ABSTRACT

Title of dissertation: STABLE PAIR THEORY ON TORIC ORBIFOLDS AND COLORED REVERSE PLANE PARTITIONS

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We give a GIT construction for the moduli space of stable pairs on projective stacks, and study PT invariants on orbiflod toric Calabi-Yau threefolds with transverse A_{n-1} singularities. The basic combinatorial object is the orbifold PT vertex $W^n_{\lambda\mu\nu}$. In the 1-leg case, $W^n_{\lambda\mu\nu}$ is the generating function for the number of $\mathbb{Z}_n$-colored reverse plane partitions, and we derive an explicit formula for $W^n_{\lambda\mu\nu}$ in terms of Schur functions. We also explicitly compute the PT partition function and verify the orbifold DT/PT correspondence for the local football $\mathrm{Tot}\left(\mathcal{O}(-p_0) \oplus \mathcal{O}(-p_\infty) \to \mathbb{P}^1_{a,b}\right)$.

STABLE PAIR THEORY ON TORIC ORBIFOLDS AND COLORED REVERSE PLANE PARTITIONS

by

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Table of Contents

Chapter 1: Introduction

Curve counting on Calabi-Yau threefolds was motivated by string theory. Let X be a smooth projective Calabi-Yau threefold over $\mathbb{C}$, i.e. $K_X = \bigwedge^3 \Omega_X \simeq \mathcal{O}_X$. Let $C \subset X$ be a nonsingular embedded curve of genus g. The Calabi-Yau condition implies that the expected dimension [32] of the space $\mathcal{M}_g(X)$ of projective nonsingular curves of genus g embedded in X is 0. To obtain a well-defined invariant, we need to compactify the space $\mathcal{M}_g(X)$. There are three main ways: Gromov-Witten (GW) theory, Donaldson-Thomas (DT) theory, and Pandharipande-Thomas (PT) theory.

In GW theory, curves are viewed as algebraic maps

$$f : C \to X.$$

The compactification strategy is to allow nodal singularities in the domain. Let $\beta \in H_2(X, \mathbb{Z})$. The moduli space of stable maps

$$\overline{\mathcal{M}}_g(X, \beta) = \left\{ f : C \to X \;\middle|\; \begin{array}{c} C \text{ is a nodal curve of arithmetic genus } g, \\ f_*[C] = \beta, \text{ and } \operatorname{Aut}(f) \text{ is finite} \end{array} \right\}.$$

is a compact Deligne-Mumford stack [15]. The moduli space admits a virtual fundamental class $[\overline{\mathcal{M}}_g(X,\beta)]^{\mathrm{vir}}$ of virtual dimension 0 [3], and integration along it defines the GW invariant

$$N_{g,\beta} = \int_{[\overline{\mathcal{M}}_g(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Q}.$$

GW invariants play a crucial role in closed type IIA string theory.

To illustrate DT theory and PT theory, we consider a family of curves

$$C_t = \{x = 0 = z\} \cup \{y = 0 = z - t\} \subset \mathbb{C}^3,\ t \neq 0.$$

Each curve C_t has two components: the y-axis, and a shift of the x-axis in the z-direction by t. The ideal of C_t is

$$I_t = (x, z) \cdot (y, z - t) = (xy, zy, x(z - t), z(z - t)),$$

which fits into a short exact sequence

$$0 \to I_t \to \mathbb{C}[x, y, z] \xrightarrow{s_t} \mathbb{C}[x, y, z]/(x, z) \oplus \mathbb{C}[x, y, z]/(y, z - t) \to 0.$$

In DT theory, we identify C_t with I_t and let $t \to 0$:

$$I_t \to I_0 = (xy, xz, yz, z^2).$$

The limit curve is $\{xy = 0 = z\}$ with a scheme-theoretic embedded point at the origin. In PT theory, we identify C_t with s_t and let $t \to 0$:

$$\mathbb{C}[x,y,z] \xrightarrow{s_0} \mathbb{C}[x,y,z]/(x,z) \oplus \mathbb{C}[x,y,z]/(y,z).$$

The kernel of s_0 is the ideal of the curve $\{xy = 0 = z\}$, and the cokernel is supported on the origin. $(\mathbb{C}[x,y,z]/(x,z) \oplus \mathbb{C}[x,y,z]/(y,z), s_0)$ is an example of stable pair.

Let X be a smooth Calabi-Yau threefold. Fix $\beta \in H_2(X,\mathbb{Z})$ and $n \in \mathbb{Z}$. The Hilbert scheme $I_n(X,\beta)$ parameterizes subschemes Z of X in the class $[Z] = \beta$ with holomorphic Euler characteristic $\chi(\mathcal{O}_Z) = n$. $I_n(X,\beta)$ is projective and has a symmetric obstruction theory [25,34] by viewing $I_n(X,\beta)$ as a moduli space of ideal sheaves I_Z. The associated virtual fundamental class $[I_n(X,\beta)]^{\mathrm{vir}}$ [5] has dimension 0, and integration along it defines the DT invariant

$$I_{n,\beta} = \int_{[I_n(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

The DT partition function is defined as

$$DT_\beta(X,q) = \sum_n I_{n,\beta} q^n$$

The degree 0 DT partition function $DT_0(X,q)$ counts 0-dimensional subschemes of X. It was conjectured in [25, Conjecture 1] and proved in [6, Theorem 4.12] that

$$DT_0(X,q) = M(-q)^{\chi_{top}(X)},$$

where $\chi_{top}(X)$ is the topological Euler characteristic of X, and

$$M(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-n}$$

is the MacMahon function counting 3D partitions. The reduced DT partition function is defined as

$$DT'_\beta(X, q) = \frac{DT_\beta(X, q)}{DT_0(X, q)}.$$

It is a Laurent series with integral coefficients.

Pandharipande and Thomas [31] introduced a new curve-counting theory via stable pairs. The moduli space $P_n(X, \beta)$ parameterizes stable pairs $\mathcal{O}_X \xrightarrow{s} F$ with $[F] = \beta$ and $\chi(F) = n$, where F is a pure 1-dimensional sheaf and $s \in H^0(X, F)$ is a section with 0-dimensional cokernel. $P_n(X, \beta)$ is a projective scheme as a special case of the work of Le Potier [22], and it has a symmetric obstruction theory [31] by viewing stable pairs as two term complexes in the derived category $D^b(X)$. The PT invariant is defined by integration of the dimension 0 virtual fundamental class,

$$P_{n,\beta} = \int_{[P_n(X,\beta)]^{\text{vir}}} 1 \in \mathbb{Z}.$$

The PT partition function is defined as

$$PT_\beta(X, q) = \sum_n P_{n,\beta} q^n.$$

Roughly speaking, we can think of $I_n(X, \beta)$ as parameterizing Cohen-Macaulay curves plus free and embedded points on X: any $Z \in I_n(X,\beta)$ contains a maximal Cohen-Macaulay curve $C \subset Z$ such that the kernel of $\mathcal{O}_Z \twoheadrightarrow \mathcal{O}_C$ is 0-dimensional. Similarly, we can think of $P_n(X,\beta)$ as parameterizing Cohen-Macaulay curves (the support of F) and free points on the curve (the cokernel of the section s). The DT/PT correspondence

$$DT'_\beta(X,q) = PT_\beta(X,q)$$

was conjectured by Pandharipande and Thomas [31, Conjecture 3.3].

In [4] Behrend associates to any scheme of finite type over $\mathbb{C}$ a constructible function

$$\nu_S : S \to \mathbb{Z},$$

with the property [4, Theorem 4.18] that if S is a proper scheme with a symmetric obstruction theory, then the associated virtual counting invariant $\int_{[S]^{\mathrm{vir}}} 1$ coincides with the weighted Euler characteristic

$$\widetilde{\chi}(S,\nu_S) := \sum_{n\in\mathbb{Z}} n\chi_{top}(\nu_S^{-1}(n)).$$

Using motivic Hall algebra, Bridgeland [7, Theorem 1.1] proved the DT/PT correspondence and showed that $DT'_\beta(X,q)$ is the Laurent expansion of a rational function invariant under the transformation $q \leftrightarrow q^{-1}$, which was conjectured in [25].

Another consequence of Behrend [4] is that we can define DT (resp., PT) invariants for smooth quasi-projective Calabi-Yau threefolds by the weighted Euler characteristic of $I_n(X,\beta)$ (resp., $P_n(X,\beta)$). When X is toric with torus $T = (\mathbb{C}^*)^3$, the T-fixed points of the Hilbert scheme $I_n(X,\beta)$ are isolated. The DT invariant is a given by a signed count of T-fixed points [6, Theorem 3.4],

$$I_{n,\beta} = \sum_{p\in I_n(X,\beta)^T} (-1)^{\dim T_p I_n(X,\beta)}.$$

An explicit formula in terms of β, n, and the geometry of X can be found in [25, Theorem 2]. The study of T-fixed curves in X naturally leads to the notion of the DT vertex $V_{\lambda\mu\nu}$ [31, Section 5.2] which enumerates monomial ideals of $\mathbb{C}[x_1,x_2,x_3]$. Here, (λ,μ,ν) is a triple of partitions. Combinatorially,

$$V_{\lambda\mu\nu}(q) = \sum_{\pi} q^{|\pi|},$$

where the sum is over all 3D partitions π asymptotic to (λ,μ,ν) (see Definition 5.16, 5.17). Okounkov, Reshetikhin and Vafa [28] derived an explicit formula in terms of Schur functions for $V_{\lambda\mu\nu}(q)$ (see Proposition 5.18). In particular,

$$V_{\emptyset\emptyset\emptyset}(q) = M(q)$$

recovers the MacMahon function.

In PT theory, each component $Q \subset P_n(X,\beta)^T$ in the T-fixed loci is a product of $\mathbb{P}^1$'s [30, Theorem 1]. Locally on $\mathbb{C}^3$, each component Q_π corresponds to a labelled box configuration $\pi \subset \mathbb{Z}^3$ [30, Section 2.5]. The corresponding PT vertex was conjectured in [30, (5-3)] to be

$$W_{\lambda\mu\nu}(q) = \sum_{Q_\pi} \chi_{top}(Q_\pi) q^{|\pi|},$$

where the sum is over all labelled box configurations with outgoing partitions (λ, μ, ν). This was proved in the 1-leg or 2-leg case, i.e. at least one of λ, μ, and ν is empty because the T-fixed loci are isolated. The DT/PT vertex correspondence was conjectured in [31, Conjecture 5.1],

$$W_{\lambda\mu\nu}(q) = \frac{V_{\lambda\mu\nu}(q)}{V_{\emptyset\emptyset\emptyset}(q)}.$$

Bryan, Cadman and Young [10] studied DT theory for a toric orbifold Calabi-Yau 3-fold $\mathcal{X}$ with transverse A_{n-1} singularities (see Definition 4.1 and Section 4.4). The local model for $\mathcal{X}$ is $[\mathbb{C}^3/\mathbb{Z}_n]$ where $\mathbb{Z}_n$ acts on $\mathbb{C}^3$ with weights $(1, -1, 0)$. Let $K(\mathcal{X})$ be the Grothendieck group of compactly supported coherent sheaves on $\mathcal{X}$ up to numerical equivalence. There is a filtration

$$F_0K(\mathcal{X}) \subset F_1K(\mathcal{X}) \subset F_2K(\mathcal{X}) \subset F_3(\mathcal{X})$$

given by the dimension of the support. Given $\beta \in F_1K(\mathcal{X})$, the moduli space $\mathrm{Hilb}^\beta(\mathcal{X})$ parameterizes substacks $\mathcal{Z} \subset \mathcal{X}$ having $[\mathcal{O}_{\mathcal{Z}}] = \beta$. It is a quasi-projective scheme [29, Theorem 1.5]. The DT invariant $DT_\beta(\mathcal{X})$ is defined as the topological Euler characteristic of $\mathrm{Hilb}^\beta(\mathcal{X})$ weighted by the Behrend's function. It is given by a signed count of T-fixed points [10, Lemma 13], and is evaluated in [10, Theorem

25]. The central object is the orbifold DT vertex $V^n_{\lambda\mu\nu}$ (see Definition 5.19), which is a generating function for 3D partitions, colored by representations of $\mathbb{Z}_n$, and asymptotic to (λ, μ, ν). An explicit formula in terms of Schur functions for $V^n_{\lambda\mu\nu}$ is given in [10, Theorem 12]. As an example, they computed the reduced DT partition function for the local football $\mathcal{X}_{a,b}$ [10, Proposition 3],

$$DT'(\mathcal{X}_{a,b}) = \prod_{k=1}^{a}\prod_{l=1}^{b}\prod_{m=1}^{\infty}(1 - vp_k \cdots p_{a-1}r_l \cdots r_{b-1}(-q)^m)^m.$$

We will study PT theory on a toric CY3 with transverse A_{n-1} singularities. We will use the orbifold PT vertex $W^n_{\lambda\mu\nu}$ to compute/conjecture the PT invariants following the work of [10,30]. In the 1-leg case, we derive an explicit formula for the orbifold PT vertex (Theorem 5.22). As an example, we compute the PT partition function $PT(\mathcal{X}_{a,b})$ (Proposition 4.20), and verify the orbifold DT/PT correspondence for the local football $\mathcal{X}_{a,b}$.

This paper is organized as follows. In Chapter 2 we review the theory of semistable sheaves on projective stacks following [26]. Let $\pi : \mathcal{X} \to X$ be a projective Deligne-Mumford stack over $\mathbb{C}$ with moduli scheme X. We fix a polarization $\mathcal{O}_X(1)$ on X and a generating sheaf $\mathcal{E}$ on $\mathcal{X}$. By Definition 2.1, $\mathcal{E}$ is a locally free sheaf on $\mathcal{X}$ whose fibre at each geometric point of $x \in \mathcal{X}$ contains the regular representation of the stabilizer group at x. Moreover, there is an exact functor

$$F_{\mathcal{E}} : \mathrm{Coh}(\mathcal{X}) \to \mathrm{Coh}(X),\ \mathcal{F} \mapsto \pi_*(\mathcal{F} \otimes \mathcal{E}^\vee).$$

In [26] Nironi introduced the modified Hilbert polynomial:

$$P_{\mathcal{E}}(\mathcal{F}, m) = \chi(X, F_{\mathcal{E}}(\mathcal{F})(m)),$$

and used this to define Gieseker stability condition in the usual way. Notice that the stability condition depends on both $\mathcal{O}_X(1)$ and $\mathcal{E}$. Nironi constructed the moduli space of semistable sheaves on $\mathcal{X}$ with modified Hilbert polynomial P as a quotient stack $[Q/GL(N)]$ [26, Theorem 5.1].

In Chapter 3 we study the moduli space of stable pairs on projective stacks. The main references are [18, 22, 23, 26]. Let P be a polynomial of degree d. A stable pair $(\mathcal{F}, s)$ consists of a pure coherent sheaf $\mathcal{F}$ with modified Hilbert polynomial $P_{\mathcal{E}}(\mathcal{F}) = P$ and a section $s : \mathcal{O}_{\mathcal{X}} \to \mathcal{F}$ with $\dim \operatorname{Coker} s < d$. When $d = 1$, this is the stable pair studied by Pandharipande and Thomas [31]. In [22] Le Potier introduced a different notion of stability. Let δ be a polynomial with $\deg \delta \geq \deg P$. The (reduced) Hilbert polynomial of a pair $(\mathcal{F}, s)$ is defined as

$$P_{\mathcal{E}}(\mathcal{F}, s) = P_{\mathcal{E}}(\mathcal{F}) + \epsilon(s)\delta \quad \left(p_{\mathcal{E}}(\mathcal{F}, s) = p_{\mathcal{E}}(\mathcal{F}) + \epsilon(s)\frac{\delta}{r_{\mathcal{E}}(\mathcal{F})}\right),$$

where $\epsilon(s) = 1$ if $s \neq 0$ and $\epsilon(s) = 0$ if $s = 0$. The δ-(semi)stability is defined as the Giesker (semi)stability with respect to $p_{\mathcal{E}}(\mathcal{F}, s)$. for nondegenerate ($s \neq 0$) pairs $(\mathcal{F}, s)$, we show that there is no strictly δ-semistable pairs and the two stability conditions are equivalent (Lemma 3.7). Using GIT, we have

Theorem 1.1. *Let $(\mathcal{X}, \mathcal{E}, \mathcal{O}_X(1))$ be a polarized smooth projective stack over $\mathbb{C}$.*

The moduli space $\mathcal{M}_{\mathcal{X}}(P)$ parameterizing stables pairs $(\mathcal{F}, s)$ with $P_{\mathcal{E}}(\mathcal{F}) = P$ is represented by a projective scheme $M_{\mathcal{X}}(P)$.

In Chapter 4 we study PT invariants on an orbifold toric CY3 with transverse A_{n-1} singularities following [10, 30]. Associated to an orbifold toric CY3 $\mathcal{X}$ is a trivalent graph whose vertices are the torus fixed points and whose edges are the torus invariant curves. There is additional data at the vertices describing the stabilizer group of the fixed points and there is additional data at the edges giving the degrees of the normal bundles to the fixed curves. The PT partition function $PT(\mathcal{X})$ is shown (see (4.6)) in the 1-leg and 2-leg cases, and conjectured in the 3-leg case to have the form

$$PT(\mathcal{X}) = \sum_{\substack{\text{edge} \\ \text{assignments}}} \prod_{e \in \text{Edges}} E(e) \prod_{v \in \text{Vertices}} \widehat{W}^{n}_{\lambda\mu\nu}(v)$$

where the sum is over all ways of assigning partitions to the edges. The edge terms $E(e)$ depend on the normal bundle of the corresponding curve and the partition assigned to the edge. The vertex terms $\widehat{W}^{n}_{\lambda\mu\nu}$ are given by the orbifold PT vertex $W^{n}_{\lambda\mu\nu}$ modified by certain signs of the variables. In the 1-leg case, $W^{n}_{\lambda\mu\nu}$ is the generating function for the number of $\mathbb{Z}_n$-colored reverse plane partitions. We have:

Theorem 1.2. *Let λ be a partition, then*

$$W^n_{\lambda\emptyset\emptyset}(q_0, \cdots, q_{n-1}) = q^{-A_\lambda}\overline{s_{\lambda^t}(\mathfrak{q})},$$

$$W^n_{\emptyset\lambda\emptyset}(q_0, \cdots, q_{n-1}) = \overline{q^{-A_{\lambda^t}}} s_\lambda(\mathfrak{q}),$$

$$W^n_{\emptyset\emptyset\lambda}(q_0, \cdots, q_{n-1}) = \prod_{\square\in\lambda} \frac{1}{1 - \prod_{a=0}^{n-1} q_a^{h_a(\square)}},$$

where

$$q^{-A_\lambda} = \prod_{a=0}^{n-1} q_a^{-A_\lambda(a,n)}, \, A_\lambda(a, n) = \sum_{(j,k)\in\lambda} \left\lfloor \frac{j+a}{n} \right\rfloor,$$

$s_\lambda(\mathfrak{q})$ *is the Schur function with* $\mathfrak{q} = (1, q_1, q_1q_2, q_1q_2q_3, \cdots)$, $h_a(\square)$ *denotes the number a-colored boxes in the hook of* $\square$, *and the overline denotes the exchange of variables* $q_a \leftrightarrow q_{-a}$.

As an example, we compute and verify the orbifold DT/PT correspondence $PT(\mathcal{X}_{a,b}) = DT'(\mathcal{X}_{a,b})$ for the local football $\mathcal{X}_{a,b}$ in Section 4.5.

In Chapter 5 we first review partitions and Schur functions following [24, 33] and vertex operators following [27]. We then prove Theorem 1.2 using vertex operators.

Chapter 2: Stable sheaves on projective stacks

2.1 Projective stacks

We work over the field of complex numbers $\mathbb{C}$. Every scheme is assumed to be Noetherian. Let S be a base scheme of finite type over $\mathbb{C}$. By Deligne-Mumford S-stack we mean a separated Noetherian Deligne-Mumford stack $\mathcal{X}$ of finite type over S. When $S = \operatorname{Spec}\mathbb{C}$, we omit the letter S. Under these assumptions, $\mathcal{X}$ has a coarse moduli space $\pi : \mathcal{X} \to X$ and the natural map $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism [20]. If X is a scheme, we call it a coarse moduli scheme. We recall the following properties of Deligne-Mumford S-stacks:

- since we work over $\mathbb{C}$, $\mathcal{X}$ is *tame*, i.e. the functor $\pi_* : \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(X)$ is exact and maps coherent sheaves to coherent sheaves [2, Lemma 2.3.4];
- if $X' \to X$ is a morphism of algebraic spaces, then X' is the coarse moduli space of $\mathcal{X} \times_X X'$ [1, Cor 3.3];
- $H^\bullet(\mathcal{X}, \mathcal{F}) \cong H^\bullet(X, \pi_*\mathcal{F})$ for any quasi-coherent sheaf $\mathcal{F}$ [26, Lemma 1.10];
- $\pi_*\mathcal{F}$ is an S-flat coherent sheaf on X whenever $\mathcal{F}$ is an S-flat coherent sheaf on $\mathcal{X}$ [26, Cor 1.3].

Let $\mathcal{X}$ be a Deligne-Mumford S-stack with coarse moduli space $\pi : \mathcal{X} \to X$. For any locally free sheaf $\mathcal{E}$ on $\mathcal{X}$, we have two functors

$$F_{\mathcal{E}} : \ \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(X), \ \mathcal{F} \mapsto \pi_*(\mathcal{F} \otimes \mathcal{E}^\vee),$$

$$G_{\mathcal{E}} : \ \mathrm{QCoh}(X) \to \mathrm{QCoh}(\mathcal{X}), \ \mathcal{G} \mapsto \pi^*\mathcal{G} \otimes \mathcal{E}.$$

The functor $F_{\mathcal{E}}$ is exact since $\mathcal{E}^\vee$ is locally free and the pushforward π_* is exact.

Definition 2.1. A locally free sheaf $\mathcal{E}$ is said to be a *generator* for the quasi-coherent sheaf $\mathcal{F}$ if the adjunction morphism (left adjoint of the identity $\mathrm{id} : \pi_*(\mathcal{F} \otimes \mathcal{E}^\vee) \to \pi_*(\mathcal{F} \otimes \mathcal{E}^\vee)$):

$$\theta_{\mathcal{E}}(\mathcal{F}) : G_{\mathcal{E}} \circ F_{\mathcal{E}}(\mathcal{F}) \to \mathcal{F}$$

is surjective. It is a *generating sheaf* for $\mathcal{X}$ if it is a generator for every quasi-coherent sheaf on $\mathcal{X}$. Equivalently, a locally free sheaf $\mathcal{E}$ on $\mathcal{X}$ is a generating sheaf if and only if the fibre of $\mathcal{E}$ at each geometric point of $\mathcal{X}$ contains the regular representation of the stabilizer group at that point [29, Theorem 5.2].

Olsson and Starr [29, Section 5] proved that a generating sheaf exists and is stable under base change for tame Deligne-Mumford stacks which are separated global quotients. In particular, we have

Proposition 2.2. (1) *Let $\mathcal{X}$ be a Deligne-Mumford S-stack which is a separated global S-quotient. Then $\mathcal{X}$ has a generating sheaf $\mathcal{E}$.*

(2) *Let $\pi : \mathcal{X} \to X$ be the moduli space of $\mathcal{X}$ and $f : X' \to X$ a morphism of*

algebraic spaces. Denote $p : \mathcal{X}' := \mathcal{X} \times_X X' \to \mathcal{X}$ the natural projection, then $p^*\mathcal{E}$ is a generating sheaf for $\mathcal{X}'$.

Now we are ready to give the definition of projective stack.

Definition 2.3 ([21, Theorem 5.3])**.** Let $\mathcal{X}$ be a Deligne-Mumford stack over $\mathbb{C}$. We say $\mathcal{X}$ is a (*quasi-*)*projective stack* if it satisfies any of the following equivalent conditions:

1. $\mathcal{X}$ admits a (locally) closed embedding into a smooth proper Deligne-Mumford stack with a projective moduli scheme.
2. $\mathcal{X}$ has a (quasi-)projective coarse moduli scheme and a generating sheaf.
3. $\mathcal{X}$ is a separated global quotient with a coarse moduli space which is a (quasi-)projective scheme.

Let $\pi : \mathcal{X} \to X$ be a projective stack. A *polarization* for $\mathcal{X}$ is a pair $(\mathcal{E}, \mathcal{O}_X(1))$, where $\mathcal{E}$ is a generating sheaf and $\mathcal{O}_X(1)$ is a very ample line bundle on X.

A relative version of the notion of projective stacks is defined as follows:

Definition 2.4. Let $p : \mathcal{X} \xrightarrow{\pi} X \xrightarrow{\rho} S$ be a Deligne-Mumford S-stack which is a separated global S-quotient with coarse moduli scheme X such that $\rho : X \to S$ is a projective morphism. We call $p : \mathcal{X} \to S$ a family of projective stacks.

Remark 2.5. For any geometric point s of S, we have the following cartesian diagram

$$\begin{array}{ccccc} \mathcal{X}_s & \xrightarrow{\pi_s} & X_s & \xrightarrow{\rho_s} & s \\ \downarrow & \square & \downarrow & \square & \downarrow \\ \mathcal{X} & \xrightarrow{\pi} & X & \xrightarrow{\rho} & S. \end{array}$$

Since $\rho : X \to S$ is projective, X_s is a projective scheme. Moreover, the properties of being a separated global quotient and being a coarse moduli space are invariant under base change, so each $\mathcal{X}_s$ is a projective stack.

2.2 Gieseker stability

In this section, we briefly recall some facts about the concept of *Gieseker stability* on projective stacks following [26, Section 3]. Let $\mathcal{X}$ be a projective stack over $\mathbb{C}$ with coarse moduli scheme $\pi : \mathcal{X} \to X$. We fix a polarization $(\mathcal{E}, \mathcal{O}_X(1))$ on $\mathcal{X}$.

Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$, we define the *support* $\mathrm{Supp}(\mathcal{F})$ of $\mathcal{F}$ to be the closed substack associated to the ideal $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_{\mathcal{X}} \to \mathscr{E}nd(\mathcal{F}))$. The dimension $\dim \mathcal{F}$ is the dimension of its support.

Definition 2.6. We say a coherent sheaf $\mathcal{F}$ is *pure* of dimension d if for any nonzero subsheaf $\mathcal{G}$ of $\mathcal{F}$ the support of $\mathcal{G}$ is of pure dimension d.

As it was shown in [26], every coherent sheaf $\mathcal{F}$ has the torsion filtration:

$$0 \subset T_0(\mathcal{F}) \subset \cdots \subset T_{\dim \mathcal{F}-1}(\mathcal{F}) \subset T_{\dim \mathcal{F}}(\mathcal{F}) = \mathcal{F}$$

where every factor $T_i(\mathcal{F})/T_{i-1}(\mathcal{F})$ is pure of dimension i or zero.

Definition 2.7. The *saturation* of a subsheaf $\mathcal{G} \subset \mathcal{F}$ is the minimal subsheaf $\bar{\mathcal{G}}$ containing $\mathcal{G}$ such that $\mathcal{F}/\bar{\mathcal{G}}$ is pure of dimension d or zero, i.e. the kernel of the surjection

$$\mathcal{F} \to \mathcal{F}/\mathcal{G} \to (\mathcal{F}/\mathcal{G})/T_{d-1}(\mathcal{F}/\mathcal{G}).$$

Lemma 2.8 ([26, Lemma 3.4]). *Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$, then we have*

$$\pi(\mathrm{Supp}(\mathcal{F})) = \pi(\mathrm{Supp}(\mathcal{F} \otimes \mathcal{E}^\vee)) \supseteq \mathrm{Supp}(F_{\mathcal{E}}(\mathcal{F})).$$

Moreover, $F_{\mathcal{E}}(\mathcal{F}) = 0$ if and only if $\mathcal{F} = 0$.

The functor $F_{\mathcal{E}}$ preserves dimension and pureness.

Proposition 2.9 ([26, Proposition 3.6]). *Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$, then*

(1) $\dim(\mathcal{F}) = \dim(F_{\mathcal{E}}(\mathcal{F}))$;

(2) *$\mathcal{F}$ is pure if and only if $F_{\mathcal{E}}(\mathcal{F})$ is pure.*

The functor $F_{\mathcal{E}}(\mathcal{F})$ preserves torsion filtration.

Corollary 2.10 ([26, Cor 3.7]). *Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$ of dimension d. Consider the torsion filtration*

$$0 = T_0(\mathcal{F}) \subset \cdots \subset T_{d-1}(\mathcal{F}) \subset T_d(\mathcal{F}) = \mathcal{F}.$$

Then

$$0 = F_{\mathcal{E}}(T_0(\mathcal{F})) \subset \cdots \subset F_{\mathcal{E}}(T_{d-1}(\mathcal{F})) \subset F_{\mathcal{E}}(T_{\dim \mathcal{F}}(\mathcal{F})) = F_{\mathcal{E}}(\mathcal{F})$$

is the torsion filtration of $F_{\mathcal{E}}(\mathcal{F})$.

Corollary 2.11 ([26, Cor 3.8]). *Let $\mathcal{F}$ be a pure sheaf on $\mathcal{X}$, then* $\pi(\mathrm{Supp}(\mathcal{F})) = \mathrm{Supp}(F_{\mathcal{E}}(\mathcal{F}))$.

