# ABSTRACT

Title of Dissertation: I: SUFFICIENT CONDITIONS FOR LOCAL SCALING LAWS IN 3-D TURBULENCE

II: WELL-POSEDNESS FOR NONLINEAR STOCHASTIC KINETIC EQUATIONS

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Incompressible fluids at high Reynolds number quickly transition to turbulence. This implies that precise predictions for many flows in the physical world are extremely difficult - if not outright impossible. The field of statistical fluid mechanics aims to mitigate this difficulty by studying averaged quantities associated with turbulent flows.

In the mathematical physics literature, turbulent flow is often described in the language of stochastic partial differential equations. The appropriate models are stochastic perturbations of well-known deterministic equations of fluid mechanics, so that the apparent randomness of turbulent flow is modeled via the tools of stochastic analysis.

In the first part of this dissertation, this point of view of stochastic fluid mechanics is employed. We focus on the three dimensional case, with the goal of obtaining a conditional theorem for Kolmogorov's celebrated 4/5 law to hold in the presence of boundaries. The dimensionality

enforces the use of a weak notion of solution to our model, in particular we work with families of "stationary martingale solutions" to the stochastic Navier-Stokes equations parametrized by the inverse Reynolds number. The main result of the first part of this dissertation provides a sufficient condition for a local version of the 4/5 law in the limit of infinite Reynolds number.

In the second part of the dissertation, the focus is shifted to kinetic theory. Kinetic equations have played a prominent role in statistical mechanics since the 19th century; typically, the kinetic viewpoint represents an intermediate step of coarse-graining between the particle level, governed by Newtonian or Hamiltonian mechanics, and the hydrodynamic level, governed by continuum or fluid mechanics.

The solution of a kinetic equation represents the normalized phase-space density of a large number of particles which might be interacting and potentially diffusing. The evolution of the density of an ensemble of particles interacting electrostatically is modeled by the Vlasov-Poisson equation. Thermal noise on the particles is modeled by the inclusion of a kinetic Fokker-Planck term.

To incorporate the effect of macroscopic fluctuating force fields into kinetic modeling, we perturb the Vlasov-Poisson-Fokker-Planck equation by a stochastic kinetic transport term. We modify and exploit a popular scheme of stochastic fluid mechanics relying on the Gyöngy-Krylov lemma and construct local strong solutions to the stochastic Vlasov-Poisson-Fokker-Planck equation.

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by

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Table of Contents

Acknowledgements ii

Table of Contents iv

**I Sufficient conditions for local scaling laws in 3-D turbulence 1**

1 Introduction to Part I . . . 2
2 Statement of the main result of Part I, discussion, and context . . . 6
2.1 Statement of the main result . . . 6
2.2 Discussion and context of this work . . . 15
3 Preliminary results . . . 24
4 Proof of the Local 4/3 Law . . . 36
5 Proof of the Local 4/5 Law . . . 43
6 Existence of stationary martingale solutions for OU driven Navier-Stokes . . . 52

**II Well-posedness of nonlinear stochastic kinetic equations 57**

7 Introduction to Part II . . . 58
8 The setup and main result of Part I . . . 61
9 Outline of the proof . . . 69
10 Very smooth solutions . . . 80
10.1 Proof of Lemma 9.4 . . . 80
10.2 Proof of Lemma 9.1 . . . 104
11 Proof of the main theorem . . . 114
12 Hypoelliptic regularization . . . 123

Bibliography 138

# Part I

# Sufficient conditions for local scaling laws in 3-D turbulence

# 1 Introduction to Part I

In the first part of this dissertation, we examine incompressible fluid flow in the turbulent regime. The velocity field of a viscous fluid which is driven by the body force $f$ evolves according to the incompressible Navier–Stokes system:

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u + f \\ \operatorname{div} u = 0 \end{cases} \tag{1.1}$$

The non-negative parameter $\nu$ corresponds to the kinematic viscosity of the fluid. Thus, when $\nu > 0$ the fluid is viscous and system (1.1) is called the Navier–Stokes equations, while when $\nu > 0$ the fluid is inviscid and (1.1) is called the Euler equations. When $x$ varies over $\mathbb{R}^d$, we need appropriate decay conditions on $u(t, x)$ as $|x| \to \infty$. For periodic flow, we take $x \in \mathbb{T}^d$. In either of these cases, $d = 2$ or 3. In what follows, for flow in a domain $D$ with boundary $\partial D$ we will be using the *no-slip* boundary condition, which is simply:

$$u\Big|_{\partial D} = 0.$$

Whether the initial value problem for (1.1) is well-posed in three spatial dimensions is an outstanding question in the theory of partial differential equations. Accordingly, the mathematical literature dedicated to the subject is very large, and one can compare the books [11, 77, 95, 99, 104] for a non exhaustive sample of the multitude of perspectives and the evolution of the subject over the past half century. In particular, there is no known class of functions that is both large enough to accommodate the existence of solutions to (1.1) and sufficiently specific to guarantee their

uniqueness. In fact, it is now known that one can construct a forcing term $f$ such that (1.1) has multiple Leray–Hopf solutions [1].

Once we determine appropriate length and velocity scales which are characteristic of the flow, denoted by $L$ and $U$ respectively, we can define the Reynolds number Re as follows:

$$\mathrm{Re} = \frac{UL}{\nu}.$$

The Reynolds number is a dimensionless parameter that allows a qualitative analysis of fluid flow: as the Reynolds number increases, fluid flow in a domain becomes increasingly unstable and transitions to turbulence; see the discussion in [56] and the references therein. By assuming that $U, L$ are fixed, the Reynolds number is simply inversely proportional to $\nu$, and thus the limit $\mathrm{Re} \to \infty$ will be replaced in what follows by the limit $\nu \to 0$.

The principal characteristics of turbulent flow are extreme sensitivity with respect to initial conditions, universality of statistics in a given scale range, and anomalous dissipation of energy. The sensitivity with respect to initial conditions is easy to conceptualize: two pieces of cork travelling downstream in a river will have drastically different trajectories even if they started close to each other, and the difference is unpredictable. This unpredictability motivates a statistical theory of hydrodynamics. In this context, universality of the statistics of turbulent flow in a wide range of scales means that while the specific evolution of the flow and the large scale structure might be difficult to predict, statistical quantities will be independent of the experimental setup. The interval of scales where universality is observed is called the *inertial range* and is bounded below by the dissipative scale and above by the integral scale (the largest scale over which the system varies). We mention the classical references [56, 85] for a reader with interest in statistical

hydrodynamics.

The last principal characteristic of turbulent flow is anomalous dissipation of energy. To describe this "anomaly", we first notice that in the limit $\nu \to 0$, the Navier–Stokes equations formally reduce to the Euler equations. It is then natural to examine the limiting behavior of a family of solutions $(u^\nu)_{\nu>0}$ in the inviscid limit.

Let $u^\nu$ be a solution to (1.1) with viscosity $\nu$. If the vector field $u^\nu$ possesses sufficient regularity, we obtain by the energy method:

$$\|u^\nu(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u^\nu(s)\|_{L^2}^2 \, \mathrm{d}s = \|u^\nu\|_{L^2}^2 \,. \tag{1.2}$$

As $\|u^\nu\|_{L^2}^2$ is a multiple of the kinetic energy, (1.2) shows that an inviscid fluid with a smooth velocity field conserves kinetic energy. Therefore, one might expect that in the limit $\nu \to 0$, the mean energy dissipation rate $\nu \|\nabla u^\nu\|_{L^2}^2$ vanishes. Anomalous dissipation is then the experimental observation that, in fact, the opposite happens, and the mean dissipation rate has a positive finite limit. Note that it is now known that weak solutions to the Euler equations can dissipate energy and therefore the $\nu \to 0$ limit might still hold validity in regimes of anomalous dissipation. See Remark 1.2.7.

Since we assume that the parameters $U, L$ stay bounded and $U$ can be taken as the r.m.s. of the velocity, anomalous dissipation paints a multi-scale picture of turbulent flow, which is observed immediately in real fluids: namely, a turbulent fluid exhibits nontrivial features across the inertial range.

Building on anomalous dissipation and some similarity hypotheses, Kolmogorov ([72, 73]) derived the $4/5$ law. This is a relation between a third order moment of the increment of the

velocity field and the mean energy dissipation rate. Besides the ease with which the $4/5$ law can be verified experimentally, the importance of Kolmogorov's law lies in the fact that it indicates a cascade of energy to small scales; see the discussion in [56].

In order to construct the foundations of a statistical theory of turbulence, the main approach in the mathematical literature is to introduce randomness using the framework of stochastic partial differential equations, e.g. through a fluctuating external force (which corresponds to *additive noise*) or by declaring that the fluid is being transported by itself as well as by a fluctuating velocity field (which corresponds to *stochastic transport* noise [82]). In the first part of this dissertation, we consider the case of an external stochastic force. Regardless of the accuracy of our model, we note that as turbulence should be driven by the nonlinearity in (1.1), the exact form of the forcing is secondary and is chosen mostly for mathematical convenience and clarity.

When the system is perturbed by an additive white-in-time external body force, as is usually done in the mathematical literature, one can construct statistically stationary martingale solutions (Definition 2.4, see [53]), which are stochastic processes in a constructed probability space which solve the stochastic Navier–Stokes equations in the sense of Leray–Hopf solutions. For a family $\{u^\nu\}_{\nu\in(0,1)}$ of such solutions, the authors of [13] showed that the condition:

$$\nu \mathbf{E} \left\| u^\nu \right\|^2 \to 0 \quad \text{as } \nu \to 0$$

is sufficient in order to derive Kolmogorov's $4/5$ law in the limit $\nu \to 0$. Thus it plays the role of a weak anomalous dissipation, and in fact it is referred to as such by the authors in [13].

The goal of the first part of this dissertation is to derive a sufficient condition for the $4/5$ law to hold in a bounded domain. We pose (1.1) on a bounded domain $D \subset \mathbb{R}^3$ and perturb

it by an Ornstein-Uhlenbeck process $Z$, i.e. a solution to a linear SPDE with additive noise. Using methods similar to [13], we can construct a global in time stationary martingale solution $u^\nu$ to the resulting system for any $\nu > 0$. The sufficient condition that we derive essentially demands that the family $\{u^\nu\}_{\nu\in(0,1)}$ is precompact in $L^3$, which is a stronger assumption than the weak anomalous dissipation. We discuss how our condition fits within the modern literature on regularity of fluid equations in the main body of the first part. The following is based on the article [90].

# 2 Statement of the main result of Part I, discussion, and context

## 2.1 Statement of the main result

We consider the Navier-Stokes equations on a sufficiently smooth bounded domain $D \subset \mathbb{R}^3$ driven by an Ornstein-Uhlenbeck process. Specifically, the set of equations we are concerned with are:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + Z_t, \\ \operatorname{div} u = 0 \text{ on } D, \\ u = 0 \text{ on } \partial D, \\ dZ_t = -LZ_t dt + dW_t. \end{cases} \tag{NSE}$$

The process $(W_t)_{t\geq 0}$ is a $Q$-Wiener process of the form:

$$W_t = \sum_{k=1}^{\infty} \sigma_k e_k(x) \beta_k(t),$$

where $(\beta_k)_{k=1}^{\infty}$ is a sequence of independent standard Brownian motions defined on the same stochastic basis, $(\sigma_k)_{k=1}^{\infty}$ is a sequence of complex numbers satisfying the coloring condition

$$\sum_{k=1}^{\infty} |\sigma_k|^2 := \epsilon < \infty$$

and $(e_k)_{k=1}^{\infty}$ is an orthonormal basis of eigenvectors of the Stokes operator on the space

$$H = \overline{\{u \in (C_c^{\infty}(D))^3 | \operatorname{div} u = 0\}},$$

the completion being taken with respect to the $L^2$ norm. The covariance operator $Q$ is defined on $H$ by

$$Qe_k = |\sigma_k|^2 e_k.$$

The linear operator $L$ satisfies

$$c_1 \langle Av, v\rangle \leq \langle Lv, v\rangle \leq c_2 \langle Av, v\rangle$$

for two positive constants $c_1, c_2$, where $A = -P\Delta$ is the Stokes operator, $P$ being the Leray projection operator. Furthermore, the couple $(L, Q)$ is chosen in such a way that the Ornstein-Uhlenbeck SPDE for $Z_t$ given in (NSE) is such that weak solutions are actually strong and unique and possesses a unique invariant measure, see for instance sufficient conditions in [34, Chapter 5]. These conditions on $L, Q$ relate to constructions given in Section 6.

The parameter $\nu > 0$ is the kinematic viscosity but may also be interpreted as the inverse Reynolds number. In particular, we may fix a reference scale and a reference velocity and treat

the vanishing viscosity and infinite Reynolds number limits as one and the same.

Notice that we do not follow the standard practice of driving the Navier-Stokes equation by a white in time process, opting instead to drive by the more regular Ornstein-Uhlenbeck process $Z_t$. Although this is not common in the mathematical literature, driving our flow by a stochastic process which is correlated in time is a practice employed in the physics literature, see for instance [105] for an application to plasma turbulence. Later, we will also make an assumption on the pressure field which seems more plausible for an external forcing with slightly better time regularity than Brownian motion. However, with minor changes in the calculations that follow and under similar assumptions to the ones we will proceed to make, one is able to adapt our results to the cases of white-in-time (i.e. driving directly by a force of the form of $W_t$ above) or an appropriate "OU tower" type forcing (see for instance [9] for a definition). This latter type of forcing ensures higher regularity in time for $Z_t$ as well as for $u$. Note also that the results we obtain here can be directly adapted to the Navier-Stokes equations on the torus $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$ with some simplifications.

In the context of the 3D Navier-Stokes equations driven by white in time forcing on a bounded domain, the only notion for which we know how to construct global in time solutions is that of *martingale solutions*. These are the weak in the probabilistic sense analogue of Leray-Hopf solutions. We refer to [53] for their definition and a proof of their existence. In the same paper, the authors also prove the existence of *stationary* martingale solutions. This very weak notion of a stationary solution provides a substitute for invariant measures for this kind of system, for which it is not even known whether a Feller Markov semigroup can be defined[1]. The main reason that this notion of solution is the best available option is that the standard a priori estimates

[1] a construction of *Markov selections* however can be found in [55]

only suffice to establish convergence in law for the Galerkin approximations of the Navier-Stokes system, so the Skorokhod embedding theorem must used - hence the stochastic basis is part of the construction. In our case, the notion of solution employed will be that of a martingale solution for the coupled process $(u_t, Z_t)$. We present the relevant definition in the next section.

We sketch the construction of stationary martingale solutions for (NSE) in Section 6, where we merely adapt the scheme used in [53] for the white-in-time case. The martingale solutions that we construct and work with will satisfy the estimates:

$$\frac{\nu}{T}\mathbf{E}\int_0^T \|\nabla u\|_{L^2_x}^2 dt \leq c_1 \nu U^2 + c_2 U^3 < \infty, \tag{2.1}$$

$$\mathbf{E}\sup_{0\leq t\leq T}\|u\|_{L^2_x}^r \leq \left(\mathbf{E}\|u_0\|_{L^2}^p + \frac{C_p}{\nu}\int_0^T \mathbf{E}\|Z_s\|_{H^{-1}}^p ds\right)\exp(\frac{C_p T}{\nu}) \text{ for any } r\in[1,\infty), \tag{2.2}$$

over any time interval $[0,T]$, where $c_1, c_2$ are positive constants depending on the geometric properties of $D$ and $Z_t$ but independent of $\nu$, while $U$ is defined as:

$$U^2 \equiv \lim_{t\to\infty}\frac{1}{t}\int_0^t \mathbf{E}\|u(\tau)\|_{L^2}^2 d\tau.$$

Inequality (2.1) can be derived following the work in [40] and for stationary solutions it simplifies to:

$$\nu\mathbf{E}\|\nabla u\|_{L^2_x}^2 dt \leq c_1\nu\mathbf{E}\|u\|_{L^2}^2 + c_2(\mathbf{E}\|u\|_{L^2}^2)^{3/2}. \tag{2.3}$$

The latter estimate will be used to deal with the dissipative terms in the proofs of our main theorems. The estimates (2.1) and (2.2) can be interpolated to obtain that for powers $p, q \in$

$[1,\infty], r \in [1,\infty)$ satisfying

$$\frac{2}{q}+\frac{3}{p} \geq \frac{3}{2} \text{ and } \frac{2}{r}+\frac{3}{p} > \frac{3}{2},$$

the $L_t^q L_x^p$ norm has finite $r$ moment over any finite time interval:

$$\mathbf{E}\|u\|^r_{L_t^q L_x^p} < \infty. \tag{2.4}$$

See [13] for a proof. A clear expression with appropriate constants for (2.4) can be obtained by carefully carrying out the appropriate interpolation procedure, but we omit it since we only need it for fixed values of $\nu$, while it is also weaker than our assumptions. Note that the upper bounds one obtains in (2.2) and (2.4) depend on $\nu$.

Given a vector $h \in \mathbb{R}^3$ and a vector field $u$, we use the following notation for the increments of $u$:

$$\delta_h u(x) := u(x+h) - u(x).$$

The classical third order structure functions of turbulence for which the $4/3$ and $4/5$ laws are usually demonstrated are defined as follows:

$$s_0(\ell) := \left\langle |\delta_{\ell\hat{n}} u|^2 (\delta_{\ell\hat{n}} u \cdot \hat{n}) \right\rangle \text{ and } s_{||}(\ell) := \left\langle (\delta_{\ell\hat{n}} u \cdot \hat{n})^3 \right\rangle,$$

where $\langle\cdot\rangle$ denotes some sort of averaging which can for instance be a combination of an ensemble, space, time and solid angle averaging, and each of the various averages can be dropped if the flow satisfies appropriate symmetries. Then, the $4/3$ and $4/5$ laws say that for any length scale $\ell$ in an

appropriate *inertial range* of scales we have:

$$s_0(\ell) \sim -\frac{4}{3}\epsilon\ell,$$

$$s_{||}(\ell) \sim -\frac{4}{5}\epsilon\ell,$$

where $\epsilon$ denotes the mean energy dissipation per unit mass. The structure functions are related to energy flux through scale $\ell$ and the negative sign in the $4/3$ and $4/5$ laws indicates a cascade of energy to small scales, see the discussion in [56].

The inertial range of scales in which these laws hold consists of scales $\ell$ such that $\ell_D \ll \ell_\nu \ll \ell_I$, where $\ell_D$ is the *dissipative scale*, at and below which viscous dissipation is dominant and $\ell_I$ is the *integral scale*, at which energy is being input into the system.

The $4/5$ law was first derived by Kolmogorov in [72] for statistically stationary, homogeneous and isotropic flows satisfying the fundamental axiom of hydrodynamic turbulence, what is called the $0$-th law or anomalous dissipation, which he previously introduced in [73]. We describe anomalous dissipation below. The $4/3$ law was already known to Yaglom (see [85, 114]), as was its equivalence to the $4/5$ law. See also the article [3] for a discussion of the relation between the two laws.

In his original theory of turbulence (known as K41), Kolmogorov postulated also what is known as his "similarity hypotheses", which led to the broader scaling prediction for the structure functions of order $p$:

$$\langle(\delta_{\ell\hat{n}}u)^p\rangle \sim C_p(\epsilon\ell)^{p/3}.$$

It was later argued (and supported by experiments and numerics) that this scaling is generally

incorrect. This phenomenon is related to the occurence of rare events in the flow and is known as *intermittency*. Thus one needs to introduce intermittency corrections $\tau(p/3)$ such that $p/3 + \tau(p/3)$ is the exponent on the right hand side above. See [56] for a discussion of intermittency, as well as [101, 107] for evidence of deviation from the exponent $p/3$. At any rate, the third order exponent (i.e. for $p = 3$) and in particular the $4/5$ law are considered to be exact. It is also worth remarking that intermittency is related to local averages of the dissipation of the form

$$\frac{\nu}{r^3} \int_{|x-y|<r} |\nabla u(y)|^2 dy,$$

see for instance [88] and [69]. Thus the local structure of a turbulent fluid is closely related to deviations from the predictions of K41.

One may observe that since the structure functions are defined in terms of increments of the velocity field, they only make sense, strictly speaking, if the domain of the flow has no boundary. Therefore, one of the points of the first part of this dissertation is to obtain $4/3$ and $4/5$ laws that make sense for flows in a bounded domain as well.

We first define our local structure functions.

**Definition 2.1.** Let $\psi \in C^\infty(D)$. The local third order structure functions (with respect to $\psi$) are defined as:

$$S_0(\ell) = \frac{1}{4\pi} \mathbf{E} \int_{S^2} \int \psi |\delta_{\ell\hat{n}} u|^2 \delta_{\ell\hat{n}} u \cdot \hat{n} dx dS(\hat{n}),$$

$$S_{||}(\ell) = \frac{1}{4\pi} \mathbf{E} \int_{S^2} \int \psi (\delta_{\ell\hat{n}} u \cdot \hat{n})^3 dx dS(\hat{n}).$$

In the above and in everything that follows, the domain $D$ of integration in $x$ has been suppressed. We proceed to state the assumptions that we will employ.

