} |q_\mu(t) - q(t)|^2 = 0. \tag{1.3}$$

Notice that here the case of an arbitrary number of degrees of freedom can be covered. The same result can be obtained also if λ is still constant, but A

is a more general matrix, whose eigenvalues have strictly positive real part, with the limiting equation (1.2) replaced by

$$dq(t) = A^{-1}b(q(t))\,dt + A^{-1}\sigma(q(t))\,dw(t), \qquad q(0) = q. \tag{1.4}$$

The case of non constant friction has been widely studied recently (see [12] and [13] for example). They have considered the following system

$$\begin{cases} \mu\,\ddot{q}_\mu(t) = b(q_\mu(t)) - \gamma(q_\mu(t))\dot{q}_\mu(t) + \sigma(q_\mu(t))\,\dot{w}_t, \\ \\ q_\mu(0) = q \in \mathbb{R}^2, \qquad \dot{q}_\mu(0) = p \in \mathbb{R}^2, \end{cases} \tag{1.5}$$

for some h-dimensional Brownian motion $w(t)$. They have assumed that the coefficients $b : \mathbb{R}^k \to \mathbb{R}^k$, $\gamma : \mathbb{R}^k \to \mathbb{R}^{k\times k}$ and $\sigma : \mathbb{R}^k \to \mathbb{R}^{h\times k}$ are smooth and uniformly bounded and the smallest eigenvalue $\lambda_1(q)$ of the symmetric matrix $\gamma(q) + \gamma^\star(q)$ is strictly positive, uniformly with respect to $q \in \mathbb{R}^k$. Namely

$$\inf_{q\in\,\mathbb{R}^k} \lambda_1(q) =: \bar{\lambda} > 0.$$

They have proved that limit (1.3) is still valid, but now $q(t)$ is the solution

of the modified equation

$$dq(t) = \left[\gamma^{-1}(q(t))b(q(t)) + S(q(t))\right]\, dt + \gamma^{-1}(q(t))\sigma(q(t))dw(t), \qquad q(0) = q, \tag{1.6}$$

where $S(q)$ is the noise-induced drift whose j-th component equals

$$S_j(q) = \sum_{i,l=1}^{k} \frac{\partial}{\partial q_i}(\gamma^{-1})_{jl}(q) J_{li}(q),$$

where J is the matrix-valued function solving the Lyapunov equation

$$J(q)\gamma^\star(q) + \gamma(q)J(q) = \sigma(q)\sigma^\star(q), \qquad q \in \mathbb{R}^k.$$

In [4], the case of a particle subject to a constant strength magnetic field orthogonal to the plane where the particle moves has been considered. The motion of the particle is governed by (1.1), with $\lambda(q) \equiv \bar{\lambda}$, for every $q \in \mathbb{R}^2$ (for semplicity of notation in what follows we shall take $\bar{\lambda} = 1$). In particular, since the eigenvalues of A are purely imaginary, it was not clear if the classical Smoluchowski-Kramer approximation was still valid. In fact, one can prove that if the stochastic term in (1.1) is replaced by a continuous function, then q_μ converges uniformly in $[0, T]$ to the solution of (1.4). But

if such continuous function is replaced by white noise, then there is no more convergence of q_μ to the solution of (1.4), as $\mu \downarrow 0$. This is a consequence of the fact that while

$$\lim_{\mu\to 0} \int_0^t \sin \frac{s}{\mu} \varphi(s)\, ds = 0,$$

for every continuous function, when $w(t)$ is a Brownian motion

$$\lim_{\mu\to 0} \int_0^t \sin \frac{s}{\mu}\, dw(s) \neq 0$$

This is due to the fact that

$$\mathrm{Var}\left(\int_0^t \sin \frac{s}{\mu}\, dw(s) \right) = \int_0^t \sin^2 \frac{s}{\mu}\, ds \to \frac{t}{2}, \qquad \text{as } \mu \downarrow 0.$$

Nevertheless, in [4] the problem has been regularized, so that a suitable counterpart of the Smoluchowski-Kramers approximation was proved.

The first regularization consisted in introducing in equation (1.1) a small friction proportional to the velocity. Namely, the following equation has been

considered

$$\begin{cases} \mu\, \ddot{q}_{\mu,\epsilon}(t) = b(q_{\mu,\epsilon}(t)) - A_\epsilon \dot{q}_{\mu,\epsilon}(t) + \sigma(q_{\mu,\epsilon}(t))\dot{w}(t), \\ \\ q_{\mu,\epsilon}(0) = q \in \mathbb{R}^2, \qquad \dot{q}_{\mu,\epsilon}(0) = p \in \mathbb{R}^2, \end{cases}$$

where $A_\epsilon = A + \epsilon\, I$ and $\epsilon > 0$ is a small parameter. It has been shown that for any $T > 0$

$$\lim_{\mu \to 0} \mathbb{E} \max_{t \in [0,T]} |q_{\mu,\epsilon}(t) - q_\epsilon(t)|^2 = 0 \tag{1.7}$$

where $q_\epsilon(t)$ is the solution of the problem

$$dq(t) = A_\epsilon^{-1} b(q(t))\, dt + A_\epsilon^{-1} \sigma(q(t)) dw(t), \quad q(0) = q.$$

Next, the limit as ϵ goes to zero has been taken and it has been proven that

$$\lim_{\epsilon \to 0} \mathbb{E} \max_{t \in [0,T]} |q_\epsilon(t) - q(t)|^2 = 0,$$

where $q(t)$ is the solution of the problem

$$dq(t) = -A\, b(q(t))\, dt - A\, \sigma(q(t))\, dw(t), \qquad q(0) = q. \tag{1.8}$$

Another approach to regularization used the fact that the white noise $\dot{w}(t)$ can be considered as an idealization of an isotropic δ-correlated smooth mean-zero Gaussian process $\dot{w}^{\delta}(t)$, with $0 < \delta << 1$, which converges to the standard white noise $w(t)$, as $\delta \downarrow 0$. In this case, it has been proven that if $q_{\mu,\delta}(t)$ is the solution of equation (1.1), with $\dot{w}(t)$ replaced by $\dot{w}^{\delta}(t)$, then

$$\lim_{\mu \to 0} \mathbb{E} \max_{t \in [0,T]} |q_{\mu,\delta}(t) - q_{\delta}(t)| = 0,$$

where $q_{\delta}(t)$ solves the equation

$$\dot{q}(t) = -Ab(q(t)) - A\sigma(q(t))\, \dot{w}^{\delta}(t), \quad q(0) = q.$$

Next, by taking the limit as $\delta \downarrow 0$, it has been proven that $q_{\delta}(t)$ converges to the solution $\hat{q}(t)$ of the problem

$$d\hat{q}(t) = -Ab(\hat{q}(t))\, dt - A\sigma(\hat{q}(t)) \circ dw(t), \quad \hat{q}(0) = q,$$

where the stochastic term has to be interpreted in Stratonovich sense.

1.2 The Smoluchowski-Kramers approximation for the regularized problem

As we mentioned in the Introduction, we are dealing here with the following equation

$$\begin{cases} \mu\, \ddot{q}_\mu(t) = b(q_\mu(t)) - \lambda(q_\mu(t)) A \dot{q}_\mu(t) + \sigma(q_\mu(t))\, \dot{w}_t, \\ \\ q_\mu(0) = q \in \mathbb{R}^2, \qquad \dot{q}_\mu(0) = p \in \mathbb{R}^2, \end{cases} \tag{1.9}$$

where μ is a small positive constant and $w(t)$ is a standard Brownian motion in $\mathbb{R}^2$. In what follows, we shall assume that the coefficients in the equation above satisfy the following conditions.

Hypothesis 1. *1. The mappings $b : \mathbb{R}^2 \to \mathbb{R}^2$ and $\sigma : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ are Lipschitz-continuous.*

2. The mapping $\lambda : \mathbb{R}^2 \to \mathbb{R}$ is locally Lipschitz-continuous and there exist $\gamma \geq 0$ and $c > 0$ such that

$$|\lambda(q)| \leq c\,(1 + |q|^\gamma)\,. \tag{1.10}$$

Moreover

$$\inf_{q\in\mathbb{R}^2} \lambda(q) =: \lambda_0 > 0.$$

Next, for every $\epsilon \geq 0$ we introduce the approximating problem

$$\begin{cases} \mu\, \ddot{q}_{\mu,\epsilon}(t) = b(q_{\mu,\epsilon}(t)) - \Lambda_\epsilon(q_{\mu,\epsilon}(t))\dot{q}_{\mu,\epsilon}(t) + \sigma(q_{\mu,\epsilon}(t))\, \dot{w}_t, \\ \\ q_{\mu,\epsilon}(0) = q \in \mathbb{R}^2, \qquad \dot{q}_{\mu,\epsilon}(0) = p \in \mathbb{R}^2, \end{cases} \tag{1.11}$$

where

$$\Lambda_\epsilon(q) = \lambda(q)A + \epsilon I = \begin{pmatrix} \epsilon & \lambda(q) \\ -\lambda(q) & \epsilon \end{pmatrix}, \qquad q \in \mathbb{R}^2.$$

Notice that for every $\epsilon > 0$ the matrix $\Lambda_\epsilon(q)$ is uniformly non-degenerate, as

$$\langle \Lambda_\epsilon(q)p, p\rangle = \epsilon\, |p|^2. \tag{1.12}$$

Moreover, when $\epsilon = 0$, equation (1.11) coincides with equation (1.9).

Theorem 1.2.1. *Under Hypothesis 1, for every $\mu > 0$ and $\epsilon \geq 0$ and for every $T > 0$ and $k \geq 1$, equation (1.11) admits a unique solution $q_{\mu,\epsilon} \in L^k(\Omega; C([0,T];\mathbb{R}^2))$.*

Proof. For every $q, p \in \mathbb{R}^2$ and $n \in \mathbb{N}$, we define

$$\beta_n(p) = \begin{cases} p, & \text{if } |p| \leq n, \\ \\ np/|p|, & \text{if } |p| \geq n, \end{cases}$$

and

$$\lambda_n(q) = \begin{cases} \lambda(q), & \text{if } |q| \leq n, \\ \\ \lambda((n+1)q/|q|), & \text{if } |q| \geq n+1, \end{cases} \qquad \Lambda_{\epsilon,n}(q) = \lambda_n(q)A + \epsilon I.$$

Notice that $\lambda_n : \mathbb{R}^2 \to \mathbb{R}$ is Lipschitz-continuous and

$$|\lambda_n(q)| \leq c\,(1 + |q|^\gamma)\,, \quad |\beta_n(p)| \leq |p|, \tag{1.13}$$

for some constant c independent of n. Moreover, since $\langle A\beta_n(p), p\rangle = 0$, and $\langle \beta_n(p), p\rangle \leq |p|^2$, for every $p \in \mathbb{R}^2$ and $n \in \mathbb{N}$, we have that

$$\langle \Lambda_{\epsilon,n}(q)\beta_n(p), p\rangle = \epsilon\,|p|^2, \tag{1.14}$$

for every $p, q \in \mathbb{R}^2$ and $n \in \mathbb{N}$.

Next, we introduce the problem

$$\begin{cases} \mu\, \ddot{q}_{\mu,\epsilon}(t) = b(q_{\mu,\epsilon}(t)) - \Lambda_{\epsilon,n}(q_{\mu,\epsilon}(t))\beta_n(\dot{q}_{\mu,\epsilon}(t)) + \sigma(q_{\mu,\epsilon}(t))\, \dot{w}_t, \\ \\ q_{\mu,\epsilon}(0) = q \in \mathbb{R}^2, \qquad \dot{q}_{\mu,\epsilon}(0) = p \in \mathbb{R}^2, \end{cases}$$

which can be rewritten as

$$\begin{cases} dq^n_{\mu,\epsilon}(t) = p^n_{\mu,\epsilon}(t)\, dt \\ \\ \mu dp^n_{\mu,\epsilon}(t) = \left[b(q^n_{\mu,\epsilon}(t)) - \Lambda_{\epsilon,n}(q^n_{\mu,\epsilon}(t))\beta_n p^n_{\mu,\epsilon}(t)\right] + \sigma(q^n_{\mu,\epsilon}(t))\, dw(t). \end{cases} \tag{1.15}$$

It is immediate to check that the mapping

$$(q, p) \in \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \Lambda_{\epsilon,n}(q)\beta_n(p) \in \mathbb{R}^2,$$

is Lipschitz-continuous, so that for every fixed $n \in \mathbb{N}$ equation (1.15) admits a unique solution $(q^n_{\mu,\epsilon}, p^n_{\mu,\epsilon}) \in L^p(\Omega; C^1([0,T];\mathbb{R}^2) \times C([0,T];\mathbb{R}^2))$.

Now, if we apply Itô's formula to the function $\Phi(q,p) = |q|^{2k} + |p|^{2k}$, for

$k \geq 2$, we obtain

$$|q^n_{\mu,\epsilon}(t)|^{2k} + |p^n_{\mu,\epsilon}(t)|^{2k} = |q|^{2k} + |p|^{2k} + k\int_0^t |q^n_{\mu,\epsilon}(s)|^{2k-2}\langle q^n_{\mu,\epsilon}(s), p^n_{\mu,\epsilon}(s)\rangle\, ds$$

$$+\frac{k}{\mu}\int_0^t |p^n_{\mu,\epsilon}(s)|^{2k-2}\langle p^n_{\mu,\epsilon}(s), b(q^n_{\mu,\epsilon}(s) - \Lambda_{\epsilon,n}(q^n_{\mu,\epsilon}(s))\beta_n(p^n_{\mu,\epsilon}(s))\rangle\, ds$$

$$+\frac{k}{2\mu^2}\int_0^t |p^n_{\mu,\epsilon}(s)|^{2k-2}\mathrm{Tr}\left[\sigma\sigma^\star(q^n_{\mu,\epsilon}(s))\right]\, ds + \frac{k(k-1)}{2\mu^2}\int_0^t |p^n_{\mu,\epsilon}(s)|^{2k-4}|\sigma(q^n_{\mu,\epsilon}(s))p^n_{\mu,\epsilon}(s)|^2\, ds$$

$$+\frac{k}{\mu}\int_0^t |p^n_{\mu,\epsilon}(s)|^{2k-2}\langle p^n_{\mu,\epsilon}(s), \sigma(q^n_{\mu,\epsilon}(s))\, dw(s)\rangle.$$

Therefore, thanks to (1.14) and the Young inequality, we have

$$|q^n_{\mu,\epsilon}(t)|^{2k} + |p^n_{\mu,\epsilon}(t)|^{2k} \leq |q|^{2k} + |p|^{2k} + c_{k,\mu}\int_0^t \left[|q^n_{\mu,\epsilon}(s)|^{2k} + |p^n_{\mu,\epsilon}(s)|^{2k}\right]\, ds$$

$$+\frac{k}{\mu}\int_0^t |p^n_{\mu,\epsilon}(s)|^{2k-2}\langle p^n_{\mu,\epsilon}(s), \sigma(q^n_{\mu,\epsilon}(s))\, dw(s)\rangle.$$

After we take expectation, due to the Gronwall lemma we obtain

$$\mathbb{E}|q^n_{\mu,\epsilon}(t)|^{2k} + \mathbb{E}|p^n_{\mu,\epsilon}(t)|^{2k} \leq c_{k,\mu}(T)\left(1 + |q|^{2k} + |p|^{2k}\right), \qquad t \in [0,T]. \quad (1.16)$$

Therefore, since

$$q^n_{\mu,\epsilon}(t) = q + \int_0^t p^n_{\mu,\epsilon}(s)\, ds,$$

and

$$p^n_{\mu,\epsilon}(t) = p + \frac{1}{\mu}\int_0^t \left[b(q^n_{\mu,\epsilon}(s)) - \Lambda_{\epsilon,n}(q^n_{\mu,\epsilon}(s))\beta_n(p^n_{\mu,\epsilon}(s))\right] ds + \frac{1}{\mu}\int_0^t \sigma(q^n_{\mu,\epsilon}(s))\, dw(s),$$

due to (1.13), from (1.16) we obtain

$$\sup_{n\in\mathbb{N}} \mathbb{E} \sup_{t\in[0,T]} \left(|q^n_{\mu,\epsilon}(t)|^{2k} + |p^n_{\mu,\epsilon}(t)|^{2k}\right) \leq c_{k,\mu}(T, |q|, |p|). \tag{1.17}$$

Now, for any $n \in \mathbb{N}$ we define

$$\tau_n = \inf\left\{t \geq 0 \ : \ |q^n_{\mu,\epsilon}(t)| \vee |p^n_{\mu,\epsilon}(t)| \geq n\right\},$$

with the usual convention that $\inf \emptyset = +\infty$. Since the sequence $\{\tau_n\}_{n\in\mathbb{N}}$ is non-decreasing, $\mathbb{P}$-a.s. we can define

$$\tau = \lim_{n\to\infty} \tau_n.$$

Due to (1.17), for every fixed $T > 0$ we have

$$\mathbb{P}\left(\sup_{t\in[0,T]}|q^n_{\mu,\epsilon}(t)\le n,\ \sup_{t\in[0,T]}|p^n_{\mu,\epsilon}(t)\le n\right)$$

$$\ge 1-\mathbb{P}\left(\sup_{t\in[0,T]}|q^n_{\mu,\epsilon}(t)> n\right)-\mathbb{P}\left(\sup_{t\in[0,T]}|p^n_{\mu,\epsilon}(t)> n\right)$$

$$\ge 1-\frac{2c_{1,\mu}(T,|q|,|p|)}{n}.$$

This implies that

$$\lim_{n\to\infty}\mathbb{P}\left(\tau_n > T\right)=1,$$

so that, due to the arbitrariness of T,

$$\mathbb{P}\left(\tau=+\infty\right)=1.$$

Therefore, since for every $n<m$ and $t\le\tau_n$

$$(q^n_{\mu,\epsilon}(t),p^n_{\mu,\epsilon}(t))=(q^m_{\mu,\epsilon}(t),p^m_{\mu,\epsilon}(t)),$$

if we set

$$(q_{\mu,\epsilon}(t), p_{\mu,\epsilon}(t)) = (q^n_{\mu,\epsilon}(t \wedge \tau_n), p^n_{\mu,\epsilon}(t \wedge \tau_n)), \qquad t \leq \tau,$$

we can conclude that there exists a unique solution $(q_{\mu,\epsilon}, p_{\mu,\epsilon})$ to problem (1.11), belonging to $L^k(\Omega; C^1([0,T];\mathbb{R}^2) \times C([0,T];\mathbb{R}^2))$, for every $k \geq 1$ and $T > 0$. □

It is immediate to check that for every $\epsilon > 0$ and $q \in \mathbb{R}^2$, the matrix $\Lambda_\epsilon(q)$ is invertible and

$$\Lambda^{-1}_\epsilon(q) = \frac{1}{\lambda^2(q) + \epsilon^2} \begin{pmatrix} \epsilon & -\lambda(q) \\ \lambda(q) & \epsilon \end{pmatrix}. \tag{1.18}$$

Now, we introduce the vector field $S^\epsilon(q)$, whose j-th component is defined by

$$S^\epsilon_j(q) = \sum_{i,l=1}^{2} \partial_i (\Lambda^{-1}_\epsilon)_{jl}(q) J^\epsilon_{li}(q), \qquad j = 1, 2, \tag{1.19}$$

where $\partial_i = \partial/\partial q_i$ and J^ϵ is the matrix-valued function solving the Lyapunov equation

$$J^\epsilon(q)\Lambda^\star_\epsilon(q) + \Lambda_\epsilon(q) J^\epsilon(q) = \sigma(q)\sigma^\star(q), \qquad q \in \mathbb{R}^2.$$

Thanks to (1.12), the equation above has a unique solution J^ϵ and it can be explicitly written as

$$J^\epsilon(q) = \int_0^\infty e^{-\Lambda_\epsilon(q)r}\sigma\sigma^\star(q)e^{-\Lambda_\epsilon^\star(q)r}\,dr$$

$$= \int_0^\infty e^{-\lambda(q)Ar}\sigma\sigma^\star(q)e^{\lambda(q)Ar}e^{-2\epsilon r}\,dr, \qquad q \in \mathbb{R}^2.$$

It is immediate to check that

$$e^{-\lambda(q)Ar} = \begin{pmatrix} \cos(\lambda(q)r) & -\sin(\lambda(q)r) \\ \\ \sin(\lambda(q)r) & \cos(\lambda(q)r) \end{pmatrix}, \qquad r \geq 0.$$

In what follows, for every $q \in \mathbb{R}^2$ we denote

$$\sigma\sigma^\star(q) = \begin{pmatrix} a_1(q) & a_0(q) \\ \\ a_0(q) & a_2(q) \end{pmatrix},$$

and

$$\beta_0(q) = \frac{a_1(q) + a_2(q)}{4}, \qquad \beta_1(q) = \frac{a_1(q) - a_2(q)}{4} \qquad \beta_2(q) = \frac{a_0(q)}{2}, \qquad (1.20)$$

Lemma 1.2.2. *Assume that $\lambda : \mathbb{R}^2 \to \mathbb{R}$ is differentiable. Then, there exist $M : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ and $R^\epsilon : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ such that for every $\epsilon > 0$*

$$S^\epsilon(q) = \frac{1}{\epsilon}\frac{\beta_0(q)}{\lambda^2(q)}\nabla^\perp\lambda(q) - M(q)\nabla\lambda(q) + R^\epsilon(q)\nabla\lambda(q), \qquad q \in \mathbb{R}^2. \quad (1.21)$$

Proof. We have

$$J^\epsilon_{11}(q) = \frac{\beta_0(q)}{\epsilon} + \beta_1(q)\int_0^\infty \cos(\lambda(q)r)e^{-\epsilon r}\,dr - \beta_2(q)\int_0^\infty \sin(\lambda(q)r)e^{-\epsilon r}\,dr$$

$$J^\epsilon_{22}(q) = \frac{\beta_0(q)}{\epsilon} - \beta_1(q)\int_0^\infty \cos(\lambda(q)r)e^{-\epsilon r}\,dr + \beta_2(q)\int_0^\infty \sin(\lambda(q)r)e^{-\epsilon r}\,dr$$

$$J^\epsilon_{12}(q) = J^\epsilon_{21}(q) = \beta_1(q)\int_0^\infty \sin(\lambda(q)r)e^{-\epsilon r}\,dr + \beta_2(q)\int_0^\infty \cos(\lambda(q)r)e^{-\epsilon r}\,dr.$$

Integrating by parts, we have

$$\int_0^\infty \cos(\lambda(q)r)e^{-\epsilon r}\,dr = \frac{\epsilon}{\lambda^2(q)+\epsilon^2},$$

and

$$\int_0^\infty \sin(\lambda(q)r)e^{-\epsilon r}\,dr = \frac{\lambda(q)}{\lambda^2(q)+\epsilon^2}.$$

This allows to conclude that

$$J^{\epsilon}_{11}(q) = \frac{\beta_0(q)}{\epsilon} + \beta_1(q)\frac{\epsilon}{\lambda^2(q)+\epsilon^2} - \beta_2(q)\frac{\lambda(q)}{\lambda^2(q)+\epsilon^2}$$

$$J^{\epsilon}_{22}(q) = \frac{\beta_0(q)}{\epsilon} - \beta_1(q)\frac{\epsilon}{\lambda^2(q)+\epsilon^2} + \beta_2(q)\frac{\lambda(q)}{\lambda^2(q)+\epsilon^2} \tag{1.22}$$

$$J^{\epsilon}_{12}(q) = J^{\epsilon}_{21}(q) = \beta_1(q)\frac{\lambda(q)}{\lambda^2(q)+\epsilon^2} + \beta_2(q)\frac{\epsilon}{\lambda^2(q)+\epsilon^2}$$

Now, due to (1.18), for every $\epsilon > 0$ and $q \in \mathbb{R}^2$ we have

$$\begin{aligned}
\partial_i \left(\Lambda_\epsilon^{-1}\right)_{11}(q) = \partial_i \left(\Lambda_\epsilon^{-1}\right)_{22}(q) &= -\frac{2\epsilon\lambda(q)}{(\lambda^2(q)+\epsilon^2)^2}\,\partial_i\lambda(q), \qquad i = 1,2, \\
\partial_i \left(\Lambda_\epsilon^{-1}\right)_{12}(q) = -\partial_i \left(\Lambda_\epsilon^{-1}\right)_{21}(q) &= \frac{\lambda^2(q)-\epsilon^2}{(\lambda^2(q)+\epsilon^2)^2}\,\partial_i\lambda(q), \qquad i = 1,2.
\end{aligned} \tag{1.23}$$

Therefore, if we replace (1.22) and (1.23) in (1.19), we obtain

$$S_1^\epsilon(q) = -\frac{2\epsilon\lambda(q)}{(\lambda^2(q)+\epsilon^2)^2}\left[\left(\frac{\beta_0(q)}{\epsilon}+\beta_1(q)\frac{\epsilon}{\lambda^2(q)+\epsilon^2}-\beta_2(q)\frac{\lambda(q)}{\lambda^2(q)+\epsilon^2}\right)\partial_1\lambda(q)\right.$$

$$\left.+\left(\beta_1(q)\frac{\lambda(q)}{\lambda^2(q)+\epsilon^2}+\beta_2(q)\frac{\epsilon}{\lambda^2(q)+\epsilon^2}\right)\partial_2\lambda(q)\right]$$

$$+\frac{\lambda^2(q)-\epsilon^2}{(\lambda^2(q)+\epsilon^2)^2}\left[\left(\beta_1(q)\frac{\lambda(q)}{\lambda^2(q)+\epsilon^2}+\beta_2(q)\frac{\epsilon}{\lambda^2(q)+\epsilon^2}\right)\partial_1\lambda(q)\right.$$

$$\left.+\left(\frac{\beta_0(q)}{\epsilon}-\beta_1(q)\frac{\epsilon}{\lambda^2(q)+\epsilon^2}+\beta_2(q)\frac{\lambda(q)}{\lambda^2(q)+\epsilon^2}\right)\partial_2\lambda(q)\right],$$

and

$$S_2^\epsilon(q) = -\frac{\lambda^2(q)-\epsilon^2}{(\lambda^2(q)+\epsilon^2)^2}\left[\left(\frac{\beta_0(q)}{\epsilon}+\beta_1(q)\frac{\epsilon}{\lambda^2(q)+\epsilon^2}-\beta_2(q)\frac{\lambda(q)}{\lambda^2(q)+\epsilon^2}\right)\partial_1\lambda(q)\right.$$

$$\left.+\left(\beta_1(q)\frac{\lambda(q)}{\lambda^2(q)+\epsilon^2}+\beta_2(q)\frac{\epsilon}{\lambda^2(q)+\epsilon^2}\right)\partial_2\lambda(q)\right]$$

$$-\frac{2\epsilon\lambda(q)}{(\lambda^2(q)+\epsilon^2)^2}\left[\left(\beta_1(q)\frac{\lambda(q)}{\lambda^2(q)+\epsilon^2}+\beta_2(q)\frac{\epsilon}{\lambda^2(q)+\epsilon^2}\right)\partial_1\lambda(q)\right.$$

$$\left.+\left(\frac{\beta_0(q)}{\epsilon}-\beta_1(q)\frac{\epsilon}{\lambda^2(q)+\epsilon^2}+\beta_2(q)\frac{\lambda(q)}{\lambda^2(q)+\epsilon^2}\right)\partial_2\lambda(q)\right],$$

Now we define

$$\Gamma_1^\epsilon(q) := \beta_1(q)\frac{\lambda(q)}{\lambda^2(q)+\epsilon^2} + \beta_2(q)\frac{\epsilon}{\lambda^2(q)+\epsilon^2}, \qquad \Gamma_1(q) := \frac{\beta_1(q)}{\lambda(q)},$$

and

$$\Gamma_2^\epsilon(q) := \beta_1(q)\frac{\epsilon}{\lambda^2(q)+\epsilon^2} - \beta_2(q)\frac{\lambda(q)}{\lambda^2(q)+\epsilon^2}, \qquad \Gamma_2(q) := -\frac{\beta_2(q)}{\lambda(q)}.$$

With these notations, we have

$$S_1^\epsilon(q) = \frac{1}{\epsilon}\frac{\beta_0(q)}{\lambda^2(q)}\,\partial_2\lambda(q) + \left[-\frac{2\beta_0(q)}{\lambda^3(q)} + \frac{\Gamma_1(q)}{\lambda^2(q)}\right]\,\partial_1\lambda(q) - \frac{\Gamma_2(q)}{\lambda^2(q)}\,\partial_2\lambda(q)$$

$$+R_{11}^\epsilon(q)\partial_1\lambda(q) + R_{12}^\epsilon(q)\partial_2\lambda(q),$$

where

$$R_{11}^\epsilon(q) := -\frac{2\epsilon\lambda(q)}{(\lambda^2(q)+\epsilon^2)^2}\Gamma_2^\epsilon(q) + \epsilon\beta_2(q)\frac{\lambda^2(q)-\epsilon^2}{(\lambda^2(q)+\epsilon^2)^3}$$

$$+2\,\beta_0(q)\left[\frac{1}{\lambda^3(q)} - \frac{\lambda(q)}{(\lambda^2(q)+\epsilon^2)^2}\right] - \beta_1(q)\left[\frac{1}{\lambda^3(q)} - \frac{\lambda(q)\,(\lambda^2(q)-\epsilon^2)}{(\lambda^2(q)+\epsilon^2)^3}\right], \tag{1.24}$$

and

$$R^{\epsilon}_{12}(q) := -\frac{2\lambda(q)\epsilon}{(\lambda^2(q)+\epsilon^2)^2}\Gamma^{\epsilon}_1(q) - \epsilon\beta_2(q)\frac{\lambda^2(q)-\epsilon^2}{(\lambda^2(q)+\epsilon^2)^3}$$

$$-\frac{\beta_0(q)}{\epsilon}\left[\frac{1}{\lambda^2(q)} - \frac{\lambda^2(q)-\epsilon^2}{(\lambda^2(q)+\epsilon^2)^2}\right] + \beta_1(q)\left[\frac{1}{\lambda^3(q)} - \frac{\lambda(q)\,(\lambda^2(q)-\epsilon^2)}{(\lambda^2(q)+\epsilon^2)^3}\right]. \tag{1.25}$$

In a similar way, we have

$$S^{\epsilon}_2(q) = -\frac{1}{\epsilon}\frac{\beta_0(q)}{\lambda^2(q)}\partial_1\lambda(q) - \frac{\Gamma_2(q)}{\lambda^2(q)}\partial_1\lambda(q) - \left[\frac{2\beta_0(q)}{\lambda^3(q)} + \frac{\Gamma_1(q)}{\lambda^2(q)}\right]\partial_2\lambda(q)$$

$$+R^{\epsilon}_{21}(q)\partial_1\lambda(q) + R^{\epsilon}_{22}(q)\partial_2\lambda(q),$$

where

$$R^{\epsilon}_{21}(q) := -\frac{2\lambda(q)\epsilon}{(\lambda^2(q)+\epsilon^2)^2}\Gamma^{\epsilon}_1(q) - \epsilon\beta_1(q)\frac{\lambda^2(q)-\epsilon^2}{(\lambda^2(q)+\epsilon^2)^3}$$

$$+\frac{\beta_0(q)}{\epsilon}\left[\frac{1}{\lambda^2(q)} - \frac{\lambda^2(q)-\epsilon^2}{(\lambda^2(q)+\epsilon^2)^2}\right] - \beta_2(q)\left[\frac{1}{\lambda^3(q)} - \frac{\lambda(q)\,(\lambda^2(q)-\epsilon^2)}{(\lambda^2(q)+\epsilon^2)^3}\right], \tag{1.26}$$

and

$$R^{\epsilon}_{22}(q) := \frac{2\epsilon\lambda(q)}{(\lambda^2(q)+\epsilon^2)^2}\Gamma^{\epsilon}_2 - \epsilon\beta_2(q)\frac{\lambda^2(q)-\epsilon^2}{(\lambda^2(q)+\epsilon^2)^3}$$

$$+2\beta_0(q)\left[\frac{1}{\lambda^3(q)} - \frac{\lambda(q)}{(\lambda^2(q)+\epsilon^2)^2}\right] + \beta_1(q)\left[\frac{1}{\lambda^3(q)} - \frac{\lambda(q)\,(\lambda^2(q)-\epsilon^2)}{(\lambda^2(q)+\epsilon^2)^3}\right]. \tag{1.27}$$

Therefore, recalling that $\Gamma_1(q) = \beta_1(q)/\lambda(q)$ and $\Gamma_2(q) = -\beta_2(q)/\lambda(q)$, if we define

$$M(q) = \frac{1}{\lambda^3(q)}\begin{pmatrix} 2\beta_0(q)-\beta_1(q) & -\beta_2(q) \\ -\beta_2(q) & 2\beta_0(q)+\beta_1(q) \end{pmatrix}, \tag{1.28}$$

and we define $R^{\epsilon}(q) = (R^{\epsilon}_{ij}(q))_{i,j=1,2}$, where $R^{\epsilon}_{ij}(q)$ are defined in (1.24), (1.25), (1.26) and (1.27), we obtain (1.21). □

In what follows we shall assume that the following condition is satisfied.

Hypothesis 2. *1. The mapping $\lambda : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable.*

2. For every $\epsilon > 0$, the mapping $S_\epsilon : \mathbb{R}^2 \to \mathbb{R}^2$ introduced in (1.19) is locally Lipschitz-continuous and has linear growth.

3. For every $\epsilon > 0$ the mappings $\Lambda_\epsilon^{-1} b : \mathbb{R}^2 \to \mathbb{R}^2$ and $\Lambda_\epsilon^{-1}\sigma : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$

are locally Lipschitz-continuous and have linear growth.

Remark 1.2.3. 1. According to the expression of $M(q)$ given in (1.28) and the expressions for the coefficients of $R_\epsilon(q)$ given in (1.24), (1.25), (1.26) and (1.27), thanks to what we have already assumed in Hypothesis 1 we can check easily that Hypothesis 2 is satisfied if we assume σ to be bounded and λ to be bounded and differentiable, with $\nabla\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ Lipschitz-continuous.

2. In the same way, if we assume that $\nabla\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ is locally Lipschitz-continuous and has linear growth and there exists $c > 0$ such that for $|q|$ large enough

$$|\lambda(q)| \geq c\,|q|^2,$$

then Hypothesis 2 is satisfied, without assuming σ to be bounded.

Theorem 1.2.4. *For every $\mu, \epsilon > 0$, let $q_{\mu,\epsilon}$ be the solution of problem (1.15). Then, under Hypotheses 1 and 2, for every $\epsilon > 0$ we have*

$$\lim_{\mu\to 0} \mathbb{E} \sup_{t\in[0,T]} |q_{\mu,\epsilon}(t) - q_\epsilon(t)| = 0, \tag{1.29}$$

where q_ϵ is the solution of the problem

$$dq_\epsilon(t) = \left[\Lambda_\epsilon^{-1} b(q_\epsilon(t)) + S_\epsilon(q_\epsilon(t))\right] dt + \Lambda_\epsilon^{-1}\sigma(q_\epsilon(t))\, dw(t), \qquad q_\epsilon(0) = q. \tag{1.30}$$

Proof. According to Hypotheses 1 and 2, we have that for every $\epsilon > 0$ and for every $k \geq 1$ and $T > 0$ problem (1.30) admits a unique solution $q_\epsilon \in L^k(\Omega; C([0,T];\mathbb{R}^2))$. As $\langle \Lambda_\epsilon(q)p, p\rangle = \epsilon\,|p|^2$, this allows to conclude thanks to [12, Theorem 2.4]. □

1.3 Some notations and further assumptions

We consider here the system

$$\dot{X}(t) = \frac{\beta_0(X(t))}{\lambda^2(X(t))}\,\nabla^\perp \lambda(X(t)). \tag{1.31}$$

Clearly, for every $t \geq 0$, we have $\lambda(X(t)) = \lambda(X(0))$. Now, if we consider the perturbed system

$$dX_\epsilon(q) = \frac{\beta_0(X_\epsilon(t))}{\lambda^2(X_\epsilon(t))}\nabla^\perp\lambda(X_\epsilon(t))\,dt$$

$$+\epsilon\left[\frac{1}{\lambda(X_\epsilon(t))}Ab(X_\epsilon(t)) - M(X_\epsilon(t))\nabla\lambda(X_\epsilon(t))\right]\,dt + \frac{\sqrt{\epsilon}}{\lambda(X_\epsilon(t))}A\sigma(X_\epsilon(t))\,dw(t)$$

$$+\epsilon^2\left[H^\epsilon(X_\epsilon(t))b(X_\epsilon(t)) + \hat{R}^\epsilon(X_\epsilon(t))\nabla\lambda(X_\epsilon(t))\right]\,dt + \epsilon H^\epsilon(X_\epsilon(t))\sigma(X_\epsilon(t))\,dw(t),$$

the quantity $\lambda(X_\epsilon(t))$ is not anymore conserved. However, for any fixed time interval $[0, T]$ we have

$$\lim_{\epsilon\to 0}\mathbb{E}\sup_{t\in\,[0,T]}|X_\epsilon(t) - X(t)|^k = 0,$$

and, in particular, the following trivial limit holds

$$\lim_{\epsilon\to 0}\mathbb{E}\sup_{t\in\,[0,T]}|\lambda(X_\epsilon(t)) - \lambda(X(0))|^k = 0.$$

Now, with the change of time $t \mapsto t/\epsilon$, we have

$$\mathcal{L}(X_\epsilon(\cdot/\epsilon)) = \mathcal{L}(q_\epsilon(\cdot)),$$

where q_ϵ is the solution of equation (1.38). Our aim is to identify the non trivial limit for the distribution of the process $\lambda(q_\epsilon(\cdot))$, as $\epsilon \downarrow 0$. To this purpose, in addition to Hypotheses 1 and 2, we assume that λ satisfies the following conditions.

Hypothesis 3. *1. If β_0 is the function defined in (1.20), we have*

$$\inf_{x \in \mathbb{R}^2} \beta_0(x) > 0. \tag{1.32}$$

2. The mapping $\lambda : \mathbb{R}^2 \to \mathbb{R}$ is four times continuously differentiable, with bounded second derivative.

3. It has only a finite number of critical points $x_1, \dots, x_n$. The matrix of second derivatives $D^2\lambda(x_i)$ is non degenerate, for every $i = 1, \dots, n$ and $\lambda(x_i) \neq \lambda(x_j)$, if $i \neq j$.

4. One of the two following conditions is satisfied.

(a) There exist three positive constants a_1, a_2, a_3 such that $\lambda(x) \geq a_1 |x|^2$, $|\nabla\lambda(x)| \geq a_2 |x|$ and $\Delta\lambda(x) \geq a_3$, for all $x \in \mathbb{R}^2$, with $|x|$ large enough.

(b) We have

$$\exists \lim_{|x|\to\infty} \lambda(x) =: \bar{\lambda} \in \mathbb{R},$$

and for every compact set $K \subset \mathbb{R}^2$

$$\bar{\lambda} > \sup_{x\in K} \lambda(x).$$

Remark 1.3.1. 1. Remember that the function β_0 was defined as $[(\sigma\sigma^\star)^2_{11} + (\sigma\sigma^\star)^2_{22}]/4$. Therefore, condition (1.32) is a non-degeneracy condition on the noisy perturbation.

2. As explained in Remark 1.2.3, both condition (a) and condition (b) in Hypothesis 3 are compatible with Hypotheses 1 and 2.

Next, for every $z \geq \lambda_0$, we denote by $C(z)$ the z-level set

$$C(z) = \left\{x \in \mathbb{R}^2 \,:\, \lambda(x) = z\right\}.$$

The set $C(z)$ may consist of several connected components

$$C(z) = \bigcup_{k=1}^{N(z)} C_k(z),$$

and for every $x \in \mathbb{R}^2$ we have

$$X(0) = x \Longrightarrow X(t) \in C_{k(x)}(\lambda(x)), \qquad t \geq 0,$$

where $C_{k(x)}(x)$ is the connected component of the level set $C(\lambda(x))$, to which the point x belongs. For every $z \geq 0$ and $k = 1, \ldots, N(z)$, we shall denote by $G_k(z)$ the domain of $\mathbb{R}^2$ bounded by the level set component $C_k(z)$.

If we identify all points in $\mathbb{R}^2$ belonging to the same connected component of a given level set $C(z)$ of the Hamiltonian λ, we obtain a graph Γ, given by several intervals $I_1, \ldots I_n$ and vertices $O_1, \ldots, O_m$. The vertices will be of two different types, external and internal vertices. External vertices correspond to local extrema of λ, while internal vertices correspond to saddle points of λ. Among external vertices, we will also include O_∞, the endpoint of the interval in the graph corresponding to the point at infinity.

In what follows, we shall denote by $\Pi : \mathbb{R}^2 \to \Gamma$ the *identification map*, that associates to every point $x \in \mathbb{R}^2$ the corresponding point $\Pi(x)$ on the graph Γ. We have $\Pi(x) = (\lambda(x), k(x))$, where $k(x)$ denotes the number of the interval on the graph Γ, containing the point $\Pi(x)$. If O_i is one of the interior vertices, the second coordinate cannot be chosen in a unique way,

as there are three edges having O_i as their endpoint. Notice that both $k(x)$ and $H(x)$ are first integrals (a discrete and a continuous one, respectively) for system (1.31).