For pure coherent sheaves on $\mathcal{X}$, the functor $F_{\mathcal{E}}$ preserves supports.

Definition 2.12. Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$ of dimension d, we define the *modified Hilbert polynomial* of $\mathcal{F}$ as

$$P_{\mathcal{E}}(\mathcal{F}, n) = \chi(\mathcal{X}, \mathcal{F} \otimes \mathcal{E}^{\vee} \otimes \pi^* \mathcal{O}_X(n)) = \chi(X, F_{\mathcal{E}}(\mathcal{F})(n)) = P(F_{\mathcal{E}}(\mathcal{F}), n).$$

Since $F_{\mathcal{E}}$ preserves dimension, the modified Hilbert polynomial can be written as

$$P_{\mathcal{E}}(\mathcal{F}, n) = \sum_{i=0}^{d} \alpha_{\mathcal{E},i}(\mathcal{F}) \frac{n^i}{i!}.$$

Moreover, the modified Hilbert polynomial is additive on short exact sequences since $F_{\mathcal{E}}$ is exact and the Euler characteristic is additive on short exact sequences. As in the scheme case, the modified Hilbert polynomial of a flat family of coherent sheaves is locally constant on the fibres.

Lemma 2.13 ([26, Lemma 3.16]). *Let $p : \mathcal{X} \to S$ be a family of projective stacks with a fixed relative polarization $(\mathcal{E}, \mathcal{O}_X(1))$. Let $\mathcal{F}$ be an $\mathcal{O}_S$-flat coherent sheaf on $\mathcal{X}$. Assume S is connected. There is a polynomial P such that for every closed point $s \in S$*

$$\chi(\mathcal{X}_s, \mathcal{F} \otimes \mathcal{E}^{\vee} \otimes \pi^* \mathcal{O}_X(m)|_{\mathcal{X}_s}) = P(m).$$

Definition 2.14. We denote by $r_{\mathcal{E}}(\mathcal{F}) = \alpha_{\mathcal{E},d}(\mathcal{F})$ the *multiplicity* of $\mathcal{F}$. The *reduced* *modified Hilbert polynomial* is then $p_{\mathcal{E}}(\mathcal{F}) = \dfrac{P_{\mathcal{E}}(\mathcal{F})}{r_{\mathcal{E}}(\mathcal{F})}$, and the *slope* is $\hat{\mu}_{\mathcal{E}}(\mathcal{F}) = \dfrac{\alpha_{\mathcal{E},d-1}(\mathcal{F})}{\alpha_{\mathcal{E},d}(\mathcal{F})}$.

Definition 2.15. A coherent sheaf $\mathcal{F}$ is *semistable* if it is pure and for every proper subsheaf $\mathcal{F}' \subset \mathcal{F}$ one has $p_{\mathcal{E}}(\mathcal{F}') \leq p_{\mathcal{E}}(\mathcal{F})$. $\mathcal{F}$ is called *stable* if it is semistable and the inequality is strict.

Definition 2.16. Let $\mathcal{F}$ be a pure sheaf on $\mathcal{X}$. A strictly ascending filtration

$$0 = HN_0(\mathcal{F}) \subset HN_1(\mathcal{F}) \subset \cdots \subset HN_l(\mathcal{F}) = F$$

is a Harder-Narasimhan filtration if it satisfies the following:

(1) the i-th graded piece $gr_i^{HN} = HN_i(\mathcal{F})/HN_{i-1}(\mathcal{F})$ is semistable for every $i = 1, \cdots, l$;

(2) denoted with $p_i = p_{\mathcal{E}}(gr_i^{HN}(\mathcal{F}))$, then

$$p_{\max}(\mathcal{F}) := p_1 > \cdots > p_l =: p_{\min}(\mathcal{F}).$$

Proposition 2.17 ([26, Theorem 3.2.2])**.** *Let $\mathcal{F}$ be a pure sheaf on $\mathcal{X}$, then $\mathcal{F}$ has a unique Harder-Narasimhan filtration.*

As pointed out by Nironi, the functor $F_{\mathcal{E}}$ doesn't preserve the Harder-Narasimhan filtration. However, we have the following relation between the maximal slopes.

Proposition 2.18. *Let $\mathcal{X}$ be a projective stack over $\mathbb{C}$. Let $\mathcal{F}$ be a pure sheaf on*

$\mathcal{X}$. Let $\widetilde{m}$ be an integer such that $F_{\mathcal{E}}(\mathcal{E})(\widetilde{m})$ is generated by global sections, i.e.

$$\mathcal{O}_X(-\widetilde{m})^{\oplus N} \twoheadrightarrow F_{\mathcal{E}}(\mathcal{E}),$$

where $N = h^0(X, F_{\mathcal{E}}(\mathcal{E})(\widetilde{m}))$. Then

$$\hat{\mu}_{\max}(\mathcal{F}) \leq \hat{\mu}_{\max}(F_{\mathcal{E}}(F)) \leq \hat{\mu}_{\max}(\mathcal{F}) + \widetilde{m}\deg(\mathcal{O}_{\mathcal{X}}(1)). \tag{2.1}$$

Proof. The proof is the same as [26, Proposition 4.24]. □

Being pure and being semistable are open conditions in flat families.

Proposition 2.19 ([26, Proposition 4.15])**.** *Let $p : \mathcal{X} \to S$ be a family of projective stacks with a fixed relative polarization $(\mathcal{E}, \mathcal{O}_X(1))$. Let $\mathcal{F}$ be an S-flat d-dimensional coherent sheaf on $\mathcal{X}$ with fixed modified Hilbert polynomial P. Then the sets*

$$\{s \in S | \mathcal{F}_s \text{ is pure of dimension } d\} \text{ and } \{s \in S | \mathcal{F}_s \text{ is semistable}\}$$

are both open in S.

2.3 Boundedness

Let m be an integer. Recall that a coherent sheaf F on X is said *m-regular* if for all $i > 0$

$$H^i(X, F(m-i)) = 0.$$

The *Mumford-Castelnuovo regularity* of F is the number

$$\operatorname{reg}(F) := \inf\{m \in \mathbb{Z} \mid F \text{ is } m\text{-regular}\}.$$

The regularity is $\operatorname{reg} F = -\infty$ if and only if F is 0-dimensional.

Definition 2.20. We define the *Mumford regularity* of a coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ to be the Mumford regularity of $F_{\mathcal{E}}(\mathcal{F})$ on X and we will denote it by $\operatorname{reg}_{\mathcal{E}}(\mathcal{F})$.

Definition 2.21. A set-theoretic family $\mathscr{F}$ of sheaves on $\mathcal{X}$ is *bounded* if there is a T of finite type and a coherent sheaf $\mathcal{H}$ on $\mathcal{X}_T$ such that every sheaf in $\mathscr{F}$ is contained in the fibers of $\mathcal{H}$.

We have the following important result on boundedness on $\mathcal{X}$.

Proposition 2.22 ([26, Cor 4.17])**.** *A set-theoretic family $\mathscr{F}$ of sheaves on $\mathcal{X}$ is bounded if and only if $F_{\mathcal{E}}(\mathscr{F})$ is bounded on X.*

We have the stacky version of the Kleiman criterion.

Proposition 2.23 ([26, Theorem 4.12])**.** *Let $\mathscr{F}$ be a family of coherent sheaves on $\mathcal{X}$. Then the following statements are equivalent:*

(1) *The family $\mathscr{F}$ is bounded.*

(2) *The set of modified Hilbert polynomials $\{P_{\mathcal{E}}(\mathcal{F}) | \mathcal{F} \in \mathscr{F}\}$ is finite and there is an integer m such that every $\mathcal{F} \in \mathscr{F}$ is m-regular.*

(3) *The set of modified Hilbert polynomials $\{P_{\mathcal{E}}(\mathcal{F}) | \mathcal{F} \in \mathscr{F}\}$ is finite and there is a coherent sheaf $\mathcal{H}$ on $\mathcal{X}$ such every $\mathcal{F} \in \mathscr{F}$ is a quotient of $\mathcal{H}$.*

We also have the stacky version of the Grothendieck lemma.

Lemma 2.24 ([26, Lemma 4.13])**.** *Let $\mathcal{X}$ be a projective stack with coarse moduli scheme $\pi : \mathcal{X} \to X$. Let P be a polynomial and ρ an integer. There is a constant $C = C(P, \rho)$ such that if $\mathcal{F}$ is a d-dimensional coherent sheaf with $P_{\mathcal{E}}(\mathcal{F}) = P$ and $\mathrm{reg}_{\mathcal{E}}(\mathcal{F}) \leq \rho$, then $\hat{\mu}_{\mathcal{E}}(\mathcal{G}) \geq C$ for every purely d-dimensional quotient $\mathcal{G}$ of $\mathcal{F}$. Moreover, the family of purely d-dimensional quotients $\mathcal{G}$ with $\hat{\mu}_{\mathcal{E}}(\mathcal{G})$ bounded from above is bounded.*

For our convenience, we list some results on boundedness on X.

Proposition 2.25 ([18, Theorem 3.3.7])**.** *Let C be a rational constant. The family of pure coherent sheaves F with Hilbert polynomial P on X such that $\hat{\mu}_{\max}(F) \leq C$ is bounded.*

Proposition 2.26 ([22, Lemma 2.13])**.** *Let F be a pure sheaf of dimension d and multiplicity r on X. Let Y be the scheme-theoretic support of F. Then the minimum slope $\hat{\mu}_{\min}(\mathcal{O}_Y)$ is bounded below by a constant determined by $\dim X$, r, and d.*

Proposition 2.27 ([18, Cor 3.3.8])**.** *Let X be a projective scheme with very ample line bundle $\mathcal{O}_X(1)$. Let F be a pure coherent sheaf of dimension d and multiplicity r. Then*

$$\frac{h^0(F(m))}{r} \leq \frac{1}{d!}\left(\frac{r-1}{r}[\hat{\mu}_{\max}(F) + C - 1 + m]_+^d + \frac{1}{r}[\hat{\mu}(F) + C - 1 + m]_+^d\right), \quad (2.2)$$

where $C = r^2 + (r+d)/2$ and $[\cdot]_+ = \max\{\cdot, 0\}$.

Chapter 3: Moduli space of stable pairs

3.1 Stable pairs

Let $\mathcal{X} \to \operatorname{Spec}\mathbb{C}$ be a projective stack with coarse moduli scheme $\pi : \mathcal{X} \to X$ and polarization $(\mathcal{E}, \mathcal{O}_X(1))$. Let P be a polynomial of degree d and multiplicity r, and δ be a polynomial with positive leading coefficient and $\deg\delta \geq \deg P$.

Definition 3.1. A *pair* $(\mathcal{F}, s)$ (of type P) consists of a coherent sheaf $\mathcal{F}$ over $\mathcal{X}$ with modified Hilbert polynomial $P_{\mathcal{E}}(\mathcal{F}) = P$ and a section $s : \mathcal{O}_{\mathcal{X}} \to \mathcal{F}$. A subpair $(\mathcal{F}', s')$ consists of a coherent subsheaf $\imath : \mathcal{F}' \subset \mathcal{F}$ and a section $s' : \mathcal{O}_{\mathcal{X}} \to \mathcal{F}'$ such that

$$\begin{cases} \imath \circ s' = s & \text{if } \operatorname{Im} s \subset \mathcal{F}' \\ s' = 0 & \text{otherwise.} \end{cases}$$

A quotient pair $(\mathcal{F}'', s'')$ consists of a coherent quotient sheaf $q : \mathcal{F} \to \mathcal{F}''$ and a section $s'' = q \circ s : \mathcal{O}_{\mathcal{X}} \to \mathcal{F}''$.

A short exact sequence of pairs

$$0 \to (\mathcal{F}', s') \to (\mathcal{F}, s) \to (\mathcal{F}'', s'') \to 0$$

consists of a short exact sequence of the underlying sheaves such that $(\mathcal{F}', s')$ is a subpair and $(\mathcal{F}'', s'')$ is the corresponding quotient pair.

Following [31], we make the following definition.

Definition 3.2. A pair $(\mathcal{F}, s)$ is a *stable pair* if $\mathcal{F}$ is pure and $\dim \operatorname{Coker}(s) < d$.

A family of stable pairs parametrized by a scheme T of finite type over $\mathbb{C}$ is a pair

$$s_T : \mathcal{O}_{\mathcal{X}_T} \to \mathscr{F}$$

such that $\mathscr{F}$ is a coherent sheaf flat over $\mathcal{X}_T$ and for all closed points $t \in T$, the restriction $(\mathscr{F}_t, s_t)$ to the fibre $\mathcal{X}_t$ is stable. We define a functor

$$\mathcal{M}_{\mathcal{X}}(P) := \mathcal{M}_{\mathcal{X}}(\mathcal{E}, \mathcal{O}_X(1), P) : (\mathrm{Sch}\,/\mathbb{C})^{\circ} \to (\mathrm{Sets})$$

which associates to any scheme T of finite type over $\mathbb{C}$ the set of isomorphism classes of flat families of stable pairs on $\mathcal{X}_T$ with Hilbert polynomial P, and associates to any morphism $T' \to T$ its pullback.

Theorem 3.3. *Let $(\mathcal{X}, \mathcal{E}, \mathcal{O}_X(1))$ be a polarized smooth projective stack over $\mathbb{C}$. Then $\mathcal{M}_{\mathcal{X}}(P)$ is represented by a projective scheme $M_{\mathcal{X}}(P)$.*

To construct the moduli scheme using GIT, we need a different notion of stability following [22].

Definition 3.4. The Hilbert polynomial of a pair $(\mathcal{F}, s)$ w.r.t. to δ is

$$P_{\mathcal{E},\delta}(\mathcal{F}, s) = P_{\mathcal{E}}(\mathcal{F}) + \epsilon(s)\delta,$$

and the reduced Hilbert polynomial of the pair is

$$p_{\mathcal{E},\delta}(\mathcal{F}, s) = p_{\mathcal{E}}(\mathcal{F}) + \epsilon(s)\frac{\delta}{r_{\mathcal{E}}(\mathcal{F})}.$$

Here,

$$\epsilon(s) = \begin{cases} 1 & \text{if } s \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.5. (1) To ease notation, we will omit δ and denote the Hilbert polynomial (resp., reduced Hilbert polynomial) of a pair $(\mathcal{F}, s)$ by $P_{\mathcal{E}}(\mathcal{F}, s)$ (resp., $p_{\mathcal{E}}(\mathcal{F}, s)$).

(2) The Hilbert polynomial of pairs is additive on short exact sequences since $\epsilon(s) = \epsilon(s') + \epsilon(s'')$ and the modified Hilbert polynomial of coherent sheaves is additive on short exact sequences.

Definition 3.6. A pair $(\mathcal{F}, s)$ is δ-*(semi)stable* if

(1) $\mathcal{F}$ is pure,

(2) $p_{(\mathcal{F}',s')}(\leq)p_{(\mathcal{F},s)}$ for every proper subpair $(\mathcal{F}', s')$.

Clearly, a pair $(\mathcal{F}, 0)$ is δ-(semi)stable if and only if $\mathcal{F}$ is (semi)stable as a coherent sheaf. We will call a pair $(\mathcal{F}, s)$ *nondegenerate* if $s \neq 0$.

Lemma 3.7. *Let $(\mathcal{F}, s)$ be a nondegenerate pair with $P_{\mathcal{E}}(\mathcal{F}) = P$. Then the following assertions are equivalent*

(1) *$(\mathcal{F}, s)$ is δ-semistable;*

(2) $(\mathcal{F}, s)$ *is* δ*-stable;*

(3) $(\mathcal{F}, s)$ *is stable, i.e.* $\mathcal{F}$ *is pure and* $\dim(\operatorname{Coker} s) < d$*.*

Proof. (1) $\implies$ (2) Suppose $(\mathcal{F}, s)$ is δ-semistable. Let $(\mathcal{F}', s')$ be a nonzero subpair of $(\mathcal{F}, s)$ such that

$$p_{\mathcal{E}}(\mathcal{F}') + \epsilon(s')\frac{\delta}{r_{\mathcal{E}}(\mathcal{F}')} = p_{\mathcal{E}}(\mathcal{F}) + \frac{\delta}{r_{\mathcal{E}}(\mathcal{F})}.$$

Since $\deg \delta \geq \deg P$, by comparing leading coefficients we obtain $\epsilon(s') = 1$ and $r_{\mathcal{E}}(\mathcal{F}') = r_{\mathcal{E}}(\mathcal{F})$. Hence, $P_{\mathcal{E}}(\mathcal{F}') = P_{\mathcal{E}}(\mathcal{F})$, which implies that $\mathcal{F}' = \mathcal{F}$. Thus, $(\mathcal{F}, s)$ has no strictly δ-semistable subpair, i.e. $(\mathcal{F}, s)$ is δ-stable.

(2) $\implies$ (3) Consider the subpair $(\operatorname{Im} s, s)$. Since $(\mathcal{F}, s)$ is δ-stable, we have

$$p_{\mathcal{E}}(\operatorname{Im} s) + \frac{\delta}{r_{\mathcal{E}}(\operatorname{Im} s)} \leq p_{\mathcal{E}}(\mathcal{F}) + \frac{\delta}{r_{\mathcal{E}}(\mathcal{F})}.$$

By comparing leading coefficients, we get $r_{\mathcal{E}}(\operatorname{Im} s) \geq r_{\mathcal{E}}(\mathcal{F})$. Since $\operatorname{Im} s \subset \mathcal{F}$, we have $r_{\mathcal{E}}(\operatorname{Im} s) \leq r_{\mathcal{E}}(\mathcal{F})$. Thus, $r_{\mathcal{E}}(\operatorname{Im} s) = r_{\mathcal{E}}(\mathcal{F})$. It follows that $\deg P_{\mathcal{E}}(\operatorname{Coker} s) < d$, i.e. $\dim(\operatorname{Coker} s) < d$.

(3) $\implies$ (1) Suppose $\mathcal{F}$ be pure and $\dim(\operatorname{Coker} s) < d$. Let $(\mathcal{F}', s')$ be a proper subpair of $(\mathcal{F}, s)$ such that

$$p_{\mathcal{E}}(\mathcal{F}') + \epsilon(s')\frac{\delta}{r_{\mathcal{E}}(\mathcal{F}')} > p_{\mathcal{E}}(\mathcal{F}) + \frac{\delta}{r_{\mathcal{E}}(\mathcal{F})}.$$

Since $\deg \delta \geq \deg P$, we have $\epsilon(s') = 1$. It follows that $r_{\mathcal{E}}(\mathcal{F}') < r_{\mathcal{E}}(\mathcal{F})$, or $r_{\mathcal{E}}(\mathcal{F}') =$

$r_{\mathcal{E}}(\mathcal{F})$ and $p_{\mathcal{E}}(\mathcal{F}') > p_{\mathcal{E}}(\mathcal{F})$. If $r_{\mathcal{E}}(\mathcal{F}') < r_{\mathcal{E}}(\mathcal{F})$, then $\operatorname{Im} s \subset \mathcal{F}'$ and the quotient $\mathcal{F}/\mathcal{F}'$ is d-dimensional, which contradicts the assumption that $\dim(\operatorname{Coker} s) < d$. If $r_{\mathcal{E}}(\mathcal{F}') = r_{\mathcal{E}}(\mathcal{F})$ and $p_{\mathcal{E}}(\mathcal{F}') > p_{\mathcal{E}}(\mathcal{F})$, then $P_{\mathcal{E}}(\mathcal{F}') > P_{\mathcal{E}}(\mathcal{F})$, which contradicts the assumption that $\mathcal{F}'$ is a proper subsheaf. Therefore, $(\mathcal{F}, s)$ is δ-semistable. □

Assumption. From now on, unless stated differently $(\mathcal{F}, s)$ is nondegenerate. As a consequence of Lemma 3.7, we will use stability and δ-stability interchangeably. Moreover, the functor $\mathcal{M}_{\mathcal{X}}(P)$ characterizes isomorphism classes of nondegenerate δ-stable pairs.

Proposition 3.8 (Harder-Narasimhan filtration)**.** *Let $(\mathcal{F}, s)$ be a nondegenerate pair where $\mathcal{F}$ is pure of dimension d. Then there is a unique filtration by subpairs*

$$0 = (\mathcal{F}_0, s_0) \subset \cdots \subset (\mathcal{F}_l, s_l) = (\mathcal{F}, s)$$

such that each $gr_i^{HN}(\mathcal{F}, s) = (\mathcal{F}_i, s_i)/(\mathcal{F}_{i-1}, s_{i-1})$ is δ-semistable of dimension d and

$$p_{\max}(\mathcal{F}, s) = p_1 > p_2 > \cdots > p_l = p_{\min}(\mathcal{F}, s),$$

where $p_i = p_{gr_i^{HN}(\mathcal{F},s)}$.

Proof. Let $\mathcal{F}_1 = \overline{\operatorname{Im}(s)}$ be the saturation of $\operatorname{Im}(s)$ in $\mathcal{F}$, then $\dim(\mathcal{F}_1/\operatorname{Im}(s)) < d$. By Lemma 3.7, $(\mathcal{F}_1, s)$ is δ-stable. Notice that the quotient pair $(\mathcal{F}, s)/(\mathcal{F}_1, s) = (\mathcal{F}/\mathcal{F}_1, 0)$ is degenerate. By Proposition 2.17, we get a Harder-Narasimhan filtration for the pure sheaf $\mathcal{F}/\mathcal{F}_1$. Combining them together, we obtain a Harder-Narasimhan filtration for the pair $(\mathcal{F}, s)$.

To prove uniqueness, it suffices to prove that $\mathcal{F}_1 = \overline{\mathrm{Im}(s)}$. Notice that

$$p_{\mathcal{E}}(\mathcal{F}_1) + \epsilon(s_1)\frac{\delta}{r_{\mathcal{E}}(\mathcal{F}_1)} > p_{\mathcal{E}}(\mathcal{F}) + \frac{\delta}{r_{\mathcal{E}}(\mathcal{F})}.$$

Since $\deg \delta \geq \deg P$, we have $\epsilon(s_1) = 1$. Hence, $\mathcal{F}_1 \supseteq \mathrm{Im}(s)$. Since $\mathcal{F}/\mathcal{F}_1$ is pure of dimension d, $\mathcal{F}_1$ contains the saturation $\overline{\mathrm{Im}(s)}$ of $\mathrm{Im}\, s$. Since $\mathcal{F}/\overline{\mathrm{Im}(s)}$ is pure of dimension d, $\mathcal{F}_1/\overline{\mathrm{Im}(s)}$ is zero or pure of dimension d. According to Lemma 3.7, $\dim(\mathcal{F}_1/\overline{\mathrm{Im}(s)}) < d$ because $(\mathcal{F}_1, s)$ is δ-stable. Thus, $\mathcal{F}_1 = \overline{\mathrm{Im}(s)}$. □

We have the following reinterpretation of δ-stability.

Lemma 3.9. *Let $(\mathcal{F}, s)$ be a nondegenerate pair where $\mathcal{F}$ is pure and $P_{\mathcal{E}}(\mathcal{F}) = P$. It is δ-stable if and only if for every proper subpair $(\mathcal{F}', s')$,*

$$\frac{P_{\mathcal{E}}(\mathcal{F}')}{2r_{\mathcal{E}}(\mathcal{F}') - \epsilon(s')} < \frac{P}{2r - 1}.$$

Proof. This is just a special case of [22, Lemma 4.3]. □

3.2 Boundedness

In order to construct the moduli space via GIT, we first prove that the family of underlying sheaves of stable pairs is bounded.

Proposition 3.10. *The family*

$$\mathscr{F} = \{\mathcal{F} | (\mathcal{F}, s) \text{ is a } \delta\text{-stable pair with } P_{\mathcal{E}}(\mathcal{F}) = P\}$$

of coherent sheaves on $\mathcal{X}$ is bounded.

Proof. According to Proposition 2.22, $\mathscr{F}$ is bounded if and only if $F_{\mathcal{E}}(\mathscr{F})$ is bounded. By Proposition 2.25, it suffices to show that there is a constant C such that

$$\hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F})) \leq C$$

for every $\mathcal{F} \in \mathscr{F}$.

Consider the pair

$$F_{\mathcal{E}}(s) : F_{\mathcal{E}}(\mathcal{O}_{\mathcal{X}}) \to F_{\mathcal{E}}(\mathcal{F}).$$

Since $F_{\mathcal{E}}$ is exact and preserves both dimension and pureness by Proposition 2.9, we obtain that $F_{\mathcal{E}}(\mathcal{O}_{\mathcal{X}})$ and $F_{\mathcal{E}}(\mathcal{F})$ are both pure and $\dim(\operatorname{Coker} F_{\mathcal{E}}(s)) < d$. Let $Y = \operatorname{Supp}(F_{\mathcal{E}}(\mathcal{F}))$ and

$$0 = HN_0(F_{\mathcal{E}}(\mathcal{F})) \subset HN_1(F_{\mathcal{E}}(\mathcal{F})) \subset \cdots \subset HN_l(F_{\mathcal{E}}(\mathcal{F})) = F_{\mathcal{E}}(\mathcal{F})$$

be the $\hat{\mu}$-Harder-Narasimhan filtration of $F_{\mathcal{E}}(\mathcal{F})$. Since $\dim(\operatorname{Coker} F_{\mathcal{E}}(s)) < d$, we have that $\operatorname{Im} F_{\mathcal{E}}(s) \nsubseteq HN_{l-1}(F_{\mathcal{E}}(\mathcal{F}))$. Hence, the composition

$$F_{\mathcal{E}}(\mathcal{O}_{\mathcal{X}}) \otimes \mathcal{O}_Y \to F_{\mathcal{E}}(\mathcal{F}) \twoheadrightarrow gr_l^{HN}(F_{\mathcal{E}}(\mathcal{F}))$$

is a non-zero morphism between pure sheaves of dimension d. This implies that

$$\hat{\mu}_{\min}(F_{\mathcal{E}}(\mathcal{F})) = \hat{\mu}(gr_l^{HN}(F_{\mathcal{E}}(\mathcal{F}))) \geq \hat{\mu}_{\min}(F_{\mathcal{E}}(\mathcal{O}_{\mathcal{X}}) \otimes \mathcal{O}_Y)$$

$$= \hat{\mu}_{\min}(F_{\mathcal{E}}(\mathcal{O}_{\mathcal{X}})) + \hat{\mu}_{\min}(\mathcal{O}_Y).$$

According to Proposition 2.26, $\hat{\mu}_{\min}(\mathcal{O}_Y)$ is bounded below by a constant A which only depends on d, r and $\dim X$. Then

$$\hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F})) \leq r\hat{\mu}(F_{\mathcal{E}}(\mathcal{F})) - (r-1)\hat{\mu}_{\min}(F_{\mathcal{E}}(\mathcal{F}))$$

$$\leq r\hat{\mu}(F_{\mathcal{E}}(\mathcal{F})) - (r-1)(\hat{\mu}_{\min}(F_{\mathcal{E}}(\mathcal{O}_{\mathcal{X}})) + A) =: C$$

as desired. □

We can rephrase the stability using global sections instead of Hilbert polynomial.

Proposition 3.11. *There is an $m_0 \in \mathbb{Z}^+$ such that for any integer $m \geq m_0$ and any nondegenerate pair $(\mathcal{F}, s)$, where $\mathcal{F}$ is pure and $P_{\mathcal{E}}(\mathcal{F}) = P$, TFAE:*

(1) *the pair $(\mathcal{F}, s)$ is stable;*

(2) *$P_{\mathcal{E}}(\mathcal{F}, m) \leq h^0(F_{\mathcal{E}}(\mathcal{F})(m))$, and for any proper subpair $(\mathcal{F}', s')$,*

$$\frac{h^0(F_{\mathcal{E}}(\mathcal{F}')(m))}{2r_{\mathcal{E}}(\mathcal{F}') - \epsilon(s')} < \frac{h^0(F_{\mathcal{E}}(\mathcal{F})(m))}{2r-1}$$

(3) *for any proper quotient pair* $(\mathcal{G}, s'')$ *where* $\dim \mathcal{G} = d$,

$$\frac{h^0(F_{\mathcal{E}}(\mathcal{G})(m))}{2r_{\mathcal{E}}(\mathcal{G}) - \epsilon(s'')} > \frac{P(m)}{2r-1}.$$

Proof. (1) $\implies$ (2) By Proposition 3.10, there is an integer m_0 such that for any integer $m \geq m_0$, we have $H^i(F_{\mathcal{E}}(\mathcal{F})(m)) = 0$ for all $i > 0$. In particular, $P(m) = h^0(F_{\mathcal{E}}(\mathcal{F})(m))$. In the proof of boundedness, we also showed that $\hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F}))$ is bounded above, say $\hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F})) \leq \mu_0$.

Since $F_{\mathcal{E}}$ is exact and preserves pureness, $F_{\mathcal{E}}(\mathcal{F}')$ is a pure subsheaf of $F_{\mathcal{E}}(\mathcal{F})$. Then we have $\hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F}')) \leq \mu_0$ and $r_{\mathcal{E}}(\mathcal{F}') \leq r$. Using Le Potier-Simpson estimate (2.2), we obtain

$$\frac{h^0(F_{\mathcal{E}}(\mathcal{F}')(m))}{r_{\mathcal{E}}(\mathcal{F}')} \leq \frac{1}{d!}\left(\frac{r-1}{r}[\mu_0 + C - 1 + m]_+^d + \frac{1}{r}[\hat{\mu}(F_{\mathcal{E}}(\mathcal{F}')) + C - 1 + m]_+^d\right),$$

where $C = r^2 + (r+d)/2$ and $[\cdot]_+ = \max\{\cdot, 0\}$.