**Assumption 1.** *There exists $C > 0$ independent of $\nu$ for which our family of stationary martingale solutions $\{u^\nu\}_{\nu>0}$ satisfies:*

$$\sup_{\nu\in(0,1)} \mathbf{E}\|u^\nu\|^3_{L^3(D)} \leq C \text{ and}$$

$$\lim_{|h|\to 0} \sup_{\nu\in(0,1)} \mathbf{E}\|\delta_h u^\nu\|^3_{L^3(D)} = 0.$$

**Assumption 2.** *There exists $C > 0$ independent of $\nu$ and a choice of pressure fields $\{p^\nu\}_{\nu>0}$ corresponding to the solutions $\{u^\nu\}_{\nu>0}$ such that:*

$$\sup_{\nu\in(0,1)} \mathbf{E}\|p^\nu\|^{3/2}_{L^{3/2}(D)} < C \text{ and}$$

$$\lim_{|h|\to 0} \sup_{\nu\in(0,1)} \mathbf{E}\|\delta_h p^\nu\|^{3/2}_{L^{3/2}(D)} = 0.$$

**Assumption 3.** *The family of stationary martingale solutions $\{u^\nu\}_{\nu>0}$ satisfies local energy equality:*

$$2\nu\mathbf{E}\int \psi|\nabla u^\nu|^2 dx = \nu\mathbf{E}\int |u^\nu|^2\Delta\psi dx + \mathbf{E}\int (|u^\nu|^2 + 2p^\nu)(u^\nu\cdot\nabla\psi)dx + 2\mathbf{E}\int \psi u\cdot Z_t dx \tag{LEE}$$

We now state our main results. The first of them relates the local structure functions to nonlinear flux terms plus a local contribution from the noise:

**Theorem 2.2.** *Let $\{u^\nu\}_{\nu>0}$ be a family of stationary martingale solutions to* (NSE) *satisfying Assumption 1 and Assumption 2. Then, for every $\nu \in (0,1)$ there exists a scale $\ell_\nu$ such that $\ell_\nu \to 0$ when $\nu \to 0$, for which we have:*

$$\lim_{\ell_I \to 0} \limsup_{\nu \to 0} \sup_{\ell \in [\ell_\nu, \ell_I]} \left| \frac{1}{\ell} S_0(\ell) + \frac{2}{3} \mathbf{E} \int (|u|^2 + 2p) u \cdot \nabla \psi dx + \frac{4}{3} \mathbf{E} \int \psi u \cdot Z dx \right| = 0, \tag{2.5}$$

$$\lim_{\ell_I \to 0} \limsup_{\nu \to 0} \sup_{\ell \in [\ell_\nu, \ell_I]} \left| \frac{1}{\ell} S_{||}(\ell) + \frac{2}{5} \mathbf{E} \int (|u|^2 + 2p) u \cdot \nabla \psi dx + \frac{4}{5} \mathbf{E} \int \psi u \cdot Z dx \right| = 0 \tag{2.6}$$

Under the additional assumption of local energy equality, the above theorem reduces to a relation between the local structure functions and the local dissipation:

**Theorem 2.3.** *Let $\{u^\nu\}_{\nu>0}$ be as in Theorem 1.1 and also satisfying Assumption 3. Then for the same $\ell_\nu$ as in Theorem 1.1, we have:*

$$\lim_{\ell_I \to 0} \limsup_{\nu \to 0} \sup_{\ell \in [\ell_\nu, \ell_I]} \left| \frac{1}{\ell} S_0(\ell) + \frac{4}{3} \nu \mathbf{E} \int \psi |\nabla u|^2 dx \right| = 0, \tag{2.7}$$

$$\lim_{\ell_I \to 0} \limsup_{\nu \to 0} \sup_{\ell \in [\ell_\nu, \ell_I]} \left| \frac{1}{\ell} S_{||}(\ell) + \frac{4}{5} \nu \mathbf{E} \int \psi |\nabla u|^2 dx \right| = 0. \tag{2.8}$$

The scale $\ell_\nu$ is only an upper bound on the dissipative scale, at which the viscous forces are dominant. The scale variable $\ell_I$ which we use in our statements should not be confused with the integral scale of the flow. Its role is to eliminate the influence of the large scale forcing and it corresponds to the variable $\ell$ in equation (6.57) in [56] (see the discussion therein).

We refer to both (2.5) and (2.7) as the *local 4/3 law*, and similarly (2.6) and (2.8) will be referred to as the *local 4/5 law*. The derivations of the above laws on the basis of Assumptions 1,2 and 3 are presented in sections 4 and spectively. Note that in order for Theorem 2.2 to be physically relevant, we would reasonably require a condition guaranteeing the existence of a

sequence $\nu_k$ of positive numbers decreasing to $0$ such that

$$\lim_{k\to\infty} \mathbf{E} \int \psi Z_0 \cdot u^{\nu_k} dx > 0,$$

which would mean that energy is being input in the system at least at a rate independent of the Reynolds number. Had we driven the Navier-Stokes equations by white-in-time forcing, this would be unnecessary since in that case it follows by Ito's formula that the rate of energy input is given by the trace of the covariance of the noise.

## 2.2 Discussion and context of this work

We present first the notion of solution we will be concerned with.

**Definition 2.4.** A *martingale solution* to (NSE) consists of a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbf{P})$ on which we have a family of independent Brownian motions $\{\beta_k\}_{k=1}^{\infty}$ and a progressively measurable process $(u, Z_t) : [0,T] \times \Omega \to H \times H$, whose paths are $\mathbf{P}-$almost surely in

$$\left(C([0,T]; D(A^{-\frac{\alpha}{2}})) \cap L^\infty(0,T;H) \cap L^2(0,T;V)\right)^2$$

for some $\alpha > 1$, for which we have:

$$dZ_t = -AZ_t dt + \sum_{k=1}^{\infty} \sigma_k e_k d\beta_k$$

and such that $\mathbf{P}$-almost surely, for every smooth divergence free vector field $v$ with compact

support in $D$ we have:

$$\langle u(t), v\rangle + \int_0^t \langle u(s)\cdot\nabla u(s), v\rangle\, ds + \nu\int_0^t \langle \nabla u(s), \nabla v\rangle\, ds = \langle u(0), v\rangle + \int_0^t \langle Z_s, v\rangle\, ds,$$

where $\langle\cdot,\cdot\rangle$ denotes the $L^2$ inner product.

A *stationary martingale solution* to (NSE) is a martingale solution such that for every $\tau > 0$, the paths $(u(\cdot+\tau), Z_{\cdot+\tau})$ and $(u(\cdot), Z_\cdot)$ coincide in law.

The reader can consult [53] to examine exactly how $D(A^s)$ is defined (for positive or negative values of $s$) in terms of series expansions with respect to an eigenbasis of $A$ in $H$. Note that $Z_t$ is acted upon by $L$, which has a specific relation to $A$ as mentioned earlier, while by taking the (equivalent) Leray projected version of the equation for $u$ we see that $A$ itself acts on $u$. This is why both $u$ and $Z_t$ can be seen in our context in terms of series expansions with respect to an eigenbasis of $A$.

We will frequently abuse the terminology of the above definition and refer to the process $u$ itself as a martingale solution. According to Definition 2.4, in constructing martingale solutions we are allowed to pick a probability space and a $Q$-Wiener process on it, for which there is a process $(u_t, Z_t)$ solving (NSE) in the weak sense, even if finding such a solution is not possible in some a priori given probability space. We note that even though the equation for $Z_t$ has a unique invariant measure, as the viscosity $\nu$ varies we obtain solutions $(u_t^\nu, Z_t^\nu)$ where the processes $Z_t^\nu$ are only equal to each other in law. This difference is not significant in what follows and we caution the reader that we will mostly ignore it, omitting the index $\nu$ and denoting all of the $Z_t^\nu$ processes by $Z_t$.

In the context of weak solutions to the deterministic Euler and Navier-Stokes equations, a

*defect measure* was introduced in [44], which is a distribution depending on the solution $u$ that accounts for loss[2] of kinetic energy not due to viscosity but due to lack of smoothness. The defect measure is a limit of expressions related to $s_0(\ell)$ and a $4/3$ law relating the structure function to the defect (but not to the local viscous dissipation) is derived in the same article assuming the existence of the limit $\lim_{\ell\to 0}(\ell^{-1}s_0(\ell))$. In a similar direction, [49] later derived scaling laws for various third order structure functions, including a $4/5$ law. These works provide relations between the structure functions and the defect measure at fixed Reynolds number (which may be infinite) and without any reference to the inertial range whereas here we study the limit of infinite Reynolds number within an appropriate range of scales, without concerning ourselves with an emerging solution of Euler as $\nu \to 0$. Other differences are that here we study a randomly driven Navier-Stokes system and that we are dealing with boundaries, complicating things for instance through the presence of terms involving the pressure which cannot be simply estimated by means of the Calderon-Zygmund inequality (see the discussion on Assumption 2 below). More recent, related but improved results for the Euler equations in bounded domains can be found in [42] and [5]. The common thread between all of these works as well as the present one is carefully performing local calculations with energy fluxes on the basis of the equation, under different assumptions each time.

More recently, it was proven in [13] that the $4/5$ law holds for stationary martingale solutions of the Navier-Stokes system on $\mathbb{T}^3$ with white in time forcing, under a very weak condition which the authors called "weak anomalous dissipation". Classically, one defines the dimensionless energy dissipation rate $\epsilon_\nu := \nu L \mathbf{E}\|\nabla u\|_{L^2}^2/U^3$ for a characteristic length scale $L$ and a

[2] or creation; it is now known that there are weak solutions attaining any energy profile; see [23]. However, the class of Leray-Hopf solutions is not known to have this pathology.

characteristic velocity $U$. For instance, $L$ is taken to be the diameter of the domain or the integral scale, while $U$ is taken to be $(\mathbf{E}\|u^\nu\|_{L^2}^2)^{1/2}$. Anomalous dissipation is the phenomenon that $\epsilon_\nu$ remains bounded below by a strictly positive number as $\nu \to 0$. If the kinetic energy (and thus $U$) remains bounded, anomalous dissipation means that $\nu\mathbf{E}\|\nabla u^\nu\|_{L^2}^2$ is also nonvanishing. Weak anomalous dissipation is then stated as a slight weakening of the boundedness of $\mathbf{E}\|u^\nu\|_{L^2}^2$ as $\nu \to 0$:

**Definition 2.5.** A family $\{u^\nu\}_{\nu>0}$ of stationary martingale solutions to (NSE) is said to satisfy *weak anomalous dissipation* if:

$$\lim_{\nu\to 0} \nu\mathbf{E}\|u^\nu\|_{L_x^2}^2 = 0. \tag{WAD}$$

This is equivalent to the vanishing, in the inviscid limit, of the so-called Taylor microscale, introduced in [103]. The derivation of the $4/3$ and $4/5$ laws in [13] uses a relation between the third order structure functions and the two-point correlation matrix of $u$ usually referred to as the von Kármán–Horwath-Monin relation. The terms involving two-point correlations descend from the viscous terms of (NSE) and can therefore be treated through (WAD), furnishing the results. The same line of reasoning does not seem to work in a manner conducive to the derivation of analogues of the $4/3$ and $4/5$ laws in our case. This is because in the analogue of the von Kármán-Horwath-Monin relation one obtains multiple terms in which $\nu$ does not appear and which unfortunately can not be treated solely on the basis of (WAD).

We now give a discussion of our assumptions. For fixed $\nu$, Assumption 1 provides nothing new. This follows from (2.4) for $p = q = r = 3$. Keeping in mind the Kolmogorov-Riesz characterization of compactness in $L^p$ over Euclidean spaces, Assumption 1 can be informally described as: *the family* $\{u^\nu\}_{\nu\in(0,1)}$ *is precompact in* $L^3(D)$ *"on average"*. This viewpoint is

followed in the proof of Proposition 2.8. Note that the uniform boundedness of $\mathbf{E}\|u^\nu\|^3_{L^3(D)}$ is already a stronger condition than the uniform boundedness of $\mathbf{E}\|u^\nu\|^2_{L^2(D)}$, which in turn is clearly stronger than (WAD). Therefore, in our proofs below, in places where we only need to use weak anomalous dissipation we will clarify it, in order to be clear about where our first assumption really comes into play.

**Remark 2.6.** In [13] an example is given of a family (parametrized by $\nu$) of Navier-Stokes systems on $\mathbb{T}^3$, driven by a degenerate white-in-time Gaussian noise, and a family of stationary martingale solutions thereof which does not satisfy (WAD). Thus in that case Assumption 1 is also not valid. To see that a similar phenomenon might still occur in the case of Ornstein-Uhlenbeck external forcing, consider (NSE) on $\mathbb{T}^3$, where the driving noise is $(e^{ix_2}, 0, 0)Z_t$, where $Z_t$ solves the one dimensional SDE $dZ_t = -Z_t dt + d\beta_t$, $\beta_t$ being a standard Brownian motion. One might construct a stationary martingale solution of

$$\partial_t u = \nu \Delta u + Z_t(e^{ix_2}, 0, 0)$$

of the form $u^\nu(t) = g(t)(e^{ix_2}, 0, 0)$, where the statistically stationary process $g(t)$ solves the (random) ODE $g'(t) = -g(t) + Z_t$. For this family, we have:

$$\nu \mathbf{E}\|u^\nu(t)\|^2_{L^2} = 2\mathbf{E} Z_t g(t),$$

whereby taking the limit $t \to \infty$ and using elementary stochastic calculus one finds that the right hand side is equal to $\frac{1}{\nu+1}$. Thus (WAD), and by extension Assumption 1, is not true. On the other hand, note that we just showed that the energy input is bounded away from $0$ in the

vanishing viscosity limit. The fact that in this situation the energy input is nonvanishing as $\nu \to 0$ (which is the most interesting case) and Assumption 1 does not hold is due to the degeneracy and low dimensionality of this particular system and does not preclude both of these being true for more complex Navier-Stokes systems. We merely showcase this example to exhibit that our assumptions may fail in the case of Ornstein-Uhlenbeck forcing in the same way as they might fail for white-in-time forcing.

Note that for the white-in-time forced Burgers equation it was shown in [45] (see also the review article [46] that the energy of the solution is bounded and the energy dissipation rate is nonvanishing as $\nu \to 0$, both of which are stronger than (WAD).

Any uniform in $\nu$ bound for $\mathbf{E}\|u\|_{B^{\alpha}_{p,\infty}}$ for some $p \geq 3, \alpha \in (0,1)$ would imply Assumption 1. Note also that via the relation between structure functions and Besov norms, numerical and experimental evidence suggests the plausibility of this assumption, see [15, 58].

**Remark 2.7.** These ideas are also related to the Onsager phenomenology of turbulence. Even though they pertain to the Euler equations, they are still instructive in the study of the Navier-Stokes equations as well. In [89] Onsager conjectured that if a weak solution of the Euler equations belongs to $C^{\alpha}$ for $\alpha \in (1/3, 1)$ then it conserves energy, and that for $\alpha \in (0, 1/3)$ there are weak solutions in $C^{\alpha}$ dissipating energy. The first part was proven in [48] for a norm stronger than $\|\cdot\|_{C^{\alpha}}$ and then settled in [29] in the sharper and more natural context of Besov spaces $B^{\alpha}_{3,\infty}$. The second part was settled completely by [68] (see also the review article [24] and references to earlier work therein). In [41] it is shown in the context of the deterministic Navier-Stokes equations that under a mild lower bound of the form $\sim \nu^{\alpha}$ on the mean dissipation rate of $u^{\nu}$ in the limit $\nu \to 0$, the family $\{u^{\nu}\}_{\nu>0}$ can not be bounded in $B^{\sigma_{\alpha}+\epsilon}_{3,\infty}$ for $\epsilon > 0$ and $\sigma_{\alpha} = \frac{1+\alpha}{3-\alpha}$. These

do not preclude the potential validity of our discussion since on-average precompactness in $L^3_x$ (which would correspond to uniform boundedness and vanishing of increments of $\{u^\nu\}$ in $L^3_t L^3_x$) is a weaker condition that may still be valid. To our knowledge, however, there are no known non-trivial, uniform in $\nu$ a priori estimates for solutions to (NSE) (i.e. ones that do not directly follow from 2.3).

With respect to Assumption 2, its role is purely technical, in that in what follows we will need some bounds on the pressure that are uniform in $\nu$. These estimates follow from Assumption 1 in the absence of boundaries: they are a consequence of the well known Calderon-Zygmund estimates. The presence of boundaries complicates things at this point. While for the Euler equations we can formally derive a Neumann problem for the pressure by taking the divergence of the equation for $u$, in the Navier-Stokes setting such a Neumann problem involves the normal trace of $\Delta u$, which requires a lot more regularity than what we are assuming here. An argument relying on the maximal regularity for the Stokes semigroup as found for instance in [95, Chapter 5] would provide us with bounds on the pressure, yet we do not know how to improve these to estimates uniform in $\nu$. Nevertheless, it is worth mentioning that in view of the aforementioned maximal regularity arguments we expect such estimates to be more readily obtained for a system driven by a process with better time regularity. This is another reason why we use an Ornstein-Uhlenbeck stochastic body force.

As for Assumption 3, we note that it is related to the notion of suitable weak solutions for the deterministic Navier-Stokes equations, see [25]. Instead of local energy inequality, here we assume equality but only for a particular spatial cutoff function $\psi$. To formally derive this one proceeds as usual, i.e. with a *space-time* dependent cutoff of the form $\psi(x)h(t)$ and uses

the stationarity to cancel the boundary (in time) contributions[3] from the integration of the term $\partial_t(\psi h)|u|^2$. We also refer to the discussion in [78] in connection to [98] for a sufficient condition for local energy equality up to the first blow-up time.

An interesting feature of our assumptions is that they imply the existence of a stationary martingale solution for the Euler equations, arising as a subsequential inviscid limit of $u^\nu$, for which the local scaling laws also hold. These solutions can be seen as a model for ideal turbulence incorporating the $4/3$ and $4/5$ laws in the local sense.

**Proposition 2.8.** *Under assumptions 1 and 2, there exists a stationary martingale solution $u^0$ of the Euler equations with associated pressure $p^0$ such that $u^\nu \to u^0$ in $L^3_{\omega,t,x}$ over any bounded time interval, and $u^\nu(0) \to u^0(0)$ in $L^3_{\omega,x}$. Moreover, we have:*

$$\lim_{\ell_I \to 0} \sup_{\ell \in (0,\ell_I)} \left| \frac{1}{\ell} S_0(\ell) + \frac{2}{3}\mathbf{E}\int (|u^0|^2 + 2p^0)u^0 \cdot \nabla\psi dx + \frac{4}{3}\mathbf{E}\int \psi u^0 \cdot Z^0 dx \right| = 0, \tag{2.9}$$

$$\lim_{\ell_I \to 0} \sup_{\ell \in (0,\ell_I)} \left| \frac{1}{\ell} S_{||}(\ell) + \frac{2}{5}\mathbf{E}\int (|u^0|^2 + 2p^0)u^0 \cdot \nabla\psi dx + \frac{4}{5}\mathbf{E}\int \psi u^0 \cdot Z^0 dx \right| = 0 \tag{2.10}$$

*Proof.* We will pass to a subsequence of $\nu \in (0,1)$ multiple times and for convenience we will denote the resulting subsequence simply by $\nu$. One may use Skorokhod's embedding theorem to construct a new stochastic basis denoted by $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ and with expectation $\mathbf{E}$, as well as processes $u^\nu, p^\nu, Z^\nu_t$ on this stochastic basis such that $(u^\nu, Z^\nu_t)$ is a stationary martingale solution of (NSE) satisfying our assumptions, as well as a process $Z^0_t$ such that $Z^\nu_t \to Z^0_t$ in $L^2_{t,x}$ $\mathbf{P}$-almost surely. Then one can follow the proof of the Kolmogorov-Riesz compactness criterion (see e.g.

[3] In using the phrase "boundary contributions" we imply that $h$ should be thought of as a placeholder for a sequence of nonnegative, compactly supported smooth functions approximating the constant function 1 from below on their common domain.

[62]) to obtain a subsequence such that:

$$\lim_{\nu\to 0}\mathbf{E}\int_0^T \|u^\nu(t)-u^0(t)\|^3_{L^3_x}dt=0,$$
$$\lim_{\nu\to 0}\mathbf{E}\|u^\nu(0)-u^0(0)\|^3_{L^3_x}=0,$$
$$\lim_{\nu\to 0}\mathbf{E}\int_0^T \|p^\nu(t)-p^0(t)\|^{3/2}_{L^{3/2}_x}dt=0,$$

as well as, $\mathbf{P}$-almost surely:

$$\lim_{\nu\to 0}\int_0^T \|u^\nu(t)-u^0(t)\|^3_{L^3_x}dt=0,$$
$$\lim_{\nu\to 0}\|u^\nu(0)-u^0(0)\|_{L^3_x}=0,$$
$$\lim_{\nu\to 0}\int_0^T \|p^\nu(t)-p^0(t)\|^{3/2}_{L^{3/2}_x}dt.$$

From the $\mathbf{P}$-a.s. convergences above it readily follows that $(u^0, Z^0)$ is a stationary martingale solution of the OU driven Euler system, i.e. $\mathbf{P}$-almost surely, for almost every $T$ and for all smooth divergence fields $v$ with compact support in $D$ we have:

$$\left\langle u^0(T),v\right\rangle+\int_0^T\left\langle u^0(s)\cdot\nabla u^0(s),v\right\rangle ds=\left\langle u^0(0),v\right\rangle+\int_0^T\left\langle Z^0_s,v\right\rangle ds.$$

From the convergences in expectation in conjunction with (2.5) and (2.6) we obtain (2.9) and (2.10). Lastly, the continuity of the generalized normal trace operator $\Gamma_N : L^3_{\text{div}}(D)\to W^{-1/3,3}(\partial D)$ (see e.g. [100]) implies the no-penetration condition for $u^0$. □

**Remark 2.9.** In the deterministic case for domains with boundary, there have been some recent investigations regarding the problem of the inviscid limit. For instance, a modulus of continuity

over a finite range of scales for the second order structure function has been shown to imply the existence of a subsequential inviscid limit in [43]. Such a modulus is not necessarily implied by Assumption 1 as the latter has been stated, but would follow (over the entire range of scales) if a uniform bound in $B^{\alpha}_{3,\infty}$ was imposed as an assumption on $\{u^{\nu}\}_{\nu}$ instead.

We finally want to discuss some aspects of our work that we deem significant. First, we believe that a considerably different approach would be necessary in order to treat turbulence *near* the boundary, since what we are doing always happens at a fixed distance away from it independently of the viscosity. Second, the conditions that we have assumed our solutions of (NSE) to satisfy seem very hard to verify, not least in light of Remark 2.6. However, see [8] for a proof of a weak anomalous dissipation condition for the problem of passive scalar advection and the earlier mentioned [45] for boundedness of the energy in the inviscid limit for the stochastic Burgers equation. At any rate, provided that one can obtain relations similar to the ones we derive in the next section, we believe that similar steps can be taken to derive local cascade laws for other systems that are expected to exhibit this sort of behavior, for instance one could attempt to adapt the derivations of the scaling laws in [12] or [8] to bounded domains in the same way that our paper adapts the results in [13].

## 3 Preliminary results

We first set some notation that will be used throughout:

$$\Gamma(t,h) = \mathbf{E}\int \psi(u \otimes T_h u)dx \tag{3.1}$$

$$D(t,h)^k = \mathbf{E}\int \psi(\delta_h u \otimes \delta_h u)\delta_h u^k dx. \tag{3.2}$$

Here and in what follows, we use $T_h f$ to denote the translation of a function (or vector field) $f$ by $-h$, i.e. $T_h f(x) = f(x+h)$. In what follows, for any order-2 tensor $\eta$, we denote the $(i,j)$ component by $\eta^{ij}$.

We first state some basic facts about the regularity of the quantities defined above. The first result states that time averages of $S_0$ and $S_{||}$ are continuous functions of $\ell$. The second one states that time averages of $\Gamma$ are $C^2$ functions of $h$.

**Lemma 3.1.** *Fix $\nu > 0$ and let $u$ be a martingale solution of (NSE). The functions:*

$$\ell \mapsto \frac{1}{T}\int_0^T S_0(t,\ell)dt,$$

$$\ell \mapsto \frac{1}{T}\int_0^T S_{||}(t,\ell)dt$$

*are continuous for $\ell \in (0,\infty)$ and satisfy the bound:*

$$\sup_{\ell\in(0,1)}\left|\frac{1}{T}\int_0^T S_0(t,\ell)dt\right| + \sup_{\ell\in(0,1)}\left|\frac{1}{T}\int_0^T S_{||}(t,\ell)dt\right| \lesssim \mathbf{E}\|u\|_{L^3_x}^3.$$

*If in addition $u$ is a stationary martingale solution of (NSE), $S_0$ and $S_{||}$ are time-independent, continuous in $\ell$ and satisfy:*

$$\lim_{\ell\to 0} S_0(\ell) = \lim_{\ell\to 0} S_{||}(\ell) = 0.$$

**Lemma 3.2.** *Let $\nu, u$ be as in Lemma 3.1. For $i,j = 1,2,3$, $\Gamma^{ij}$ is uniformly bounded and*

*continuous. Moreover, the time averages*

$$h \mapsto \frac{1}{T}\int_0^T \Gamma^{ij}(t,h)dt$$

*are in* $C^2(\mathbb{R}^3)$. *If* $u$ *is stationary,* $\Gamma^{ij}(t,h) \equiv \Gamma^{ij}(h) \in C^2(\mathbb{R}^3)$.