On the graph Γ, a distance can be introduced in the following way. If $y_1 = (z_1, k)$ and $y_2 = (z_2, k)$ belong to the same edge I_k, then $d(y_1, y_2) = |z_1 - z_2|$. In the case y_1 and y_2 belong to different edges, then

$$d(y_1, y_2) = \min \left\{ d(y_1, O_{i_1}) + d(O_{i_1}, O_{i_2}) + \cdots + d(O_{i_j}, y_2) \right\},$$

where the minimum is taken over all possible paths from y_1 to y_2, through every possible sequence of vertices $O_{i_1}, \ldots, O_{i_j}$, connecting y_1 to y_2.

If z is not a critical value, then each $C_k(z)$ consists of one periodic trajectory of the vector field $\nabla^{\perp}\lambda(x)$. If z is a local extremum of $\lambda(x)$, then, among the components of $C(z)$ there is a set consisting of one point, the rest point of the flow. If $\lambda(x)$ has a saddle point at some point x_0 and $\lambda(x_0) = z$, then $C(z)$ consists of three trajectories, the equilibrium point x_0 and the two trajectories that have x_0 as their limiting point, as $t \to \pm\infty$.

Now, for every $(z,k) \in \Gamma$, we define

$$T_k(z) = \oint_{C_k(z)} \frac{\lambda^2(x)}{\beta_0(x)|\nabla\lambda(x)|}\, dl_{z,k}, \tag{1.33}$$

where $dl_{z,k}$ is the length element on $C_k(z)$. Notice that $T_k(z)$ is the period of the motion along the level set $C_k(z)$.

As we have seen above, if $X(0) = x \in C_k(z)$, then $X(t) \in C_k(z)$, for every $t \geq 0$. As known, for every $(z,k) \in \Gamma$ the probability measure

$$d\mu_{z,k} := \frac{1}{T_k(z)} \frac{\lambda^2(x)}{\beta_0(x)|\nabla\lambda(x)|}\, dl_{z,k} \tag{1.34}$$

is invariant for system (1.31) on the level set $C_k(z)$.

1.4 The averaging limit

For every $\epsilon > 0$ we have

$$\Lambda_\epsilon^{-1}(q) = \frac{1}{\lambda(q)} A + \epsilon\, H^\epsilon(q), \tag{1.35}$$

where

$$H^\epsilon(q) := -\frac{\epsilon}{\lambda(q)\,(\lambda^2(q)+\epsilon^2)} A + \frac{1}{\lambda^2(q)+\epsilon^2} I.$$

Notice that

$$\sup_{\epsilon>0} |H^\epsilon(q)| < \infty, \qquad q \in \mathbb{R}^2. \tag{1.36}$$

Lemma 1.4.1. *There exists* $\hat{R}^\epsilon : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ *such that*

$$R^\epsilon(q) = \epsilon \hat{R}^\epsilon(q), \qquad \sup_{\epsilon>0} |\hat{R}^\epsilon(q)| < \infty, \qquad q \in \mathbb{R}^2. \tag{1.37}$$

Proof. We have

$$\frac{1}{\lambda^3(q)} - \frac{\lambda(q)}{(\lambda^2(q)+\epsilon^2)^2} = \epsilon^2 \left[\frac{2\lambda^2(q)+\epsilon^2}{\lambda^3(q)(\lambda^2(q)+\epsilon^2)^2} \right],$$

and

$$\frac{1}{\lambda^3(q)} - \frac{\lambda(q)\left(\lambda^2(q)-\epsilon^2\right)}{(\lambda^2(q)+\epsilon^2)^3} = \epsilon^2 \left[\frac{4\lambda^4(q)+3\epsilon^2\lambda^2(q)+\epsilon^4}{\lambda^3(q)\left(\lambda^2(q)+\epsilon^2\right)^3} \right]$$

and

$$\frac{1}{\lambda^2(q)} - \frac{\lambda^2(q)-\epsilon^2}{(\lambda^2(q)+\epsilon^2)^2} = \epsilon^2 \left[\frac{3\lambda^2(q)+\epsilon^2}{\lambda^2(q)\left(\lambda^2(q)+\epsilon^2\right)^2} \right].$$

Therefore, recalling how $R^\epsilon(q)$ was defined in (1.24), (1.25), (1.26) and (1.27), we can conclude.

□

Now, according to (1.21), (1.35), (1.36) and (1.37), equation (1.30) can

be rewritten as

$$dq_\epsilon(t) = \frac{1}{\epsilon}\frac{\beta_0(q_\epsilon(t))}{\lambda^2(q_\epsilon(t))}\nabla^\perp\lambda(q_\epsilon(t))\,dt + B(q_\epsilon(t))\,dt + \Sigma(q_\epsilon(t))\,dw(t),$$

$$+\epsilon\,\left[B_\epsilon(q_\epsilon(t))\,dt + \Sigma_\epsilon(q_\epsilon(t))\,dw(t)\right], \qquad q_\epsilon(0) = q \tag{1.38}$$

where

$$B(q) = \frac{1}{\lambda(q)}Ab(q) - M(q)\nabla\lambda(q), \quad \Sigma(q) = \frac{1}{\lambda(q)}A\sigma(q),$$

and

$$B_\epsilon(q) = H^\epsilon(q)b(q) + \hat{R}^\epsilon(q)\nabla\lambda(q), \quad \Sigma_\epsilon(q) = H^\epsilon(q)\sigma(q).$$

This means that, as $\epsilon \downarrow 0$, part of the coefficients are of order $O(\epsilon^{-1})$, part of order $O(1)$ and part of order $O(\epsilon)$.

With the notations introduced in the previous section, in what follows, we want to investigate the limiting behavior of the Γ-valued process $\Pi(q_\epsilon(\cdot)) = (\lambda(q_\epsilon(\cdot)), k(q_\epsilon(\cdot)))$, as $\epsilon \downarrow 0$.

If we apply Itô's formula to $\lambda(q_\epsilon(t))$, we get

$$d\lambda(q_\epsilon(t)) = \mathcal{G}\lambda(q_\epsilon(t))\,dt + \mathcal{A}\lambda(q_\epsilon(t))\,dw(t) + \epsilon\,\mathcal{G}_\epsilon\lambda(q_\epsilon(t))\,dt + \epsilon\,\mathcal{A}_\epsilon\lambda(q_\epsilon(t))\,dw(t),$$

where for every $f \in C^2(\mathbb{R}^2)$ and $q \in \mathbb{R}^2$

$$\mathcal{G}f(q) = \frac{1}{2}\mathrm{Tr}\left[\Sigma\Sigma^\star(q)D^2f(q)\right] + \langle Df(q), B(q)\rangle,$$

$$\mathcal{A}f(q) = \Sigma(q)^\star Df(q),$$

$$\mathcal{G}_\epsilon f(q) = \frac{1}{2}\mathrm{Tr}\left[\left(\epsilon\,\Sigma_\epsilon\Sigma_\epsilon^\star(q) + \Sigma\Sigma_\epsilon^\star(q) + \Sigma_\epsilon\Sigma^\star(q)\right)D^2f(q)\right] + \langle Df(q), B_\epsilon(q)\rangle,$$

and

$$\mathcal{A}_\epsilon f(q) = \Sigma_\epsilon^\star(q)Df(q).$$

We recall that the graph Γ is made of n intervals $I_1, \ldots, I_n$ and m verteces $O_1, \ldots, O_m$. For every $j = 1, \ldots, n$ and for every f that is twice differentiable in the interior of the edge I_j, we denote

$$\mathcal{L}_j f(z) = \frac{1}{2}\alpha_j(z)f''(x) + \gamma_j(z)f'(z), \tag{1.39}$$

where

$$\alpha_j(z) = \oint_{C_j(z)} |\mathcal{A}\lambda(x)|^2\, d\mu_{z,j}(x) = \oint_{C_j(z)} |\Sigma^\star(x)\nabla\lambda(x)|^2\, d\mu_{z,j}(x)$$

and

$$\gamma_j(z) = \oint_{C_j(z)} \mathcal{G}\lambda(x)\, d\mu_{z,j}(x).$$

Definition 1.4.2. *For each interior vertex O_k and any segment I_j meeting at O_k (notation $I_j \sim O_k$), let ρ_{kj} be the positive constant defined by*

$$\rho_{kj} = \oint_{C_{kj}} \frac{\lambda^2(x)}{\beta_0(x)|\nabla\lambda(x)|} |\Sigma^\star(x)\nabla\lambda(x)|^2\, dl(x).$$

We denote by $D(L) \subset C(\Gamma)$ the set consisting all continuous functions f defined on the graph Γ such that $\mathcal{L}_j f$ is well defined in the interior of the edge I_j and for every $I_j \sim O_k$ there exists finite

$$\lim_{x \to O_k} \mathcal{L}_j f(x)$$

and the limit is independent of the edge I_j. Moreover, for each interior vertex O_k

$$\sum_{j\,:\,I_j \sim O_k} \pm \rho_{kj} f_j'(\lambda(O_k)) = 0,$$

where f_j' denotes the derivative of f with respect to the local coordinate λ, along the edge I_j and the sign $\pm$ are taken if $\lambda > \lambda(O_k)$ or $\lambda < \lambda(O_k)$.

Next, for every $f \in D(L)$, we define

$$Lf(x) = \begin{cases} \mathcal{L}_j f(x), & \text{if } x \text{ is an interior point of } I_j, \\ \lim_{x \to O_k} \mathcal{L}_j f(x), & \text{if } x \text{ is the vertex } O_k \text{ and } I_j \sim O_k. \end{cases}$$

As proven in [11, Theorem 8.2.1], in case $\Sigma(q) = I$ the operator L defined on the domain $D(L)$, as described in Definition 1.4.2, is the generator of a strong Markov process Y_t on Γ with continuous trajectories. here the same result holds, because of the non-degeneracy condition (1.32) satisfied by the diffusion coefficient $\Sigma(q)$.

Theorem 1.4.3. *Under Hypotheses 1, 2 and 3, for every fixed $T > 0$ the Γ-valued process $\Pi(q_\epsilon(\cdot))$ converges weakly in $C([0,T];\Gamma)$ to the Markov process Y generated by the oiperator $(L, D(L))$, as described in Definition 1.4.2.*

Proof. In case in equation (1.38) we have $B(q) = B_\epsilon(q) = \Sigma_\epsilon(q) = 0$, $\Sigma(q) = I$ and λ satisfies condition (b) in Hypothesis 3, the result above is what is proven in [11, Theorem 8.2.2]. In the present situation we are dealing with the more general situation in which we have a coefficient $B(q)$ of order $O(1)$ and coefficients $B_\epsilon(q)$ and $\Sigma_\epsilon(q)$ of order $O(\epsilon)$. Moreover we allow a non-constant diffusion coefficient $\Sigma(q)$. Under this more general assumptions, an

averaging principle of the same type of the one described in [11, Theorem 8.2.2] holds. This of course has required to introduce a suitable generalization of the operator $(L, D(L))$. We will give a proof of this generalized result in Chapter 2.

□

Chapter 2: A generaliztion of the Freidlin-Wentcell theorem on averaging of Hamiltonian systems

In order to prove Theorem 1.4.3, we generalize the classical Freidlin-Wentzell's theorem for random perturbations of Hamiltonian systems in this Chapter. In stead of the two-dimensional standard Brownian motion, the coefficient for the noise term is no longer the identity matrix but a state-dependent matrix plus a state-dependent matrix that converges uniformly to 0 on any compact sets as ϵ tends to 0. We also take the drift term into consideration where the drfit term also contains two parts, the state-dependent mapping and a state-dependent mapping that converges uniformly to 0 on any compact sets as ϵ tends to 0. In the proof, we use the result of generalized differential operator. We also adapt a new way to prove the weak convergence inside the edge by constructing an auxiliary process and apply Girsanov's theorem in the proof of gluing condition.

2.1 Introduction

To generalize the theorem to a large extent, we will not use the notation in the previous chapter and consider the following system

$$\begin{cases} dq_t^\epsilon = \frac{1}{\epsilon} g(q_t^\epsilon)dt + [b(q_t^\epsilon) + b^\epsilon(q_t^\epsilon)]dt + [\sigma(q_t^\epsilon) + \sigma^\epsilon(q_t^\epsilon)]dW_t, \\ q_0^\epsilon = q \in \mathbb{R}^2, \end{cases} \tag{2.1}$$

where ϵ is a small positive constant and W_t is a standard two-dimensional Brownian motion defined on the stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. We assume that $b, b^\epsilon : \mathbb{R}^2 \to \mathbb{R}^2$ and $\sigma, \sigma^\epsilon : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ are differentiable mappings having bounded derivatives and the mappings b^ϵ and σ^ϵ converge to zero, as ϵ goes to zero, uniformly on any compact subset of $\mathbb{R}^2$.

In what follows, we shall assume that there exists a function $H : \mathbb{R}^2 \mapsto \mathbb{R}^2$ such that

$$g(x) \cdot \nabla H(x) = 0, \qquad x \in \mathbb{R}^2. \tag{2.2}$$

This means in particular that if we denote by X_t^ϵ the solution of the unperturbated system

$$\frac{dX_t^\epsilon}{dt} = \frac{1}{\epsilon} g(X_t^\epsilon),$$

then $H(X_t^\epsilon) = H(X_0^\epsilon)$, for every $t \geq 0$ and $\epsilon > 0$. This means that X_t^ϵ remains on the same level set of H, for every $t \geq 0$. Moreover, if we define

$$a(x) = |g(x)|/|\nabla H(x)|, \tag{2.3}$$

then it can be easily proved that $a^{-1}(x)$ is the density of the invariant measure for X_t^ϵ .

Now, for every $x \geq \inf_{z\in\mathbb{R}^2} H(z)$, we denote by $C(x)$ the x-level set of H, that is

$$C(x) = \left\{ z \in \mathbb{R}^2 \,:\, H(z) = x \right\}.$$

The set $C(x)$ may consist of several connected components

$$C(x) = \bigcup_{k=1}^{N(x)} C_k(x),$$

and for every $z \in \mathbb{R}^2$ we will denote by $C_{k(z)}(H(z))$ the connected component of the level set $C(H(z))$, to which the point z belongs. If we identify all points in $\mathbb{R}^2$ belonging to the same connected component of a given level set $C(x)$ of the Hamiltonian H, we obtain a graph Γ, given by several intervals $I_1, \ldots I_n$ and vertices $O_1, \ldots, O_m$. In what follows, we shall denote by $\Pi : \mathbb{R}^2 \to \Gamma$ the

identification map, that associates to every point $z \in \mathbb{R}^2$ the corresponding point $\Pi(z)$ on the graph Γ. We have $\Pi(z) = (H(z), i(z))$, where $i(z)$ denotes the number of the interval on the graph Γ, containing the point $\Pi(z)$.

In the present paper we are interested in the asymptotic behavior of the Γ-valued process $\Pi(q^\epsilon)$. Namely, we want to show that $\Pi(q^\epsilon)$ converges in distribution in the space $C([0,T];\Gamma)$, as $\epsilon \to 0$, to a Markov process in Γ, whose generator is explicitly described in terms of suitable differential operators in the interior of every edge and suitable gluing conditions at each interior vertex.

If we define $x_t^\epsilon = H(q_t^\epsilon)$, as an immediate consequence of the Itô's formula we have

$$dx_t^\epsilon = \mathcal{L}_0 H(q_t^\epsilon)dt + \mathcal{R}_0 H(q_t^\epsilon)dW_t + \mathcal{L}_0^\epsilon H(q_t^\epsilon)dt + \mathcal{R}_0^\epsilon H(q_t^\epsilon)dW_t$$

where

$$\mathcal{L}_0 f(x) = \nabla f(x)^t \cdot b(x) + \frac{1}{2}\sum_{i,j}(\sigma\sigma^*)_{i,j}(x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j},$$

$$\mathcal{R}_0 f(x) = \nabla f(x)^t \sigma(x),$$

$$\mathcal{L}_0^\epsilon f(x) = \nabla f(x)^t \cdot b^\epsilon(x) + \frac{1}{2}\sum_{i,j}[\sigma(\sigma^\epsilon)^* + \sigma^\epsilon\sigma^* + \sigma^\epsilon(\sigma^\epsilon)^*]_{i,j}(x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j},$$

and

$$\mathcal{R}_0^\epsilon f(x) = \nabla f(x)^t \sigma^\epsilon(x).$$

Next, for every $x \in I_i$, we define

$$A_i(x) = \frac{1}{T_i(x)} \oint_{C_i(x)} \mathcal{R}_0 H(u) \mathcal{R}_0 H(u)^* \frac{dl}{|g(u)|}$$

and

$$B_i(x) = \frac{1}{T_i(x)} \oint_{C_i(x)} \mathcal{L}_0 H(u) \frac{dl}{|g(u)|},$$

where

$$T_i(x) = \oint_{C_i(x)} \frac{dl}{|g(u)|}.$$

Moreover, we define

$$\mathcal{L}_i f(x) = B_i(x) \frac{d}{dx} f + \frac{1}{2} A_i(x) \frac{d^2}{dx^2} f. \tag{2.4}$$

With these notations, we can introduce the following operator acting on functions defined on the graph Γ.

Definition 2.1.1. *For $I_i \sim O_k$, let*

$$p_{ki} = \frac{\beta_{ki}}{\sum_{i:I_i \in O_k} \beta_{ki}}, \tag{2.5}$$

where

$$\beta_{ki} = \oint_{C_{ki}} \frac{|\nabla H(x)\sigma\sigma^*(x)\nabla H(x)|}{|g(x)|} dl.$$

We denote by $D(L) \subseteq C(\Gamma;\mathbb{R})$ the set consisting all functions f defined on the graph Γ such that $\mathcal{L}_j f$ is well defined in the interior of the edge I_j and for every $I_j \sim O_k$ there exists finite

$$\lim_{x \to O_k} \mathcal{L}_j f(x)$$

and the limit is independent of the edge I_j. Moreover, for each interior vertex O_k

$$\sum_{j\,:\,I_j \sim O_k} \pm \rho_{kj} f_j'(H(O_k)) = 0,$$

where f_j' denotes the derivative of f with respect to the local coordinate λ, along the edge I_j and the signs $\pm$ are taken if $H > H(O_k)$ or $H < H(O_k)$.

Next, for every $f \in D(L)$, we define

$$Lf(x) = \begin{cases} \mathcal{L}_j f(x), & \text{if } x \text{ is an interior point of } I_j, \\ \lim_{x \to O_k} \mathcal{L}_j f(x), & \text{if } x \text{ is the vertex } O_k \text{ and } I_j \sim O_k. \end{cases}$$

The main result of this chapter is given by the following theorem.

Theorem 2.1.2. *Suppose q_t^ϵ satisfies the following stochastic differential equation*

$$dq_t^\epsilon = \frac{1}{\epsilon} g(q_t^\epsilon)dt + b(q_t^\epsilon)dt + \sigma(q_t^\epsilon)dW_t + b^\epsilon(q_t^\epsilon)dt + \sigma^\epsilon(q_t^\epsilon)dW_t$$

with initial condition $q_0^\epsilon = q$. Assume the coefficients satisfy Hypothesis 8 and the Hamiltonian $H : \mathbb{R}^2 \to \mathbb{R}$ introduced in (2.2) satisfies Hypothesis 9. Then the process $\Pi(q_t^\epsilon) = (x_t^\epsilon, i(q_t^\epsilon))$ converges weakly in $C([0,T];\Gamma)$ to the Markov process Y generated by the operator $(L, D(L))$, as described in Definition 2.1.1.

In the present paper, we generalize the well known result by Freidlin-Wentcell on the validity of an averaging principle for Hamiltonian sustems (see [11, Chapter 8])to a more general case and introduce a new method which simplifies some steps in the proof. Compared with the original Freidlin-

Wentcell theorem, here we can cover the case of a state dependent diffusion coefficient and we can also deal with a the drift term. Moreover, both the diffusion coefficient and the drift term are given by the sum of a term of order one and a term of order ϵ. For the terms of order ϵ, we assume that, as ϵ goes to zero, they converge uniformly to zero over any compact sets in $\mathbb{R}^2$.

In the proof of the weak convergence in the interior of every edge and the analysis of the behavior of the process near the exterior vertices, we introduce a new proof, based on the construction of a suitable auxiliary process. What is remarkable is that, unlike the original proof, this new method unifies the two cases together. In the proof of the svalidity of the gluing conditions, when dealing with the extra terms of order ϵ in the drift and in the diffusion coefficient, we first introduce an auxiliary vector field in order to be able to apply the classical results based on generalized differential operators. Then we apply Girsanov's theorem to get rid of the compensated drift term.

2.2 Some Preliminaries

2.2.1 Hypotheses and notations

Concerning the coefficients in equation (2.1), we assume the following conditions

Hypothesis 4. *1. The mappings* $g,\ b,\ b_\epsilon : \mathbb{R}^2 \to \mathbb{R}^2$ *and* $\sigma,\ \sigma_\epsilon : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ *are all continuously differentiable with uniformly bounded derivatives.*

2. The functions b^ϵ *and* σ^ϵ *converge to zero as* ϵ *goes to zero, uniformly on any compact set in* $\mathbb{R}^2$*.*

The Hamiltonian H satisfies the following conditions.

Hypothesis 5. *H belongs to $\boldsymbol{C}^1(\mathbb{R}^2)$ and has bounded second derivatives. Moreover*

1. H has finite many critical points and for any two critical points x_1 and x_2, $H(x_1) \neq H(x_2)$.

2. For any critical point of H, the matrix σ is invertible in some neighbor of it.

3. *The matrix of second order derivative is non-degenerate at any critical point of H.*

4. *There exists a constant $C_1 > 0$ such that $H(x) \geq C_1(|x|^2+1)$, $\nabla H(x) \geq C_1|x|$, and $\Delta H(x) \geq C_1$, for all $x \in \mathbb{R}^2$ such that $|x|$ large enough.*

As we mentioned in the Introduction, if we identify all points in $\mathbb{R}^2$ belonging to the same connected component of a given level set $C(x)$ of the Hamiltonian H, we obtain a graph Γ, given by several intervals $I_1, \dots I_n$ and vertices $O_1, \dots, O_m$. The vertices will be of two different types, external and internal vertices. External vertices correspond to local extrema of H, while internal vertices correspond to saddle points of H. Among external vertices, we will also include O_∞, the endpoint of the interval in the graph corresponding to the point at infinity.

We have seen that the *identification map* associates to every point $z \in \mathbb{R}^2$ the corresponding point $\Pi(z)$ on the graph Γ. Thus, if $\Pi(z) = (H(z), i(z))$, then $i(z)$ denotes the number of the edge on the graph Γ, containing the point $\Pi(z)$. If O_i is one of the interior vertices, the second coordinate cannot be chosen in a unique way, as there are three edges having O_i as their endpoint.

On the graph Γ, a distance can be introduced in the following way. If $y_1 = (x_1, i)$ and $y_2 = (x_2, i)$ belong to the same edge I_i, then $d(y_1, y_2) =$

$|x_1 - x_2|$. In the case y_1 and y_2 belong to different edges, then

$$d(y_1, y_2) = \min \left\{ d(y_1, O_{i_1}) + d(O_{i_1}, O_{i_2}) + \cdots + d(O_{i_j}, y_2) \right\},$$

where the minimum is taken over all possible paths from y_1 to y_2, through every possible sequence of vertices $O_{i_1}, \ldots, O_{i_j}$, connecting y_1 to y_2.

If x is not a critical value of H, then each $C_k(x)$ consists of one periodic trajectory of the vector field $g(z)$. If x is a local extremum of $H(z)$, then, among the components of $C(x)$ there is a set consisting of one point, the rest point of the flow. If $H(x)$ has a saddle point at some point z_0 and $H(z_0) = x$, then $C(x)$ consists of three trajectories, the equilibrium point z_0 and the two trajectories that have z_0 as their limiting point, as $t \to \pm\infty$.

We introduce some other notations that will be used through out the paper. Letter D is used to denote domain, while letter C is for the level set of the Hamiltonian system. For any $A \subseteq \mathbb{R}$, $D(A)$ is the interior of $\{z \in \mathbb{R}^2; H(z) \in A\}$

$$D_i = \{z \in \mathbb{R}^2; \Pi(z) = (\cdot, i)\}$$

$$D_i(H_1, H_2) = D_i \cap D((H_1, H_2))$$

$$D_k(\pm\delta) = D((H(O_k) - \delta, H(O_k) + \delta)),$$

and

$$C(x) = \partial D((-\infty, x])$$

$$C_i(x) = \bar{D}_i \cap C(x)$$

$$C_k = C(H(O_k))$$

$$C_{ki} = C(H(O_k)) \cap \partial D_i$$

$$C_{ki}(\pm\delta) = \partial D_k(\pm\delta) \cap I_i.$$

Finally, we write $I_i \sim O_k$ if and only if one end of the edge I_i is O_k.

Throughout this paper, we shall denote

$$\tilde{q}_t^\epsilon = q_{\epsilon t}^\epsilon, \qquad \tilde{X}_t^\epsilon = X_{\epsilon t}^\epsilon.$$

2.2.2 Generalized Differential Operator

In the proof of Theorem 3.3.3, we will need to rewrite each operator $\mathcal{L}_i$ in the form of generalized differential operator. That is we want to find two

measures u_i and v_i such that

$$\mathcal{L}_i f = \frac{d}{dv_i}(\frac{df}{du_i}).$$

Let u_i' and v_i' be the Radon-Nikodym derivative of u_i and v_i with respect to the Lebesgue measure respectively. For a reason which will be clear lator, we want to choose

$$u_i'(x) = (\oint_{C_i(x)} \frac{[(\nabla H)^t \sigma\sigma^* \nabla H](u)}{|g(u)|} dl)^{-1}$$

and

$$v_i'(x) = \oint_{C_i(x)} \frac{dl}{|g(u)|}.$$

However, with this choice of u_i and u_i, $d/du_i d/v_i$ does not generally equal to $\mathcal{L}_i$. In fact

$$\begin{aligned}
&\frac{d}{dv_i(x)}(\frac{d}{du_i(x)}f(x)) \\
=&\frac{d}{dx}(f'(x)\cdot\oint_{C_i(x)}\frac{[(\nabla H)^t\sigma\sigma^*\nabla H](u)}{|g(u)|}dl)\cdot\frac{dx}{dv_i(x)} \\
=&f''(x)\oint_{C_i(x)}\frac{[(\nabla H)^t\sigma\sigma^*\nabla H](u)}{|g(u)|}dl\cdot(\oint_{C(x)}\frac{dl}{|g(u)|})^{-1}+ \\
&+f'(x)\frac{d}{dx}\oint_{C_i(x)}\frac{[(\nabla H)^t\sigma\sigma^*\nabla H](u)}{|g(u)|}dl\cdot(\oint_{C(x)}\frac{dl}{|g(u)|})^{-1} \\
=&A_i(x)f''(x)+f'(x)\frac{d}{dx}\oint_{C_i(x)}\frac{[(\nabla H)^t\sigma\sigma^*\nabla H](u)}{|g(u)|}dl\cdot(\oint_{C(x)}\frac{dl}{|g(u)|})^{-1}.
\end{aligned}$$

To calculate

$$\frac{d}{dx}\oint_{C_i(x)}\frac{[(\nabla H)^t\sigma\sigma^*\nabla H](u)}{|g(u)|}dl,$$

by equation (2.3),

$$\begin{aligned}
\frac{d}{dx}\oint_{C_i(x)}\frac{[(\nabla H)^t\sigma\sigma^*\nabla H](u)}{|g(u)|}dl=&\frac{d}{dx}\oint_{C_i(x)}\frac{\sigma\sigma^*\nabla H}{|g|}\cdot\frac{\nabla H}{|\nabla H|}|\nabla H|dl \\
=&\frac{d}{dx}\oint_{C_i(x)}\frac{\sigma\sigma^*\nabla H}{a}\cdot\vec{\nu}dl
\end{aligned}$$

where $\vec{\nu}$ is the unit normal vector of $\partial D_i(x)$. Apply Divergence Theorem, we have

$$\oint_{C_i(x)} \frac{\sigma\sigma^*\nabla H}{a} \cdot \vec{\nu} dl = \oint_{D_i(x)} \nabla \cdot (\frac{\sigma\sigma^*\nabla H}{a}) du.$$

To deal with

$$\frac{d}{dx} \oint_{D_i(x)} \nabla \cdot (\frac{\sigma\sigma^*\nabla H}{a}) du,$$

we have the following lemma

Lemma 2.2.1. *If we assume that $|\nabla H| \geq c_0 > 0$, for any $f \in \boldsymbol{C}^1(\mathbb{R}^2)$, we have*

$$\frac{d}{dx} \int_{D_i(x)} f(u) du = \oint_{C_i(x)} f(u) \frac{dl}{|\nabla H(u)|}. \tag{2.6}$$

Proof. Let z_t be the solution to the following ordinary differential equation

$$\begin{cases} dz_t = \frac{\nabla H(z_t)}{|\nabla H(z_t)|^2} dt \\ z_0 = \theta \in C_i(x). \end{cases}$$

Then

$$dH(z_t) = \nabla H(z_t) dz_t = \nabla H(z_t) \cdot \frac{\nabla H(z_t)}{|\nabla H(z_t)|^2} dt = dt,$$

which means H can be served as the time and

$$H(z_t) = t = H,$$

$$d|z_t| = \frac{1}{|\nabla H(z_t)|}dt = \frac{dH}{|\nabla H(z_t)|}.$$

Let dl denote the unit length on the level set $C_i(x)$, and notice that z_t is orthogonal to the normal vector of the curve $C_i(x)$, $dl \cdot d|z_t|$ is the Lebesgue measure on $\mathbb{R}^2$.

$$\int_{D_i(x)} f(u)du = \int_{D_i(x)} f(u)dld|z_t| = \int_{D_i(x)} f(u)\frac{dl}{|\nabla H(z_t)|}dH.$$

The Lemma follows easily by

$$\int_{D_i(x)} f(u)\frac{dl}{|\nabla H(z_t)|}dH = \int_0^x \int_{C_i(H)} f(u)\frac{dl}{|\nabla H(u)|}dH.$$

□

Now we apply equation (2.6) and we get

$$\frac{d}{dx}\oint_{D_i(x)} \nabla \cdot (\frac{\sigma\sigma^*\nabla H}{a})du = \oint_{C_i(x)} \nabla \cdot (\frac{\sigma\sigma^*\nabla H}{a})\frac{dl}{|\nabla H|}.$$

Moreover,

$$\begin{aligned}
\nabla \cdot (\frac{\sigma\sigma^* \nabla H}{a}) =& \sum_i \frac{\partial}{\partial x_i}(\sigma\sigma^*\nabla H)_i a^{-1} + (\sigma\sigma^*\nabla H)_i \frac{\partial}{\partial x_i} a^{-1} \\
=& \sum_i \frac{\partial}{\partial x_i}(\sum_{k,j} \sigma_{ik}\sigma_{jk}\frac{\partial H}{\partial x_j}) a^{-1} + (\sigma\sigma^*\nabla H)_i \frac{\partial}{\partial x_i} a^{-1} \\
=& (\sum_i \sum_{k,j} \sigma_{ik}\sigma_{jk}\frac{\partial^2 H}{\partial x_i \partial x_j} + \sum_i \sum_{j,k} \frac{\partial}{\partial x_i}(\sigma_{ik}\sigma_{jk})\frac{\partial H}{\partial x_j} a^{-1} + \sum_i (\sigma\sigma^*\nabla H)_i \frac{\partial}{\partial x_i} a^{-1} \\
=& (\sum_{i,j} \sum_k \sigma_{ik}\sigma_{jk}\frac{\partial^2 H}{\partial x_i \partial x_j} + 2b \cdot \nabla H) a^{-1} + \nabla \cdot (\frac{\sigma\sigma^*}{a}) \cdot \nabla H - \frac{2b}{a} \cdot \nabla H \\
=& 2\mathcal{L}_0 H a^{-1} + [\nabla \cdot (\frac{\sigma\sigma^*}{a}) - \frac{2b}{a}] \cdot \nabla H.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{d}{dx} \oint_{D_i(x)} \nabla \cdot (\frac{\sigma\sigma^*\nabla H}{a}) du =& 2 \oint_{C_i(x)} \mathcal{L}_0 H \frac{dl}{a|\nabla H|} + \oint_{C_i(x)} [\nabla \cdot (\frac{\sigma\sigma^*}{a}) - \frac{2b}{a}] \nabla H \frac{dl}{|\nabla H|} \\
=& 2 \oint_{C_i(x)} \mathcal{L}_0 H \frac{dl}{|g(x)|} + \oint_{C_i(x)} [\nabla \cdot (\frac{\sigma\sigma^*}{a}) - \frac{2b}{a}] \cdot \vec{\nu} dl.
\end{aligned}$$

If we apply the Divergence Theorem again, we get

$$\oint_{C_i(x)} [\nabla \cdot (\frac{\sigma\sigma^*}{a}) - \frac{2b}{a}] \cdot \vec{\nu} dl = \int_{D_i(x)} \nabla \cdot [\nabla \cdot (\frac{\sigma\sigma^*}{a}) - \frac{2b}{a}] du = 2 \int_{D_i(x)} \mathcal{L}_0^* a^{-1} du,$$

where $\mathcal{L}_0^*$ is the formal adjoint of the operator $\mathcal{L}_0$. Therefore

$$\frac{d}{dx}\oint_{C_i(x)}\frac{(\nabla H)^t\sigma\sigma^*\nabla H(u)}{|g(u)|}dl = 2B_i(x)(\oint_{C_i(x)}\frac{dl}{|g(u)|}) + 2\int_{D_i(x)}\mathcal{L}_0^* a^{-1}du \tag{2.7}$$

and the following Theorem follows.

Theorem 2.2.2. *Let H and f satisfy the condition in Lemma 2.2.1, then*

$$\frac{d}{dv_i(x)}(\frac{d}{du_i(x)}f(x)) = 2\mathcal{L}_i f(x) + \frac{2}{T_i(x)}\int_{D_i(x)}\mathcal{L}_0^* a^{-1}du f'(x).$$

2.2.3 A-priori Estimates

Consider the stopping time

$$T_q^\epsilon(H_0) := \inf\{t; H(q_t^\epsilon) \geq H_0\}. \tag{2.8}$$

We have the following Lemma.

Lemma 2.2.3. *Under Hypothesis 8 and 9, for any fixed $T > 0$, and arbitrary $\eta > 0$, there exists a constant H_0 such that*

$$\mathbb{P}[T_q^\epsilon(H_0) < T] < \eta.$$

Proof. Recall that $x_t^\epsilon = H(q_t^\epsilon)$ satisfies the following stochastic differential equation

$$dx_t^\epsilon = \mathcal{L}^\epsilon H(q_t^\epsilon)dt + \mathcal{R}^\epsilon H(q_t^\epsilon)dW_t$$

where $\mathcal{L}^\epsilon = \mathcal{L}_0 + \mathcal{L}_0^\epsilon$, $\mathcal{R}^\epsilon = \mathcal{R}_0 + \mathcal{R}_0^\epsilon$. By our assumption that ∇H is of linear growth, b, b^ϵ, σ, σ^ϵ are all Lipschitz continuous. There exists a constant C, such that

$$\mathcal{L}^\epsilon H(x) \leq C(1 + |x|^2),$$

$$\mathcal{R}^\epsilon H(x) \leq C(1 + |x|^2).$$

Therefore

$$\mathbb{E}[x_t^\epsilon] \leq x_0^\epsilon + C\int_0^t 1 + \mathbb{E}[|q_s^\epsilon|^2]ds,$$

and since $H(x) \geq a|x|^2$ for x large enough, there exists a constant C such that

$$\mathbb{E}[x_t^\epsilon] \leq C(1+T) + C\int_0^T \mathbb{E}[x_s^\epsilon]ds, \qquad t \leq T.$$

If we apply the Gronwall's inequality, this implies

$$\mathbb{E}[x_t^\epsilon] \leq C(1+T)e^{Ct}, \quad t \leq T.$$

Also,

$$\sup_{0\le t\le T}\int_0^t \mathcal{L}^\epsilon H(q_s^\epsilon)ds \le \int_0^T C(1+|q_s^\epsilon|^2)ds.$$

So that,

$$\begin{aligned}\mathbb{P}[\sup_{0\le t\le T}\int_0^t \mathcal{L}^\epsilon H(q_s^\epsilon)ds \ge R] &\le \frac{C}{R}\mathbb{E}[\int_0^T (1+|q_s^\epsilon|^2)ds] \le \frac{C}{R}\mathbb{E}[\int_0^T (1+\frac{1}{a}H(q_s^\epsilon))ds] \\ &\le \frac{C}{R}\int_0^T C(1+\frac{1}{a}C(1+s)e^{Cs}ds \le \frac{C_T}{R}.\end{aligned} \tag{2.9}$$

Now pick R such that $C_T/R < \eta$, and $H_0 > x_0 + R$

$$\begin{aligned}\mathbb{P}[\sup_{0\le t\le T} x_t^\epsilon \ge H_0] &= \mathbb{P}[\sup_{0\le t\le T}\int_0^t \mathcal{L}^\epsilon H(q_s^\epsilon)ds + \int_0^t \mathcal{R}^\epsilon H(q_s^\epsilon)dW_s \ge H_0 - x_0] \\ &\le \mathbb{P}[\sup_{0\le t\le T}\int_0^t \mathcal{L}^\epsilon H(q_s^\epsilon)ds + \int_0^t \mathcal{R}^\epsilon H(q_s^\epsilon)dW_s \ge H_0 - x_0; \sup_{0\le t\le T}\int_0^t \mathcal{L}^\epsilon H(q_s^\epsilon)ds \\ &\qquad \le R] + \mathbb{P}[\sup_{0\le t\le T}\int_0^t \mathcal{L}^\epsilon H(q_s^\epsilon)ds > R] \\ &\le \mathbb{P}[\sup_{0\le t\le T}\int_0^t \mathcal{R}^\epsilon H(q_s^\epsilon)dW_s \ge H_0 - x_0 - R] + \mathbb{P}[\sup_{0\le t\le T}\int_0^t \mathcal{L}^\epsilon H(q_s^\epsilon)ds > R].\end{aligned}$$

Due to (2.9), the second term above is smaller than $\eta/2$ by our choice of R.

For the first term, we have

$$\begin{aligned}
&\mathbb{P}[\sup_{0\le t\le T}\int_0^t \mathcal{R}^\epsilon H(q_s^\epsilon)dW_s \ge H_0 - x_0 - R] \\
\le&\frac{1}{(H_0-x_0-R)^2}\mathbb{E}[\int_0^T |\mathcal{R}^\epsilon H(q_s^\epsilon)|^2 ds] \\
\le&\frac{C}{(H_0-x_0-R)^2}\mathbb{E}[\int_0^T C(1+|q_t^\epsilon|^4)dt] \\
\le&\frac{C}{(H_0-x_0-R)^2}\mathbb{E}[\int_0^T (1+|H(q_t^\epsilon)|^2)dt] \\
\le&\frac{C_T}{a^2(H_0-x_0-R)^2} + \frac{C}{(H_0-x_0-R)^2}\int_0^T \mathbb{E}[|x_t^\epsilon|^2]dt.
\end{aligned} \tag{2.10}$$

Now,

$$\begin{aligned}
\mathbb{E}[|x_t^\epsilon|^2] &\le 3x_0^2 + 3\mathbb{E}[(\int_0^t C(1+H(q_s^\epsilon))ds)^2] + 3\mathbb{E}[\int_0^t C(1+|x_s^\epsilon|^2)ds] \\
&\le 3x_0^2 + 3\mathbb{E}[\int_0^t C(1+|x_s^\epsilon|^2)dsT] + 3\mathbb{E}[\int_0^t |x_s^\epsilon|^2 ds] + 3CT \\
&\le C(x_0,T) + C(T)\mathbb{E}[\int_0^t |x_s^\epsilon|^2 ds],
\end{aligned}$$

so that

$$\mathbb{E}[|x_t^\epsilon|^2] \le C(x_0,T)e^{C(T^2+T)} = C(T,x_0). \tag{2.11}$$

Therefore, we can pick H_0 large enough in (2.10) so that

$$\mathbb{P}[\sup_{0\le t\le T} x_t^\epsilon \ge H_0] < \eta.$$

□

2.2.4 Lipschitz Continuity

Lemma 2.2.4. *Under Hypothesis 8 and 9, for any continous function f, A_i, B_i and $\mathcal{L}_0 f$ are Liptichitz continuous in $D(H_1, H_2) \subseteq \mathbb{R}^2$ where H_1 and H_2 are inside the interior of some edge I_i.*

Proof. The tool we use here Lemma 1.1 in Chapter 8 of [11]. We first calculate the derivative of T_i.

$$\frac{d}{dx}T_i(x) = \frac{d}{dx}\oint_{C_i(x)} \frac{dl}{|g(u)|} = \frac{d}{dx}\oint_{C_i(x)} \frac{|\nabla H(u)|}{|g(u)||\nabla H(u)|}dl$$
$$= \oint_{C_i(x)} \nabla(\frac{1}{|g(u)||\nabla H(u)|}) \cdot \frac{\nabla H(u)}{|\nabla H(u)|} + \frac{1}{|g(u)||\nabla H(u)|} \cdot \frac{\Delta H(u)}{|\nabla H(u)|}dl.$$

Let

$$\bar{A}_i(x) = \oint_{C_i(x)} \mathcal{R}_0[H](u)\mathcal{R}_0[H](u)^* \frac{dl}{|g(u)|}, \quad \bar{B}_i(x) = \oint_{C_i(x)} \mathcal{L}_0[H](u)\frac{dl}{|g(u)|}.$$

By equation (2.7)

$$\frac{d}{dx}\bar{A}_i(x) = 2\bar{B}_i(x) + 2\int_{D_i(x)} \mathcal{L}_0^*[a^{-1}]du. \tag{2.12}$$

Next, we apply Lemma 1.1 in Chapter 8 of [11] again to calculate $\bar{B}_i'(x)$

$$\frac{d}{dx}\bar{B}_i(x) = \oint_{C_i(x)} \nabla\left(\frac{\mathcal{L}_0[H](u)}{|g(u)||\nabla H(u)|}\right)\cdot\frac{\nabla H(u)}{|\nabla H(u)|} + \frac{\mathcal{L}_0[H](u)}{|g(u)||\nabla H(u)|}\cdot\frac{\Delta H(u)}{|\nabla H(u)|}dl.$$

Also we apply Lemma 2.2.1 to calculate the derivative of the residue in (2.12)

$$\frac{d}{dx}\int_{D_i(x)} \mathcal{L}_0^*[a^{-1}]du = \oint_{C_i(x)} \mathcal{L}_0^*[a^{-1}]\frac{dl}{|\nabla H(x)|}.$$

By our assumptions, it can be easily checked that $T_i'(x)$ is both bounded and bounded below above 0, and both $\bar{A}_i'(x)$ and $\bar{B}_i'(x)$ are bounded. Therefore $A_i'(x)$ and $B_i'(x)$ are bounded in D_i, which implies their Lipschitz continuity. The Lipschitz conitnuity for $\mathcal{L}_0 f$ is obvious. □

2.3 Proof of Theorem 2.1.2

We first define the following sequence of stopping times. In the definition below, δ and δ' are to be determined later.