Let $A > 0$ be an integer such that A is larger than all roots of P. Replace m_0 by $\max\{m_0, A\}$. Then

$$h^0(F_{\mathcal{E}}(\mathcal{F})(m)) = P(m) > \frac{r}{d!}(m-A)^d, \text{ for all } m \geq m_0.$$

Suppose μ_1 is an integer such that

$$C - 1 + \mu_0\left(1 - \frac{1}{r}\right) + \frac{\mu_1}{r} < -A.$$

Enlarging m_0 if necessary, we have

$$\frac{1}{d!}\left(\frac{r-1}{r}[\mu_0+C-1+m]_+^d+\frac{1}{r}[\mu_1+C-1+m]_+^d\right)<\frac{(m-A)^d}{d!}<\frac{P(m)}{r}$$

by considering the coefficient of m^{d-1}.

If $\hat{\mu}(F_{\mathcal{E}}(\mathcal{F}'))\leq\mu_1$, then for $m\geq m_0$ we get

$$h^0(F_{\mathcal{E}}(\mathcal{F}')(m))<\frac{r_{\mathcal{E}}(\mathcal{F}')}{r}h^0(F_{\mathcal{E}}(\mathcal{F})(m))\leq\frac{2r_{\mathcal{E}}(\mathcal{F}')-\epsilon(s')}{2r-1}h^0(F_{\mathcal{E}}(\mathcal{F})(m)).$$

The reason for the last inequality is as follows. Since $(\mathcal{F},s)$ is stable, we have

$$p_{\mathcal{E}}(\mathcal{F}')+\epsilon(s')\frac{\delta}{r_{\mathcal{E}}(\mathcal{F}')}<p+\frac{\delta}{r}.$$

In particular, if $\epsilon(s')=1$, then $r_{\mathcal{E}}(\mathcal{F}')=r$. Thus,

$$\frac{r_{\mathcal{E}}(\mathcal{F}')}{r}\leq\frac{2r_{\mathcal{E}}(\mathcal{F}')-\epsilon(s')}{2r-1}.$$

We are left to consider the case where $\hat{\mu}(F_{\mathcal{E}}(\mathcal{F}'))\geq\mu_1$. We can assume $\mathcal{F}'$ is saturated. By Grothendieck's lemma [18, Lemma 1.7.9], the family of such $\mathcal{F}'$ is bounded. Thus, there are only finitely many modified Hilbert polynomials $P_{\mathcal{E}}(\mathcal{F}')$. We can enlarge m_0, if necessary, such that for $m\geq m_0$, $P_{\mathcal{E}}(\mathcal{F}',m)=h^0(F_{\mathcal{E}}(\mathcal{F}')(m))$ and

$$\frac{P_{\mathcal{E}}(\mathcal{F}')}{2r_{\mathcal{E}}(\mathcal{F}')-\epsilon(s')}<\frac{P}{2r-1}\iff\frac{P_{\mathcal{E}}(\mathcal{F}',m)}{2r_{\mathcal{E}}(\mathcal{F}')-\epsilon(s')}<\frac{P(m)}{2r-1}.$$

Combining this with Lemma 3.9, we finish the proof.

(2) $\implies$ (3) Given a proper quotient pair, we can form the short exact sequence

$$0 \to (\mathcal{F}', s') \to (\mathcal{F}, s) \to (\mathcal{G}, s'') \to 0.$$

Thus, we obtain an exact sequence,

$$0 \to H^0(F_{\mathcal{E}}(\mathcal{F}')(m)) \to H^0(F_{\mathcal{E}}(\mathcal{F})(m)) \to H^0(F_{\mathcal{E}}(\mathcal{G})(m)).$$

Notice that $r = r_{\mathcal{E}}(\mathcal{F}') + r_{\mathcal{E}}(\mathcal{G})$ and $1 = \epsilon(s') + \epsilon(s'')$. By condition (2),

$$\begin{aligned}\frac{h^0(F_{\mathcal{E}}(\mathcal{G})(m))}{2r_{\mathcal{E}}(\mathcal{G}) - \epsilon(s'')} &\geq \frac{h^0(F_{\mathcal{E}}(\mathcal{F})(m)) - h^0(F_{\mathcal{E}}(\mathcal{F}')(m))}{2r - 1 - (2r_{\mathcal{E}}(\mathcal{F}') - \epsilon(s'))} \\ &> \frac{h^0(F_{\mathcal{E}}(\mathcal{F})(m))}{2r-1} \geq \frac{P(m)}{2r-1}.\end{aligned}$$

(3) $\implies$ (1) We first show that the family of coherent sheaves satisfying condition (3) is bounded. Let $\mathcal{F}_{\min} = gr_l^{HN}(\mathcal{F})$ be the last factor in the $\hat{\mu}$-Harder-Narasimhan filtration of $\mathcal{F}$ with respect to slope. By Le Potier-Simpson estimate (2.2) and (2.1), we have

$$\begin{aligned}\frac{h^0(F_{\mathcal{E}}(\mathcal{F}_{\min})(m))}{r_{\mathcal{E}}(\mathcal{F}_{\min})} &\leq \frac{1}{d!}\left([\hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F}_{\min})) + m - 1 + C]_+\right)^d \\ &\leq \frac{1}{d!}\left([\hat{\mu}_{\mathcal{E}}(\mathcal{F}_{\min}) + \widetilde{m}\deg \mathcal{O}_{\mathcal{X}}(1) + m - 1 + C]_+\right)^d \\ &= \frac{1}{d!}\left([\hat{\mu}_{\mathcal{E},\min}(\mathcal{F}) + \widetilde{m}\deg \mathcal{O}_{\mathcal{X}}(1) + m - 1 + C]_+\right)^d,\end{aligned}$$

where $C = r^2 + (r+d)/2$. Let $(\mathcal{F}_{\min}, s'')$ be the induced quotient pair. If $\epsilon(s'') = 0$, by assumption, we have

$$\begin{aligned}\frac{P(m)}{r} < \frac{2P(m)}{2r-1} < \frac{h^0(F_{\mathcal{E}}(\mathcal{F}_{\min})(m))}{r_{\mathcal{E}}(\mathcal{F}_{\min})} \\ \leq \frac{1}{d!}\left([\hat{\mu}_{\mathcal{E},\min}(\mathcal{F}) + \widetilde{m}\deg \mathcal{O}_{\mathcal{X}}(1) + m - 1 + C]_+\right)^d.\end{aligned}$$

Since $P(m)/r \geq (m-A)^d/d!$, we have

$$\hat{\mu}_{\mathcal{E},\min}(\mathcal{F}) > -\widetilde{m}\deg \mathcal{O}_{\mathcal{X}}(1) - C - A - 1,$$

which is bounded below. If $\epsilon(s'') \neq 0$, then the composition

$$F_{\mathcal{E}}(\mathcal{O}_{\mathcal{X}}) \otimes \mathcal{O}_Y \to F_{\mathcal{E}}(\mathcal{F}) \twoheadrightarrow F_{\mathcal{E}}(\mathcal{F}_{\min})$$

is a non-zero morphism between pure sheaves of dimension d. Hence,

$$\begin{aligned}\hat{\mu}_{\min}(F_{\mathcal{E}}(\mathcal{O}_{\mathcal{X}})) + \hat{\mu}_{\min}(\mathcal{O}_Y) &= \hat{\mu}_{\min}(F_{\mathcal{E}}(\mathcal{O}_{\mathcal{X}}) \otimes \mathcal{O}_Y) \\ &\leq \hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F}_{\min})) \leq \hat{\mu}(\mathcal{F}_{\min}) + \widetilde{m}\deg \mathcal{O}_X(1) \\ &= \hat{\mu}_{\mathcal{E},\min}(\mathcal{F}) + \widetilde{m}\deg \mathcal{O}_X(1).\end{aligned}$$

Since $\hat{\mu}_{\min}(\mathcal{O}_Y)$ is bounded below, $\hat{\mu}_{\mathcal{E},\min}(\mathcal{F})$ is also bounded below. Thus, in both cases, $\hat{\mu}_{\mathcal{E},\max}(\mathcal{F})$ is bounded above. Using (2.1) again, we have that $\hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F}))$ is bounded above. Therefore, the family of coherent sheaves satisfying condition (3) is bounded.

Let $(\mathcal{F}_1, s)$ be the first factor in the Harder-Narasimhan filtration of $(\mathcal{F}, s)$. From Proposition 3.8, we know $(\mathcal{F}_1, s)$ is a nondegenerate stable pair. By Proposition 3.10, the family of the underlying sheaves $\{\mathcal{F}_1\}$ is bounded. Therefore, the family of sheaves $\{\mathcal{F}/\mathcal{F}_1\}$, where $(\mathcal{F}, s)$ satisfies condition (3) and $(\mathcal{F}_1, s)$ is the first factor in the Harder-Narasimhan filtration of $(\mathcal{F}, s)$, is also bounded. In particular, the set of reduced modified Hilbert polynomial $p_{\mathcal{E}}(\mathcal{F}/\mathcal{F}_1)$ is finite. Let $gr_l^{HN}(\mathcal{F}, s) = (\mathcal{G}, s'')$ be the last factor in the Harder-Narasimhan filtration of $(\mathcal{F}, s)$. We can assume $\epsilon(s'') = 0$; otherwise, $(\mathcal{F}, s)$ is stable according to Proposition 3.8. Then $\mathcal{G}$ is actually the last factor of the Harder-Narasimhan filtration of $\mathcal{F}/\mathcal{F}_1$. Hence, $p_{\mathcal{E}}(\mathcal{G}) < p_{\mathcal{E}}(\mathcal{F}/\mathcal{F}_1)$. This implies $\hat{\mu}(F_{\mathcal{E}}(\mathcal{G}))$ is bounded above because there are only finitely many $p_{\mathcal{E}}(\mathcal{F}/\mathcal{F}_1)$. By Grothendieck's lemma [18, Lemma 1.7.9], the family of such $\{F_{\mathcal{E}}(\mathcal{G})\}$ is bounded. Enlarging m_0 if necessary, we can assume that, for all $m \geq m_0$, $P_{\mathcal{E}}(\mathcal{G}, m) = h^0(F_{\mathcal{E}}(\mathcal{G})(m))$ and

$$\frac{P_{\mathcal{E}}(\mathcal{G}, m)}{2r_{\mathcal{E}}(\mathcal{G}) - \epsilon(s'')} > \frac{P(m)}{2r-1} \iff \frac{P_{\mathcal{E}}(\mathcal{G})}{2r_{\mathcal{E}}(\mathcal{G}) - \epsilon(s'')} > \frac{P}{2r-1}.$$

Now according to condition (3), the last inequality holds. Thus, $\epsilon(s'') \geq r_{\mathcal{E}}(\mathcal{G})/r$, which forces $\epsilon(s'') = 1$, which is a contradiction. Therefore, $(\mathcal{F}, s)$ is stable.

□

3.3 Construction of the moduli space

By Proposition 3.10 and Proposition 3.11, there is an integer m_0 such that for all $m \geq m_0$, the following conditions are satisfied:

(1) $F_{\mathcal{E}}(\mathcal{F})(m)$ is globally generated and $H^i(F_{\mathcal{E}}(\mathcal{F})(m)) = 0$ when $i > 0$ for every nondegenerate stable pair $(\mathcal{F}, s)$;

(3) the three conditions in Proposition 3.11 are equivalent.

Fix such an m and let V be a vector space of dimension equal to $P(m)$.

Let $(\mathcal{F}, s)$ is a stable pair, then we get a quotient

$$q : V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m) \twoheadrightarrow \mathcal{F}$$

obtained by applying the functor $G_{\mathcal{E}}$ to

$$V(-m) \simeq H^0(F_{\mathcal{E}}(F)(m))(-m) \twoheadrightarrow F_{\mathcal{E}}(\mathcal{F})$$

and composing with $\theta_{\mathcal{E}}(\mathcal{F}) : G_E \circ F_{\mathcal{E}}(\mathcal{F}) \twoheadrightarrow \mathcal{F}$. The morphism q corresponds to a closed point of $Q := \mathrm{Quot}(V \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m), P)$, which is a projective scheme according to [26, Proposition 4.20]. Similarly, let $U = H^0(F_{\mathcal{E}}(\mathcal{O}_{\mathcal{X}})(m))$, we have the quotient

$$\mathrm{ev} : U \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m) \twoheadrightarrow \mathcal{O}_{\mathcal{X}}.$$

The section s gives rise to a linear map

$$\sigma : U \to H^0(F_{\mathcal{E}}(\mathcal{F})(m)) \simeq V,$$

which corresponds a closed point of $N := \mathbb{P}(\mathrm{Hom}(U, V))$. Thus, any stable pair $(\mathcal{F}, s)$ determines a point $(\sigma, q) \in N \times Q$ and the following commutative diagram

$$\begin{array}{ccc} U \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(-m) & \xrightarrow{\mathrm{ev}} & \mathcal{O}_{\mathcal{X}} \\ {\scriptstyle \widetilde{\sigma} := \sigma \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle s} \\ V \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(-m) & \xrightarrow{q} & \mathcal{F}. \end{array}$$

Conversely, given a pair $(\sigma, q) \in N \times Q$, we obtain a pair if $q \circ \widetilde{\sigma}(\ker(\mathrm{ev})) = 0$.

Lemma 3.12. *There is a closed subscheme $W \subseteq N \times Q$ such that for every point $(\sigma, q) \in N \times Q$ the composition $q \circ \widetilde{\sigma}$ factors through* ev *if and only if $(\sigma, q) \in W$.*

Proof. Same as [35, Proposition 3.4]. □

Definition 3.13. We define Z to be the open locus of points (σ, q) in W such that $\mathcal{F}$ is pure and q induces an isomorphism of vector spaces $V \simeq H^0(F_{\mathcal{E}}(\mathcal{F})(m))$. Let $\overline{Z}$ denote the closure of Z.

Remark 3.14. Z is indeed open because being pure is open [26, Proposition 5.15] and the semicontinuity theorem for cohomology holds for projective stacks [26, Theorem 1.8].

We now come to the GIT construction of the moduli space of stable pairs.

Consider the natural action of $GL(V)$ on $N \times Q$:

$$(\sigma, q) \cdot g = (g^{-1} \circ \sigma, q \circ g).$$

for $g \in GL(V)$. We observe that $\mathbb{C}^* \subset GL(V)$ acts trivially on both N and Q. We can consider the actions of $PGL(V)$ or $SL(V)$. Indeed, the line bundles linearized for the actions of these two groups are the same up to taking finite tensor powers since $PGL(V)$ is a quotient of $SL(V)$ by a finite group. We consider the $SL(V)$ action. It is clear that Z is invariant under this action. The closure $\overline{Z}$ is invariant as well.

By [26, Proposition 4.20], the functor $F_{\mathcal{E}}$ induces a closed embedding

$$Q \hookrightarrow \mathrm{Quot}\,(V \otimes F_{\mathcal{E}}(\mathcal{E})(-m), P).$$

For $l \in \mathbb{N}$ big enough, there is a closed embedding into the Grassmannian

$$\mathrm{Quot}\,(V \otimes F_{\mathcal{E}}(\mathcal{E})(-m), P) \hookrightarrow \mathrm{Grass}\,(V \otimes H^0(F_{\mathcal{E}}(\mathcal{E})(l-m)), P(l)).$$

Consider the very ample line bundle $\det(\mathcal{S})$ where $\mathcal{S}$ is the universal quotient bundle on the Grassmannian. Let L_l be its pull back to Q. According to [26, Lemma 6.3], L_l is $SL(V)$-linearized. The line bundle $\mathcal{O}_N(1)$ is also $SL(V)$-linearized. For positive integers n_1 and n_2, the following line bundle is $SL(V)$-linearized:

$$L = \mathcal{O}_N(n_1) \boxtimes L_l^{\otimes n_2}.$$

Let $\lambda : \mathbb{C}^* \to SL(V)$ be a 1-parameter subgroup. We have a weight decomposition

$$V = \bigoplus_n V_n$$

such that $\lambda(t) \cdot v = t^n \cdot v$ for every $t \in \mathbb{C}^*, v \in V_n$. This gives us an ascending filtration $V_{\leq n} = \bigoplus_{i \leq n} V_i$.

For a point $\xi = (\sigma, q) \in N \times Q$. Let $n(\sigma)$ be the smallest integer n such that $\mathrm{Im}(\sigma) \subset V_{\leq n}$. Then the Hilbert-Mumford weight of λ at ξ with respect to $\mathcal{O}_N(1)$ is

$$\mu^{\mathcal{O}_N(1)}(\xi, \lambda) = n(\sigma).$$

The filtration on V produces a filtration on $\mathcal{F}$ with subsheaves $\mathcal{F}_{\leq n} = q(V_{\leq n} \otimes \mathcal{E}(-m))$. We have an induced surjection $q_n : V_n \otimes \mathcal{E}(-m) \to \mathcal{F}_{\leq n}/\mathcal{F}_{\leq n-1} =: \mathcal{F}_n$. Taking the sum of all weights we obtain a new quotient sheaf:

$$\overline{q} : V \otimes \mathcal{E}(-m) \to \bigoplus_n \mathcal{F}_n =: \overline{\mathcal{F}}.$$

By [26, Lemma 6.11],

$$\lim_{t \to 0} \lambda(t) \cdot q = \overline{q}.$$

Moreover, according to [26, Lemma 6.12], the Hilbert-Mumford weight of λ at ξ with respect to L_l is

$$\mu^{L_l}(\xi, \lambda) = -\sum_n n P_{\mathcal{E}}(\mathcal{F}_n, l).$$

So we have:

Lemma 3.15. *The Hilbert-Mumford weight of λ at ξ with respect to L is*

$$\mu^L(\xi, \lambda) = n_1 \cdot n(\sigma) - n_2 \sum_n n P_{\mathcal{E}}(\mathcal{F}_n, l).$$

An application of Hilbert-Mumford criterion shows the following lemma.

Lemma 3.16. *For l sufficiently large, let $(\sigma, q) \in \overline{Z}$ be a closed point. Then the following two conditions are equivalent:*

(1) *(σ, q) is GIT-(semi)stable with respect to L;*

(2) *For any nontrivial proper subspace $W < V$, let*

$$\mathcal{F}_W = q(W \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m)).$$

Then

$$P_{\mathcal{E}}(\mathcal{F}_W, l)(\geq) \frac{n_1}{n_2} \left(\epsilon_W(\sigma) - \frac{\dim W}{\dim V} \right) + P(l) \frac{\dim W}{\dim V}. \tag{3.1}$$

Here, $\epsilon_W(\sigma)$ is either 1 or 0 depending on whether W contains $\mathrm{Im}(\sigma)$ or not.

Proof. Same as [23, Lemma 4.1]. □

Now, let

$$\frac{n_1}{n_2} = \frac{P(l)}{2r}.$$

Since the family of such $\mathcal{F}_W$ that is generated by a linear subspace of V is bounded.

We can fix l such that (3.1) is equivalent to

$$P_{\mathcal{E}}(\mathcal{F}_W)(\geq)\frac{P}{2r}\left(\epsilon_W(\sigma)-\frac{\dim W}{\dim V}\right)+P\frac{\dim W}{\dim V}. \tag{3.2}$$

Remark 3.17. Let $(\sigma,q)\in\overline{Z}$ be GIT-(semi)stable, and let $(\mathcal{F},s)$ be the associated pair. Then $(\mathcal{F},s)$ is nondegenerate. Indeed, let $W=\mathrm{Im}(\sigma)$, then $\mathcal{F}_W=\mathrm{Im}(s)$. If $W=V$, then $\mathrm{Im}(s)=\mathcal{F}$; otherwise, according to (3.2), $P_{\mathcal{E}}(\mathrm{Im}(s))>\left(\frac{1}{2r}\left(1-\frac{\dim W}{\dim V}\right)+\frac{\dim W}{\dim V}\right)P>0$. Hence, $\mathrm{Im}(s)\neq 0$.

Lemma 3.18. *Let $(\sigma,q)\in\overline{Z}$ be GIT-(semi)stable with associated pair $(\mathcal{F},s)$. For any coherent subsheaf $\mathcal{F}'\subset\mathcal{F}$, let $(\mathcal{F}',s')$ denote the induced subpair and $W=V\cap H^0(F_{\mathcal{E}}(\mathcal{F}')(m))$, then*

$$P_{\mathcal{E}}(\mathcal{F}')(\geq)\frac{P}{2r}\left(\epsilon(\mathcal{F}')-\frac{\dim W}{\dim V}\right)+P\frac{\dim W}{\dim V}, \tag{3.3}$$

where $\epsilon(\mathcal{F}')=1$ if $\mathrm{Im}(s)\subset\mathcal{F}'$; 0 otherwise.

Proof. The proof is similar to [35, Proposition 4.3]. According to [26, Rem 6.14], we obtain a natural injection $\mathcal{F}_W\hookrightarrow\mathcal{F}'$. If $\epsilon_W(\sigma)=1$, i.e. $\mathrm{Im}(\sigma)\subset W$, then $\mathrm{Im}(s)\subset\mathcal{F}_W\subset\mathcal{F}'$. Thus, $\epsilon(\mathcal{F}')=1$ and (3.3) is the same as (3.2).

We only need to consider the case when $\epsilon_W(\sigma)=0$ and $\mathrm{Im}(s)\subset\mathcal{F}'$. Let

$W' = W \oplus \operatorname{Im}(\sigma)$. Clearly, $\mathcal{F}_{W'} \subset \mathcal{F}'$ and $\epsilon_{W'}(\sigma) = 1$. By (3.2), we have

$$\begin{aligned} P_{\mathcal{E}}(\mathcal{F}') \geq P_{\mathcal{E}}(\mathcal{F}_{W'})(\geq) \frac{P}{2r}\left(1 - \frac{\dim W'}{\dim V}\right) + P\frac{\dim W'}{\dim V} \\ \geq \frac{P}{2r}\left(1 - \frac{\dim W}{\dim V}\right) + P\frac{\dim W}{\dim V}. \end{aligned}$$

□

When defining Z, we require $\mathcal{F}$ to be pure and q induces an isomorphism $V \simeq H^0(F_{\mathcal{E}}(\mathcal{F})(m))$. When taking closure, these may no longer be true. The following two Corollaries impose restrictions.

Corollary 3.19. *If $(\sigma, q) \in \overline{Z}$ is GIT-semistable with associated pair $(\mathcal{F}, s)$, then the induced map $V \to H^0(F_{\mathcal{E}}(\mathcal{F})(m))$ is injective and for any coherent subsheaf $\mathcal{G} \subset \mathcal{F}$ such that $\dim \mathcal{G} \leq d-1, H^0(F_{\mathcal{E}}(\mathcal{G})(m)) \cap V = 0$.*

Proof. Same as [26, Lemma 6.16] and [18, Cor 4.4.7] using (3.2). □

Corollary 3.20. *If $(\sigma, q) \in \overline{Z}$ is GIT-semistable with associated pair $(\mathcal{F}, s)$. Then there exists a pure coherent sheaf $\mathcal{H}$ such that*

$$0 \to T_{d-1}(\mathcal{F}) \to \mathcal{F} \xrightarrow{\phi} \mathcal{H}$$

is exact and $P_{\mathcal{E}}(\mathcal{H}) = P_{\mathcal{E}}(\mathcal{F})$. Moreover, the induced pair $(\mathcal{H}, \phi \circ s)$ is nondegenerate.

Proof. The first part is just [26, Lemma 6.10]. For the second part, let $W = \operatorname{Im}(\sigma)$, then $\mathcal{F}_W = \operatorname{Im}(s)$. By looking at the leading coefficients in (3.2), we see that $\operatorname{Im}(s)$ has dimension d. Thus, $\operatorname{Im}(s) \subsetneq T_{d-1}(\mathcal{F})$ and $\phi \circ s \neq 0$.

□

Now we are ready to compare δ-stability and GIT-stability.

Proposition 3.21. *Let $(\sigma, q) \in \overline{Z}$ with associated pair $(\mathcal{F}, s)$. The following two assertions are equivalent:*

(1) *(σ, q) is GIT-(semi)stable with respect to L.*

(2) *$(\mathcal{F}, s)$ is (semi)stable and q induces an isomorphism $V \simeq H^0(F_{\mathcal{E}}(\mathcal{F})(m))$.*

Proof. Let $(\sigma, q) \in \overline{Z}$ be GIT-(semi)stable. Let $\phi : \mathcal{F} \to \mathcal{H}$ be as in Corollary 3.20. Then $(\mathcal{H}, \phi \circ s)$ is nondegenerate. Since $\ker \phi = T_{d-1}(\mathcal{F})$, according to Corollary 3.19, the induced map

$$V \hookrightarrow H^0(F_{\mathcal{E}}(\mathcal{F})(m)) \to H^0(F_{\mathcal{E}}(\mathcal{H})(m)) \tag{3.4}$$

is injective. For any dimension d quotient $\rho : \mathcal{H} \to \mathcal{G}$, let $\mathcal{K} = \ker \rho \circ \phi$. We obtain an exact sequence

$$0 \to \mathcal{K} \to \mathcal{F} \xrightarrow{\rho \circ \phi} \mathcal{G}.$$

Let $W = V \cap H^0(F_{\mathcal{E}}(\mathcal{K})(m))$. Then we have

$$h^0(F_{\mathcal{E}}(\mathcal{G})(m)) \geq h^0(F_{\mathcal{E}}(\mathcal{F})(m)) - h^0(F_{\mathcal{E}}(\mathcal{K})(m)) \geq \dim V - \dim W. \tag{3.5}$$

By taking the leading coefficients in (3.3) we get

$$(2r_{\mathcal{E}}(\mathcal{K}) - \epsilon(\mathcal{K})) \dim V \geq (2r - 1) \dim W. \tag{3.6}$$

Since $T_{d-1}(\mathcal{F}) \subset \mathcal{K}$, we have

$$\mathcal{K}/T_{d-1}(\mathcal{F}) \hookrightarrow \mathcal{F}/T_{d-1}(\mathcal{F}) \hookrightarrow \mathcal{H} \twoheadrightarrow \mathcal{G}.$$

It follows that $r = r_{\mathcal{E}}(\mathcal{H}) \geq r_{\mathcal{E}}(\mathcal{K}/T_{d-1}(\mathcal{F})) + r_{\mathcal{E}}(\mathcal{G}) = r_{\mathcal{E}}(\mathcal{K}) + r_{\mathcal{E}}(\mathcal{G})$. Combining this with (3.5) and (3.6), we have

$$\frac{h^0(F_{\mathcal{E}}(\mathcal{G})(m))}{2r_{\mathcal{E}}(\mathcal{G}) - \epsilon(\rho \circ \phi \circ s)} \geq \frac{\dim V}{2r-1} \cdot \frac{2r_{\mathcal{E}}(\mathcal{G}) - (1-\epsilon(\mathcal{K}))}{2r_{\mathcal{E}}(\mathcal{G}) - \epsilon(\rho \circ \phi \circ s)}$$

If $\epsilon(\rho \circ \phi \circ s) = 0$, then $\mathrm{Im}(s) \subset \mathcal{K}$. Hence, $\epsilon(\mathcal{K}) = 1$. Then the above inequality becomes

$$\frac{h^0(F_{\mathcal{E}}(\mathcal{G})(m))}{2r_{\mathcal{E}}(\mathcal{G}) - \epsilon(\rho \circ \phi \circ s)} \geq \frac{\dim V}{2r-1} = \frac{P(m)}{2r-1}.$$

According to Proposition 3.11, the pair $(\mathcal{H}, \phi \circ s)$ is (semi)stable. In particular, $h^0(F_{\mathcal{E}}(\mathcal{H})(m)) = P(m)$. By a dimension reason, the induced map (3.4) is an isomorphism and

$$V \simeq H^0(F_{\mathcal{E}}(\mathcal{F}(m))).$$

Moreover, we obtain the following commutative diagram:

$$\begin{array}{ccc}
V \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(-m) & \overset{q}{\searrow} & \\
\downarrow \wr & & \\
H^0(F_{\mathcal{E}}(\mathcal{F}(m))) \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(-m) & \twoheadrightarrow & \mathcal{F} \\
\downarrow \wr & & \downarrow \phi \\
H^0(F_{\mathcal{E}}(\mathcal{H}(m))) \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(-m) & \twoheadrightarrow & \mathcal{H}.
\end{array}$$

Hence, ϕ is surjective. Since $P_{\mathcal{E}}(\mathcal{F}) = P_{\mathcal{E}}(\mathcal{H})$, ϕ is an isomorphism. Thus, $(\mathcal{F}, s)$ is (semi)stable.