The proofs of the preceding lemmata closely follow those of Lemma 2.7 and Lemma 2.8 respectively in [13] with only minor modifications and are therefore omitted.

In order to establish local versions of the relevant third-order scaling laws we will give a localized variation of a traditional recipe. Kolmogorov himself in [72] derived the $4/5$ law using a relation that according to Frisch's discussion in [56] was described for isotropic flows in [35] and later generalized to the anisotropic case in [85], attributed mainly to Monin. This is a relation between the third order structure matrix and the correlation matrix of $u$, where these are given by (3.2) and (3.1) respectively when we take $\psi \equiv 1$. Following Frisch, we call this relation the von Kármán-Howarth-Monin relation, in short KHM. The main ingredient of our take on Kolmogorov's recipe is a local version of the KHM relation. This is the content of Lemma 3.4.

Since our localization proceeds by means of multiplying the integrands of the classical structure functions by a smooth cutoff function $\psi \in C^\infty(D)$ of the spatial variable (cf. (3.2)), we begin by calculating an expression for fluxes in the local structure matrix:

**Lemma 3.3.** *Let* $u \in L^3(D)$ *be a divergence-free vector field. Define* $\kappa = d(\mathrm{supp}(\psi), \partial D)$. *Then, in the sense of distributions in the* $h$ *variable for* $|h| < \kappa$ *we have:*

$$\sum_k \partial_{h_k} \int \psi(\delta_h u \otimes \delta_h u)\delta_h u^k dx = \sum_k \partial_{h_k} \int \psi \left[(u \otimes T_h u)u^k + (T_h u \otimes u)u^k\right] dx \tag{3.3}$$

$$- \sum_k \partial_{h_k} \int \psi \left[ (u \otimes T_h u) T_h u^k + (T_h u \otimes u) T_h u^k \right] dx$$
$$- \int T_h(u \otimes u) \delta_h u \cdot \nabla \psi dx.$$

*Proof.* Expanding the left hand side of (3.3) we get:

$$\sum_k \partial_{h_k} \int \psi \left[ (u \otimes T_h u) u^k + (T_h u \otimes u) u^k - (u \otimes T_h u) T_h u^k - (T_h u \otimes u) T_h u^k \right] dx$$
$$+ \sum_k \partial_{h_k} \int \psi \left[ (T_h u \otimes T_h u) \delta_h u^k + (u \otimes u) T_h u^k - (u \otimes u) u^k \right] dx.$$

One easily sees that:

$$\sum_k \partial_{h_k} \int \psi (u \otimes u) u^k dx = \sum_k \partial_{h_k} \int \psi (u \otimes u) T_h u^k dx = 0,$$

where the second equality comes from the incompressibility of $u$. Next, notice that for $|h| < \kappa$ we can apply the operator $T_{-h}$ inside the integrand and then compute directly:

$$\sum_k \partial_{h_k} \int \psi (T_h u \otimes T_h u) \delta_h u^k dx = - \sum_k \partial_{h_k} \int T_{-h} \psi (u \otimes u) \delta_{-h} u^k dx =$$
$$\int (u \otimes u) \delta_{-h} u \cdot T_{-h} \nabla \psi dx = - \int (T_h u \otimes T_h u) \delta_h u \cdot \nabla \psi dx.$$

Combining these calculations we arrive at the result. □

In what follows, we will need the double inner product of two tensors $\eta_1^{ij}, \eta_2^{ij}$, which we shall denote by $\eta_1 : \eta_2 \equiv \sum_{i,j} \eta_1^{ij} \eta_2^{ij}$. Our local version of the KHM relation is the following:

**Lemma 3.4** (Local KHM relation)**.** *Let $u^\nu$ be a stationary martingale solution to* (NSE) *on the*

*time interval* $[0,T]$ *as postulated in Assumptions 1 and 2. Let* $\eta(h)=(\eta^{ij}(h))_{i,j=1}^{3}$ *be a smooth order* 2 *test function of the form:*

$$\eta(h)=\phi(|h|)I+\varphi(|h|)\hat{h}\otimes\hat{h},$$

*where* $\phi(\ell),\varphi(\ell)$ *are smooth functions compactly supported on* $(0,\kappa)$ *and* $\hat{h}=h/|h|$*. In component notation:*

$$\eta^{ij}(h)=\phi(|h|)\delta^{ij}+\varphi(|h|)\frac{h_ih_j}{|h|^2}.$$

*We then have the following equality:*

$$\begin{aligned}
&\frac{1}{2}\int\partial_{h_k}\eta:D(t,h)^k dh+\frac{1}{2}\mathbf{E}\int\int\eta:T_h(u\otimes u)(\delta_h u\cdot\nabla\psi)dxdh=\\
&2\nu\int\Delta\eta:\Gamma(h)dh-\nu\mathbf{E}\int\int\left(2(\partial_{x_k}\psi\partial_{h_k}\eta):(u\otimes T_hu)+\Delta\psi\eta:(u\otimes T_hu)\right)dxdh\\
&+\mathbf{E}\int\int p\eta:(\nabla\psi\otimes T_hu)dxdh-\mathbf{E}\int\int\psi(\nabla(T_hp)\otimes u):\eta dxdh\\
&+2\mathbf{E}\int\int\eta:(u\otimes T_hZ_t+Z_t\otimes T_hu)dxdhdt+\mathbf{E}\int\int u\cdot\nabla\psi\left(\eta:(u\otimes T_hu)\right)dxdh.
\end{aligned}$$

Here and in what follows, $I$ denotes the identity matrix.

*Proof.* Note that for any stationary process $X_t$, one has

$$\frac{1}{T}\mathbf{E}\int_0^T\|X_t\|_{L^p}^p dt=\mathbf{E}\|X_t\|_{L^p}^p.$$

Therefore Assumptions 1 and 2 give uniform bounds for $L^3_{\omega,t,x}$ norms of $u$ and $L^{3/2}_{\omega,t,x}$ norms of

$p$ and their respective increments over fixed finite intervals. We will first prove an intermediate balance relation with an additional integration over a finite time interval and subsequently use Lemma 3.3 and the stationarity of $u$.

In order to prove the KHM relation, our strategy is to smoothen the Navier-Stokes equations to get a similar relation and then pass to the limit. Given that we are considering functions on a bounded domain, convolving with a standard mollifier might be problematic since differential operators and convolution do not simply commute. We thus opt for a slightly different approach which is however still rather direct.

We refer to [47] for the complete definition of the smoothing operators $K_\delta$. We merely mention that for appropriate smooth maps $\phi_\delta$ and (sufficiently small) fixed $r > 0$, these operators are given by:

$$K_\delta f(x) = \int_{B(0,1)} \rho(y) f(\phi_\delta(x) + r\delta y) dy,$$

$\rho$ being a $C^\infty$ positive and radially symmetric function with integral $1$ and support in $B(0,1)$. If $f$ is in any $L^p$ space for some $p \in [1, \infty)$, $K_\delta f$ is $C^\infty$ and thus we can apply differential operators to it. The main benefit of the operators $K_\delta$ is that one is able to control their commutators with differential operators in the limit $\delta \to 0$.

By applying $K_\delta$ to (NSE), we obtain:

$$\partial_t K_\delta u + K_\delta \operatorname{div}(u \otimes u) + K_\delta \nabla p = \nu K_\delta \Delta u + K_\delta Z_t \iff$$

$$\partial_t K_\delta u + \operatorname{div} K_\delta(u \otimes u) + \nabla K_\delta p - \nu \Delta K_\delta u - K_\delta Z_t = -[K_\delta, \operatorname{div}](u \otimes u) - [K_\delta, \nabla] p - \nu [K_\delta, \Delta] u.$$

The product rule gives:

$$\partial_t(\psi K_\delta u \otimes T_h K_\delta u) = \psi \partial_t K_\delta u \otimes T_h K_\delta u + \psi K_\delta u \otimes \partial_t T_h K_\delta u,$$

therefore:

$$\begin{aligned}
\partial_t \int \psi K_\delta u \otimes T_h K_\delta u dx &= \int \psi [K_\delta \partial_t u \otimes T_h K_\delta u + K_\delta u \otimes T_h K_\delta \partial_t u] dx \\
&= -\int \psi [\partial_k K_\delta (u^k u) \otimes T_h K_\delta u + K_\delta u \otimes \partial_k T_h K_\delta (u^k u)] dx \\
-\int_D \psi [\nabla K_\delta p \otimes T_h K_\delta u + K_\delta u \otimes \nabla T_h K_\delta p] dx &+ \nu \int_D \psi [\Delta K_\delta u \otimes T_h K_\delta u + K_\delta u \otimes \Delta T_h K_\delta u] dx \\
&+ \int \psi [K_\delta Z_t \otimes T_h K_\delta u + K_\delta u \otimes T_h K_\delta Z_t] dx + I_C,
\end{aligned}$$

where $I_C$ is the contribution of the commutator terms:

$$\begin{aligned}
I_C := &-\int \psi \left( [K_\delta, \mathrm{div}](u \otimes u) + [K_\delta, \nabla] p + \nu [K_\delta, \Delta] u \right) \otimes T_h K_\delta u dx \\
&-\int \psi K_\delta u \otimes \left( [K_\delta, \mathrm{div}](u \otimes u) + [K_\delta, \nabla] p + \nu [K_\delta, \Delta] T_h u \right) dx,
\end{aligned}$$

and should not be confused with the identity matrix $I$. The contribution of the nonlinearity becomes:

$$\int (K_\delta u \cdot \nabla \psi) K_\delta u \otimes T_h K_\delta u dx + \partial_{h_k} \int \psi K_\delta u \otimes (u^k T_h K_\delta u - T_h K_\delta (u^k u)) dx.$$

The contribution of the pressure terms:

$$\int K_\delta p \nabla\psi \otimes T_h K_\delta u dx + \nabla_h \int \psi K_\delta p T_h K_\delta u dx - \nabla_h^\perp \int \psi K_\delta u T_h K_\delta p dx.$$

The contribution of the viscous terms:

$$\int \left(\Delta\psi K_\delta u \otimes T_h K_\delta u dx + 2\psi K_\delta u \otimes (\nabla\psi \cdot \nabla T_h K_\delta u) + 2\psi K_\delta u \otimes T_h \Delta K_\delta u\right) dx.$$

The integrations by parts performed above give no boundary terms because $\psi$ is compactly supported inside $D$. For brevity's sake, from now on we use the abbreviations $u_\delta = K_\delta u, p_\delta = K_\delta p$ and $\Gamma_\delta(t,h) = \int \psi u_\delta \otimes T_h u_\delta dx$. Upon contracting by the tensor $\eta$ and integrating (by parts) in $h$, in $t$ and in $\omega$, as well as using the above calculations, we obtain:

$$\int \eta : \Gamma_\delta(T,h)dh - \int \eta : \Gamma_\delta(0,h)dh = \tag{3.4}$$

$$-\mathbf{E}\int_0^T \int\int \psi \left(\partial_{h_k}\eta : (u_\delta \otimes T_h u_\delta)\right)(u_\delta^k - T_h u_\delta^k)dxdhdt \tag{3.5}$$

$$+\nu\mathbf{E}\int_0^T \int\int \eta : (u_\delta \otimes T_h u_\delta)\Delta\psi dxdhdt - 2\nu\mathbf{E}\int_0^T \int\int (\partial_{x_k}\psi\partial_{h_k}\eta) : (u_\delta \otimes T_h u_\delta)dxdhdt \tag{3.6}$$

$$+2\nu\mathbf{E}\int_0^T \int\int \psi\Delta\eta : (u_\delta \otimes T_h u_\delta)dxdhdt \tag{3.7}$$

$$+\mathbf{E}\int_0^T \int\int p_\delta\eta : (\nabla\psi \otimes T_h u_\delta)dxdhdt \tag{3.8}$$

$$-\mathbf{E}\int_0^T \int\int \psi \operatorname{div}\eta \cdot (T_h p_\delta u_\delta - p_\delta T_h u_\delta)dxdhdt \tag{3.9}$$

$$+\mathbf{E}\int_0^T \int\int (u_\delta \cdot \nabla\psi)\left(\eta : (u_\delta \otimes T_h u_\delta)\right)dxdhdt \tag{3.10}$$

$$+\mathbf{E}\int_0^T \int\int \psi(x)\eta : (K_\delta Z_t \otimes T_h u_\delta + u_\delta \otimes T_h K_\delta Z_t)dxdhdt \tag{3.11}$$

$$+ \mathbf{E} \int_0^T \int I_C : \eta dh dt. \tag{3.12}$$

We now want to take the limit $\delta \to 0$ and produce a relation for the solution $u$.

First of all, note that $\|u\|_{L^3_{\omega,x}}$ is finite. Since $\|u_\delta - u\|_{L^3_x} \leq C\|u\|_{L^3_x}$ and $\|u_\delta - u\|_{L^3_x} \to 0$ as $\delta \to 0$, by the dominated convergence theorem it follows that $\|u_\delta - u\|_{L^3_{\omega,x}} \to 0$ as $\delta \to 0$. For convenience and clarity, we henceforth will freely use component notation and employ the Einstein summation convention. Thus, observe that for the terms on the left hand side of (3.4) we have for fixed $t$:

$$\begin{aligned}
\left| \int \eta(h) : (\Gamma_\delta(t,h) - \Gamma(t,h))\, dh \right| &= \left| \mathbf{E} \int \int \eta^{ij} \psi \left( u^i_\delta T_h u^j_\delta - u^i T_h u^j \right) dx dh \right| \\
&\leq C\|\eta\|_{L^\infty} \left( \mathbf{E}\|\psi(u^i_\delta - u^i) T_h u^j_\delta\|_{L^1} + \mathbf{E}\|T_{-h}(\psi u^i)(u^j_\delta - u^k)\|_{L^1} \right) \\
&\leq C\|\eta\|_{L^\infty} \|\psi\|_{L^3_x} \|u_\delta - u\|_{L^3_{\omega,x}} \|u\|_{L^3_{\omega,x}} \to 0 \text{ as } \delta \to 0. \quad (3.13)
\end{aligned}$$

Similarly, the following also hold:

$$\mathbf{E} \int_0^T \int \int \Delta\psi \eta(h) : (u_\delta \otimes T_h u_\delta - u \otimes T_h u)\, dx dh dt \to 0 \text{ as } \delta \to 0, \tag{3.14}$$

$$\mathbf{E} \int_0^T \int \int (\partial_{x_k} \psi \partial_{h_k} \eta(h)) : (u_\delta \otimes T_h u_\delta - u \otimes T_h u)\, dx dh dt \to 0 \text{ as } \delta \to 0 \tag{3.15}$$

$$\mathbf{E} \int_0^T \int \int \psi \Delta \eta(h) : (u_\delta \otimes T_h u_\delta - u \otimes T_h u)\, dx dh dt \to 0 \text{ as } \delta \to 0, \tag{3.16}$$

For the term in (3.5), we only need to modify the previous argument in order to account for the

fact that there are three $u$ factors in the products and therefore we have:

$$\left|\mathbf{E}\int_0^T \iint \partial_k \psi \eta^{ij}(h)\left(u_\delta^k u_\delta^i T_h u_\delta^j - u^k u^i T_h u^j\right) dxdhdt\right| \leq C\|\psi\|_{L^\infty}\|\eta\|_{L^\infty}\|u\|^2_{L^3_{\omega,t,x}}\|u_\delta - u\|_{L^3_{\omega,t,x}}$$

$$\to 0 \text{ as } \delta \to 0. \tag{3.17}$$

The same argument suffices to show:

$$\mathbf{E}\int_0^T \int \psi \partial_{h_k}\eta^{ij}(h)\left(u_\delta^i T_h u_\delta^j (u_\delta^k - T_h u_\delta^k) - u^i T_h u^j (u^k - T_h u^k)\right) dxdhdt \to 0, \text{ as } \delta \to 0. \tag{3.18}$$

We now discuss the terms involving the pressure. For the term in (3.8), we have:

$$\begin{aligned}\left|\mathbf{E}\int_0^T \int\int \eta^{ij}\partial_i\psi(p_\delta T_h u_\delta^j - pT_h u^j)dxdhdt\right| &\leq C\|\eta\|_{L^\infty}\|\psi\|_{L^\infty}\|p_\delta - p\|_{L^{3/2}_{\omega,t,x}}\|u\|_{L^3_{\omega,t,x}} \\ &+ C\|\eta\|_{L^\infty}\|\psi\|_{L^\infty}\|p\|_{L^{3/2}_{\omega,t,x}}\|u_\delta - u\|_{L^3_{\omega,t,x}} \\ &\to 0 \text{ as } \delta \to 0, \end{aligned} \tag{3.19}$$

following the same lines of reasoning as before but now exploiting Assumption 2 as well. We similarly obtain for the term in (3.9):

$$\begin{aligned}\left|\mathbf{E}\int_0^T \int\int \psi \operatorname{div}\eta \cdot (p_\delta T_h u_\delta - pT_h u)dxdhdt\right| &\leq C\|\psi\|_{L^\infty}\|\operatorname{div}\eta\|_{L^\infty}\|p_\delta - p\|_{L^{3/2}_{\omega,t,x}}\|u\|_{L^3_{\omega,t,x}} \\ &+ C\|\psi\|_{L^\infty}\|\operatorname{div}\eta\|_{L^\infty}\|p\|_{L^{3/2}_{\omega,t,x}}\|u_\delta - u\|_{L^3_{\omega,t,x}} \\ &\to 0 \text{ as } \delta \to 0 \end{aligned} \tag{3.20}$$

as above. However, the limit can be rewritten as:

$$\begin{aligned}\mathbf{E}\int_0^T\int\int\psi\,\mathrm{div}\,\eta\cdot T_h u p dxdhdt &= \mathbf{E}\int_0^T\int\int_{\ell=0}^{\infty}\int_{|h|=\ell}\Phi(\ell)\psi p(T_h u\cdot\hat{h})dS(h)d\ell dxdt\\ &= \mathbf{E}\int_0^T\int\int_{\ell=0}^{\infty}\int_{|h|\le\ell}\Phi(\ell)\psi p\,\mathrm{div}_h\,T_h u dhd\ell dxdt\\ &= 0,\end{aligned}\tag{3.21}$$

where $\Phi(\ell)=\phi'(\ell)+\varphi'(\ell)+2\ell^{-1}\varphi'(\ell)$ comes from the calculation $\mathrm{div}\,\eta(h)=\Phi(|h|)\hat{h}$ and we have used the divergence theorem and incompressibility of $u$. Thus this term does not give any contribution in the limit.

The last pressure term is treated in the same way but does not seem to vanish in the limit, giving instead a term that will later be seen to give rise to $\frac{1}{2}$ times the pressure term in the local energy inequality:

$$\mathbf{E}\int_0^T\int\int\psi\,\mathrm{div}\,\eta(T_h p_\delta u_\delta)dxdhdt\to\mathbf{E}\int_0^T\int\int\psi\,\mathrm{div}\,\eta\cdot(T_h p u)dxdhdt\tag{3.22}$$

as $\delta\to 0$. Notice that the limit can be integrated by parts in $h$ (which can be formally justified by approximation) to obtain:

$$\mathbf{E}\int_0^T\int\int\psi\,\mathrm{div}\,\eta\cdot(T_h p u)dxdhdt=\mathbf{E}\int_0^T\int\int\psi\eta:(u\otimes T_h\nabla p)dxdhdt\tag{3.23}$$

As for the energy input term, we have:

$$\left|\mathbf{E}\int_0^T\int\int\psi\eta:(u_\delta\otimes T_h K_\delta Z_t+K_\delta Z_t\otimes T_h u_\delta-u\otimes T_h Z_t-Z_t\otimes T_h u)dxdhdt\right|\le$$

$$C\|\psi\|_{L^\infty}\|\eta\|_{L^\infty}\left(\|u_\delta - u\|_{L^2_{\omega,t,x}}\|Z_t\|_{L^2_{\omega,t,x}} + \|K_\delta Z_t - Z_t\|_{L^2_{\omega,t,x}}\|u\|_{L^2_{\omega,t,x}}\right) \to 0 \quad (3.24)$$

as $\delta \to 0$. The last thing to deal with is the commutator terms, however those will all be of lower order and tend to zero in the limit $\delta \to 0$, as is shown in [47]. This fact, the equality in the lines (3.4) through (3.12) and the information on the limits collected between (3.13) and (3.24) give the following balance relation:

$$\begin{aligned}
&\int \eta : \Gamma(T,h)dh - \int \eta : \Gamma(0,h)dh = \mathbf{E}\int_0^T \int\int (u \otimes T_h u) : \left((u\cdot\nabla\psi)\eta + \psi\partial_{h_k}\eta\delta_h u^k\right) dxdhdt \\
&+ \nu\mathbf{E}\int_0^T \int\int (u \otimes T_h u) : \left[\Delta\psi\eta - 2\partial_{x_k}\psi\partial_{h_k}\eta + 2\psi\Delta\eta\right] dxdhdt \\
&+ \mathbf{E}\int_0^T \int\int p\eta : (\nabla\psi \otimes T_h u)dxdhdt - \mathbf{E}\int_0^T \int\int \psi(u\otimes\nabla(T_h p)) : \eta dxdhdt \\
&+ \mathbf{E}\int_0^T \int\int \psi\eta : (u\otimes T_h Z_t + Z_t \otimes T_h u)\, dxdhdt.
\end{aligned}$$

The left hand side above vanishes, as $\Gamma$ is independent of time by Lemma 3.2, using the stationarity of $u$. Using Lemma 3.3 we get:

$$\begin{aligned}
\mathbf{E}\int\int \psi(u\otimes T_h u) : \partial_{h_k}\eta\delta_h u^k dxdh = &-\frac{1}{2}\int \partial_{h_k}\eta : D(h)^k dh \\
&-\frac{1}{2}\mathbf{E}\int\int \eta : T_h(u\otimes u)(\delta_h u\cdot\nabla\psi)dxdh.
\end{aligned}$$

Plugging this in the relation above and using the stationarity of $u$ to drop the integration in time completes the proof. □

# 4 Proof of the Local 4/3 Law

In this section we prove (2.5) and (2.7) under the assumptions of their respective theorems. Lemma 3.4 comes into play here, as well as in the next section. Our use of it consists in plugging in appropriate forms of isotropic tensors $\eta^{ij}$ as suggested by the statement of the local KHM relation, then switching to polar coordinates and deriving a differential equation for the structure function in the sense of distributions. After that point we can start using our assumptions to deal with the nonlinear flux terms and obtain the scaling laws we desire.