Definition 2.3.1. *Let S be the set of integers $k \leq m$, such that O_k is a saddle point. For any $\epsilon > 0$ and $0 < \delta' < \delta$, let $0 = \tau_0^\epsilon \leq \sigma_1^\epsilon \leq \tau_1^\epsilon \leq ... \leq$*

$\sigma_n^\epsilon \leq \tau_n^\epsilon \leq ...$ *be a sequence of stopping times with*

$$\sigma_n^\epsilon = \inf\{t \geq \tau_{n-1}^\epsilon; q_t^\epsilon \notin \bigcup_{i \in S} D_i(\pm\delta)\} \wedge T_q^\epsilon(H_0)$$

and

$$\tau_n^\epsilon = \inf\{t \geq \sigma_n^\epsilon; q_t^\epsilon \notin \bigcup_{i \in S} C_i(\pm\delta')\} \wedge T_q^\epsilon(H_0).$$

Where $T_q^\epsilon(H_0)$ *is the stopping time defined in* (2.8)*. Moreover, we define*

$$\tau_k^\epsilon(\pm\delta) := \inf\{t \geq 0; q_t^\epsilon \notin D_k(\pm\delta)\}.$$

Recall that we denote by L the infinitesimal operator of the process (x_t, i_t) on Γ. If the Poisson problem $(\alpha I - L)f = u$ has a unique solution then this solution has the representation

$$f(x,i) = \mathbb{E}_{(x,i)} \int_0^{+\infty} e^{-\alpha t} u(x_t, i_t) dt,$$

Replacing u by $(\alpha I - L)f$, gives

$$f(x,i) = \mathbb{E}_{(x,i)} \int_0^{+\infty} e^{-\alpha t} (\alpha I - L)[f](x_t, i_t) dt.$$

If we can prove that for all $u \in D(L)$,

$$\lim_{\epsilon \to 0} \mathbb{E}_{(x,i)} \int_0^{+\infty} e^{-\alpha t}(\alpha f - Lf)(x_t^\epsilon, i_t^\epsilon)dt = f(x,i),$$

then the tightness of the family $\{x_t^\epsilon, i_t^\epsilon\}_{\epsilon>0}$, Prokhorov theorem and its corollary, and the fact that the range of the operator $\alpha I - L$ uniquely determines a measure guarantee that $\{(x_t^\epsilon, i_t^\epsilon)\}_{\epsilon>0}$ converges weakly to (x_t, i_t) as $\epsilon \to 0$ in $\mathbf{C}([0,+\infty);\Gamma)$.

By Freidin and Wenztell's procedure, the tightness of the measure on $\mathbf{C}([0,+\infty);\Gamma)$ follows easily from apriori estimate proved in Lemma 2.2.3. Hence it is sufficient to prove that, for any $\eta > 0$ and $H_0 > 0$, there exists ϵ_0 such that for all $0 < \epsilon < \epsilon_0$

$$|\mathbb{E}_{(x,i)} \int_0^{T_q^\epsilon(H_0)} e^{-\alpha t}(\alpha I - L)f(x_t, i_t)dt - f(x,i)| < \eta$$

which conatins two parts: the part where $(x_t^\epsilon, i_t^\epsilon)$ remains in the same edge

and the part where $(x_t^\epsilon, i_t^\epsilon)$ approaches an inner vertex of the graph. We have

$$
\begin{aligned}
&\mathbb{E}_{(x,i)}[\int_0^{T_q^\epsilon(H_0)} e^{-\alpha t}(\alpha f - Lf)(x_t^\epsilon, i_t^\epsilon)dt - f(x,i)] \\
=&\mathbb{E}_{(x,i)}[\sum_{n=0}^{\infty} e^{-\alpha\sigma_{n+1}^\epsilon} f(x_{\sigma_{n+1}^\epsilon}^\epsilon, i_{\sigma_{n+1}^\epsilon}^\epsilon) - e^{-\alpha\tau_n} f(x_{\tau_n^\epsilon}^\epsilon, i_{\tau_n^\epsilon}^\epsilon) + \int_{\tau_n^\epsilon}^{\sigma_{n+1}^\epsilon} e^{-\alpha t}(\alpha I - L)f(x_t^\epsilon, i_t^\epsilon)dt] \\
&+ \mathbb{E}_{(x,i)}[\sum_{n=1}^{\infty} e^{-\alpha\tau_n^\epsilon} f(x_{\tau_n^\epsilon}^\epsilon, i_{\tau_n^\epsilon}^\epsilon) - e^{-\alpha\sigma_n^\epsilon} f(x_{\sigma_n^\epsilon}^\epsilon, i_{\sigma_n^\epsilon}^\epsilon) + \int_{\sigma_n^\epsilon}^{\tau_n^\epsilon} e^{-\alpha t}(\alpha I - L)f(x_t^\epsilon, i_t^\epsilon)dt] \\
=&\mathbb{E}_{(x,i)}[\sum_{n=0}^{\infty} \mathbb{E}_{(x,i)}[e^{-\alpha\sigma_{n+1}^\epsilon} f(x_{\sigma_{n+1}^\epsilon}^\epsilon, i_{\sigma_{n+1}^\epsilon}^\epsilon) - e^{-\alpha\tau_n} f(x_{\tau_n^\epsilon}^\epsilon, i_{\tau_n^\epsilon}^\epsilon) + \int_{\tau_n^\epsilon}^{\sigma_{n+1}^\epsilon} e^{-\alpha t}(\alpha I - L)f(x_t^\epsilon, i_t^\epsilon)dt|\mathcal{F}_{\tau_n^\epsilon}]] \\
&+ \mathbb{E}_{(x,i)}[\sum_{n=1}^{\infty} \mathbb{E}_{(x,i)}[e^{-\alpha\tau_n^\epsilon} f(x_{\tau_0^\epsilon, i_{\tau_0^\epsilon}}^\epsilon) - e^{-\alpha\sigma_n^\epsilon} f(x_{\sigma_n^\epsilon}^\epsilon, i_{\sigma_n^\epsilon}^\epsilon) + \int_{\sigma_n^\epsilon}^{\tau_n^\epsilon} e^{-\alpha t}(\alpha I - L)f(x_t^\epsilon, i_t^\epsilon)dt|\mathcal{F}_{\sigma_n^\epsilon}]] \\
=&\mathbb{E}_{(x,i)}[\sum_{n=0}^{\infty} e^{-\alpha\tau_n^\epsilon} \mathbb{E}_{(x_{\tau_n^\epsilon}^\epsilon, i_{\tau_n^\epsilon}^\epsilon)}[e^{-\alpha\sigma_1^\epsilon} f(x_{\sigma_1^\epsilon}^\epsilon, i_{\sigma_1^\epsilon}^\epsilon) - f(x_0^\epsilon, i_0^\epsilon) + \int_0^{\sigma_1^\epsilon} e^{-\alpha t}(\alpha I - L)f(x_t^\epsilon, i_t^\epsilon)dt]] \\
&+ \mathbb{E}_{(x,i)}[\sum_{n=1}^{\infty} e^{-\alpha\sigma_n^\epsilon} \mathbb{E}_{(x_{\sigma_n^\epsilon}^\epsilon, i_{\sigma_n^\epsilon}^\epsilon)}[e^{-\alpha\tau_1^\epsilon} f(x_{\tau_1^\epsilon}^\epsilon, i_{\tau_1^\epsilon}^\epsilon) - f(x_0^\epsilon, i_0^\epsilon) + \int_0^{\tau_1^\epsilon} e^{-\alpha t}(\alpha I - L)f(x_t^\epsilon, i_t^\epsilon)dt]].
\end{aligned}
$$

If we define

$$
\Phi_1^\epsilon(x,i) = \mathbb{E}_{(x,i)}^\epsilon[e^{-\alpha\sigma_1^\epsilon} f(x_{\sigma_1^\epsilon}^\epsilon, i_{\sigma_1^\epsilon}^\epsilon) + \int_0^{\sigma_1^\epsilon} e^{-\alpha t}(\alpha I - L)f(x_t^\epsilon, i_t^\epsilon)dt] - f(x,i),
$$

and

$$
\Phi_2^\epsilon(x,i) = \mathbb{E}_{(x,i)}^\epsilon[e^{-\alpha\tau_1^\epsilon} f(x_{\tau_1^\epsilon}^\epsilon, i_{\tau_1^\epsilon}^\epsilon) + \int_0^{\tau_1^\epsilon} e^{-\alpha t}(\alpha I - L)f(x_t^\epsilon, i_t^\epsilon)dt] - f(x,i).
$$

We have

$$
\begin{aligned}
&|\mathbb{E}_{(x,i)}[\int_0^{T_q^\epsilon(H_0)} e^{-\alpha t}(\alpha f - Lf)(x_t^\epsilon, i_t^\epsilon)dt - f(x,i)]| \\
\leq &\mathbb{E}_{(x,i)} \sum_{n=0}^{\infty} e^{-\alpha \tau_n^\epsilon} \sup_{(x,i)\in \bigcup_{i\in S} C_i(\pm\delta')} |\Phi_1^\epsilon(x,i)| + \mathbb{E}_{(x,i)} \sum_{n=1}^{\infty} e^{-\alpha \sigma_n^\epsilon} \sup_{(x,i)\in \bigcup_{i\in S} C_i(\pm\delta)} |\Phi_2^\epsilon(x,i)|.
\end{aligned}
$$

Follow the same procedure in [11] and get, for sufficiently small ϵ,

$$
\mathbb{E}_{(x,i)}[\sum_{n=1}^{\infty} e^{-\alpha \sigma_n^\epsilon}] \leq \mathbb{E}_{(x,i)}[\sum_{n=0}^{\infty} e^{-\alpha \tau_n^\epsilon}] \leq C\delta^{-1},
$$

so that for sufficiently small ϵ,

$$
\begin{aligned}
&|\mathbb{E}_{(x,i)}[\int_0^{T_q^\epsilon(H_0)} e^{-\alpha t}(\alpha f - Lf)(x_t^\epsilon, i_t^\epsilon)dt - f(x,i)]| \\
\leq &C\delta^{-1}(\sup_{(x,i)\in \bigcup_{i\in S} C_i(\pm\delta')} |\Phi_1^\epsilon(x,i)| + \sup_{(x,i)\in \bigcup_{i\in S} C_i(\pm\delta)} |\Phi_2^\epsilon(x,i)|).
\end{aligned}
$$

In order to study the term Φ_2, we need the following result.

Lemma 2.3.2. *Let β_t be a one dimensional standard Brownian motion, and let x_t be the solution to the following equation*

$$
dx_t = B_i(x_t)dt + A_i^{\frac{1}{2}}(x_t)d\beta_t,
$$

Assume $H_1 < H_2$, and let τ_{H_1,H_2}^ϵ be the stopping time when q_t^ϵ leaves the

region $D_i(H_1, H_2)$. If either one of these three cases holds,

1. *I_i is an edge such that both of its vertex are interior vertex, and $H_1 < H_2$ are any fixed values belonging to the interior of the interval.*

2. *I_i is an edge such that one of its vertex is an interior vertex while the other vertex is an exterior vertex, where the Hamiltonian takes the value H_1, H_2 belonging to the interior of the interval.*

3. *I_i is an edge that has only one vertex, and $H_1 < H_2$ are any fixed values belonging to the interior of the interval.*

Then under Hypothesis 8 and 9, for every function f on $[H_1, H_2]$ that is three times continuously differentiable and for every $\lambda > 0$,

$$\lim_{\epsilon \to 0} \mathbb{E}_{x_0}[e^{-\lambda \tau^{\epsilon}_{H_1,H_2}} f(x^{\epsilon}_{\tau^{\epsilon}_{H_1,H_2}}) - \int_0^{\tau^{\epsilon}_{H_1,H_2}} e^{-\lambda s}(-\lambda I + \mathcal{L}_i)[f](x^{\epsilon}_s) ds] = f(x_0)$$

uniformly with respect to $x_0 \in [H_1, H_2]$.

In order to study the term Φ_1, we need the following result.

Lemma 2.3.3. *Let O_k be an interior vertex, then for every positive α and*

κ, there exists $\delta_0 >$ such that for all $0 < \delta < \delta_0$ and sufficiently small ϵ,

$$\sup_{z \in D_k(\pm\delta)} \mathbb{E}_z^\epsilon [\int_0^{\tau_k^\epsilon(\pm\delta)} e^{-\alpha t} dt] < \kappa\delta,$$

and there exists constant C such that

$$\sup_{q \in D_k(\pm\delta)} \mathbb{E}_z^\epsilon [\tau_k^\epsilon(\pm\delta)] \leq C\delta^2 \ln(\delta).$$

(see remark on page 310 in [11])

We can use Lemma 3.5 in [11] and apply the change of random time to prove this Lemma easily. Now if we take

$$\kappa = \frac{\eta}{\alpha||f||_\infty + ||\alpha f - Lf||_\infty}$$

in Lemma 2.3.3, we have

$$\begin{aligned}
|\Phi_1^\epsilon(x,i)| \leq & |\mathbb{E}_{(x,i)}[f(x_{\sigma_1^\epsilon}^\epsilon, i_{\sigma_1^\epsilon}^\epsilon) - f(x,i)]| + (\alpha||f||_\infty + ||\alpha f - Lf||_\infty) \cdot \frac{\delta}{\alpha||f||_\infty + ||\alpha f - Lf||_\infty}\eta \\
& \leq \sum_{I_j \sim O_k} |\mathbb{E}_{q_0}[f_j(H(q_{\sigma_1^\epsilon}^\epsilon)) - f_i(H(q)); q_{\sigma_1^\epsilon}^\epsilon \in I_j]| + \delta\eta \\
& = \sum_{I_j \sim O_k} |f_j(H(O_k) + \delta) - f_i(x)| \cdot \mathbb{P}_{q_0}[q_{\sigma_1^\epsilon}^\epsilon \in I_j] + \delta\eta.
\end{aligned}$$

We will prove the following Lemma in Section 2.5 to handle the term

$$\sup_{q_0 \in \bigcup_{I_j \sim O_k} C_j(\pm\delta')\}} \mathbb{P}_{q_0}[q^\epsilon_{\sigma^\epsilon_1} \in I_j]$$

The following lemma will be proved in section 5.

Lemma 2.3.4. *For every $\kappa > 0$ there exists a positve $\delta_0 > 0$ such that for $0 < \delta < \delta_0$ there exists $\delta_0' = \delta_0'(\delta)$ such that for sufficiently small ϵ,*

$$\sup_{x \in \bar{D}_k(\pm\delta_0')} |\mathbb{P}_x[\hat{q}^\epsilon_{\tau^\epsilon_k(\pm\delta)} \in C_{ki}(\delta)] - p_{ki}| < \kappa,$$

where p_{ki} is the constant defined in (2.5).

If we take $\kappa < \eta$, then for sufficiently small ϵ,

$$\begin{aligned}
\Phi^\epsilon_1(x,i) \leq & \sum_{I_j \sim O_k} |f_j(H(O_k)+\delta) - f_i(x)| \cdot \mathbb{P}_{q_0}[q^\epsilon_{\sigma^\epsilon_1} \in I_j] + \delta\eta \\
\leq & \sum_{I_j \sim O_k} ||f_j(H(O_k)+\delta) - f_i(x)| - \delta D_{jk} f(x)| \cdot |\mathbb{P}_{q_0}[q^\epsilon_{\sigma^\epsilon_1} \in I_j] - p_{kj}| + \\
& \sum_{I_j \sim O_k} ||f_j(H(O_k)+\delta) - f_i(x)| - \delta D_{jk} f(x)| \cdot p_{kj} \\
& \qquad + \sum_{I_j \sim O_k} \delta |D_{jk} f(x)| \cdot |\mathbb{P}_{q_0}[q^\epsilon_{\sigma^\epsilon_1} \in I_j] - p_{kj}| \\
\leq & C(\kappa\delta^2 + \delta^2 + \delta\kappa) \leq C\delta\eta.
\end{aligned}$$

To this point, we first pick $\kappa < \eta$ and then determine δ and δ'according to Lemma 2.3.4. After that, we let ϵ sufficiently small so that Lemma 2.3.4 holds for $\kappa < \eta$ and Lemma 2.3.2 holds for

$$\sup_{(x,i)\in\bigcup_{i\in S} C_i(\pm\delta)} |\Phi_2^\epsilon(x,i)| \leq \delta\eta,$$

Therefore

$$\mathbb{E}_{(x,i)}[\int_0^{T_q^\epsilon(H_0)} e^{-\alpha t}(\alpha f - Lf)(x_t^\epsilon, i_t)dt - f(x,i)] \leq \eta + C\eta,$$

which proves the result.

2.4 Weak Convergence Inside the Edge

We first prove the weak convergence of x_t^ϵ inside an edge of finite length. For $H_1 < H_2 < \infty$, recall that τ_{H_1,H_2}^ϵ is the stopping time when the process q_t^ϵ leaves the region $D_i(H_1, H_2)$. Noticed that the coefficients are all bounded in $D_i(H_1, H_2)$. Let $T^\epsilon(z)$ be the period of X_t^ϵ starting at z. Suppose $\underline{C}\epsilon \leq T^\epsilon(z) \leq \bar{C}\epsilon$ for all $z \in D_i(H_1, H_2)$. Where we should remark that the constant $\underline{C}$ and $\bar{C}$ depends on the constant H_1 and H_2. In this section,

we consider the process q_t^ϵ starting from $q_0 \in D_i(H_1, H_2)$ and denote $\tau_T^\epsilon = \tau_{H_1,H_2}^\epsilon \wedge T$. We will first prove a weaker version of Lemma 2.3.2.

Lemma 2.4.1. *Let β_t and x_t be defined as in Lemma 2.3.2. Under the same condition as in Lemma 2.3.2, for every function f on $[H_1, H_2]$ that is three times continuously differentiable and for every $\lambda > 0$,*

$$\lim_{\epsilon \to 0} \mathbb{E}_{x_0}[e^{-\lambda \tau_T^\epsilon} f(x_{\tau_T^\epsilon}^\epsilon) - \int_0^{\tau_T^\epsilon} e^{-\lambda s}(-\lambda f + \mathcal{L}_i f)(x_s^\epsilon) ds] = f(x_0)$$

uniformly with respect to $x_0 \in [H_1, H_2]$.

In what follows we shall define a sequence of stopping times T_k^ϵ defined by $T_0^\epsilon = T^\epsilon(q)$, $T_1 = T^\epsilon(q_{T_0^\epsilon}^\epsilon) + T_0^\epsilon$, ..., $T_k^\epsilon = T^\epsilon(q_{T_{k-1}^\epsilon}^\epsilon) + T_{k-1}^\epsilon$,... We first define an auxilliary process ξ_t^ϵ trajectory by trajectory.

Definition 2.4.2. *Let $V_s^\epsilon = \mathcal{R}_0 H / |\mathcal{R}_0 H|(q_s^\epsilon)$, and*

$$\beta_t^\epsilon = \int_0^t V_s^\epsilon dW_s.$$

For every $k \in \mathbb{N}$, we define $\xi_{T_k}^\epsilon = x_{T_k}^\epsilon$ and for $t \in [T_k^\epsilon \wedge \tau_T^\epsilon, T_{k+1}^\epsilon \wedge \tau_T^\epsilon)$,

$$\xi_t^\epsilon = \xi_{T_k^\epsilon} + B_i(x_{T_k^\epsilon}^\epsilon)(t - T_k^\epsilon) + A_i^{\frac{1}{2}}(x_{T_k^\epsilon}^\epsilon)(\beta_t^\epsilon - \beta_{T_k^\epsilon}^\epsilon).$$

Remark 2.4.3. *We should notice that β_t^ϵ is a standard Brownian motion on $\mathbb{R}$.*

We should also define the auxilliary process $\tilde{x}_t^\epsilon$ which has the same distribution as the limiting process of the slow motion.

Definition 2.4.4. *For $t \in [0, \tau_T^\epsilon]$, $\tilde{x}_t^\epsilon$ is the solution of the problem*

$$d\tilde{x}_t^\epsilon = B_i(\tilde{x}_t^\epsilon)dt + A_i^{\frac{1}{2}}(\tilde{x}_t^\epsilon)d\beta_t^\epsilon,$$

with the initial condition $x_0 = H(q) \in [H_1, H_2]$.

2.4.1 Closeness of the q_t^ϵ and $\bar{X}_s^{\epsilon,k}$

Let $\bar{X}_t^\epsilon$ satisfies the equation $d\bar{X}_t^\epsilon = \epsilon^{-1} g(\bar{X}_t^\epsilon)dt$ with the initial condition $\bar{X}_0^\epsilon = q_0$. It can be easily seen that

$$|q_s^\epsilon - \bar{X}_s^\epsilon| \leq \int_0^s \frac{L}{\epsilon}|q_u^\epsilon - \bar{X}_u^\epsilon|du + \int_0^s C(1+L)|q_u^\epsilon|du + |\int_0^s (\sigma + \sigma^\epsilon)(q_u^\epsilon)dW_u|.$$

Therefore, for $t < T$,

$$\begin{aligned}
\mathbb{E}[\sup_{0\le s\le t}|q_s^\epsilon - \bar{X}_s^\epsilon|^4] \le & \frac{cL^4}{\epsilon^4}\mathbb{E}[(\int_0^t \sup_{0\le u\le s}|q_u^\epsilon - \bar{X}_u^\epsilon|ds)^4] + ct^3\int_0^t C^2(1+L)^2\mathbb{E}[|q_u^\epsilon|^4]du \\
& + c\mathbb{E}[|\sup_{0\le s\le t}\int_0^s (\sigma+\sigma^\epsilon)(q_u^\epsilon)dW_u|^4] \\
\le & \frac{cL^4}{\epsilon^4}\mathbb{E}[\int_0^t \sup_{0\le u\le s}|q_u^\epsilon - \bar{X}_u^\epsilon|^4 ds\cdot t^3] + ct^3\int_0^t C^2(1+L)^2\mathbb{E}[|x_u^\epsilon|^2]du \\
& + c(\mathbb{E}[\int_0^t \mathrm{Tr}[(\sigma+\sigma^\epsilon)^*(\sigma+\sigma^\epsilon)](q_s^\epsilon)ds])^2 \\
\le & \frac{Ct^3}{\epsilon^4}\mathbb{E}[\int_0^t \sup_{0\le u\le s}|q_u^\epsilon - \bar{X}_u^\epsilon|^4 ds] + C(1+t+t^3)\int_0^t \mathbb{E}[|x_u^\epsilon|^2]du.
\end{aligned}$$

Due to equation (2.11), $\mathbb{E}[|x_u^\epsilon|^2] \le C(T, x_0)$, we have

$$\mathbb{E}[\sup_{0\le s\le t}|q_s^\epsilon - \bar{X}_s^\epsilon|^4] \le \frac{Ct^3}{\epsilon^4}\mathbb{E}[\int_0^t \sup_{0\le u\le s}|q_u^\epsilon - \bar{X}_u^\epsilon|^4 ds] + C(T,x_0)(t+t^2+t^4).$$

Apply Gronwall's inequality, we have

$$\mathbb{E}[\sup_{0\le s\le t}|q_s^\epsilon - \bar{X}_s^\epsilon|^4] < C(T,x_0)t(1+t+t^3)\exp(\frac{Ct^4}{\epsilon^4}). \tag{2.13}$$

2.4.2 $\Delta_b^k(t)$ and $\Delta_\sigma^k(t)$

In this subsection, we will introduce two critical term in the estimation of the closeness of the trajectories. For $t \in [T_{k-1}^\epsilon \wedge \tau_T^\epsilon, T_k^\epsilon \wedge \tau_T^\epsilon]$, let

$$\Delta_b^{k-1}(t) = | \int_{T_{k-1}^\epsilon \wedge \tau_T^\epsilon}^t \mathcal{L}_0 H(q_s^\epsilon) ds - (t - T_{k-1}^\epsilon \wedge \tau_T^\epsilon) B_i(x^\epsilon_{T_{k-1}^\epsilon \wedge \tau_T^\epsilon})|,$$

and

$$\Delta_b^{k-1}(T_k^\epsilon \wedge \tau_T^\epsilon) = \Delta_b^{k-1}.$$

It is easy to see that $\Delta_b^{k-1}(t) = 0$, if $\tau_T^\epsilon < T_{k-1}^\epsilon$. Otherwise,

$$\begin{aligned}
|\Delta_b^{k-1}(t)| &= | \int_{T_{k-1}^\epsilon}^t \mathcal{L}_0 H(q_s^\epsilon) ds - (t - T_{k-1}^\epsilon \wedge \tau_T^\epsilon) B_i(x^\epsilon_{T_{k-1}^\epsilon}) | \\
&= | \int_{T_{k-1}^\epsilon}^t \mathcal{L}_0 H(q_s^\epsilon) ds - \frac{t - T_{k-1}^\epsilon}{T^\epsilon(q^\epsilon_{T_{k-1}^\epsilon})} \oint_{C_i(x^\epsilon_{T_{k-1}^\epsilon})} \mathcal{L}_0(u) \frac{\epsilon dl}{|g(u)|} | \\
&= | \int_{T_{k-1}^\epsilon}^t \mathcal{L}_0[H](q_s^\epsilon) ds - \frac{t - T_{k-1}^\epsilon}{T_k^\epsilon - T_{k-1}^\epsilon} \int_{T_{k-1}^\epsilon}^{T_k^\epsilon} \mathcal{L}_0(\bar{X}_s^{\epsilon,k}) ds | \\
&\le \int_{T_{k-1}^\epsilon}^{T_k^\epsilon \wedge \tau} |\mathcal{L}_0 H(q_s^\epsilon) - \mathcal{L}_0 H(\bar{X}_s^{\epsilon,k})| ds + \int_t^{T_k^\epsilon} |\mathcal{L}_0 H(\bar{X}_s^{\epsilon,k})| ds \\
&\qquad + \frac{T_k^\epsilon - t}{T_k^\epsilon - T_{k-1}^\epsilon} \int_{T_{k-1}^\epsilon}^{T_k^\epsilon} |\mathcal{L}_0 H(\bar{X}_s^{\epsilon,k})| ds.
\end{aligned}$$

Where $\bar{X}_s^{\epsilon,k}$ satisfies the equation $d\bar{X}_t^{\epsilon,k} = \frac{1}{\epsilon} g(\bar{X}_t^{\epsilon,k})dt,\ \bar{X}_0^{\epsilon,k} = q^\epsilon_{T^\epsilon_{k-1}}$. Notice that the last two terms in the summation appears only once in all $\{\Delta_b^k(t)\}_{k\in\mathbb{N}}$. For the first term, we have

$$\begin{aligned}\int_{T^\epsilon_{k-1}}^{T^\epsilon_k\wedge\tau^\epsilon_T} |\mathcal{L}_0 H(q_s^\epsilon) - \mathcal{L}_0 H(\bar{X}_s^\epsilon)|ds &\le \int_{T^\epsilon_{k-1}}^{T^\epsilon_k\wedge\tau^\epsilon_T} |\mathcal{L}_0 H|_{Lip}|q_s^\epsilon - \bar{X}_s^\epsilon|ds \\ &\le C(\mathcal{L}_0, I_i)\int_{T^\epsilon_{k-1}}^{T^\epsilon_k\wedge\tau^\epsilon_T} |q_s^\epsilon - \bar{X}_s^\epsilon|ds.\end{aligned}$$

Thus, if we set, for $t \in [T^\epsilon_{k-1}\wedge\tau^\epsilon_T, T^\epsilon_k\wedge\tau^\epsilon_T]$ and every k except one, we get

$$\begin{aligned}\mathbb{E}[|\Delta_b^{k-1}(t)|^4|\mathcal{F}_{T^\epsilon_{k-1}}] &\le C\mathbb{E}[(\int_{T^\epsilon_{k-1}}^{T^\epsilon_k\wedge\tau^\epsilon_T} |q_s^\epsilon - \bar{X}_s^{\epsilon,k}|ds)^4|\mathcal{F}_{T^\epsilon_{k-1}}] \\ &\le C\mathbb{E}_{q^\epsilon_{T^\epsilon_{k-1}}}[\int_0^{T_0^\epsilon} |q_s^\epsilon - \bar{X}_s^{\epsilon,k}|^4 ds (T_0^\epsilon\wedge\tau_T^\epsilon)^3] \\ &\le C(\bar{C}\epsilon)^4\mathbb{E}_{q^\epsilon_{T^\epsilon_{k-1}}}[\sup_{0\le s\le\bar{C}\epsilon}|q_s^\epsilon - \bar{X}_s^\epsilon|^4].\end{aligned}$$

Similarlly, for $t \in [T^\epsilon_{k-1}\wedge\tau^\epsilon_T, T^\epsilon_k\wedge\tau^\epsilon_T]$, let

$$\Delta_\sigma^{k-1}(t) = \int_{T_{k-1}\wedge\tau^\epsilon_T}^t |A_i^{\frac{1}{2}}(x^\epsilon_{T^\epsilon_{k-1}\wedge\tau^\epsilon_T})V_s^\epsilon - \mathcal{R}_0 H(q_s^\epsilon)|^2 ds$$

and

$$\Delta_\sigma^{k-1}(T_k^\epsilon\wedge\tau_T^\epsilon) = \Delta_\sigma^{k-1}.$$

Due to the inequality that $(x-y)^2 \leq 2|x^2-y^2|$, for $x, y \in \mathbb{R}^+$, we have for $t \in [T^\epsilon_{k-1} \wedge \tau^\epsilon_T, T^\epsilon_k \wedge \tau^\epsilon_T]$

$$\Delta^{k-1}_\sigma(t) \leq 2 \int_{T_{k-1}}^{T_k \wedge \tau^\epsilon_T} |A_i(x^\epsilon_{T_{k-1}}) - \mathcal{R}^*_0[H]\mathcal{R}_0[H](q^\epsilon_s)|ds.$$

By perceeding as for $\Delta^{k-1}_b(t)$, we have for $t \in [T^\epsilon_{k-1} \wedge \tau^\epsilon_T, T^\epsilon_k \wedge \tau^\epsilon_T]$ and every k except one,

$$\mathbb{E}[\Delta^k_\sigma(t)|\mathcal{F}_{T^\epsilon_k}] \leq C_1 \bar{C}\epsilon \mathbb{E}_{q^\epsilon_{T^\epsilon_k}}[\sup_{0 \leq s \leq \bar{C}\epsilon} |q^\epsilon_s - \bar{X}^{\epsilon,k}_s|].$$

2.4.3 Closeness of ξ^ϵ_t and x^ϵ_t

Let

$$h(\epsilon) = \sup_{x \in \mathbb{R}^2} \max\{|b^\epsilon(x)|, |\sigma^\epsilon(x)|\},$$

then $\lim_{\epsilon\to 0} h(\epsilon) = 0$ as we assumed in Hypothesis 8. For $t \in [T_k^\epsilon \wedge \tau_T^\epsilon, T_{k+1}^\epsilon \wedge \tau_T^\epsilon)$, we have

$$
\begin{aligned}
&|\xi_t^\epsilon - x_t^\epsilon| \\
\leq &|B_i(x^\epsilon_{T_k^\epsilon})(t - T_k^\epsilon) - \int_{T_k^\epsilon}^t \mathcal{L}_0 H(q_s^\epsilon) ds| + |A_i^{1/2}(x^\epsilon_{T_k^\epsilon})(\beta_t - \beta_{T_k^\epsilon}) - \int_{T_k^\epsilon}^t \mathcal{R}_0 H(q_s^\epsilon) dW s| \\
&+ |\int_{T_k^\epsilon}^t \mathcal{L}_0^\epsilon H(q_s^\epsilon) ds| + |\int_{T_k^\epsilon}^t \mathcal{R}_0^\epsilon H(q_s^\epsilon) dW s| \\
\leq &|B_i(x^\epsilon_{T_k^\epsilon})(t - T_k^\epsilon) - \int_{T_k^\epsilon}^t \mathcal{L}_0 H(q_s^\epsilon) ds| + |A_i^{1/2}(x^\epsilon_{T_k^\epsilon})(\beta_t - \beta_{T_k^\epsilon}) - \int_{T_k^\epsilon}^{T_{k+1}^\epsilon \wedge \tau} \mathcal{R}_0 H(q_s^\epsilon) dW s| \\
&+ \bar{C}\epsilon h(\epsilon) + |\int_{T_k^\epsilon}^t \mathcal{R}_0^\epsilon H(q_s^\epsilon) dW s|
\end{aligned}
$$

.

If $T_k^\epsilon > \tau_T^\epsilon$, then

$$\sup_{T_k^\epsilon \wedge \tau_T^\epsilon \leq t \leq T_{k+1}^\epsilon \wedge \tau_T^\epsilon} |\xi_t^\epsilon - x_t^\epsilon|^4 = 0.$$

So we consider the case when $T_k^\epsilon > \tau_T^\epsilon$,

$$\sup_{T_k^\epsilon \wedge \tau_T^\epsilon \leq t \leq T_{k+1}^\epsilon \wedge \tau_T^\epsilon} |\xi_t^\epsilon - x_t^\epsilon|^4 = \sup_{T_k^\epsilon \leq t \leq T_{k+1}^\epsilon \wedge \tau_T^\epsilon} |\xi_t^\epsilon - x_t^\epsilon|^4,$$

below.

$$
\begin{aligned}
&\sup_{T_k^\epsilon \le t \le T_{k+1}^\epsilon \wedge \tau_T^\epsilon} |\xi_t^\epsilon - x_t^\epsilon|^4 \\
\le & C(\Delta_b^k (T_{k+1}^\epsilon \wedge \tau_T^\epsilon))^4 + C \sup_{T_k^\epsilon \le t \le T_{k+1}^\epsilon \wedge \tau_T^\epsilon} | \int_{T_k^\epsilon}^t A_i^{1/2}(x_{T_k^\epsilon}^\epsilon \wedge \tau_T^\epsilon) V_s^\epsilon - \mathcal{R}_0[H](q_s^\epsilon) dW_s|^4 \\
&+ C\bar{C}^4 \epsilon^4 h^4(\epsilon) + C \sup_{T_k^\epsilon \le t \le T_{k+1}^\epsilon \wedge \tau_T^\epsilon} | \int_{T_k^\epsilon}^t \mathcal{R}_0^\epsilon[H](q_s^\epsilon) dWs|^4 \\
\le & C(\Delta_b^k)^4 + C \sup_{T_k^\epsilon \le t \le T_{k+1}^\epsilon \wedge \tau_T^\epsilon} | \int_{T_k^\epsilon}^t A_i^{1/2}(x_{T_k^\epsilon}^\epsilon) V_s^\epsilon - \mathcal{R}_0 H(q_s^\epsilon) dW_s|^4 \\
&+ C\bar{C}^4 \epsilon^4 h^4(\epsilon) + C \sup_{T_k^\epsilon \le t \le T_{k+1}^\epsilon \wedge \tau_T^\epsilon} | \int_{T_k^\epsilon}^t \mathcal{R}_0^\epsilon H(q_s^\epsilon) dWs|^4.
\end{aligned}
$$

Therefore, by Kolmogorov's inequality

$$
\begin{aligned}
&\mathbb{E}[\sup_{T_k^\epsilon \le t \le T_{k+1}^\epsilon \wedge \tau} |\xi_t^\epsilon - x_t^\epsilon|^4 | \mathcal{F}_{T_k^\epsilon}] \\
\le & C\mathbb{E}[(\Delta_b^k)^4 | \mathcal{F}_{T_k^\epsilon}] + C\mathbb{E}_{q_{T_k^\epsilon}^\epsilon}[\int_0^{T_0^\epsilon \wedge \tau_T^\epsilon} |A_i(x_{T_k^\epsilon}^\epsilon) - \mathcal{R}_0^* H \mathcal{R}_0 H(q_s^\epsilon)| ds]^2 + C\bar{C}^4 \epsilon^4 h^4(\epsilon) + 2\bar{C}^2 \epsilon^2 h(\epsilon)^2 \\
\le & C\mathbb{E}[(\Delta_b^k)^4 | \mathcal{F}_{T_k^\epsilon}] + C\mathbb{E}[\Delta_\sigma^k | \mathcal{F}_{T_k^\epsilon}]^2 + C\bar{C}^4 \epsilon^4 h^4(\epsilon) + C\bar{C}^2 \epsilon^2 h(\epsilon)^2 \\
\le & C^2 (\bar{C}\epsilon)^4 \mathbb{E}[\sup_{0 \le s \le \bar{C}\epsilon} |q_s^\epsilon - \bar{X}_s^\epsilon|^4] + C\bar{C}^2 \epsilon^2 \mathbb{E}[\sup_{0 \le s \le \bar{C}\epsilon} |q_s^\epsilon - \bar{X}_s^\epsilon|]^2 + 2\bar{C}^4 \epsilon^4 h^4(\epsilon) + 2\bar{C}^2 \epsilon^2 h(\epsilon)^2 \\
\le & C^4 (\bar{C}\epsilon)^4 \mathbb{E}[\sup_{0 \le s \le \bar{C}\epsilon} |q_s^\epsilon - \bar{X}_s^\epsilon|^4] + C\bar{C}^2 \epsilon^2 \mathbb{E}[\sup_{0 \le s \le \bar{C}\epsilon} |q_s^\epsilon - \bar{X}_s^\epsilon|^4]^{1/2} + 2\bar{C}^4 \epsilon^4 h^4(\epsilon) + 2\bar{C}^2 \epsilon^2 h(\epsilon)^2.
\end{aligned}
$$

Therefore, thanks to (2.13)

$$\mathbb{E}[\sup_{T_k^\epsilon \le t \le T_{k+1}^\epsilon \wedge \tau_T^\epsilon} |\xi_t^\epsilon - x_t^\epsilon|^4 | \mathcal{F}_{T_k^\epsilon}] \le C((\bar{C}\epsilon)^5 + (\bar{C}\epsilon)^{2.5}) + 2\bar{C}^4\epsilon^4 h^4(\epsilon) + 2\bar{C}^2\epsilon^2 h^2(\epsilon).$$

Finally, since there are at most $N := [T/\underline{C}\epsilon] + 1$ T_k's before time time T, we get

$$\begin{aligned}
\mathbb{E}[\sup_{0 \le t \le \tau_T^\epsilon} |\xi_t^\epsilon - x_t^\epsilon|^4] \le & \mathbb{E}[\sum_{k=0}^{\infty} \sup_{T_k^\epsilon \wedge \tau_T^\epsilon \le t \le T_{k+1}^\epsilon \wedge \tau_T^\epsilon} |\xi_t^\epsilon - x_t^\epsilon|^4] \\
\le & \mathbb{E}[\sum_{k=0}^{N} \sup_{T_k^\epsilon \le t \le T_{k+1}^\epsilon \wedge \tau_T^\epsilon} |\xi_t^\epsilon - x_t^\epsilon|^4] + \mathbb{E}[\sup_{T_N^\epsilon \wedge \tau_T^\epsilon \le t \le T_{N+1}^\epsilon \wedge \tau_T^\epsilon} |\xi_t^\epsilon - x_t^\epsilon|^4 | \mathcal{F}_{T_N}]] \\
\le & N(C((\bar{C}\epsilon)^5 + (\bar{C}\epsilon)^{2.5}) + 2\bar{C}^4\epsilon^4 h^4(\epsilon) + 2\bar{C}^2\epsilon^2 h^2(\epsilon)) \\
\le & C\frac{T}{\epsilon}(C((\bar{C}\epsilon)^5 + (\bar{C}\epsilon)^{2.5}) + 2\bar{C}^4\epsilon^4 h^4(\epsilon) + 2\bar{C}^2\epsilon^2 h^2(\epsilon)),
\end{aligned}$$