Conversely, assume $(\mathcal{F}, s)$ is (semi)stable and $V \simeq H^0(F_{\mathcal{E}}(\mathcal{F}(m)))$. For any nontrivial proper subspace $W < V$, let $\mathcal{F}' = q(W \otimes \mathcal{E} \otimes \pi^* \mathcal{O}_X(-m))$ and $(\mathcal{F}', s')$ the corresponding subpair. If $(\mathcal{F}', s') = (\mathcal{F}, s)$, then (3.2) is obviously satisfied. Assume that $(\mathcal{F}', s')$ is a proper subpair. By Proposition 3.11, we have

$$\frac{h^0(F_{\mathcal{E}}(\mathcal{F}')(m))}{2r_{\mathcal{E}}(\mathcal{F}') - \epsilon(s')} < \frac{h^0(F_{\mathcal{E}}(\mathcal{F})(m))}{2r - 1}.$$

The following commutative diagram

$$\begin{array}{ccc} W & \longrightarrow & H^0(F_{\mathcal{E}}(\mathcal{F}'(m))) \\ \big\downarrow & & \big\downarrow \\ V & \xrightarrow{\sim} & H^0(F_{\mathcal{E}}(\mathcal{F}(m))). \end{array}$$

implies that $\dim W \leq h^0(F_{\mathcal{E}}(\mathcal{F}'(m)))$. Hence,

$$\frac{\dim W}{2r_{\mathcal{E}}(\mathcal{F}') - \epsilon(s')} < \frac{\dim V}{2r - 1}.$$

Therefore,

$$r_{\mathcal{E}}(\mathcal{F}') > \frac{1}{2}\epsilon(s') - \frac{1}{2} \cdot \frac{\dim W}{\dim V} + r\frac{\dim W}{\dim V}.$$

Notice that $\mathrm{Im}(\sigma) \subset W$ implies $\mathrm{Im}(s) \subset \mathcal{F}'$, we have $\epsilon(s') \geq \epsilon_W(\sigma)$. Combining this with (3.2), (σ, q) is GIT-(semi)stable. □

Proof of Theorem 3.3. Let R denote the locus of stable points such that q induces an isomorphism $V \simeq H^0(F_{\mathcal{E}}(\mathcal{F})(m))$. By Proposition 3.21, $R = Z^s$, the GIT-stable points. Using a similar argument as in [18, Lemma 4.3.1] or the more detailed version on projective stacks [9, Theorem 4.12], we get $\mathcal{M}_{\mathcal{X}}(P) \simeq [Z^s/GL(V)]$. Let M^s be the GIT-quotient, then M^s corepresents $\mathcal{M}_{\mathcal{X}}(P)$. Moreover, M^s is a projective scheme because we don't have any strictly semistable points.

By a similar argument as [18, Cor 1.2.8, Lem 4.3.2], we can show that the stabilizer in $PGL(V)$ of a closed point in Z^s is trivial. By Luna's étale slice Theorem [18, Theorem 4.2.12], $Z^s \to M^s$ is a principal $PGL(V)$-bundle. Since the universal family on Z^s is $PGL(V)$-linearized, it descends to M^s according to [18, Theorem 4.2.14]. Thus, M^s is a fine moduli space. □

Chapter 4: Curve counting via stable pairs

4.1 PT invariants

Definition 4.1. An *orbifold* Calabi-Yau 3-fold (CY3) is a smooth, quasi-projective, Deligne-Mumford stack $\mathcal{X}$ over $\mathbb{C}$ of dimension three having generically trivial stabilizers and trivial canonical bundle,

$$K_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}.$$

The definition implies that the local model for $\mathcal{X}$ at a point p is $[\mathbb{C}^3/G_p]$ where $G_p \subset SL(3, \mathbb{C})$ is the (finite) group of automorphisms of p.

Let $K_c(\mathcal{X})$ be the Grothendieck group of compactly supported coherent sheaves on $\mathcal{X}$. We say $\mathcal{F}_1, \mathcal{F}_2 \in K_c(\mathcal{X})$ are numerically equivalent,

$$\mathcal{F}_1 \sim_{num} \mathcal{F}_2$$

if

$$\chi(\mathcal{G} \otimes \mathcal{F}_1) = \chi(\mathcal{G} \otimes \mathcal{F}_2)$$

for all locally free sheaves $\mathcal{G}$ on $\mathcal{X}$. In particular, $P_{\mathcal{E}}(\mathcal{F}_1) = P_{\mathcal{E}}(\mathcal{F}_2)$. We define

$$K(\mathcal{X}) = K_c(\mathcal{X}) / \sim_{num} .$$

There is a natural filtration

$$F_0K(\mathcal{X}) \subset F_1K(\mathcal{X}) \subset F_2K(\mathcal{X}) \subset K(\mathcal{X})$$

given by the dimension of the support.

Given $\beta \in F_1K(\mathcal{X})/F_0K(\mathcal{X})$, the moduli space $P(\mathcal{X}, \beta)$ parameterizes stable pairs

$$\mathcal{O}_{\mathcal{X}} \xrightarrow{s} \mathcal{F}$$

where $[\mathcal{F}] = \beta$. The two stability conditions are:

1. the sheaf $\mathcal{F}$ is pure with compact support,

2. the section s has 0-dimensional cokernel.

By Definition 2.3, we can embed $\mathcal{X}$ into a projective stack. It follows from Theorem 3.3 that $P(\mathcal{X}, \beta)$ is a quasi-projective scheme.

Let

$$\mathcal{C}_{\mathcal{F}} = \mathrm{Supp}(\mathcal{F}) = V(\mathrm{Ann}(\mathcal{F}))$$

be the support of $\mathcal{F}$.

Lemma 4.2. *For a stable pair* $(\mathcal{F}, s)$*,*

$$\mathrm{Supp}(\mathrm{Im}(s)) = \mathcal{C}_{\mathcal{F}}.$$

Proof. This is the stacky version of [31, Lemma 1.6]. It suffices to show that $\mathrm{Ann}(\mathrm{Im}(s)) \subset \mathrm{Ann}(\mathcal{F})$. Let $a \in \mathrm{Ann}(\mathrm{Im}(s))$. If $a \notin \mathrm{Ann}(\mathcal{F})$, let f be a section for which af is not 0. Let Z be the 0-dimensional support of $\mathrm{Coker}(s)$ and U be its complement. Since $\mathcal{F}|_U = \mathrm{Im}(s)|_U$, we obtain $(af)|_U = a|_U f|_U = 0$. Hence, the subsheaf generated by af has dimension 0 support, which violates the purity of $\mathcal{F}$. □

Since $\mathrm{Im}(s)$ is a quotient of $\mathcal{O}_{\mathcal{X}}$, $\mathrm{Im}(s)$ is a structure sheaf. By Lemma 4.2, $\mathcal{O}_{\mathcal{C}_{\mathcal{F}}} \simeq \mathrm{Im}(s)$ is pure. We have the following exact sequence,

$$0 \to \mathcal{I}_{\mathcal{C}_{\mathcal{F}}} \to \mathcal{O}_{\mathcal{X}} \xrightarrow{s} \mathcal{F} \to \mathcal{Q} \to 0.$$

The cokernel $\mathcal{Q}$ has dimension 0 support. The reduced support stack, $\mathrm{Supp}^{red}(\mathcal{Q})$, is called the *zero locus* of the pair. The zero locus lies on $\mathcal{C}_{\mathcal{F}}$.

Let $\mathcal{C} \subset \mathcal{X}$ be a fixed curve with compact support and pure structure sheaf $\mathcal{O}_{\mathcal{C}}$. Let $\mathfrak{m} \subset \mathcal{O}_{\mathcal{C}}$ be the ideal sheaf of a 0-dimensional reduced substack. Since $\mathfrak{m}^r/\mathfrak{m}^{r+1}$ has dimension 0 support and $\mathcal{O}_{\mathcal{C}}$ is pure, $\mathscr{H}om(\mathfrak{m}^r/\mathfrak{m}^{r+1}, \mathcal{O}_{\mathcal{C}}) = 0$. Applying $\mathscr{H}om(\cdot, \mathcal{O}_{\mathcal{C}})$ to the following exact sequence

$$0 \to \mathfrak{m}^{r+1} \to \mathfrak{m}^r \to \mathfrak{m}^r/\mathfrak{m}^{r+1} \to 0$$

yields the inclusion

$$\mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_\mathcal{C}) \hookrightarrow \mathscr{H}om(\mathfrak{m}^{r+1}, \mathcal{O}_\mathcal{C}).$$

In particular, the inclusion $\mathfrak{m}^r \hookrightarrow \mathcal{O}_\mathcal{C}$ induces a canonical section

$$\mathcal{O}_\mathcal{C} \hookrightarrow \mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_\mathcal{C}).$$

Let $(\mathcal{F}, s)$ be a stable pair with support $\mathcal{C}$ satisfying

$$\mathrm{Supp}^{red}(\mathcal{Q}) \subset \mathrm{Supp}(\mathcal{O}_\mathcal{C}/\mathfrak{m}).$$

Notice that $\mathscr{H}om(\mathcal{Q}, \mathcal{O}_\mathcal{C}) = 0$ by purity of $\mathcal{O}_\mathcal{C}$. Applying $\mathscr{H}om(\cdot, \mathcal{O}_\mathcal{C})$ to the following exact sequence

$$0 \to \mathcal{O}_\mathcal{C} \to \mathcal{F} \to \mathcal{Q} \to 0$$

yields the inclusion

$$0 \to \mathscr{H}om(\mathcal{F}, \mathcal{O}_\mathcal{C}) \to O_\mathcal{C}.$$

Let $\mathcal{O}_\mathcal{Z}$ be the cokernel, then $\mathcal{I}_\mathcal{Z} = \mathscr{H}om(\mathcal{F}, \mathcal{O}_\mathcal{C})$. Since $\mathcal{F}$ is isomorphic to $\mathcal{O}_\mathcal{C}$ away from the support of $\mathcal{Q}$, we have $\mathcal{Z}$ is 0-dimensional and

$$\mathcal{Z}^{red} \subset \mathrm{Supp}^{red}(\mathcal{Q}) \subset \mathrm{Supp}(\mathcal{O}_\mathcal{C}/\mathfrak{m}).$$

For $r \gg 0$, there is an inclusion $\mathfrak{m}^r \subset \mathcal{I}_\mathcal{Z}$ with 0-dimensional cokernel. By purity, we get

$$\mathscr{H}om(\mathcal{I}_{\mathcal{Z}}, \mathcal{O}_{\mathcal{C}}) \subset \mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_{\mathcal{C}}).$$

The obvious double dual

$$\mathcal{F} \to \mathscr{H}om(\mathscr{H}om(\mathcal{F}, \mathcal{O}_{\mathcal{C}}), \mathcal{O}_{\mathcal{C}}) = \mathscr{H}om(\mathcal{I}_{\mathcal{Z}}, \mathcal{O}_{\mathcal{C}})$$

is isomorphic away from the support of $\mathcal{Q}$, so is an injection by the purity of $\mathcal{F}$. Therefore, we obtain

$$\mathcal{O}_{\mathcal{C}} \hookrightarrow \mathcal{F} \hookrightarrow \mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_{\mathcal{C}}).$$

with composition the canonical section. Dividing by $\mathcal{O}_{\mathcal{C}}$, we get

$$\mathcal{Q} \subset \mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_{\mathcal{C}})/\mathcal{O}_{\mathcal{C}}. \tag{4.1}$$

Conversely, given (4.1), let $\mathcal{F}$ be the preimage of $\mathcal{Q}$ in $\mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_{\mathcal{C}})$. Since $\mathcal{O}_{\mathcal{C}}$ is pure, $\mathcal{F}$ is also pure. Moreover, $\mathcal{F}$ fits into an exact sequence

$$\mathcal{O}_{\mathcal{X}} \twoheadrightarrow \mathcal{O}_{\mathcal{C}} \to \mathcal{F} \to \mathcal{Q} \to 0.$$

Let s denote the section. By Lemma 4.2, $(\mathcal{F}, s)$ is stable with support $\mathcal{C}$. We obtain the following stacky version of [31, Proposition 1.8].

Lemma 4.3. *A stable pair $(\mathcal{F}, s)$ with support $\mathcal{C}$ and*

$$\mathrm{Supp}^{red}(\mathcal{Q}) \subset \mathrm{Supp}(\mathcal{O}_{\mathcal{C}}/\mathfrak{m})$$

is equivalent to a coherent subsheaf $\mathcal{Q} \subset \varinjlim \mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_{\mathcal{C}})/\mathcal{O}_{\mathcal{C}}$.

Let $D^b(\mathcal{X})$ be the bounded derived category of coherent sheaves on $\mathcal{X}$. To each stable pair

$$[\mathcal{O}_X \xrightarrow{s} \mathcal{F}] \in P(\mathcal{X}, \beta)$$

we associate a complex

$$\mathcal{I}^\bullet = \{\mathcal{O}_X \to \mathcal{F}\} \in D^b(\mathcal{X}).$$

As in [31], $P(\mathcal{X}, \beta)$ can be viewed as a component of the moduli space of complexes with trivial determinant in $D^b(\mathcal{X})$. Using the stability condition and same argument as in [31, Lemma 1.14], we obtain

$$\mathscr{E}xt^{\leq -1}(\mathcal{I}^\bullet, \mathcal{I}^\bullet) = 0, \ \mathscr{H}om(\mathcal{I}^\bullet, \mathcal{I}^\bullet) = \mathcal{O}_{\mathcal{X}}.$$

In particular, $\mathcal{I}^\bullet$ is simple. Using the result of [19] or a similar argument as in [16, Proposition 2.2.1] for moduli space of stable sheaves with fixed determinant, we obtain a symmetric perfect obstruction theory on $P(\mathcal{X}, \beta)$ with tangent space governed by $\mathrm{Ext}^1_0(\mathcal{I}^\bullet, \mathcal{I}^\bullet)$ and obstruction space governed by $\mathrm{Ext}^2_0(\mathcal{I}^\bullet, \mathcal{I}^\bullet)$ where the subscript 0 denotes trace-free Ext. By [5, Section 5], it gives rise to a virtual fundamental class $[P(\mathcal{X}, \beta)]^{\mathrm{vir}} \in A_0(\mathcal{X})$. When $\mathcal{X}$ is projective, $P(\mathcal{X}, \beta)$ is also projective, and the virtual fundamental class $[P(\mathcal{X}, \beta)]^{\mathrm{vir}}$ can be integrated to an integer

$$\#^{\mathrm{vir}}(P(\mathcal{X}, \beta)) = \int_{[P(\mathcal{X}, \beta)]^{\mathrm{vir}}} 1.$$

In [4], Behrend defined an integer-valued constructible function

$$v_S : S \to \mathbb{Z}$$

associated to any scheme S over $\mathbb{C}$. The weighted Euler characteristic is defined to be

$$\widetilde{\chi}(S) = \chi(S, v_S) = \sum_{k \in \mathbb{Z}} k \chi_{\text{top}}(v_S^{-1}(k))$$

where $\chi_{\text{top}}(\cdot)$ is the topological Euler characteristic. If S is endowed with a symmetric obstruction theory and assume that S is proper. Behrend [4, Theorem 4.18] proved that

$$\#^{\text{vir}}(S) = \widetilde{\chi}(S).$$

Definition 4.4 (PT invariants)**.** The *PT invariant* of $\mathcal{X}$ in the class $\beta \in F_1K(\mathcal{X})$ is given by

$$PT_\beta(\mathcal{X}) = \widetilde{\chi}(P(\mathcal{X}, \beta)).$$

Notice that this is well defined for non-compact geometries.

We define the *PT partition function* by

$$PT(\mathcal{X}) = \sum_{\beta \in F_1K(\mathcal{X})} PT_\beta(\mathcal{X}) q^\beta.$$

With an appropriate choice of a basis $e_1, \cdots, e_r$ for $F_1K(\mathcal{X})$, we can regard $PT(\mathcal{X})$

as a formal Laurent series in the variables $q_1, \cdots, q_r$, where

$$q^\beta = q_1^{d_1} \cdots q_r^{d_r}$$

for $\beta = \sum_{i=1}^r d_i e_i$.

We end this section with some facts about the Behrend function v_S.

- If S is smooth at P, then $v_S(P) = (-1)^{\dim S}$ [4, Section 1.2].
- If S admits a $\mathbb{G}_m$-action with isolated fixed points and a $\mathbb{G}_m$-equivariant symmetric obstruction theory, then for each fixed point P,

$$v_S(P) = (-1)^{\dim T_{S|P}}.$$

In particular,

$$\widetilde{\chi}(S) = \sum_P (-1)^{\dim T_P S}, \tag{4.2}$$

where the sum is over the $\mathbb{G}_m$-fixed points [6, Theorem 3.4].

4.2 Orbifold DT crepant resolution conjecture (CRC) and DT/PT correspondence

Let $\mathcal{X}$ be an orbifold CY3 and let X be its coarse space. Given $\alpha \in K(\mathcal{X})$, let $\mathrm{Hilb}^\alpha(\mathcal{X})$ be the category of families of substacks $\mathcal{Z} \subset \mathcal{X}$ having $[\mathcal{O}_\mathcal{Z}] = \alpha$. By [29, Theorem 1.5], $\mathrm{Hilb}^\alpha(\mathcal{X})$ is represented by a quasi-projective scheme.

Definition 4.5. The *DT invariant* of $\mathcal{X}$ in the class $\beta \in F_1K(\mathcal{X})$ is given by

$$DT_\beta(\mathcal{X}) = \widetilde{\chi}(\mathrm{Hilb}^\beta(\mathcal{X})).$$

where $\widetilde{\chi}(\cdot)$ is the weighted Euler characteristic.

The *DT partition function* is defined as

$$DT(\mathcal{X}) = \sum_{\beta \in F_1K(\mathcal{X})} DT_\beta(\mathcal{X}) q^\beta.$$

The *degree zero* DT partition function is

$$DT_0(\mathcal{X}) = \sum_{\alpha \in F_0K(\mathcal{X})} DT_\alpha(\mathcal{X}) q^\alpha,$$

and the *reduced* DT partition function is

$$DT'(\mathcal{X}) = \frac{DT(\mathcal{X})}{DT_0(\mathcal{X})}.$$

Let $Y = \mathrm{Hilb}^{[\mathcal{O}_p]}(\mathcal{X})$ be the Hilbert scheme parameterizing substacks in the class $[\mathcal{O}_p] \in F_0K(\mathcal{X})$. According to [8], Y is a smooth CY3 and Y is a *crepant resolution* of X, i.e. there is resolution of singularities $\pi : Y \to X$ such that $\pi^* K_X = K_Y$. Moreover, there is a Fourier-Mukai isomorphism

$$\Phi : K(\mathcal{X}) \to K(Y),\ \mathcal{F} \mapsto Rq_* p^* \mathcal{F},$$

where $p : Z \to \mathcal{X}, q : Z \to Y$ are the projections from the universal substack $Z \subset \mathcal{X} \times Y$ onto each factor. This isomorphism doesn't respect the filtration $F_\bullet K(\mathcal{X})$ and $F_\bullet K(Y)$. However, if $\mathcal{X}$ satisfies the *hard Lefschetz condition* [12, Def 1.1], which in this case is equivalent [11, Lemma 24] to the condition that all G_p are finite subgroups of $SO(3) \subset SU(3)$ or $SU(2) \subset SU(3)$, then the image of $F_0K(\mathcal{X})$ under Φ is contained in $F_1K(Y)$. Let $F_{exc}K(Y) = \Phi(F_0K(\mathcal{X}))$, whose elements can be represented by formal differences of sheaves supported on the exceptional fibers of $\pi : Y \to X$, and $F_{mr}K(\mathcal{X}) = \Phi^{-1}(F_1K(Y))$, whose elements can be represented by formal differences of sheaves supported in dimension one where at the generic point of each curve in the support, the associated representation of the stabilizer of that point is a multiple of the regular representation. We have the following commutative diagram

$$\begin{array}{ccccccc} & & F_0K(\mathcal{X}) & \hookrightarrow & F_{mr}K(\mathcal{X}) & \hookrightarrow & F_1K(\mathcal{X}) \\ & & \downarrow \wr\, \Phi & & \downarrow \wr\, \Phi & & \\ F_0K(Y) & \hookrightarrow & F_{exc}K(Y) & \hookrightarrow & F_1K(Y). & & \end{array}$$

Define the *exceptional* DT partition function of Y and *multi-regular* DT partition function of $\mathcal{X}$ to be:

$$DT_{exc}(Y) = \sum_{\alpha \in F_{exc}K(Y)} DT_\alpha(Y) q^\alpha,$$

$$DT_{mr}(\mathcal{X}) = \sum_{\beta \in F_{mr}K(\mathcal{X})} DT_\beta(\mathcal{X}) q^\beta$$

Jim Bryan and David Steinberg [13, Conjecture 1.1] made the following conjecture:

Conjecture 4.6 (CRC)**.** *Let $\mathcal{X}$ be an orbifold CY3 satisfying the hard Lefschetz condition. Let Y be the CY resolution of X described above. Then using Φ to identify the variables we have*

$$\frac{DT_{mr}(\mathcal{X})}{DT_0(\mathcal{X})} = \frac{DT(Y)}{DT_{exc}(Y)}.$$

Conjecture 4.7 (Orbifold DT/PT correspondence)**.** *Let $\mathcal{X}$ be an orbifold CY3 satisfying the hard Lefschetz condition. Then*

$$PT_{mr}(\mathcal{X}) = \frac{DT_{mr}(\mathcal{X})}{DT_0(\mathcal{X})}.$$

4.3 Orbifold toric CY3s and web diagrams

Let $\mathcal{X}$ be an orbifold toric CY3. By [10, Lemma 40], $\mathcal{X}$ is uniquely determined by its coarse moduli space X, a toric variety with Gorenstein finite quotient singularities and trivial canonical bundle. The combinatorial data determining an orbifold toric CY3 is expressed as the data of a web diagram, which is essentially dual to the data of a fan.

Definition 4.8. A *web diagram* consists of the following data.

- A finite trivalent graph Γ.
- A marking $\{x_{v,e}\}$, which consists of a non-zero vector $x_{v,e} \in \mathbb{Z}^2$ for each pair (v, e) where e is an edge incident to a vertex v.

- For each compact edge e with bounding vertices v and v',

$$x_{v,e} + x_{v',e} = 0.$$

- For each vertex v with incident edges (e_1, e_2, e_3),

$$x_{v,e_1} + x_{v,e_2} + x_{v,e_3} = 0.$$

Two markings $\{x_{v,e}\}$ and $\{x'_{v,e}\}$ are equivalent if there exists $g \in SL_2(\mathbb{Z})$ such that $g \cdot x_{v,e} = x'_{v,e}$ for all (v, e).

Let $\mathcal{X}$ be an orbifold toric CY3 with coarse moduli space X. Such an X determines a simplicial fan $\Sigma \subset N \otimes \mathbb{Q}$ where $N \simeq \mathbb{Z}^3$. Since the canonical divisor is trivial, there is a linear function $l : N \to \mathbb{Z}$ such that $l(v_i) = 1$ for all the generators of 1-dimensional cones of Σ. Let $\widehat{\Gamma}$ be the intersection of Σ with the plane $\{v : l(v) = 1\}$. $\widehat{\Gamma}$ is a triangulation with integral vertices. Let Γ be graph dual to $\widehat{\Gamma}$ in the plane $\{v : l(v) = 1\}$. Under duality, a vertex v with incident edge e corresponds to a triangle $\hat{v}$ in $\widehat{\Gamma}$ and a bounding edge $\hat{e}$. We define a marking on Γ as follows. Fixing an orientation on the plane, the edge $\hat{e}$ inherits an orientation from the triangle $\hat{v}$. The oriented edge defines an integral vector $x_{v,e}$ in $\{v : l(v) = 0\}$. The set $\{x_{v,e}\}$ makes the graph Γ a web diagram.

Remark 4.9. The vertices of Γ correspond to the torus fixed points in $\mathcal{X}$, the edges correspond to torus invariant curves, and the regions in the plane delineated by the graph correspond to torus invariant divisors. Γ will necessarily have some non-

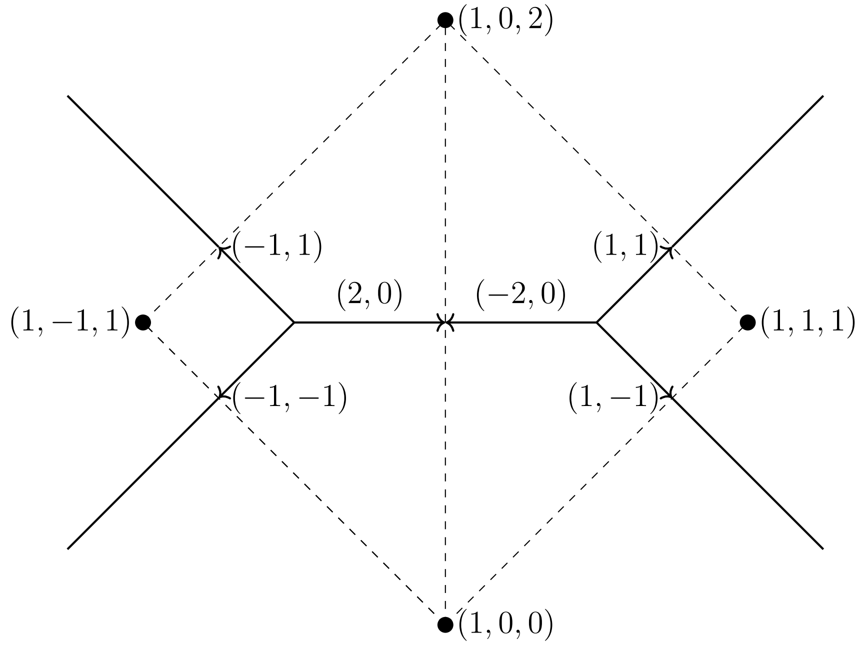


Figure 4.1: The web diagram for local $\mathbb{P}^1 \times B\mathbb{Z}_2$

compact edges; these correspond to edges incident to only one vertex. We denote the set of compact edges by Edges^{cpt}.

Example 4.10. Let $\mathcal{X}$ be the local $\mathbb{P}^1 \times B\mathbb{Z}_2$, namely the global quotient of the resolved conifold $\text{Tot}(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1)$ by $\mathbb{Z}_2$ acting fiberwise by -1. The web diagram of $\mathcal{X}$ is given in Figure 4.1.

Example 4.11. Let a, b be positive integers. Let

$$\mathcal{X}_{a,b} = \text{Tot}\left(\mathcal{O}(-p_0) \oplus \mathcal{O}(-p_\infty) \to \mathbb{P}^1_{a,b}\right)$$

be the total space of the bundle $\mathcal{O}(-p_0) \oplus \mathcal{O}(-p_\infty)$ over the football $\mathbb{P}^1_{a,b}$ which is by definition $\mathbb{P}^1$ with root construction [14] of order a and b at the points p_0 and p_∞. The web diagram of $\mathcal{X}_{a,b}$ is given in Figure 4.2.

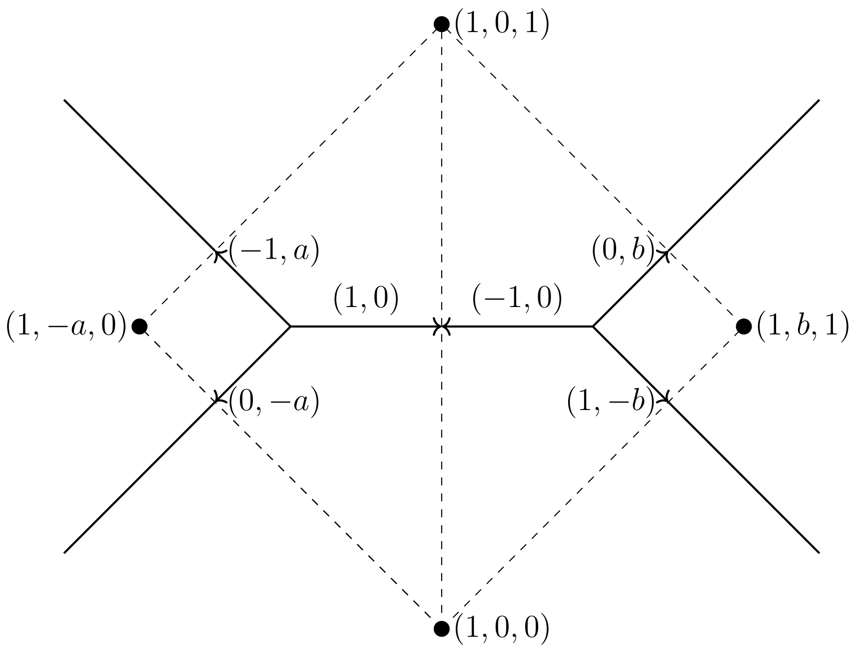


Figure 4.2: The web diagram for local football $\mathcal{X}_{a,b}$

Locally, $\mathcal{X}$ is of the form $[\mathbb{C}^3/G]$ where G is a finite subgroup of the torus $T = (\mathbb{C}^*)^3$.