We use the following shorthand notations:

$$\bar{\Gamma}(\ell) = \frac{1}{4\pi}\int_{S^2} I : \Gamma(\ell\hat{n})dS(\hat{n}),$$

$$\bar{Z}(\ell) = \frac{1}{4\pi}\mathbf{E}\int_{S^2}\int (u\cdot T_{\ell\hat{n}}Z_t + T_{\ell\hat{n}}u\cdot Z_t)dS(\hat{n}),$$

where $I$ is again the 3 by 3 identity matrix.

*Proof of* (2.5). We start by picking our test function to be of the form $\eta^{ij}(h) = \delta^{ij}\phi(|h|)$. This in particular implies that:

$$\partial_{h_k}\eta^{ij}(h) = \delta^{ij}\frac{h_k}{|h|}\phi'(|h|).$$

Using the fact that $\bar{\Gamma}$ is spherically symmetric and $C^2$ (Lemma 3.2) one obtains:

$$\int \Delta\eta : \Gamma(h)dh = \int \eta : \Delta\Gamma(h)dh = 4\pi\int_{\ell=0}^{\infty}\phi(\ell)(\ell^2\bar{\Gamma}''(\ell) + 2\ell\bar{\Gamma}'(\ell))d\ell,$$

whereby using Lemma 3.4 we obtain:

$$\frac{1}{2}\mathbf{E}\int_0^\infty \int_{S^2}\int \phi'(\ell)\ell^2|\delta_{\ell\hat{n}}u|^2\delta_{\ell\hat{n}}u\cdot\hat{n}dxdS(\hat{n})d\ell$$
$$+\frac{1}{2}\mathbf{E}\int_0^\infty \int_{S^2}\int \phi(\ell)\ell^2|T_{\ell\hat{n}}u|^2\delta_{\ell\hat{n}}u\cdot\nabla\psi dxdS(\hat{n})d\ell =$$
$$2\nu\cdot 4\pi\int_0^\infty \phi(\ell)\ell^2(\bar{\Gamma}''(\ell)+\frac{2}{\ell}\bar{\Gamma}'(\ell))d\ell+\nu\mathbf{E}\int_0^\infty\int_{S^2}\int \ell^2\phi(\ell)\Delta\psi(u\cdot T_{\ell\hat{n}}u)dxdS(\hat{n})d\ell$$
$$-2\nu\mathbf{E}\int_0^\infty \int_{S^2}\int \phi'(\ell)\ell^2(u\cdot T_{\ell\hat{n}}u)(\nabla\psi\cdot\hat{n})dxdS(\hat{n})d\ell$$
$$+\mathbf{E}\int_0^\infty \int_{S^2}\int \phi(\ell)\ell^2 pT_{\ell\hat{n}}u\cdot\nabla\psi dxdS(\hat{n})d\ell-\mathbf{E}\int_0^\infty\int_{S^2}\int\phi(\ell)\ell^2\psi T_{\ell\hat{n}}(\nabla p)\cdot udxdS(\hat{n})d\ell$$
$$+\mathbf{E}\int_0^\infty \int_{S^2}\int \phi(\ell)\ell^2(u\cdot\nabla\psi)(u\cdot T_{\ell\hat{n}}u)dxdS(\hat{n})d\ell+4\pi\int_0^\infty\phi(\ell)\ell^2\bar{Z}(\ell)d\ell.$$

Set some further notation:

$$\bar{F}(\ell)=\frac{1}{4\pi}\mathbf{E}\int_{S^2}\int|T_{\ell\hat{n}}u|^2\delta_{\ell\hat{n}}u\cdot\nabla\psi dxdS(\hat{n}),$$

$$\bar{G}(\ell)=\frac{1}{4\pi}\mathbf{E}\int_{S^2}\int\Delta\psi(u\cdot T_{\ell\hat{n}}u)dxdS(\hat{n}),$$

$$\bar{Q}(\ell)=\frac{1}{4\pi}\mathbf{E}\int_{S^2}\int(u\cdot T_{\ell\hat{n}}u)(\nabla\psi\cdot\hat{n})dxdS(\hat{n}),$$

$$\bar{H}(\ell)=\frac{1}{4\pi}\mathbf{E}\int_{S^2}\int(u\cdot\nabla\psi)(u\cdot T_{\ell\hat{n}}u)dxdS(\hat{n})$$

$$\bar{P}_1(\ell)=\frac{1}{4\pi}\mathbf{E}\int_{S^2}\int pT_{\ell\hat{n}}u\cdot\nabla\psi dxdS(\hat{n}),$$

$$\bar{P}_2(\ell)=\frac{1}{4\pi}\mathbf{E}\int_{S^2}\int\psi u\cdot\nabla(T_{\ell\hat{n}}p)dxdS(\hat{n}).$$

A fact that we will use is that $\bar{P}_2(0)=-\bar{P}_1(0)$, which can be seen by an integration by parts. We

thus have the following equation in the sense of distributions:

$$\begin{aligned}-(S_0(\ell)\ell^2)' &= 4\nu(\ell^2\bar{\Gamma}'(\ell))' + 2\ell^2\bar{Z}(\ell)\\ &\quad + 2\nu\ell^2\bar{G}(\ell) + 4\nu(\ell^2\bar{Q}(\ell))'\\ &\quad + 2\ell^2\bar{P}_1(\ell) - 2\ell^2\bar{P}_2(\ell)\\ &\quad + 2\ell^2\bar{H}(\ell) - \ell^2\bar{F}(\ell).\end{aligned}$$

By Lemma 3.1 (resp. Lemma 3.2) we obtain that $S_0(\ell)\ell^2 \to 0$ (resp. $\bar{\Gamma}(\ell)\ell^2 \to 0$) as $\ell \to 0$. Thus, after integrating with respect to $\ell$ and dividing by $\ell^3$, we have:

$$\begin{aligned}-\frac{S_0(\ell)}{\ell} &= 4\frac{\nu\bar{\Gamma}'(\ell)}{\ell} + \frac{2}{\ell^3}\int_0^\ell \tau^2\bar{Z}(\tau)d\tau\\ &\quad + \frac{2\nu}{\ell^3}\int_0^\ell \tau^2\bar{G}(\tau)d\tau + \frac{4\nu}{\ell}\bar{Q}(\ell)\\ &\quad + \frac{2}{\ell^3}\int_0^\ell \tau^2\bar{P}_1(\tau)d\tau - \frac{2}{\ell^3}\int_0^\ell \tau^2\bar{P}_2(\tau)d\tau\\ &\quad + \frac{2}{\ell^3}\int_0^\ell \tau^2\bar{H}(\tau)d\tau - \frac{1}{\ell^3}\int_0^\ell \tau^2\bar{F}(\tau)d\tau.\end{aligned}$$

Now, for each fixed $\nu$ we have:

$$\frac{2}{\ell^3}\int_0^\ell \tau^2(\bar{P}_1(\tau) - \bar{P}_1(0))d\tau = \frac{2}{\ell^3}\frac{1}{4\pi}\mathbf{E}\int_0^\ell\int_{S^2}\int \tau^2 p\delta_{\tau\hat{n}}u\cdot\nabla\psi dx dS(\hat{n})d\tau,$$

$$\frac{2\nu}{\ell^3}\int_0^\ell \tau^2(\bar{G}(\tau) - \bar{G}(0))d\tau = \frac{2\nu}{\ell^3}\frac{1}{4\pi}\mathbf{E}\int_0^\ell\int_{S^2}\int \tau^2\Delta\psi u\cdot\delta_{\tau\hat{n}}u dx dS(\hat{n})d\tau,$$

$$\frac{2}{\ell^3}\int_0^\ell \tau^2(\bar{H}(\tau) - \bar{H}(0))d\tau = \frac{2}{\ell^3}\frac{1}{4\pi}\mathbf{E}\int_0^\ell\int_{S^2}\int \tau^2(u\cdot\nabla\psi)(u\cdot\delta_{\tau\hat{n}}u)dx dS(\hat{n})d\tau,$$

$$\frac{2}{\ell^3}\int_0^\ell \tau^2(\bar{P}_2(\tau)-\bar{P}_2(0))d\tau = \frac{2}{\ell^3}\frac{1}{4\pi}\mathbf{E}\int_0^\ell\int_{S^2}\int \tau^2\psi u\cdot\nabla(\delta_{\tau\hat{n}}p)dxdS(\hat{n})d\tau.$$

In what follows, we will consistently use the little-$o$ notation with the understanding that everything vanishes *uniformly in* $\nu$, i.e. we write

$$f(\ell)=o(g(\ell)) \text{ if and only if } \lim_{\nu\to 0}\lim_{\ell\to 0}\frac{f(\ell)}{g(\ell)}=0.$$

For the stochastic forcing term, it is immediate by the estimates on $Z$ that:

$$\frac{1}{\ell^3}\int_0^\ell \tau^2\bar{Z}(\tau)d\tau = \frac{1}{3}\bar{Z}(0)+o(1). \tag{4.1}$$

We now begin to make use of Assumption 1, pointing out where (WAD) is sufficient. By (WAD) there is $\ell_\nu\to 0$ with $\nu$ such that $\sqrt{\nu\mathbf{E}\|u\|^2_{L^2_x}}=o(\ell_\nu)$. For the $\bar{\Gamma}'(\ell)$ term, notice that:

$$\left|\frac{\nu\bar{\Gamma}'(\ell)}{\ell}\right| \le \|\psi\|_\infty\frac{\nu}{4\pi\ell}\mathbf{E}\int_{S^2}\int |u(x)||\nabla T_{\ell\hat{n}}u(x)|dxdS(\hat{n}) \le C\frac{\nu}{\ell}(\mathbf{E}\|u\|^2_{L^2_x})^{1/2}(\mathbf{E}\|\nabla u\|^2_{L^2_x})^{1/2},$$

so that:

$$\begin{aligned}\limsup_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in(\ell_\nu,\ell_I)}\left|\frac{\nu\bar{\Gamma}(\ell)}{\ell}\right| &\le C\limsup_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in(\ell_\nu,\ell_I)}\frac{\nu^{1/2}(\mathbf{E}\|u\|^2_{L^2_x})^{1/2}}{\ell_\nu}\left(\nu^{1/2}(\mathbf{E}\|\nabla u\|^2_{L^2_x})^{1/2}\right)\\ &\le C\limsup_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in(\ell_\nu,\ell_I)}\frac{(\nu\mathbf{E}\|u\|^2_{L^2_x})^{1/2}}{\ell_\nu}=0,\end{aligned} \tag{4.2}$$

where we have used (2.3). Similarly, one sees that:

$$\limsup_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in(\ell_\nu,\ell_I)}\left|\frac{\nu}{\ell}\bar{Q}(\ell)\right| \le C\limsup_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in(\ell_\nu,\ell_I)}\frac{\nu\mathbf{E}\|u\|^2_{L^2_x}}{\ell_\nu}=0 \tag{4.3}$$

As for the $\bar{G}$ term, we note:

$$\begin{aligned}\frac{\nu}{\ell^3}\int_0^\ell \tau^2 \bar{G}(\tau)d\tau &= \frac{\nu}{\ell^3}\int_0^\ell \tau^2(\bar{G}(\tau)-\bar{G}(0))d\tau + \frac{\nu}{3}\bar{G}(0)\\ &= \frac{\nu}{\ell^3}\frac{1}{4\pi}\mathbf{E}\int_0^\ell \int_{S^2}\int \tau^2 \Delta\psi u\cdot \delta_{\tau\hat{n}}u dx dS(\hat{n})d\tau + \frac{\nu}{3}\bar{G}(0).\end{aligned}$$

We now use the straightforward bound:

$$\begin{aligned}\left|\frac{\nu}{\ell^3}\frac{1}{4\pi}\mathbf{E}\int_0^\ell \int_{S^2}\int \tau^2 \Delta\psi u\cdot \delta_{\tau\hat{n}}u dx dS(\hat{n})d\tau\right| &\leq C\frac{\nu}{\ell^3}\int_0^\ell \tau^2 \mathbf{E}\int_{S^2}\int |\Delta\psi||u||\delta_{\tau\hat{n}}u| dx dS(\hat{n})d\tau\\ &\leq C\nu\mathbf{E}\|u\|_{L^2_x}^2,\end{aligned}$$

so we can use (WAD) to see that:

$$\limsup_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in(\ell_\nu,\ell_I)}\left|\frac{\nu}{\ell^3}\int_0^\ell \tau^2(\bar{G}(\tau)-\bar{G}(0))d\tau\right| = 0, \tag{4.4}$$

which can also be rewritten as $\nu\ell^{-3}\int_0^\ell \tau^2\bar{G}(\tau)d\tau = \frac{\nu}{3}\bar{G}(0)+o(1)$. The reader might notice that $\nu\bar{G}(0)$ vanishes in the inviscid limit by (WAD). The reason that we choose to keep track of it is that it will allow us, in the next proof, to use Assumption 3 to establish an expression for $\ell^{-1}S_0(\ell)$ in terms of the local dissipation.

We now turn to the nonlinear terms. For the $\bar{F}$ term, we have:

$$\begin{aligned}\left|\frac{1}{\ell^3}\int_0^\ell \tau^2\bar{F}(\tau)d\tau\right| &\leq C\frac{1}{4\pi\ell^3}\int_0^\ell \tau^2\mathbf{E}\int_{S^2}\int |T_{\ell\hat{n}}u|^2|\delta_{\ell\hat{n}}u||\nabla\psi| dx dS(\hat{n})d\tau\\ &\leq C(\mathbf{E}\|u\|_{L^3_x}^3)^{2/3}\sup_{|h|\leq\ell}(\mathbf{E}\|\delta_h u\|_{L^3_x}^3)^{1/3},\end{aligned}$$

and therefore:

$$\limsup_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in(\ell_\nu,\ell_I)}\left|\frac{1}{\ell^3}\int_0^\ell \tau^2\bar{F}(\tau)d\tau\right| \le C(\mathbf{E}\|u\|_{L^3_x}^3)^{2/3}\limsup_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in(\ell_\nu,\ell_I)}\sup_{|h|\le\ell}(\mathbf{E}\|\delta_h u\|_{L^3_x}^3)^{1/3}$$
$$\le C(\mathbf{E}\|u\|_{L^3_x}^3)^{2/3}\limsup_{\ell\to 0}\sup_{|h|\le\ell}(\mathbf{E}\|\delta_h u\|_{L^3_x}^3)^{1/3} = 0 \quad (4.5)$$

because of Assumption 1. In other words, $\ell^{-3}\int_0^\ell \tau^2\bar{F}(\tau)d\tau = o(1)$.

Notice that since

$$\left|\frac{1}{\ell^3}\int_0^\ell \tau^2(\bar{H}(\tau)-\bar{H}(0))d\tau\right| \le C\frac{1}{\ell^3}\int_0^\ell \tau^2\mathbf{E}\int_{S^2}\int |\nabla\psi||u|^2|\delta_{\tau\hat{n}}u|dxdS(\hat{n})d\tau,$$

an argument entirely parallel to the one regarding the $\bar{F}$ term will give:

$$\limsup_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in(\ell_\nu,\ell_I)}\left|\frac{1}{\ell^3}\int_0^\ell \tau^2(\bar{H}(\tau)-\bar{H}(0))d\tau\right| = 0, \quad (4.6)$$

or equivalently $\ell^{-3}\int_0^\ell \tau^2\bar{H}(\tau)d\tau = \frac{1}{3}\bar{H}(0)+o(1)$

For the $\bar{P}_1$ term, we have:

$$\left|\frac{2}{\ell^3}\frac{1}{4\pi}\mathbf{E}\int_0^\ell\int_{S^2}\int \tau^2 p\delta_{\tau\hat{n}}u\cdot\nabla\psi dxdS(\hat{n})d\tau\right| \le C\frac{1}{\ell}\int_0^\ell \sup_{\hat{n}\in S^2}(\mathbf{E}\|\delta_{\tau\hat{n}}u\|_{L^3_x}^3)^{1/3}(\mathbf{E}\|p\|_{L^{3/2}_x}^{3/2})^{2/3}d\tau$$
$$\le C(\mathbf{E}\|u\|_{L^3_x}^3)^{2/3}\sup_{|h|\le\ell}(\mathbf{E}\|\delta_h u\|_{L^3_x}^3)^{1/3}d\tau,$$

where we have used Assumption 2. This bound creates the same sort of situation that we had

with the $\bar{F}$ term, so we can conclude that

$$\left| \limsup_{\ell_I \to 0} \limsup_{\nu \to 0} \sup_{\ell \in (\ell_\nu, \ell_I)} \frac{1}{\ell^3} \int_0^\ell \tau^2 (\bar{P}_1(\tau) - \bar{P}_1(0)) d\tau \right| = 0, \tag{4.7}$$

or equivalently $\ell^{-3} \int_0^\ell \tau^2 \bar{P}_1(\tau) d\tau = \frac{1}{3} \bar{P}_1(0) + o(1)$. Finally, for the $\bar{P}_2$ term we will first integrate by parts in $x$, so that we get a situation similar to the one for $\bar{P}_1$:

$$\frac{1}{\ell^3} \int_0^\ell \tau^2 (\bar{P}_2(\tau) - \bar{P}_2(0)) d\tau = -\frac{1}{\ell^3} \frac{1}{4\pi} \mathbf{E} \int_0^\ell \int_{S^2} \int \tau^2 (u \cdot \nabla \psi) \delta_{\tau \hat{n}} p dx dS(\hat{n}) d\tau,$$

which we can estimate as:

$$\begin{aligned} \left| \frac{1}{\ell^3} \int_0^\ell \tau^2 (\bar{P}_2(\tau) - \bar{P}_2(0)) d\tau \right| &\leq C \frac{1}{\ell} \int_0^\ell (\mathbf{E} \|u\|_{L^3_x}^3)^{1/3} \sup_{\hat{n} \in S^2} (\mathbf{E} \|\delta_{\tau \hat{n}} p\|_{L^{3/2}_x}^{3/2})^{2/3} d\tau \\ &\leq C (\mathbf{E} \|u\|_{L^3_x}^3)^{1/3} \sup_{|h| \leq \ell} (\mathbf{E} \|\delta_h u\|_{L^3_x}^3)^{2/3}, \end{aligned}$$

where we have again used Assumption 2. The above gives:

$$\left| \limsup_{\ell_I \to 0} \limsup_{\nu \to 0} \sup_{\ell \in (\ell_\nu, \ell_I)} \frac{1}{\ell^3} \int_0^\ell \tau^2 (\bar{P}_2(\tau) - \bar{P}_2(0)) d\tau \right| = 0, \tag{4.8}$$

or equivalently $\ell^{-3} \int_0^\ell \tau^2 \bar{P}_2(\tau) d\tau = \frac{1}{3} \bar{P}_2(0) + o(1)$.

Gathering the information we have collected above in equations (4.1) through (4.8), we get:

$$-\frac{S_0(\ell)}{\ell} = \frac{2}{3} \left( 2\bar{P}_1(0) + \bar{H}(0) + \bar{Z}(0) + \nu \bar{G}(0) \right) + o(1), \tag{4.9}$$

where as we have explained above the $o(1)$ vanishes as $\ell \to 0$, uniformly in $\nu$. By the definitions of $\bar{P}_1, \bar{H}, \bar{Z}$ and since $\nu\bar{G}(0)$ is also $o(1)$ by (WAD), this completes the proof of (2.5). □

*Proof of* (2.7). We merely need to notice that the terms inside the parentheses in the right hand side of (4.9) are precisely the ones seen in the right hand side of (LEE). □

## 5 Proof of the Local 4/5 Law

Our strategy here is similar to that in the previous section. However we will also need to use the $4/3$ law itself.

*Proof of* (2.6). While in the proof of the local $4/3$ law our isotropic test function $\eta$ was of the form $\eta^{ij}(h) = \varphi(|h|)\delta^{ij}$, now we will take it to be of the other form suggested by the statement of the local KHM relation, namely we take:

$$\eta^{ij}(h) = \frac{h^i h^j}{|h|^2}\varphi(|h|),$$

or equivalently:

$$\eta(h) = \varphi(|h|)\hat{h} \otimes \hat{h}, \text{ where } \hat{h} = \frac{h}{|h|}.$$

We have

$$\partial_{h_k}\left(\varphi(|h|)\hat{h} \otimes \hat{h}\right) = (\varphi'(|h|) - 2|h|^{-1}\varphi(|h|))\hat{h} \otimes \hat{h}\hat{h}^k + |h|^{-1}\varphi(|h|)(e^k \otimes \hat{h} + \hat{h} \otimes e^k),$$

where $e^k$ is the unit vector in the $k$-th spatial direction[4]. In this context, Lemma 3.4 gives:

[4]not to be confused with $e_k$, the $k$-th eigenvector of the Stokes operator

$$\int \partial_{h_k}\left(\varphi(|h|)(\hat{h}\otimes\hat{h})\right) : \left(\frac{1}{2}D^k(h) + 2\nu\partial_{h_k}\Gamma(h) + 2\nu\mathbf{E}\int \partial_{x_k}\psi(u\otimes T_h u)dx\right)dh$$

$$+\frac{1}{2}\mathbf{E}\int\int \varphi(|h|)(\hat{h}\otimes\hat{h}) : T_h(u\otimes u)\delta_h u\cdot\nabla\psi dxdh =$$

$$\mathbf{E}\int\int \varphi(|h|)(\hat{h}\otimes\hat{h}) : (T_h u\otimes Z_t + u\otimes T_h Z_t)dh$$

$$+\mathbf{E}\int\int p\varphi(|h|)\hat{h}\otimes\hat{h} : (\nabla\psi\otimes T_h u)dxdh + \mathbf{E}\int\int \psi\varphi(|h|)(\hat{h}\otimes\hat{h})(\nabla(T_h p)\otimes u)dxdh$$

$$+\nu\mathbf{E}\int\int \Delta\psi\varphi(|h|)(\hat{h}\otimes\hat{h}) : (u\otimes T_h u)dxdh + \mathbf{E}\int\int u\cdot\nabla\psi\varphi(|h|)(\hat{h}\otimes\hat{h}) : (u\otimes T_h u)dxdh.$$

We now wish to rewrite everything in terms of the variable $\ell$ as in the proof of the local $4/3$ law.