So that

$$\mathbb{E}[\sup_{0 \le t \le \tau_T^\epsilon} |\xi_t^\epsilon - x_t^\epsilon|^4] \le CT(\bar{C}^5\epsilon^4 + \bar{C}^{2.5}\epsilon^{1.5} + 2\bar{C}^4\epsilon^3 h^4(\epsilon) + 2\bar{C}^2\epsilon h^2(\epsilon)).$$

Therefore, we have proved the following Lemma

Lemma 2.4.5. *Under Hypothesis 8 and 9, for any intitial condition* (x_0, i)

on edge I_i and $H_1 \le x_0 \le H_2$ such that $(H_1, i), (H_2, i) \in I_i$, we have

$$\sup_{(x_0,i)\in I_i} \mathbb{E}_{x_0}[\sup_{0\le s\le \tau^\epsilon_{H_1,H_2}\wedge T} |\xi^\epsilon_t - x^\epsilon_t|^4] \le CT(\bar{C}^5\epsilon^4 + \bar{C}^{2.5}\epsilon^{1.5} + 2\bar{C}^4\epsilon^3 h^4(\epsilon) + 2\bar{C}^2\epsilon h^2(\epsilon)). \tag{2.14}$$

2.4.4 Closeness of ξ^ϵ_t and $\tilde{x}^\epsilon_t$

Let $|\xi^\epsilon_{T^\epsilon_k} - \tilde{x}^\epsilon_{T^\epsilon_k}| = \delta_k$. For $t \in [T^\epsilon_{k-1}, T^\epsilon_k)$, we have

$$\begin{aligned}
|\xi^\epsilon_t - \tilde{x}^\epsilon_t| \le & |\xi^\epsilon_{T^\epsilon_{k-1}} - \tilde{x}^\epsilon_{T^\epsilon_{k-1}}| + \int_{T^\epsilon_{k-1}}^t |B_i(x^\epsilon_{T^\epsilon_{k-1}}) - B_i(\tilde{x}^\epsilon_s)| ds + |\int_{T^\epsilon_{k-1}}^t (A_i(x^\epsilon_{T^\epsilon_{k-1}}) - A_i(\tilde{x}^\epsilon_s)) d\beta^\epsilon_s| \\
\le & \delta_{k-1} + C\int_{T^\epsilon_{k-1}}^t |x^\epsilon_{T^\epsilon_{k-1}} - \tilde{x}^\epsilon_s| ds + |\int_{T^\epsilon_{k-1}}^t (A_i(x^\epsilon_{T^\epsilon_{k-1}}) - A_i(\tilde{x}^\epsilon_s)) d\beta^\epsilon_s| \\
\le & \delta_{k-1} + C\int_{T^\epsilon_{k-1}}^t |\xi^\epsilon_{T^\epsilon_{k-1}} - \xi^\epsilon_s| ds + C\int_{T^\epsilon_{k-1}}^t |\xi^\epsilon_s - \tilde{x}^\epsilon_s| ds + |\int_{T^\epsilon_{k-1}}^t (A_i(\xi^\epsilon_{T^\epsilon_{k-1}}) - A_i(\xi^\epsilon_s)) d\beta^\epsilon_s| \\
& + |\int_{T^\epsilon_{k-1}}^t (A_i(\xi^\epsilon_s) - A_i(\tilde{x}^\epsilon_s)) d\beta^\epsilon_s|.
\end{aligned}$$

Let $\beta^{\epsilon,s}_t = \beta^\epsilon_{t+s} - \beta^\epsilon_s$. For $s \in [T^\epsilon_{k-1}, T^\epsilon_k)$,

$$|\xi^\epsilon_{T^\epsilon_{k-1}} - \xi^\epsilon_s| \le |B_i(x^\epsilon_{T^\epsilon_{k-1}})(s - T^\epsilon_{k-1})| + |A_i^{\frac{1}{2}}(x^\epsilon_{T^\epsilon_{k-1}})(\beta^\epsilon_s - \beta^\epsilon_{T^\epsilon_{k-1}})| \le C|s - T^\epsilon_{k-1}| + C|\beta^{\epsilon,T^\epsilon_{k-1}}_{s-T^\epsilon_{k-1}}|.$$

Since $T^\epsilon_k - T^\epsilon_{k-1} \le \bar{C}\epsilon$, we define $\bar{\xi}^\epsilon_s$ to be the extension of ξ^ϵ_s in the interval $[T^\epsilon_{k-1}, T^\epsilon_{k-1} + \bar{C}\epsilon]$ such that $\bar{\xi}^\epsilon_s = \xi^\epsilon_s$, $s \in [T^\epsilon_{k-1}, T^\epsilon_k)$ and $\bar{\xi}^\epsilon_s = 0$, $T^\epsilon_k \le s \le$

$T^\epsilon_{k-1}+\bar{C}\epsilon$. The process $\bar{\tilde{x}}^\epsilon_s$ is defined similarly. Then for $T^\epsilon_{k-1} < t_0 < T^\epsilon_{k-1}+\bar{C}\epsilon$,

$$\sup_{T^\epsilon_{k-1}\le s\le T^\epsilon_k\wedge t_0}|\xi^\epsilon_s-\tilde{x}^\epsilon_s| = \sup_{T^\epsilon_{k-1}\le s\le t_0}|\bar{\xi}^\epsilon_s-\bar{\tilde{x}}^\epsilon_s|,$$

and

$$\begin{aligned}
&\sup_{T^\epsilon_{k-1}\le s\le t_0\wedge T^\epsilon_k}|\xi^\epsilon_s-\tilde{x}^\epsilon_s| \\
\le&\delta_{k-1}+C\int_{T^\epsilon_{k-1}}^{T^\epsilon_k\wedge t_0}(|s-T^\epsilon_{k-1}|+|\beta^{\epsilon,T^\epsilon_{k-1}}_{s-T^\epsilon_{k-1}}|)ds+C\int_{T^\epsilon_{k-1}}^{T^\epsilon_k\wedge t_0}\sup_{0\le u\le s\wedge T^\epsilon_k}|\xi^\epsilon_u-\tilde{x}^\epsilon_u|ds \\
&+|\int_{T^\epsilon_{k-1}}^{T^\epsilon_k\wedge t_0}(A_i(\xi^\epsilon_{T^\epsilon_{k-1}})-A_i(\xi^\epsilon_s))d\beta^{\epsilon,T^\epsilon_{k-1}}_s|+|\int_{T^\epsilon_{k-1}}^{T^\epsilon_k\wedge t_0}(A_i(\xi^\epsilon_s)-A_i(\tilde{x}^\epsilon_s))d\beta^{\epsilon,T^\epsilon_{k-1}}_s|.
\end{aligned}$$

So

$$\begin{aligned}
&\mathbb{E}[\sup_{T^\epsilon_{k-1}\le s\le t_0\wedge T^\epsilon_k}|\xi^\epsilon_s-\tilde{x}^\epsilon_s|^4|\mathcal{F}_{T^\epsilon_{k-1}}] \\
\le&c\delta^4_{k-1}+c\mathbb{E}_{q^\epsilon_{T^\epsilon_{k-1}}}[|\int_0^{T^\epsilon_0}sds|^4]+c\mathbb{E}_{q^\epsilon_{T^\epsilon_{k-1}}}[|\int_0^{T^\epsilon_0}|\beta^{\epsilon,T^\epsilon_{k-1}}_s|ds|^4] \\
&+c\mathbb{E}_{q^\epsilon_{T^\epsilon_{k-1}}}[|\int_0^{T^\epsilon_0\wedge\tau}\sup_{0\le u\le s\wedge T^\epsilon_0}|\xi^\epsilon_u-x_u|ds|^4]+c\mathbb{E}_{q^\epsilon_{T^\epsilon_{k-1}}}[|\int_0^{T^\epsilon_0\wedge t_0}(A_i(\xi^\epsilon_{T^\epsilon_{k-1}})-A_i(\xi^\epsilon_s))d\beta^\epsilon_s|^4] \\
&+c\mathbb{E}_{q^\epsilon_{T^\epsilon_{k-1}}}[|\int_0^{T^\epsilon_0\wedge t_0}(A_i(\xi^\epsilon_s)-A_i(\tilde{x}^\epsilon_s))d\beta^\epsilon_s|^4].
\end{aligned}$$

Let $r = t_0 - T^\epsilon_{k-1}$, then $r \le \bar{C}\epsilon$ and

$$\begin{aligned}
&\mathbb{E}[\sup_{T^\epsilon_{k-1}\le s\le t_0\wedge T^\epsilon_k}|\xi^\epsilon_s - \tilde{x}^\epsilon_s|^4|\mathcal{F}_{T^\epsilon_{k-1}}] = \mathbb{E}[\sup_{T^\epsilon_{k-1}\le s\le t_0}|\bar{\xi}^\epsilon_s - \bar{\tilde{x}}^\epsilon_s|^4|\mathcal{F}_{T^\epsilon_{k-1}}] \\
\le&\delta^4_{k-1} + (\bar{C}\epsilon)^8 + \bar{C}^4\epsilon^6 + \mathbb{E}_{q^\epsilon_{T^\epsilon_{k-1}}}[|\int_0^r \sup_{0\le u\le s}|\bar{\xi}^\epsilon_u - \bar{\tilde{x}}^\epsilon_u|ds|^4] \\
&+ c\mathbb{E}_{q^\epsilon_{T^\epsilon_{k-1}}}[|\int_0^{T^\epsilon_0\wedge r}(A_i(\bar{\xi}^\epsilon_{T^\epsilon_{k-1}}) - A_i(\bar{\xi}^\epsilon_s))d\beta^{\epsilon,T^\epsilon_{k-1}}_s|^4] + c\mathbb{E}_{q^\epsilon_{T^\epsilon_{k-1}}}[|\int_0^{T^\epsilon_0\wedge r}(A_i(\bar{\xi}^\epsilon_s) - A_i(\bar{\tilde{x}}^\epsilon_s))d\beta^{\epsilon,T^\epsilon_{k-1}}_s|^4] \\
\le&\delta^4_{k-1} + (\bar{C}\epsilon)^8 + \bar{C}^4\epsilon^6 + \mathbb{E}_{q^\epsilon_{T^\epsilon_{k-1}}}[|\int_0^r \sup_{0\le u\le s}|\bar{\xi}^\epsilon_u - \bar{\tilde{x}}^\epsilon_u|ds|^4] \\
&+ c\mathbb{E}_{q^\epsilon_{T^\epsilon_{k-1}}}[|\int_0^r(A_i(\bar{\xi}^\epsilon_{T^\epsilon_{k-1}}) - A_i(\bar{\xi}^\epsilon_s))d\beta^{\epsilon,T^\epsilon_{k-1}}_s|^4] + c\mathbb{E}_{q^\epsilon_{T^\epsilon_{k-1}}}[|\int_0^r(A_i(\bar{\xi}^\epsilon_s) - A_i(\bar{\tilde{x}}^\epsilon_s))d\beta^{\epsilon,T^\epsilon_{k-1}}_s|^4].
\end{aligned}$$

Since

$$\begin{aligned}
\mathbb{E}[|\int_0^r(A_i(\bar{\xi}^\epsilon_{T^\epsilon_k}) - A_i(\bar{\xi}^\epsilon_s))d\beta^\epsilon_s|^4] \le&\bar{C}\epsilon\mathbb{E}[\int_0^r|A_i(\bar{\xi}^\epsilon_{T^\epsilon_k}) - A_i(\bar{\xi}^\epsilon_s)|^4ds] \\
\le&L^4\bar{C}\epsilon\mathbb{E}[\int_0^r|\bar{\xi}^\epsilon_{T^\epsilon_k} - \bar{\xi}^\epsilon_s|^4ds] \\
\le&C\bar{C}\epsilon\int_0^r\mathbb{E}[|\bar{\xi}^\epsilon_{T^\epsilon_k} - \bar{\xi}^\epsilon_s|^4]ds \\
\le&C\bar{C}\epsilon\int_0^r\mathbb{E}[s^4 + |\beta^\epsilon_s|^4]ds \\
\le&C(\bar{C}\epsilon)^4,
\end{aligned}$$

we get

$$\mathbb{E}[|\int_0^r(A_i(\bar{\xi}^\epsilon_s) - A_i(\bar{\tilde{x}}^\epsilon_s))d\beta^\epsilon_s|^4] \le \bar{C}\epsilon\int_0^r\mathbb{E}[|\bar{\xi}^\epsilon_s - \bar{\tilde{x}}^\epsilon_s|^4]ds \le \bar{C}\epsilon\int_0^r\mathbb{E}[\sup_{0\le u\le s}|\bar{\xi}^\epsilon_u - \bar{\tilde{x}}^\epsilon_u|^4]ds$$

and

$$\mathbb{E}[\sup_{T^\epsilon_{k-1}\le s\le t_0}|\bar{\xi}^\epsilon_s-\bar{x}_s|^4|\mathcal{F}_{T^\epsilon_{k-1}}]=\mathbb{E}_{q^\epsilon_{T^\epsilon_{k-1}}}[\sup_{0\le s\le t}|\bar{\xi}^\epsilon_s-\bar{x}_s|^4]].$$

Therefore

$$\mathbb{E}_{q^\epsilon_{T^\epsilon_{k-1}}}[\sup_{0\le s\le r}|\bar{\xi}^\epsilon_s-\bar{\tilde{x}}^\epsilon_s|^4]]\le\delta^4_{k-1}+(\bar{C}\epsilon)^8+\bar{C}^4\epsilon^6+(\bar{C}\epsilon)^4+((\bar{C}\epsilon)^4+\bar{C}\epsilon)\int_0^r\mathbb{E}_{q^\epsilon_{T^\epsilon_{k-1}}}[\sup_{0\le u\le s}|\bar{\xi}^\epsilon_u-\bar{\tilde{x}}^\epsilon_u|^4].$$

If we apply Gronwall's inequality, we have

$$\mathbb{E}_{q^\epsilon_{T^\epsilon_{k-1}}}[\sup_{0\le s\le r}|\bar{\xi}^\epsilon_s-\bar{\tilde{x}}^\epsilon_s|^4]]\le(\delta^4_{k-1}+(\bar{C}\epsilon)^8+\bar{C}^4\epsilon^6+(\bar{C}\epsilon)^4)\exp(((\bar{C}\epsilon)^4+\bar{C}\epsilon)t_0)$$

and thus

$$\mathbb{E}[\sup_{T^\epsilon_{k-1}\le s\le t_0}|\xi^\epsilon_s-\tilde{x}^\epsilon_s|^4|\mathcal{F}_{T^\epsilon_{k-1}}]ds\le(\delta^4_{k-1}+(\bar{C}\epsilon)^8+\bar{C}^4\epsilon^6+(\bar{C}\epsilon)^4)\exp((\bar{C}\epsilon)^5+(\bar{C}\epsilon)^2)$$

All this implies,

$$\begin{aligned}
&\mathbb{E}[\sup_{T^\epsilon_{k-1}\wedge\tau^\epsilon_T\le s\le T^\epsilon_k\wedge\tau^\epsilon_T}|\bar{\xi}^\epsilon_s-\bar{\tilde{x}}^\epsilon_s|^4]=\mathbb{E}[\mathbb{E}[\sup_{T^\epsilon_{k-1}\wedge\tau^\epsilon_T\le s\le T^\epsilon_k\wedge\tau^\epsilon_T}|\bar{\xi}^\epsilon_s-\bar{\tilde{x}}^\epsilon_s|^4]|\mathcal{F}_{T^\epsilon_{k-1}}]\\
\le&\mathbb{E}[(\delta^4_{k-1}+(\bar{C}\epsilon)^8+\bar{C}^4\epsilon^6+(\bar{C}\epsilon)^6+(\bar{C}\epsilon)^4)\exp((\bar{C}\epsilon)^4+(\bar{C}\epsilon)^2)]\\
\le&\mathbb{E}[\sup_{T^\epsilon_{k-2}\wedge\tau^\epsilon_T\le s\le T^\epsilon_{k-1}\wedge\tau^\epsilon_T}|\bar{\xi}^\epsilon_s-\bar{x}_s|^4]]\exp((\bar{C}\epsilon)^4+(\bar{C}\epsilon)^2)\\
&+((\bar{C}\epsilon)^8+\bar{C}^4\epsilon^6+(\bar{C}\epsilon)^6+(\bar{C}\epsilon)^4)\exp((\bar{C}\epsilon)^4+(\bar{C}\epsilon)^2)].
\end{aligned}$$

Now, let $a_k=\mathbb{E}[\sup_{T^\epsilon_{k-1}\wedge\tau^\epsilon_T\le s\le T^\epsilon_k\wedge\tau^\epsilon_T}|\bar{\xi}^\epsilon_s-\bar{\tilde{x}}^\epsilon_s|^4]]$. The calculation above implies the following recursive ,

$$a_k\le\exp(\bar{C}\epsilon)(a_{k-1}+(\bar{C}\epsilon)^4).$$

By induction, we can easily deduce that

$$a_n\le(\bar{C}\epsilon)^4\sum_{k=1}^{n}\exp(k\bar{C}\epsilon)=(\bar{C}\epsilon)^4\frac{1-\exp((n+1)\bar{C}\epsilon)}{1-\exp(\bar{C}\epsilon)}\le C\bar{C}^4\epsilon^3.$$

Finally, let $N = [t/\underline{C}\epsilon] + 1$,

$$\mathbb{E}[\sup_{0\le s\le\tau_T^\epsilon}|\xi_s^\epsilon - \tilde{x}_s^\epsilon|^4] \le \mathbb{E}[\sup_k \sup_{T_{k-1}^\epsilon\wedge\tau_T^\epsilon\le s\le T_k^\epsilon\wedge\tau_T^\epsilon}|\bar{\xi}_s^\epsilon - \bar{\tilde{x}}_s^\epsilon|^4] \le \mathbb{E}[\sum_{k=1}^N \sup_{T_{k-1}^\epsilon\wedge\tau_T^\epsilon\le s\le T_k^\epsilon\wedge\tau_T^\epsilon}|\bar{\xi}_s^\epsilon - \bar{\tilde{x}}_s^\epsilon|^4]$$
$$\le \sum_{k=1}^N \mathbb{E}[\sup_{T_{k-1}^\epsilon\wedge\tau_T^\epsilon\le s\le T_k^\epsilon\wedge\tau_T^\epsilon}|\bar{\xi}_s^\epsilon - \bar{\tilde{x}}_s^\epsilon|^4] = \sum_{k=1}^N a_k \le CT\bar{C}^4\epsilon^2.$$

Therefore, we have proved the following Lemma

Lemma 2.4.6. *Under Hypothesis 8 and 9, for any intitial condition (x_0, i) on edge I_i and $H_1 \le x_0 \le H_2$ such that $(H_1, i), (H_2, i) \in I_i$, we have*

$$\sup_{(x_0,i)\in I_i}\mathbb{E}_{x_0}[\sup_{0\le s\le\tau_T^\epsilon}|\xi_s^\epsilon - \tilde{x}_s^\epsilon|^4] \le CT\bar{C}^4\epsilon^2. \tag{2.15}$$

2.4.5 Proof of Lemma 2.4.1

By Ito's formula, for any $\epsilon > 0$,

$$\mathbb{E}_{x_0}[e^{-\lambda\tau_T^\epsilon}f(x_{\tau_T^\epsilon}) - \int_0^{+\infty} e^{-\lambda s}(-\lambda I + \mathcal{L}_i)[f](x_s)\mathcal{X}_{\tau_T^\epsilon > s}ds] = f(x_0).$$

Since $\tilde{x}_t^\epsilon$ and x_t have the same distribution,

$$\mathbb{E}_{x_0}[e^{-\lambda\tau_T^\epsilon}f(\tilde{x}_{\tau_T^\epsilon}^\epsilon) - \int_0^{+\infty} e^{-\lambda s}(-\lambda I + \mathcal{L}_i)[f](\tilde{x}_s^\epsilon)\mathcal{X}_{\tau_T^\epsilon > s}ds] = f(x_0).$$

Moreover,

$$\begin{aligned}
&\mathbb{E}_{x_0}[e^{-\lambda\tau_T^\epsilon}(f(x_{\tau_T^\epsilon}^\epsilon)-f(\tilde{x}_{\tau_T^\epsilon}^\epsilon))]^2\\
\leq&\mathbb{E}_{x_0}[|f(x_{\tau_T^\epsilon}^\epsilon)-f(\tilde{x}_{\tau_T^\epsilon}^\epsilon))|^2]\leq|f|_{Lip}^2\mathbb{E}_{x_0}[|x_{\tau_T^\epsilon}^\epsilon-\tilde{x}_{\tau_T^\epsilon}^\epsilon|^2]\\
\leq&|f|_{Lip}^2(\mathbb{E}_{x_0}[|\xi_{\tau_T^\epsilon}-\tilde{x}_{\tau_T^\epsilon}^\epsilon|^4]^{\frac{1}{2}}+\mathbb{E}_{x_0}[|x_{\tau_T^\epsilon}^\epsilon-\xi_{\tau_T^\epsilon}^\epsilon|^4]^{\frac{1}{2}})\\
\leq&|f|_{Lip}^2(\mathbb{E}_{x_0}[\sup_{0\leq t\leq\tau_T^\epsilon}|\xi_t-\tilde{x}_t^\epsilon|^4]^{\frac{1}{2}}+\mathbb{E}_{x_0}[\sup_{0\leq t\leq\tau_T^\epsilon}|x_t^\epsilon-\xi_t^\epsilon|^4]^{\frac{1}{2}}).
\end{aligned}$$

Apply Lemma 2.4.5 and 2.4.6, $\mathbb{E}_{x_0}[e^{-\lambda\tau_T^\epsilon}(f(x_{\tau_T^\epsilon}^\epsilon)-f(\tilde{x}_{\tau_T^\epsilon}^\epsilon))]$ converges to 0 uniformly with respect to x_0. And

$$\begin{aligned}
&\mathbb{E}_{x_0}[|\int_0^{\tau_T^\epsilon}e^{-\lambda s}((-\lambda I+\mathcal{L}_i)[f](x_s^\epsilon)-(-\lambda I+\mathcal{L}_i)[f](\tilde{x}_s^\epsilon))ds|]^2\\
\leq&\mathbb{E}_{x_0}[\int_0^{\tau_T^\epsilon}e^{-\lambda s}C_f|x_s^\epsilon-\tilde{x}_s^\epsilon|ds]^2\leq C_f^2T^2\mathbb{E}_{x_0}[\sup_{0\leq t\leq\tau_T^\epsilon}|x_t^\epsilon-\tilde{x}_t^\epsilon|^2])\\
\leq&C_f^2T^2(\mathbb{E}_{x_0}[\sup_{0\leq t\leq\tau_T^\epsilon}|\xi_t-\tilde{x}_t^\epsilon|^4]^{\frac{1}{2}}+\mathbb{E}_{x_0}[\sup_{0\leq t\leq\tau_T^\epsilon}|x_t^\epsilon-\xi_t^\epsilon|^4]^{\frac{1}{2}}).
\end{aligned}$$

Similarly, $\mathbb{E}_{x_0}[|\int_0^{\tau_T^\epsilon}e^{-\lambda s}((-\lambda I+\mathcal{L}_i)[f](x_s^\epsilon)-(-\lambda I+\mathcal{L}_i)[f](\tilde{x}_s^\epsilon))ds|]$ converges to 0 uniformly with respect to x_0. So

$$\begin{aligned}
&\lim_{\epsilon\to0}\mathbb{E}_{x_0}[e^{-\lambda\tau_T^\epsilon}f(x_{\tau_T^\epsilon})-\int_0^{\tau_T^\epsilon}e^{-\lambda s}(-\lambda I+\mathcal{L}_i)[f](x_s)ds]\\
=&\lim_{\epsilon\to0}\mathbb{E}_{x_0}[e^{-\lambda\tau_T^\epsilon}f(\tilde{x}_{\tau_T^\epsilon}^\epsilon)-\int_0^{\tau_T^\epsilon}e^{-\lambda s}(-\lambda I+\mathcal{L}_i)[f](\tilde{x}_s^\epsilon)ds]=f(x_0).
\end{aligned}$$

2.4.6 Proof of Lemma 2.3.2

We start with the following Lemma

Lemma 2.4.7. *Under Hypothesis 8 and 9, and let the stopping time $\tau^{\epsilon}_{H_1,H_2}$ be defined as before, where H_1, H_2 are in the interior of $H(I_i)$, then*

$$\sup_{\epsilon>0} \sup_{z\in D_i([H_1,H_2])} \mathbb{E}_z[\tau^{\epsilon}_{H_1,H_2}] < \infty.$$

Proof. Let f satisfies $\mathcal{L}_i[f] = -1$ and the boundary conditions $f(H_1) = f(H_2) = 0$. Then apply Lemma 2.3.2 with f and have

$$\mathbb{E}[\tau^{\epsilon}_T] = f(x_0) - \mathbb{E}[f(x^{\epsilon}_{\tau^{\epsilon}_T})]$$

Since the second order differential operator $\mathcal{L}_i$ is uniformly elliptic on $[H_1, H_2]$, we can apply the maximum principle, and we get

$$|f|_\infty \leq C,$$

where C is a constant depending on the coefficients in $\mathcal{L}_i$ and $[H_1, H_2]$. In

particular, it is independent of T. Therefore, by Fatou's Lemma

$$\mathbb{E}_z[\tau^\epsilon_{H_1,H_2}] = \mathbb{E}_z[\lim_{T\to\infty} \tau^\epsilon_T] \leq \liminf_{T\to\infty} \mathbb{E}[\tau^\epsilon_T] \leq 2C < \infty.$$

Since this inequality holds for all $z \in D_i([H_1, H_2])$ and $\epsilon > 0$. □

Lemma 2.4.8. *Under Hypothesis 8 and 9, for any intitial condition (x_0, i) on the edge I_i and $H_1 \leq x_0 \leq H_2$ such that $(H_1, i), (H_2, i)$ are inside the interior of I_i, we have*

$$\lim_{\epsilon\to 0} \sup_{(x_0,i)\in I_i} \mathbb{E}_{x_0}[\sup_{0\leq s\leq \tau^\epsilon_{H_1,H_2}} |\xi^\epsilon_t - x^\epsilon_t|^2] = 0.$$

Proof. Unlike the its previous counterpart, the number of intervals here is random. Let N be the largest k such that $T^\epsilon_k \leq \tau^\epsilon_{H_1,H_2}$. For simplicity, we write $\tau^\epsilon_{H_1,H_2}$ as τ^ϵ_T, eliminate the edge coordinate i and let $T_{-1} = 0$ in this proof. Thanks to (2.14) and apply Fatou's Lemma and strong Markov

property,

$$\begin{aligned}
\mathbb{E}_{x_0}[\sup_{0\le t\le\tau_T^\epsilon}|\xi_t^\epsilon - x_t^\epsilon|^2] \le& \mathbb{E}_{x_0}[\sum_{k=-1}^{\infty}\sup_{T_k\wedge\tau_T^\epsilon\le t\le T_{k+1}\wedge\tau_T^\epsilon}|\xi_t^\epsilon - x_t^\epsilon|^2] \\
&\le \sum_{k=-1}^{\infty}\mathbb{E}_{x_0}[\sup_{T_k\le t\le T_{k+1}\wedge\tau_T^\epsilon}|\xi_t^\epsilon - x_t^\epsilon|^2\mathcal{X}_{k\le N}] \\
&\le \sum_{k=-1}^{\infty}\mathbb{E}_{x_0}[\mathbb{E}_{x_0}[\sup_{T_k\le t\le T_{k+1}\wedge\tau_T^\epsilon}|\xi_t^\epsilon - x_t^\epsilon|^2\mathcal{X}_{T_k\le\tau_T^\epsilon}|\mathcal{F}_{T_k}]] \\
&= \sum_{k=-1}^{\infty}\mathbb{E}_{x_0}[\mathbb{E}_{x_0}[\sup_{T_k\le t\le T_{k+1}\wedge\tau_T^\epsilon}|\xi_t^\epsilon - x_t^\epsilon|^2|\mathcal{F}_{T_k}]\mathcal{X}_{T_k\le\tau_T^\epsilon}] \\
&\le \sum_{k=-1}^{\infty}\mathbb{E}_{x_0}[\mathbb{E}_{H(q_{T_k}^\epsilon)}[\sup_{0\le t\le T_0\wedge\tau_T^\epsilon}|\xi_t^\epsilon - x_t^\epsilon|^2]\mathcal{X}_{T_k\le\tau_T^\epsilon}] \\
&\le \sum_{k=-1}^{\infty}\mathbb{E}_{x_0}[\sup_{(x_0,i)\in I_i}\mathbb{E}_{x_0}[\sup_{0\le t\le T_0\wedge\tau_T^\epsilon}|\xi_t^\epsilon - x_t^\epsilon|^2]\mathcal{X}_{T_k\le\tau_T^\epsilon}] \\
&= \sum_{k=-1}^{\infty}\sup_{(x_0,i)\in I_i}\mathbb{E}_{x_0}[\sup_{T_k\le t\le T_{k+1}\wedge\tau_T^\epsilon}|\xi_t^\epsilon - x_t^\epsilon|^2]\mathbb{E}_{x_0}[\mathcal{X}_{T_k\le\tau_T^\epsilon}] \\
&\le (C((\bar{C}\epsilon)^5 + (\bar{C}\epsilon)^{2.5}) + 2\bar{C}^4\epsilon^4h^4(\epsilon) + 2\bar{C}^2\epsilon^2h^2(\epsilon))\sum_{k=0}^{\infty}\mathbb{E}[\mathcal{X}_{k\le N}].
\end{aligned}$$

Moreover,

$$\sum_{k=0}^{\infty}\mathbb{E}[\mathcal{X}_{k\le N}] \le 1 + \mathbb{E}[N] = 1 + \mathbb{E}[\frac{\tau_{H_1,H_2}^\epsilon}{\underline{C}\epsilon}] \le \frac{C}{\epsilon}$$

and thus prove for all $T > 0$,

$$\mathbb{E}_{x_0}[\sup_{0\le t\le\tau_T^\epsilon}|\xi_t^\epsilon - x_t^\epsilon|^2] \le C(C((\bar{C}\epsilon)^4 + (\bar{C}\epsilon)^{3/2}) + 2\bar{C}^4\epsilon^3h^4(\epsilon) + 2\bar{C}^2\epsilon h^2(\epsilon)).$$

Finally, we take the sup over T and prove the result. □

Lemma 2.4.9. *Under Hypothesis 8 and 9, for any intitial condition (x_0, i) on edge I_i and $H_1 \leq x_0 \leq H_2$ such that $(H_1, i), (H_2, i)$ inside the interior of I_i, we have*

$$\lim_{\epsilon \to 0} \sup_{(x_0,i)\in I_i} \mathbb{E}_{x_0}[\sup_{0\leq s\leq \tau^{\epsilon}_{H_1,H_2}} |\xi^{\epsilon}_s - \tilde{x}^{\epsilon}_s|^2] = 0.$$

Proof. Similarly, we have

$$\begin{aligned}
\mathbb{E}_{x_0}[\sup_{0\leq s\leq \tau^{\epsilon}_T} |\xi^{\epsilon}_s - \tilde{x}^{\epsilon}_s|^2] \leq & \mathbb{E}_{x_0}[\sup_k \sup_{T_{k-1}\wedge\tau^{\epsilon}_T \leq s\leq T_k\wedge\tau^{\epsilon}_T} |\bar{\xi}^{\epsilon}_s - \bar{\tilde{x}}^{\epsilon}_s|^2] \\
\leq & \mathbb{E}_{x_0}[\sum_{k=0}^{\infty} \sup_{T_{k-1}\wedge\tau^{\epsilon}_T \leq s\leq T_k\wedge\tau^{\epsilon}_T} |\bar{\xi}^{\epsilon}_s - \bar{\tilde{x}}^{\epsilon}_s|^2 \mathcal{X}_{k\leq N}] \\
\leq & \sum_{k=0}^{\infty} \mathbb{E}_{x_0}[\sup_{T_{k-1}\wedge\tau^{\epsilon}_T \leq s\leq T_k\wedge\tau^{\epsilon}_T} |\bar{\xi}^{\epsilon}_s - \bar{\tilde{x}}^{\epsilon}_s|^2 \mathcal{X}_{k\leq N}] \\
\leq & \sum_{k=0}^{\infty} \sup_{(x_0,i)\in I_i} \mathbb{E}_{x_0}[\sup_{T_{k-1}\wedge\tau^{\epsilon}_T \leq s\leq T_k\wedge\tau^{\epsilon}_T} |\bar{\xi}^{\epsilon}_s - \bar{\tilde{x}}^{\epsilon}_s|^4] \mathbb{E}[\mathcal{X}_{k\leq N}].
\end{aligned}$$

Now by the estimation (2.15), and process as the previous Lemma, the result is obvious. □

Now if we apply exactly the same proof as in Lemma 2.4.1, Lemma 2.3.2 is proved.

2.5 Properties near the Saddle Point

Since the gluing condition is given by local property, we can assume without loss of generality in this section that

Hypothesis 6. *The coefficients $g(x)$, $b(x)$, $b^\epsilon(x)$, $\sigma(x)$ and $\sigma^\epsilon(x)$, $x \in \mathbb{R}^2$ in equation (2.1) are uniformly bounded.*

2.5.1 Averaging the Measure

We first remark that: the probability for q_t^ϵ, starting from an initial point $q \in D_k(\pm\delta')$ to reach the level set $C_{ki}(\delta)$ is approximately the same. The following Lemma is a consequence of Krylov and Safonov's theory (see [16]).

Lemma 2.5.1. *There exists a $\delta^{2.5.1} > 0$ such that for every $0 < \delta' < \delta < \delta^{2.5.1}$ and and any $\kappa > 0$, the following estimate holds for sufficiently small ϵ,*

$$\sup_{q_1,q_2\in\bar{D}_k(\pm\delta')} |\mathbb{E}_{q_1}[f(q^\epsilon_{\tau_k^\epsilon(\pm\delta)})] - \mathbb{E}_{q_2}[f(q^\epsilon_{\tau_k^\epsilon(\pm\delta)})]| < \kappa.$$

If the function f in the Lemma is $\mathcal{X}_{C_{kj}(\delta)}(I_j \sim O_k)$, we have $\mathbb{E}_{q_i}[f(q^\epsilon_{\tau_k^\epsilon(\pm\delta)})] = \mathbb{P}_{q_i}[q^\epsilon_{\tau_k^\epsilon(\pm\delta)} \in C_{kj}(\delta)]$. Now we can average the probability $\mathbb{P}_q[q^\epsilon_{\sigma_1} \in C_{kj(\delta)}]$ by a measure μ on the level set $C_k(\pm\delta')$.

Lemma 2.5.2. *Let μ be a measure on the level set $C_k(\pm\delta')$. For any $\kappa > 0$ and sufficiently small ϵ,*

$$\sup_{x \in C_k(\pm\delta')} |\mathbb{P}_x[q^\epsilon_{\tau^\epsilon_k(\pm\delta) \in C_{ki}(\delta)}] - \int_{C_k(\pm\delta')} \frac{\mu(dx)\mathbb{P}_x[q^\epsilon_{\tau^\epsilon_k(\pm\delta)} \in C_{ki}(\delta)]}{\mu(C_k(\pm\delta'))}| < \kappa. \quad (2.16)$$

The measure μ will be determined by the long term behavior of q^ϵ_t as to be shown in the next subsection.

2.5.2 Representation of the Invariant Measure

Let

$$C_\delta = \bigcup_{k \in S} \bigcup_{I_i \sim O_k} C_{ki}(\pm\delta) \qquad C_{\delta'} = \bigcup_{k \in S} \bigcup_{I_i \sim O_k} C_{ki}(\pm\delta').$$

The sequence $\{q^\epsilon_{\tau^\epsilon_k}\}_{k=0}^\infty$ is a Markov chain and for all $k > 1$, $q^\epsilon_{\tau^\epsilon_k} \in C_{\delta'}$. Moreover if $q^\epsilon_0 \in C_\delta$, $\{q^\epsilon_{\sigma^\epsilon_k}\}_{k=0}^\infty$ is a Markov chain on C_δ. Then every invariant measure μ^ϵ of the process $(q^\epsilon_t, \mathbb{P}^\epsilon_q)$ can be represented in the form

$$\mu^\epsilon(A) = \int_{C_\delta} \nu^\epsilon(dz)\mathbb{E}^\epsilon_z[\int_0^{\sigma^\epsilon_1} \mathbb{1}_A(q^\epsilon_t)dt] = \int_{C_{\delta'}} \nu'^\epsilon(dz)\mathbb{E}^\epsilon_z[\int_0^{\tau^\epsilon_1} \mathbb{1}_A(q^\epsilon_t)dt], \quad (2.17)$$

where ν^ϵ and ν'^ϵ are measures on C_δ and $C_{\delta'}$ such that

$$\nu^\epsilon(B) = \int_{C_{\delta'}} \nu'^\epsilon(dz)\mathbb{P}_z^\epsilon[q^\epsilon_{\sigma_0^\epsilon} \in B], \quad \nu'^\epsilon(C) = \int_{C_\delta} \nu^\epsilon(dz)\mathbb{P}_z^\epsilon[q^\epsilon_{\tau_1^\epsilon} \in C]. \quad (2.18)$$

See [14]. Notice that ν^ϵ and ν'^ϵ are the invariant measures for the Markov chains $\{q^\epsilon_{\sigma_k^\epsilon}\}_{k=0}^\infty$ and $\{q^\epsilon_{\tau_k^\epsilon}\}_{k=0}^\infty$ on C_δ and $C_{\delta'}$ respectively. Now we pick the measure μ in equation (2.16) to be ν'^ϵ and we get

$$\int_{C_k(\pm\delta')} \frac{\mu(dz)\mathbb{P}_z[q^\epsilon_{\tau_k^\epsilon(\pm\delta)} \in C_{ki}(\delta)]}{\mu(C_k(\pm\delta'))} = \frac{\nu^\epsilon(C_{ki}(\delta)}{\nu'^\epsilon(C_k(\pm\delta'))}.$$

Hence, we apply equation (2.18) with $C = C_k(\pm\delta')$ and since $\mathbb{P}_z^\epsilon[q^\epsilon_{\tau_1} \in C] = 1$, for all $z \in C_k(\pm\delta`)$, we have $\nu'^\epsilon(C_k(\pm\delta')) = \nu^\epsilon(C_k(\pm\delta'))$. Therefore

$$|\mathbb{P}_z[q^\epsilon_{\tau^\epsilon(\pm\delta)\in C_{ki}(\delta)}] - \frac{\nu^\epsilon(C_{ki}(\delta)}{\nu^\epsilon(C_k(\pm\delta))}| < \kappa.$$

Finally, apply equation (2.17) with a bounded measurable function G and get

$$\int_{\mathbb{R}^2} G(z)\mu^\epsilon(dz) = \int_{C_\delta} \nu^\epsilon(dz)\mathbb{E}_z^\epsilon[\int_0^{\sigma_1} G(q_t^\epsilon)dt]. \quad (2.19)$$

2.5.3 Proof of Lemma 2.3.4: A Special Case

In this section we first assume that the following Hypothesis holds,

Hypothesis 7. *Let function $a(x)$ be the function defined by (2.3). We assume that a satisfies the following relation for all $\epsilon > 0$.*

$$(\mathcal{L}_0)^*[a^{-1}](z) = 0, \quad (\mathcal{L}_0^\epsilon)^*[a^{-1}](z) = 0 \quad \forall z \in \mathbb{R}^2.$$

Then $a^{-1}(z)dz$ is the invariant measure for the the system $\{q_t^\epsilon\}$ as $\epsilon \downarrow 0$ and $\mathcal{L}_i$ is a generalized differential operator. Apply Lemma 2.2.1 again, the left hand side of equation (2.19) becomes

$$\int_{\mathbb{R}^2} G(z)\mu^\epsilon(dz) = \int_{\mathbb{R}^2} G(z)\frac{dz}{a(z)} = \int\int_{C(H)} G(z)\frac{dl}{|\nabla H(z)|a(z)}dH,$$

Let the window function $G(z)$ depend only on the Hamiltonian,

$$G(z) = f \circ H(z),$$

where $f \in \mathbf{C}^\infty(\mathbb{R})$ with support in $(H_{k1} + \delta, H_{k2} - \delta)$ and $H_{ki} = H(O_{ki})$,

$i = 1, 2$, $O_{ki} (i = 1, 2)$ are the two end points of an edge I_k. Then

$$\int_{\mathbb{R}^2} G(z)\mu^\epsilon(dz) = \int_{H_1+\delta}^{H_2-\delta} f(H)v_k'(H)dH. \tag{2.20}$$

Finally, we apply Lemma 2.3.2 to the right hand side of equation (2.19),

$$\lim_{\epsilon\to 0} \mathbb{E}_z^\epsilon[\int_0^{\sigma_1^\epsilon} G(q_t^\epsilon)dt] = \lim_{\epsilon\to 0} \mathbb{E}_x^\epsilon[\int_0^{\sigma_1^\epsilon} f(x_t^\epsilon)dt] = \lim_{\epsilon\to 0} \mathbb{E}_x^\epsilon[\int_0^{\tau_1^\epsilon} f(x_t^\epsilon)dt] = u(x),$$

where $x = H(z)$ and $u(x)$ is the solution to

$$\begin{cases} \mathcal{L}_i[u] = f \\ \\ u(H_1 + \delta') = u(H_2 - \delta') = 0. \end{cases}$$

It can be easily seen that $u(x)$ can be solved by the following formula,

$$\begin{aligned} u(x) =& \frac{u_k(H_{k2} - \delta') - u_k(x)}{u_k(H_{k2} - \delta') - u_k(H_{k1} + \delta')} \int_{H_{k1}+\delta'}^{x} (u_k(x) - u_k(H_{k1} + \delta'))f(h)dv_k(h) \\ &+ \frac{u_k(x) - u_k(H_1 + \delta')}{u_k(H_{k2} - \delta') - u_k(H_{k1} + \delta')} \int_{x}^{H_{k2}-\delta'} (u_k(H_{k2} - \delta') - u_k(x))f(h)dv_k(h). \end{aligned}$$

Notice that f has support in $(H_{k1} + \delta, H_{k2} - \delta)$, so if $x \in C_{k1}(\delta)$, then

$$u(H_{k1}+\delta) = \frac{u_k(H_1 + \delta) - u_k(H_1 + \delta')}{u_k(H_{k2} - \delta') - u_k(H_{k1} + \delta')} \int_{H_{k1}+\delta}^{H_{k2}-\delta'} (u_k(H_{k2}-\delta')-u_k(H_{k1}+\delta))f(h)dv_k(h),$$

and if $x \in C_{k2}(-\delta)$, then

$$u(H_{k2}-\delta) = \frac{u_k(H_{k2}-\delta') - u_k(H_{k2}-\delta)}{u_k(H_{k2}-\delta') - u_k(H_{k1}+\delta')} \int_{H_{k1}+\delta'}^{H_{k2}-\delta} (u_k(H_{k2}-\delta) - u_k(H_{k1}+\delta')) f(h) dv_k(h).$$

Now we combine equation (2.19) and equation (2.20),

$$\int_{H_1+\delta}^{H_2-\delta} f(H) v_k'(H) dH = \lim_{\epsilon \to 0} \nu^\epsilon(C_{k1}(\delta))(u(H_{k1}+\delta)+\mathcal{O}(\epsilon)) + \nu^\epsilon(C_{k2}(\delta))(u(H_{k2}-\delta)+\mathcal{O}(\epsilon)).$$

which is an equation holds for all f, therefore we have:

$$\lim_{\epsilon \to 0} \nu^\epsilon(C_{k1}(\delta)) = \frac{1}{u_i(H_1+\delta) - u_i(H_2+\delta')},$$

$$\lim_{\epsilon \to 0} \nu^\epsilon(C_{k2}(\delta)) = \frac{1}{u_i(H_2-\delta') - u_i(H_2-\delta)}.$$

Finally, letting $\delta \to \delta'$ and $\delta' \to 0$ gives us Lemma 2.3.4 under Hypothesis 11.