Lemma 4.12 ([10, Lemma 46]). *Let v be the vertex of Γ, let (e_1, e_2, e_3) be the three edges incident to v, and let $x_{v,e_i} = (a_i, b_i)$ be the markings. Then $\mathcal{X}$ has an open neighbourhood about the torus fixed point corresponding to v given by $[\mathbb{C}^3/G]$ where G is the subgroup of the torus $T = (\mathbb{C}^*)^3$ given by*

$$t_1 t_2 t_3 = 1,\ t_i^{a_j} = t_j^{a_i},\ t_i^{b_j} = t_j^{b_i}.$$

The action of G on $\mathbb{C}^3$ is given by

$$(z_1, z_2, z_3) \mapsto (t_1 z_1, t_2 z_2, t_3 z_3)$$

where the z_i coordinate axis is the T invariant curve corresponding to the edge e_i. Moreover, the order of G is given by

$$|G| = x_1 \wedge x_2 = x_2 \wedge x_3 = x_3 \wedge x_1,$$

where $x_i \wedge x_j = a_i b_j - a_j b_i$. The order of H_i, the stabilizer group of a generic point on the T invariant curve corresponding to e_i is given by

$$|H_i| = \gcd(a_i, b_i).$$

4.4 Orbifold toric CY3 with transverse A_{n-1} singularities

Let $\mathcal{X}$ be an orbifold toric CY3 whose orbifold structure is supported on a disjoint union of smooth curves. By Lemma 4.12, the local model is $[\mathbb{C}^3/\mathbb{Z}_n]$ where $\mathbb{Z}_n$ acts on $\mathbb{C}^3$ with weights $(1, -1, 0)$. The coarse space X has transverse A_{n-1} singularities along the curves (where n can vary from curve to curve). In particular, such $\mathcal{X}$ satisfies the hard Lefschetz condition.

Let Γ be the web diagram of $\mathcal{X}$. For each edge e, let $\mathcal{C}(e)$ be the corresponding torus invariant curve. Define $n := n(e)$ such that $\mathbb{Z}_n$ is the local group of $\mathcal{C}(e)$. It will be convenient to choose an orientation on Γ.

Definition 4.13. Let Γ be the web diagram associated to an orbifold toric CY3 with transverse A_{n-1} singularities. An *orientation* is a choice of directions for each edge and an ordering $(e_1(v), e_2(v), e_e(v))$ of the edges incident to each vertex v which

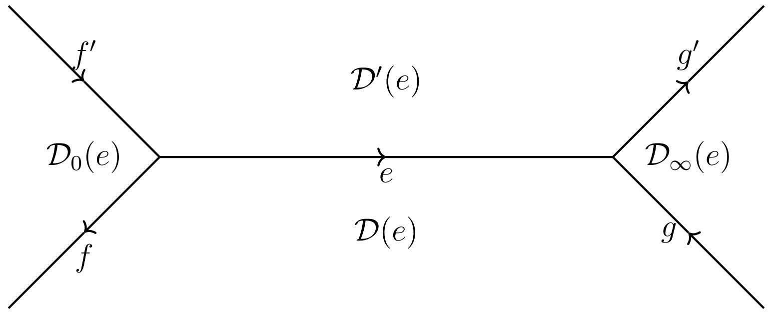


Figure 4.3: The edge e with orientation chosen for adjacent edges

is compatible with the counterclockwise cyclic ordering. If any of the $n(e_i(v)) \neq 1$, we make this (necessarily unique) edge $e_3(v)$. We will call such an edge the *special edge* and denote it as simply $e(v)$.

Given an orientation on Γ and a compact edge e corresponding to $\mathcal{C}(e)$, let $\mathcal{D}(e)$ and $\mathcal{D}'(e)$ denote the two regions incident to e with the convention that $\mathcal{D}(e)$ lies to the right of e. We also use $\mathcal{D}(e)$ and $\mathcal{D}'(e)$ to denote the corresponding torus invariant divisors. Notice that $\mathcal{C}(e) = \mathcal{D}(e) \cap \mathcal{D}'(e)$. Let $p_0(e)$ and $p_\infty(e)$ denote the torus fixed points corresponding to the initial and final vertices incident to e. Let $\mathcal{D}_0(e)$ and $\mathcal{D}_\infty(e)$ denote the torus invariant divisors meeting $\mathcal{C}(e)$ transversely at $p_0(e)$ and $p_\infty(e)$. Given a vertex v, let $\mathcal{D}_1(v), \mathcal{D}_2(v), \mathcal{D}_3(v)$ denote the regions and the corresponding torus invariant divisors opposite the edges $e_1(v), e_2(v), e_3(v)$. The oriented web diagram near the edge e is given in Figure 4.3.

Let e be a compact edge and let $\mathcal{C} = \mathcal{C}(e)$, $\mathcal{D} = \mathcal{D}(e)$, $\mathcal{D}' = \mathcal{D}'(e)$. The normal bundle of $\mathcal{C} \subset \mathcal{X}$ is

$$N_{\mathcal{C}/\mathcal{X}} = \mathcal{O}_{\mathcal{C}}(\mathcal{D}) \oplus \mathcal{O}_{\mathcal{C}}(\mathcal{D}').$$

Let

$$m = \deg \mathcal{O}_{\mathcal{C}}(\mathcal{D}),\ m' = \deg \mathcal{O}_{\mathcal{C}}(\mathcal{D}'). \tag{4.3}$$

If $n = n(e) > 1$, the $\mathcal{C}$ is a $B\mathbb{Z}_n$ gerbe over $\mathbb{P}^1$ and

$$m, m' \in \frac{1}{n}\mathbb{Z}.$$

By Calabi-Yau condition,

$$m + m' = -2.$$

If $n = 1$, then in Figure 4.3, one of

$$a = n(f),\ a' = n(f')$$

and/or one of

$$b = n(g),\ b' = n(g')$$

is possibly greater than 1 and $\mathcal{C}$ is a football: a $\mathbb{P}^1$ with root constructions of order $\max(a, a')$ and $\max(b, b')$ at 0 and ∞.

We define

$$\delta_0 = \begin{cases} 1 & \text{if } a > 1, \\ 0 & \text{if } a = 1, \end{cases}$$

and similarly for δ_0', δ_∞, and δ_∞'. By [10, Lemma 48] and the Calabi-Yau condition

$\mathcal{O}_{\mathcal{C}}(\mathcal{D}+\mathcal{D}') = K_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}}(-p_0 - p_\infty)$, we can write

$$\mathcal{O}_{\mathcal{C}}(\mathcal{D}) = \mathcal{O}_{\mathcal{C}}(\widetilde{m}p - \delta_0 p_0 - \delta_\infty p_\infty), \tag{4.4}$$

$$\mathcal{O}_{\mathcal{C}}(\mathcal{D}') = \mathcal{O}_{\mathcal{C}}(\widetilde{m}'p - \delta_0' p_0 - \delta_\infty' p_\infty), \tag{4.5}$$

where p is a generic point on $\mathcal{C}$, $\widetilde{m}, \widetilde{m}' \in \mathbb{Z}$, and

$$\widetilde{m} + \widetilde{m}' = \delta_0 + \delta_0' + \delta_\infty + \delta_\infty' - 2.$$

Notice that

$$m = \widetilde{m} - \frac{\delta_0}{a} - \frac{\delta_\infty}{b},$$
$$m = \widetilde{m}' - \frac{\delta_0'}{a'} - \frac{\delta_\infty'}{b'}.$$

By convention, we define $\widetilde{m} = m$ and $\widetilde{m}' = m'$ if $n = n(e) > 1$.

As in [10, Section 3.3], we will use the following generators for $F_1K(\mathcal{X})$. Let $p \in \mathcal{X}$ be a generic point and let $p(e) \simeq B\mathbb{Z}_{n(e)}$ be a generic point on the curve $\mathcal{C}(e)$. Let ρ_a, $a \in \{0, \cdots, n(e)-1\}$ be the irreducible representations of $\mathbb{Z}_{n(e)}$ with indexing chosen so that

$$\mathcal{O}_{p(e)}(-a\mathcal{D}(e)) \simeq \mathcal{O}_{p(e)} \otimes \rho_a.$$

We have the following classes in $F_1K(\mathcal{X})$ and their associated variables (see Table 4.1).

Table 4.1: Generators for $F_1K(\mathcal{X})$

Class in $F_1K(\mathcal{X})$	Associated variable	Indexing set
$[\mathcal{O}_p]$	q	
$[\mathcal{O}_{p(e)} \otimes \rho_a]$	$q_{e,a}$	$e \in \text{Edges}, a \in \{0, \cdots, n(e)-1\}$
$[\mathcal{O}_{\mathcal{C}(e)}(-1) \otimes \rho_a]$	$v_{e,a}$	$e \in \text{Edges}^{cpt}, a \in \{0, \cdots, n(e)-1\}$

Remark 4.14. (1) If $\mathcal{C}(e) \simeq \mathbb{P}^1 \times B\mathbb{Z}_{n(e)}$, then $\mathcal{O}_{\mathcal{C}(e)}(-1)$ is the pull back of $\mathcal{O}_{\mathbb{P}^1}(-1)$ and ρ_a is the pullback from $B\mathbb{Z}_{n(e)}$. In general, let $\pi : \widetilde{\mathcal{C}}(e) \to \mathcal{C}(e)$ be the degree $n(e)$ cover obtained from the base change $\mathbb{P}^1 \to \mathbb{P}^1, z \mapsto z^{n(e)}$. Then $\widetilde{\mathcal{C}}(e)$ is a trivial $B\mathbb{Z}_{n(e)}$ gerbe and $[\mathcal{O}_{\mathcal{C}(e)}(-1) \otimes \rho_a]$ is defined to be the class $\frac{1}{n(e)}\pi_*[\mathcal{O}_{\widetilde{\mathcal{C}(e)}}(-1) \otimes \rho_a]$.
(2) The above classes generate $F_1K(\mathcal{X})$ over $\mathbb{Q}$ but there are relations. In particular, for each edge e, there is the relation

$$[\mathcal{O}_p] = [\mathcal{O}_{p(e)} \otimes R_{reg}]$$

where $R_{reg} = \sum_a \rho_a$ denotes the regular representation of $\mathbb{Z}_{n(e)}$. This relation gives rise to the relation

$$q = \prod_{k=0}^{n(e)-1} q_{e,a}.$$

Given a partition $\lambda \subset \mathbb{Z}^2$ and an integer n, let

$$\lambda[a,n] = \{(i,j) \in \lambda : i - j = a \mod n\}$$

denote the set of boxes in λ of color $a \mod n$. Let

$$|\lambda|_a = |\lambda[a,n]|$$

denote the number of boxes of color $a \mod n$ in λ.

Definition 4.15. Let Γ be the web diagram of $\mathcal{X}$ and assume that Γ has an orientation. An *edge assignment* on Γ is a choice of a partition $\lambda(e)$ for each edge e such that $\lambda(e) = \emptyset$ for every non-compact edge. Given a vertex v, we get a triple of partitions $(\lambda_1(v), \lambda_2(v), \lambda_3(v))$ by setting

$$\lambda_i(v) = \begin{cases} \lambda(e_i(v)) & e_i(v) \text{ is oriented outward,} \\ \\ \lambda^t(e_i(v)) & e_i(v) \text{ is oriented inward,} \end{cases}$$

where λ^t is the transpose of λ. An edge assignment is called *multi-regular* if each λ_3 satisfies $|\lambda_3|_a = |\lambda_3|/n(e_3)$ for all a

The action of the torus T on $\mathcal{X}$ induces a T action on $P(\mathcal{X}, \beta)$. Let $\mathbf{Q} \subset P(\mathcal{X}, \beta)^T$ be a connected T-fixed locus. By Lemma 4.21,

$$\mathbf{Q} = \prod_v Q_{\pi(v)},$$

where each $Q_{\pi(v)}$ is a product of $\mathbb{P}^1$'s. $\mathbf{Q}$ corresponds with sets $\{\lambda(e), \pi(v)\}$ where $\{\lambda(e) : e \in \text{Edge}^{cpt}\}$ is an edge assignment and $\{\pi(v) : v \in \text{Vertices}\}$ is a collection of labelled box configurations with outgoing partitions $(\lambda_1(v), \lambda_2(v), \lambda_3(v))$. Here, each $\pi(v)$ is a subset of $\mathbb{Z}^3$ depending on $(\lambda_1(v), \lambda_2(v), \lambda_3(v))$. The complete description of $\pi(v)$ will be given in Section 4.6.

Proposition/Conjecture 4.16. *Let $\mathcal{X}$ be an orbifold toric CY3 with transverse*

A_{n-1} orbifold structure. Then

$$PT_\beta(\mathcal{X}) = \sum_{\mathbf{Q}\subset \mathbf{P}(\mathcal{X},\beta)^{\mathbf{T}}} \chi_{\mathrm{top}}(\mathbf{Q})(-1)^{\dim T_{\mathcal{I}^\bullet} P(\mathcal{X},\beta)},$$

where $\mathcal{I}^\bullet \in \mathbf{Q} \subset P(\mathcal{X},\beta)^T$ is a T-fixed stable pair.

Remark 4.17. To get an explicit formula for $PT_\beta(\mathcal{X})$, we will follow [10] to give a combinatorial description of the T-fixed substacks in Section 4.6 and to calculate the parity of the tangent space to a T-fixed point in Section 4.7.

In the 1-leg or 2-leg case, that is at most 2 of $\lambda_i(v)$'s are nonempty for each v, the T-fixed points are isolated (See Section 4.6). We can prove Proposition 4.16 using (4.2). The 3-leg case is conjectural.

The $\mathbb{Z}_n$ PT vertex (See Definition 4.27)

$$W^n_{\lambda_1\lambda_2\lambda_3}(q_0,\cdots,q_n) = \sum_{Q_\pi} \chi_{\mathrm{top}}(Q_\pi) q_0^{|\pi|_0}\cdots q_{n-1}^{|\pi|_{n-1}}$$

counts colored labelled box configurations with outgoing partitions $(\lambda_1,\lambda_2,\lambda_3)$. We color the boxes in a labelled box configuration π according to the rule that a box $(i,j,k)\in\pi$ has color $i-j \bmod n$. In the 1-leg case we have the following explicit formula for the $\mathbb{Z}_n$ PT vertex. For any partition λ, define

$$q^{-A_\lambda} = \prod_{a=0}^{n-1} q_a^{-A_\lambda(a,n)}$$

where

$$A_\lambda(a,n) = \sum_{(j,k)\in\lambda} \left\lfloor \frac{j+a}{n} \right\rfloor.$$

Theorem 4.18. *Let λ be a partition, then*

$$W^n_{\lambda\emptyset\emptyset}(q_0,\cdots,q_{n-1}) = q^{-A_\lambda}\overline{s_{\lambda^t}(\mathfrak{q})},$$

$$W^n_{\emptyset\lambda\emptyset}(q_0,\cdots,q_{n-1}) = \overline{q^{-A_{\lambda^t}}}s_\lambda(\mathfrak{q}),$$

$$W^n_{\emptyset\emptyset\lambda}(q_0,\cdots,q_{n-1}) = \prod_{\square\in\lambda}\frac{1}{1-\prod_{a=0}^{n-1}q_a^{h_a(\square)}},$$

where $s_\lambda(\mathfrak{q})$ is the Schur function with $\mathfrak{q} = (1, q_1, q_1q_2, q_1q_2q_3, \cdots)$, $h_a(\square)$ denotes the number a-colored boxes in the hook of $\square$, and the overline denotes the exchange of variables $q_a \leftrightarrow q_{-a}$.

We will prove this in Chapter 5.

Given a triple of partitions $(\lambda_1, \lambda_2, \lambda_3)$, we define

$$\lambda_1[a,n] = \{(j,k)\in\lambda_1 | -j \equiv a \bmod n\},$$

$$\lambda_2[a,n] = \{(k,i)\in\lambda_2 | i \equiv a \bmod n\},$$

$$\lambda_3[a,n] = \{(i,j)\in\lambda_3 | i-j \equiv a \bmod n\}$$

to be the set of boxes in λ_i with color a. Let

$$|\lambda_i|_a = |\lambda_i[a,n]|, i = 1,2,3$$

be the number of boxes with color a. Let

$$C^{\lambda}_{\widetilde{m},\widetilde{m}'} = \sum_{(i,j)\in\lambda} (-\widetilde{m}i - \widetilde{m}'j + 1)$$

and let

$$C^{\lambda}_{\widetilde{m},\widetilde{m}'}[a,n] = \sum_{(i,j)\in\lambda[a,n]} (-\widetilde{m}i - \widetilde{m}'j + 1).$$

Let $e = e_3(v)$, $n = n(e)$, and

$$q_v = \begin{cases} (q_{e,0}, q_{e,1}, \cdots, q_{e,n-1}) & e \text{ is oriented outward,} \\ (q_{e,0}, q_{e,n-1}, \cdots, q_{e,1}) & e \text{ is oriented inward.} \end{cases}$$

We define

$$(-1)^{s(\lambda_3)} q_v$$

to be the same as q_v but with each $q_{e,a}$ multiplied by the sign $(-1)^{s_a(\lambda_3)}$ where

$$s_a(\lambda_3) = |\lambda_3|_{a-1} + |\lambda_3|_{a+1} + \delta_{a,0}.$$

We also define

$$v_e^{|\lambda|} := \prod_{a=0}^{n(e)-1} v_{e,a}^{|\lambda|_{a,n(e)}},$$
$$q_e^{C^{\lambda}_{\widetilde{m},\widetilde{m}'}} := \prod_{a=0}^{n(e)-1} ((-1)^{\delta_{a,0}} q_{e,a})^{C^{\lambda}_{\widetilde{m},\widetilde{m}'}[a,n(e)]},$$
$$q_e^{A_\lambda} := \prod_{a=0}^{n(e)-1} ((-1)^{\delta_{a,0}} q_{e,a})^{A_\lambda(a,n(e))}.$$

Finally, let $\lambda = \lambda(e), n = n(e)$, and let

$$\mathrm{SE}_\lambda = \begin{cases} \sum_{a=0}^{n-1} C^\lambda_{m,m'}[a,n](|\lambda|_{a-1} - |\lambda|_{a+1}) + |\lambda|_a(|\lambda|_a + (1+m)|\lambda|_{a-1}), & n > 1 \\ |\lambda|(\widetilde{m} + \delta_0 + \delta_\infty), & n = 1. \end{cases}$$

Let

$$\Sigma_{\pi(v)} = \sum_{a=0}^{n-1} |\lambda_3|_a(|\lambda_1|_a + |\lambda_2|_a + |\lambda_1|_{a+1} + |\lambda_2|_{a-1}).$$

Theorem/Conjecture 4.19. *Let $\mathcal{X}$ be an orbifold toric CY3 with transverse A_{n-1} singularities and let Γ be the diagram of $\mathcal{X}$. Then*

$$PT(\mathcal{X}) = \sum_{\substack{\text{edge} \\ \text{assignments}}} \prod_{e \in \text{Edges}} E_{\lambda(e)} \prod_{v \in \text{Vertices}} (-1)^{\Sigma_{\pi(v)}} W^{n(e_3(v))}_{\lambda_1(v)\lambda_2(v)\lambda_3(v)}((-1)^{s(\lambda_3(v))} q_v) \quad (4.6)$$

where

$$E_\lambda(e) = (-1)^{\mathrm{SE}_\lambda} v_e^{|\lambda|} q_e^{C^\lambda_{\widetilde{m},\widetilde{m}'}} \left(q_f^{A_\lambda}\right)^{\delta_0} \left(q_{f'}^{A_{\lambda^t}}\right)^{\delta_0'} \left(q_g^{A_\lambda}\right)^{\delta_\infty} \left(q_{g'}^{A_{\lambda^t}}\right)^{\delta_\infty'}$$

and (f, f', g, g') are as in Figure 4.3.

4.5 Example: the local football

The graph of the local football $\mathcal{X}_{a,b}$ is in Figure 4.3. Since

$$\mathcal{O}(\mathcal{D}) = \mathcal{O}(-p_0),\ \mathcal{O}(\mathcal{D}') = \mathcal{O}(-p_\infty),$$

we have

$$n(f)=a, n(g')=b, n(f')=n(g)=n(e)=1, \widetilde{m}=\widetilde{m}'=0.$$

To ease notation, let

$$\begin{aligned} v &= v_e, \\ p_k &= q_{f,k},\ k=0,\cdots,a-1, \\ r_l &= q_{g',l},\ k=0,\cdots,b-1, \end{aligned}$$

and

$$q=p_0\cdots p_{a-1}=r_0\cdots r_{b-1}.$$

Here, v and q keep track of the degree and the holomorphic Euler characteristic of the curve, and $p_k \in \widehat{\mathbb{Z}}_a$ and $r_l \in \widehat{\mathbb{Z}}_b$ keep track of the embedded stacky points at p_0 and p_∞. By Theorem 4.19, we have

$$PT(\mathcal{X}_{a,b})=\sum_\lambda E_\lambda \cdot W^a_{\lambda\emptyset\emptyset}((-p_0),p_1,\cdots,p_{a-1})\cdot W^b_{\lambda^t\emptyset\emptyset}((-r_0),r_1,\cdots,r_{b-1})$$

where

$$\begin{aligned} E_\lambda =&(-1)^{|\lambda|}v^{|\lambda|}(-q)^{|\lambda|} \\ &\cdot(-p_0)^{A_\lambda(0,a)}p_1^{A_\lambda(1,a)}\cdots p_{a-1}^{A_\lambda(a-1,a)}(-r_0)^{A_{\lambda^t}(0,b)}r_1^{A_{\lambda^t}(1,b)}\cdots r_{b-1}^{A_{\lambda^t}(b-1,b)}. \end{aligned}$$

Applying the formula in Theorem 4.18, we get

$$W^a_{\lambda\emptyset\emptyset}((-p_0),p_1,\cdots,p_{a-1}) = (-p_0)^{-A_\lambda(0,a)}p_1^{-A_\lambda(1,a)}\cdots p_{a-1}^{-A_\lambda(a-1,a)}s_{\lambda^t}(1,p_{a-1},p_{a-1}p_{a-2},\cdots)$$
$$W^b_{\lambda\emptyset\emptyset}((-r_0),r_1,\cdots,r_{b-1}) = (-r_0)^{-A_{\lambda^t}(0,b)}r_1^{-A_{\lambda^t}(1,b)}\cdots r_{b-1}^{-A_{\lambda^t}(b-1,b)}s_{\lambda}(1,r_{b-1},r_{b-1}r_{b-2},\cdots).$$

Let $Q=(1,q,q^2,\cdots)$ and use the homogeneity of Schur functions

$$w^{|\lambda|}s_\lambda(x_1,x_2,\cdots) = s_\lambda(wx_1,wx_2,\cdots)$$

we get

$$PT(\mathcal{X}_{a,b}) = \sum_\lambda s_{\lambda^t}(-v(-q)Q,-vp_{a-1}(-q)Q,\cdots,-vp_{a-1}p_{a-2}\cdots p_1(-q)Q)$$
$$\cdot\, s_\lambda(Q,r_{b-1}Q,\cdots,r_{b-1}r_{b-2}\cdots r_1Q).$$

Using the orthogonality of Schur functions [24, Section 1.4 (4.3')]

$$\sum_\lambda s_\lambda(x)s_{\lambda^t}(y) = \prod_{i,j}(1+x_iy_j)$$

we arrive the following

Proposition 4.20. *The PT partition function of the local football $\mathcal{X}_{a,b}$ is given by*

$$PT(\mathcal{X}_{a,b}) = \prod_{k=1}^{a}\prod_{l=1}^{b}\prod_{m=1}^{\infty}(1-vp_k\cdots p_{a-1}r_l\cdots r_{b-1}(-q)^m)^m. \qquad (4.7)$$

Since the only stacky curves in $\mathcal{X}_{a,b}$ are non-compact, the edge assignments are multi-regular. Thus,

$$PT_{mr}(\mathcal{X}_{a,b}) = PT(\mathcal{X}_{a,b})$$

The Calabi-Yau resolution $Y \to X$ has a single $(-1,-1)$-curve given by the proper transform of the football to which are attached two chains of $(0,-2)$-curves having $a-1$ and $b-1$ components each. According to Proposition 4.20 and [10, Proposition 3], we have

$$PT_{mr}(\mathcal{X}_{a,b}) = DT'_{mr}(\mathcal{X}_{a,b}) = \frac{DT(Y)}{DT_{exc}(Y)},$$

which verifies the CRC 4.6 and DT/PT correspondence 4.7. Notice on Y the variables $p_1, \cdots, p_{a-1}$ and $r_1, \cdots, r_{b-1}$ corresponds to the classes of the curves in each of the chains and v corresponds to the class of the $(-1,-1)$-curve.

4.6 T-fixed points and the $\mathbb{Z}_n$ PT vertex

Let $\mathcal{X}$ be a toric CY3 orbifold with web diagram Γ. Let $v \in \Gamma$ be a vertex. By Lemma 4.12, $\mathcal{X}$ has an open neighbourhood about the torus fixed point corresponding to v given by $\mathcal{U}_v = [\mathbb{C}^3/G]$ where G is a finite subgroup of the torus $T = (\mathbb{C}^*)^3$.

Let

$$\mathcal{I}^\bullet = [\mathcal{O}_{\mathcal{X}} \xrightarrow{s} \mathcal{F}] \in P(\mathcal{X}, \beta)^T$$

be a T-fixed stable pair. Let

$$\mathcal{I}_v^\bullet = [\mathcal{O}_{\mathcal{U}_v} \xrightarrow{s_v} \mathcal{F}_v]$$

be the restriction of this stable pair on $\mathcal{U}_v$. Notice that $\mathcal{I}_v^\bullet$ is the same as

$$I_v^\bullet = [\mathcal{O}_{\mathbb{C}^3} \xrightarrow{s_v} F_v]$$

where s_v is an G-invariant section of the G-equivariant sheaf F_v on $\mathbb{C}^3$. Thus, we get a T-fixed stable pair on $\mathbb{C}^3$. Conversely, given a T-fixed stable pair $I_v^\bullet$ on $\mathbb{C}^3$. By [30, Section 2.1], s_v is a T-invariant section of the T-equivariant sheaf F_v. Since G is a subgroup of T, we obtain a T-fixed stable pair on $\mathcal{U}_v$.

The restricted data (F_v, s_v) can be characterized as certain labelled box configurations [30, Section 2]. Let C_v be the support of F_v. The subscheme $C_v \subset \mathbb{C}^3$ is T-invariant of pure dimension 1 and is defined by a monomial ideal

$$I_{C_v} \subset \mathbb{C}[x_1, x_2, x_3].$$

The localizations

$$(I_{C_v})_{x_i} \subset \mathbb{C}[x_1, x_2, x_3]_{x_i}, i = 1, 2, 3$$

are all T-fixed and correspond to a triple of partitions $\vec{\mu} = (\mu^1, \mu^2, \mu^3)$. Since C_v has dimension 1, at least one of the μ^i is non-empty.

Conversely, consider a triple $\vec{\mu} = (\mu^1, \mu^2, \mu^3)$ of partitions such that they are not all empty. Let

$$I_{\mu^1} = \mu^1[x_2, x_3] \cdot \mathbb{C}[x_1, x_2, x_3],$$

$$I_{\mu^2} = \mu^2[x_3, x_1] \cdot \mathbb{C}[x_1, x_2, x_3],$$

$$I_{\mu^3} = \mu^3[x_1, x_2] \cdot \mathbb{C}[x_1, x_2, x_3]$$

and let

$$I_{\vec{\mu}} = \bigcap_{i=1}^{3} I_{\mu^i}.$$

The curve $C_{\vec{\mu}}$ with ideal sheaf $I_{\vec{\mu}}$ is easily seen to be the unique T-fixed pure curve with partitions $\vec{\mu}$.

Consider the exact sequence associated to (F_v, s_v),

$$0 \to I_{C_v} \to \mathcal{O}_{\mathbb{C}^3} \xrightarrow{s_v} F_v \to Q_v \to 0.$$

We conclude that $C_v = C_{\vec{\mu}}$ for some $\vec{\mu}$.