With that in mind, we introduce the following notation:

$$\tilde{\Gamma}(\ell) := \frac{1}{4\pi}\mathbf{E}\int_{S^2}\int \psi(x)(\hat{n}\cdot u)(\hat{n}\otimes\hat{n} : T_{\ell\hat{n}}\nabla u)dxdS(\hat{n}), \tag{5.1}$$

$$\tilde{Q}_1(\ell) := \frac{1}{4\pi}\mathbf{E}\int_{S^2}\int (\nabla\psi\cdot\hat{n})(u\cdot\hat{n})(T_{\ell\hat{n}}u\cdot\hat{n})dxdS(\hat{n}), \tag{5.2}$$

$$\tilde{Q}_2(\ell) := \frac{1}{4\pi}\mathbf{E}\int_{S^2}\int (u\cdot\nabla\psi)(T_{\ell\hat{n}}u\cdot\hat{n})dxdS(\hat{n}), \tag{5.3}$$

$$\tilde{Q}_3(\ell) := \frac{1}{4\pi}\mathbf{E}\int_{S^2}\int (u\cdot\hat{n})(T_{\ell\hat{n}}u\cdot\nabla\psi)dxdS(\hat{n}), \tag{5.4}$$

$$\tilde{F}(\ell) := \frac{1}{4\pi}\mathbf{E}\int_{S^2}\int |T_{\ell\hat{n}}u\cdot\hat{n}|^2\delta_{\ell\hat{n}}u\cdot\nabla\psi dxdS(\hat{n}), \tag{5.5}$$

$$\tilde{Z}(\ell) := \frac{1}{4\pi}\mathbf{E}\int_{S^2}\int (\hat{n}\otimes\hat{n}) : (T_{\ell\hat{n}}u\otimes Z_t + u\otimes T_{\ell\hat{n}}Z_t)dS(\hat{n}), \tag{5.6}$$

$$\tilde{P}_1(\ell) := \frac{1}{4\pi}\mathbf{E}\int_{S^2}\int p(\nabla\psi\cdot\hat{n})(T_{\ell\hat{n}}u\cdot\hat{n})dxdS(\hat{n}), \tag{5.7}$$

$$\tilde{G}(\ell) := \frac{1}{4\pi}\mathbf{E}\int_{S^2}\int \Delta\psi(u\cdot\hat{n})(T_{\ell\hat{n}}u\cdot\hat{n})dxdS(\hat{n}), \tag{5.8}$$

$$\tilde{H}(\ell) := \frac{1}{4\pi}\mathbf{E}\int_{S^2}\int (u\cdot\nabla\psi)(u\cdot\hat{n})(T_{\ell\hat{n}}u\cdot\hat{n})dxdS(\hat{n}). \tag{5.9}$$

Notice that in the above we have omitted an expression for the second pressure term on the right hand side of the KHM relation. The reasons for this are the following. To begin with, a term involving $T_h(\nabla p)$ cannot be estimated on the basis of Assumption 2. In addition, we cannot merely integrate by parts and use the divergence free condition on $u$ in order to obtain an expression only involving $\nabla\psi$ as we did in the proof of the local $4/3$ law, since any such manipulation produces a term involving $\nabla u : (\hat{h}\otimes\hat{h})$, which in turn cannot be estimated on the basis of Assumption 1. Instead, we will recast this pressure term in a different form involving more convenient expressions. To this end, we recall the following elementary identity:

$$\hat{h}_i\hat{h}_j = \delta^{ij} - |h|\partial_{h_i}\hat{h}_j.$$

We now use this to rewrite:

$$\begin{aligned}
\mathbf{E}\int_{\mathbb{R}^3}\int \psi\varphi(|h|)(\hat{h}\otimes\hat{h}) : (u\otimes\nabla(T_hp))dxdh &= \mathbf{E}\int_{\mathbb{R}^3}\int \psi(x)\varphi(|h|)\hat{h}_i\hat{h}_ju^i\partial_jT_hpdxdh \\
&= \mathbf{E}\int_{\mathbb{R}^3}\int \psi(x)\varphi(|h|)(\delta^{ij}-|h|\partial_{h_i}\hat{h}_j)u^i\partial_jT_hpdxdh \\
&= -\mathbf{E}\int_{\mathbb{R}^3}\int \psi(x)\varphi(|h|)|h|\partial_{h_i}\hat{h}_ju^i\partial_jT_hpdxdh \\
&\quad +4\pi\int_{\ell=0}^{\infty}\ell^2\varphi(\ell)\bar{P}_2(\ell)d\ell.
\end{aligned} \tag{5.10}$$

For the term in (5.10), we integrate by parts in $h$ to dispose of the pressure gradient:

$$-\mathbf{E}\int_{\mathbb{R}^3}\int \psi(x)\varphi(|h|)|h|\partial_{h_i}\hat{h}_ju^i\partial_{h_j}T_hpdxdh = \mathbf{E}\int_{\mathbb{R}^3}\int \partial_{h_j}(\varphi(|h|)|h|)\partial_{h_i}\hat{h}_j\psi u^iT_hpdxdh$$

$$+\mathbf{E}\int_{\mathbb{R}^3}\int \varphi(|h|)|h|\partial_{h_j}\partial_{h_i}\hat{h}_j\psi u^i T_h p dx dh. \tag{5.11}$$

Notice that for the function $g(\ell) := \varphi(\ell)\ell$, we have $\partial_{h_j}\left(g(|h|)\right)\partial_{h_i}\hat{h}_j = g'(|h|)\hat{h}_j\partial_{h_i}\hat{h}_j$, which vanishes because $\hat{h}_j\partial_{h_i}\hat{h}_j = \frac{1}{2}\partial_{h_i}(\hat{h}_j\hat{h}_j) = \frac{1}{2}\partial_{h_i}1 = 0$. On the other hand, for the term in (5.11), notice that since $\partial_{h_i}\hat{h}_j = \partial_{h_j}\hat{h}_i$, we get:

$$\begin{aligned}
\mathbf{E}\int_{\mathbb{R}^3}\int \varphi(|h|)|h|\partial_{h_j}\partial_{h_i}\hat{h}_j\psi u^i T_h p dx dh &= \mathbf{E}\int_{\mathbb{R}^3}\int \varphi(|h|)|h|\partial_{h_j}\partial_{h_j}\hat{h}_i\psi u^i T_h p dx dh \\
&= -2\mathbf{E}\int_{\mathbb{R}^3}\int \frac{\varphi(|h|)}{|h|}\hat{h}_i T_{-h}(\psi u^i) p dx dh \\
&= -2\int_{\ell=0}^{\infty}\frac{\varphi(\ell)}{\ell}\mathbf{E}\int\int_{|h|=\ell} T_{-h}(\psi u)\cdot\hat{h}p(x)dS(|h|)dx d\ell \\
&= 2\int_{\ell=0}^{\infty}\frac{\varphi(\ell)}{\ell}\mathbf{E}\int\int_{|y|\leq\ell} T_{-y}(u\cdot\nabla\psi)p(x)dy dx d\ell \\
&= 8\pi\int_{\ell=0}^{\infty}\frac{\varphi(\ell)}{\ell}\tilde{P}_2(\ell)d\ell,
\end{aligned}$$

where we denote:

$$\tilde{P}_2(\ell) := \frac{1}{4\pi}\mathbf{E}\int\int_{|y|\leq\ell}(u\cdot\nabla\psi)T_y p dy dx. \tag{5.12}$$

As in the proof of the local $4/3$ law, the expressions (5.1) through (5.9) and (5.12) will now be used to rewrite the integrals of the KHM relation in spherical coordinates. As an example, we only show how $\tilde{\Gamma}$ appears, the other changes of variables being simpler:

$$\begin{aligned}
&\int_{\mathbb{R}^3}\partial_{h_k}(\phi(|h|)\hat{h}\otimes\hat{h}) : 2\nu\partial_{h_k}\Gamma(h)dh \\
&= 2\nu\mathbf{E}\int_{\mathbb{R}^3}\int \psi(x)((\phi'(|h|)-2|h|^{-1}\phi(|h|))(u^i\hat{h}_i)(\partial_{h_k}T_h u^j\hat{h}_j\hat{h}_k)dx dh \\
&\quad +2\nu\mathbf{E}\int_{\mathbb{R}^3}|h|^{-1}\phi(|h|)(\delta^{i,k}\hat{h}_j+\hat{h}_i\delta^{j,k})\int\psi(x)u^i\partial_k T_h u^j dx dh
\end{aligned} \tag{5.13}$$

$$= 8\pi\nu \int_{\ell=0}^{\infty} \ell^2(\phi'(\ell) - 2\ell^{-1}\phi(\ell))\tilde{\Gamma}(\ell)d\ell,$$

the expression in line (5.13) being equal to 0 because:

$$\mathbf{E}\int_{\mathbb{R}^3} |h|^{-1}\phi(|h|)(\delta^{i,k}\hat{h}_j + \hat{h}_i\delta^{j,k}) \int \psi(x)u^i\partial_k T_h u^j dxdh =$$
$$\mathbf{E}\int_{\ell=0}^{\infty} \ell\phi(\ell) \int\int_{S^2} \psi(x)(u^i\partial_i T_{\ell\hat{n}}u^j\hat{n}_j + \hat{n}_i\partial_j T_{\ell\hat{n}}u^j)dS(\hat{n})dxd\ell =$$
$$\mathbf{E}\int_{\ell=0}^{\infty} \ell\phi(\ell) \int\int_{|y|\leq 1} \psi(x)u^i\partial_{y_j}T_{\ell y}u^j dydxd\ell = 0,$$

where we have used the divergence theorem and the incompressibility of $u$.

Using that $\ell^2\phi'(\ell) - 2\ell\phi(\ell) = \ell^4(\ell^{-2}\phi(\ell))'$, we rewrite our KHM relation as an equality of integrals in $\ell$:

$$-\frac{1}{2}\int_{\ell=0}^{\infty} \ell^{-2}\phi(\ell)(\ell^4 S_{||}(\ell))'d\ell + \int_{\ell=0}^{\infty} \ell S_0(\ell)d\ell - 2\nu\int_{\ell=0}^{\infty} \ell^{-2}\phi(\ell)(\ell^4\tilde{\Gamma}(\ell))'d\ell$$
$$+2\nu\int_{\ell=0}^{\infty} \left(-\ell^{-2}\phi(\ell)(\ell^4\tilde{Q}_1(\ell))' + \ell\phi(\ell)(\tilde{Q}_2(\ell) + \tilde{Q}_3(\ell))\right)d\ell =$$
$$\int_{\ell=0}^{\infty} \ell^2\phi(\ell)\left(\tilde{Z}(\ell) + \tilde{P}_1(\ell) + \bar{P}_2(\ell) + \nu\tilde{G}(\ell) + \tilde{H}(\ell) - \frac{1}{2}\tilde{F}(\ell)\right)d\ell + 2\int_{\ell=0}^{\infty} \frac{\phi(\ell)}{\ell}\tilde{P}_2(\ell)d\ell.$$

In the sense of distributions on $(0,\infty)$, this translates to the following ODE:

$$-\frac{1}{2}(\ell^4 S_{||}(\ell))' + \ell^3 S_0(\ell) - 2\nu(\ell^4\tilde{\Gamma}(\ell))' + 2\nu\left(-(\ell^4\tilde{Q}_1(\ell))' + \ell^3(\tilde{Q}_2(\ell) + \tilde{Q}_3(\ell))\right) =$$
$$\ell^4\left(\tilde{Z}(\ell) + \tilde{P}_1(\ell) + \bar{P}_2(\ell) + \nu\tilde{G}(\ell) + \tilde{H}(\ell) - \frac{1}{2}\tilde{F}(\ell)\right) + 2\ell\tilde{P}_2(\ell).$$

As in the proof of the $4/3$ law, we now integrate from 0 to $\ell$ and multiply by $2\ell^{-5}$ in order to

obtain an expression for the quantity of interest $\ell^{-1}S_{||}(\ell)$ :

$$-\frac{S_{||}(\ell)}{\ell}+\frac{2}{\ell^5}\int_0^\ell \tau^3 S_0(\tau)d\tau-\frac{4\nu\tilde{\Gamma}(\ell)}{\ell}-\frac{4\nu\tilde{Q}_1(\ell)}{\ell}+\frac{4\nu}{\ell^5}\int_0^\ell \tau^3(\tilde{Q}_2(\tau)+\tilde{Q}_3(\tau))d\tau=$$
$$\frac{2}{\ell^5}\int_0^\ell \tau^4\left(\tilde{Z}(\tau)+\tilde{P}_1(\tau)+\bar{P}_2(\tau)+\nu\tilde{G}(\tau)+\tilde{H}(\tau)\right)d\tau+\frac{1}{\ell^5}\int_0^\ell \tau^4\tilde{F}(\tau)d\tau+\frac{4}{\ell^5}\int_0^\ell \tau\tilde{P}_2(\tau)d\tau.$$

Again, the $\ell\to 0$ boundary contributions are not present because

$$\lim_{\ell\to 0}\ell^4 S_{||}(\ell)=\lim_{\ell\to 0}\ell^4\tilde{\Gamma}(\ell)=\lim_{\ell\to 0}\ell^4\tilde{Q}_1(\ell)=0,$$

by Lemma 3.1, Lemma 3.2 and the bound $\mathbf{E}\|u\|^2_{L^2_x}<\infty$ used for $\tilde{Q}_1$. We now start using our assumptions. Once again, in what follows the little-$o$ notation denotes *uniform in* $\nu$ vanishing at the limit $\ell\to 0$. For the $\tilde{Z}$ term, it follows by the estimates for the Ornstein-Uhlenbeck SPDE that:

$$\frac{1}{\ell^5}\int_0^\ell \tau^4\tilde{Z}(\tau)d\tau=\frac{1}{5}\tilde{Z}(0)+o(1). \tag{5.14}$$

For the $\tilde{\Gamma}$ term, we have:

$$\left|\frac{\nu\tilde{\Gamma}(\ell)}{\ell}\right|\leq C\frac{\nu}{\ell}\mathbf{E}\int_{S^2}\int|u||T_{\ell\hat{n}}\nabla u|dxdS(\hat{n})\leq C\frac{\nu}{\ell}(\mathbf{E}\|u\|^2_{L^2_x})^{1/2}(\mathbf{E}\|\nabla u\|^2_{L^2_x})^{1/2},$$

so we pick our scales $\ell_\nu$ such that $(\nu\mathbf{E}\|u\|^2_{L^2_x})^{1/2}=o(\ell_\nu)$, which is possible according to (WAD), and we use (2.3) to obtain:

$$\limsup_{\nu\to 0}\sup_{\ell\in(\ell_\nu,\ell_I)}\left|\frac{\nu}{\ell}\tilde{\Gamma}(\ell)\right|\leq C\limsup_{\nu\to 0}\frac{(\nu\mathbf{E}\|u\|^2_{L^2_x})^{1/2}}{\ell_\nu}=0. \tag{5.15}$$

Similarly, observe that:

$$\left|\frac{\nu}{\ell}\tilde{Q}_1(\ell)\right| \leq C\frac{\nu}{\ell}\mathbf{E}\int_{S^2}\int |u||T_{\ell\hat{n}}u|dxdS(\hat{n}) \leq \frac{\nu\mathbf{E}\|u\|_{L^2_x}^2}{\ell},$$

so by the same procedure as above, we obtain:

$$\limsup_{\nu\to 0}\sup_{\ell\in(\ell_\nu,\ell_I)}\left|\frac{\nu}{\ell}\tilde{Q}_1(\ell)\right| \leq C\limsup_{\nu\to 0}\frac{\nu}{\ell_\nu}\mathbf{E}\|u\|_{L^2_x}^2 \leq C\limsup_{\nu\to 0}\frac{(\nu\mathbf{E}\|u\|_{L^2_x}^2)^{1/2}}{\ell_\nu} = 0. \tag{5.16}$$

Next, for $\mu = 2, 3$, we have:

$$\left|\frac{\nu}{\ell^5}\int_0^\ell \tau^3\tilde{Q}_\mu(\tau)d\tau\right| \leq C\frac{\nu}{\ell}\mathbf{E}\|u\|_{L^2_x}^2,$$

so once again in the same way as before (WAD) gives:

$$\limsup_{\nu\to 0}\sup_{\ell\in(\ell_\nu,\ell_I)}\left|\frac{\nu}{\ell^5}\int_0^\ell \tau^3\tilde{Q}_\mu(\tau)d\tau\right| \leq C\limsup_{\nu\to 0}\frac{\nu\mathbf{E}\|u\|_{L^2_x}^2}{\ell_\nu} = 0. \tag{5.17}$$

For the term involving $\tilde{G}$, we have:

$$\left|\frac{\nu}{\ell^5}\int_0^\ell \tau^4(\tilde{G}(\tau)-\tilde{G}(0))d\tau\right| \leq C\nu\mathbf{E}\|u\|_{L^2_x}^2,$$

so that by (WAD):

$$\limsup_{\nu\to 0}\sup_{\ell\in(\ell_\nu,\ell_I)}\left|\frac{\nu}{\ell^5}\int_0^\ell \tau^4(\tilde{G}(\tau)-\tilde{G}(0))d\tau\right| = 0. \tag{5.18}$$

In other words, we have $\nu\ell^{-5}\int_0^\ell \tau^4\tilde{G}(\tau)d\tau = \frac{\nu}{5}\tilde{G}(0) + o(1)$. At this point our treatment of the

viscous terms is complete. We turn to the nonlinear terms.

For the $\tilde{F}$ term, we have:

$$|\tilde{F}(\tau)| \leq C(\mathbf{E}\|u\|_{L^3_x}^3)^{2/3}(\mathbf{E}\|\delta_{\tau\hat{n}}\|_{L^3_x}^3)^{1/3},$$

which vanishes uniformly in $\nu$ when $\tau \to 0$ by Assumption 1, implying that:

$$\limsup_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in(\ell_\nu,\ell_I)}\left|\ell^{-5}\int_0^\ell \tau^4\tilde{F}(\tau)d\tau\right| \leq C(\mathbf{E}\|u\|_{L^3_x}^3)^{2/3}\limsup_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{|h|\leq\ell_I}(\mathbf{E}\|\delta_h\|_{L^3_x}^3)^{1/3}$$
$$= 0. \tag{5.19}$$

In an entirely analogous manner, i.e. using exactly the same bounds as for $\tilde{F}$, one obtains:

$$\limsup_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in(\ell_\nu,\ell_I)}\left|\frac{1}{\ell^5}\int_0^\ell \tau^4(\tilde{H}(\tau)-\tilde{H}(0))d\tau\right| = 0, \tag{5.20}$$

or in other words $\ell^{-5}\int_0^\ell \tau^4\tilde{G}(\tau) = \frac{1}{5}\tilde{G}(0) + o(1)$. For the pressure term involving $\tilde{P}_1$, we have:

$$|\tilde{P}_1(\tau)-\tilde{P}_1(0)| \leq C(\mathbf{E}\|p\|_{L^{3/2}_x}^{3/2})^{2/3}\sup_{\hat{n}\in S^2}(\mathbf{E}\|\delta_{\tau\hat{n}}u\|_{L^3_x}^3)^{1/3} \leq C\sup_{\hat{n}\in S^2}(\mathbf{E}\|\delta_{\tau\hat{n}}u\|_{L^3_x}^3)^{1/3},$$

where we have used Assumption 2 on the boundedness of the pressure. The above implies:

$$\limsup_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in(\ell_\nu,\ell_I)}\left|\ell^{-5}\int_0^\ell \tau^4(\tilde{P}_1(\tau)-\tilde{P}_1(0))d\tau\right| \leq$$
$$C\limsup_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in(\ell_\nu,\ell_I)}\ell^{-1}\int_0^\ell \sup_{\hat{n}\in S^2}(\mathbf{E}\|\delta_{\tau\hat{n}}u\|_{L^3_x}^3)^{1/3}d\tau \leq$$
$$C\limsup_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{|h|\leq\ell_I}(\mathbf{E}\|\delta_h u\|_{L^3_x}^3)^{1/3} = 0, \tag{5.21}$$

again by Assumption 1. This can also be restated as $\ell^{-5}\int_0^\ell \tau^4 \tilde{P}_1(\tau)d\tau = \frac{1}{5}\tilde{P}_1(0) + o(1)$. Dealing with the $\bar{P}_2$ term follows an absolutely identical argument to the one in the proof of the local $4/3$ law, save for the different powers of $\ell$ and $\tau$ that appear. We simply remark that as for the previous terms, we again have $\ell^{-5}\int_0^\ell \tau^4 \bar{P}_2(\tau)d\tau = \frac{1}{5}\bar{P}_2(0) + o(1)$. It remains to deal with the $\tilde{P}_2$ term. We have:

$$\begin{aligned}
\frac{1}{\ell^5}\int_0^\ell \tau \tilde{P}_2(\tau)d\tau &= \frac{1}{\ell^5}\int_0^\ell \frac{\tau}{4\pi}\mathbf{E}\int\int_{|y|\leq\tau}(u\cdot\nabla\psi)T_y p dy dx d\tau \\
&= \frac{1}{\ell^5}\int_0^\ell \tau^4\left(\frac{1}{4\pi\tau^3}\mathbf{E}\int\int_{|y|\leq\tau}(u\cdot\nabla\psi)T_y p dy dx - \frac{1}{3}\mathbf{E}\int(u\cdot\nabla\psi)p dx\right)d\tau \\
&+ \frac{1}{3\ell^5}\int_0^\ell \tau^4\mathbf{E}\int(u\cdot\nabla\psi)p dx d\tau \\
&= \frac{1}{\ell^5}\int_0^\ell \tau^4\left(\frac{1}{4\pi\tau^3}\mathbf{E}\int\int_{|y|\leq\tau}(u\cdot\nabla\psi)\delta_y p dy dx\right)d\tau + \frac{1}{15}\bar{P}_1(0),
\end{aligned}$$

and also:

$$\begin{aligned}
\limsup_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in(\ell_\nu,\ell_I)}&\left|\ell^{-5}\int_0^\ell \tau^4\left(\frac{1}{4\pi\tau^3}\mathbf{E}\int\int_{|y|\leq\tau}(u\cdot\nabla\psi)\delta_y p dy dx\right)d\tau\right| \\
&\leq C(\mathbf{E}\|u\|_{L^3_x}^3)^{1/3}\limsup_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{|y|\leq\ell_I}(\mathbf{E}\|\delta_y p\|_{L^{3/2}_x}^{3/2})^{2/3} \\
&= 0, \qquad (5.22)
\end{aligned}$$

where we have used both Assumption 1 and Assumption 2. In this case similarly to before we have: $\frac{1}{\ell^5}\int_0^\ell \tau\tilde{P}_2(\tau)d\tau = \frac{1}{15}\bar{P}_1(0) + o(1)$. Collecting the information from (5.14) through (5.22),

we have:

$$-\frac{S_{||}(\ell)}{\ell}+\frac{2}{\ell^5}\int_0^\ell \tau^3 S_0(\tau)d\tau=\frac{2}{5}\left(\tilde{Z}(0)+\tilde{P}_1(0)+\bar{P}_2(0)+\nu\tilde{G}(0)+\tilde{H}(0)\right)+\frac{4}{15}\bar{P}_1(0)+o(1). \tag{5.23}$$

Noticing that $\int_{S^2}\hat{n}\otimes\hat{n}dS(\hat{n})=\frac{4\pi}{3}I$, we have:

$$\tilde{Z}(0)=\frac{1}{3}\bar{Z}(0),\tilde{P}_1(0)=\frac{1}{3}\bar{P}_1(0),\tilde{G}(0)=\frac{1}{3}\bar{G}(0),\tilde{H}(0)=\frac{1}{3}\bar{H}(0),$$

so combining (5.23) with (2.5), we obtain:

$$\begin{aligned}-\frac{S_{||}(\ell)}{\ell}&=\frac{2}{15}\left(\bar{Z}(0)+5\bar{P}_1(0)+3\bar{P}_2(0)+\bar{H}(0)+\nu\bar{G}(0)\right)\\&\quad+\frac{4}{3\ell^5}\int_0^\ell\tau^4\left(\bar{H}(0)+2\bar{P}_1(0)+2\bar{Z}(0)+\nu\bar{G}(0)+o(1)\right)d\tau+o(1)\\&=\frac{2}{5}\left(\bar{Z}(0)+2\bar{P}_1(0)+\bar{H}(0)+\nu\bar{G}(0)\right)+o(1),\end{aligned} \tag{5.24}$$

and since (WAD) implies that $\nu\bar{G}(0)$ vanishes as $\nu\to 0$, we see that (5.24) furnishes (2.6). □

*Proof of* (2.8). The proof is the same as the proof of (2.7), namely, by noticing that the terms in the parentheses in the right hand side of (5.24) are exactly the right hand side of (LEE). □

## 6 Existence of stationary martingale solutions for OU driven Navier-Stokes

We sketch the construction of stationary martingale solutions to (NSE), which closely follows the approach in [53] for the case of white-in-time forcing.