2.5.4 Proof of Lemma 2.3.4: The General Case

Now we consider the case without Hypothesis 11. Instead of considering the process q_t^ϵ whose invariant measure is unknown when $\epsilon \downarrow 0$, we consider

the system by compensating a smooth vector filed $\hat{b}$ and $\hat{b}^\epsilon$ satisfying

$$(\mathcal{L}_0 + \hat{b} \cdot \nabla)^*[a^{-1}] = 0, \quad (\mathcal{L}_0^\epsilon + \hat{b}^\epsilon \cdot \nabla)^*[a^{-1}] = 0 \tag{2.21}$$

and $\mathcal{L}_0 + \hat{b}\nabla$, $(\mathcal{L}_0^\epsilon + \hat{b}^\epsilon\nabla)^*$ are the corresponding formal adjoint operator. We first claim that both such $\hat{b}$ and $\hat{b}^\epsilon$ exists under Hypothesis 8, 9, and 10. Indeed

$$(\mathcal{L}_0 + \hat{b} \cdot \nabla)^*[a^{-1}] = \mathcal{L}_0[a^{-1}] - \sum_{i=1}^{2} \frac{\partial}{\partial x_i}(\hat{b}_i a^{-1}).$$

Let $\hat{b}_2 = 0$ and $\hat{b}_1$ satisfy the first-order partial differential equation

$$\frac{\partial}{\partial x_1}(\hat{b}_1 a^{-1}) = \mathcal{L}_0[a^{-1}](x).$$

Then

$$\hat{b}_1(x_1, x_2) = a(x_1, x_2) \int_0^{x_1} \mathcal{L}_0[a^{-1}](u_1, x_2) du_1. \tag{2.22}$$

By Hypothesis 10, $\hat{b}$ is well-defined, bounded and globaly Lipschitz continuous. Similarly, such $\hat{b}^\epsilon$ also exists. Therefore the stochastic process $\hat{q}_t^\epsilon$ is

well-defined for every fixed $\epsilon > 0$:

$$\begin{cases} d\hat{q}_t^\epsilon = \frac{1}{\epsilon} g(\hat{q}_t^\epsilon)dt + b(\hat{q}_t^\epsilon)dt + \sigma(\hat{q}_t^\epsilon)dW_t + b^\epsilon(\hat{q}_t^\epsilon)dt + \sigma^\epsilon(\hat{q}_t^\epsilon)dW_t + \hat{b}(\hat{q}_t^\epsilon)dt + \hat{b}^\epsilon(\hat{q}_t^\epsilon)dt \\ \hat{q}_0^\epsilon = q, \end{cases}$$

and $\hat{q}_t^\epsilon$ satisfies Hypothesis 11. Therefore, the gluing condition holds for $\hat{q}_t^\epsilon$.

For the process q_t^ϵ, the drift term $\hat{b}$ and $\hat{b}^\epsilon$ need to be killed from $\hat{q}_t^\epsilon$. I apply Girsanov Theorem up to any fixed time T.

The measure $\mathbb{P}_q^\epsilon$ induced by the process q_t^ϵ on $\mathbf{C}([0,T];\mathbb{R}^2)$ is absolutely continuous to the measure $\hat{\mathbb{P}}_q^\epsilon$ induced by $\hat{q}_t^\epsilon$ on $\mathbf{C}([0,T];\mathbb{R}^2)$ with the density

$$\frac{d\mathbb{P}_q^\epsilon}{d\hat{\mathbb{P}}_q^\epsilon}|_{\mathcal{F}_t} = \exp\{\int_0^t M^\epsilon(\hat{q}_s^\epsilon)dW_s - \frac{1}{2}\int_0^t |M^\epsilon(\hat{q}_s^\epsilon)|^2 ds\}, \quad 0 \le t \le T$$

where

$$M^\epsilon(x) = -(\sigma + \sigma^\epsilon)^{-1}(x)(\hat{b} + \hat{b}^\epsilon)(x).$$

Compare the two measures of a set $A \in \mathcal{F}_T$, where the expectation, without

specification, is taken with respect to the measure $\hat{\mathbb{P}}_q^\epsilon$.

$$\begin{aligned}|\mathbb{P}_q^\epsilon[A] - \hat{\mathbb{P}}_q^\epsilon[A]| =& |\int_A [\exp\{\int_0^T M^\epsilon(\hat{q}_s^\epsilon)dW_s - \frac{1}{2}\int_0^T |M^\epsilon(\hat{q}_s^\epsilon)|^2 ds\} - 1] d\hat{\mathbb{P}}_q^\epsilon[\omega]| \\ \leq& (\hat{\mathbb{P}}_q^\epsilon[A])^{1/2} \mathbb{E}_q^\epsilon[[\int_0^T M^\epsilon(\hat{q}_s^\epsilon)dW_s - \frac{1}{2}\int_0^T |M^\epsilon(\hat{q}_s^\epsilon)|^2 ds - 1]^2]^{1/2}.\end{aligned}$$

For simplicity, we denote

$$Z_t^\epsilon = \int_0^t M^\epsilon(\hat{q}_s^\epsilon)dW_s - \frac{1}{2}\int_0^t |M^\epsilon(\hat{q}_s^\epsilon)|^2 ds.$$

Apply Ito's formula to $[\exp(Z_t) - 1]^2$:

$$\begin{aligned}&\mathbb{E}[\exp(Z_t^\epsilon) - 1]^2 \\ =& \mathbb{E}[0 + \int_0^t 2[\exp(Z_s^\epsilon) - 1]\exp(Z_s^\epsilon) dZ_s^\epsilon \\ &+ \frac{1}{2}\int_0^t [2\exp(Z_s^\epsilon)^2 + 2\exp(Z_s^\epsilon)(\exp(Z_s^\epsilon) - 1)] d\langle Z_s^\epsilon, Z_s^\epsilon\rangle] \\ =& \mathbb{E}[\int_0^t \exp(2Z_s^\epsilon)|M(\hat{q}_s^\epsilon)|^2 ds].\end{aligned}$$

By Hypothesis 10 and 11,

$$\sup_{\epsilon>0, x\in\mathbb{R}^2} |M^\epsilon(x)|^2 = M$$

and

$$\mathbb{E}[[\exp(Z_t^\epsilon)-1]^2] = \mathbb{E}[\int_0^t \exp(2Z_s^\epsilon)|M^\epsilon(\hat{q}_s^\epsilon)|^2 ds] \leq M\mathbb{E}[\int_0^t \exp(2Z_s^\epsilon) ds]$$

$$= M\mathbb{E}[\int_0^t \exp(2\int_0^s M^\epsilon(\hat{q}_u^\epsilon)dW_u - \int_0^s |M^\epsilon(\hat{q}_u^\epsilon)|^2 du)]ds$$

$$= M\mathbb{E}[\int_0^t \exp(\int_0^s 2M^\epsilon(\hat{q}_u^\epsilon)dW_u - \frac{1}{2}\int_0^s |2M^\epsilon(\hat{q}_u^\epsilon)|^2 du)\exp(\int_0^s |M^\epsilon(\hat{q}_u^\epsilon)|^2 du)ds].$$

Then apply Girsanov theorem again to the last equation

$$\mathbb{E}[\int_0^t \exp(\int_0^s 2M^\epsilon(\hat{q}_u^\epsilon)dW_u - \frac{1}{2}\int_0^s |2M^\epsilon(\hat{q}_u^\epsilon)|^2 du)\exp(\int_0^s |M^\epsilon(\hat{q}_u^\epsilon)|^2 du)ds]$$

$$= \int_0^t \tilde{\mathbb{E}}[\exp\int_0^s |M^\epsilon(\hat{q}_u^\epsilon)|^2 du]ds$$

$$\leq C(e^{MT}-1).$$

This implies

$$\hat{\mathbb{E}}_q^\epsilon[[\int_0^T M^\epsilon(\hat{q}_s^\epsilon)dW_s - \frac{1}{2}\int_0^T |M^\epsilon(\hat{q}_s^\epsilon)|^2 ds - 1]^2] \leq M_0 T.$$

Then let $T = \delta$ and for $q \in \bar{D}_k(\pm\delta'_g)$ as in Lemma 2.3.4,

$$
\begin{aligned}
&|\mathbb{P}^\epsilon_q[q^\epsilon_{\tau^\epsilon_k(\pm\delta)} \in C_{ki}(\delta)] - p_{ki}| \\
\leq &|\mathbb{P}^\epsilon_q[q^\epsilon_{\tau^\epsilon_k(\pm\delta)} \in C_{ki}(\delta)] - \hat{\mathbb{P}}^\epsilon_q[q^\epsilon_{\tau^\epsilon_k(\pm\delta)} \in C_{ki}(\delta)]| + |\hat{\mathbb{P}}^\epsilon_q[q^\epsilon_{\tau^\epsilon_k(\pm\delta)} \in C_{ki}(\delta)] - p_{ki}| \\
\leq &|\mathbb{P}^\epsilon_q[q^\epsilon_{\tau^\epsilon_k(\pm\delta)} \in C_{ki}(\delta); \hat{\tau}^\epsilon_k(\pm\delta) < \delta] - \hat{\mathbb{P}}^\epsilon_q[\hat{q}^\epsilon_{\tau^\epsilon_k(\pm\delta)} \in C_{ki}(\delta); \hat{\tau}^\epsilon_k(\pm\delta) < \delta] \\
&+ 2\mathbb{P}^\epsilon_q[\hat{\tau}^\epsilon_k(\pm\delta) > \delta] + \kappa \\
\leq &\hat{\mathbb{P}}^\epsilon_q[\hat{q}^\epsilon_{\tau^\epsilon_k(\pm\delta)} \in C_{ki}(\delta)]^{1/2} \cdot M\delta^{1/2} + 2\mathbb{P}^\epsilon_q[\hat{\tau}^\epsilon_k(\pm\delta) > \delta] + \kappa,
\end{aligned}
$$

and thanks to Lemma 2.3.3, we have

$$\mathbb{E}^\epsilon_q[\hat{\tau}^\epsilon_k(\pm\delta)] \leq M\delta^2 \ln(\delta).$$

Therefore, for sufficiently small δ,

$$|\mathbb{P}^\epsilon_q[q^\epsilon_{\tau^\epsilon(\pm\delta)} \in C_{ki}(\delta)] - p_{ki}| \leq M\delta^{1/2} + M\delta\ln(\delta) + \kappa < 3\kappa.$$

Therefore, we proved the general case.

Chapter 3: Averaging principle for slow-fast systems of stochastic PDEs with rough coefficients

3.1 Introduction

We consider the following system of stochastic PDEs on a separable Hilbert space H, endowed with the scalar product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$,

$$\begin{cases} du_\epsilon(t) = [A_1 u_\epsilon(t) dt + F_1(t, u_\epsilon(t), v_\epsilon(t))]\ dt + Q_1\, dW_t^1, \qquad u_\epsilon(0) = u_0, \\ dv_\epsilon(t) = \dfrac{1}{\epsilon}\left[A_2 v_\epsilon(t) + F_2(u_\epsilon(t), v_\epsilon(t))\right]\ dt + \dfrac{1}{\sqrt{\epsilon}}\, Q_2\, dW_t^2, \qquad v_\epsilon(0) = v_0. \end{cases} \tag{3.1}$$

The stochastic perturbations are given by two independent cylindrical Wiener processes in H, which are white in time and colored in space, with covariance operators Q_1^2 and Q_2^2, respectively. The operators A_1 and A_2 are the generators of two analytic semigroups $S_1(t)$ and $S_2(t)$, respectively.

The non-linearity F_2 in the fast equation is Lipschitz continuous, but the non-linearity F_1 in the slow equation is not assumed to be continuous nor bounded or linearly bounded. More precisely, we assume that $F_1 : [0,T] \times D(F_1) \subseteq [0,T] \times H \times H \to H$ is measurable and there exists a family of bounded and measurable mappings F_1^θ, depending on a parameter $\theta \in (0,1)$ and defined on $[0,T] \times H \times H$ with values in H, such that

C1. for all $t \in [0,T]$ and $h \in H$ the mapping

$$(x,y) \in H \times H \mapsto \langle F_1^\theta(t,x,y), h\rangle \in \mathbb{R},$$

is continuous;

C2. there exists a convex and lower semicontinuous mapping $V : H \times H \to [1,\infty]$ such that

$$\|F_1^\theta(t,x,y)\|^2 \le \|F_1(t,x,y)\|^2 \le V(x,y), \qquad (x,y) \in H \times H,$$

and

$$\|F_1(t,x,y) - F_1^\theta(t,x,y)\| \le \theta\, V(x,y), \qquad (x,y) \in D(F_1)$$

for all $t \in [0,T]$ and $\theta \in (0,1)$.

We will show that these conditions are satisfied for example when $H = L^2(\mathcal{O})$, for some bounded and smooth domain $\mathcal{O} \subset \mathbb{R}^d$, and

$$F_1(t,x,y)(\xi) = b(t,\xi,x(\xi),y(\xi)), \qquad t \in [0,T], \quad \xi \in \mathcal{O},$$

for some function $b : [0,T] \times \mathcal{O} \times \mathbb{R}^2 \to \mathbb{R}$ which is just continuous and such that $b(t,\xi,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ has polynomial growth and the following one-sided growth condition holds

$$\sup_{t \in [0,T]} b(t,\xi,\sigma+\rho,\lambda)\sigma \leq c\left(\alpha(\xi) + |\sigma|^2 + |\lambda|^{\kappa_1} + |\rho|^{\kappa_2}\right),$$

for all $(t,\xi) \in [0,T] \times \mathcal{O}$ and $(\lambda,\sigma,\rho) \in \mathbb{R}^3$ and for some positive constants c, κ_1, κ_2 and some function $\alpha \in L^1(\mathcal{O})$. In this regard, it is worth noting that in [32], systems of reaction-diffusion equations with coefficients having polynomial growth were considered. However, in that case, the coefficients were assumed to be regular, stronger dissipativity conditions were imposed, and the coefficient of the slow equation had polynomial growth only in the slow variable, while the coefficient in the fast equation could have polynomial growth only in the fast variable. In contrast, in the example we consider in

this paper, the reaction term in the slow equation is only continuous and has polynomial growth, both in the slow and in the fast variables.

Conditions C1. and C2. are not new in the current literature. They have been introduced in several papers (see e.g. [27], [28] and [29]) in order to study the existence of measure-valued solutions to the Fokker Planck equations associated with a wide class of stochastic differential equations both in finite and in infinite dimensional spaces. Here, we are proving how the same kind of assumptions imply the existence of $C([0,T];H)$-valued martingale solutions for the following class of stochastic evolution equations in arbitrary separable Hilbert spaces

$$dX(t) = [AX(t) + F(t, X(t))]\ dt + Q\, dW_t, \qquad X(0) = x.$$

This result applies also to systems like (3.1), once we rewrite them as

$$dX^\epsilon(t) = [A^\epsilon X^\epsilon(t) + F^\epsilon(t, X^\epsilon(t))]\ dt + Q^\epsilon\, dW_t, \qquad X^\epsilon(0) = (u_0, v_0),$$

where

$$A_\epsilon := (A_1, \epsilon^{-1}A_2), \quad F^\epsilon := (F_1, \epsilon^{-1}F_2), \quad Q^\epsilon := (Q_1, \epsilon^{-1/2}Q_2),$$

and where $W_t := (W_t^1, W_t^2)$. This means that for every $(u_0, v_0) \in D_V := \{V < \infty\}$ and $\epsilon \in (0,1)$ there exists a martingale solution. That is there exists a stochastic basis, $(\Omega^\epsilon, \mathcal{F}^\epsilon, \{\mathcal{F}_t^\epsilon\}_{t\geq 0}, \mathbb{P}^\epsilon)$, an adapted cylindrical Wiener process $W_t^\epsilon = (W_t^{1,\epsilon}, W_t^{2,\epsilon})$ in $H \times H$, and an adapted process $X^\epsilon = (u_\epsilon, v_\epsilon) \in L^2(\Omega^\epsilon; C([0,T]; D(F_\epsilon))$ such that

$$X^\epsilon(t) = S^\epsilon(t)(u_0, v_0) + \int_0^t S^\epsilon(t-s)F^\epsilon(s, X^\epsilon(s))ds + \int_0^t S^\epsilon(t-s)Q^\epsilon dW_s^\epsilon, \quad t \geq 0,$$

where $S^\epsilon(t)$ is the semigroup generated by A^ϵ.

Next, we assume that there exists some constant $c > 0$ such that for every initial condition $(u_0, v_0) \in D_V$

$$\sup_{\epsilon \in (0,1)} \int_0^T \mathbb{E}^\epsilon V(X^\epsilon(t))\, dt \leq c\, V(u_0, v_0), \tag{3.2}$$

and we prove that under this condition the family $\{\mathcal{L}(u_\epsilon)\}_{\epsilon \in (0,1)}$ is tight in $C([0,T]; H)$. Our purpose is showing that any weak limit point μ_{slow} of $\{\mathcal{L}(u_\epsilon)\}_{\epsilon \in (0,1)}$ in $C([0,T]; H)$ is a martingale solution of a suitable limiting stochastic equation in H whose non-linearity is obtained by averaging F_1 with respect to the invariant measure of the fast equation.

We would like to stress that the roughness of F_1 makes the definition of

the averaged coefficient $\bar{F}_1$ quite delicate and requires some approximation procedures. Moreover, $\bar{F}_1$ inherits from F_1 the same roughness and the proof of the averaging limit requires several new ideas compared to what is available in the existing literature on the averaging for slow-fast systems of SPDEs with regular coefficients.

Before concluding this introduction, we would like to describe the plan of the paper. After introducing in Section 2 all the notations and assumptions that will used throughout the paper, in Section 3, we describe the ergodic properties of the fast equation, define the average of the nonlinearity F_1 in terms of the invariant measure of the fast equation, introduce the corresponding averaged limiting equation, and state the main theorem of the paper. In Section 4, we prove several a-priori bounds for the solution of system (3.1) and we give a proof of the tightness of the laws of the solutions of the slow equation. In Section 5, we prove the validity of the averaging limit, by using a localization- in-time argument. In Section 6, we provide an example of a class of systems of stochastic reaction-diffusion equations in bounded domains of $\mathbb{R}^d$, where the slow equation has a reaction coefficient that is only continuous and has polynomial growth, both in the slow and in the fast variable. Finally, in Appendix A, we give a proof of the existence

of a $C([0,T];H)$-valued martingale solution for a general class of stochastic evolution equations with rough coefficients, that cover also the system (3.1).

3.2 Notations and assumptions

Let H be a separable Hilbert space, endowed with the scalar product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. We denote by $B_b(H)$ the Banach space of Borel bounded functions $\varphi : H \to \mathbb{R}$, endowed with the norm

$$\|\varphi\|_0 := \sup_{x \in H} |\varphi(x)|.$$

We denote by $C_b(H)$ the subspace of continuous functions. Moreover, we denote by $\mathrm{Lip}_b(H)$ the subspace of all Lipschitz continuous functions. $\mathrm{Lip}_b(H)$ is a Banach space, endowed with the norm

$$\|\varphi\|_{\mathrm{Lip}} := \|\varphi\|_0 + [\varphi]_{\mathrm{Lip}} := \|\varphi\|_0 + \sup_{\substack{x,y \in H \\ x \neq y}} \frac{|\varphi(x) - \varphi(y)|}{\|x - y\|}.$$

In the present paper, we are dealing with the following system of stochas-

tic equations in the space H

$$\begin{cases} du_\epsilon(t) = [A_1 u_\epsilon(t)dt + F_1(t, u_\epsilon(t), v_\epsilon(t))]\, dt + Q_1\, dW_t^1, \qquad u_\epsilon(0) = u_0, \\ \\ dv_\epsilon(t) = \dfrac{1}{\epsilon}\,[A_2 v_\epsilon(t) + F_2(u_\epsilon(t), v_\epsilon(t))]\, dt + \dfrac{1}{\sqrt{\epsilon}}\, Q_2\, dW_t^2, \qquad v_\epsilon(0) = v_0. \end{cases} \tag{3.3}$$

Throughout the present section, we will introduce the assumptions we make on the coefficients and the noise and we will introduce some notations.

Hypothesis 8. *1. The operators $(A_1, D(A_1))$ and $(A_2, D(A_2))$ generate the analytic semigroups $S_1(t)$ and $S_2(t)$ in H, respectively. Moreover, there exist two complete orthonormal systems, $\{e_{1,k}\}_{k\in\mathbb{N}}$ and $\{e_{2,k}\}_{k\in\mathbb{N}}$ in H, and two non-decreasing sequences of positive real numbers $\{\alpha_{1,k}\}_{k\in\mathbb{N}}$ and $\{\alpha_{2,k}\}_{k\in\mathbb{N}}$, such that*

$$A_1 e_{1,k} = -\alpha_{1,k} e_{1,k}, \qquad A_2 e_{2,k} = -\alpha_{2,k} e_{2,k}, \qquad k \in \mathbb{N}.$$

2. The bounded linear operators Q_1 and Q_2 share the same set of eigenvectors with the operators A_1 and A_2, respectively. Namely, there exist

two sequences of real numbers $\{\lambda_{1,k}\}_{k\in\mathbb{N}}$ *and* $\{\lambda_{2,k}\}_{k\in\mathbb{N}}$ *such that*

$$Q_1 e_{1,k} = \lambda_{1,k} e_{1,k}, \qquad Q_2 e_{2,k} = \lambda_{2,k} e_{2,k}, \qquad k \in \mathbb{N}.$$

3. *There exist two strictly positive constants* γ_1 *and* γ_2 *such that*

$$\sum_{k\in\mathbb{N}} \lambda_{1,k}^2 \alpha_{1,k}^{2\gamma_1 - 1} < \infty, \qquad \sum_{k\in\mathbb{N}} \lambda_{2,k}^2 \alpha_{2,k}^{2\gamma_2 - 1} < \infty. \tag{3.4}$$

Remark 3.2.1. 1. In view of Condition 1. in Hypothesis 8, we have

$$\inf_{k\in\mathbb{N}} \alpha_{1,\,k} = \alpha_{1,1} > 0, \qquad \inf_{k\in\mathbb{N}} \alpha_{2,\,k} = \alpha_{2,1} > 0. \tag{3.5}$$

2. Thanks to (3.5), for every $\delta \geq 0$ we can define the norms

$$\|x\|_{D((-A_i)^\delta)} := \|(-A_i)^\delta x\|, \qquad i = 1, 2.$$

Notice that for every $\delta > 0$ the space $D((-A_i)^\delta)$, endowed with the norm above, is compactly embedded in H, for $i = 1, 2$.

In what follows, we will denote by $\mathcal{H}_2$ the product space $H \times H$, endowed

the inner product and the corresponding norm

$$\langle (x_1, y_1), (x_2, y_2) \rangle_{\mathcal{H}_2} := \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle, \qquad \|(x, y)\|_{\mathcal{H}_2} := \sqrt{\|x\|^2 + \|y\|^2}.$$

Clearly, the family of vectors

$$\Lambda := \{(e_{1,j}, 0)\}_{j \in \mathbb{N}} \cup \{(0, e_{2,j})\}_{j \in \mathbb{N}}$$

is an orthonormal basis for the Hilbert space $\mathcal{H}_2$.

For every $\epsilon > 0$, we define the operators A^ϵ, Q^ϵ by setting

$$A^\epsilon(x, y) = (A_1 x, \epsilon^{-1} A_2 y), \qquad (x, y) \in D(A_1) \times D(A_2) \subset \mathcal{H}_2,$$

and

$$Q^\epsilon(x, y) = (Q_1 x, \epsilon^{-1/2} Q_2 y), \qquad (x, y) \in \mathcal{H}_2.$$

It is immediate to check that Λ is a set of eigenvectors which diagonalizes the operators A^ϵ and Q^ϵ simultaneously and $\{\alpha_{i,k}\}_{i=1,2,\ k \in \mathbb{N}}$ are eignevalues

of A^ϵ, while $\{\lambda_{i,k}\}_{i=1,2,\ k\in\mathbb{N}}$ are eigenvalues of Q^ϵ. Due to (3.4)

$$\sum_{k\in\mathbb{N}} \lambda_{1,k}^2 \alpha_{1,k}^{2\gamma_1-1} + \lambda_{2,k}^2 \alpha_{2,k}^{2\gamma_2-1} < \infty.$$

Moreover, if $S^\epsilon(t)$ is the analytic semigroup generated by A^ϵ on the space $\mathcal{H}_2$, we have

$$S^\epsilon(t)(x,y) = (S_1(t)\,x, S_2(t/\epsilon)\,y), \qquad (x,y) \in \mathcal{H}_2.$$

Concerning the nonlinearity F_1, we shall assume the following conditions.

Hypothesis 9. *The mapping $F_1 : [0,T] \times D(F_1) \subseteq [0,T] \times \mathcal{H}_2 \to H$ is measurable. Moreover, there exists a family of mappings $\{F_1^\theta\}_{\theta\in(0,1)}$, defined on $[0,T] \times \mathcal{H}_2$ with values in H, such that the following conditions hold.*

1. *For every $\theta \in (0,1)$, the mapping $F_1^\theta : [0,T] \times \mathcal{H}_2 \to H$ is measurable and bounded. Moreover, for all $t \in [0,T]$ and $h \in H$ the mapping*

$$(x,y) \in \mathcal{H}_2 \mapsto \langle F_1^\theta(t,x,y), h\rangle \in \mathbb{R},$$

is continuous.

2. *There exists a convex and lower semicontinuous mapping* $V : \mathcal{H}_2 \to [1, \infty]$, *with* $D_V := \{V < \infty\} \subseteq D(F_1)$, *such that for all* $\theta \in (0, 1)$ *and* $t \in [0, T]$

$$\|F_1^\theta(t, x, y)\|^2 \leq \|F_1(t, x, y)\|^2 \leq V(x, y), \qquad (x, y) \in \mathcal{H}_2, \tag{3.6}$$

and

$$\|F_1(t, x, y) - F_1^\theta(t, x, y)\| \leq \theta\, V(x, y), \qquad (x, y) \in D(F_1). \tag{3.7}$$

Remark 3.2.2. 1. When $(x, y) \notin D(F_1)$, we take $\|F_1(t, x, y)\| = +\infty$. This allows to interpret (3.6), for every $(x, y) \in \mathcal{H}_2$.

2. In Hypothesis 8, we have assumed that $\alpha_{1,1} > 0$. However, this is not necessary. Actually if we define $\hat{A}_1 := A_1 - \delta I$, and $\hat{F}_1(t, \cdot) = F_1(t, \cdot) + \delta I$, for some $\delta > -\alpha_{1,1}$, we have that

$$\hat{A}_1 x + \hat{F}_1(t, x, y) = A_1 x + F_1(t, x, y), \quad x \in D(A),\ (x, y) \in D(F_1),\ \ t \in [0, T], \tag{3.8}$$

and $\hat{A}_1$ is a negative operator. Now, for every $\theta \in (0,1)$ we define

$$\hat{F}_1^\theta(t,x,y) := F_1^\theta(t,x,y) + \frac{\delta\, x}{1+\theta\ \|x\|}, \qquad (t,x,y) \in [0,T] \times \mathcal{H}_2,$$

we have that $\hat{F}_1^\theta$ is measurable and bounded and for every $h \in H$ and $t \in [0,T]$ the mapping $(x,y) \in \mathcal{H}_2 \mapsto \langle F_1^\theta(t,x,y), h\rangle \in \mathbb{R}$ is continuous. Moreover,

$$\|\hat{F}_1^\theta(t,x,y)\|^2 \leq 2\,\|F_1^\theta(t,x,y)\|^2 + \delta^2\|x\|^2.$$

Therefore, if we define

$$\hat{V}(x,y) = c\,(V(x,y) + \|(x,y)\|^2_{\mathcal{H}_2}), \tag{3.9}$$

due to (4.4) there exists some sufficiently large constant $c > 0$ such that

$$\|\hat{F}_1^\theta(t,x,y)\|^2 \leq \hat{V}(x,y), \qquad (t,x,y) \in [0,T] \times \mathcal{H}_2.$$

Moreover,

$$\begin{aligned}\|\hat{F}_1(t,x,y)-\hat{F}_1^\theta(t,x,y)\| &\le \|F_1(t,x,y)-F_1^\theta(t,x,y)\|+\delta\,\|x-x/(1+\theta\|x\|)\|\\ &\le \theta V(x,y)+\theta\frac{\delta\,\|x\|}{1+\theta\|x\|}\le\theta\left(V(x,y)+\delta\|x\|\right)\le\theta\hat{V}(x,y),\end{aligned}$$

if c is sufficiently large. As for Hypothesis 15, thanks to estimate (3.46) below and (4.10) we get

$$\int_0^T \mathbb{E}^n\hat{V}(X_n(t))\,dt\le c\int_0^T\left(\mathbb{E}^nV(X_n(t))+\mathbb{E}\|X_n(t)\|^2\right)\,dt\le c\left(V(x,y)+\|(x,y)\|^2_{\mathcal{H}_2}\right). \tag{3.10}$$

So we can recover (4.10) for $\hat{V}$, once we take c large enough.

For the nonlinear coefficient F_2, we make the following assumptions.

Hypothesis 10. *The mapping $F_2:\mathcal{H}_2\to H$ is Lipschitz continuous. Moreover, if we define*

$$L_2:=\sup_{\substack{x\in H\\ y_1,\,y_2\in H}}\frac{\|F_2(x,y_1)-F_2(x,y_2)\|}{\|y_1-y_2\|},$$

we have

$$\omega:=\alpha_{2,1}-L_2>0. \tag{3.11}$$

Next, for every $\epsilon > 0$, we define the nonlinear mapping $F^\epsilon : [0,T] \times D(F^\epsilon) \to \mathcal{H}_2$, by setting

$$F^\epsilon(t,x,y) := \left(F_1(t,x,y), \epsilon^{-1}F_2(x,y)\right), \qquad t \in [0,T], \quad (x,y) \in D(F^\epsilon),$$

with $D(F^\epsilon) := D(F_1) \subseteq \mathcal{H}_2$. By using the notations we have introduced above, we can rewrite system (3.3) as the following stochastic evolution equation on the Hilbert space $\mathcal{H}_2$

$$dX(t) = [A^\epsilon X(t) + F^\epsilon(t, X(t))]\, dt + Q^\epsilon dW_t, \qquad X_\epsilon(0) = (u_0, v_0), \qquad (3.12)$$

where $W_t = (W_t^1, W_t^2)$ is a cylindrical Wiener process in $\mathcal{H}_2$.

In what follows, we shall make the following fundamental assumption.

Hypothesis 11. *For every $\epsilon \in (0,1)$ and every initial condition $(u_0, v_0) \in D_V$, there exists a martingale solution for (3.12). This means that once fixed $(u_0, v_0) \in D_V$, for every $\epsilon \in (0,1)$ there exist a stochastic base, $(\Omega^\epsilon, \mathcal{F}^\epsilon, \{\mathcal{F}_t^\epsilon\}_{t\geq 0}, \mathbb{P}^\epsilon)$, an adapted cylindrical Wiener process $W_t^\epsilon = (W_t^{1,\epsilon}, W_t^{2,\epsilon})$ in $\mathcal{H}_2$, and an*

adapted process $X^\epsilon \in L^2(\Omega^\epsilon; C([0,T]; D(F^\epsilon))$ *such that*

$$X^\epsilon(t) = S^\epsilon(t)(u_0, v_0) + \int_0^t S^\epsilon(t-s)F^\epsilon(s, X^\epsilon(s))ds + W^\epsilon_{A^\epsilon}(t), \qquad t \in [0,T],$$

where $W^\epsilon_{A^\epsilon}(t)$ *is the stochastic convolution*

$$W^\epsilon_{A^\epsilon}(t) := \int_0^t S^\epsilon(t-s)Q^\epsilon dW^\epsilon_s, \qquad t \geq 0.$$

Moreover, there exists a constant $M > 0$*, independent of* $(u_0, v_0) \in D_V$*, such that*

$$\sup_{\epsilon \in (0,1)} \int_0^T \mathbb{E}^\epsilon V(X^\epsilon(t))\, dt \leq M\, V(u_0, v_0). \tag{3.13}$$

Remark 3.2.3. In fact, the existence of a martingale solution for the slow-fast system (3.12), for every initial condition $(u_0, v_0) \in D_V$ and every parameter $\epsilon \in (0,1)$ is a consequence of Theorem

In what follows, we shall denote

$$W^\epsilon_{A_1}(t) = \int_0^t S_1(t-s)Q_1 dW^{1,\epsilon}_s, \qquad W^\epsilon_{A_2}(t) = \frac{1}{\sqrt{\epsilon}} \int_0^t S_2\left((t-s)/\epsilon\right) Q_2 dW^{2,\epsilon}_s.$$

With these notations, the slow and fast components of $X^\epsilon(t) = (u^\epsilon(t), v^\epsilon(t))$,

solve the equations

$$u^\epsilon(t) = S_1(t)u_0 + \int_0^t S_1(t-s)F_1(s, X^\epsilon(s))ds + W^\epsilon_{A_1}(t), \tag{3.14}$$

and

$$v^\epsilon(t) = S_2\left(t/\epsilon\right) v_0 + \frac{1}{\epsilon}\int_0^t S_2\left(\left(t-s\right)/\epsilon\right) F_2(X^\epsilon(s))ds + W^\epsilon_{A_2}(t). \tag{3.15}$$

3.3 Preliminaries and statement of the main result

In this section, we will first review some properties of the asymptotic behavior of the fast motion with frozen slow component. Then we will introduce the coefficients of the candidate averaged equation and finally we will state the main result of the present paper.

3.3.1 The fast motion equation with frozen slow variable

We recall here some known facts about the ergodic properties of the fast motion equation with frozen slow variable. For all details we refer e.g. to [33].

For any fixed $x, y \in H$, we consider the following equation

$$\begin{cases} dv(t) = [A_2 v(t) + F_2(x, v(t))]\, dt + Q_2 dW_t, \\ \\ v(0) = y \in H, \end{cases} \tag{3.16}$$

where W_t is a cylindrical Wiener process on some stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$. Under Hypotheses 8 and 10, equation (3.16) admits a unique mild solution $v^{x,y}$ belonging to $L^2(\Omega; C([0,T]; H))$ such that for every $p \geq 1$

$$\mathbb{E} \sup_{t\in[0,T]} \|v^{x,y}(t)\|^p < \infty.$$

This allows us to introduce the transition semigroup P_t^x associated with equation (3.16), which is defined by

$$P_t^x \varphi(y) := \mathbb{E}\, \varphi(v^{x,y}(t)), \quad t \geq 0, \quad y \in H,$$

for any $\varphi \in B_b(H)$.

It is possible to show that for every $0 < \gamma < \gamma_2^\star$

$$\sup_{t\geq 0} \mathbb{E}\, \|v^{x,0}(t)\|_{D((-A_2)^\gamma)} \leq c\,(1 + \|x\|), \tag{3.17}$$

(see also the proof of Lemma 3.4.4). Hence, thanks to the Krylov-Bogoliubov theorem, for every $x \in H$ the semigroup P_t^x admits an invariant measure μ^x. Moreover, by using again arguments analogous to those we will use in the proof of Lemma 3.4.4, we can show that for every $p \geq 1$,

$$\mathbb{E}\,\|v^{x,y}(t)\|^p \leq c_p\,\left(1 + \|x\|^p + e^{-\omega p t}\|y\|^p\right), \qquad t \geq 0. \tag{3.18}$$

In particular, this implies that for every $p \geq 1$

$$\int_H \|y\|^p \mu^x(dy) \leq c_p\,(1 + \|x\|^p). \tag{3.19}$$

Now, we fix $x_1, x_2, y \in H$ and we define $\Gamma_i(t) := v^{x_i,y}(t) - W_{A_2}(t)$, for $i = 1, 2$, where

$$W_{A_2}(t) := \int_0^t S_2(t-s)Q_2 dW_s.$$

We have $\Gamma(t) := v^{x_1,y}(t) - v^{x_2,y}(t) = \Gamma_1(t) - \Gamma_2(t)$, and Γ satisfies the following equation

$$\frac{d\Gamma(t)}{dt} = A_2\Gamma(t) + F_2(x_1, v^{x_1,y}(t)) - F_2(x_2, v^{x_2,y}(t)), \qquad \Gamma(0) = 0.$$

Thanks to Hypothesis 10, we have

$$\frac{1}{2}\frac{d}{dt}\|\Gamma(t)\|^2 \leq -\alpha_{2,0}\|\Gamma(t)\|^2 + \left([F_2]_{\mathrm{Lip}}\|x_1 - x_2\| + L_2\,\|\Gamma(t)\|\right)\|\Gamma(t)\|,$$

so that

$$\frac{d}{dt}\|\Gamma(t)\|^2 \leq -\omega\|\Gamma(t)\|^2 + c\,\|x_1 - x_2\|^2.$$

By comparison, this gives

$$\|\Gamma(t)\|^2 \leq c\int_0^t e^{-\omega(t-s)}\|x_1 - x_2\|^2 ds \leq c\,\|x_1 - x_2\|^2,$$

which implies

$$\mathbb{E}\sup_{t>0}\|v^{x_1,y}(t) - v^{x_2,y}(t)\|^2 \leq c\,\|x_1 - x_2\|^2. \tag{3.20}$$

In a similar way, it is possible to prove that for every $x, y_1, y_2 \in H$

$$\sup_{x\in H}\mathbb{E}\|v^{x,y_1}(t) - v^{x,y_2}(t)\|^2 \leq c\,e^{-\omega t}\|y_1 - y_2\|^2, \qquad t \geq 0,$$

for some constant c independent of t and $y_1, y_2 \in H$, so that for every

$\varphi \in \mathrm{Lip}_b(H)$ we have

$$|P_t^x \varphi(y_1) - P_t^x \varphi(y_2)| \leq c\,[\varphi]_{\mathrm{Lip}} e^{-\frac{\omega}{2}t} \|y_1 - y_2\|, \qquad t \geq 0. \tag{3.21}$$

This implies that μ^x is the unique invariant measure for P_t^x and for every $x, y \in H$ and $\varphi \in \mathrm{Lip}_b(H)$

$$\left| P_t^x \varphi(y) - \int_H \varphi(z)\,\mu^x(dz) \right| = \left| \int_H \left(P_t^x \varphi(y) - P_t^x \varphi(z)\right)\,\mu^x(dz) \right|$$
$$\leq c\,[\varphi]_{\mathrm{Lip}} e^{-\frac{\omega}{2}t} \int_H \|y - z\|\,\mu^x(dz) \leq c\,[\varphi]_{\mathrm{Lip}} e^{-\frac{\omega}{2}t} \left(\|y\| + \int_H \|z\|\,\mu^x(dz) \right).$$

Therefore, thanks to (3.19), we obtain

$$\left| P_t^x \varphi(y) - \int_H \varphi(z)\,\mu^x(dz) \right| \leq c\,[\varphi]_{\mathrm{Lip}} e^{-\frac{\omega}{2}t} \left(\|y\| + \|x\| + 1 \right). \tag{3.22}$$

3.3.2 The averaged nonlinear coefficient and its approximation

In what follows, in addition to Hypotheses 8 to 11, we shall assume the following condition.