Since $\mathrm{Supp}(Q_v)$ is both 0-dimensional by stability and T-fixed, Q_v must be supported at the origin. By Lemma 4.3, the pair (F_v, s_v) corresponds to a T-invariant coherent subsheaf

$$Q_v \subset \varinjlim \mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_{C_{\vec{\mu}}})/\mathcal{O}_{C_{\vec{\mu}}},$$

where $\mathfrak{m} = \langle x_1, x_2, x_3 \rangle$ is the ideal sheaf of the origin. Let

$$M_i = (\mathcal{O}_{C_{\mu^i}})_{x_i} = (\mathbb{C}[x_1, x_2, x_3]/I_{\mu^1})_{x_i}, i = 1, 2, 3.$$

For example,

$$M_1 = \mathbb{C}[x_1, x_1^{-1}] \otimes \frac{\mathbb{C}[x_2, x_3]}{\mu^1[x_2, x_3]},$$

which can be viewed as a cylinder

$$\mathrm{Cyl}_1 = \{(i, j, k) | (j, k) \in \mu^1\} \subset \mathbb{Z}^3$$

in the space of T-weights. By simple calculation,

$$\varinjlim \mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_{C_{\vec{\mu}}}) = \bigoplus_{i=1}^{3} \varinjlim \mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_{C_{\mu^i}}) = \bigoplus_{i=1}^{3} M_i := M.$$

The canonical section $\mathcal{O}_{C_{\vec{\mu}}}$ corresponds to $(1, 1, 1) \in M$ and is T-invariant. Hence, the T-fixed stable pair (F_v, s_v) is equivalent to a finitely generated T-invariant $\mathbb{C}[x_1, x_2, x_3]$-submodule

$$Q_v \subset M/\langle(1, 1, 1)\rangle.$$

For every weight $w \in \mathbb{Z}^3$, let $\mathbf{1}_w, \mathbf{2}_w$ and $\mathbf{3}_w$ be three independent vectors. A $\mathbb{C}$-basis for M is

$$\{\mathbf{i}_w | w \in \mathrm{Cyl}_i\}.$$

The $\mathbb{C}[x_1, x_2, x_3]$-module structure is given by

$$x_1 \cdot \mathbf{i}_w = \mathbf{i}_{w+(1,0,0)}, x_2 \cdot \mathbf{i}_w = \mathbf{i}_{w+(0,1,0)}, x_3 \cdot \mathbf{i}_w = \mathbf{i}_{w+(0,0,1)}.$$

A $\mathbb{C}$-basis for the submodule $\mathcal{O}_{C_{\vec{\mu}}} \subset M$ is given by the set

$$\{\mathbf{1}_w + \mathbf{2}_w + \mathbf{3}_w | w \in \mathbb{Z}^3_{\geq 0}\}$$

where $\mathbf{i}_w = 0$ if $w \notin \mathrm{Cyl}_i$. We can decompose the union of the cylinders Cyl_i into 4 disjoint parts

$$\bigcup_{i=1}^{3} \mathrm{Cyl}_i = \mathrm{I}^+ \cup \mathrm{II} \cup \mathrm{III} \cup \mathrm{I}^-$$

where

$$\begin{aligned}
\mathrm{I}^+ &= \{w | w \text{ has nonnegative coordinates and lies in exactly one of the cylinders}\}, \\
\mathrm{II} &= \{w | w \text{ lies in exactly two of the cylinders}\}, \\
\mathrm{III} &= \{w | w \text{ lies in all three of the cylinders}\}, \\
\mathrm{I}^- &= \{w | w \text{ has at least one negative coordinate}\}.
\end{aligned}$$

The quotient $M/\mathcal{O}_{C_{\vec{\mu}}}$ is supported on $\mathrm{II} \cup \mathrm{III} \cup \mathrm{I}^-$ and has the following $\mathbb{C}$-basis

- If $w \in \mathrm{I}^-$ is supported on Cyl_i, then

$$\mathbb{C} \cdot \mathbf{i}_w \subset M/\mathcal{O}_{C_{\vec{\mu}}}.$$

- If $w \in \mathrm{II}$ is supported on Cyl_i and Cyl_j, then

$$\frac{\mathbb{C} \cdot \mathbf{i}_w \oplus \mathbb{C} \cdot \mathbf{j}_w}{\mathbb{C} \cdot (\mathbf{1}_w + \mathbf{2}_w + \mathbf{3}_w)} = \frac{\mathbb{C} \cdot \mathbf{i}_w \oplus \mathbb{C} \cdot \mathbf{j}_w}{\mathbb{C} \cdot (\mathbf{i}_w + \mathbf{j}_w)} \simeq \mathbb{C} \subset M/\mathcal{O}_{C_{\vec{\mu}}}.$$

- If $w \in \mathrm{III}$, then

$$\frac{\mathbb{C} \cdot \mathbf{1}_w \oplus \mathbb{C} \cdot \mathbf{2}_w \oplus \mathbb{C} \cdot \mathbf{3}_w}{\mathbb{C} \cdot (\mathbf{1}_w + \mathbf{2}_w + \mathbf{3}_w)} \simeq \mathbb{C}^2 \subset M/\mathcal{O}_{C_{\vec{\mu}}}.$$

A finitely generated T-invariant $\mathbb{C}[x_1, x_2, x_3]$-submodule

$$Q \subset M/\mathcal{O}_{C_{\vec{\mu}}}$$

yields the following *labelled box configuration* [30, Section 2.5]: a finite number of boxes supported on $\mathrm{II} \cup \mathrm{III} \cup \mathrm{I}^-$ satisfying the following rules:

1. If $w \in \mathrm{I}^-$ and if any of

$$(w_1 - 1, w_2, w_3), (w_1, w_2 - 1, w_3), (w_1, w_2, w_3 - 1)$$

   support a box then w must support a box.

2. If $w \in \mathrm{II}, w \notin \mathrm{Cyl}_i$, and if any of

$$(w_1 - 1, w_2, w_3), (w_1, w_2 - 1, w_3), (w_1, w_2, w_3 - 1)$$

   support a box other than a type III box labelled by the 1-dimensional subspace

$\mathbb{C} \cdot \mathbf{i}_w$, then w must support a box.

3. If $w \in \mathrm{III}$ and the subspace of

$$\frac{\mathbb{C} \cdot \mathbf{1}_w \oplus \mathbb{C} \cdot \mathbf{2}_w \oplus \mathbb{C} \cdot \mathbf{3}_w}{\mathbb{C} \cdot (\mathbf{1}_w + \mathbf{2}_w + \mathbf{3}_w)}$$

induced by boxes supported on

$$(w_1 - 1, w_2, w_3), (w_1, w_2 - 1, w_3), (w_1, w_2, w_3 - 1)$$

is nonzero, the w must support a box. If the subspace has dimension 1, then w is labelled by the corresponding point in

$$\mathbb{P}^1 = \mathbb{P}\left(\frac{\mathbb{C} \cdot \mathbf{1}_w \oplus \mathbb{C} \cdot \mathbf{2}_w \oplus \mathbb{C} \cdot \mathbf{3}_w}{\mathbb{C} \cdot (\mathbf{1}_w + \mathbf{2}_w + \mathbf{3}_w)}\right)$$

or unlabelled. If the subspace has dimension 2, then w is unlabelled.

We will use $\pi = \pi(v) = \pi(\overrightarrow{\mu})$ to denote such a labelled box configuration.

Lemma 4.21. *Let $\mathcal{X}$ be a toric CY3 orbifold with web diagram Γ. Let $\boldsymbol{Q} \subset P(\mathcal{X}, \beta)^T$ be a connected component of T-fixed locus. Then $\boldsymbol{Q}$ is a product of $\mathbb{P}^1$'s and corresponds with sets $\{\lambda(e), \pi(v)\}$ where $\{\lambda(e) : e \in \mathrm{Edge}^{cpt}\}$ is an edge assignment and $\{\pi(v) : v \in \mathrm{Vertices}\}$ is a collection of labelled box configurations with outgoing partitions $(\lambda_1(v), \lambda_2(v), \lambda_3(v))$.*

Proof. For each vertex $v \in \Gamma$, we have seen that the T-fixed restricted data

$$\mathcal{I}_v^\bullet = [\mathcal{O}_{\mathcal{U}_v} \xrightarrow{s_v} \mathcal{F}_v]$$

locally on each open chart $\mathcal{U}_v = [\mathbb{C}^3/G]$ corresponds to a labelled box configuration $\pi(v)$. The gluing condition is simply the matching of edge partitions. We conclude that

$$\mathbf{Q} = \prod_v Q_v,$$

where each Q_v is a component of the moduli space of T-invariant $\mathbb{C}[x_1, x_2, x_3]$-submodules of $M/\mathcal{O}_{C_{\vec{\mu}}}$. By [30, Proposition 3], each Q_v, as a reduced variety, is a product of $\mathbb{P}^1$'s which is obtained by assigning different labels to each unrestricted path component of labelled type III boxes in $\pi(v)$. By [30, (3-1)], the global to local restriction map of the T-weight 0 part of the Zariski tangent space

$$\mathrm{Ext}^0(\mathcal{I}^\bullet, \mathcal{F})^T = \bigoplus_v \mathrm{Ext}^0(\mathcal{I}_v^\bullet, \mathcal{F}_v)^T$$

is an isomorphism. Since G is a subgroup of T, we have

$$\mathrm{Ext}^0(\mathcal{I}_v^\bullet, \mathcal{F}_v)^T = \left(\mathrm{Ext}^0(I_v^\bullet, F_v)^G\right)^T = \mathrm{Ext}^0(I_v^\bullet, F_v)^T.$$

By [30, Proposition 4],

$$\dim \mathrm{Ext}^0(I_v^\bullet, F_v)^T = \dim Q_v.$$

Therefore, $\mathbf{Q}$ is nonsingular. □

To compute the PT invariant $PT_\beta(\mathcal{X})$ using (4.2), we consider the Calabi-Yau subtorus

$$T_0 = \{(t_1, t_2, t_3) | t_1 t_2 t_3 = 1\} \subset T.$$

Since T_0 acts trivially on $K_{\mathcal{X}}$, we obtain

$$\mathrm{Ext}^1_0(\mathcal{I}^\bullet, \mathcal{I}^\bullet)^\vee \simeq \mathrm{Ext}^2_0(\mathcal{I}^\bullet, \mathcal{I}^\bullet)$$

as T_0-representation. Hence, $P(\mathcal{X}, \beta)$ carries a T_0-equivariant symmetric obstruction theory.

Let $\mathcal{I}^\bullet_v = [\mathcal{O}_{\mathcal{U}_v} \xrightarrow{s_v} \mathcal{F}_v]$ be a T-fixed restricted data, it is also T_0-fixed. By Lemma 4.12,

$$G \subset T_0 \subset T.$$

Hence,

$$\mathrm{Ext}^0(\mathcal{I}^\bullet_v, \mathcal{F}_v)^{T_0} = \left(\mathrm{Ext}^0(I^\bullet_v, F_v)^G\right)^{T_0} = \mathrm{Ext}^0(I^\bullet_v, F_v)^{T_0}.$$

In the 1-leg or 2-leg case, we have (see [30, Section 3.3])

$$\mathrm{Ext}^0(I^\bullet_v, F_v)^{T_0} = 0.$$

It follows that $P(\mathcal{X}, \beta)^{T_0}$ is no larger and consists of a finite number of isolated points. Now we use the fact in the proof of Lemma 4.1 in [6] to find a one-parameter subgroup $\mathbb{G}_m \subset T_0$ with respect to which all weights of all tangent spaces at all fixed points are nonzero. Thus, all $\mathbb{G}_m$-fixed points are also isolated. Using (4.2), we obtain

Proposition 4.22. *In the 1-leg or 2-leg case, the T-action on $P(\mathcal{X},\beta)$ has isolated fixed points, and we have*

$$PT_\beta(\mathcal{X}) = \sum_{\mathcal{I}^\bullet \in P(\mathcal{X},\beta)^T} (-1)^{\dim T_{\mathcal{I}^\bullet} P(\mathcal{X},\beta)}.$$

In the 3-leg case, let X be a smooth toric Calabi-Yau 3 fold , then the loci $P(X,\beta)^{T_0}$ are conjectured to be nonsingular [30, Conjecture 2]. Assuming this conjecture, Pandharipande and Thomas use the localization formula [17] to prove that

$$PT_\beta(X) = \sum_{Q \subset P(X,\beta)^T} \chi_{\mathrm{top}}(Q)(-1)^{\dim T_{I^\bullet} P(X,\beta)}$$

in the localized T-equivariant chow ring $\mathbb{Q}[s_1,s_2,s_3]_{(s_1,s_2,s_3)}$. Combining this with the previous proposition we make the following conjecture

Conjecture 4.23. *Let $\mathcal{X}$ be an orbifold toric CY3. Then*

$$PT_\beta(\mathcal{X}) = \sum_{\mathbf{Q} \subset P(\mathcal{X},\beta)^T} \chi_{\mathrm{top}}(\mathbf{Q})(-1)^{\dim T_{\mathcal{I}^\bullet} P(\mathcal{X},\beta)},$$

where $\mathcal{I}^\bullet \in \mathbf{Q} \subset P(\mathcal{X},\beta)^T$ is a T-fixed stable pair.

Remark 4.24. Let S be a scheme with a $\mathbb{G}_m$-equivariant symmetric obstruction theory and nonsingular fixed loci. Let P be a fixed point (not necessarily isolated), then Conjecture 4.23 suggests that the Behrend function takes value

$$v_S(P) = (-1)^{\dim T_{S|P}}.$$

Next, we write the K-theory class of the underlying sheaf $\mathcal{F}$ of a T-fixed stable pair $(\mathcal{F}, s)$ as a sum over edge and vertex terms. It will be convenient to identify an element $(i, j, k) \in \mathcal{S}_\pi(v)$ with the corresponding divisor. Thus if we write $\mathcal{D} \in \mathcal{S}_\pi(v)$ we will mean

$$\mathcal{D} = i\mathcal{D}_1(v) + j\mathcal{D}_2(v) + k\mathcal{D}_3(v)$$

for the corresponding $(i, j, k) \in \mathcal{S}_\pi(v)$. Similarly, $\mathcal{D} \in \lambda(e)$ means

$$\mathcal{D} = i\mathcal{D}(e) + j\mathcal{D}'(e)$$

for the corresponding $(i, j) \in \lambda(e)$.

Given a triple of partitions $\overrightarrow{\mu} = (\mu^1, \mu^2, \mu^3)$. Let Q be a component of the moduli space of T-fixed $\mathbb{C}[x_1, x_2, x_3]$-submodules of $M/\mathcal{O}_{C_{\overrightarrow{\mu}}}$. By Lemma 4.21, Q is a product of $\mathbb{P}^1$. For each labelled box configuration $\pi \in Q$, consider the set of boxes

$$\mathcal{S}_\pi = \pi \cup \mathrm{II} \cup \mathrm{III}.$$

For each box $w \in \mathcal{S}_\pi$, we define

$$\eta(w) = \xi(w) + \ell(w) \tag{4.8}$$

where

$$\ell(w) = \begin{cases} 0 & \text{if } w \notin \pi \\ 2 & \text{if } w \text{ is a unlabelled box of type III,} \\ 1 & \text{otherwise,} \end{cases}$$

and

$$\xi(w) = 1 - \#\text{cylinders containing } w.$$

Notice that every $\pi \in Q$ has the same type of boxes when forgetting the exact labels and $\eta(w)$ depends on the type of w. We will use Q_π to denote the component containing π.

Proposition 4.25. *Let $\mathcal{O}_\mathcal{X} \to \mathcal{F}$ be a T-invariant stable pair on $\mathcal{X}$ with associated data $\{\lambda(e), \pi(v)\}$. Then in T-equivariant compactly supported K-theory we have*

$$\mathcal{F} = \sum_{e \in \text{Edges}^{cpt}} \sum_{\mathcal{D} \in \lambda(e)} \mathcal{O}_{\mathcal{C}(e)}(-\mathcal{D}) + \sum_{v \in \text{Vertices}} \sum_{\mathcal{D} \in \mathcal{S}_\pi(v)} \eta(\mathcal{D}) \mathcal{O}_{p(v)}(-\mathcal{D})$$

where $\eta(\mathcal{D})$ is defined in (4.8).

Proof. By Lemma 4.2, we obtain the following short exact sequence

$$0 \to \mathcal{O}_\mathcal{C} \to \mathcal{F} \to \mathcal{Q} \to 0,$$

where $\mathcal{C}$ is the support of $\mathcal{F}$. Hence,

$$\mathcal{F} = \mathcal{O}_{\mathcal{C}} + \mathcal{Q}.$$

By [10, Proposition 4],

$$\mathcal{O}_{\mathcal{C}} = \sum_{e \in \text{Edges}^{cpt}} \sum_{\mathcal{D} \in \lambda(e)} \mathcal{O}_{\mathcal{C}(e)}(-\mathcal{D}) + \sum_{v \in \text{Vertices}} \left(-\sum_{\mathcal{D} \in \text{II}} \mathcal{O}_{p(v)}(-\mathcal{D}) - \sum_{\mathcal{D} \in \text{III}} 2\mathcal{O}_{p(v)}(-\mathcal{D}) \right).$$

Notice that $\mathcal{Q}$ is supported entirely at the T-fixed points $p(v)$. We have

$$\mathcal{Q} = \sum_{v \in \text{Vertices}} \mathcal{Q}_v = \sum_{v \in \text{Vertices}} \sum_{\mathcal{D} \in \mathcal{S}_\pi(v)} \ell(\mathcal{D}) \mathcal{O}_{p(v)}(-\mathcal{D}).$$

The proposition follows immediately. □

In the case where $\mathcal{X}$ has transverse A_{n-1} orbifold structure, we can write the decomposition of $\mathcal{F}$ into the generators described in Table 4.1.

Proposition 4.26. (1) *The vertex terms decompose as*

$$\sum_{\mathcal{D} \in \mathcal{S}_\pi(v)} \eta(\mathcal{D}) \mathcal{O}_{p(v)}(-\mathcal{D}) = \begin{cases} \sum_{(i,j,k) \in \mathcal{S}_{\pi(v)}} \eta(\mathcal{D}) [\mathcal{O}_{p(v)} \otimes \rho_{i-j}] & e_3(v) \text{ is oriented outward,} \\ \sum_{(i,j,k) \in \mathcal{S}_{\pi(v)}} \eta(\mathcal{D}) [\mathcal{O}_{p(v)} \otimes \rho_{j-i}] & e_3(v) \text{ is oriented inward.} \end{cases}$$

(2) *Using the notation as in Section 4.4 and Figure 4.3, we have*

$$\sum_{(i,j)\in\lambda} \mathcal{O}_{\mathcal{C}}(-i\mathcal{D} - j\mathcal{D}') = \sum_{k=0}^{n-1} |\lambda|_k \cdot [\mathcal{O}_{\mathcal{C}}(-1) \otimes \rho_k] + \sum_{k=0}^{n-1} C^{\lambda}_{\widetilde{m},\widetilde{m}'}[k,n] \cdot [\mathcal{O}_{p(e)} \otimes \rho_k]$$
$$+ \delta_0 \sum_{k=0}^{a-1} A_\lambda(k,a) \cdot [\mathcal{O}_{p(f)} \otimes \rho_k] + \delta_0' \sum_{k=0}^{a'-1} A_{\lambda^t}(k,a') \cdot [\mathcal{O}_{p(f')} \otimes \rho_k]$$
$$+ \delta_\infty \sum_{k=0}^{b-1} A_\lambda(k,b) \cdot [\mathcal{O}_{p(g)} \otimes \rho_k] + \delta_\infty' \sum_{k=0}^{b'-1} A_{\lambda^t}(k,b') \cdot [\mathcal{O}_{p(g')} \otimes \rho_k].$$

Proof. See [10, Lemma 15 & Prop 5]. □

Proposition 4.25, Proposition 4.26, and Table 4.1 suggest the following definition.

Definition 4.27. The $\mathbb{Z}_n$ *PT vertex* $W^n_{\vec{\mu}}$ is defined by

$$W^n_{\vec{\mu}} = \sum_{Q_\pi} \chi_{\text{top}}(Q_\pi) q_0^{|\pi|_0} \cdots q_{n-1}^{|\pi|_{n-1}} \tag{4.9}$$

where the sum is taken over the components Q_π of the moduli space of T-fixed $\mathbb{C}[x_1, x_2, x_3]$-submodules of $M/\mathcal{O}_{C_{\vec{\mu}}}$ and

$$|\pi|_a = \sum_{\substack{w=(i,j,k)\in\mathcal{S}_\pi \\ i-j\equiv a \bmod n}} \eta(w)$$

is the (normalized) number of boxes of color a in $\mathcal{S}_\pi$.

Given a labelled box configuration $\pi = \pi(\vec{\mu})$. We can view

$$\pi \cap \text{Cyl}_i^-$$

as a reverse plane partition (RPP) of shape μ^i, an array of nonnegative integers of shape μ^i that is weakly increasing in both rows and columns, by summing over the boxes sitting on top of each $\square \in \mu^i$ along the i-axis.

In the 1-leg case, i.e. only one μ^i is nonempty, the T-fixed points are isolated and are in bijective correspondence with RPPs of shape μ^i by Lemma 4.21. The $\mathbb{Z}_n$ PT vertex $W^n_{\vec{\mu}}$ simplifies to

$$\sum_{\pi \in \mathrm{RPP}(\mu^i)} q_0^{|\pi|_0} \cdots q_{n-1}^{|\pi|_{n-1}}, i = 1, 2, 3,$$

where

$$|\pi|_a = \sum_{\substack{(i,j,k) \in \pi \\ i-j \equiv a \bmod n}} 1.$$

We will find an explicit formula for the PT vertex in Section 5.

4.7 The sign formula

Let

$$\mathcal{I}^\bullet = [\mathcal{O}_\mathcal{X} \to \mathcal{F}] \in P(\mathcal{X}, \beta)^T$$

be a T-fixed stable pair. The Zariski tangent space to $\mathcal{I}^\bullet$ in $P(\mathcal{X}, \beta)$ is isomorphic to $\mathrm{Ext}^1_0(\mathcal{I}^\bullet, \mathcal{I}^\bullet)$. We want to compute the sign $(-1)^{\dim \mathrm{Ext}^1_0(\mathcal{I}^\bullet, \mathcal{I}^\bullet)}$ and arrange them into vertex and edge terms. The calculation is adapted from [10, Section 6].

The Calabi-Yau condition on $\mathcal{X}$ implies that

$$K_\mathcal{X} \simeq \mathcal{O}_\mathcal{X} \otimes \mathbb{C}[\mu]$$

as a T-equivariant line bundle for some primitive weight $\mu \in \mathrm{Hom}(T, \mathbb{C}^*)$ [10, Lemma 18]. The shifted dual of a T-representation V is defined by the formula

$$V^* = V^\vee \otimes \mathbb{C}[-\mu].$$

Proposition 4.28 ([10, Proposition 6]). *The shifted dual satisfies the following properties.*

(1) *For any T-equivariant sheaf $\mathcal{F}$ and $\mathcal{G}$,*

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{G})^* \simeq \mathrm{Ext}^{3-i}(\mathcal{G}, \mathcal{F})$$

and likewise for traceless Ext*.*

(2) *Let V and W be virtual T-representations such that*

$$V - V^* = W - W^*.$$

Then the virtual dimension of V and W are equal modulo 2.

Let V be a virtual T-representation. We define $s(V) \in \mathbb{Z}/2\mathbb{Z}$ to be the dimension modulo 2 of V. If V is an anti-self shifted dual virtual representation, i.e.

$$V = W - W^*$$

for some W. We define

$$\sigma(V) = s(W).$$

By Proposition 4.28 above, $\sigma(V)$ is independent of the choice of W.

Consider as T-representations, we have that

$$\begin{aligned}
&\mathrm{Ext}^1_0(\mathcal{I}^\bullet,\mathcal{I}^\bullet) - \mathrm{Ext}^2_0(\mathcal{I}^\bullet,\mathcal{I}^\bullet)\\
=&\chi(\mathcal{O}_\mathcal{X},\mathcal{O}_\mathcal{X}) - \chi(\mathcal{I}^\bullet,\mathcal{I}^\bullet)\\
=&\chi(\mathcal{O}_\mathcal{X},\mathcal{O}_\mathcal{X}) - \chi(\mathcal{O}_\mathcal{X}-\mathcal{F},\mathcal{O}_\mathcal{X}-\mathcal{F})\\
=&\chi(\mathcal{O}_\mathcal{X},\mathcal{F}) + \chi(\mathcal{F},\mathcal{O}_\mathcal{X}) - \chi(\mathcal{F},\mathcal{F}).
\end{aligned}$$

By Proposition 4.28, we have

$$\mathrm{Ext}^1_0(\mathcal{I}^\bullet,\mathcal{I}^\bullet) = \mathrm{Ext}^2_0(\mathcal{I}^\bullet,\mathcal{I}^\bullet)^*$$

and

$$\chi(\mathcal{F},\mathcal{O}_\mathcal{X}) = -\chi(\mathcal{O}_\mathcal{X},\mathcal{F})^*.$$

Hence,

$$\begin{aligned}
s(\mathrm{Ext}^1_0(\mathcal{I}^\bullet,\mathcal{I}^\bullet)) &= \sigma(\mathrm{Ext}^1_0(\mathcal{I}^\bullet,\mathcal{I}^\bullet) - \mathrm{Ext}^2_0(\mathcal{I}^\bullet,\mathcal{I}^\bullet))\\
&= \sigma(\chi(\mathcal{O}_\mathcal{X},\mathcal{F}) - \chi(\mathcal{O}_\mathcal{X},\mathcal{F})^* - \chi(\mathcal{F},\mathcal{F}))\\
&= s(\chi(\mathcal{O}_\mathcal{X},\mathcal{F})) + \sigma(\chi(\mathcal{F},\mathcal{F}))\\
&= \chi(\mathcal{F}) + \sigma(\chi(\mathcal{F},\mathcal{F})).
\end{aligned}$$

Given any decomposition $\mathcal{F} = \sum_i K_i$ in $K_T(\mathcal{X})$, we have

$$\begin{aligned}
\chi(\mathcal{F},\mathcal{F}) &= \sum_{i,j} \chi(K_i, K_j) \\
&= \sum_i [(\mathrm{Ext}^0(K_i,K_i) - \mathrm{Ext}^1(K_i,K_i)) - ((\mathrm{Ext}^0(K_i,K_i) - \mathrm{Ext}^1(K_i,K_i))^*] \\
&\quad + \sum_{i<j} [\chi(K_i,K_j) - \chi(K_i,K_j)^*].
\end{aligned}$$

Therefore,

$$s(\mathrm{Ext}^1_0(\mathcal{I}^\bullet,\mathcal{I}^\bullet)) = \chi(\mathcal{F}) + \sum_i s(\mathrm{Hom}(K_i,K_i) - \mathrm{Ext}^1(K_i,K_i)) + \sum_{i<j} s(\chi(K_i,K_j)).$$

Let's first compute

$$\sum_i s(\mathrm{Hom}(K_i,K_i) - \mathrm{Ext}^1(K_i,K_i)).$$

We call these the diagonal terms. It can be divided into edge terms and vertex terms.

By Proposition 4.21, we have

$$\mathcal{F} = \sum_{e \in \mathrm{Edges}^{cpt}} \sum_{\mathcal{D} \in \lambda(e)} \mathcal{O}_{\mathcal{C}(e)}(-\mathcal{D}) + \sum_{v \in \mathrm{Vertices}} \sum_{\mathcal{D} \in \mathcal{S}_\pi(v)} \eta(\mathcal{D}) \mathcal{O}_{p(v)}(-\mathcal{D}).$$

Let e be a compact edge and let $\mathcal{C} = \mathcal{C}(e)$, $\mathcal{D} = \mathcal{D}(e)$, and $\mathcal{D}' = \mathcal{D}'(e)$ so that

$\mathcal{C} = \mathcal{D} \cap \mathcal{D}'$. We have the following exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}}(-\mathcal{D} - \mathcal{D}') \to \mathcal{O}_{\mathcal{X}}(-\mathcal{D}) \oplus \mathcal{O}_{\mathcal{X}}(-\mathcal{D}') \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{C}} \to 0. \tag{4.10}$$

For $\mathcal{A} \in \lambda(e)$, we tensor (4.10) by $\mathcal{O}_{\mathcal{X}}(-\mathcal{A})$ and then apply $\mathscr{H}om(\cdot, \mathcal{O}_{\mathcal{C}}(-\mathcal{A}))$ to obtain

1. $\mathscr{H}om(\mathcal{O}_{\mathcal{C}}(-\mathcal{A}), \mathcal{O}_{\mathcal{C}}(-\mathcal{A})) = \mathcal{O}_{\mathcal{C}}$,
2. $\mathscr{E}xt^1(\mathcal{O}_{\mathcal{C}}(-\mathcal{A}), \mathcal{O}_{\mathcal{C}}(-\mathcal{A})) = \mathcal{O}_{\mathcal{C}}(\mathcal{D}) \oplus \mathcal{O}_{\mathcal{C}}(\mathcal{D}') = N_{\mathcal{C}/\mathcal{X}}$,
3. $\mathscr{E}xt^2(\mathcal{O}_{\mathcal{C}}(-\mathcal{A}), \mathcal{O}_{\mathcal{C}}(-\mathcal{A})) = \mathcal{O}_{\mathcal{C}}(\mathcal{D} + \mathcal{D}') = \bigwedge^2 N_{\mathcal{C}/\mathcal{X}}$.

By local-to-global spectral sequence

$$\begin{aligned} \mathrm{Hom}(\mathcal{O}_{\mathcal{C}}(-\mathcal{A}), \mathcal{O}_{\mathcal{C}}(-\mathcal{A})) &= H^0(\mathcal{O}_{\mathcal{C}}) \\ \mathrm{Ext}^1(\mathcal{O}_{\mathcal{C}}(-\mathcal{A}), \mathcal{O}_{\mathcal{C}}(-\mathcal{A})) &= H^0(N_{\mathcal{C}/\mathcal{X}}) \oplus H^1(\mathcal{O}_{\mathcal{C}}). \end{aligned}$$

Since $h^0(\mathcal{O}_{\mathcal{C}}) = 1$ and $h^1(\mathcal{O}_{\mathcal{C}}) = 0$, we deduce that each edge e contributes

$$\sum_{\mathcal{A} \in \lambda(e)} (1 + h^0(N_{\mathcal{C}/\mathcal{X}})) = |\lambda(e)|(1 + h^0(N_{\mathcal{C}/\mathcal{X}})) \tag{4.11}$$

to the diagonal terms.