Given $k \in \mathbb{N}$, we write Galerkin approximations to (NSE):

$$\begin{cases} \partial_t u^k + P^k B(u^k) + \nu A u^k = Z_t^k \\ dZ_t^k = -LZ_t^k + P^k dW_t \end{cases}$$

where $P^k$ is the projection onto the subspace of $H$ generated by $(e_j)_{j=1}^k$, $B(u) = P(\mathrm{div}(u \otimes u))$, $Au = -P\Delta u$ and $P$ is the Leray projection operator.

This system is a coupling of an Ornstein-Uhlenbeck SDE with a finite dimensional (random) differential equation with Lipschitz coefficients.

We can completely adapt the derivation of the estimates found for the process $u_t^k$ in [53] to the joint process $(u_t^k, Z_t^k)$ in our setting. To begin with, notice that by Ito's formula it follows that for $p \geq 2$:

$$\begin{gathered} d\|Z_t^k\|_H^p \leq p\|Z_t^k\|_H^{p-2} \left\langle Z_t^k, dZ_t^k \right\rangle + \frac{p(p-1)}{2}\|Z_t^k\|_H^{p-2}\epsilon dt \implies \\ d\|Z_t^k\|_H^p + p\|Z_t^k\|_H^{p-2} \left\langle Z_t^k, LZ_t^k \right\rangle \leq p\|Z_t^k\|_H^{p-2} \left\langle Z_t^k, QdW_t \right\rangle + \frac{p(p-1)}{2}\|Z_t^k\|_H^{p-2}\epsilon dt \implies \\ d\|Z_t^k\|_H^p + pc_1\|Z_t^k\|_H^{p-2}\|Z_t^k\|_V^2 \leq p\|Z_t^k\|_H^{p-2} \left\langle Z_t^k, QdW_t \right\rangle + C\left(\|Z_t^k\|_H^p dt + \epsilon^{p/2}\right) dt. \quad (6.1) \end{gathered}$$

where $\langle \cdot, \cdot \rangle$ is the inner product of $H$ and the constant $C$ depends only on $p$. Hence, by taking expectations we obtain:

$$\mathbf{E}\|Z_t^k\|_H^p + pc_1\mathbf{E}\int_0^t \|Z_s^k\|_H^{p-2}\|Z_s^k\|_V^2 ds \leq \mathbf{E}\|Z_0^k\|_H^p + C\int_0^t \mathbf{E}\|Z_s^k\|_H^p ds + C\epsilon^{p/2}t$$

which implies by Gronwall's inequality that for every time interval $[0, T]$ there exists a finite

positive constant which we again denote by $C$, depending only on $p$ and $T$, such that

$$\mathbf{E}\|Z_t^k\|_H^p \leq C$$

for all $t$ in $[0, T]$.

This in turn implies the existence of another positive constant (again denoted by $C$) depending on $p, T$ and $c_1$, such that:

$$\mathbf{E}\int_0^T \|Z_t^k\|_H^{p-2}\|Z_t^k\|_V^2 \leq C < \infty.$$

Plugging in $p = 2$, we obtain the fundamental estimate:

$$\mathbf{E}\int_0^T \|Z_t^k\|_V^2 \leq C < \infty.$$

Note that none of the bounds above depend on $k$, because we have assumed that $Z_0^k = 0$ for all $k \geq 1$. The only difference between the above and the estimates carried out in [53] is that we need to assume coercivity of the bilinear form $\langle Lu, v\rangle$ in the space $V$ (ensuring which is the role of the constant $c_1$) whereas for $A$ this is immediate since one just takes $\langle Av, v\rangle = \|v\|_V^2$ by definition.

Next, we apply the Burkholder-Davis-Gundy inequality to get that for an absolute constant $\tilde{C}$:

$$\mathbf{E}\sup_{0\leq t\leq T}\left|\int_0^t \|Z_s^k\|_H^{p-2}\left\langle Z_s^k, QdW_s\right\rangle\right| \leq \tilde{C}\mathbf{E}\left(\int_0^T \|Z_s^k\|_H^{2p-2}\epsilon ds\right)^{\frac{1}{2}}$$

$$\leq \tilde{C}\mathbf{E}\left[\sup_{0\leq t\leq T}\|Z_s^k\|_H^{\frac{p}{2}}\left(\int_0^T \|Z_s^k\|_H^{p-2}\epsilon ds\right)^{\frac{1}{2}}\right]$$
$$\leq \frac{1}{2p}\mathbf{E}\sup_{0\leq t\leq T}\|Z_s^k\|_H^p + \frac{p}{2}\tilde{C}^2\mathbf{E}\int_0^T \|Z_s^k\|_H^{p-2}\epsilon ds$$
$$\leq \frac{1}{2p}\mathbf{E}\sup_{0\leq t\leq T}\|Z_s^k\|_H^p + \frac{pC\tilde{C}^2}{2}\mathbf{E}\int_0^T \|Z_s^k\|_H^p ds + \frac{pC\tilde{C}^2}{2}\epsilon^{p/2}t,$$

where the constant $C$ is the one that appears in line (6.1).

From (6.1) and the inequality just derived, it follows that for a new constant $C$:

$$\mathbf{E}\sup_{0\leq t\leq T}\|Z_t^k\|_H^p \leq \mathbf{E}\|Z_0^k\|_H^p + \frac{1}{2}\mathbf{E}\sup_{0\leq t\leq T}\|Z_t^k\|_H^p + C\mathbf{E}\int_0^T \|Z_s^k\|_H^p ds + C\epsilon^{p/2}t,$$

hence by Gronwall's we obtain that there exists a positive constant $C$ depending on $p$ and $T$ such that:

$$\mathbf{E}\sup_{0\leq t\leq T}\|Z_t^k\|_H^p \leq C < \infty.$$

The same analysis can be carried out for the Galerkin equation governing $u_t^k$ using classical rather than stochastic calculus, i.e. this case is more elementary than the one treated in the Appendix in [53].

We therefore arrive at estimates of the form:

$$\mathbf{E}\int_0^T \|(u_t^k, Z_t^k)\|_{V\times V}^2 dt \leq C < \infty, \tag{6.2}$$

$$\mathbf{E}\sup_{0\leq t\leq T}\|(u_t^k, Z_t^k)\|_{H\times H}^p \leq C < \infty. \tag{6.3}$$

The rest of the work done in [53] can be repeated in exactly the same way. One uses the Skorokhod embedding theorem and ends up with a stochastic basis $(\tilde{\Omega}, \tilde{F}, \{\tilde{F}_t\}_{t\geq 0}, \tilde{P}, \tilde{W}_t)$, $\tilde{W}_t$ being a $Q$-Wiener process on which we will also have a process $(\tilde{u}_t, \tilde{Z}_t)$ satisfying (NSE) in the sense of Definition 2.4.

Note that a priori it is only shown that $\tilde{Z}_t$ satisfies the equation

$$d\tilde{Z}_t = -L\tilde{Z}_t dt + d\tilde{W}_t$$

in the weak sense. However, it is easy to see that under conditions specified in [34, Chapter 5] this weak solution is in fact strong and in the stationary case corresponds to an invariant measure of the Ornstein-Uhlenbeck SPDE above. Note that the estimates (6.2) and (6.3) are used to derive stationary martingale solutions satisfying (2.1) and (2.2).

# Part II

## Well-posedness of nonlinear stochastic kinetic equations

# 7 Introduction to Part II

In the second part of this dissertation, we examine the short time behavior of a cloud of particles subjected to a fluctuating external field. These particles interact according to an inverse square law (i.e. Coulomb or Newtonian interactions). We will ignore collisions in what follows, or at best model their effect via a linear (degenerate) diffusion operator.

After appropriate rescaling, the system of differential equations describing the motion of $N$ identical particles in 3 spatial dimensions (ignoring for the time being the fluctuating external field) is the following:

$$\begin{aligned} \mathrm{d}X^i &= V^i \mathrm{d}t \\ \mathrm{d}V^i &= \frac{\alpha}{N} \sum_{j \neq i} \frac{X^i - X^j}{|X^i - X^j|^3} \mathrm{d}t. \end{aligned} \tag{7.1}$$

Here, $\alpha > 0$ corresponds to repulsive (i.e. Coulomb) interactions, while $\alpha < 0$ corresponds to attractive (i.e. Newtonian) interactions. Notice the prefactor $\frac{1}{N}$, which indicates that we are in the mean-field scaling regime, i.e. the total interaction strength is bounded in $N$. It is well known that this system is chaotic for $N > 2$, and computationally prohibitive in the large $N$ regime; it is thus important to find effective models for the evolution of these systems in the large $N$ limit.

This is the role of kinetic equations: assuming that particles are indistinguishable, the object of study of kinetic theory is the normalized number density of particles or single-particle distribution function in phase space. That is, we define $f(t, x, v)$ to be the probability density function for a particle to be in location $x$ with velocity $v$ at time $t$. In the limit $N \to \infty$, the equa-

tions of motion above are replaced by a partial differential equation that looks at the system at a "mesoscopic" level, which is an intermediate level of coarse-graining between particle dynamics and hydrodynamics.

The PDE which results when we take the mean field limit is called the *Vlasov-Poisson equation*, which reads:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = 0 \\ \rho = \int f \mathrm{d}v \\ F = \alpha K *_x \rho, \quad K(x) = \frac{x}{|x|^3} \end{cases} \tag{7.2}$$

The passage to from (7.1) to (7.2) is a very delicate mathematical issue on its own, and the relevant literature is large. We refer to the articles [18, 70, 76, 97] and the references therein for a review of recent progress in the mean field limit problem for both second order and first order dynamics.

The Vlasov-Poisson equation (similarly to other kinetic equations) is a PDE in 7 variables; however, from the point of view of computational expenses this is significantly preferable to (7.1), a system of $6N$ ODEs where in typical situations $N$ can be extremely large (e.g. $10^9$ electrons per cm$^3$ in the ionosphere [94]). As a coarse-grained model, the benefit of kinetic equations over hydrodynamics is that they encapsulate the dynamics of a system *before* it reaches thermodynamic equilibrium (cf [92]).

We are also interested in the diffusive variant of (7.2) called the *Vlasov–Poisson–Fokker–*

*Planck equation*:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = \nu \Delta_v f + \beta \operatorname{div}_v (fv) \\ \rho = \int f \mathrm{d}v \\ F = \alpha K *_x \rho. \end{cases} \tag{7.3}$$

In fact, we will be taking $\alpha = 1$ and studying kinetic equations with Vlasov–type self-consistent forces on the torus for simplicity, which necessitates subtracting $\int \rho \mathrm{d}x = 1$ from $\rho$ in the definition of $F$, so that $F$ is the solution of the Poisson problem:

$$\operatorname{div}_x F = \rho - 1.$$

Subtracting $1$ can be thought of as incorporating the influence of a homogeneous neutralizing background for the charged particles.

On the level of particles, (7.3) corresponds to changing the second equation in (7.1) to:

$$\mathrm{d}V^i = -\beta V^i \mathrm{d}t + \frac{\alpha}{N} \sum_{j \neq i} K(X^i - X^j) \mathrm{d}t + \sqrt{2\nu} \mathrm{d}W_t^i.$$

The family $\{W_t^i\}_i$ are independent standard Brownian motions, that play the role of thermal noise acting on particles. The well-posedness theory for (7.2) and (7.3) has been understood since the 1990's, and we recall the relevant bibliography in what follows.

The thermal noise that results in the diffusion term $\nu \Delta_v f$ in (7.3) is spatially homogeneous. In order to incorporate fluctuating force fields in kinetic theory, we perturb (7.3) by a stochastic

transport term as follows:

$$\mathrm{d}f + v \cdot \nabla_x f \mathrm{d}t + F \cdot \nabla_v f \mathrm{d}t + \nabla_v f \circ \mathrm{d}W_t = \nu \Delta_v f \mathrm{d}t + \beta \operatorname{div}_v(fv)\mathrm{d}t.$$

Linear stochastic kinetic equations can be approached via the Feynman–Kac formula in pretty much the same way as non-degenerate advection equations are studied in [75]. The literature on nonlinear stochastic kinetic theory is very sparse (see the discussion below), and to the best of our knowledge the second part of this thesis is the first construction of local strong (in the sense of PDE as well as probability) solutions to a nonlinear kinetic equation. This is based on the preprint [10], which is joint work with my advisor, Jacob Bedrossian.

## 8 The setup and main result of Part I

In the second part of this dissertation we prove the local-in-time existence and uniqueness of (probabilistically strong) solutions of the Vlasov and Vlasov-Fokker-Planck equations for a distribution of charged particles subjected to a stochastic external electric field

$$\begin{cases} \mathrm{d}f + v \cdot \nabla_x f \mathrm{d}t + E \cdot \nabla_v f \mathrm{d}t + \nabla_v f \odot \mathrm{d}W_t = \nu \nabla_v \cdot (\nabla_v f + fv)\mathrm{d}t \\ \rho = \int f \mathrm{d}v \\ E = \nabla_x (-\Delta_x)^{-1}(\rho - 1) \\ \rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\mathrm{d}v, \end{cases} \tag{8.1}$$

where $\nu \geq 0$ is the collision frequency – we treat both the case $\nu > 0$ and $\nu = 0$ (i.e. the Vlasov–Poisson equations). Below we denote the Fokker-Planck operator

$$\mathcal{L}f := \Delta_v f + \nabla_v \cdot (vf),$$

which is a commonly used simplification for collisions of charged particles against a background (see e.g. [17]).

Here, we consider the problem in the periodic box $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$, although the case $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$ could be approached with similar arguments. The process $W_t$ is a white-in-time, colored-in-space, vector-valued Gaussian process which plays the role of an external fluctuating electric field which we describe in more detail below. Our analysis works for general $1 \leq d \leq 3$ and also applies to external magnetic fields. For simplicity, we will take initial conditions in the velocity-weighted Sobolev space $H^\sigma_m$ defined by the norm:

$$\|f\|^2_{H^\sigma_m} := \sum_{|\alpha|+|\beta|\leq\sigma} \iint_{\mathbb{T}^d\times\mathbb{R}^d} |\partial^\alpha_x \partial^\beta_v f(x, v)|^2 (1 + |v|^2)^{m/2} \mathrm{d}v\mathrm{d}x.$$

Stochastic and randomly fluctuating electromagnetic fields are a classical topic in the plasma physics literature where they are used as a model for studying various dynamics in "turbulent"-like situations in both confined fusion and astrophysical applications. Much of the work is on studying the motion of charged particles (i.e. Lagrangian trajectories or passive scalars) subjected to stochastic electromagnetic fields of various kinds; see e.g. [4, 60, 61, 63, 71, 74, 108, 112, 113] and the references therein for a tiny fraction of the existing work on the topic. Another line of work regards subjecting gyrokinetic equations or other macroscopic models to randomly fluc-

tuating external force fields of this type for the purpose of studying plasma turbulence; see e.g. [86, 105, 106] and the references therein. The purpose of this work is to begin laying down some rigorous mathematical theory for studying nonlinear kinetic theory models of plasmas in these kinds of settings.

It is sometimes useful to make a concrete representation of $W_t$ and for simplicity we will show how to do this in $d = 3$; the extension to other dimensions is straightforward and is omitted. To make this concrete representation of $W_t$, we define a real Fourier basis on $L^2(\mathbb{T}^3; \mathbb{R}^3)$ by defining for each $k = (\ell, i) \in \mathbb{K} := \mathbb{Z}_0^d \times \{1, 2, 3\}$

$$e_k(x) = \begin{cases} c_d \gamma_\ell^i \sin(\ell \cdot x), & \ell \in \mathbb{Z}_+^d \\ c_d \gamma_\ell^i \cos(\ell \cdot x), & \ell \in \mathbb{Z}_-^d, \end{cases}$$

where $\mathbb{Z}_0^d := \mathbb{Z}^d \setminus \{0, \dots, 0\}$, $\mathbb{Z}_+^d = \{\ell \in \mathbb{Z}_0^d : \ell^{(d)} > 0\} \cup \{\ell \in \mathbb{Z}_0^d : \ell^{(1)} > 0, \ell^{(d)} = 0\}$ and $\mathbb{Z}_-^d = -\mathbb{Z}_+^d$, and for each $\ell \in \mathbb{Z}_0^d$, $\{\gamma_\ell^i\}_{i=1}^3$ is a set of three orthonormal vectors with $\{\gamma_\ell^1, \gamma_\ell^2\}$ spanning the plane perpendicular to $\ell \in \mathbb{R}^3$ with the property that $\gamma_{-\ell}^i = -\gamma_\ell^i$ and $\gamma_\ell^3$ parallel to $\ell$. The constant $c_d = \sqrt{2}(2\pi)^{-d/2}$ is a normalization factor so that $e_k(x)$ are a complete orthonormal basis on $L^2$. With this, we define our external electric field as

$$W_t(x) = \sum_{k \in \mathbb{K}} \sigma_k e_k(x) W_t^{(k)},$$

with $\left\{W_t^{(k)}\right\}_{k \in \mathbb{K}}$ is a family of independent standard Wiener processes defined on a given stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$. The $\sigma_k$ are coloring coefficients satisfying at least $\sum_{k \in \mathbb{K}} |\sigma_k|^2 < \infty$, however, more stringent regularity requirements will be assumed below (here we make the nat-

ural definition $|k| = |\ell|$ for $k = (\ell, i) \in \mathbb{K}$). We can also treat the case of fluctuating magnetic fields; see Remark 8.4 below.

Local well-posedness of strong solutions for the deterministic problem is classical; see e.g. [65, 66] for the Vlasov equations and [87, 109] for the Vlasov-Fokker-Planck equation. Global existence for the deterministic problems was proved in [91] (see also [6, 96]) for the Vlasov equations and [16] for the Vlasov-Fokker-Planck equations; we will consider global existence for (8.1) in a follow up work. Notice that in Itô form, the SPDE becomes

$$\mathrm{d}f + v \cdot \nabla_x f \mathrm{d}t + E \cdot \nabla_v f \mathrm{d}t + \nabla_v f \cdot \mathrm{d}W_t = \nu \mathcal{L} f \mathrm{d}t + \frac{1}{2} \sum_k (\sigma_k e_k \cdot \nabla_v)^2 f \mathrm{d}t,$$

so it is clear that stochastic transport cannot be treated perturbatively with respect to the deterministic evolution, as the Stratonovich-Itô correction term is of second order. However, this correction term is subelliptic, and so stochastic transport enjoys a special structure that makes it possible to develop a strong well-posedness theory, and in fact, it is sometimes possible to produce a better well-posedness theory for stochastic transport than for deterministic transport [54]. Due to this special structure and the many physical applications, there have been a great number of works studying stochastic transport equations recently; see for example [7, 27, 50, 51, 84] and also [36, 52] in the kinetic case.

There have been many works on fluid equations subjected to multiplicative or transport-type stochastic forcing. For the Navier-Stokes equations see for example [19, 21, 26, 82, 83]. The 2D Euler equations in vorticity form subjected to transport noise was studied in, for example [20, 31], where strong solutions with bounded vorticity were constructed (see also [30]). The aforementioned papers [30, 31] belong to the so-called theory of "Stochastic Advection by Lie

Transport" (SALT), see the foundational paper [64], as well as [2, 32]. The work [22] studies the 3D primitive equations with transport noise.

The works [37, 38] provide a fairly general framework to study a wide class of dissipative fluid equations forced with multiplicative noise, such as the Navier-Stokes equations or the primitive equations. For the 2D Euler equations with various types of general multiplicative noise, see [57] and the references therein.

In comparison to stochastic fluid dynamics, the work on nonlinear, stochastically forced kinetic equations is significantly thinner. The paper [93] constructs global-in-time renormalized martingale (probabilistically weak) solutions of the Boltzmann equations with external stochastic forcing similar to that used in (8.1). In work with a clear relationship with our own, [39] constructs global solutions of interacting point charges (i.e. Vlasov–Poisson with solutions given by a finite number of Dirac masses) subjected to stochastic external electric fields; see also [28].

Here, we continue the study of stochastic kinetic theory by proving local existence and pathwise uniqueness of strong solutions. Let us recall some standard notions for probabilistically strong solutions of SPDEs that may experience finite-time blow up (we follow the presentation used in [38, 57]), which are nothing more than the natural stochastic analogues of the deterministic notions of local-in-time existence, uniqueness, and maximally-extended solutions.

**Definition 8.1.** A *local pathwise* solution of (8.1) is a pair $(f, \tau)$ with $\tau$ an almost-surely strictly positive stopping time and $f$ an adapted stochastic process satisfying the regularity

$$f(\cdot \wedge \tau) \in C([0, \infty); H^{\sigma}_{m})$$

and for $t \geq 0$ satisfies,

$$f(t \wedge \tau) - f(0) + \int_0^{t \wedge \tau} (v \cdot \nabla_x f(s) + E(s) \cdot \nabla_v f - \nu \mathcal{L} f(s))\, d\tau = \int_0^{t \wedge \tau} \nabla_v f(s) \circ \mathrm{d}W_s.$$

Moreover, we say such pathwise solutions are *unique* if for any pair $(f_1, \tau_1)$, $(f_2, \tau_2)$ we have

$$\mathbb{P}\left(f_1(t) - f_2(t) = 0 \quad \forall 0 \leq t < \tau_1 \wedge \tau_2 | f_1(0) = f_2(0)\right) = 1.$$

In this case, $f_1$ and $f_2$ are called *indistinguishable*.

The following definition of maximal pathwise solution provides a continuation criterion for strong solutions. For this we will use $H^{d/2+1+}_{d/2+}$; sharper continuation criteria will be considered in future work. That is, we show that local $H^\sigma_m$ solutions can be uniquely extended provided some $H^{s_0}_{m_0}$ norm remains finite for $s_0 > 1 + d/2$ and $m_0 > d$ fixed and arbitrary.