Hypothesis 12. *If $\mu^x(dy)$ is the invariant measure of the fast motion with*

frozen slow component x introduced in Section 3.3.1, then

$$\bar{V}(x) := \int_H V(x,y)\,\mu^x(dy) < \infty, \qquad x \in \Pi_1 D_V, \tag{3.23}$$

where

$$\Pi_1 D_V := \{\, x \in H \,:\, (x,y) \in D_V, \text{ for some } y \in H \,\}.$$

In particular, Hypothesis 12 implies that for every $x \in \Pi_1 D_V$ the support of the invariant measure μ^x is contained in

$$\Pi_2 D_V(x) := \{\, y \in H \,:\, (x,y) \in D_V \,\}.$$

Now, we define the averaged coefficient $\bar{F}_1$ as

$$\bar{F}_1(t,x) := \int_H F_1(t,x,y)\mu^x(dy), \qquad (t,x) \in [0,T] \times \Pi_1 D(F_1),$$

where

$$\Pi_1 D(F_1) := \{\, x \in H \,:\, (x,y) \in D(F_1) \text{ for some } y \in H \,\}.$$

Moreover, for any $\theta \in (0,1)$, we define the approximating averaged coefficient

$\bar{F}_1^\theta$ as

$$\bar{F}_1^\theta(t,x) := \int_H F_1^\theta(t,x,y)\mu^x(dy), \qquad (t,x) \in [0,T] \times H.$$

Next, for every $n \in \mathbb{N}$ and $(t,x,y) \in [0,T] \times \mathcal{H}_2$ we define

$$F_{1,n}^\theta(t,x,y) := \int_{\mathbb{R}^n} \rho_n(\xi - \Pi_n(x,y)) F_1^\theta(t, R_n\,\xi)\, d\xi,$$

where $\{\rho_n\}_{n\in\mathbb{R}^n}$ is a sequence of non-negative smooth functions such that

$$\operatorname{supp}\rho_n \subset \{\, \xi \in \mathbb{R}^x \,:\, |\xi| \le 1/n \,\}, \qquad \int_{\mathbb{R}^n} \rho_n(\xi)\, d\xi = 1,$$

and the mappings $\Pi_n : \mathcal{H}_2 \to \mathbb{R}^n$ and $R_n : \mathbb{R}^n \to \mathcal{H}_2$ are defined by

$$\Pi_n(x,y) = (\langle (x,y), f_1\rangle_{\mathcal{H}_2}, \cdots, \langle (x,y), f_n\rangle_{\mathcal{H}_2}), \qquad R_n\xi = \sum_{i=1}^n \xi_i f_i,$$

for some orthonormal basis $\{f_i\}_{i\in\mathbb{N}} \subset \mathcal{H}_2$. Clearly, $F_{1,n}^\theta : [0,T] \times \mathcal{H}_2 \to H$ is measurable and bounded, and $F_{1,n}^\theta(t,\cdot) \in C_b^\infty(\mathcal{H}_2; H)$, for every fixed $t \in [0,T]$, with

$$\sup_{t\in[0,T]} \|D_x^j F_{1,n}^\theta(t,\cdot)\|_0 =: c_{j,n,\theta} < \infty, \tag{3.24}$$

for every $j \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ and $\theta \in (0,1)$. Moreover, for every $h \in H$

$$\lim_{n\to\infty} \langle F^{\theta}_{1,n}(t,x,y), h\rangle = \langle F^{\theta}_{1}(t,x,y), h\rangle, \quad (t,x,y) \in [0,T] \times \mathcal{H}_2, \tag{3.25}$$

and

$$\sup_{t\in[0,T]} \|F^{\theta}_{1,n}\|_0 \leq \sup_{t\in[0,T]} \|F^{\theta}_{1}\|_0, \qquad n \in \mathbb{N}. \tag{3.26}$$

In what follows, we shall denote

$$\bar{F}^{\theta}_{1,n}(t,x) := \int_H F^{\theta}_{1,n}(t,x,y)\, \mu^x(dy), \qquad (t,x) \in [0,T] \times H. \tag{3.27}$$

Lemma 3.3.1. *Under Hypotheses 8, 9 and 10, the mapping $\bar{F}^{\theta}_{1,n} : [0,T] \times H \to H$ is measurable, for every $\theta \in (0,1)$ and $n \in \mathbb{N}$. Moreover, $\bar{F}^{\theta}_{1,n}(t,\cdot) : H \to H$ is Lipschitz-continuous, uniformly with respect to $t \in [0,T]$, and*

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \|\bar{F}^{\theta}_{1,n}(t,\cdot)\|_0 < \infty. \tag{3.28}$$

Proof. According to (3.24), the function $F^{\theta}_{1,n}(t,\cdot) : \mathcal{H}_2 \to H$ is Lipschitz-continuous, uniformly with respect to $t \in [0,T]$. In particular, for every

$h \in H$ the mapping

$$(x, y) \in \mathcal{H}_2 \mapsto \langle F^{\theta}_{1,n}(t, x, y), h\rangle \in \mathbb{R},$$

is Lipschitz continuous, uniformly with respect to $t \in [0, T]$. Hence, since

$$\langle \bar{F}^{\theta}_{1,n}(t, x), h\rangle = \int_H \langle F^{\theta}_{1,n}(t, x, y), h\rangle \, \mu^x(dy),$$

according to (3.22), we have

$$\begin{aligned}
&\langle \bar{F}^{\theta}_{1,n}(t, x_1) - \bar{F}^{\theta}_{1,n}(t, x_2), h\rangle \\
&= \lim_{T\to\infty} \left(\mathbb{E}\, \langle F^{\theta}_{1,n}(t, x_1, v^{x_1,0}(T)), h\rangle - \mathbb{E}\, \langle F^{\theta}_{1,n}(t, x_2, v^{x_2,0}(T)), h\rangle \right).
\end{aligned}$$

Now, in view of (3.20) we have

$$\begin{aligned}
&\left| \mathbb{E}\, \langle F^{\theta}_{1,n}(t, x_1, v^{x_1,0}(T)), h\rangle - \mathbb{E}\, \langle F^{\theta}_{1,n}(t, x_2, v^{x_2,0}(T)), h\rangle \right| \\
&\leq [F^{\theta}_{1,n}(t, \cdot)]_{\text{Lip}} \left(\|x_1 - x_2\| + \mathbb{E}\, \|v^{x_1,0}(T) - v^{x_2,0}(T)\| \right) \|h\| \\
&\leq c\, [F^{\theta}_{1,n}(t, \cdot)]_{\text{Lip}} \|x_1 - x_2\| \, \|h\|,
\end{aligned}$$

and, due to the arbitrariness of $h \in H$, this implies

$$\|\bar{F}^{\theta}_{1,n}(t,x_1) - \bar{F}^{\theta}_{1,n}(t,x_2)\| \leq c\,[F^{\theta}_{1,n}(t,\cdot)]_{\text{Lip}}\,\|x_1 - x_2\|. \tag{3.29}$$

This means that the mapping $x \in H \mapsto \bar{F}^{\theta}_{1,n}(t,x) \in H$ is Lipschitz-continuous, uniformly with respect to $t \in [0,T]$.

□

Lemma 3.3.2. *Under Hypotheses 8, 9, 10 and 12, for every $\theta \in (0,1)$ the mapping $\bar{F}^{\theta}_1 : [0,T] \times H \to H$ is measurable and bounded, and for every $h \in H$ and $t \in [0,T]$ the mapping*

$$x \in \mathcal{H}_2 \mapsto \langle \bar{F}^{\theta}_1(t,x), h\rangle \in \mathbb{R}, \tag{3.30}$$

is continuous. Moreover,

$$\|\bar{F}_1(t,x) - \bar{F}^{\theta}_1(t,x)\|^2 \leq \theta\,\bar{V}(x), \qquad t \in [0,T], \quad x \in \Pi_1 D(F_1). \tag{3.31}$$

In particular, the mapping $\bar{F}_1 : [0,T] \times \Pi_1 D(F_1) \to H$ is measurable.

Proof. In view of (3.25) and (3.26), from the dominated convergence theo-

rem, for every $h \in H$ we have

$$\lim_{n\to\infty} |\langle \bar{F}^{\theta}_{1,n}(t,x), h\rangle - \langle \bar{F}^{\theta}_{1}(t,x), h\rangle| = 0, \qquad (t,x) \in [0,T]\times H.$$

Thanks to Lemma 3.3.1, all $\bar{F}^{\theta}_{1,n}$ are measurable, and since H is separable due to Petti's theorem the limit above implies that the mapping $\bar{F}^{\theta}_{1} : [0,T]\times H \to H$ is measurable, for every $\theta \in (0,1)$.

Now, thanks to condition 1. in Hypothesis 9, for every $h \in H$ and $t \in [0,T]$ the mapping

$$(x,y) \in \mathcal{H}_2 \mapsto \langle F^{\theta}_{1}(t,x,y), h\rangle \in \mathbb{R},$$

is continuous and bounded. Then, as proved in [40], there is a sequence $\{\Phi^{\theta}_{h,n}(t,\cdot)\}_{n\in\mathbb{N}}$ in $\mathrm{Lip}_b(\mathcal{H}_2)$ such that

$$\lim_{n\to\infty} \sup_{(x,y)\in\mathcal{H}_2} |\langle F^{\theta}_{1}(t,x,y), h\rangle - \Phi^{\theta}_{h,n}(t,x,y)| = 0, \qquad t \in [0,T].$$

As shown in the proof of Lemma 3.3.1, the function

$$x \in H \mapsto \bar{\Phi}^{\theta}_{h,n}(t,x) := \int_H \Phi^{\theta}_{h,n}(t,x,y)\, \mu^x(dy) \in \mathbb{R},$$

is Lipschitz continuous. Therefore, since

$$\lim_{n\to\infty}\sup_{x\in H}\left|\int_H \left(\langle F_1^\theta(t,x,y),h\rangle - \Phi_{h,n}^\theta(t,x,y)\right)\,\mu^x(dy)\right| = 0,$$

we can conclude that function (3.30) is continuous.

Finally, due to (3.7) we have

$$\begin{aligned}\|\bar F_1(t,x)-\bar F_1^\theta(t,x)\|^2 \le & \int_H \|F_1(t,x,y)-F_1^\theta(t,x,y)\|^2\,\mu^x(dy)\\ &\le \theta\int_H V(x,y)\,\mu^x(dy) = \theta\,\bar V(x), \qquad t\in[0,T], \quad x\in \Pi_1 D(F_1),\end{aligned}$$

and (3.31) follows. □

3.3.3 The main theorem

Now, we can state the main result of this paper.

Theorem 3.3.3. *Under Hypotheses 8 to 11, for every initial condition* $(u_0,v_0)\in \mathcal{H}_2$, *with* $u_0\in D((-A_1)^\beta)$, *for some* $\beta>0$, *the family* $\{\mathcal{L}(u^\epsilon)\}_{\epsilon\in(0,1)}$ *is tight in* $C([0,T];H)$.

Moreover, if Hypothesis 12 holds and

$$\sup_{\epsilon\in(0,1)}\int_0^T \mathbb{E}^\epsilon\,\bar V(u^\epsilon(t))\,dt<\infty, \tag{3.32}$$

any weak limit μ_{slow} *of* $\{\mathcal{L}(u^\epsilon)\}_{\epsilon\in(0,1)}$ *in* $C([0,T];H)$ *solves the martingale problem with data* A_1, $\bar{F}_1$, Q_1, *and* u_0, *in the Hilbert space* H. *Namely there exists a stochastic basis* $(\bar{\Omega},\bar{\mathcal{F}},\{\bar{\mathcal{F}}_t\}_{t\geq 0},\bar{\mathbb{P}})$, *a cylindrical Wiener process* $\bar{W}_t$ *and an adapted process* $\bar{u}\in L^2(\Omega;C([0,T];D(\bar{F}_1)))$ *such that* $\mathcal{L}(\bar{u})=\mu_{slow}$ *and*

$$\bar{u}(t)=S_1(t)u_0+\int_0^t S_1(t-s)\bar{F}_1(s,\bar{u}(s))ds+\int_0^t S_1(t-s)Q_1 d\bar{W}_s. \quad (3.33)$$

In the next two sections we will give a proof of Theorem 3.3.3: in Section 3.4 we will prove tightness and in Section 3.5 we will prove the validity of the averaging principle.

3.4 Tightness

We start with some a-priori bounds for the martingale solutions of equation (3.12).

3.4.1 Estimates for the stochastic convolution

As a consequence of (3.4), it is possible to prove that under Hypothesis 8 for all $p\geq 1$ and $0\leq\gamma<\gamma_1$

$$\sup_{\epsilon\in(0,1)} \mathbb{E}^{\epsilon} \sup_{t\in[0,T]} \|W^{\epsilon}_{A_1}(t)\|^p_{D((-A_1)^{\gamma})} < \infty. \tag{3.34}$$

(for a proof see e.g. [36, Section 5.4]). Moreover, we have the following uniform bound.

Lemma 3.4.1. *Under Hypothesis 8, for every $p \geq 1$ we have*

$$\sup_{\epsilon\in(0,1)} \sup_{t\geq 0} \mathbb{E}^{\epsilon} \|W^{\epsilon}_{A_2}(t)\|^p < \infty. \tag{3.35}$$

Proof. Let $\{\beta^{2,\epsilon}_k\}_{k\in\mathbb{N}}$ be a sequence of independent Brownian motions such that

$$W^{2,\epsilon}_t = \sum_{k=1}^{\infty} e_{2,k}\beta^{2,\epsilon}_k(t), \qquad t \geq 0.$$

Thanks to the Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}\mathbb{E}^{\epsilon} \|W^{\epsilon}_{A_2}(t)\|^p =& \epsilon^{-p/2}\, \mathbb{E}^{\epsilon} \left\| \int_0^t S_2((t-s)/\epsilon) \sum_{k=1}^{\infty} \lambda_{2,k}\, e_{2,k}\, d\beta^{2,\epsilon}_k(s) \right\|^p \\ &\leq c_p\, \epsilon^{-p/2} \left(\sum_{k=1}^{\infty} \int_0^t e^{-2\alpha_{2,k}\left(\frac{t-s}{\epsilon}\right)} \lambda^2_{2,k} ds \right)^{p/2} \leq c_p \left(\sum_{k=1}^{\infty} \alpha^{-1}_{2,k} \lambda^2_{2,k} \right)^{p/2}.\end{aligned}$$

Now, thanks to (3.5), we have

$$\sum_{k=1}^{\infty} \alpha_{2,k}^{-1}\, \lambda_{2,k}^2 < \alpha_{2,1}^{-2\gamma_2} \sum_{k=1}^{\infty} \alpha_{2,k}^{2\gamma_2 - 1} \lambda_{2,k}^2 < +\infty,$$

and (3.35) follows. □

Notice that we also have tha for every $p \geq 1$

$$\sup_{\epsilon \in (0,1)} \mathbb{E}^{\epsilon} \sup_{t \in [0,T]} \|W_{A_2}^{\epsilon}\|^p < \infty. \tag{3.36}$$

Next, we want to investigate the time-continuity of the stochastic convolution. Thanks to Hypothesis 8, for every $p \geq 1$ and $0 \leq \gamma < \gamma_1^\star := \gamma_1 \wedge 1/2$ we have

$$\sup_{\epsilon \in (0,1)} \mathbb{E}^{\epsilon}\, \|W_{A_1}^{\epsilon}(t+h) - W_{A_1}^{\epsilon}(t)\|_{D((-A_1)^\gamma)}^p \leq c\, h^{(\gamma_1^\star - \gamma)\, p}, \qquad t \geq 0, \quad h \in (0,1). \tag{3.37}$$

The Garcia-Rademich-Rumsey theorem, together with (3.37), imply that there exists $\beta > 0$ such that

$$\sup_{\epsilon \in (0,1)} \mathbb{E}^{\epsilon} \sup_{t,s, \in [0,T]} \|W_{A_1}^{\epsilon}(t) - W_{A_1}^{\epsilon}(s)\|_{D((-A_1)^\gamma)}^2\, |t-s|^{-\beta} < \infty. \tag{3.38}$$

Lemma 3.4.2. *Assume Hypothesis 8 hold and fix* $0 \leq \gamma < \gamma_2^\star := \gamma_2 \wedge 1/2$.

Then, we have

$$\sup_{\epsilon \in (0,1)} \epsilon^{(\gamma_2^\star - \gamma)2}\, \mathbb{E}^\epsilon \left\| W_{A_2}^\epsilon(t+h) - W_{A_2}^\epsilon(t) \right\|^2_{D((-A_2)^\gamma)} \leq c\, h^{(\gamma_2^\star - \gamma)2}, \quad t \geq 0, \;\; h \in (0,1). \tag{3.39}$$

Proof. We recall that

$$W_{A_2}^\epsilon(t) = \frac{1}{\sqrt{\epsilon}} \int_0^t S_2\left((t-s)/\epsilon\right) Q_2\, dW_s^{2,\epsilon}.$$

Therefore, since

$$\mathcal{L}\left(\frac{1}{\sqrt{\epsilon}}\, W_{A_2}^{2,\epsilon}(\cdot) \right) = \mathcal{L}\left(W_{A_2}^{2,\epsilon}(\cdot/\epsilon) \right),$$

we have

$$\mathcal{L}\left(W_{A_2}^\epsilon(\cdot) \right) = \mathcal{L}\left(\int_0^{\cdot/\epsilon} S_2\left(\cdot/\epsilon - s\right) Q_2\, dW_s^{2,\epsilon} \right).$$

In particular, since A_2 satisfies the same assumptions as A_1, we can apply (3.37), and we get (3.39). □

3.4.2 Estimates for the slow motion $u^\epsilon(t)$

For every $\epsilon \in (0,1)$, we define

$$\Psi^\epsilon(t) := \int_0^t S_1(t-s)F_1(s, X^\epsilon(s))ds, \qquad t \in [0,T].$$

By using the same arguments used in the proof of Lemma 4.1.1 and Lemma 4.1.2, it is possible to prove that under Hypotheses 8, 9 and 11, for every $\gamma \in [0, 1/2)$

$$\sup_{\epsilon\in(0,1)} \mathbb{E}^\epsilon \sup_{t\in[0,T]} \|\Psi^\epsilon(t)\|^2_{D((-A_1)^\gamma)} \leq c_{T,\gamma} V(u_0, v_0), \quad t \in [0,T]. \tag{3.40}$$

Moreover, there exists some $\beta > 0$ such that

$$\sup_{\epsilon\in(0,1)} \mathbb{E}^\epsilon \sup_{t,s\in[0,T]} \|\Psi^\epsilon(t) - \Psi^\epsilon(s)\|^2_{D((-A_1)^\gamma)} |t-s|^{-\beta} \leq c_T\, V(u_0, v_0). \tag{3.41}$$

Therefore, it is possible to prove the following result.

Lemma 3.4.3. *Assume that Hypotheses 8, 9 and 11 hold. Then for any*

initial condition $(u_0, v_0) \in D_V$ *and* $0 \leq \gamma < \gamma_1^\star$, *we have*

$$\sup_{\epsilon\in(0,1)} \mathbb{E}^\epsilon \|u^\epsilon(t)\|^2_{D((-A_1)^\gamma)} \leq c\left(t^{-2\gamma}+1\right)\left(1+\|u_0\|^2+V(u_0,v_0)\right), \quad t \in [0,T]. \tag{3.42}$$

Moreover,

$$\sup_{\epsilon\in(0,1)} \mathbb{E}^\epsilon \|u^\epsilon(t)-u^\epsilon(s)\|^2_{D((-A_1)^\gamma)} \leq c\,\rho_\gamma(s,t)\left(1+\|u_0\|^2+V(u_0,v_0)\right), \qquad 0 \leq s \leq t \leq T, \tag{3.43}$$

where

$$\rho_\gamma(s,t) := \left(\int_s^t r^{-(\gamma+1)} dr\right)^2 + (t-s)^\beta + (t-s)^{2(\gamma_1^\star-\gamma)}, \qquad s \leq t, \tag{3.44}$$

and β *is the constant introduced in* (3.41). *Finally, if* $u_0 \in D((-A_1)^\delta)$, *for some* $\delta > \gamma$, *then there exists* $\eta = \eta(\gamma,\delta) > 0$ *such that*

$$\sup_{\epsilon\in(0,1)} \mathbb{E}^\epsilon \|u^\epsilon(t)-u^\epsilon(s)\|^2_{D((-A_1)^\gamma)} \leq c\,(t-s)^\eta\left(1+\|u_0\|^2_{D((-A_1)^\delta)}+V(u_0,v_0)\right), \qquad 0 \leq s \leq t \leq T. \tag{3.45}$$

Proof. Since

$$\|S_1(t)u_0\|_{D((-A_1)^\gamma} \leq c\,t^{-\gamma}\|u_0\|,$$

(3.42) follows from (3.34) and (3.40). Moreover, for every $\delta \in [0, \gamma+1]$, we have

$$\|S_1(t)u_0 - S_1(s)u_0\|_{D((-A_1)^\gamma)} \leq \int_s^t \left\|(-A_1)^{\gamma+1-\delta} S_1(r)(-A_1)^\delta u_0\right\| dr$$

$$\leq \int_s^t r^{\delta-(\gamma+1)} dr \ \|u_0\|_{D((-A_1)^\delta)}.$$

Therefore, thanks to (3.37), and (3.41), if we take $\delta = 0$ we get (3.43).

Now, since

$$a^\lambda - b^\lambda \leq (a-b)^\lambda, \qquad \lambda \in (0,1), \qquad 0 \leq b \leq a,$$

if we take $\delta \in (\gamma, \gamma+1]$ we have

$$\int_s^t r^{\delta-(\gamma+1)}\, dr = \frac{1}{\delta-\gamma}\left(t^{\delta-\gamma} - s^{\delta-\gamma}\right) \leq (t-s)^{\delta-\gamma}, \qquad t > s.$$

Therefore, (3.45) holds for any $\gamma < \gamma_1^\star$ and $\delta > \gamma$. □

Notice that from (3.34) and (3.40) we have also

$$\sup_{\epsilon \in (0,1)} \mathbb{E}^\epsilon \sup_{t \in [0,T]} \|u^\epsilon(t)\| \leq c_T \left(1 + \|u_0\|^2 + V(u_0, v_0)\right). \tag{3.46}$$

3.4.3 Estimates for the fast motion $v^\epsilon(t)$

Lemma 3.4.4. *Under Hypotheses 8 to 11 we have*

$$\sup_{\epsilon\in(0,1)} \mathbb{E}^\epsilon \sup_{t\in[0,T]} \|v^\epsilon(t)\|^2 < \infty. \tag{3.47}$$

Proof. If we define $\Gamma^\epsilon(t) := v^\epsilon(t) - W^\epsilon_{A_2}(t)$, we have

$$\begin{aligned}\frac{d\Gamma^\epsilon(t)}{dt} =& \frac{1}{\epsilon} A_2\Gamma^\epsilon(t) + \frac{1}{\epsilon}(F_2(u^\epsilon(t), \Gamma^\epsilon(t) + W^\epsilon_{A_2}(t)) - F_2(u^\epsilon(t), W^\epsilon_{A_2}(t)))\\ &+ \frac{1}{\epsilon} F_2(u^\epsilon(t), W^\epsilon_{A_2}(t)).\end{aligned}$$

Thanks to Hypothesis 10, we have

$$\frac{1}{2}\frac{d}{dt}\|\Gamma^\epsilon(t)\|^2 \le -\frac{\omega}{\epsilon}\|\Gamma^\epsilon(t)\|^2 + \frac{1}{\epsilon}\|F_2(u^\epsilon(t), W^\epsilon_{A_2}(t))\|\|\Gamma^\epsilon(t)\|,$$

and the Young's inequality gives

$$\frac{d}{dt}\|\Gamma^\epsilon(t)\|^2 \le -\frac{\omega}{2\epsilon}\|\Gamma^\epsilon(t)\|^2 + \frac{c}{\epsilon}(1 + \|u^\epsilon(t)\|^2 + \|W^\epsilon_{A_2}(t)\|^2).$$

By comparison, this implies

$$\begin{aligned}\|v^\epsilon(t)\|^2 &\leq 2\,\|\Gamma^\epsilon(t)\|^2 + 2\,\|W^\epsilon_{A_2}(t)\|^2 \\ &\leq 2\,e^{-\frac{t\omega}{\epsilon}}\|v_0\|^2 + \frac{c}{\epsilon}\int_0^t e^{-\frac{(t-s)\omega}{2\epsilon}}(1+\|u^\epsilon(s)\|^2 + \|W^\epsilon_{A_2}(s)\|^2)ds + c\,\|W^\epsilon_{A_2}(t)\|^2.\end{aligned}$$

Therefore, in view of (3.36) and (3.46), this gives

$$\mathbb{E}^\epsilon \sup_{t\in[0,T]} \|v^\epsilon(t)\|^2 \leq c\left(1 + \|(u_0,v_0)\|^2_{\mathcal{H}_2} + V(u_0,v_0)\right),$$

where c is a constant independent of $\epsilon \in (0,1)$. □

3.4.4 Proof of the tightness

If we define $y^\epsilon(t) := u^\epsilon(t) - S_1(t)u_0$, thanks to (3.34) and (3.40), we have that for $\gamma < \gamma_1^\star$

$$\sup_{\epsilon\in(0,1)} \mathbb{E}^\epsilon \sup_{t\in[0,T]} \|y^\epsilon(t)\|^2_{D((-A_1)^\gamma)} < \infty. \tag{3.48}$$

Moreover, thanks to (3.38) and (3.41), there exists some $\beta > 0$ sufficiently small, we have

$$\sup_{\epsilon\in(0,1)} \mathbb{E}^\epsilon \sup_{t,s\in[0,T]} \|y^\epsilon(t) - y^\epsilon(s)\|^2 ds\ |t-s|^{-\beta} < \infty. \tag{3.49}$$

Lemma 3.4.5. *Under Hypotheses 8 to 11, the family of measures* $\{\mathcal{L}(u^\epsilon)\}_{\epsilon\in(0,1)}$ *is tight in* $C([0,T];H)$.

Proof. Let γ and β be the same constants as in (3.48) and (3.49). If we fix an arbitrary $R>0$ and introduce the compact set of $C([0,T];H)$

$$K_R := \left\{ u\in C([0,T];H) \,:\, \|u\|_{C^\gamma([0,T];H)} + \sup_{t,s\in[0,T]} \frac{\|u(t)-u(s)\|^2}{|t-s|^\beta} \le R \right\},$$

thanks to (3.48) and (3.49) we have

$$\sup_{\epsilon\in(0,1)} \mathbb{P}^\epsilon \left(y^\epsilon \in K_R^c\right) \le \frac{c_T}{R}.$$

Therefore, for every $\eta>0$ we can fix $R_\eta>0$ such

$$\sup_{\epsilon\in(0,1)} \mathbb{P}^\epsilon \left(u^\epsilon \in S_1(\cdot)u_0 + K_{R_\eta}^c\right) = \sup_{\epsilon\in(0,1)} \mathbb{P}^\epsilon \left(y^\epsilon \in K_{R_\eta}^c\right) \le \eta.$$

Since $K_{R_\eta} + S_1(\cdot)u_0$ is also compact in $C([0,T];H)$, due to the arbitrariness of $\eta>0$ we conclude that the family of measures $\{\mathcal{L}(u^\epsilon)\}_{\epsilon\in(0,1)}$ is tight on $C([0,T];H)$. □

3.5 The averaging limit

In this section we conclude the proof of Theorem 3.3.3 by proving that any weak limit of $\mathcal{L}(u^\epsilon)$ is a martingale solution of problem (3.33).

3.5.1 A time-discretization of the fast motion

For every $\delta > 0$ and $k = 1, 2, ..., [T/\delta]$, we have that v^ϵ satisfies the following equation

$$v^\epsilon(t) = S_2\left((t-k\delta)/\epsilon\right) v^\epsilon(k\delta) + \frac{1}{\epsilon}\int_{k\delta}^{t} S_2\left((t-s)/\epsilon\right) F_2(u^\epsilon(s), v^\epsilon(s))ds$$
$$+ \frac{1}{\sqrt{\epsilon}}\int_{k\delta}^{t} S_2\left((t-s)/\epsilon\right) Q_2 dW_s^{2,\epsilon}, \quad t \in [k\delta, (k+1)\delta).$$

Now, we denote by $v^{\epsilon,\delta}$ the solution to the following equation

$$v^{\epsilon,\delta}(t) = S_2\left((t-k\delta)/\epsilon\right) v^\epsilon(k\delta) + \frac{1}{\epsilon}\int_{k\delta}^{t} S_2\left((t-s)/\epsilon\right) F_2(u^\epsilon(k\delta), v^{\epsilon,\delta}(s))ds$$
$$+ \frac{1}{\sqrt{\epsilon}}\int_{k\delta}^{t} S_2\left((t-s)/\epsilon\right) Q_2 dW_s^{2,\epsilon}, \qquad t \in [k\delta, (k+1)\delta). \tag{3.50}$$

Moreover, we denote by $u^{\epsilon,\delta}$ the process defined by

$$u^{\epsilon,\delta}(t) = \sum_{k=0}^{[T/\delta]} u^{\epsilon}(k\delta)\mathbb{1}_{\{k\delta \le t < (k+1)\delta\}}, \qquad t \in [0,T]. \tag{3.51}$$

Lemma 3.5.1. *Assume Hypotheses 8 to 11 and fix $(u_0, v_0) \in D_V$. Then, if ρ_0 is the function defined in (3.44) (when $\gamma = 0$), for any $k \le [T/\delta]$ and $t \in [k\delta, (k+1)\delta)$ we have*

$$\sup_{\epsilon\in(0,1)} \mathbb{E}^{\epsilon} \left\| u^{\epsilon}(t) - u^{\epsilon,\delta}(t) \right\|^2 \le c\, \rho_0(t, k\delta) \left(1 + \|u_0\|^2 + V(u_0, v_0)\right), \tag{3.52}$$

and

$$\mathbb{E}^{\epsilon} \left\| v^{\epsilon,\delta}(t) - v^{\epsilon}(t) \right\|^2 \le \frac{c}{\epsilon} \left(1 + \|u_0\|^2 + V(u_0, v_0)\right) \exp\left(\frac{c\,\delta}{\epsilon}\right) \int_{k\delta}^{t} \rho_0(s, k\delta) ds, \qquad \epsilon \in (0,1). \tag{3.53}$$

Proof. In order to prove (3.52), we notice that as a consequence of Lemma 3.4.3

$$\mathbb{E}^{\epsilon} \left\| u^{\epsilon}(t) - u^{\epsilon,\delta}(t) \right\|^2 = \mathbb{E}^{\epsilon} \left\| u^{\epsilon}(t) - u^{\epsilon}(k\delta) \right\|^2 \le c\rho_0(t, k\delta) \left(1 + \|u_0\|^2 + V(u_0, v_0)\right). \tag{3.54}$$

In order to prove (3.53), we notice that for $t \in [k\delta, (k+1)\delta)$ we have

$$\begin{aligned}
&\mathbb{E}^\epsilon \left\| v^{\epsilon,\delta}(t) - v^\epsilon(t) \right\|^2 \\
&\leq \frac{1}{\epsilon^2} \mathbb{E}^\epsilon \left(\int_{k\delta}^t \left\| S_2\left((t-s)/\epsilon\right) \left(F_2(u^\epsilon(k\delta), v^{\epsilon,\delta}(s)) - F_2(u^\epsilon(s), v^\epsilon(s)) \right) \right\| ds \right)^2 \\
&\leq \frac{c}{\epsilon^2} \int_{k\delta}^t e^{-\frac{2\alpha_{2,0}}{\epsilon}(t-s)} ds \left(\int_{k\delta}^t \mathbb{E}^\epsilon \left\| u^\epsilon(s) - u^\epsilon(k\delta) \right\|^2 ds + \int_{k\delta}^t \mathbb{E}^\epsilon \left\| v^{\epsilon,\delta}(s) - v^\epsilon(s) \right\|^2 ds \right).
\end{aligned} \tag{3.55}$$

Hence, thanks to (3.54), we get

$$\mathbb{E}^\epsilon \left\| v^{\epsilon,\delta}(t) - v^\epsilon(t) \right\|^2 \leq \frac{c}{\epsilon} \int_{k\delta}^t \rho_0(s, k\delta) ds + \frac{c}{\epsilon} \int_{k\delta}^t \mathbb{E}^\epsilon \left\| v^{\epsilon,\delta}(s) - v^\epsilon(s) \right\|^2 ds, \tag{3.56}$$

and by Gronwall's inequality, this implies

$$\mathbb{E}^\epsilon \left\| v^{\epsilon,\delta}(t) - v^\epsilon(t) \right\|^2 \leq \frac{c}{\epsilon} \left(1 + \|u_0\|^2 + V(u_0, v_0) \right) \exp\left(\frac{c\,\delta}{\epsilon} \right) \int_{k\delta}^t \rho_0(s, k\delta) ds. \tag{3.57}$$

□

3.5.2 Some preliminary results

For any $\xi \in C([0,T];H)$, we define

$$G_\xi(t,x,y) := \langle F_1(t,x,y), \xi(t)\rangle, \quad t \in [0,T], \quad (x,y) \in D(F_1),$$

and for every $\theta \in (0,1)$ we define

$$G^\theta_\xi(t,x,y) := \langle F^\theta_1(t,x,y), \xi(t)\rangle \quad t \in [0,T], \quad (x,y) \in \mathcal{H}_2.$$

The mapping $G_\xi : [0,T] \times \mathcal{H}_2 \to H$ is measurable, and, due to Condition 1. in Hypothesis 9, it is easy to check that the mapping $G^\theta_\xi : [0,T] \times \mathcal{H}_2 \to H$ is continuous and bounded. Moreover, as a consequence of (3.7), we have

$$|G_\xi(t,x,y) - G^\theta_\xi(t,x,y)| \le \theta \sup_{t\in[0.T]} \|\xi(t)\|\, V(x,y), \quad t \in [0,T], \quad (x,y) \in D(F_1). \tag{3.58}$$

Next we define $\bar{G}_\xi$ and $\bar{G}^\theta_\xi$ by setting

$$\bar{G}_\xi(t,x) := \langle \bar{F}_1(t,x), \xi(t)\rangle, \qquad (t,x) \in [0,T] \times \Pi_1 D(F_1),$$

and

$$\bar{G}^{\theta}_{\xi}(t,x) := \langle \bar{F}^{\theta}_{1}(t,x), \xi(t) \rangle, \qquad (t,x) \in [0,T] \times H.$$

Thanks to (3.31), we have

$$|\bar{G}_{\xi}(t,x) - \bar{G}^{\theta}_{\xi}(t,x)|^2 \leq \theta \sup_{t \in [0,T]} \|\xi(t)\|^2 \, \bar{V}(x), \qquad (t,x) \in [0,T] \times \Pi_1 D(F_1). \tag{3.59}$$

According to Lemma 3.3.2, we have that the mapping $\bar{G}_{\xi} : [0,T] \times D \to \mathbb{R}$ is measurable and for every $\theta \in (0,1)$ the mapping $\bar{G}^{\theta}_{\xi} : [0,T] \times H \to \mathbb{R}$ is continuous.