Let v be a vertex. Let $p = p(v)$ and $\mathcal{D}_i = \mathcal{D}_i(v), i = 1, 2, 3$. We have the following exact sequence

$$
\begin{aligned}
0 &\to \mathcal{O}_{\mathcal{X}}(-\mathcal{D}_1-\mathcal{D}_2-\mathcal{D}_3) \to \bigoplus_{1\le i<j\le 3} \mathcal{O}_{\mathcal{X}}(-\mathcal{D}_i-\mathcal{D}_j) \\
&\to \bigoplus_{i=1}^{3} \mathcal{O}_{\mathcal{X}}(-\mathcal{D}_i) \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_p \to 0.
\end{aligned} \tag{4.12}
$$

For $\mathcal{A} \in \pi(v)$, we tensor (4.12) by $\mathcal{O}_{\mathcal{X}}(-\mathcal{A})$ and then apply $\mathscr{H}om(\cdot, \mathcal{O}_p(-\mathcal{A}))$ to obtain

$$
\mathscr{E}xt^i(\mathcal{O}_p(-\mathcal{A}), \mathcal{O}_p(-\mathcal{A})) = \mathcal{O}_p \otimes \bigwedge^i N_{p/\mathcal{X}}, i = 0,1,2,3.
$$

By local-to-global spectral sequence

$$
\begin{aligned}
&\mathrm{Hom}(\mathcal{O}_p(-\mathcal{A}), \mathcal{O}_p(-\mathcal{A})) = H^0(\mathcal{O}_p) \\
&\mathrm{Ext}^1(\mathcal{O}_p(-\mathcal{A}), \mathcal{O}_p(-\mathcal{A})) = H^0(N_{p/\mathcal{X}}) \oplus H^1(\mathcal{O}_p).
\end{aligned}
$$

Since $h^0(\mathcal{O}_p) = 1$ and $h^1(\mathcal{O}_p) = 0$, we deduce that each vertex v contributes

$$
\sum_{\mathcal{A} \in \mathcal{S}_\pi(v)} \eta^2(\mathcal{A})(1 + h^0(N_{p/\mathcal{X}})) \tag{4.13}
$$

to the diagonal terms, where $\eta(\mathcal{A})$ is defined in (4.8).

Next, we compute the off-diagonal terms

$$
\sum_{i<j} s(\chi(K_i, K_j)).
$$

It is convenient to introduce an arbitrary total order on each $\lambda(e)$ and $\pi(v)$. Let

$\mathcal{C} = \mathcal{C}(e)$, $\mathcal{C}' = \mathcal{C}(e')$, and $p = p(v)$. The off-diagonal terms can be divided into edge terms,

$$\mathcal{O}_{\mathcal{C}}(-\mathcal{A}) \text{ and } \mathcal{O}_{\mathcal{C}}(-\mathcal{B})$$

for $\mathcal{A} < \mathcal{B}$ in $\lambda(e)$, and vertex terms, which come in three types:

1. $\mathcal{O}_p(-\mathcal{A})$ and $\mathcal{O}_p(-\mathcal{B})$ for $\mathcal{A} < \mathcal{B}$ in $\mathcal{S}_\pi(v)$.

2. $\mathcal{O}_{\mathcal{C}}(-\mathcal{A})$ and $\mathcal{O}_p(-\mathcal{B})$ for $\mathcal{A} \in \lambda(e)$ and $\mathcal{B} \in \mathcal{S}_\pi(v)$ where e is incident to v.

3. $\mathcal{O}_{\mathcal{C}}(-\mathcal{A})$ and $\mathcal{O}_{\mathcal{C}'}(-\mathcal{B})$ for $\mathcal{A} \in \lambda(e)$ and $\mathcal{B} \in \lambda(e')$, where $e \neq e'$ have common vertex v.

For the vertex term, we tensor (4.10) by $\mathcal{O}_{\mathcal{X}}(-\mathcal{A})$ and then apply $\mathscr{H}om(\cdot, \mathcal{O}_{\mathcal{C}}(-\mathcal{B}))$ to obtain

$$\mathscr{E}xt^i(\mathcal{O}_{\mathcal{C}}(-\mathcal{A}), \mathcal{O}_{\mathcal{C}}(-\mathcal{B})) = \mathcal{O}_{\mathcal{C}}(\mathcal{A} - \mathcal{B}) \otimes \bigwedge^i N_{\mathcal{C}/\mathcal{X}}, i = 0, 1, 2.$$

It follows that each edge e contributes

$$\sum_{\substack{\mathcal{A},\mathcal{B} \in \lambda(e) \\ \mathcal{A} < \mathcal{B}}} \chi(\mathcal{O}_{\mathcal{C}}(\mathcal{A} - \mathcal{B}) \otimes \lambda_{-1} N_{\mathcal{C}/\mathcal{X}}) \tag{4.14}$$

to the off-diagonal terms. Here,

$$\lambda_{-1} N_{\mathcal{C}/\mathcal{X}} = \sum_{i=0}^{2} (-1)^i \bigwedge^i N_{\mathcal{C}/\mathcal{X}} = \mathcal{O}_{\mathcal{C}} - \mathcal{O}_{\mathcal{C}}(\mathcal{D}) - \mathcal{O}_{\mathcal{C}}(\mathcal{D}') + \mathcal{O}_{\mathcal{C}}(\mathcal{D} + \mathcal{D}').$$

For the type (1) vertex terms, let $\mathcal{A} < \mathcal{B}$ in $\mathcal{S}_\pi(v)$, we tensor (4.12) by $\mathcal{O}_{\mathcal{X}}(-\mathcal{A})$

and then apply $\mathscr{H}om(\cdot, \mathcal{O}_p(-\mathcal{B}))$ to obtain

$$\mathscr{E}xt^i(\mathcal{O}_p(-\mathcal{A}), \mathcal{O}_p(-\mathcal{B})) = \mathcal{O}_p(\mathcal{A} - \mathcal{B}) \otimes \bigwedge^i N_{p/\mathcal{X}}, i = 0, 1, 2, 3.$$

It follows that the contribution is

$$\sum_{\substack{\mathcal{A}, \mathcal{B} \in \mathcal{S}_\pi(v) \\ \mathcal{A} < \mathcal{B}}} \eta(\mathcal{A})\eta(\mathcal{B}) h^0(\mathcal{O}_p(\mathcal{A} - \mathcal{B}) \otimes \lambda_{-1} N_{p/\mathcal{X}}), \tag{4.15}$$

where

$$\lambda_{-1} N_{p/\mathcal{X}} = \sum_{i=0}^{3} (-1)^i \bigwedge^i N_{p/\mathcal{X}}.$$

Let $\mathcal{A} \in \lambda(e)$, and $\mathcal{B} \in \mathcal{S}_\pi(v)$, where e is incident to v. We tensor (4.10) by $\mathcal{O}_{\mathcal{X}}(-A)$ and apply $\mathscr{H}om(\cdot, \mathcal{O}_p(-\mathcal{B}))$ to obtain

$$\mathscr{E}xt^i(\mathcal{O}_{\mathcal{C}}(-\mathcal{A}), \mathcal{O}_p(-\mathcal{B})) = \mathcal{O}_p(\mathcal{A} - \mathcal{B}) \otimes \bigwedge^i N_{\mathcal{C}/\mathcal{X}}, i = 0, 1, 2.$$

It follows that the contribution of the type (2) vertex terms is

$$\sum_{i=1}^{3} \sum_{\mathcal{A} \in \lambda(e_i)} \sum_{\mathcal{B} \in \mathcal{S}_\pi(v)} \eta(\mathcal{B}) h^0(\mathcal{O}_p(\mathcal{A} - \mathcal{B}) \otimes \lambda_{-1} N_{\mathcal{C}(e_i)/\mathcal{X}}). \tag{4.16}$$

Finally, let $\mathcal{C} = \mathcal{C}(e) = \mathcal{D} \cap \mathcal{D}'$, $\mathcal{C}' = \mathcal{C}(f') = \mathcal{D}' \cap \mathcal{D}_0$, and $p = p(v) = \mathcal{C} \cap \mathcal{C}'$ (see Figure. 4.3). Let $\mathcal{A} \in \lambda(e)$ and $\mathcal{B} \in \lambda(f')$. We tensor (4.10) by $\mathcal{O}_{\mathcal{X}}(-\mathcal{A})$ and then apply $\mathscr{H}om(\cdot, \mathcal{O}_{\mathcal{C}'}(-\mathcal{B}))$ to obtain the complex

$$0 \to \mathcal{O}_{\mathcal{C}'}(\mathcal{A}-\mathcal{B}) \to \mathcal{O}_{\mathcal{C}'}(\mathcal{A}-\mathcal{B}+\mathcal{D}) \oplus \mathcal{O}_{\mathcal{C}'}(\mathcal{A}-\mathcal{B}+\mathcal{D}') \to \mathcal{O}_{\mathcal{C}'}(\mathcal{A}-\mathcal{B}+\mathcal{D}+\mathcal{D}') \to 0.$$

Using the fact that

$$0 \to \mathcal{O}_{\mathcal{C}'} \to \mathcal{O}_{\mathcal{C}'}(\mathcal{D}) \to \mathcal{O}_p(\mathcal{D}) \to 0$$

is exact and $\mathcal{O}_{\mathcal{C}'} \to \mathcal{O}_{\mathcal{C}'}(\mathcal{D}')$ is 0, we get

1. $\mathscr{H}om(\mathcal{O}_{\mathcal{C}}(-\mathcal{A}), \mathcal{O}_{\mathcal{C}'}(-\mathcal{B})) = 0$,
2. $\mathscr{E}xt^1(\mathcal{O}_{\mathcal{C}}(-\mathcal{A}), \mathcal{O}_{\mathcal{C}'}(-\mathcal{B})) = \mathcal{O}_p(\mathcal{A}-\mathcal{B}+\mathcal{D})$,
3. $\mathscr{E}xt^2(\mathcal{O}_{\mathcal{C}}(-\mathcal{A}), \mathcal{O}_{\mathcal{C}'}(-\mathcal{B})) = \mathcal{O}_p(\mathcal{A}-\mathcal{B}+\mathcal{D}+\mathcal{D}')$.

By Calabi-Yau condition, $\mathcal{O}_p(\mathcal{D}+\mathcal{D}') = \mathcal{O}_p(-\mathcal{D}_0)$. Hence,

$$\begin{aligned} s(\chi(\mathcal{O}_{\mathcal{C}}(-\mathcal{A}, \mathcal{O}_{\mathcal{C}}(-\mathcal{B})) &= h^0(\mathcal{O}_p(\mathcal{A}-\mathcal{B}+\mathcal{D})) + h^0(\mathcal{O}_p(\mathcal{A}-\mathcal{B}+\mathcal{D}+\mathcal{D}')) \\ &= h^0(\mathcal{O}_p(\mathcal{A}-\mathcal{B}+\mathcal{D})) + h^0(\mathcal{O}_p(\mathcal{B}-\mathcal{A}+\mathcal{D}_0)), \end{aligned}$$

and the contribution of type (3) vertex terms is

$$\sum_{i\neq j} \sum_{\mathcal{A}\in\lambda(e_i(v))} \sum_{\mathcal{B}\in\lambda(e_j(v))} h^0(\mathcal{O}_p(\mathcal{A}-\mathcal{B}+\mathcal{D}_j)) \tag{4.17}$$

Putting (4.11), (4.13), (4.14), (4.15), (4.16), and (4.17) all together yields

$$s(\mathrm{Ext}^1_0(\mathcal{I}^\bullet, \mathcal{I}^\bullet)) = \chi(\mathcal{F}) + \sum_{e\in\mathrm{Edges}} \mathrm{SE}_{\lambda(e)} + \sum_{v\in\mathrm{Vertices}} \mathrm{SV}_{\pi(v)}$$

where

$$\mathrm{SE}_{\lambda(e)} = |\lambda(e)|(1 + h^0(N_{\mathcal{C}(e)/\mathcal{X}})) + \sum_{\substack{\mathcal{A},\mathcal{B}\in\lambda(e)\\ \mathcal{A}<\mathcal{B}}} \chi(\mathcal{O}_{\mathcal{C}(e)}(\mathcal{A}-\mathcal{B}) \otimes \lambda_{-1} N_{\mathcal{C}(e)/\mathcal{X}}) \tag{4.18}$$

and

$$\begin{aligned}\mathrm{SV}_{\pi(v)} = &\sum_{\mathcal{A}\in\mathcal{S}_\pi(v)} \eta^2(\mathcal{A})(1 + h^0(N_{p/\mathcal{X}})) + \sum_{\substack{\mathcal{A},\mathcal{B}\in\mathcal{S}_\pi(v)\\ \mathcal{A}<\mathcal{B}}} \eta(\mathcal{A})\eta(\mathcal{B})h^0(\mathcal{O}_p(\mathcal{A}-\mathcal{B}) \otimes \lambda_{-1} N_{p/\mathcal{X}})\\ &+ \sum_{i=1}^{3} \sum_{\mathcal{A}\in\lambda(e_i)} \sum_{\mathcal{B}\in\mathcal{S}_\pi(v)} \eta(\mathcal{B})h^0(\mathcal{O}_p(\mathcal{A}-\mathcal{B}) \otimes \lambda_{-1} N_{\mathcal{C}(e_i)/\mathcal{X}})\\ &+ \sum_{i\neq j} \sum_{\mathcal{A}\in\lambda(e_i(v))} \sum_{\mathcal{B}\in\lambda(e_j(v))} h^0(\mathcal{O}_p(\mathcal{A}-\mathcal{B}+\mathcal{D}_j)).\end{aligned} \tag{4.19}$$

Proposition 4.29. *Let $\mathcal{X}$ be an orbifold toric CY3 with transverse A_{n-1} orbifold structure. Let*

$$\mathcal{I}^\bullet = [\mathcal{O}_\mathcal{X} \to \mathcal{F}] \in P(\mathcal{X}, \beta)^T$$

be a T-fixed stable pair. The parity of the dimension of the Zariski tangent space of $\mathcal{I}^\bullet$ in $P(\mathcal{X}, \beta)$ is given by

$$s(\mathrm{Ext}^1_0(\mathcal{I}^\bullet, \mathcal{I}^\bullet)) = \chi(\mathcal{F}) + \sum_{e\in\mathrm{Edges}} \mathrm{SE}_{\lambda(e)} + \sum_{v\in\mathrm{Vertices}} \mathrm{SV}_{\pi(v)}$$

where

$$\mathrm{SE}_\lambda = \begin{cases} \sum_{a=0}^{n-1} C^\lambda_{m,m'}[a,n](|\lambda_{a-1}| - |\lambda_{a+1}|) + |\lambda|_a(|\lambda|_a + (1+m)|\lambda|_{a-1}), & n > 1 \\ |\lambda|(\widetilde{m} + \delta_0 + \delta_\infty), & n = 1, \end{cases}$$

and

$$\mathrm{SV}_\pi = \sum_{a=0}^{n-1} |\pi|_a(|\lambda_3|_{a-1} + |\lambda_3|_{a+1}) + \sum_{a=0}^{n-1} |\lambda_3|_a(|\lambda_1|_a + |\lambda_2|_a + |\lambda_1|_{a+1} + |\lambda_2|_{a-1}).$$

Proof. We first treat the edge term (4.18). If $n = n(e) > 1$, then $\mathcal{C} = \mathcal{C}(e)$ is a $B\mathbb{Z}_n$ gerbe. We resymmetrize it as follows. Since $N_{\mathcal{C}/\mathcal{X}} = \mathcal{O}_C(\mathcal{D}) + \mathcal{O}_{\mathcal{C}}(\mathcal{D}')$, we get $K_C = \mathcal{O}_{\mathcal{C}}(\mathcal{D} + \mathcal{D}')$ and

$$\lambda_{-1} N_{\mathcal{C}/\mathcal{X}} = \mathcal{O}_C - \mathcal{O}_C(\mathcal{D}) - \mathcal{O}_{\mathcal{C}}(\mathcal{D}') + K_{\mathcal{C}}.$$

It follows that

$$\begin{aligned} \mathrm{SE}_\lambda = & |\lambda|(1 + h^0(\mathcal{O}_{\mathcal{C}}(\mathcal{D})) + h^0(\mathcal{O}_{\mathcal{C}}(\mathcal{D}'))) \\ & + \sum_{\substack{\mathcal{A},\mathcal{B}\in\lambda \\ \mathcal{A}<\mathcal{B}}} \chi(\mathcal{O}_{\mathcal{C}}(\mathcal{A}-\mathcal{B})) - \chi(\mathcal{O}_{\mathcal{C}}(\mathcal{A}-\mathcal{B}) \otimes K_{\mathcal{C}}) \\ & + \sum_{\substack{\mathcal{A},\mathcal{B}\in\lambda \\ \mathcal{A}<\mathcal{B}}} \chi(\mathcal{O}_{\mathcal{C}}(\mathcal{A}-\mathcal{B}+\mathcal{D})) - \chi(\mathcal{O}_{\mathcal{C}}(\mathcal{A}-\mathcal{B}+\mathcal{D}')). \end{aligned}$$

By Serre duality,

$$h^0(\mathcal{O}_\mathcal{C}(\mathcal{D}'))) = h^1(\mathcal{O}_\mathcal{C}(\mathcal{D}))),$$

$$\chi(\mathcal{O}_\mathcal{C}(\mathcal{A} - \mathcal{B}) \otimes K_\mathcal{C}) = -\chi(\mathcal{O}_\mathcal{C}(\mathcal{B} - \mathcal{A})),$$

$$\chi(\mathcal{O}_\mathcal{C}(\mathcal{A} - \mathcal{B} + \mathcal{D}')) = -\chi(\mathcal{O}_\mathcal{C}(\mathcal{B} - \mathcal{A} + \mathcal{D})).$$

Hence,

$$\begin{aligned}
\mathrm{SE}_\lambda = & \sum_{\mathcal{A},\mathcal{B}\in\lambda} \chi(\mathcal{O}_\mathcal{C}(\mathcal{A} - \mathcal{B})) + \chi(\mathcal{O}_\mathcal{C}(\mathcal{A} - \mathcal{B} + \mathcal{D})) \\
= & \sum_{a=0}^{n-1} \sum_{\mathcal{A},\mathcal{B}\in\lambda[a,n]} \deg(\mathcal{A}) - \deg(\mathcal{B}) + 1 \\
& + \sum_{\substack{\mathcal{A}\in\lambda[a,n] \\ \mathcal{B}\in\lambda[a+1,n]}} \deg(\mathcal{A}) - \deg(\mathcal{B}) + \deg(\mathcal{D}) + 1 \\
= & \sum_{a=0}^{n-1} |\lambda|_a^2 + |\lambda_{a+1}| \sum_{\mathcal{A}\in\lambda[a,n]} (\deg(\mathcal{A}) - 1) \\
& - |\lambda_a| \sum_{\mathcal{B}\in\lambda[a+1,n]} (\deg(\mathcal{B}) - 1) + |\lambda|_a|\lambda|_{a+1}(\deg(\mathcal{D}) + 1).
\end{aligned}$$

Recall that $m = \deg \mathcal{O}_\mathcal{C}(\mathcal{D})$ and $m' = \deg \mathcal{O}_\mathcal{C}(\mathcal{D}')$ (See (4.3)). Therefore,

$$\begin{aligned}
\mathrm{SE}_\lambda &= \sum_{a=0}^{n-1} |\lambda|_a^2 - |\lambda_{a+1}| C^\lambda_{m,m'}[a,n] + |\lambda_a| C^\lambda_{m,m'}[a+1,n] + (1+m)|\lambda|_a|\lambda|_{a+1} \\
&= \sum_{a=0}^{n-1} C^\lambda_{m,m'}[a,n](|\lambda_{a-1}| - |\lambda_{a+1}|) + |\lambda|_a(|\lambda|_a + (1+m)|\lambda|_{a-1}).
\end{aligned}$$

If $n = 1$, then $\mathcal{C}$ is a football. Since $\lambda_{-1} N_{\mathcal{C}/\mathcal{X}}$ has rank and degree zero, it is trivial in K-theory. Hence, (4.18) becomes

$$\begin{aligned}\mathrm{SE}_\lambda &= |\lambda|(1 + h^0(\mathcal{O}_{\mathcal{C}}(\mathcal{D})) + h^0(\mathcal{O}_{\mathcal{C}}(\mathcal{D}'))) \\ &= |\lambda|(1 + h^0(\mathcal{O}_{\mathcal{C}}(\mathcal{D})) + h^1(\mathcal{O}_{\mathcal{C}}(-\mathcal{D}' + K_{\mathcal{C}}))) \\ &= |\lambda|(1 + \chi(\mathcal{O}_{\mathcal{C}}(\mathcal{D}))).\end{aligned}$$

Since $\mathcal{O}_{\mathcal{C}}(\mathcal{D}) = \mathcal{O}_{\mathcal{C}}(\widetilde{m}p - \delta_0 p_0 - \delta_\infty p_\infty)$ (see (4.4)), by [10, Lemma 39],

$$\begin{aligned}\chi(\mathcal{O}_{\mathcal{C}}(\mathcal{D})) &= 1 + \widetilde{m} + \left\lfloor \frac{-\delta_0}{\max\{n(f), n(f')\}} \right\rfloor + \left\lfloor \frac{-\delta_\infty}{\max\{n(g), n(g')\}} \right\rfloor \\ &= 1 + \widetilde{m} - \delta_0 - \delta_\infty\end{aligned}$$

Hence, $\mathrm{SE}_\lambda = |\lambda|(\widetilde{m} + \delta_0 + \delta_\infty)$.

Writing $\lambda_i = \lambda(e_i)$ and using the facts that

$$\lambda_{-1} N_{p/\mathcal{X}} = \sum_{i=1}^{3} (\mathcal{O}_p(-\mathcal{D}_i) - \mathcal{O}_p(\mathcal{D}_i)) = 0$$

and

$$\lambda_{-1} N_{\mathcal{C}(e_i)/\mathcal{X}} = 0, \;\; i = 1, 2,$$

the vertex term (4.19) becomes

$$
\begin{aligned}
\mathrm{SV}_\pi = &\sum_{\mathcal{A}\in\mathcal{S}_\pi} \eta^2(\mathcal{A})(1+h^0(\mathcal{O}_p(\mathcal{D}_1))+h^0(\mathcal{O}_p(\mathcal{D}_2))+h^0(\mathcal{O}_p(\mathcal{D}_3))) \\
&+\sum_{\mathcal{A}\in\lambda_3}\sum_{\mathcal{B}\in\mathcal{S}_\pi} \eta(\mathcal{B})(h^0(\mathcal{O}_p(\mathcal{A}-\mathcal{B}+\mathcal{D}_1))+h^0(\mathcal{O}_p(\mathcal{A}-\mathcal{B}+\mathcal{D}_2))) \\
&+\sum_{i\neq j}\sum_{\mathcal{A}\in\lambda(e_i(v))}\sum_{\mathcal{B}\in\lambda(e_j(v))} h^0(\mathcal{O}_p(\mathcal{A}-\mathcal{B}+\mathcal{D}_j)) \\
=&\sum_{a=0}^{n-1}\sum_{\mathcal{B}\in\mathcal{S}_\pi[a,n]}\eta(\mathcal{B})\left(\sum_{\mathcal{A}\in\lambda_3[a-1,n]}1+\sum_{\mathcal{A}\in\lambda_3[a+1,n]}1\right)+\sum_{\mathcal{A}\in\lambda_1[a,n]}\left(\sum_{\mathcal{B}\in\lambda_2[a-1,n]}1+\sum_{\mathcal{B}\in\lambda_3[a,n]}1\right) \\
&+\sum_{\mathcal{A}\in\lambda_2[a,n]}\left(\sum_{\mathcal{B}\in\lambda_1[a+1,n]}1+\sum_{\mathcal{B}\in\lambda_3[a,n]}1\right)+\sum_{\mathcal{A}\in\lambda_3[a,n]}\left(\sum_{\mathcal{B}\in\lambda_1[a+1,n]}1+\sum_{\mathcal{B}\in\lambda_2[a-1,n]}1\right) \\
=&\sum_{a=0}^{n-1}|\pi|_a(|\lambda_3|_{a-1}+|\lambda_3|_{a+1})+|\lambda_1|_a(|\lambda_2|_{a-1}+|\lambda_3|_a) \\
&+|\lambda_2|_a(|\lambda_1|_{a+1}+|\lambda_3|_a)+|\lambda_3|_a(|\lambda_1|_{a+1}+|\lambda_2|_{a-1}) \\
=&\sum_{a=0}^{n-1}|\pi|_a(|\lambda_3|_{a-1}+|\lambda_3|_{a+1})+\sum_{a=0}^{n-1}|\lambda_3|_a(|\lambda_1|_a+|\lambda_2|_a+|\lambda_1|_{a+1}+|\lambda_2|_{a-1}).
\end{aligned}
$$

□

Theorem/Conjecture 4.19 is now easily proved assuming Proposition/Conjecture 4.16. Using Proposition 4.25, Proposition 4.26, and Table 4.1, the variables in (4.6) are assigned. The sign of each term is determined by Proposition 4.29. The $\chi(\mathcal{F})$ term is accounted for by multiplying the variables q and $q_{e,0}$ by -1. The edge term is multiplied by $(-1)^{\mathrm{SE}_\lambda}$. The first term in SV_π is accounted for by multiplying the variables $q_{e,a}$ by $(-1)^{|\lambda_3|_{a-1}+|\lambda_3|_{a+1}}$, and the second term in SV_π is accounted for by the sign $(-1)^{\Sigma_\pi}$.

Chapter 5: Generation functions for colored reverse plane partitions

5.1 Partitions and Schur functions

In this section, we review some facts about partitions and Schur functions. The main references are [24] and [33].

Definition 5.1. A *partition* is any sequence

$$\lambda = (\lambda_1, \lambda_2, \cdots) \tag{5.1}$$

of non-negative integers in decreasing order:

$$\lambda_1 \geq \lambda_2 \geq \cdots$$

and containing only finitely many non-zero terms.

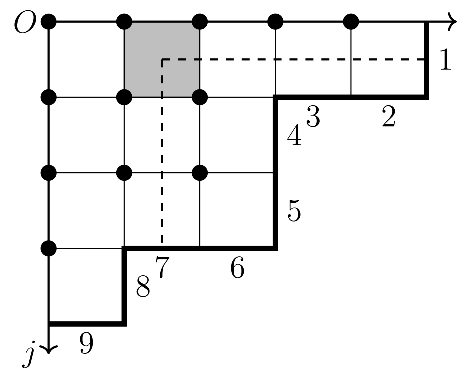
The non-zero λ_i in (5.1) are called the parts of λ. The number of parts is the length of λ, denoted $l(\lambda)$. We define the weight of λ to be

$$|\lambda| = \lambda_1 + \cdots + \lambda_{l(\lambda)}.$$

The *Young diagram* of of λ is obtained by drawing a left-justified array of juxtaposed squares with λ_i squares in the ith row. Alternatively, we can view a partition as a subset of $\mathbb{Z}^2_{\geq 0}$ in the ij-plane with points being placed at the upper-left corner of each square.

The *conjugate* of a partition λ is the partition λ^t whose Young diagram is the transpose of the Young diagram of λ.

Example 5.2. The Young diagram of $\lambda = (5331)$ is



.

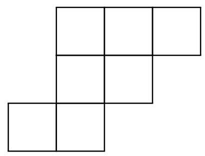
Its weight is $|\lambda| = 12$, and its conjugate is $\lambda^t = (43311)$.

For any $\square \in \lambda$, the *hook length* $h(\square)$ is defined to be the sum of one plus the number of boxes horizontally to the right and vertically below the box. In Example 5.2, the hook length of the shaded square is $h(\blacksquare) = 6$.

We label the boundaries of the Young diagram of λ from the upper right-hand corner to the lower left-hand corner by 1 to $\lambda_1 + \lambda_1^t$. Let $\mathcal{B}_h(\lambda)$ denote the set of horizontal boundaries, and $\mathcal{B}_v(\lambda)$ denote the set of vertical boundaries. In Example 5.2, we have

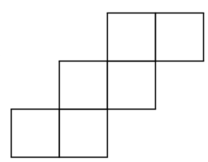
$$\mathcal{B}_v(\lambda) = \{1, 4, 5, 8\} \text{ and } \mathcal{B}_h(\lambda) = \{2, 3, 6, 7, 9\}.$$

Definition 5.3. Given two partitions λ and μ, we write $\lambda \supset \mu$ to mean that the Young diagram of λ contains the Young diagram of μ, i.e. $\lambda_i \geq \mu_i$ for all $i \geq 1$. The set-theoretic difference is called a *skew* Young diagram, denoted λ/μ.

Example 5.4.  is a skew Young diagram of shape (432)/(11).

Definition 5.5. We say λ/μ is a *border strip* if it is connected and contains no 2×2 block of squares, i.e. successive rows of λ/μ overlap by exactly one square. The height of a border strip λ/μ is defined to be one less than the number of rows its Young diagram occupies, denoted by $\mathrm{ht}(\lambda/\mu)$.

Example 5.6. Let $\lambda = (432)$ and $\mu = (21)$, then

$\lambda/\mu =$ 

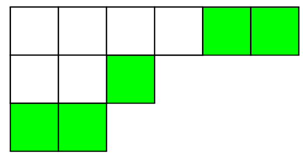
is a border strip of height $\mathrm{ht}(\lambda/\mu) = 2$.

Definition 5.7. Let λ and μ be two partitions. We say λ *interlaces* with μ, denoted $\lambda \succ \mu$, if $\lambda \supset \mu$ and they satisfy the Pieri's relation

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \cdots .$$

Equivalently, the skew Young diagram λ/μ contains at most one box in each column.

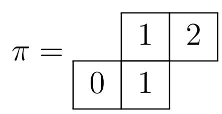
Example 5.8. Let $\lambda = (632)$ and $\mu = (42)$. The skew Young diagram of λ/μ is



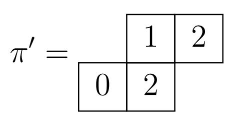
which contains 0 or 1 box in each column. Thus, $\lambda \succ \mu$.