**Definition 8.2.** Fix $s_0 > d/2 + 1$ an integer and $m_0 > d$. We call a *maximal pathwise solution* a triple of a solution $f$, an increasing sequence of almost-surely positive stopping times $\{\tau_n\}_{n \geq 0}$, and a limiting stopping time $\xi$ such that each pair $(f, \tau_n)$ is a local pathwise solution, $\lim_{n \to \infty} \tau_n = \xi$, and

$$\sup_{0 \leq t \leq \tau_n} \|f(t)\|_{H^{s_0}_{m_0}} \geq n \quad \text{on the set } \{\xi < \infty\}.$$

We prove the following local existence and uniqueness theorem. Global existence of these strong solutions will be considered in a future work.

**Theorem 8.3.** *Let $1 \leq d \leq 3$. Let $\sigma > s_0$ and $m > m_0$ be fixed integers and assume that*

$$\sum_{k \in \mathbb{K}} |k|^{2\sigma'} |\sigma_k|^2 < \infty$$

*for some $\sigma' > \sigma + 4$ (integer). Suppose that the initial condition $f_0$ is an $\mathcal{F}_0$-measurable random variable such that $f_0 \in H_m^\sigma$ almost-surely. Then, there exists a unique, maximal pathwise solution to (8.1) for any $\nu \geq 0$.*

**Remark 8.4.** Our proof also applies when there is a stochastic magnetic field. We may similarly treat the case of independent electric and magnetic fields as the following, for example (denoting $c > 0$ the speed of light),

$$\begin{aligned} W_t(x) \mapsto & \sum_{k \in \mathbb{K}} \sigma_k^{(E)} e_k(x) W_t^{k;E} \\ & + \frac{1}{c} v \times \sum_{\ell \in \mathbb{Z}_0^3} \sigma_\ell^{(B;1)} e_{(\ell,1)}(x) W_t^{(\ell,1);B} + \frac{1}{c} v \times \sum_{\ell \in \mathbb{Z}_0^3} \sigma_\ell^{(B;2)} e_{(\ell,2)}(x) W_t^{(\ell,2);B}, \end{aligned}$$

with

$$\sum_{k=(\ell,j) \in \mathbb{K}} |k|^{2\sigma'} \left( \left| \sigma_k^{(E)} \right|^2 + \left| \sigma_\ell^{(B;1)} \right|^2 + \left| \sigma_\ell^{(B;2)} \right|^2 \right) < \infty,$$

or when electromagnetic fields are correlated, for example one could use forcing of the following potentially natural form

$$W_t \mapsto \sum_{\ell \in \mathbb{Z}_0^3} \sigma_\ell^{(1)} \left( e_{(\ell,1)}(x) + \frac{v \times e_{(\ell,2)}(x)}{c} \right) W_t^{(\ell,1)} + \sum_{\ell \in \mathbb{Z}_0^3} \sigma_\ell^{(2)} \left( e_{(\ell,2)}(x) + \frac{v \times e_{(\ell,1)}(x)}{c} \right) W_t^{(\ell,2)}.$$

Sufficiently regular-in-space deterministic external electromagnetic fields or random fields that are smoother in time than white noise (for example, Ornstein-Uhlenbeck processes and variations thereof as in the Langevin antenna forcing used in the plasma physics literature [105]) can also be easily included in the analysis without any significant changes. For simplicity of presentation, we will mainly focus on the case of external electric fields and simply make comments about what changes when considering an additional magnetic field.

**Remark 8.5.** The methods we use can also deal with more general mean-field interactions, replacing the self-consistent electric field with:

$$E = \nabla_x K * (\rho - 1),$$

for any kernel $K$ such that $\|\nabla_x^{j+1} E\|_{L^p} \lesssim_p \left\| \langle \nabla_x \rangle^j \rho \right\|_{L^p}$ for all $1 < p < \infty$.

**Remark 8.6.** It should be straightforward to extend to $d \geq 4$. It should also be possible to treat non-integer $\sigma$, $s_0$, and $\sigma'$, however, this would require more delicate (anisotropic) commutator estimates.

**Remark 8.7.** We believe our methods could be extended to the Landau collision operators for initial data $f_0$ sufficiently close to a global Maxwellian to prove local-in-time existence and uniqueness of strong solutions to e.g. the Vlasov–Poisson–Landau equations with stochastic external electromagnetic fields. This extension may be considered in future work.

**Remark 8.8.** In light of the classical deterministic theory of bounded solutions of the Vlasov equations (see e.g. [79, 80]), it is natural to expect an analogue of [20] in kinetic theory. Similarly, we expect local (and global) existence and uniqueness of the Vlasov-Fokker-Planck equations

using only e.g. $f_0 \in L^2_m$. These extensions may be considered in future work.

Finally, in Section 12 we present a proof of the following hypoelliptic regularization result. This is proved using a time-dependent hypocoercivity norm in the spirit of [111].

**Theorem 8.9.** *Let $f$ be a maximal pathwise solution to (8.1) as in Theorem 8.3. Suppose that for all $N$ there holds*

$$|\sigma_k| \lesssim_N |k|^{-N}.$$

*Then if $\nu > 0$, then $f(t) \in C^\infty_{x,v}$ for all $t \in (0, \xi)$.*

# 9 Outline of the proof

Let us outline the general idea of how to prove Theorem 8.3 and then provide the details in the main body of the text. See Section 12 for how to prove Theorem 8.9.

As in e.g. [38, 57], we first construct solutions to (8.1) with smoother initial data $H^{\sigma'}_{m'}$ with $\sigma' > \sigma + 4$ and $m' > m + 3$ (both integers) with trajectories in $L^\infty_{t,loc} H^{\sigma'}_{m'} \cap C_{t,loc} H^\sigma_m$. This procedure is done in Section 10.2. Then we regularize the initial condition and pass to the limit to obtain solutions with initial data in $H^\sigma_m$ that take values in $C_{t,loc} H^\sigma_m$. In addition to obtaining solutions with lower regularity, what is more important for many purposes, is that this constructs solutions which take values continuously in the highest regularity available. This latter procedure is done in Section 11.

To construct maximal pathwise solutions to (8.1) we first introduce a standard trick for regularizing the nonlinearity in a way which allows to close necessary probabilistic moment

estimates. Consider a smooth nonnegative and nonincreasing function $\theta : [0, \infty) \to \mathbb{R}$ such that:

$$\theta(x) = \begin{cases} 1 & \text{if } x \leq 1, \\ \\ 0 & \text{if } x \geq 2, \end{cases}$$

and define:

$$\theta_R(x) \equiv \theta\left(\frac{x}{R}\right).$$

Then we define the regularized SPDE

$$\mathrm{d}f + v \cdot \nabla_x f \mathrm{d}t + \theta_R(\|f\|_{H^{s_0}_{m_0}}) E \cdot \nabla_v f \mathrm{d}t + \nabla_v f \odot \mathrm{d}W_t = \nu \mathcal{L} f \mathrm{d}t, \tag{9.1}$$

We show in Section 10.2 that this SPDE admits global-in-time, unique, pathwise solutions (i.e. $\xi = \infty$ with probability 1 in the definition of maximal solutions) starting from $H^{\sigma'}_{m'}$ initial conditions. Specifically, we prove the following.

**Lemma 9.1.** *Let $f_0$ be a $\mathcal{F}_0$-measurable random variable such that $\forall p \geq 2$,*

$$\mathbf{E} \|f_0\|^p_{H^{\sigma'}_{m'}} < \infty.$$

*Then, there exists an $f \in C([0, \infty); H^{\sigma'-4}_{m'-3}) \cap L^\infty_{t,loc}([0, \infty); H^{\sigma'-1}_{m'-1})$ $\mathbf{P}$–a.s. which is a solution to (9.1) in the sense that*

$$f(t) = f_0 + \int_0^t \left(-v \cdot \nabla_x f(s) - \theta_R(\|f\|_{H^{s_0}_{m_0}}) E(s) \cdot \nabla_v f(s) + \nu \mathcal{L} f(s)\right) \mathrm{d}s - \int_0^t \nabla_v f(s) \circ \mathrm{d}W_s,$$

*where the equality holds* $\mathbf{P}$*–a.s. in* $C([0,\infty); H^{\sigma'-4}_{m'-3})$*. Moreover, if* $\tilde{f}$ *is any other solution in the above sense, then* $f = \tilde{f}$ *almost surely in the sense that*

$$\mathbb{P}\left(f(t) - \tilde{f}(t) = 0 \quad \forall 0 \leq t < \infty | f(0) = \tilde{f}(0)\right) = 1.$$

It is clear that solutions of (9.1) are also solutions to (8.1) for as long as $\|f\|_{H^{s_0}_{m_0}} < R$, and so by considering the increasing sequence of $R = n$ and defining the stopping times

$$\tau_n = \inf\left\{t \geq 0 : \|f(t)\|_{H^{s_0}_{m_0}} > n\right\},$$

we may use (9.1) to construct local pathwise solutions to (8.1). A standard cutting procedure (described below) also shows how to remove the moment requirement on the initial data.

**Lemma 9.2.** *Let* $f_0$ *be an* $\mathcal{F}_0$*-measurable random variable with* $f_0 \in H^{\sigma'}_{m'}$ *almost surely. Then, Lemma 9.1 implies the pathwise existence and uniqueness of a maximal solution* $(f, \tau)$ *to* (8.1) *with initial data* $f_0$ *with trajectories* $f$ *satisfying*

$$f(\cdot \wedge \tau) \in L^{\infty}_{t,loc}(0,\infty; H^{\sigma'-1}_{m'-1}) \cap C_{t,loc}([0,\infty); H^{\sigma'-4}_{m'-3}).$$

*Proof.* First consider the case that $\|f_0\|_{H^{\sigma'}_{m'}} < M$ almost-surely. Then, we choose $R = M + 1$ in (9.1), and define the stopping time:

$$\tau = \inf\{t \geq 0 : \ \|f(t)\|_{H^{\sigma}_{m}} > R\},$$

where $f$ is the solution to (9.1) with initial data $f_0$, guaranteed to exist and be unique from

Lemma 9.1. Note that up to time $\tau$, the process $f$ also solves (8.1), since for $t < \tau$ we have $\|f(t)\|_{H^{s_0}_{m_0}} \leq \|f(t)\|_{H^{\sigma}_{m}} \leq R$ and therefore $\theta_R(\|f(t)\|_{H^{s_0}_{m_0}}) = 1$. Clearly $\tau > 0$ almost surely since $H^{\sigma'-2}_{m'-2} \subset H^{\sigma}_{m}$ and $f$ takes values continuously in $H^{\sigma'-2}_{m'-2}$. The pair $(f, \tau)$ is thus a local solution of (8.1) within the higher regularity framework of this lemma, which is unique by Lemma 10.13 below. Now we will extend $f$ to a maximal solution.

Let $\mathcal{T}$ be the collection of all stopping times corresponding to a local solution and define $\xi = \sup \mathcal{T}$. Define also:

$$\tau_n := \inf\{t \geq 0 : \ \|f(t)\|_{H^{s_0}_{m_0}} > n\}.$$

Fix $T > 0$ finite but arbitrary and assume $\mathbf{P}(\xi = \tau_n \wedge T) > 0$ for some $n$. This implies that

$$\sup_{t \leq \xi} \|f(t)\|_{H^{s_0}_{m_0}} \leq n \text{ on the set } \{\xi = \tau_n \wedge T\},$$

and thus $f$ can be continued to a solution of (9.1) with $R = n + 1$ and thus of (8.1) up to a stopping time past $\xi$ - contradicting $\xi$'s maximality. Since $T$ was arbitrary, we either have $\xi = \infty$, or $\tau_n < \xi < \infty$ for all $n$. In the latter case, we also get $\sup_{t<\xi} \|f(t)\|_{H^{s_0}_{m_0}} \geq n$ for all $n$ and thus $\sup_{t<\xi} \|f(t)\|_{H^{s_0}_{m_0}} = \infty$.

Now we drop the almost-sure uniform boundedness requirement. If $\|f_0\|_{H^{\sigma'}_{m'}} < \infty$ almost surely, we decompose $f_0 = \sum_{k=0}^{\infty} f_{0,k}$, where $f_{0,k} := \mathbb{1}_{\{k \leq \|f_0\|_{H^{\sigma'}_{m'}} < k+1\}} f_0$. Now each $f_{0,k}$ generates a maximal solution $(f_k, \tau_k)$ where $\tau_k$ is the corresponding maximal existence time, and we define the "total" maximal solution (in high regularity) of (8.1) as $(\bar{f}, \bar{\tau})$ with:

$$\bar{f} = \sum_{k=0}^{\infty} \mathbb{1}_{\{k \leq \|f_0\|_{H^{\sigma'}_{m'}} < k+1\}}(\omega) f_k,$$

$$\bar{\tau} = \sum_{k=0}^{\infty} \mathbb{1}_{\{k \leq \|f_0\|_{H^{\sigma'}_{m'}} < k+1\}}(\omega)\tau_k.$$

□

Solutions to (9.1) are constructed using a two-step procedure. First, we regularize the non-linearity again and use an iteration procedure to construct a solution to the regularized SPDE and then second, we pass to the limit in the additional regularization parameter. Let $\varphi \in C_c^\infty(B(0,2))$ with $\int_{\mathbb{R}^n} \varphi \mathrm{d}x = 1$ and define $\varphi_\epsilon = \epsilon^{-d}\varphi(\epsilon^{-1}\cdot)$. Specifically, we seek a solution to the following regularized SPDE (here the convolution in $x$ has been periodized),

$$\begin{aligned} &\mathrm{d}\tilde{f} + v \cdot \nabla_x \tilde{f}\mathrm{d}t + \nabla_v \tilde{f} \circ \mathrm{d}W_t + \theta_R(\left\|\tilde{f}\right\|_{H^{s_0}_{m_0}})(\varphi_\epsilon * \tilde{E}) \cdot \nabla_v \tilde{f} = \nu \mathcal{L}\tilde{f}\mathrm{d}t \\ &\tilde{E} = \nabla_x(-\Delta_x)^{-1}\left(\int_{\mathbb{R}^d} \tilde{f}(t,\cdot,v)\mathrm{d}v - 1\right). \end{aligned} \tag{9.2}$$

This is done by an iteration method, specifically the following

$$f^j(0) = f_0$$

$$\mathrm{d}f^0 + v \cdot \nabla_x f^0 \mathrm{d}t + \nabla_v f^0 \odot \mathrm{d}W_t = \nu \mathcal{L} f^0 \mathrm{d}t \tag{9.3}$$

$$\mathrm{d}f^{j+1} + v \cdot \nabla_x f^{j+1}\mathrm{d}t + \theta_R(\left\|f^j\right\|_{H^{s_0}_{m_0}})(\varphi_\epsilon * E[f^j]) \cdot \nabla_v f^{j+1}\mathrm{d}t + \nabla_v f^{j+1} \odot \mathrm{d}W_t = \nu \mathcal{L} f^{j+1}\mathrm{d}t, \tag{9.4}$$

where we denote

$$E[f] := \nabla_x(-\Delta_x)^{-1}\left(\int_{\mathbb{R}^d} f(t,\cdot,v)\mathrm{d}v - 1\right).$$

The solutions to (9.3) - (9.4) are constructed by the method of characteristics. Indeed, (9.4) is the forward Kolmogorov equation associated to the SDE

$$\begin{cases} \mathrm{d}X_t = V_t \mathrm{d}t \\ \mathrm{d}V_t = -\nu V_t \mathrm{d}t + \theta_R(\|f^j\|_{H^{s_0}_{m_0}})(\varphi_\epsilon * E[f^j])(t, X_t)\mathrm{d}t + \sqrt{2\nu}\mathrm{d}\tilde{W}_t + \sum_k \sigma_k e_k(X_t) \circ \mathrm{d}W_t^{(k)}, \end{cases} \tag{9.5}$$

where $(\tilde{W}_t)$ is a $d$-dimensional Brownian motion defined in a new stochastic basis $(\Omega', \mathcal{F}', \mathbf{P}')$ (independent of the original basis). That is, (9.5) are the *stochastic characteristics* corresponding to (9.4), which generates a global stochastic flow of volume-preserving diffeomorphisms $\phi_t$ on $\mathbb{T}^d \times \mathbb{R}^d$, defined on the product space

$$(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbf{P} \times \mathbf{P}'),$$

which map $H^{\sigma'}_{m'}$ back to itself for all finite times almost-surely. The multiplicative (linear!) SPDE (9.4) is then solved by a "partial Feynman-Kac" formula with respect to $\mathbf{P}'$ :

$$f^{j+1} = \mathbf{E}_{\mathbf{P}'} f_0 \circ \phi_t^{-1}.$$

See [75] for more details.

**Remark 9.3.** Note that this type of regularization procedure has the added benefit of retaining non-negativity of $f$ as well as the preservation of the Casimir conservation laws, e.g. if $\nu = 0$ then $\|f^j\|_{L^p} = \|f_0\|_{L^p}$ and for $\nu > 0$ one at least has $\|f^j\|_{L^p} \leq e^{d\nu t}\|f_0\|_{L^p}$. However, these properties do not play an important role here.

Next, we need uniform a priori estimates to enable passing $j \to \infty$. These are obtained via Eulerian energy methods and come out as $\forall p < \infty$, $\alpha \in (0, 1/2)$, and $T < \infty$,

$$\sup_{j \geq 1} \mathbf{E} \left\| f^j \right\|^p_{L^\infty(0,T;H^{\sigma'}_{m'})} \lesssim_{p,T,\epsilon} 1$$

$$\sup_{j \geq 1} \mathbf{E} \left\| f^j \right\|^p_{W^{\alpha,p}(0,T;H^{\sigma'-2}_{m'-1})} \lesssim_{p,T,\epsilon,\alpha} 1.$$

See Lemma 10.1 for the proof of these estimates. Several previous works, for example [22, 38, 57] have used compactness to pass to similar limits, extract martingale solutions (i.e. probabilistically weak) using the Skorokhod embedding theorem, and then subsequently upgrade these solutions using a Gyöngy-Krylov lemma [59] argument and pathwise uniqueness. However, this technique seems not to apply in a clear manner to the Lagrangian iteration (9.4). Instead, we prove directly that there is a stopping time $\xi$ which is almost-surely greater than 1 such that $\{f^j\}_{j \geq 0}$ forms a Cauchy sequence in $L^2(\Omega; C_t([0, \xi); H^{s_0}_{m_0}))$, at which point it is not hard to pass to the limit, iterate in $t$, and construct global solutions to (9.2) in the desired regularity classes. This is proved in Lemma 10.6, where, in analogy with a classical Picard iteration, we show that $f^{j+1} - f^j$ is nearly comparable in size to the $j$-th term of a power series in powers of $\sqrt{t}$ of the solution. This procedure finally yields

**Lemma 9.4.** *Let $f_0$ be an $\mathcal{F}_0$-measurable random variable such that $\forall p \geq 2$,*

$$\mathbf{E} \left\| f_0 \right\|^p_{H^{\sigma'}_{m'}} < \infty.$$

*Then, there exists an $f \in C([0, \infty); H^{\sigma'-3}_{m'-2}) \cap L^\infty_{t,loc}([0, \infty); H^{\sigma'}_{m'})$ $\mathbf{P}$–a.s. which is a solution to*

(9.2) *in the sense that ,* $\mathbf{P}$*–a.s.:*

$$f(t) = f_0 + \int_0^t \Big( -v \cdot \nabla_x f(s) - \theta_R(\|f\|_{H^{s_0}_{m_0}}) \varphi_\epsilon * E(s) \cdot \nabla_v f(s) + \nu \mathcal{L} f(s) \Big) \, \mathrm{d}s - \int_0^t \nabla_v f(s) \circ \mathrm{d}W_s,$$

*where the equality holds in* $C([0,\infty); H^{\sigma'-3}_{m'-2})$*. Moreover, if* $\tilde{f}$ *is any other solution in the above sense, then* $f = \tilde{f}$ *almost surely in the sense that*

$$\mathbb{P}\Big( f(t) - \tilde{f}(t) = 0 \quad \forall 0 \le t < \infty | f(0) = \tilde{f}(0) \Big) = 1.$$

The next step in the proof of Lemma 9.1 is to remove the superfluous mollifier $\varphi_\epsilon$, which begins with obtaining $\epsilon$-independent estimates (now indexing solutions to (9.2) by $\epsilon$),

$$\sup_{\epsilon \in (0,1)} \mathbf{E} \, \|f_\epsilon\|^p_{L^\infty(0,T;H^{\sigma'}_{m'})} \lesssim_{p,T,R} 1$$

$$\sup_{\epsilon \in (0,1)} \mathbf{E} \, \|f_\epsilon\|^p_{W^{\alpha,p}(0,T;H^{\sigma'-2}_{m'-2})} \lesssim_{p,T,R,\alpha} 1.$$

See Lemma 10.10 for the proof of these estimates. These estimates can be considered the probabilistic analogue of the common deterministic method of sharpening a continuation criterion a posteriori, specifically, the thrust of the estimates is to show that the $H^{s_0}_{m_0}$ norm controls all $H^{\sigma'}_{m'}$ norms for $m' > m_0$ and $\sigma' > m_0$. At this step, it does not seem straightforward to prove that $\{f_\epsilon\}_{\epsilon \in (0,1)}$ is Cauchy, and so we follow the martingale approach. Specifically, we use these uniform bounds to apply the Skorokhod embedding theorem to produce probabilistically weak solutions to (9.1) (see Proposition 10.11 below). These solutions are subsequently upgraded to probabilistically strong solutions by proving pathwise uniqueness (Lemma 10.13) and an applica-

tion of the Gyöngy-Krylov lemma (from [59]; see Lemma 10.12 below). This general procedure is rather standard at this point; see for example [22, 38, 57]. This step completes the proof of Lemma 9.1.

The final step in the proof of Theorem 8.3 is to pass to a suitable limit in order to construct solutions in $C_t H^\sigma_m$ from $H^\sigma_m$ initial data, which is done in Section 11. We perform a regularization procedure on the initial data by defining a sequence of initial conditions

$$f_{0;n} = \mathcal{R}^n f_0 \equiv \theta_n(v) n^{2d} \eta\left(\frac{\cdot}{n}\right) *_{x,v} f_0,$$

where $\eta \in C_c^\infty(\mathbb{R}^{2d})$ and satisfies $\eta \geq 0$ and $\int_{\mathbb{R}^{2d}} \eta \mathrm{d}x\mathrm{d}v = 1$. Note these have been both mollified and cut-off in velocity (to improve both regularity and localization). For all $f_0 \in H^\sigma_m \cap L^1_+$, we hence have $f_{0;n} \in H^{\sigma'}_{m'} \cap L^1_+$ for all $\sigma', m'$. Subsequently, there are unique local pathwise solutions to (8.1) $(f_n, \tau_n)$ with

$$f_n(\cdot \wedge \tau_n) \in L^\infty_{loc}(0,\infty; H^{\sigma'}_{m'}) \cap C([0,\infty); H^\sigma_m).$$

By obtaining suitable uniform-in-$n$ upper bounds on the $C_t H^\sigma_m$ norm, we may pass to the limit $n \to \infty$ and hence extract local pathwise solutions to the original problem in $C_t H^\sigma_m$; see Lemmas 11.2 and 11.3 for details.