As we already mentioned above, in [40] it is shown that the space $\mathrm{Lip}_b([0,T] \times H)$ is dense in $C_b([0,T] \times H)$. Then, we can fix a sequence $\{G^{\theta}_{\xi,n}\}_{n \in \mathbb{N}} \subset \mathrm{Lip}_b([0,T] \times H)$ such that

$$\lim_{n \to \infty} \sup_{(t,x,y) \in [0,T] \times \mathcal{H}_2} |G^{\theta}_{\xi,n}(t,x,y) - G^{\theta}_{\xi}(t,x,y)| = 0. \tag{3.60}$$

Furthermore, if we define

$$\bar{G}^{\theta}_{\xi,n}(t,x) := \int_H G^{\theta}_{\xi,n}(t,x,y) \mu^x(dy), \qquad t \in [0,T], \quad x \in H,$$

by using arguments analogous to those used in Lemma 3.3.1 for $\bar{F}_n^\theta$, we can show that each mapping $\bar{G}_{\xi,n}^\theta : [0,T] \times H \to \mathbb{R}$ is Lipschitz-continuous. Moreover,

$$\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times H} |\bar{G}_{\xi,n}^\theta(t,x) - \bar{G}_\xi^\theta(t,x)| = 0. \tag{3.61}$$

Lemma 3.5.2. *Under Hypotheses 8 to 12, we have*

$$\lim_{\epsilon\to 0} \mathbb{E}^\epsilon \sup_{t\in[0,T]} \left| \int_0^t \langle F^1(s,u^\epsilon(s),v^\epsilon(s)),\xi(s)\rangle ds - \int_0^t \langle \bar{F}^1(s,u^\epsilon(s)),\xi(s)\rangle ds \right| = 0. \tag{3.62}$$

Proof. By using the notations we have just introduced, (3.62) can be rewritten as

$$\lim_{m\to\infty} \mathbb{E}^\epsilon \sup_{t\in[0,T]} \left| \int_0^t \left[G_\xi(s,u^\epsilon(s),v^\epsilon(s)) - \bar{G}_\xi(s,u^\epsilon(s)) \right] ds \right| = 0. \tag{3.63}$$

For every $\theta \in (0,1)$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\mathbb{E}^\epsilon \sup_{t\in[0,T]} \left| \int_0^t \left[G_\xi(s, u^\epsilon(s), v^\epsilon(s)) - \bar{G}_\xi(s, u^\epsilon(s)) \right] ds \right| \\
&\leq \mathbb{E}^\epsilon \sup_{t\in[0,T]} \left| \int_0^t \left[G_\xi(s, u^\epsilon(s), v^\epsilon(s)) - G^\theta_{\xi,n}(s, u^\epsilon(s), v^\epsilon(s)) \right] ds \right| \\
&\qquad + \mathbb{E}^\epsilon \sup_{t\in[0,T]} \left| \int_0^t \left[G^\theta_{\xi,n}(s, u^\epsilon(s), v^\epsilon(s)) - \bar{G}^\theta_{\xi,n}(s, u^\epsilon(s)) \right] ds \right| \\
&\qquad + \mathbb{E}^\epsilon \sup_{t\in[0,T]} \left| \int_0^t \left[\bar{G}^\theta_{\xi,n}(s, u^\epsilon(s)) - \bar{G}_\xi(s, u^\epsilon(s)) \right] ds \right| =: \sum_{k=1}^3 I_{\epsilon,k}(\theta, n).
\end{aligned} \tag{3.64}$$

For the term $I_{\epsilon,1}(\theta, n)$, due to (3.58), we have

$$\begin{aligned}
I_{\epsilon,1}(\theta,n) &\leq \int_0^T \mathbb{E}^\epsilon \left| G_\xi(s, u^\epsilon(s), v^\epsilon(s)) - G^\theta_\xi(s, u^\epsilon(s), v^\epsilon(s)) \right| ds \\
&+ \int_0^T \mathbb{E}^\epsilon \left| G^\theta_\xi(s, u^\epsilon(s), v^\epsilon(s)) - G^\theta_{\xi,n}(s, u^\epsilon(s), v^\epsilon(s)) \right| ds \\
&\leq \ \theta \int_0^T \mathbb{E}^\epsilon V(u^\epsilon(s), v^\epsilon(s))\, ds + \int_0^T \mathbb{E}^\epsilon \left| G^\theta_\xi(s, u^\epsilon(s), v^\epsilon(s)) - G^\theta_{\xi,n}(s, u^\epsilon(s), v^\epsilon(s)) \right| ds.
\end{aligned} \tag{3.65}$$

In view of Hypothesis 11, this gives

$$I_{\epsilon,1}(\theta,n) \leq \ \ c_T(u_0, v_0)\, \theta + c_T \sup_{(t,x,y)\in\, [0,T]\times\mathcal{H}_2} |G^\theta_\xi(t,x,y) - G^\theta_{\xi,n}(t,x,y)|. \tag{3.66}$$

For the term $I_{\epsilon,3}(\theta, n)$, according to (3.59) we have

$$I_{\epsilon,3}(\theta,n) \leq \int_0^T \mathbb{E}^\epsilon \left|\bar{G}_\xi(s,u^\epsilon(s)) - \bar{G}^\theta_\xi(s,u^\epsilon(s))\right| ds + \int_0^T \mathbb{E}^\epsilon \left|\bar{G}^\theta_\xi(s,u^\epsilon(s)) - \bar{G}^\theta_{\xi,n}(s,u^\epsilon(s))\right| ds$$

$$\leq \theta \int_0^T \mathbb{E}^\epsilon \bar{V}(u^\epsilon(s))\, ds + \int_0^T \mathbb{E}^\epsilon \left|\bar{G}^\theta_\xi(s,u^\epsilon(s)) - \bar{G}^\theta_{\xi,n}(s,u^\epsilon(s))\right| ds.$$

Thanks to Hypothesis 12 and (3.32), this implies

$$I_{\epsilon,3}(\theta,n) \leq c\,\theta + T \sup_{(t,x)\in[0,T]\times H} |\bar{G}^\theta_\xi(t,x) - \bar{G}^\theta_{\xi,n}(t,x)|.$$

Due to (3.60) and (3.61), the inequality above, together with (3.66), implies that for every $\eta > 0$ there exist $\theta_\eta \in (0,1)$ and $n_\eta \in \mathbb{N}$ such that

$$\sup_{\epsilon\in(0,1)} \left(I_{\epsilon,1}(\theta_\eta, n_\eta) + I_{\epsilon,3}(\theta_\eta, n_\eta)\right) < \eta. \tag{3.67}$$

Finally, concerning $I_{\epsilon,2}(\theta_\eta, n_\eta)$, if $u^{\epsilon,\delta}$ and $v^{\epsilon,\delta}$ are the auxiliary process

introduced in Section 3.5.1, we have

$$
\begin{aligned}
I_{\epsilon,2}(\theta_\eta, n_\eta) \leq & \int_0^T \mathbb{E}^\epsilon \left| G^{\theta_\eta}_{\xi,n_\eta}(s, u^\epsilon(s), v^\epsilon(s)) - G^{\theta_\eta}_{\xi,n_\eta}(s, u^{\epsilon,\delta}(t), v^{\epsilon,\delta}(t)) \right| ds \\
& + \int_0^T \mathbb{E}^\epsilon \left| \bar{G}^{\theta_\eta}_{\xi,n_\eta}(s, u^\epsilon(s)) - \bar{G}^{\theta_\eta}_{\xi,n_\eta}(t, u^{\epsilon,\delta}(s)) \right| ds \\
& + \int_0^T \mathbb{E}^\epsilon \left| G^{\theta_\eta}_{\xi,n_\eta}(s, u^{\epsilon,\delta}(s), v^{\epsilon,\delta}(s)) - \bar{G}^{\theta_\eta}_{\xi,n_\eta}(s, u^{\epsilon,\delta}(s)) \right| ds =: \sum_{k=1}^3 J_{\epsilon,\eta,k}.
\end{aligned} \tag{3.68}
$$

Since both $G^{\theta_\eta}_{\xi,n_\eta}$ and $\bar{G}^{\theta_\eta}_{\xi,n_\eta}$ are Lipschitz continuous, by Lemma 3.5.1, we have

$$
\begin{aligned}
& J_{\epsilon,\eta,1} + J_{\epsilon,\eta,2} \\
& \leq \left([G^{\theta_\eta}_{\xi,n_\eta}]_{\text{Lip}} + [\bar{G}^{\theta_\eta}_{\xi,n_\eta}]_{\text{Lip}} \right) \sum_{k=0}^{[T/\delta]+1} \int_{k\delta}^{(k+1)\delta} \mathbb{E}^\epsilon \left(\|u^\epsilon(s) - u^{\epsilon,\delta}(s)\| + \|v^\epsilon(s) - v^{\epsilon,\delta}(s)\| \right) ds \\
& \leq c_{u,\eta}(u_0, v_0) \sum_{k=0}^{[T/\delta]+1} \int_{k\delta}^{(k+1)\delta} \left(\rho_0(s, k\delta)^{1/2} + \left(\frac{1}{\epsilon} \exp\left(\frac{c\,\delta}{\epsilon} \right) \int_{k\delta}^s \rho_0(\zeta, k\delta) d\zeta \right)^{1/2} \right) ds.
\end{aligned} \tag{3.69}
$$

Moreover, we have

$$
\begin{aligned}
J_{\epsilon,\eta,3} &= \sum_{k=0}^{[T/\delta]+1} \int_{k\delta}^{(k+1)\delta} \mathbb{E}^{\epsilon} \left| G^{\theta_\eta}_{\xi,n_\eta}(s, u^{\epsilon}(k\delta), v^{\epsilon,\delta}(s)) - \bar{G}^{\theta_\eta}_{\xi,n_\eta}(s, u^{\epsilon}(k\delta)) \right| ds \\
&\leq \sum_{k=0}^{[T/\delta]+1} \int_{k\delta}^{(k+1)\delta} \mathbb{E}^{\epsilon} \left| G^{\theta_\eta}_{\xi,n_\eta}(s, u^{\epsilon}(k\delta), v^{\epsilon,\delta}(s)) - G^{\theta_\eta}_{\xi,n_\eta}(k\delta, u^{\epsilon}(k\delta), v^{\epsilon,\delta}(s)) \right| ds \\
&+ \sum_{k=0}^{[T/\delta]+1} \int_{k\delta}^{(k+1)\delta} \mathbb{E}^{\epsilon} \left| G^{\theta_\eta}_{\xi,n_\eta}(k\delta, u^{\epsilon}(k\delta), v^{\epsilon,\delta}(s)) - \bar{G}^{\theta_\eta}_{\xi,n_\eta}(k\delta, u^{\epsilon}(k\delta)) \right| ds \\
&+ \sum_{k=0}^{[T/\delta]+1} \int_{k\delta}^{(k+1)\delta} \mathbb{E}^{\epsilon} \left| \bar{G}^{\theta_\eta}_{u,n_\eta}(k\delta, u^{\epsilon}(k\delta)) - \bar{G}^{\theta_\eta}_{u,n_\eta}(s, u^{\epsilon}(k\delta)) \right| ds =: \sum_{k=1}^{3} K_{\epsilon,\eta,k}.
\end{aligned}
\tag{3.70}
$$

Thanks again to the Lipschitz continuity of $\bar{G}^{\theta_\eta}_{\xi,n_\eta}$ and $G^{\theta_\eta}_{\xi,n_\eta}$, there exists some constant $c_\eta > 0$ such that

$$
K_{\epsilon,\eta,1} + K_{\epsilon,\eta,3} \leq c_{u,\eta} \sum_{k=0}^{[T/\delta]+1} \int_{k\delta}^{(k+1)\delta} (s - k\delta)\, ds \leq c_\eta\, T\delta. \tag{3.71}
$$

Finally, by proceeding as in [31, Lemma 2.3] and [33, Theorem 3.5], as a consequence of (3.22) we have

$$
K_{\epsilon,\eta,2} = \sum_{k=0}^{[T/\delta]+1} \int_{k\delta}^{(k+1)\delta} \mathbb{E}^{\epsilon} \left| G^{\theta_\eta}_{\xi,n_\eta}(k\delta, u^{\epsilon}(k\delta), v^{\epsilon,\delta}(s)) - \bar{G}^{\theta_\eta}_{\xi,n_\eta}(k\delta, u^{\epsilon}(k\delta)) \right| ds \leq c_\eta \sqrt{\frac{\epsilon}{\delta}}. \tag{3.72}
$$

Therefore, in view of (3.64), from (3.67), (3.69), (3.71) and (3.72), we

obtain

$$\mathbb{E}^{\epsilon} \sup_{t\in[0,T]} \left| \int_0^T \langle F^1(s,u^{\epsilon}(s),v^{\epsilon}(s)),\xi(s)\rangle ds - \int_0^t \langle \bar{F}^1(s,u^{\epsilon}(s)),\xi(s)\rangle ds \right|$$
$$\leq c_{\eta}(x_0) \sum_{k=0}^{[T/\delta]+1} \int_{k\delta}^{(k+1)\delta} \left(\rho_0(t,k\delta)^{1/2} + \left(\frac{1}{\epsilon} \exp\left(\frac{c\,\delta}{\epsilon}\right) \int_{k\delta}^{t} \rho_0(t,k\delta) ds \right)^{1/2} \right) dt$$
$$+ I_{\epsilon,3}(\theta_{\eta}, n_{\eta}) + c_{\eta}\, T\delta + c_{\eta} \sqrt{\frac{\epsilon}{\delta}} + \eta. \tag{3.73}$$

Under the assumption that $u_0 \in D((-A_1)^{\beta})$, for some $\beta > 0$, according to (3.45), there exists some $\lambda > 0$ such that

$$\rho_u(t,s) \leq c(x_0)(t-s)^{\lambda},$$

so that

$$\sum_{k=0}^{[T/\delta]+1} \int_{k\delta}^{(k+1)\delta} \left(\rho_0(t,k\delta)^{1/2} + \left(\frac{1}{\epsilon} \exp\left(\frac{c\,\delta}{\epsilon}\right) \int_{k\delta}^{s} \rho_u(\zeta,k\delta) d\zeta \right)^{1/2} \right) ds$$
$$\leq c \sum_{k=0}^{[T/\delta]+1} \int_{k\delta}^{(k+1)\delta} \left(\delta^{\lambda/2} + \left(\frac{1}{\epsilon} \exp\left(\frac{c\,\delta}{\epsilon}\right) \delta^{1+\lambda} \right)^{1/2} \right) ds$$
$$\leq c\, t \left(\delta^{\lambda/2} + \frac{1}{\epsilon^{1/2}} \exp\left(\frac{c\,\delta}{2\,\epsilon}\right) \delta^{(1+\lambda)/2} \right).$$

Therefore, we have

$$\mathbb{E}^{\epsilon} \sup_{t\in[0,T]} \left| \int_0^t \langle F^1(s,u^{\epsilon}(s),v^{\epsilon}(s)),\xi(s)\rangle ds - \int_0^t \langle \bar{F}^1(s,u^{\epsilon}(s)),\xi(s)\rangle ds \right|$$
$$\leq c\,T\left(\delta^{\lambda/2} + \frac{1}{\epsilon^{1/2}} \exp\left(\frac{c\,\delta}{2\,\epsilon}\right) \delta^{(1+\lambda)/2}\right) + I_{\epsilon,3}(\theta_\eta, n_\eta) + c_\eta\, T\delta + c_\eta \sqrt{\frac{\epsilon}{\delta}} + \eta. \tag{3.74}$$

Now if we take

$$\delta_\epsilon := \frac{2}{c}\,\epsilon |\ln \epsilon|^{\lambda/2},$$

we have

$$\lim_{\epsilon\to 0} \delta_\epsilon = 0, \qquad \lim_{\epsilon\to 0} \frac{\epsilon}{\delta_\epsilon} = \frac{c}{2} \lim_{\epsilon\to 0} |\ln \epsilon|^{-\lambda/2} = 0. \tag{3.75}$$

Moreover, we have

$$\frac{1}{\epsilon^{1/2}} \exp\left(\frac{c\,\delta_\epsilon}{2\,\epsilon}\right) \delta_\epsilon^{(1+\lambda)/2} = \epsilon^{\eta} |\ln \epsilon|^{-\eta(1+\eta)/2} \left(\frac{1}{\epsilon}\right)^{\eta/2} = \epsilon^{\eta/2} |\ln \epsilon|^{-\eta(1+\eta)/2},$$

so that

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon^{1/2}} \exp\left(\frac{c\,\delta_\epsilon}{2\,\epsilon}\right) \delta_\epsilon^{(1+\lambda)/2} = 0. \tag{3.76}$$

Hence, due to (3.75) and (3.76), from (3.74) we get

$$\limsup_{\epsilon\to 0} \mathbb{E}^{\epsilon} \sup_{t\in[0,T]} \left| \int_0^t \langle F^1(s,u^{\epsilon}(s),v^{\epsilon}(s)),\xi(s)\rangle ds - \int_0^t \langle \bar{F}^1(s,u^{\epsilon}(s)),\xi(s)\rangle ds \right| \leq \eta,$$

and the arbitrariness of $\eta > 0$ allows to obtain (3.62). $\square$

3.5.3 Conclusion

Let $\{\epsilon_m\}_{m\in\mathbb{N}}$ a sequence in $(0,1)$ converging to 0 such that

$$\lim_{m\to\infty} \mathcal{L}(u^{\epsilon_m}) = \mu_{\text{slow}}, \qquad \text{weakly in } C([0,T];H). \tag{3.77}$$

In particular the sequence $\{(u^{\epsilon_m}, W^{\epsilon_m})\}_{m\in\mathbb{N}} \subset C([0,T];H) \times C([0,T];\mathcal{D}')$ is tight. Therefore there exists a subsequence of $\{\epsilon_m\}_{m\in\mathbb{N}}$ that we will still denote $\{\epsilon_m\}_{m\in\mathbb{N}}$ and $\nu \in \mathcal{P}(C([0,T];H) \times C([0,T];\mathcal{D}'))$ such that

$$\lim_{m\to\infty} \mathcal{L}(u^{\epsilon_m}, W^{\epsilon_m}) = \nu, \qquad \text{weakly in } C([0,T];H) \times C([0,T];\mathcal{D}').$$

Now, as a consequence of the Skorokhod theorem, we can fix a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, a sequence $\{(u^m, W^m)\}_{m\in\mathbb{N}} \subset C([0,T];H)\times C([0,T];\mathcal{D}')$ and $(\bar{u}, \bar{W}) \in C([0,T];H) \times C([0,T];\mathcal{D}')$ such that

$$\mathcal{L}(u^m, W^m) = \mathcal{L}(u^{\epsilon_m}, W^{\epsilon_m}), \qquad \mathcal{L}(\bar{u}, \bar{W}) = \nu, \tag{3.78}$$

and

$$\lim_{m\to\infty}(u^m, W^m) = (\bar{u}, \bar{W}), \qquad \bar{\mathbb{P}} - \text{a.s. in } C([0,T];H)\times C([0,T];\mathcal{D}'). \tag{3.79}$$

Notice that, due to (3.77), we have $\Pi_1\nu = \mu_{\text{slow}}$. Moreover, since all W^ϵ are space-time white noises in H, we have that $\Pi_2\nu$ is a space-time white noise. All this implies that $\mathcal{L}(\bar{u}) = \mu_{\text{slow}}$ and $\bar{W}$ is a space-time white noise.

Next, for every $\epsilon \in (0,1)$ and $\xi \in C^1([0,T];D(A_1^\star))$, we define

$$R^\epsilon_\xi(t) := \langle u^\epsilon(t),\xi(t)\rangle - \langle u_0,\xi(0)\rangle - \int_0^t \langle \bar{F}_1(s,u^\epsilon(s)),\xi(s)\rangle\, ds - \int_0^t \langle Q_1 dW^\epsilon(s),\xi(s)\rangle.$$

According to Lemma 3.5.2, we have that

$$\lim_{\epsilon\to 0}\mathbb{E}^\epsilon \sup_{t\in[0,T]} |R^\epsilon_\xi(t)| = 0. \tag{3.80}$$

Therefore, if we define

$$R^m_\xi(t) := \langle u^m(t),\xi(t)\rangle - \langle u_0,\xi(0)\rangle - \int_0^t \langle \bar{F}_1(s,u^m(s)),\xi(s)\rangle\, ds - \int_0^t \langle Q_1 dW^m(s),\xi(s)\rangle, \tag{3.81}$$

due to (3.78) we have

$$\lim_{m\to\infty} \bar{\mathbb{E}} \sup_{t\in[0,T]} |R_\xi^m(t)| = 0.$$

This, together with (3.79), implies that if we take the limit as $m \to \infty$ in both sides of (3.81), and if we show that

$$\lim_{m\to\infty} \bar{\mathbb{E}} \sup_{t\in[0,T]} \left| \int_0^t \langle \bar{F}_1(s, u^m(s)), \xi(s)\rangle \, ds - \int_0^t \langle \bar{F}_1(s, \bar{u}(s)), \xi(s)\rangle \, ds \right| = 0, \tag{3.82}$$

we obtain that $\bar{u}$ satisfies the equation

$$\langle \bar{u}(t), \xi(t)\rangle = \langle u_0, \xi(0)\rangle + \int_0^t \langle \bar{F}_1(s, \bar{u}(s)), \xi(s)\rangle \, ds - \int_0^t \langle Q_1 d\bar{W}(s), \xi(s)\rangle.$$

By using the notations introduced in Subsection 3.5.2, we can rewrite (3.82) as

$$\lim_{m\to\infty} \bar{\mathbb{E}} \sup_{t\in[0,T]} \left| \int_0^t \left[\bar{G}_\xi(s, u^m(s)) - \bar{G}_\xi(s, \bar{u}(s))\right] ds \right| = 0. \tag{3.83}$$

We have

$$\bar{\mathbb{E}} \sup_{t\in[0,T]} \left| \int_0^t \left[\bar{G}_\xi(s, u^m(s)) - \bar{G}_\xi(s, \bar{u}(s)) \right] ds \right| \leq \int_0^T \bar{\mathbb{E}} \left| \bar{G}_\xi(s, u^m(s)) - \bar{G}^\theta_\xi(s, u^m(s)) \right| ds$$

$$+ \int_0^T \bar{\mathbb{E}} \left| \bar{G}^\theta_\xi(s, u^m(s)) - \bar{G}^\theta_\xi(s, \bar{u}(s)) \right| ds + \int_0^T \bar{\mathbb{E}} \left| \bar{G}^\theta_\xi(s, \bar{u}(s)) - \bar{G}_\xi(s, \bar{u}(s)) \right| ds =: \sum_{k=1}^3 I^\theta_{m,k}.$$

According to (3.59), we have

$$I^\theta_{m,1} + I^\theta_{m,3} \leq c_T\, \theta \int_0^T \left[\bar{\mathbb{E}} \bar{V}(u^m(t)) + \bar{\mathbb{E}} \bar{V}(\bar{u}(t)) \right] dt.$$

As a consequence of (3.78) and (3.32), we have

$$\sup_{m\in\mathbb{N}} \int_0^T \bar{\mathbb{E}}\, \bar{V}(u^m(t))\, dt = \sup_{m\in\mathbb{N}} \int_0^T \mathbb{E}^{\epsilon_m} \bar{V}(u^{\epsilon_m}(t))\, dt < \infty.$$

Moreover, Fatou's lemma implies

$$\begin{aligned} \int_0^T \bar{\mathbb{E}}\, \bar{V}(\bar{u}(t))\, dt &\leq \liminf_{R\to\infty} \int_0^T \bar{\mathbb{E}} \left(\bar{V}(\bar{u}(t)) \wedge R \right) dt \\ &= \liminf_{R\to\infty} \lim_{m\to\infty} \int_0^T \bar{\mathbb{E}} \left(\bar{V}(\bar{u}(t)) \wedge R \right) dt \leq \sup_{m\in\mathbb{N}} \int_0^T \bar{\mathbb{E}}\, \bar{V}(u^m(t))\, dt < \infty. \end{aligned}$$

Therefore, for every $\eta > 0$, we can find $\theta_\eta > 0$ such that

$$\sup_{m\in\mathbb{N}} \left(I^{\theta_\eta}_{m,1} + I^{\theta_\eta}_{m,3} \right) < \eta.$$

In particular, due to the arbitrariness of $\eta > 0$, (3.83) follows once we show that

$$\lim_{m\to\infty} \int_0^T \bar{\mathbb{E}} \left| \bar{G}^{\theta_\eta}_\xi(s, u^m(s)) - \bar{G}^{\theta_\eta}_\xi(s, \bar{u}(s)) \right| ds = 0.$$

But this is clearly a consequence of (3.79), as $\bar{G}^{\theta_\eta}_\xi$ is bounded and continuous.

3.6 An example

In this section, we are going to discuss some examples of slow-fast systems of SPDEs which satisfy the conditions we have assumed in the previous sections and hence can be treated with our theory.

Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded open set, with a smooth boundary and let H be the Hilbert space $L^2(\mathcal{O})$, endowed with the norm $\|\cdot\|$ and the scalar product $\langle \cdot, \cdot \rangle$. As in Section 3.2, we denote by $\mathcal{H}_2$ the product sopace $H \times H$.

3.6.1 About Hypothesis 8

We assume that for every $p \in (1,\infty)$ the operators $(A_{1,p}, D(A_{1,p}))$ and $(A_{2,p}, D(A_{2,p}))$ are the realizations in $L^p(\mathcal{O})$ of second order uniformly elliptic operators $\mathcal{A}_i$, $i = 1, 2$, satisfying Hypothesis 8. All these operators are consistent, in the sense that for every $p, r \in (0,\infty)$

$$A_{i,p}x = A_{i,r}x, \qquad x \in D(A_{i,p}) \cap D(A_{i,r}), \quad i = 1, 2,$$

and for this reason, whenever there is no room for confusion, for the sake of simplicity we shall denote all of them as $(A_1, D(A_1))$ and $(A_2, D(A_2))$. In what follows, we will use the fact that for every $p \in (1,\infty)$ there exists $\lambda_p > 0$ such that

$$\langle A_i x, x|x|^{p-2}\rangle \le -\lambda_p \, \|(-A_i)^{1/2}(x|x|^{\frac{p-2}{2}})\|^2, \qquad x \in D(A_{i,p}), \quad i = 1,\, 2.$$

In particular, this implies that

$$\langle A_i x, x|x|^{p-2}\rangle \le -\alpha_{1,i}\, \lambda_p \, \|x\|^p_{L^p}, \qquad x \in D(A_{i,p}), \quad i = 1, 2, \tag{3.84}$$

(for all details, we refer e.g. to [42]).

As far as the Gaussian perturbations in the slow and in the fast equations are concerned, we assume that the operators Q_1 and Q_2 satisfy Hypothesis 8. Moreover, we assume that one of the following two conditions are satisfied:

C1. both Q_1 and Q_2 are invertible, with $Q_1^{-1}, Q_2^{-1} \in \mathcal{L}(H)$,

C2. both Q_1 and Q_2 are Hilbert-Schmidt operators.

Under reasonable conditions of the coefficients of the differential operators $\mathcal{A}_i$ and the domain $\mathcal{O}$, we have that

$$\alpha_{i,k} \sim k^{\frac{d}{2}}, \qquad k \in \mathbb{N}.$$

Hence, Condition C1. is satisfied only if $d = 1$. In particular, if we want to treat the case of SPDEs of reaction-diffusion type in space dimension $d > 1$, we can only consider Gaussian noise with trace-class covariance.

As we did in Section 3.2, for every $\epsilon \in (0, 1)$ we define

$$A^\epsilon(x_1, x_2) = (A_1 x_1, \epsilon^{-1} A_2 x_2), \qquad (x_1, x_2) \in D(A^\epsilon) := D(A_1) \times D(A_2) \subset \mathcal{H}_2, \tag{3.85}$$

and

$$Q^{\epsilon}(x_1, x_2) = (Q_1 x_1, \epsilon^{-1/2} Q_2 x_2), \qquad (x_1, x_2) \in \mathcal{H}_2. \tag{3.86}$$

Notice that if Condition C1. holds, then Q^{ϵ} is invertible with $(Q^{\epsilon})^{-1} \in \mathcal{L}(\mathcal{H}_2)$, for each $\epsilon \in (0,1)$. In the same way, if Condition C2. holds, then $\mathrm{Tr}\,[(Q^{\epsilon})^2] < \infty$, for each $\epsilon \in (0,1)$.

3.6.2 About Hypothesis 9

We assume that $b : [0,T] \times \mathcal{O} \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function satisfying the following conditions.

1. There exist $c_1 > 0$ and $m_1, m_2 \geq 1$ and a function $a_1 \in L^4(\mathcal{O})$ such that

$$\sup_{t \in [0,T]} |b(t, \xi, \sigma, \lambda)| \leq c_1 \left(a_1(\xi) + |\sigma|^{m_1} + |\lambda|^{m_2}\right), \tag{3.87}$$

   for all $\xi \in \mathcal{O}$ and $\sigma, \lambda \in \mathbb{R}$.

2. There exist $c_2 > 0$, $\kappa_1 \leq 2m_2$ and $\kappa_2 \geq 0$ and a function $a_2 \in L^1(\mathcal{O})$

such that

$$\sup_{t\in[0,T]} b(t,\xi,\sigma+\rho,\lambda)\sigma \leq c_2\left(a_2(\xi)+|\sigma|^2+|\lambda|^{\kappa_1}+|\rho|^{\kappa_2}\right), \tag{3.88}$$

for all $\xi \in \mathcal{O}$ and $\sigma,\rho,\lambda \in \mathbb{R}$.

Now, given any two functions $x, y : \mathcal{O} \to \mathbb{R}$, we define

$$F_1(t,x,y)(\xi) = b(t,\xi,x(\xi),y(\xi)), \quad t \in [0,T], \quad \xi \in \mathcal{O}. \tag{3.89}$$

According to (3.87), if we define

$$D(F_1) := L^{2m_1}(\mathcal{O}) \times L^{2m_2}(\mathcal{O}),$$

we have that $F_1 : [0,T] \times D(F_1) \to H$ and

$$\|F_1(t,x,y)\| \leq c\left(1+\|x\|^{m_1}_{L^{2m_1}}+\|y\|^{m_2}_{L^{2m_2}}\right), \quad (x,y) \in D(F_1).$$

Moreover, since we are assuming $\kappa_1 \leq 2m_2$, according to (3.88), we have that

$$\langle F_1(t,x+z,y),x\rangle \leq c\left(1+\|x\|^2+\|y\|^{\kappa_1}_{L^{\kappa_1}}+\|z\|^{\kappa_2}_{L^{\kappa_2}}\right), \tag{3.90}$$

for every $(x, y) \in D(F_1)$ and $z \in L^{\kappa_2}(\mathcal{O})$.

Next, for every $\theta \in (0, 1)$, we define

$$b_\theta(t, \xi, \sigma, \lambda) := \frac{b(t, \xi, \sigma, \lambda)}{1 + \theta\, |b(t, \xi, \sigma, \lambda)|}, \qquad t \in [0, T], \quad \xi \in \mathcal{O}, \quad (\sigma, \rho) \in \mathbb{R}^2.$$

The function $b_\theta : [0, T] \times \mathcal{O} \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and bounded. Therefore, if for every $x, y \in H$ we define

$$F_1^\theta(t, x, y)(\xi) := b_\theta(t, \xi, x(\xi), y(\xi)), \qquad t \in [0, T], \quad \xi \in \mathcal{O},$$

we have that $F_1^\theta : [0, T] \times \mathcal{H}_2 \to H$ is continuous and bounded. Since

$$|b_\theta(t, \xi, \sigma, \lambda)| \leq |b(t, \xi, \sigma, \lambda)|,$$

according to (3.87) we have

$$\begin{aligned} \|F_1^\theta(t, x, y)\|^2 &\leq \|F_1(t, x, y)\|^2 \leq c \int_{\mathcal{O}} \left(|a_2(\xi)|^2 + |x(\xi)|^{2m_1} + |y(\xi)|^{2m_2} \right) d\xi \\ &\leq c \left(1 + \|x\|_{L^{2m_1}}^{2m_1} + \|y\|_{L^{2m_2}}^{2m_2} \right). \end{aligned} \tag{3.91}$$

Moreover, since

$$|b_\theta(t,\xi,\sigma,\lambda) - b(t,\xi,\sigma,\lambda)| = \theta\,|b(t,\xi,\sigma,\lambda)|^2,$$

thanks again to (3.87) we have

$$\begin{aligned}
&\|F_1^\theta(t,x,y) - F_1(t,x,y)\|^2 \le \theta^2 \int_{\mathcal{O}} |b(t,\xi,x(\xi),y(\xi))|^4\,d\xi \\
&\le c\,\theta^2 \int_{\mathcal{O}} \left(|a_2(\xi)|^4 + |x(\xi)|^{4m_1} + |y(\xi)|^{4m_2}\right)\,d\xi \le c\,\theta^2 \left(1 + \|x\|_{L^{4m_1}}^{2m_1} + \|y\|_{L^{4m_2}}^{2m_2}\right)^2.
\end{aligned} \tag{3.92}$$

This implies that if we define

$$V(x,y) := c_V \left(1 + \|x\|_{L^{4m_1}}^{2m_1} + \|y\|_{L^{4m_2}}^{2m_2} + \|y\|_{L^{2\kappa_1 m_1}}^{\kappa_1 m_1}\right), \tag{3.93}$$

for some $c_V > 0$ sufficiently large, as a consequence of (3.91) and (3.92) we obtain

$$\|F_1^\theta(t,x,y)\|^2 \le \|F_1(t,x,y)\|^2 \le V(x,y), \tag{3.94}$$

and

$$\|F_1^\theta(t,x,y) - F_1(t,x,y)\| \le \theta\,V(x,y). \tag{3.95}$$

The function $V : \mathcal{H}_2 \to [0, +\infty]$ is convex and lower semi-continuous and

$$D_V := \{V < \infty\} \subset D \times H.$$

Remark 3.6.1. In (3.95) and (3.94) we only need

$$V(x, y) := c_V \left(1 + \|x\|_{L^{4m_1}}^{2m_1} + \|y\|_{L^{4m_2}}^{2m_2}\right).$$

The reason why we define V as in (3.93) is because we will need it when we will prove that condition (3.13) holds.

3.6.3 About Hypothesis 10

We introduce a function $g : [0, T] \times \mathcal{O} \times \mathbb{R}^2 \to \mathbb{R}$ continuous, such that $g(t, \xi, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ is Lipschitz continuous, uniformly with respect to $(t, \xi) \in [0, T] \times \mathcal{O}$. We assume that there exists $L_2 < \alpha_{2,1}$ that

$$\sup_{\substack{(t,\xi) \in [0,T] \times \mathcal{O} \\ \rho \in \mathbb{R}}} |g(t, \xi, \rho, \sigma_1) - g(t, \xi, \rho, \sigma_2)| \leq L_2 |\sigma_1 - \sigma_2|. \tag{3.96}$$

Thus, if we define

$$F_2(t,x)(\xi) := g(t,\xi,x(\xi)), \qquad \xi \in \mathcal{O},$$

for every $x \in \mathcal{H}_2$, we have that F_2 satisfies Hypothesis 10.

3.6.4 About Hypothesis 11

As we did in Section 3.2, for every $\epsilon \in (0,1)$ we define $D(F^\epsilon) := D(F_1) \subseteq \mathcal{H}_2$ and

$$F^\epsilon(t,x) := \left(F_1(t,x), \epsilon^{-1}F_2(x)\right), \qquad t \in [0,T], \quad x \in D(F^\epsilon).$$

Moreover, for every $\theta \in (0,1)$ we define

$$F^{\epsilon,\theta}(t,x) := \left(F_1^\theta(t,x), \epsilon^{-1}F_2(x)\right), \qquad t \in [0,T], \quad x \in \mathcal{H}_2.$$

Since F_1^θ and F_2 are both continuous, the mapping $F^{\epsilon,\theta}(t,\cdot) : \mathcal{H}_2 \to \mathcal{H}_2$ is continuous. Moreover, due to the linear growth of $F_2(t,\cdot)$, which is uniform

with respect to $t \in [0,T]$, according to (3.95) we have

$$\begin{aligned}\|F^{\epsilon,\theta}(t,x)\|^2_{\mathcal{H}_2} &= \|F_1^{\theta}(t,x)\|^2 + \epsilon^{-2}\|F_2(t,x)\|^2 \leq \|F_1(t,x)\|^2 + \epsilon^{-2}\|F_2(t,x)\|^2 \\ &\leq V(x) + \frac{c}{\epsilon^2}\left(1 + \|x\|^2_{\mathcal{H}_2}\right) =: V_\epsilon(x),\end{aligned}$$

and the function $V_\epsilon : \mathcal{H}_2 \to [0,+\infty]$ is convex and lower semi-continuous. Finally,

$$\|F^{\epsilon,\theta}(t,x) - F^{\epsilon}(t,x)\|_{\mathcal{H}_2} = \|F_1^{\theta}(t,x) - F_1(t,x)\| \leq \theta\, V(x) \leq \theta\, V_\epsilon(x).$$

The operators $(A^\epsilon, D(A^\epsilon))$ and Q^ϵ and the functions F^ϵ, $F^{\epsilon,\theta}$ and V_ϵ that we have just constructed satisfy the all the conditions assumed Chapter 4, for each $\epsilon \in (0,1)$. Therefore, once we have fixed $(u_0, v_0) \in D_V$ for every $\epsilon \in (0,1)$ there exists a martingale solution

$$(\Omega^\epsilon, \mathcal{F}^\epsilon, \{\mathcal{F}_t^\epsilon\}_{t\in[0,T]}, \mathbb{P}^\epsilon, \{W_t^\epsilon\}_{t\in[0,T]}, \{X^\epsilon(t)\}_{t\in[0,T]}),$$

for system (3.3). Now, we check that condition (3.13) holds.

Lemma 3.6.2. *Assume that the function a_2 introduced in (3.88) belongs to*

$L^{2m_1}(\mathcal{O})$ *and there exists* $c_T > 0$ *such that*

$$\sup_{t\in[0,T]} \mathbb{E}^\epsilon \|W_{A_1}(t)\|_{L^{\bar{p}}}^{\bar{p}} + \sup_{t\in[0,T]} \mathbb{E}^\epsilon \|W_{A_2}(t)\|_{L^{\bar{q}}}^{\bar{q}} \leq c_T, \qquad \epsilon \in (0,1), \tag{3.97}$$

where $\bar{p} := 2\kappa_2 m_1$ *and* $\bar{q} := 2\kappa_1 m_1 \vee 4m_2$ *and where* κ_2, κ_1, m_1 *and* m_2 *are the constants introduced in* (3.87) *and* (3.88). *Moreover, assume that*

$$\sup_{(t,\xi)\in[0,T]\times\mathcal{O}} |g(t,\xi,\sigma,\rho)| \leq c\left(1 + |\sigma|^{\frac{2}{\kappa_1}\wedge\frac{m_1}{m_2}\wedge 1} + |\rho|\right), \qquad (\sigma,\rho) \in \mathbb{R}^2, \tag{3.98}$$

and

$$L_2 < \alpha_{2,1}\lambda_{\bar{q}}, \tag{3.99}$$

where L_2 *is the constant introduced in* (3.96). *Then condition* (3.13) *is satisfied.*

Proof. If we define $\Gamma_1^\epsilon(t) := u^\epsilon(t) - W_{A_1}^\epsilon(t)$, due to (3.84), we have

$$\begin{aligned}\frac{1}{4m_1}\frac{d}{dt}\|\Gamma_1^\epsilon(t)\|_{L^{4m_1}}^{4m_1} &= \langle A_1\Gamma_1^\epsilon(t) + F_1(t,\Gamma_1^\epsilon(t) + W_{A_1}^\epsilon(t), v^\epsilon(t)), \Gamma_1^\epsilon(t)\,|\Gamma_1^\epsilon(t)|^{4m_1-2}\rangle \\ &\leq -\alpha_{1,1}\lambda_{4m_1}\|\Gamma_1^\epsilon(t)\|_{L^{4m_1}}^{4m_1} + \langle F_1(t,\Gamma_1^\epsilon(t) + W_{A_1}^\epsilon(t), v^\epsilon(t)), \Gamma_1^\epsilon(t)\,|\Gamma_1^\epsilon(t)|^{4m_1-2}\rangle.\end{aligned}$$

Thanks to (3.88), we have

$$\begin{aligned}
&\langle F_1(t,\Gamma_1^\epsilon(t)+W_{A_1}^\epsilon(t),v^\epsilon(t)),\Gamma_1^\epsilon(t)\Gamma_1^\epsilon(t)\,|\Gamma_1^\epsilon(t)|^{4m_1-2}\rangle\\
&\leq \int_{\mathcal{O}}\left(a_2(\xi)+|\Gamma_1^\epsilon(t,\xi)|^2+|v^\epsilon(t,\xi)|^{\kappa_1}+|W_{A_1}^\epsilon(t,\xi)|^{\kappa_2}\right)\,|\Gamma_1^\epsilon(t,\xi)|^{4m_1-2}\,d\xi\\
&\leq \int_{\mathcal{O}}\left(|a_2(\xi)|^{2m_1}+|\Gamma_1^\epsilon(t,\xi)|^{4m_1}+|v^\epsilon(t,\xi)|^{\bar q}+|W_{A_1}^\epsilon(t,\xi)|^{\bar p}\right)\,d\xi,
\end{aligned}$$

so that

$$\frac{d}{dt}\|\Gamma_1^\epsilon(t)\|_{L^{4m_1}}^{4m_1}\quad\leq c\,\|\Gamma_1^\epsilon(t)\|_{L^{4m_1}}^{4m_1}+c\left(1+\|W_{A_1}^\epsilon(t)\|_{L^{\bar p}}^{\bar p}+\|v^\epsilon(t)\|_{L^{\bar q}}^{\bar q}\right).$$

By comparison, we obtain

$$\mathbb{E}^\epsilon\,\|\Gamma_1^\epsilon(t)\|_{L^{4m_1}}^{4m_1}\leq c_T\left(1+\|u_0\|_{L^{4m_1}}^{4m_1}+\sup_{t\in[0,T]}\mathbb{E}^\epsilon\,\|W_{A_1}^\epsilon(t)\|_{L^{\bar p}}^{\bar p}+\int_0^T\mathbb{E}^\epsilon\|v^\epsilon(t)\|_{L^{\bar q}}^{\bar q}\,dt\right),$$

and, in view of (3.97), this yields

$$\mathbb{E}^\epsilon\,\|u^\epsilon(t)\|_{L^{4m_1}}^{4m_1}\leq c_T\left(1+\|u_0\|_{L^{4m_1}}^{4m_1}\right)+c_T\int_0^t\mathbb{E}^\epsilon\,\|v^\epsilon(s)\|_{L^{\bar q}}^{\bar q}\,ds,\qquad t\in\,[0,T]. \tag{3.100}$$

As we have done above for u^ϵ, we denote $\Gamma_2^\epsilon(t):=v^\epsilon(t)-W_{A_2}^\epsilon(t)$. Due to

(3.84), we have

$$\frac{1}{\bar{q}}\frac{d}{dt}\|\Gamma_2^\epsilon(t)\|_{L^{\bar{q}}}^{\bar{q}} = \frac{1}{\epsilon}\langle A_2\Gamma_2^\epsilon(t) + F_2(t,u^\epsilon(t),\Gamma_2^\epsilon(t)+W_{A_2}^\epsilon(t)),\Gamma_2^\epsilon(t)\,|\Gamma_2^\epsilon(t)|^{\bar{q}-2}\rangle$$

$$\leq -\frac{\alpha_{2,1}\lambda_{\bar{q}}}{\epsilon}\,\|\Gamma_2^\epsilon(t)\|_{L^{\bar{q}}}^{\bar{q}} + \frac{1}{\epsilon}\langle F_2(t,u^\epsilon(t),\Gamma_2^\epsilon(t)+W_{A_2}^\epsilon(t)),\Gamma_2^\epsilon(t)\,|\Gamma_2^\epsilon(t)|^{\bar{q}-2}\rangle. \tag{3.101}$$

Since

$$\bar{q}\left(\frac{2}{\kappa_1}\wedge\frac{m_1}{m_2}\wedge 1\right) = (2\kappa_1 m_1\vee 4m_2)\left(\frac{2}{\kappa_1}\wedge\frac{m_1}{m_2}\wedge 1\right)\leq 4m_1,$$

in view of (3.96) and (3.98), for every $\delta>0$ we have

$$|\langle F_2(t,u^\epsilon(t),\Gamma_2^\epsilon(t)+W_{A_2}^\epsilon(t)),\Gamma_2^\epsilon(t)\,|\Gamma_2^\epsilon(t)|^{\bar{q}-2}\rangle|$$

$$\leq L_2\int_{\mathcal{O}}|\Gamma_2^\epsilon(t,\xi)|^{\bar{q}}\,d\xi + c\int_{\mathcal{O}}|\Gamma_2^\epsilon(t,\xi)|^{\bar{q}-1}\left(1+|u^\epsilon(t,\xi)|^{\frac{2}{\kappa_1}\wedge\frac{m_1}{m_2}\wedge 1}+|W_{A_2}^\epsilon(t,\xi)|\right)d\xi$$

$$\leq (L_2+\delta)\,\|\Gamma_2^\epsilon(t)\|_{L^{\bar{q}}}^{\bar{q}} + c_\delta\left(1+\|u^\epsilon(t)\|_{L^{4m_1}}^{4m_1}+\|W_{A_2}^\epsilon(t)\|_{L^{\bar{q}}}^{\bar{q}}\right). \tag{3.102}$$

Hence, if we put together (3.101) and (3.102), we get

$$\frac{d}{dt}\|\Gamma_2^\epsilon(t)\|_{L^{\bar{q}}}^{\bar{q}} \leq -\frac{1}{\epsilon}\left(\alpha_{2,1}\lambda_{\bar{q}}-(L_2+\delta)\right)\bar{q}\,\|\Gamma_2^\epsilon(t)\|_{L^{\bar{q}}}^{\bar{q}} + \frac{c_\delta\bar{q}}{\epsilon}\left(1+\|u^\epsilon(t)\|_{L^{4m_1}}^{4m_1}+\|W_{A_2}^\epsilon(t)\|_{L^{\bar{q}}}^{\bar{q}}\right).$$

Now, thanks to (3.99) we can fix $\bar{\delta} > 0$ such that

$$\vartheta := \left(\alpha_{2,1}\lambda_{\bar{q}} - (L_2 + \bar{\delta})\right)\bar{q} > 0,$$

and by comparison, we have

$$\mathbb{E}^\epsilon \|\Gamma_2^\epsilon(t)\|_{L^{\bar{q}}}^{\bar{q}} \leq e^{-\frac{\vartheta}{\epsilon}t}\|v_0\|_{L^{\bar{q}}}^{\bar{q}} + \frac{c}{\epsilon}\int_0^t e^{-\frac{\vartheta}{\epsilon}(t-s)}\left(1 + \mathbb{E}^\epsilon\|u^\epsilon(s)\|_{L^{4m_1}}^{4m_1} + \mathbb{E}^\epsilon\|W_{A_2}^\epsilon(s)\|_{L^{\bar{q}}}^{\bar{q}}\right) ds.$$

As $\vartheta > 0$, thanks to (3.97) we get

$$\begin{aligned}
\mathbb{E}^\epsilon \|v_2^\epsilon(t)\|_{L^{\bar{q}}}^{\bar{q}} &\leq e^{-\frac{\vartheta}{\epsilon}t}\|v_0\|_{L^{\bar{q}}}^{\bar{q}} + \mathbb{E}^\epsilon \|W_{A_2}^\epsilon(t)\|_{L^{\bar{q}}}^{\bar{q}} \\
&\quad + c\left(1 + \sup_{s\in[0,t]} \mathbb{E}^\epsilon \|u^\epsilon(s)\|_{L^{4m_1}}^{4m_1} + \sup_{s\in[0,t]} \mathbb{E}^\epsilon \|W_{A_2}^\epsilon(s)\|_{L^{\bar{q}}}^{\bar{q}}\right) \\
&\leq e^{-\frac{\vartheta}{\epsilon}t}\|v_0\|_{L^{\bar{q}}}^{\bar{q}} + c\left(1 + \sup_{s\in[0,t]} \mathbb{E}^\epsilon \|u^\epsilon(s)\|_{L^{4m_1}}^{4m_1}\right) + c_T.
\end{aligned} \tag{3.103}$$

Thus, if we plug (3.103) into (3.100), we obtain

$$\sup_{s\in[0,t]} \mathbb{E}^\epsilon \|u^\epsilon(s)\|_{L^{4m_1}}^{4m_1} \leq c_T\left(1 + \|u_0\|_{L^{4m_1}}^{4m_1} + \|v_0\|_{L^{\bar{q}}}^{\bar{q}}\right) + c_T\int_0^t \sup_{r\in[0,s]} \mathbb{E}^\epsilon \|u^\epsilon(r)\|_{L^{4m_1}}^{4m_1} ds,$$

and Gronwall's lemma implies

$$\sup_{s\in[0,t]} \mathbb{E}^{\epsilon} \|u^{\epsilon}(s)\|^{4m_1}_{L^{4m_1}} \leq c_T \left(1 + \|u_0\|^{4m_1}_{L^{4m_1}} + \|v_0\|^{\bar{q}}_{L^{\bar{q}}}\right). \tag{3.104}$$

Thanks to (3.103), this gives

$$\sup_{s\in[0,t]} \mathbb{E}^{\epsilon} \|v^{\epsilon}(s)\|^{\bar{q}}_{L^{\bar{q}}} \leq c_T \left(1 + \|u_0\|^{4m_1}_{L^{4m_1}} + \|v_0\|^{\bar{q}}_{L^{\bar{q}}}\right). \tag{3.105}$$

Therefore, recalling how V was defined in (3.93), from (3.104) and (3.105) we obtain (3.13). □

Remark 3.6.3. When $\kappa_1 \leq 2$ and $m_1 \geq m_2$, condition (3.98) is always satisfied, as we are assuming that $g(t,\xi,\cdot)$ has linear growth, uniformly with respect to $(t,\xi) \in [0,T]\times\mathcal{O}$.