Definition 5.9. A *reverse plane partition* (RPP) of skew shape λ/μ is an array $\{\pi_{ij}\}$ of nonnegative integers of shape λ/μ that is weakly increasing in both rows and columns. A *semistandard Young tableau* (SSYT) is a RPP that is strictly increasing in columns. The size of π is the sum of its entries, denoted by $|\pi|$.

Example 5.10. Let $\lambda = (32)$ and $\mu = (1)$. Then



is a RPP of shape λ/μ with size $|\pi| = 4$ and



is a SSYT of shape λ/μ with size $|\pi'| = 5$.

Let $x = (x_0, x_1, x_2, \cdots)$ be an infinite set of variables.

Definition 5.11. The *skew Schur function* of shape λ/μ can be defined as

$$s_{\lambda/\mu}(x) = \sum_{\pi \in \mathrm{SSYT}(\lambda/\mu)} x^{\pi},$$

where $x^\pi = x_0^{\#0s \text{ in } \pi} x_1^{\#1s \text{ in } \pi} \cdots$.

The *principle specialization* is

$$s_{\lambda/\mu}(q) := s_{\lambda/\mu}(q)(1, q, q^2, \cdots) = \sum_{\pi \in \mathrm{SSYT}(\lambda/\mu)} q^{|\pi|},$$

which is the generating function for SSYT of shape λ/μ. Similarly, we define

$$\mathrm{RPP}_{\lambda/\mu}(q) = \sum_{\pi \in \mathrm{RPP}(\lambda/\mu)} q^{|\pi|}.$$

When $\mu = \emptyset$, we have the following beautiful formula.

Proposition 5.12 ([33, Theorem 7.22.1])**.** *We have*

$$\mathrm{RPP}_\lambda(q) = \prod_{\square \in \lambda} \frac{1}{1 - q^{h(\square)}},$$

and

$$s_\lambda(q) = q^{a(\lambda)} \prod_{\square \in \lambda} \frac{1}{1 - q^{h(\square)}},$$

where $a(\lambda) = \sum_i (i-1)\lambda_i$.

5.2 Vertex operators

In this section, we review vertex operators following [27, Appendix A]. Let V be a linear space with basis $\{\underline{k}\}$, $k \in \mathbb{Z}+\frac{1}{2}$. The *Fock space* $\Lambda^{\frac{\infty}{2}}V$ is spanned by vectors

$$v_S = \underline{s_1} \wedge \underline{s_2} \wedge \underline{s_3} \wedge \cdots ,$$

where $S = \{s_1 > s_2 > s_3 > \cdots\} \subset \mathbb{Z} + \frac{1}{2}$ is such a subset that both sets

$$S_+ = S \backslash (\mathbb{Z}_{\leq 0} - 1/2), \; S_- = (\mathbb{Z}_{\leq 0} - 1/2) \backslash S$$

are finite. $\Lambda^{\frac{\infty}{2}}V$ is equipped with the inner product such that the basis $\{v_S\}$ is orthonormal.

For any $k \in \mathbb{Z} + 1/2$, let ψ_k be the operator

$$\psi_k(v_S) = \underline{k} \wedge v_S$$

and let ψ_k^* be its adjoint. Explicitly, let i be the largest index such that $s_i > k$. Then

$$\psi_k(v_S) = \begin{cases} (-1)^i v_{S \cup \{k\}} & k \notin S, \\ 0 & k \in S. \end{cases}$$

$$\psi_k^*(v_S) = \begin{cases} (-1)^i v_{S - \{k\}} & k \in S, \\ 0 & k \notin S. \end{cases}$$

It is clear that

$$\psi_k^*\psi_k(v_S) = \begin{cases} v_S & k \notin S, \\ 0 & k \in S. \end{cases}, \quad \psi_k\psi_k^*(v_S) = \begin{cases} v_S & k \in S, \\ 0 & k \notin S. \end{cases}$$

and they satisfy the anti-commutation relations

$$\psi_k\psi_l^* + \psi_l^*\psi_k = \delta_{kl}.$$

Define the *energy* and *charge* operator by

$$H = \sum_k k : \psi_k\psi_k^* :, \; C = \sum_k : \psi_k\psi_k^* :,$$

where $: \psi_k\psi_k^* := \begin{cases} \psi_k\psi_k^* & k > 0, \\ -\psi_k^*\psi_k & k < 0. \end{cases}$. Noting that

$$Cv_S = (|S_+| - |S_-|)v_S.$$

Let $\Lambda_0^{\frac{\infty}{2}} V := \ker(C)$ be the *charge zero* Fock space.

Let $\lambda = (\lambda_1, \lambda_2, \cdots)$ be a partition. Define

$$v_\lambda = v_{S(\lambda)}, \; S(\lambda) = \{\lambda_i - i + 1/2\} \subset \mathbb{Z} + 1/2.$$

Let d be the number of boxes along the main diagonal of the Young diagram of shape λ. Then

$$S(\lambda)_+ = \{\lambda_i - i + 1/2\}_{i=1}^d, S(\lambda)_- = \{-(\lambda_i^t - i + 1/2)\}_{i=1}^d.$$

Hence, $|S(\lambda)_+| = |S(\lambda)_-| = d$. Conversely, for any $S \subset \mathbb{Z} + 1/2$, if $|S_+| = |S_-| < \infty$, we have $S = S(\lambda)$ for some partition λ. Clearly, $\Lambda_0^{\frac{\infty}{2}} V$ is spanned by the vectors v_λ where λ runs over all partitions.

The energy operator H acts on v_λ by

$$Hv_\lambda = \sum_{i=1}^{d} (\lambda_i - i + 1/2 + \lambda_i^t - i + 1/2) v_\lambda = |\lambda| v_\lambda,$$

and so the operator q^H acts by

$$q^H v_\lambda = q^{|\lambda|} v_\lambda$$

where q is a formal parameter. We call q^H the *weight* operator.

For $0 \neq n \in \mathbb{Z}$ define

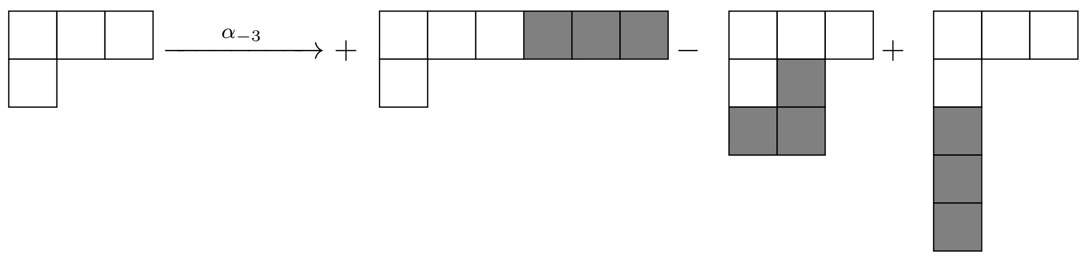
$$\alpha_n = \sum_k \psi_{k-n} \psi_k^*.$$

Evidently, $\alpha_n^* = \alpha_{-n}$. For $n > 0$, an easy calculation yields the following

$$\alpha_{-n} v_\mu = \sum_\lambda (-1)^{\mathrm{ht}(\lambda/\mu)} v_\lambda$$

summed over all partitions $\lambda \supset \mu$ for which λ/μ is a border strip of size n.

Example 5.13.



The operators α_n satisfy the Heisenberg commutation relations

$$[\alpha_n, \alpha_{-m}] = n\delta_{m,n} \tag{5.2}$$

Let $x = (x_1, x_2, x_3, \cdots)$ be an infinite set of variables and let

$$p_n(x) = \sum_i x_i^n$$

be the power sum symmetric function.

Definition 5.14. The *vertex operators* $\Gamma_\pm(x)$, which are operators on $\Lambda_0^{\frac{\infty}{2}} V$ over the coefficient ring given by symmetric functions in x_j's, are defined as

$$\Gamma_\pm(x) = \exp\left(\sum_{n=1}^{\infty} \frac{p_n}{n}\alpha_{\pm n}\right).$$

By the Heisenberg commutation relations (5.2), we observe that $\Gamma_\pm^*(x) = \Gamma_\mp(x)$. The matrix coefficients of the vertex operators $\Gamma_\pm(x)$ with respect to the basis $\{v_\lambda\}$ are given by skew Schur functions.

Proposition 5.15 ([27, A.15]). *We have*

$$\langle \Gamma_-(x)v_\mu, v_\lambda\rangle = \langle v_\mu, \Gamma_+(x)v_\lambda\rangle = s_{\lambda/\mu}(x). \tag{5.3}$$

Let $y = (y_1, y_2, \cdots)$ be another infinite set of variables. By [24, (1) Pg.93], we have the following orthogonality of skew Schur functions

$$\sum_\lambda s_{\lambda/\mu}(x)s_{\lambda/\nu}(y) = \prod_{i,j}(1 - x_i y_j)^{-1}\sum_\eta s_{\nu/\eta}(x)s_{\mu/\eta}(y).$$

Combining this with (5.3) gives rise to the following commutation equation:

$$\Gamma_+(x)\Gamma_-(y) = \prod_{i,j}(1 - x_i y_j)^{-1}\Gamma_-(y)\Gamma_+(x). \tag{5.4}$$

By the homogeneity of skew Schur functions, that is $s_\lambda(qx) = q^{|\lambda|}s_\lambda(x)$, we obtain that the vertex operator $\Gamma_\pm(x)$ and the weight operator q^H satisfy the following commutation equations

$$\Gamma_+(x)q^H = q^H\Gamma_+(xq),\ q^H\Gamma_-(x) = \Gamma_-(xq)q^H. \tag{5.5}$$

We consider the following important specialization of $\Gamma_\pm(x)$ which create interlacing partitions. Let $\Gamma_\pm(1)$ be obtained by the specialization $x_1 \mapsto 1,\ x_i \mapsto 0$ for $i > 1$. Explicitly,

$$\Gamma_\pm(1) = \exp\left(\sum_n \frac{1}{n}\alpha_{\pm n}\right).$$

Under this specialization, the skew Schur function $s_{\lambda/\mu}(x)$ becomes

$$s_{\lambda/\mu}(1,0,0,\cdots) = \begin{cases} 1 & \text{if } \lambda \succ \mu \\ 0 & \text{otherwise.} \end{cases}$$

By (5.3), we obtain

$$\Gamma_{-}(1)\mu = \sum_{\lambda \succ \mu} \lambda,\ \ \Gamma_{+}(1)\lambda = \sum_{\mu \prec \lambda} \mu.$$

As a motivating example, we derive MacMahon's generating function for plane partitions using vertex operators as in [28]. Recall that a plane partition π is an array $\{\pi_{ij}\}$ of positive integers that is weakly increasing in both rows and columns. Let π_t be the $t = i - j$ diagonal slice. It is clear that

$$\emptyset \cdots \prec \pi_{-2} \prec \pi_{-1} \prec \pi_0 \succ \pi_1 \succ \pi_2 \succ \cdots \emptyset.$$

The generating function for plane partitions is defined by

$$M(q) = \sum_{\pi} q^{|\pi|}.$$

By interlacing plane partitions along the diagonals, we have

$$
\begin{aligned}
M(q) &= \left\langle \emptyset \middle| \prod (\Gamma_+(1) q^H) \prod (\Gamma_-(1) q^H) \middle| \emptyset \right\rangle \\
&= \left\langle \emptyset \middle| \prod_{i=1}^{\infty} \Gamma_+(q^i) \prod_{j=0}^{\infty} \Gamma_-(q^j) \middle| \emptyset \right\rangle && \text{apply (5.5)} \\
&= \prod_{i>0} \prod_{j\geq 0} (1-q^{i+j})^{-1} && \text{apply (5.4)} \\
&= \prod_{n=1}^{\infty} (1-q^n)^{-n},
\end{aligned}
$$

which is the MacMahon function.

We also call a plane partition a 3D Young diagram, or 3D diagram for short. It is a stable pile of cubical boxes that sit in the corner of a large cubical room. More formally, a 3D Young diagram is a finite set π of $\mathbb{Z}^3_{\geq 0}$ such that if any of

$$(i+1, j, k), \quad (i, j+1, k), \quad (i, j, k+1)$$

is in π, then $(i, j, k) \in \pi$. Each ordered triple is a box; the condition means that boxes are stacked stably in the positive octant with gravity pulling them in the direction $(-1, -1, -1)$.

Definition 5.16. Let (λ, μ, ν) be a triple of partitions. A 3D partition π asymptotic to (λ, μ, ν) is a subset $\pi \subset \mathbb{Z}^3_{\geq 0}$ satisfying

(1) if any of $(i+1, j, k), (i, j+1, k)$ and $(i, j, k+1)$ is in π, then $(i, j, k) \in \pi$.

(2) (a) $(j, k) \in \lambda \Leftrightarrow (i, j, k) \in \pi$ for $i \gg 0$,

(b) $(k, i) \in \mu \Leftrightarrow (i, j, k) \in \pi$ for $j \gg 0$,

(c) $(i,j) \in \nu \Leftrightarrow (i,j,k) \in \pi$ for $k \gg 0$.

Let

$$\xi_\pi(i,j,k) = 1 - \#\text{ of legs containing } (i,j,k).$$

The normalized size of π is defined by

$$|\pi| = \sum_{(i,j,k)\in\pi} \xi_\pi(i,j,k).$$

Definition 5.17. The *topological vertex* $V_{\lambda\mu\nu}$ is defined to be

$$V_{\lambda\mu\nu}(q) = \sum_\pi q^{|\pi|}$$

where the sum is over all 3D partitions π asymptotic to (λ,μ,ν).

Okounkov, Reshetikhin and Vafa derive an explicit formula for $V_{\lambda\mu\nu}$ using vertex operators.

Proposition 5.18 ([28, Eqs (3.18)-(3.20)])**.**

$$\begin{aligned} V_{\lambda,\mu,\nu}(q) &= M(q) q^{-\binom{\lambda}{2}-\binom{\mu^t}{2}-\binom{\nu}{2}-|\lambda|/2-|\mu|/2-|\nu|/2} \\ &\times s_{\nu^t}(q^{-\rho}) \sum_\eta s_{\lambda^t/\eta}(q^{-\nu-\rho}) s_{\mu/\eta}(q^{-\nu^t-\rho}) \end{aligned}$$

where $\binom{\lambda}{2} = \sum_i \binom{\lambda_i}{2}$ *and* $\rho = (-1/2, -3/2, \cdots)$.

Definition 5.19. The $\mathbb{Z}_n$ *DT vertex* $V^n_{\lambda\mu\nu}$ is defined by

$$V^n_{\lambda\mu\nu}(q_0, q_1, \cdots, q_{n-1}) = \sum_{\pi} q_0^{|\pi|_0} \cdots q_{n-1}^{|\pi|_{n-1}}$$

where the sum is over all 3D partitions π asymptotic to (λ, μ, ν) and

$$|\pi|_a = \sum_{\substack{(i,j,k)\in\pi \\ i-j\equiv a \bmod n}} \xi_\pi(i, j, k).$$

We refer to [10, Theorem 12] for a closed formula for $V^n_{\lambda\mu\nu}$ in terms of Schur function.

Let $W^n_{\lambda\mu\nu}$ be the $\mathbb{Z}_n$ PT vertex (Definition 4.27). We have the following conjecture:

Conjecture 5.20 (Orbifold DT/PT vertex correspondence)**.** *If ν is multi-regular, then*

$$W^n_{\lambda\mu\nu} = \frac{V^n_{\lambda\mu\nu}}{V^n_{\emptyset\emptyset\emptyset}}.$$

5.3 Reverse plane partitions with $\mathbb{Z}_n$-coloring

Let λ be a partition. To give a natural coloring to $\pi = (\pi_{ij}) \in \mathrm{RPP}(\lambda)$, we consider it as a subset of $\mathbb{Z}^3$ in the following way: we put the Young diagram of λ on the ij-plane, then

$$\pi = \{(i, j, k) \in \mathbb{Z}^3 | (i, j) \in \lambda, k = -1, -2, \cdots, -\pi_{ij}\}.$$

Intuitively, we stack cubical boxes on the Young diagram of λ along the direction of negative k-axis.

Definition 5.21. A $\mathbb{Z}_n$-coloring is a homomorphism

$$K : \mathbb{Z}^3 \to \mathbb{Z}_n.$$

For $i = 0, 1, \cdots, n-1$, let q_i be the variable representing color i. Let π be a RPP of shape λ, then each point in π comes with a color. Let $|\pi|_a$ be the number of a-colored points in π,

$$|\pi|_a = |K^{-1}(a) \cap \pi|.$$

We will study the following $\mathbb{Z}_n$-colored generating function

$$W^n(q_0, \cdots, q_{n-1}) = \sum_{\pi \in \mathrm{RPP}(\lambda)} \prod_{a=0}^{n-1} q_a^{|\pi|_a}.$$

Notice that we can also place λ on the jk-plane or the ki-plane. We use the following notation:

$$W^n_{\lambda\emptyset\emptyset} : \lambda \text{ is placed on } jk\text{-plane},$$

$$W^n_{\emptyset\lambda\emptyset} : \lambda \text{ is placed on } ki\text{-plane},$$

$$W^n_{\emptyset\emptyset\lambda} : \lambda \text{ is placed on } ij\text{-plane}.$$

For a fixed coloring K, this will lead to different generating functions.

We adopt the notation in [10]. For any partition λ, define

$$q^{-A_\lambda} = \prod_{a=0}^{n-1} q_a^{-A_\lambda(a,n)}$$

where

$$A_\lambda(a,n) = \sum_{(i,j)\in\lambda} \left\lfloor \frac{i+a}{n} \right\rfloor .$$

For any function $f(q_0, q_1, \cdots, q_{n-1})$, we use $\overline{f(q_0, q_1, \cdots, q_{n-1})}$ to denote the function obtained by making the change of variables $q_m \leftrightarrow q_{-m}$, where we use q_{-m} and q_{n-m} interchangeably. Finally, let $\mathfrak{q} = (1, q_1, q_1q_2, q_1q_2q_3, \cdots)$.

Theorem 5.22. *Let the coloring K be given by*

$$K(i,j,k) \equiv i - j \bmod n.$$

Then

$$W^n_{\lambda\emptyset\emptyset} = q^{-A_\lambda}\overline{s_{\lambda^t}(\mathfrak{q})}. \tag{5.6}$$

$$W^n_{\emptyset\lambda\emptyset} = \overline{q^{-A_{\lambda^t}}} s_\lambda(\mathfrak{q}). \tag{5.7}$$

$$W^n_{\emptyset\emptyset\lambda} = \prod_{\square\in\lambda} \frac{1}{1 - \prod_{a=0}^{n-1} q_a^{h_a(\square)}}. \tag{5.8}$$

where $h_a(\square)$ denotes the number a-colored boxes in the hook of $\square$.

Remark 5.23. 1. When $n = 1$, i.e. there is no coloring, we have

$$W^1_{\lambda\emptyset\emptyset} = W^1_{\emptyset\lambda\emptyset} = W^1_{\emptyset\emptyset\lambda} = \mathrm{RPP}_\lambda .$$

In this case, $A_\lambda = \sum_{(i,j)\in\lambda} i = \sum_i (i-1)\lambda_i^t$ and $\mathfrak{q} = \{1, q, q^2, \cdots\}$. Thus,

$$\mathrm{RPP}_\lambda(q) = \prod_{\square\in\lambda} \frac{1}{1-q^{h(\square)}} = q^{-\sum_i (i-1)\lambda_i} s_\lambda(q),$$

which gives a different proof for Proposition 5.12.

2. Theorem 5.22 and [10, Theorem 12] together verify the orbifold DT/PT vertex correspondence 5.20 in the 1-leg case.

Proof. We first consider $W^n_{\lambda\emptyset\emptyset}$. By definition, λ is placed on the first quadrant of jk-plane. Let $(j,k) \in \lambda$. For any $\pi \in \mathrm{RPP}(\lambda)$, let π_{jk} be the integer in position (j,k). The points (i,j,k) in π with this (j,k) are

$$(-1, j, k), (-2, j, k), \cdots, (-\pi_{jk}, j, k).$$

Hence, the contribution of $(j,k) \in \lambda$ to $W^n_{\lambda\emptyset\emptyset}$ is $q_{-j-1}q_{-j-2}\cdots q_{-j-\pi_{jk}}$. Thus, the contribution of π is

$$\begin{aligned} &\prod_{(j,k)\in\lambda} q_{-j-1}q_{-j-2}\cdots q_{-j-\pi_{jk}} \\ = &\prod_{(j,k)\in\lambda} (q_{-1}\cdots q_{-j})^{-1} \cdot \prod_{(j,k)\in\lambda} q_{-1}\cdots q_{-j}q_{-j-1}\cdots q_{-j-\pi_{jk}}. \end{aligned}$$

Notice that the first factor depends only on λ. We can write it as

$$q^{-A_\lambda} := \prod_{a=0}^{n-1} q_a^{-A_\lambda(a,n)}, \text{ where } A_\lambda(a,n) = \sum_{(j,k)\in\lambda} \left\lfloor \frac{a+j}{n} \right\rfloor.$$

Recall that we have the following bijective map

$$\phi : \mathrm{RPP}(\lambda) \to \mathrm{SSYT}(\lambda^t)$$
$$(\pi_{jk}) \mapsto (\pi_{jk} + j).$$

Hence, the second factor is just a term in the Schur function $s_{\lambda^t}(x)$ under the specialization $x_m = q_{-1} \cdots q_{-m}$. Therefore,

$$\begin{aligned} W^n_{\lambda\emptyset\emptyset} &= q^{-A_\lambda} \sum_{\pi \in \mathrm{RPP}(\lambda)} \prod_{(j,k)\in\lambda} q_{-1} \cdots q_{-j} q_{-j-1} \cdots q_{-j-\pi_{jk}}. \\ &= q^{-A_\lambda} s_{\lambda^t}(1, q_{-1}, q_{-1}q_{-2}, \cdots) \\ &= q^{-A_\lambda} \overline{s_{\lambda^t}(\mathfrak{q})}, \end{aligned}$$

which is (5.6).

By same argument, we prove (5.7)

$$W^n_{\emptyset\lambda\emptyset} = \overline{q^{-A_{\lambda^t}}} s_\lambda(\mathfrak{q}).$$

It remains to prove (5.8) for $W^n_{\emptyset\emptyset\lambda}$. In this case, λ is placed on the first quadrant of the ij-plane. Let $(i, j) \in \lambda$ and $\pi = (\pi_{ij}) \in \mathrm{RPP}(\lambda)$. The points $(i, j, k) \in \pi$ with this (i, j) are

$$(i, j, -1), (i, j, -2), \cdots , (i, j, -\pi_{ij}).$$

Hence, the contribution of (i, j) is $q_{i-j}^{\pi_{ij}}$. In particular, each diagonal slice π_{i-j} is only

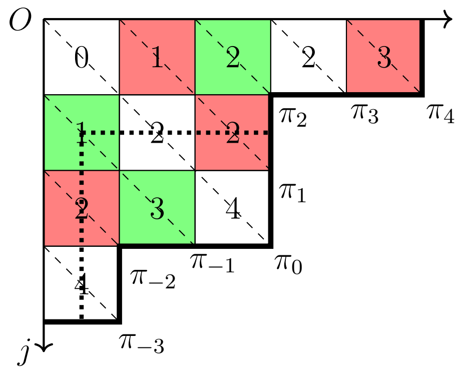


Figure 5.1: A $\mathbb{Z}_3$-colored RPP of shape (5331)

$(i-j)$-colored. The adjacent diagonal slices of π interlace in a way depending on the boundary of λ. We will use an example to illustrate this. Let $n = 3, \lambda = (5331)$, and $\pi \in \mathrm{RPP}(\lambda)$ be as in Figure 5.1. It is clear that

$$\emptyset \overset{\Gamma_+(1)}{\underset{9}{\prec}} \pi_{-3} \overset{\Gamma_-(1)}{\underset{8}{\succ}} \pi_{-2} \overset{\Gamma_+(1)}{\underset{7}{\prec}} \pi_{-1} \overset{\Gamma_+(1)}{\underset{6}{\prec}} \pi_0 \overset{\Gamma_-(1)}{\underset{5}{\succ}} \pi_1 \overset{\Gamma_-(1)}{\underset{4}{\succ}} \pi_2 \overset{\Gamma_+(1)}{\underset{3}{\prec}} \pi_3 \overset{\Gamma_+(1)}{\underset{2}{\prec}} \pi_4 \overset{\Gamma_-(1)}{\underset{1}{\succ}} \emptyset.$$

where $\Gamma_\pm(1)$ acts from right to left and we label them from 1 to $\lambda_1 + \lambda_1^t = 9$. Recall that the boundaries are

$$\mathcal{B}_v(\lambda) = \{1, 4, 5, 8\} \text{ and } \mathcal{B}_h(\lambda) = \{2, 3, 6, 7, 9\}.$$

We observe that the pattern of the $\Gamma_\pm(1)$ coincides with that of the boundary of λ. More specifically, if h represents a horizontal boundary then the vertex operator at position h is $\Gamma_+(1)$; if v represents a vertical boundary then the vertex operator at position v is $\Gamma_-(1)$.

For $t = 1, \cdots, \lambda_1 + \lambda_1^t$, define

$$\Gamma_t(1) = \begin{cases} \Gamma_-(1), & t \in \mathcal{B}_v(\lambda) \\ \\ \Gamma_+(1), & t \in \mathcal{B}_h(\lambda). \end{cases}$$

For $a = 0, \cdots, n-1$, let q_a^H denote the a-colored weight operator. We obtain the following vertex operator expression of $W^n_{\emptyset\emptyset\lambda}$:

$$W^n_{\emptyset\emptyset\lambda} = \left\langle \emptyset, \prod_{s=1}^{\lambda_1^t - 1} \Gamma_{\lambda_1+s+1}(1) q_{-s}^H \cdot \Gamma_{\lambda_1+1}(1) \cdot \prod_{t=1}^{\lambda_1} q_{\lambda_1 - t}^H \Gamma_t(1) \emptyset \right\rangle$$

We commute all q_t^H to the right and all q_{-s}^H to the left using (5.5) to get

$$W^n_{\emptyset\emptyset\lambda} = \left\langle \emptyset, \prod_{s=1}^{\lambda_1^t - 1} \Gamma_{\lambda_1+s+1}\left(\mathfrak{q}_{\lambda_1+s+1}^{\mathrm{sgn}(\lambda_1+s+1)}\right) \cdot \Gamma_{\lambda_1+1}(\mathfrak{q}_{\lambda_1+1}) \cdot \prod_{t=1}^{\lambda_1} \Gamma_t\left(\mathfrak{q}_t^{\mathrm{sgn}(t)}\right) \emptyset \right\rangle.$$

Here,

$$\mathfrak{q}_r = \begin{cases} q_0 q_1 \cdots q_{\lambda_1 - r}, & r = 1, \cdots, \lambda_1, \\ \\ 1, & l = \lambda_1 + 1, \\ \\ q_{-1} \cdots q_{-(r-\lambda_1-1)}, & r = \lambda_1 + 2, \cdots, \lambda_1 + \lambda_1^t, \end{cases}$$

and

$$\mathrm{sgn}(r) = \begin{cases} 1, & r = \lambda_1 + 1, \\ \\ (-1)^{\mathbf{1}_{\{r > \lambda_1\}}} (-1)^{\mathbf{1}_{\{r \in \mathcal{B}_h(\lambda)\}}}, & r = 1, \cdots, \lambda_1, \lambda_1 + 2, \cdots, \lambda_1 + \lambda_1^t \end{cases}$$

where $\mathbf{1}_A$ denotes the indicator function.

We then commute all Γ_+ to the right and all Γ_- to the left using (5.4) to obtain

$$W^n_{\emptyset\emptyset\lambda} = \prod_{\substack{(h,v)\in(\mathcal{B}_h(\lambda),\mathcal{B}_v(\lambda))\\ h>v}} \frac{1}{1-\mathfrak{q}_h^{\mathrm{sgn}(h)}\mathfrak{q}_v^{\mathrm{sgn}(v)}}.$$

Finally, notice that each pair $(h,v)\in(\mathcal{B}_h(\lambda),\mathcal{B}_v(\lambda))$ with $h>v$ uniquely determines a $\square\in\lambda$. For example, in Figure 5.1 above, let $h=9$ and $v=4$, the corresponding $\square$ and its hook are labeled by dotted lines. Clearly, $h_0(\square)=h_1(\square)=2$ and $h_2(\square)=1$. Hence,

$$\mathfrak{q}_9^{\mathrm{sgn}(9)}\cdot\mathfrak{q}_4^{\mathrm{sgn}(4)} = (q_{-1}q_{-2}q_{-3})\cdot(q_0q_1) = q_0^2q_1^2q_2 = q_0^{h_0(\square)}q_1^{h_1(\square)}q_2^{h_2(\square)}.$$

By similar argument, we obtain

$$\mathfrak{q}_h^{\mathrm{sgn}(h)}\mathfrak{q}_v^{\mathrm{sgn}(v)} = \prod_{a=0}^{n-1} q_a^{h_a(\square)}.$$

Therefore,

$$W^n_{\emptyset\emptyset\lambda} = \prod_{\square\in\lambda}\frac{1}{1-\prod_{a=0}^{n-1} q_a^{h_a(\square)}}.$$

$\square$

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