3.6.5 About Hypothesis 12

We need to show that if $\mu^x(dy)$ is the invariant measure of the fast motion with frozen slow component x introduced in Section 3.3.1, then condition (3.23) is satisfied. Namely we need to show that

$$\bar{V}(x) := \int_H V(x,y)\,\mu^x(dy) < \infty, \qquad x \in \Pi_1 D_V.$$

Recalling how V was defined in (3.93), for every $x \in L^{4m_1}(\mathcal{O})$, we have

$$\bar{V}(x) \leq c_V \int_H \left(1 + \|x\|_{L^{4m_1}}^{2m_1} + \|y\|_{L^{\bar{q}}}^{\bar{q}/2}\right) \mu^x(dy) \leq c_V \left(1 + \|x\|_{L^{4m_1}}^{2m_1}\right) + c_V \int_H \|y\|_{L^{\bar{q}}}^{\bar{q}/2} \mu^x(dy), \tag{3.106}$$

where $\bar{q} = 2\kappa_1 m_1 \vee 4m_2$. This means that (3.23) follows once we show that

$$\int_H \|y\|_{L^{\bar{q}}}^{\bar{q}/2} \mu^x(dy) < \infty, \qquad x \in L^{4m_1}(\mathcal{O}).$$

Lemma 3.6.4. *Under the same assumptions of Lemma 3.6.2 we have*

$$\int_H \|y\|_{L^{\bar{q}}}^{\bar{q}} \mu^x(dy) < \infty, \qquad x \in L^{4m_1}(\mathcal{O}). \tag{3.107}$$

Proof. Due to the invariance of μ^x, for every $t \geq 0$, we have

$$\begin{aligned} \int_H \|y\|_{L^{\bar{q}}}^{\bar{q}} \mu^x(dy) \leq & \liminf_{R\to\infty} \int_H \left(\|y\|_{L^{\bar{q}}}^{\bar{q}} \wedge R\right) \mu^x(dy) \\ & = \liminf_{R\to\infty} \int_H P_t^x \left(\|\cdot\|_{L^{\bar{q}}}^{\bar{q}} \wedge R\right)(y)\, \mu^x(dy) \leq \int_H P_t^x \left(\|\cdot\|_{L^{\bar{q}}}^{\bar{q}}\right)(y)\, \mu^x(dy), \end{aligned}$$

where P_t^x is the semigroup associated with equation (3.16).

By proceeding as in the proof of Lemma 3.6.2, we have

$$P_t^x\left(\|\cdot\|_{L^{\bar q}}^{\bar q}\right)(y)=\mathbb{E}\,\|v^{x,y}(t)\|_{L^{\bar q}}^{\bar q}\le e^{-\vartheta t}\|y\|_{L^{\bar q}}^{\bar q}+c\left(1+\|x\|_{L^{4m_1}}^{4m_1}\right)+\sup_{s\in[0,t]}\mathbb{E}\|W_{A_2}(s)\|_{L^{\bar q}}^{\bar q},$$

and this implies

$$\int_H\|y\|_{L^{\bar q}}^{\bar q}\,\mu^x(dy)\le e^{-\vartheta t}\int_H\|y\|_{L^{\bar q}}^{\bar q}\,\mu^x(dy)+c_t\left(1+\|x\|_{L^{4m_1}}^{4m_1}\right).$$

Thus, if we pick $\bar t>0$ such that $e^{-\vartheta\bar t}\le 1/2$, we conclude that

$$\int_H\|y\|_{L^{\bar q}}^{\bar q}\,\mu^x(dy)\le 2c_{\bar t}\left(1+\|x\|_{L^{4m_1}}^{4m_1}\right),$$

and (3.107) follows.

□

3.6.6 About Condition (3.32)

According to (3.106) and (3.107), we have that

$$\bar V(x)\le c\left(1+\|x\|_{L^{4m_1}}^{4m_1}\right),\qquad x\in L^{4m_1}(\mathcal{O}).$$

Then, condition (3.32) holds if

$$\sup_{\epsilon\in(0,1)} \int_0^T \mathbb{E}^\epsilon \|u^\epsilon(s)\|^{4m_1}_{L^{4m_1}} dt < \infty,$$

and this is true because of (3.104).

Chapter 4: Existence of martingale solution of stochastic PDEs with rough nonlinear coefficient

In what follows, H is a separable Hilbert space, endowed with the scalar product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. We are considering the following equation in the Hilbert space H

$$dX(t) = [AX(t) + F(t, X(t))]\, dt + Q\, dW_t, \qquad X(0) = x, \tag{4.1}$$

where $W(t)$, $t \geq 0$, is a space-time white noise. In what follows, we shall assume that the linear operators A and Q and the non-linearity F satisfy the following assumptions.

Hypothesis 13. *1. The self-adjoint operator A generates an analytic semi-group $S(t)$.*

2. There exists a complete orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$ and two sequences

of positive real numbers $\{\alpha_k\}_{k\in\mathbb{N}}$ *and* $\{\lambda_k\}_{k\in\mathbb{N}}$ *such that*

$$Ae_k = -\alpha_k e_k, \qquad Qe_k = \lambda_k e_k, \qquad k \geq 1. \tag{4.2}$$

Moreover $\inf_{k\in\mathbb{N}} \alpha_k > 0$.

3. *There exists some* $\gamma_0 > 0$ *such that*

$$\sum_{k\in\mathbb{N}} \lambda_k^2 \alpha_k^{2\gamma_0 - 1} < \infty. \tag{4.3}$$

For any $\gamma > 0$, the operator $(-A)^{-\gamma}$ is compact and thus the embedding $D((-A)^\gamma) \hookrightarrow H$ is compact. Moreover, by the closed graph theorem, the set $\{x \in H : \|(-A)^\gamma x\| \leq R\}$ is not only precompact but also compact for any $\gamma > 0$ and $R > 0$. We denote the graph norm of the operator $(-A)^\gamma$ by $\|\cdot\|_\gamma$.

Hypothesis 14. *There exists a family of measurable and bounded mappings* $\{F^\theta\}_{\theta\in(0,1)}$ *defined on* $[0,T] \times H$ *with values in* H*, such that the following conditions hold.*

1. *For every* $\theta \in (0,1)$ *and* $h \in H$ *and for every* $t \in [0,T]$*, the mapping*

$$x \in H \mapsto \langle F^\theta(t,x), h\rangle \in \mathbb{R}$$

is continuous.

2. *There exists a convex and lower semi-continuous mapping* $V : H \to [1 + \infty]$ *such that for all* $\theta \in (0, 1)$

$$\|F^\theta(t, x)\|_H^2 \leq V(x), \qquad x \in H, \tag{4.4}$$

and

$$\|F(t, x) - F^\theta(t, x)\| \leq \theta V(x), \qquad x \in D(F). \tag{4.5}$$

Here and in what follows, we shall define $\|F(t, x)\| = +\infty$, whenever $x \notin D(F)$. In particular $D_V := \{x \in H : V(x) < \infty\} \subseteq D(F)$.

In what follows, we shall prove the following result.

Theorem 4.0.1. *Under Hypotheses 13 and 14, and Hypothesis 15 below, for every initial condition* $x \in D_V$ *there exists a martingale solution in* $C([0, T]; H)$ *for problem (4.1). That is, there exist some filtered probability space* $(\bar{\Omega}, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t\}_{t\in[0,T]}, \bar{\mathbb{P}})$*, a space-time white noise* $\bar{W}_t$*,* $t \in [0, T]$*, adapted to the filtration* $\{\bar{\mathcal{F}}_t\}_{t\in[0,T]}$*, and a* $C([0, T]; H)$*-valued process* $\bar{X}$ *such that*

$$\bar{X}(t) = S(t)x + \int_0^t S(t - s)F(s, \bar{X}(s, x))ds + \int_0^t S(t - s)Q\, d\bar{W}_s.$$

4.1 The approximating problem

Let $\{\theta_n\}_{n\in\mathbb{N}} \subset (0,1)$ be a sequence which converges to 0. We denote F^{θ_n} by F_n for simplicity and consider the following approximating problem

$$dX_n(t) = [AX_n(t)dt + F_n(t, X_n(t))]\, dt + QdW_t, \qquad X_n(0) = x \in D(F). \tag{4.6}$$

As a consequence of Hypotheses 13 and 14, equation (4.6) admits a martingale solution in $C([0,T];H)$ (see [39] and [36] for a proof). That is, there exist some probability space $(\Omega^n, \mathcal{F}^n, \{\mathcal{F}_t^n\}_{t\in[0,T]}, \mathbb{P}^n)$, a space-time white noise W_t^n, $t \in [0,T]$, adapted to the filtration $\{\mathcal{F}_t^n\}_{t\in[0,T]}$ and a $C([0,T];H)$-valued process X_n such that

$$X_n(t) = S(t)x + \int_0^t S(t-s)F_n(s, X_n(s,x))ds + W_A^n(t), \tag{4.7}$$

where $W_A^n(t)$ is the stochastic convolution

$$W_A^n(t) := \int_0^t S(t-s)Q\, dW_s^n. \tag{4.8}$$

Equivalently, X_n satisfies the following equation

$$\begin{aligned}\langle X_n(t), \xi(t)\rangle =& \langle x, \xi(0)\rangle + \int_0^t \langle X_n(s), \xi'(s) + A^*\xi(s)\rangle \, ds \\ &+ \int_0^t \langle F_n(s, X_n(s)), \xi(s)\rangle \, ds + \int_0^t \langle Q dW_s^n, \xi(s)\rangle,\end{aligned} \tag{4.9}$$

for any $\xi \in C^1([0,T]; D(A^*))$.

Hypothesis 15. *For every $x \in D(F)$, we have*

$$\sup_{n\in\mathbb{N}} \int_0^T \mathbb{E}^n V(X_n(t)) \, dt \leq c\, V(x) \tag{4.10}$$

Notice that, as a consequence of Hypothesis 15, we have that $\mathbb{P}^n(X_n(t) \in D(F),\ t \in [0,T]) = 1$, for all $n \in \mathbb{N}$. Moreover, as a consequence of Hypothesis 13, it is possible to show that for every $p \geq 1$ and $\gamma \in [0, \gamma_0)$

$$\sup_{n\in\mathbb{N}} \mathbb{E}^n \sup_{t\in[0,T]} \|W_A^n(t)\|_\gamma^p < \infty. \tag{4.11}$$

Moreover,

$$\sup_{n\in\mathbb{N}} \mathbb{E}^n \|W_A^n(t+h) - W_A^n(t)\|_\gamma^p \leq c\, h^{p\gamma_\star}, \tag{4.12}$$

where $\gamma_\star := (\gamma_0 - \gamma) \wedge 1/2$. In particular, from the Garsia-Rademich-Rumsey

Theorem we obtain that for every $\gamma \in [0, \gamma_0)$ there exists some $\beta > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}^n \|W_A^n\|^2_{C^\beta([0,T];D((-A)^\gamma))} < \infty. \tag{4.13}$$

For all details see e.g. [36, Section 5.4].

Now, we define

$$\Psi_n(t) := \int_0^t S(t-s)F_n(s, X_n(s))ds.$$

Lemma 4.1.1. *Let Hypotheses 13, 14 and 15 hold and assume that* $\gamma \in [0, 1/2)$. *Then*

$$\sup_{n\in\mathbb{N}} \mathbb{E}^n \sup_{t\in[0,T]} \|\Psi_n(t)\|^2_\gamma \le c_T V(x), \qquad x \in D_V. \tag{4.14}$$

Proof. We have

$$\begin{aligned}\mathbb{E}^n \sup_{t\in[0,T]} \|\Psi_n(t)\|^2_\gamma &\le c \sup_{t\in[0,T]} \left(\int_0^t (t-s)^{-2\gamma}\, ds\right) \mathbb{E}^n \sup_{t\in[0,T]} \int_0^t \|F_n(s, X_n(s))\|^2 ds \\ &\le c_T\, \mathbb{E}^n \int_0^T \|F_n(s, X_n(s))\|^2 ds.\end{aligned}$$

Then (4.14) follows from (4.4) and (4.10). □

As for the time continuity for Ψ_n, we have the following result.

Lemma 4.1.2. *Let Hypotheses 13, 14 and 15 hold and fix $\gamma \in [0, 1/2)$. Then there exists $\beta > 0$ such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}^n \left[\sup_{\delta\in(0,1)} \sup_{t,s\in[0,T],\ |t-s|\le\delta} \left(\frac{\|(\Psi_n(t) - \Psi_n(s))\|_\gamma}{|t-s|^\beta} \right)^2 \right] \le c\, V(x), \qquad x \in D_V.$$

Proof. For every $\gamma \in (0, 1/2)$, $h \in (0, \delta)$ and $t \in [0, T - \delta]$, we have

$$\begin{aligned} \|\Psi_n(t+h) - \Psi_n(t)\|_\gamma \le \quad & c \int_t^{t+h} \|S(t+h-s)F_n(s, X_n(s))\|_\gamma \, ds \\ & + c \int_0^t \|(S(t+h-s) - S(t-s))F_n(s, X_n(s))\|_\gamma \, ds, \end{aligned}$$

so that

$$\begin{aligned} & \|(\Psi_n(t+h) - \Psi_n(t))\|_\gamma^2 \le c \int_t^{t+h} (t+h-s)^{-2\gamma} \, ds \int_t^{t+h} \|F_n(s, X_n(s))\|^2 ds \\ & + c \left(\int_0^t \int_{t-s}^{t+h-s} \|AS(u)du \ F_n(s, X_n(s))\|_\gamma \, ds \right)^2 \\ & \le c \int_t^{t+h} (t+h-s)^{-2\gamma} ds \int_0^T \|F_n(s, X_n(s))\|^2 ds \\ & + c \int_0^t \left(\int_{t-s}^{t+h-s} \|(-A)^{\gamma+1} S(r)\|_{\mathcal{L}(H)} \, dr \right)^2 ds \int_0^T \|F_n(s, X_n(s))\|^2 \, ds. \end{aligned}$$

Therefore, in view of Hypothesis 15, we have

$$\begin{aligned}
&\mathbb{E}^n\left(\sup_{\delta\in(0,1)}\sup_{t\in[0,T-\delta]}\sup_{h\in(0,\delta)}\|(\Psi_n(t+h)-\Psi_n(t))\|_\gamma^2 h^{-2\beta}\right)\\
&\leq c_T\sup_{\delta\in(0,1)}\sup_{t\in[0,T-\delta]}\sup_{h\in(0,\delta)}h^{-2\beta}\int_t^{t+h}(t+h-s)^{-2\gamma}\,ds\,V(x)\\
&+c_T\sup_{h_0\in(0,1)}\sup_{t\in[0,T-h_0]}\sup_{h\in(0,h_0)}h^{-2\beta}\int_0^t\left(\int_{t-s}^{t+h-s}r^{-(\gamma+1)}\,dr\right)^2 ds\,V(x).
\end{aligned}$$

Since

$$\left((t-s)^{-\gamma}-(t+h-s)^{-\gamma}\right)^2\leq(t-s)^{-2\gamma}-(t+h-s)^{-2\gamma},$$

we have

$$\begin{aligned}
&\mathbb{E}^n\left(\sup_{\delta\in(0,1)}\sup_{t\in[0,T-\delta]}\sup_{h\in(0,\delta)}\|(\Psi_n(t+h)-\Psi_n(t))\|_\gamma^2\,h^{-2\eta}\right)\\
&\leq c_T\,V(x)\sup_{h\in(0,1)}\left(h^{-2\beta}h^{-2\gamma+1}+h^{-2\beta}\int_0^t\left((t+h-s)^{-2\gamma}-(t-s)^{-2\gamma}\right)ds\right).
\end{aligned}$$

Moreover, since $1-2\gamma<1$, we have

$$(t+h)^{-2\gamma+1}-t^{-2\gamma+1}\leq h^{-2\gamma+1}.$$

Therefore, if we take $\beta < 1/2 - \gamma$, we obtain

$$\mathbb{E}^n \left(\sup_{h_0 \in (0,1)} \sup_{t \in [0, T-h_0]} \sup_{h \in (0,h_0)} \|(\Psi_n(t+h) - \Psi_n(t))\|_\gamma^2 \, h^{-2\beta} \right)$$

$$\leq c_T \, V(x) \sup_{h \in (0,1)} h^{1-2\gamma-2\beta} \leq c_T \, V(x).$$

Notice that when $\gamma = 0$, the result holds for all $\beta \in (0, 1/2)$. □

For every $n \in \mathbb{N}$, we define

$$y_n(t) := \Psi_n(t) + W_A^n(t), \qquad t \in [0, T].$$

As a consequence of (4.13) and Lemmas 4.1.1 and 4.1.2, there exist $\gamma, \eta > 0$ such that

$$\sup_{n \in \mathbb{N}} \mathbb{E}^n \sup_{t \in [0,T]} \|y_n(t)\|_\gamma^2 < \infty, \tag{4.15}$$

and

$$\sup_{n \in \mathbb{N}} \mathbb{E}^n \sup_{t,s, \in [0,T]} \left(\frac{\|y_n(t) - y_n(s)\|}{|t-s|^\beta} \right)^2 < \infty. \tag{4.16}$$

This implies the following tightness result.

Lemma 4.1.3. *Under Hypotheses 13, 14 and 15, for every $x \in D_V$ the family of measures $\{\mathcal{L}(X_n)\}_{n \in \mathbb{N}}$ is tight in $C([0,T]; H))$.*

Proof. Let γ and η be the same as the ones in (4.15) and (4.16). For every $R > 0$, we introduce the set

$$K_R := \{u \in C([0,T];H) \ : \ \|u\|_{C([0,T];D((-A)^\gamma))} + \sup_{t,s\in[0,T]} \frac{\|u(t)-u(s)\|}{|t-s|^\beta} \leq R\}.$$

By Ascoli-Arzela theorem, K_R is compact in $C([0,T];H)$. Moreover, according to (4.15) and (4.16), we have

$$\sup_{n\in\mathbb{N}} \mathbb{P}^n \left(y_n \in K_R^c\right) \leq c_T R^{-1}.$$

Then, for every $\eta > 0$ we can find $R_\eta > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{P}^n \left(y_n \in K_{R_\eta}^c\right) \leq \eta.$$

Since $X_n = y_n + S(\cdot)x$, we have

$$\mathbb{P}^n \left(X_n \in K_M^c + S(\cdot)x\right) = \mathbb{P}^n \left(y_n \in K_M^c\right) \leq \eta.$$

Therefore, since $K_{R_\eta} + S(\cdot)x$ is a compact set in $C([0,T];H)$, due to the arbitrariness of η we conclude that the family of measures $\{\mathcal{L}(X_n)\}_{n\in\mathbb{N}}$ is

tight in $C([0,T];H)$. $\square$

4.2 Construction of a martingale solution

In view of Lemma 4.1.3, the sequence $\{(X_n, W_n)\}_{n\in\mathbb{N}} \subset C([0,T];H) \times C([0,T];\mathcal{D}')$ is tight. Due to the Prokhorov theorem, there exists a subsequence, still denoted by $\{(X_n, W_n)\}_{n\in\mathbb{N}}$, that converges weakly to some measure μ on $C([0,T];H) \times C([0,T];\mathcal{D}')$. By Skorohod's theorem we can find a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ and a sequence of random variables $\{(\bar{X}_n, \bar{W}^n)\}_{n\in\mathbb{N}}$, and $(\bar{X}, \bar{W})$ with values in $C([0,T];H) \times C([0,T];\mathcal{D}')$, such that

$$\mathcal{L}(\bar{X}_n, \bar{W}^n) = \mathcal{L}(X_n, W^n), \qquad \mathcal{L}(\bar{X}, \bar{W}) = \mu, \tag{4.17}$$

and

$$\lim_{n\to\infty} (\bar{X}_n, \bar{W}^n) = (\bar{X}, \bar{W}), \quad \bar{\mathbb{P}} - \text{a.s. in } C([0,T];H) \times C([0,T];\mathcal{D}'). \tag{4.18}$$

Notice that since $\mathcal{L}(\bar{W}^n) = \mathcal{L}(W^n)$ and W^n is a space-time white noise, for every $n \in \mathbb{N}$, then $\bar{W}$ is a space-time white noise.

For any $\xi \in C^1([0,T];D(A^*))$ and $t \in [0,T]$, we define

$$\begin{aligned}\bar{W}_\xi^n(t) := \langle \bar{X}_n(t),\xi(t)\rangle - \langle x,\xi(0)\rangle - \int_0^t \langle \bar{X}_n(s), \xi'(s) + A^*\xi(s)\rangle ds \\ - \int_0^t \langle F_n(s,\bar{X}_n(s)),\xi(s)\rangle ds.\end{aligned} \tag{4.19}$$

Recalling that X_n satisfies equation (4.9), thanks to (4.17), we have

$$\bar{W}_\xi^n(t) = \int_0^t \langle Q d\bar{W}_s^n, \xi(s)\rangle, \qquad t \in [0,T].$$

Hence, thanks to (4.18), we have

$$\lim_{n\to\infty} \bar{W}_\xi^n = \int_0^t \langle Q d\bar{W}_s, \xi(s)\rangle, \qquad \text{in } C([0,T]), \quad \bar{\mathbb{P}} - \text{a.s.} \tag{4.20}$$

Therefore, if we define $\bar{\mathcal{F}}_t := \sigma(\bar{W}_s,\ s \le t)$, for $t \in [0,T]$, we conclude that

$$(\bar{\Omega}, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t\}_{t\in[0,T]}, \bar{\mathbb{P}}, \bar{W}, \bar{X})$$

is a martingale solution for equation (4.1), once we prove the following result.

Lemma 4.2.1. *There exists a subsequence of $\{\bar{W}_\xi^n\}_{n\in\mathbb{N}}$, still indexed by n,*

such that

$$\lim_{n\to\infty} \bar{W}_\xi^n = \langle \bar{X}, \xi\rangle - \langle x, \xi(0)\rangle \\ - \int_0^{\cdot} \langle \bar{X}(t), \xi'(s) + A^*\xi(s)\rangle ds - \int_0^{\cdot} \langle F(s, \bar{X}(s)), \xi(s)\rangle ds, \tag{4.21}$$

in $C([0,T];H)$*,* $\bar{\mathbb{P}}$*-almost surely.*

Proof. Due to (4.18), we only need to prove the convergence of the fourth term on the right hand side of equation (4.19). We have

$$\liminf_{n\to\infty} \bar{\mathbb{E}} \sup_{t\in[0,T]} \left| \int_0^t \langle F_n(s, \bar{X}_n(s)), \xi(s)\rangle ds - \int_0^t \langle F(s, \bar{X}(s)), \xi(s)\rangle \, ds \right| \\ \leq \liminf_{n\to\infty} \bar{\mathbb{E}} \int_0^T |\langle F_n(s, \bar{X}_n(s)) - F(s, \bar{X}(s)), \xi(s)\rangle| \, ds.$$

Due to (4.5), for every $n \in \mathbb{N}$ and $s \in [0,T]$, we have

$$|\langle F_n(s, \bar{X}_n(s)) - F(s, \bar{X}(s)), \xi(s)\rangle|$$

$$\leq |\langle F_n(s, \bar{X}_n(s)) - F(s, \bar{X}_n(s)), \xi(s)\rangle| + |\langle F(s, \bar{X}_n(s)) - F(s, \bar{X}(s)), \xi(s)\rangle|$$

$$\leq c_\xi \, \theta_n \, V(\bar{X}_n(s)) + |\langle F(s, \bar{X}_n(s)) - F(s, \bar{X}(s)), \xi(s)\rangle|.$$

Therefore, thanks to (4.10) and (4.17),

$$\begin{aligned}&\liminf_{n\to\infty} \bar{\mathbb{E}} \sup_{t\in[0,T]} \left| \int_0^t \langle F_n(s,\bar{X}_n(s)),\xi(s)\rangle ds - \int_0^t \langle F(s,\bar{X}(s)),\xi(s)\rangle ds \right| \\ &\leq c_\xi\, \theta_n\, V(x) + \liminf_{n\to\infty} \bar{\mathbb{E}} \int_0^T \left| \langle F(s,\bar{X}_n(s)) - F(s,\bar{X}(s)),\xi(s)\rangle \right| ds.\end{aligned} \tag{4.22}$$

Now, for every $n, m \in \mathbb{N}$ and $s \in [0,T]$, we have

$$\langle F(s,\bar{X}_n(s)) - F(s,\bar{X}(s)),\xi(s)\rangle = \langle F(s,\bar{X}_n(s)) - F_m(s,\bar{X}_n(s)),\xi(s)\rangle$$

$$+\langle F_m(s,\bar{X}_n(s)) - F_m(s,\bar{X}(s)),\xi(s)\rangle + \langle F_m(s,\bar{X}(s)) - F(s,\bar{X}(s)),\xi(s)\rangle,$$

so that, thanks to (4.5)

$$\begin{aligned}&\int_0^T \left| \langle F(s,\bar{X}_n(s)) - F(s,\bar{X}(s)),\xi(s)\rangle \right| ds \leq c_\xi\, \theta_m \left(\int_0^T V(\bar{X}_n(s))\, ds + \int_0^T V(\bar{X}(s))\, ds \right) \\ &+ \int_0^T \left| \langle F_m(s,\bar{X}_n(s)) - F_m(s,\bar{X}(s)),\xi(s)\rangle \right| ds.\end{aligned} \tag{4.23}$$

In view of Hypothesis 15, for every $t \in [0,T]$ and $x \in D_V$, we have

$$\int_0^T \bar{\mathbb{E}}\, V(\bar{X}(t))\, dt \leq c\, V(x). \tag{4.24}$$

Actually, since V is lower semicontinuous, for every $t \in [0,T]$ we have

$$V(\bar{X}(t)) = V(\liminf_{n\to\infty} \bar{X}_n(t)) \leq \liminf_{n\to\infty} V(\bar{X}_n(t)).$$

Therefore,

$$\int_0^T \bar{\mathbb{E}}\, V(\bar{X}(t))\, dt \leq \liminf_{n\to\infty} \int_0^T \bar{\mathbb{E}}\, V(\bar{X}_n(t))\, dt = \liminf_{n\to\infty} \int_0^T \mathbb{E}^n\, V(X_n(t))\, dt,$$

and thanks to Hypothesis 15, this implies (4.24).

Now, (4.24), together with (4.23) and (4.22), imply

$$\liminf_{n\to\infty} \bar{\mathbb{E}} \sup_{t\in[0,T]} \left| \int_0^t \langle F_n(s, \bar{X}_n(s)), \xi(s)\rangle ds - \int_0^t \langle F(s, \bar{X}(s)), \xi(s)\rangle ds \right|$$

$$\leq c_\xi\, (\theta_n + \theta_m)\, V(x) + \liminf_{n\to\infty} \bar{\mathbb{E}} \int_0^T \left| \langle F_m(s, \bar{X}_n(s)) - F_m(s, \bar{X}(s)), \xi(s)\rangle \right| ds.$$

If we fix an arbitrary $\eta > 0$, we can find $m_\eta \in \mathbb{N}$ such that $c_\xi \theta_{m_\eta} V(x) < \eta$, so that

$$\liminf_{n\to\infty} \bar{\mathbb{E}} \sup_{t\in[0,T]} \left| \int_0^t \langle F_n(s, \bar{X}_n(s)), \xi(s)\rangle ds - \int_0^t \langle F(s, \bar{X}(s)), \xi(s)\rangle ds \right|$$

$$\leq \eta + c_\xi\, \theta_n\, V(x) + \liminf_{n\to\infty} \bar{\mathbb{E}} \int_0^T \left| \langle F_{m_\eta}(s, \bar{X}_n(s)) - F_{m_\eta}(s, \bar{X}(s)), \xi(s)\rangle \right| ds.$$

Since $F_{m_\eta}(t,\cdot) : H \to H$ is strong-weak continuous and bounded and (4.18) holds, we can take the limit above, as $n \to \infty$, and, due to the arbitrariness of η we conclude

$$\liminf_{n\to\infty} \bar{\mathbb{E}} \sup_{t\in[0,T]} \left| \int_0^t \langle F_n(s, \bar{X}_n(s)), \xi(s)\rangle ds - \int_0^t \langle F(s, \bar{X}(s)), \xi(s)\rangle ds \right| = 0.$$

Finally, by extracting possibly another subsequence again, we have

$$\lim_{n\to\infty} \sup_{t\in[0,T]} | \int_0^t \langle F_n(s, \bar{X}_n(s)), \xi(s)\rangle ds - \int_0^t \langle F(s, \bar{X}(s)), \xi(s)\rangle ds| = 0, \quad (4.25)$$

$\bar{\mathbb{P}}-$almost surely, and our lemma follows. □

Bibliography

[1] J. Birrell, S. Hottovy, G. Volpe, J. Wehr, *Small mass limit of a Langevin equation on a manifold*, Annales Henri Poincaré 18 (2017), pp. 707–755.

[2] S. Cerrai, M. Freidlin, *On the Smoluchowski-Kramers approximation for a system with an infinite number of degrees of freedom*, Probability Theory and Related Fields 135 (2006), pp. 363-394.

[3] S. Cerrai, M. Freidlin, *Smoluchowski-Kramers approximation for a general class of SPDE's*, Journal of Evolution Equations 6 (2006), pp. 657-689.

[4] S. Cerrai, M. Freidlin, *Small mass asymptotics for a charged particle in a magnetic field and longtime influence of small perturbations*, Journal of Statistical Physics 144 (2011), pp. 101–123.

[5] S. Cerrai, M. Freidlin, M. Salins, *On the Smoluchowski-Kramers approximation for SPDEs and its interplay with large deviations and long time behavior*, Discrete and Continuous Dynamical Systems, Series A, 37 (2017), pp. 33–76.

[6] S. Cerrai, M. Salins, *Smoluchowski-Kramers approximation and large deviations for infinite dimensional gradient systems*, Asymptotic Analysis 88 (2013), pp. 201-215.

[7] S. Cerrai, M. Salins, *Smoluchowski-Kramers approximation and large deviations for infinite dimensional non-gradient systems with applications to the exit problem*, Annals of Probability 44 (2016), pp. 2591–2642.

[8] S. Cerrai, M. Salins, *On the Smoluchowski-Kramers approximation for a system with an infinite number of degrees of freedom subject to a magnetic field*, Stochastic Processes and their Applications 127 (2017) pp. 273–303.

[9] M. Freidlin, *Some remarks on the Smoluchowski-Kramers approximation*, J. Statist. Phys. 117 (2004), pp. 617–634.

[10] M. Freidlin, W. Hu, *Smoluchowski–Kramers approximation in the case of variable friction*, Journal of Mathematical Sciences 179 (2011), pp. 184–207.

[11] M. Freidlin, A. Wentzell, Random perturbations of dynamical systems, third edition, Springer Verlag 2012.

[12] D. Herzog, S. Hottovy, G. Volpe, *The small-mass limit for Langevin dynamics with unbounded coefficients and positive friction*, Journal of Statistical Physics 163 (2016), pp. 659–673.

[13] S. Hottovy, A. McDaniel, G. Volpe, J. Wehr, *The Smoluchowski-Kramers limit of stochastic differential equations with arbitrary state-dependent friction*, Communications in Mathematical Physics 336 (2015), pp. 1259–1283.

[14] Khas'minskii, R.Z. *Ergodic properties of recurrent diffusion processed and stabilization of the solutions of the Cauchy problem for parabolic equations*, Teor.Veroatn.Ee Primen,.5(1960), 179-196.

[15] H. Kramers, *Brownian motion in a field of force and the diffusion model of chemical reactions*, Physica 7 (1940), pp. 284–304.

[16] Krylov, N.Y. and Safonov, M.V. *On a problem suggested by A.D. Wentzell. In The Dynkin Festschrift. Markov Processes and Their Applications, M.I.Freidlin*, Ed.Birkhauser: Boston,1994,209-220.

[17] J. J. Lee, *Small mass asymptotics of a charged particle in a variable magnetic field*, Asymptotic Analysis 86 (2014), pp. 99–121.

[18] Y. Lv, A. Roberts, *Averaging approximation to singularly perturbed nonlinear stochastic wave equations*, Journal of Mathematical Physics 53 (2012), pp. 1–11.

[19] Y. Lv, A. Roberts, *Large deviation principle for singularly perturbed stochastic damped wave equations*, Stochastic Analysis and Applications 32 (2014), pp. 50-60.

[20] Y. Lv, A. Roberts, W. Wang, *Approximation of the random inertial manifold of singularly perturbed stochastic wave equations*, Stochastics and Dynamics 32, 2014.

[21] Y. Lv, W. Wang, *Limiting dynamics for stochastic wave equations*, Journal of Differential Equations 244, (2008), pp. 1–23.

[22] H. Nguyen, *The small-mass limit and white-noise limit of an infinite dimensional Generalized Langevin Equation*, arXiv:1804.09682 (2018).

[23] M. Salins, *Smoluchowski-Kramers approximation for the damped stochastic wave equation with multiplicative noise in any spatial dimension*, arXiv:1801.10538 (2018).

[24] M. Smoluchowski, *Drei Vortage über Diffusion Brownsche Bewegung und Koagulation von Kolloidteilchen*, Physik Zeit. 17 (1916), pp. 557-585.

[25] K. Spiliopoulos, *A note on the Smoluchowski-Kramers approximation for the Langevin equation with reflection*, Stochastics and Dynamics 7 (2007), pp. 141–152.

[26] Y. Zhu, *A generalization of the Freidlin-Wentcell theorem on averaging for Hamiltonian systems*

[27] V.I. Bogachev, G. Da Prato, M. Rockner, *Fokker-Planck equations and maximal dissipativity for Kolmogorov operators with time dependent singular drifts in Hilbert spaces*, Journal of Functional Analysis, 256 (2009), pp. 1269–1298.

[28] V.I. Bogachev, G. Da Prato, M. Rockner, *Existence and uniqueness of solutions for Fokker-Planck equations on Hilbert spaces*, Journal of Evolutional Equations, 10 (2010), pp. 487–509.

[29] V.I. Bogachev, G. Da Prato, M. Rockner, *Uniqueness for solutions of Fokker-Planck equations on infinite dimensional spaces*, Communications in Partial Differential Equations, 36 (2011), pp. 925–939.

[30] S. Cerrai, Second Order PDE's in Finite and Infinite Dimension, A Probabilistic Approach, Springer Lecture Notes in Mathematics, (2001).

[31] S. Cerrai, *A Khasminskii type averaging principle for stochastic reaction-diffusion equations*, The Annals of Applied Probability, 19 (2009), pp. 899–948.

[32] S. Cerrai, *Averaging principle for systems of RDEs with polynomial nonlinearities perturbed by multiplicative noise*, Siam Journal of Mathematical Analysis, 43 (2011), pp. 2482–2518.

[33] S. Cerrai and M.Freidlin, *Averaging principle for a class of stochastic reaction-diffusion equations*, Probability Theory and Related Fields, 144 (2009) pp.137–177.

[34] S. Cerrai, A. Lunardi, *Averaging principle for non autonomous slow-fast systems of stochastic RDEs: the almost periodic case*, Siam Journal of Mathematical Analysis, 49 (2017), pp. 2843–2884.

[35] G. Da Prato, M. Rockner, *Singular dissipative stochastic equations in Hilbert spaces*, Probability Theory Related Fields, 143 (2009), pp. 659–664.

[36] G. Da Prato, J, Zabczyk, Stochastic Equations in Infinite dimensions, Cambridge University Press, (2014).

[37] A. Figalli, *Existence and uniqueness of martingale solutions for SDEs with rough or degenerate coefficients*, Journal of Functional Analysis, 254 (2008), pp. 109–153.

[38] B. Goldys, M. Rockner, X. Zhang, *Martingale solutions and Markov selections for stochastic partial differential equations*, Stochastic Processes and their Applications, 119 (2009), pp. 1725–1764.

[39] D. Gatarek, B. Goldys, *On weak solutions of stochastic equations in Hilbert spaces*, Stochastic Stochastics Reports, 46 (1992), pp. 41–51.

[40] J.L. Lasry and P.L. Lions, *A remark on regularization in Hilbert spaces*, Israel Journal of Mathematics, 55 (1996), pp. 257–266.

[41] C. Le Bris, P.L. Lions, *Existence and uniqueness of solutions to Fokker-Planck type equations with irregular coefficients*, Communications of Partial Differential Equations, 33 (2008), pp. 1272–1317.

[42] A. Lunardi, Analytic semigroups and optimal regularity in parabolic problems, Birkhäuser, 1995.

[43] L. Manca, *Fokker-Planck equation for Kolmogorov operators with unbounded coefficients*, Stochastic Analysis and Applications, 27 (2009), pp. 747–769.

[44] L. Manca, *Kolmogorov equations for measures*, Journal of Evolution Equations, 8 (2008), pp. 231–262.