## Notation and conventions

For technical reasons, it is sometimes necessary (particularly when passing to the limit in the proofs of Lemmas 10.11 and 9.1) to view the fluctuating field as coming from a *cylindri-*

*cal* Wiener process. Specifically, let $\mathfrak{U}$ be a separable Hilbert space, with an orthonormal basis $(g_k)_{k\in\mathbb{K}}$. We can formally define a cylindrical Wiener process $\mathcal{W}_t$ on $\mathfrak{U}$ by the formula

$$\mathcal{W}_t := \sum_{k\in\mathbb{K}} g_k W_t^{(k)}.$$

Since this sum is divergent on $\mathfrak{U}$, one frequently employs the larger Hilbert space:

$$\mathfrak{U}_0 := \left\{ \sum_k \alpha_k g_k : \sum_k k^{-2}\alpha_k^2 < \infty \right\},$$
$$\left\| \sum_k \alpha_k g_k \right\|_{\mathfrak{U}_0}^2 = \sum_k k^{-2}\alpha_k^2,$$

where it can be shown that the formal sum for $\mathcal{W}_t$ converges and defines a process whose paths are almost surely in $C([0,T);\mathfrak{U}_0)$. Moreover, the embedding $\mathfrak{U} \subset \mathfrak{U}_0$ is Hilbert–Schmidt. For any separable Hilbert space $X$, we denote the space of all Hilbert–Schmidt operators from $\mathfrak{U}$ to $X$ by $L_2(\mathfrak{U};X)$; the definition of this space is:

$$L_2(\mathfrak{U};X) := \left\{ T \in L(\mathfrak{U};X) : \sum_k \|Tg_k\|_X^2 < \infty \right\},$$
$$\|T\|_{L_2(\mathfrak{U};X)}^2 = \sum_k \|Tg_k\|_X^2.$$

For more details on cylindrical Wiener processes and the relevant functional analytic setting, see [34].

At various points, we use the notation $f \in L^{p-}$ to signify that $f$ is in any $L^q$ space for $q < p$.

We often employ the common notation:

$$A(f) \lesssim_{p_1,p_2,\ldots} B(f)$$

which means that there exists a constant $C > 0$ depending only on the parameters $p_1, p_2, \ldots$ but *not* on the argument $f$, such that $A(f) \leq CB(f)$ for all relevant $f$. We omit the parameters if they are unimportant or clear from the context.

For the velocity-weighted $L^2$ norms and inner products, we set:

$$\langle f, g \rangle_m := \iint_{\mathbb{T}^d \times \mathbb{R}^d} f(x,v) g(x,v) \langle v \rangle^m \, \mathrm{d}v \mathrm{d}x$$
$$\|f\|_{L^2_m}^2 := \langle f, f \rangle_m .$$

Finally, at various points we use the mixed weighted norms:

$$\|f\|_{L^2_{v,n} L^p_x} := \left( \int_{\mathbb{R}^d} \left[ \int_{\mathbb{T}^d} |f(x,v)|^p \mathrm{d}x \right]^{2/p} \langle v \rangle^n \, \mathrm{d}v \right)^{1/2},$$
$$\|f\|_{L^2_{v,n} H^s_x} := \left( \sum_{|\alpha| \leq s} \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} |\partial_x^\alpha f(x,v)|^2 \langle v \rangle^n \, \mathrm{d}x \mathrm{d}v \right)^{1/2}.$$

# 10 Very smooth solutions

## 10.1 Proof of Lemma 9.4

As discussed in Section 9. a key step in proving Lemma 9.4 is constructing a convergent sequence of approximate solutions derived from a Lagrangian iteration scheme for (9.2). In particular, consider a sequence $f^j$ defined inductively as:

$$\begin{cases} \mathrm{d}f^0 + v \cdot \nabla_x f^0 \mathrm{d}t + \nabla_v f^0 \circ \mathrm{d}W_t = \nu \mathcal{L} f^0 \mathrm{d}t, \\ \mathrm{d}f^{j+1} + v \cdot \nabla_x f^{j+1} \mathrm{d}t + \nabla_v f^{j+1} \circ \mathrm{d}W_t = \nu \mathcal{L} f^{j+1} \mathrm{d}t - \theta_R(\|f^j\|_{H^{s_0}_{m_0}})(\varphi_\epsilon * E^j) \cdot \nabla_v f^{j+1} \mathrm{d}t, \\ f^j(0) = f_0. \end{cases} \tag{10.1}$$

As discussed in Section 9, for a given $f^j \in f \in C([0,\infty); H^{\sigma'-3}_{m'-2}) \cap L^\infty_{t,loc}([0,\infty), H^{\sigma'}_{m'})$, the solution $f^{j+1}$ is constructed via the method of stochastic characteristics.

First, we provide $j$-independent estimates in order to pass to the limit $j \to \infty$, for which we need appropriate compactness estimates for the iterates $f^j$ defined above. We remark that studying the stochastic flow of diffeomorphisms could show that $f^j \in f \in C([0,\infty); H^{\sigma'-3}_{m'-2}) \cap L^\infty_{t,loc}([0,\infty), H^{\sigma'}_{m'})$, however, providing $j$-independent (and eventually $\epsilon$-independent) bounds seems to be significantly more complicated than an Eulerian energy method approach, which is hence the approach we take. The main ingredient is provided by the following lemma:

**Lemma 10.1.** *Let $(f^j)_{j\geq 1}$ be a sequence of global solutions to the iterative scheme (10.1) with*

$\mathbf{E}\,\|f_0\|^p_{H^{\sigma'}_{m'}} < M_p < \infty$ *for all* $p \geq 2$. *For* $\alpha \in (0, \frac{1}{2}), p \geq 2$, *we have the uniform estimates:*

$$\sup_{j\geq 1} \mathbf{E} \sup_{t\leq T} \|f^j(t)\|^p_{H^{\sigma'}_{m'}} \leq C_{T,R,\epsilon,M} \tag{10.2}$$

*and*

$$\sup_{j\geq 1} \mathbf{E}\|f^j\|^p_{W^{\alpha,p}([0,T];H^{\sigma'-2}_{m'-1})} \leq C_{T,R,\epsilon,M}. \tag{10.3}$$

Before we begin, let us begin by recalling a few standard estimates. The first shows how to estimate the density in terms of the distribution function using sufficiently many velocity moments.

**Lemma 10.2.** *For any* $m > d$ *there exists a constant* $C_{m,d}$ *such that*

$$\left\| \int f(x,v) \mathrm{d}v \right\|_{L^2_x} \leq C \|f\|_{L^2_m}$$

Next, we recall the following Gagliardo-Nirenberg-Sobolev estimate: for all integers $0 \leq i \leq \sigma$ and functions $f \in H^\sigma$ (in $\mathbb{T}^d$ or $\mathbb{R}^n$) there holds

$$\sum_{|\alpha|=i} \|\partial^\alpha f\|_{L^{\frac{2\sigma}{i}}} \leq C \|f\|^{1-\frac{i}{\sigma}}_{L^\infty} \left( \sum_{|\alpha|=\sigma} \|\partial^\alpha f\|_{L^2} \right)^{\frac{i}{\sigma}}. \tag{10.4}$$

The next estimate recalls how to adapt Sobolev space product rules to the anisotropic nonlinearity. We give a proof for the readers' convenience.

**Lemma 10.3.** *Let* $g \in H^{s-1}_{n_0}, f \in H^s_n$ *for some* $n_0 > d$, $s > \frac{d}{2} + 1$, *and* $n \geq 0$ *arbitrary. Denote*

$E^g := \nabla_x \Delta_x^{-1} \left( \int g \mathrm{d}v - 1 \right)$. *Then, for a constant* $C$ *that does not depend on* $g$ *or* $f$*:*

$$\|E^g \cdot \nabla_v f\|_{H^s_n} \leq C \|g\|_{H^{s-1}_{n_0}} \|\nabla_v f\|_{H^s_n} \tag{10.5}$$

*and*

$$\sum_{|\alpha|+|\beta|\leq s} \left\langle \partial_x^\alpha \partial_v^\beta (E^g \cdot \nabla_v f), \partial_x^\alpha \partial_v^\beta f \right\rangle_n \leq C(\|E^g\|_{W^{1,\infty}} + \|g\|_{H^{s-1}_{n_0}}) \|f\|^2_{H^s_n}. \tag{10.6}$$

*Proof of Lemma 10.3.* In what follows, $\eta \in (0, \frac{1}{2})$ will be fixed, which implies $H_x^{\frac{d}{2}+\eta} \subset L_x^\infty$. Denote the multivariate binomial coefficients by

$$\begin{pmatrix} \alpha \\ \alpha' \end{pmatrix} := \frac{\prod_{j=1}^d \alpha_j!}{\prod_{j=1}^d (\alpha'_j)! \prod_{j=1}^d (\alpha_j - \alpha'_j)!}.$$

We begin the proof of (10.5) by using the product rule and the triangle inequality:

$$\|E^g \cdot \nabla_v f\|_{H^s_n} \leq \sum_{\substack{|\alpha|+|\beta|\leq s \\ \alpha_1 \leq \alpha}} \begin{pmatrix} \alpha \\ \alpha_1 \end{pmatrix} \left\| \partial_x^{\alpha-\alpha_1} E^g \cdot \nabla_v \partial_x^{\alpha_1} \partial_v^\beta f \right\|_{L^2_n}.$$

Now, we split into four separate cases:

Case 1 $\alpha_1 = \alpha$: In this case, we use Hölder's inequality, the embedding $H_x^{d/2+\eta} \subset L_x^\infty$, and Lemma 10.2

$$\begin{aligned} \|E^g \cdot \nabla_v \partial_x^\alpha \partial_v^\beta f\|_{L^2_n} \leq & \|E^g\|_{L^\infty} \|\nabla_v \partial_x^\alpha \partial_v^\beta f\|_{L^2_n} \\ \leq & C \|g\|_{H^{d/2-1+\eta}_{n_0}} \|\nabla_v f\|_{H^s_n}. \end{aligned} \tag{10.7}$$

In the following cases, $\alpha_1 < \alpha$.

Case 2 $|\alpha_1| + |\beta| = |\alpha| + |\beta| - 1$ or $|\alpha| + |\beta| = 1$: Here, we have $|\alpha - \alpha_1| = 1$, and hence similarly to the previous case

$$\begin{aligned}\|\partial_x^{\alpha-\alpha_1} E^g \cdot \nabla_v \partial_x^{\alpha_1} \partial_v^\beta f\|_{L^2_n} \leq & \|\nabla_x E^g\|_{L^\infty} \|\nabla_v f\|_{H^s_n} \\ \leq & C\|g\|_{H^{\frac{d}{2}+\eta}_{n_0}} \|\nabla_v f\|_{H^s_n}.\end{aligned}$$

Case 3 $|\alpha - \alpha_1| = |\alpha| + |\beta| \geq 2$: Here, we necessarily have $\alpha_1 = \beta = 0$, and hence (using the Sobolev embedding now on $\nabla_v f$),

$$\begin{aligned}\|\partial_x^\alpha E^g \cdot \nabla_v f\|_{L^2_n} \leq & \|\partial_x^\alpha E^g\|_{L^2} \left\| \|\nabla_v f\|_{L^\infty_x} \right\|_{L^2_{v,m'}} \\ \leq & C\|g\|_{H^{s-1}_{n_0}} \|\nabla_v f\|_{H^{d/2+\eta}_n}.\end{aligned}$$

Case 4 $|\alpha_1| + |\beta| \leq |\alpha| + |\beta| - 2$: Here $|\alpha - \alpha_1| \geq 2$, so:

$$\begin{aligned}\|\partial_x^{\alpha-\alpha_1} E^g \cdot \nabla_v \partial_x^{\alpha_1} \partial_v^\beta f\|_{L^2_n} \leq & \|\partial_x^{\alpha-\alpha_1} E^g\|_{L^2_x} \left\| \left\| \nabla_v \partial_x^{\alpha_1} \partial_v^\beta f \right\|_{L^\infty_x} \right\|_{L^2_{v,n}} \\ \leq & C\|g\|_{H^{|\alpha-\alpha_1|-1}_{n_0}} \|\nabla_v \partial_x^{\alpha_1} \partial_v^\beta f\|_{H^{d/2+\eta}_n} \\ \leq & C\|g\|_{H^{s-1}_{n_0}} \|\nabla_v f\|_{H^s_n}. \qquad (10.8)\end{aligned}$$

Summing over the various cases, (10.5) follows.

The proof of (10.6) is similar but just slightly more subtle after using the cancellation that occurs when all of the derivatives land on $\nabla_v f$. First, distribute the derivatives with Leibniz's

rule.

$$\sum_{|\alpha|+|\beta|\le s}\left\langle \partial_x^\alpha\partial_v^\beta(E^g\cdot\nabla_v f),\partial_x^\alpha\partial_v^\beta f\right\rangle_n \le \sum_{|\alpha|+|\beta|\le s}\sum_{\alpha_1\le\alpha}\binom{\alpha}{\alpha_1}\left|\left\langle \partial_x^{\alpha-\alpha_1}E^g\cdot\nabla_v\partial_x^{\alpha_1}\partial_v^\beta f,\partial_x^\alpha\partial_v^\beta f\right\rangle_n\right|.$$

We distinguish the following cases:

Case 1 $\alpha_1=\alpha$: by integrating by parts the $\nabla_v$ onto the weight, we have

$$\begin{aligned}\left|\left\langle E^g\cdot\nabla_v\partial_x^\alpha\partial_v^\beta f,\partial_x^\alpha\partial_v^\beta f\right\rangle_n\right| =&\frac{1}{2}\iint_{\mathbb{T}^d\times\mathbb{R}^d}E^g\cdot\nabla_v|\partial_x^\alpha\partial_v^\beta f|^2(\langle v\rangle^m)\mathrm{d}v\mathrm{d}x\\ \le& C\|E^g\|_{L^\infty}\|\partial_x^\alpha\partial_v^\beta f\|_{L^2_n}^2.\end{aligned}$$

Case 2 $|\alpha-\alpha_1|=1$: Cauchy-Schwarz gives

$$\begin{aligned}\left|\left\langle \partial_x^{\alpha-\alpha_1}E^g\cdot\nabla_v\partial_x^{\alpha_1}\partial_v^\beta f,\partial_x^\alpha\partial_v^\beta f\right\rangle_n\right| \le&\|\partial_x^{\alpha-\alpha_1}E^g\|_{L^\infty}\|f\|_{H^s_n}^2\\ \le& C\|\nabla_x E^g\|_{L^\infty}\|f\|_{H^s_n}^2.\end{aligned}$$

Case 3 $|\alpha-\alpha_1|\ge 2$: We have:

$$\begin{aligned}&\left|\left\langle \partial_x^{\alpha-\alpha_1}E^g\cdot\nabla_v\partial_x^{\alpha_1}\partial_v^\beta f,\partial_x^\alpha\partial_v^\beta f\right\rangle_n\right|\\ \le&\|\partial_x^{\alpha-\alpha_1}E^g\|_{L^{2\frac{s-1}{|\alpha-\alpha_1|-1}}}\left\|\nabla_v\partial_x^{\alpha_1}\partial_v^\beta f\right\|_{L^2_{v,n}L_x^{2\frac{s-1}{|\alpha_1|+|\beta|}}}\|\partial_x^\alpha\partial_v^\beta f\|_{L^2_n}\\ \le& C\|\nabla_x E^g\|_{L^\infty}^{\frac{|\alpha_1|+|\beta|}{s-1}}\|\nabla_x E^g\|_{H^s_x}^{\frac{|\alpha-\alpha_1|-1}{s-1}}\|f\|_{H^s_n}^2.\end{aligned}$$

In the above we used Hölder's inequality, Gagliardo-Nirenberg interpolation (10.4) on $\nabla_x E^g$ and Sobolev embedding on the $\nabla_v \partial_x^{\alpha_1} \partial_v^\beta f$ term, where we note that the order of integrability $p := 2\frac{s-1}{|\alpha_1|+|\beta|}$ corresponds to the embedding of $H_x^{\tilde{\sigma}}$ in $L_x^p$ for $\tilde{\sigma}$ given by:

$$\tilde{\sigma} = \frac{d}{2}\frac{|\alpha - \alpha_1| - 1}{s-1}$$

which satisfies $\tilde{\sigma} < |\alpha - \alpha_1| - 1$ exactly if $s > \frac{d}{2} + 1$.

Summing over the above cases, we obtain (10.6), which completes the proof of Lemma 10.3. □

Next, we prove Lemma 10.1.

*Proof of Lemma 10.1.* We begin with an estimate for $\|f\|^2_{H^{\sigma'}_{m'}}$, which we then use to derive an estimate for $\|f\|^p_{H^{\sigma'}_{m'}}$, for $p > 2$. Applying Itô's formula to $\|\partial_x^\alpha \partial_v^\beta f^j\|^2_{L^2_{m'}}$, we have:

$$\begin{aligned}
\mathrm{d}\|\partial_x^\alpha \partial_v^\beta f^j\|^2_{L^2_{m'}} =& -2\left\langle \partial_x^\alpha \partial_v^\beta (v \cdot \nabla_x f^j), \partial_x^\alpha \partial_v^\beta f^j \right\rangle_{m'} \mathrm{d}t \\
&+ 2\left\langle \Delta_v \partial_x^\alpha \partial_v^\beta f^j, \partial_x^\alpha \partial_v^\beta f^j \right\rangle_{m'} \mathrm{d}t \\
&+ 2\left\langle \partial_x^\alpha \partial_v^\beta (\mathrm{div}_v(f^j v)), \partial_x^\alpha \partial_v^\beta f^j \right\rangle_{m'} \mathrm{d}t \\
&- 2\left\langle \theta_R \partial_x^\alpha \partial_v^\beta (\varphi_\epsilon * E^{j-1} \cdot \nabla_v f^j), \partial_x^\alpha \partial_v^\beta f^j \right\rangle_{m'} \mathrm{d}t \\
&- 2\left\langle \partial_x^\alpha \partial_v^\beta (\nabla_v f^j \cdot \mathrm{d}W_t), \partial_x^\alpha \partial_v^\beta f^j \right\rangle_{m'} \\
&+ \sum_k \left\langle \partial_x^\alpha \partial_v^\beta [(\sigma_k e_k \cdot \nabla_v)^2 f^j], \partial_x^\alpha \partial_v^\beta f^j \right\rangle_{m'} \mathrm{d}t \\
&+ \sum_k \|\partial_x^\alpha \partial_v^\beta (\sigma_k e_k \cdot \nabla_v f^j)\|^2_{L^2_{m'}} \mathrm{d}t \\
=:& \, \mathcal{T}_{\alpha,\beta}(f^j) + \mathcal{D}_{\alpha,\beta}(f^j) + \mathcal{F}_{\alpha,\beta}(f^j) \\
&+ \mathcal{N}_{\alpha,\beta}(f^j) + \mathcal{M}_{\alpha,\beta}(f^j) + \mathcal{C}_{\alpha,\beta}(f^j),
\end{aligned} \tag{10.9}$$

where

$$\mathcal{C}_{\alpha,\beta}(f^j) = \sum_k \left\langle \partial_x^\alpha \partial_v^\beta [(\sigma_k e_k \cdot \nabla_v)^2 f^j], \partial_x^\alpha \partial_v^\beta f^j \right\rangle_{m'} \mathrm{d}t + \sum_k \|\partial_x^\alpha \partial_v^\beta (\sigma_k e_k \cdot \nabla_v f^j)\|_{L^2_{m'}}^2 \mathrm{d}t.$$

Here, the $\mathcal{T}, \mathcal{D}, \mathcal{F}, \mathcal{N}, \mathcal{M}, \mathcal{C}$ terms abbreviate transport, dissipation, friction, nonlinear electric field, martingale, and correction contributions, respectively. We begin by observing that by integration by parts,

$$\mathcal{T}_{\alpha,\beta}(f^j) + \mathcal{F}_{\alpha,\beta}(f^j) \leq C\|f^j\|^2_{H^{|\alpha|+|\beta|}_{m'}} \mathrm{d}t. \tag{10.10}$$

Similarly, for the dissipative term, integrating by parts gives:

$$\begin{aligned}
\mathcal{D}_{\alpha,\beta}(f^j) =& -2\|\nabla_v \partial_x^\alpha \partial_v^\beta f^j\|^2_{L^2_{m'}} \mathrm{d}t + \iint_{\mathbb{T}^d \times \mathbb{R}^d} |\partial_x^\alpha \partial_v^\beta f^j|^2 \Delta_v \langle v \rangle^m \, \mathrm{d}v \mathrm{d}x \mathrm{d}t \\
\leq& -2\|\nabla_v \partial_x^\alpha \partial_v^\beta f^j\|^2_{L^2_{m'}} \mathrm{d}t + C\|f\|^2_{H^{|\alpha|+|\beta|}_{m'}} \mathrm{d}t
\end{aligned}$$

Next, turn to the Itô correction terms, which need to be treated carefully in order to not lose derivatives. Distributing derivatives gives

$$\begin{aligned}
\mathcal{C}_{\alpha,\beta}(f^j) =& \sum_k \left( \|\partial_x^\alpha \partial_v^\beta (\sigma_k e_k \cdot \nabla_v) f^j\|^2_{L^2_{m'}} + \left\langle \partial_x^\alpha \partial_v^\beta (\sigma_k e_k \cdot \nabla_v)^2 f^j, \partial_x^\alpha \partial_v^\beta f^j \right\rangle_{m'} \right) \mathrm{d}t \\
=& \sum_k \sum_{\alpha' < \alpha} \binom{\alpha}{\alpha'} \sigma_k^2 \left\langle \partial_x^{\alpha - \alpha'} (e_k \otimes e_k) : \nabla_v^2 \partial_x^{\alpha'} \partial_v^\beta f^j, \partial_x^\alpha \partial_v^\beta f^j \right\rangle_{m'} \mathrm{d}t \qquad (10.11) \\
&+ \sum_k \sigma_k^2 \left\langle (e_k \otimes e_k) : \nabla_v^2 \partial_x^\alpha \partial_v^\beta f^j, \partial_x^\alpha \partial_v^\beta f^j \right\rangle_{m'} \mathrm{d}t \qquad (10.12) \\
&+ \sum_k \|\partial_x^\alpha \partial_v^\beta (\sigma_k e_k \cdot \nabla_v) f^j\|^2_{L^2_{m'}} \mathrm{d}t. \qquad (10.13)
\end{aligned}$$

Now, (10.12) provides a term of highest order that cancels the Itô correction $\|\partial_x^\alpha \partial_v^\beta (\sigma_k e_k \cdot \nabla_v) f^j\|_{L^2_{m'}}^2$, and terms of lower order that can either be readily controlled by $\|f^j\|_{H^{\sigma'}_{m'}}^2$ or cancel out with a corresponding term in (10.11). Integrating by parts in (10.12) we have,

$$
\begin{aligned}
(10.12) &= \sigma_k^2 \left\langle (e_k \otimes e_k) : \nabla_v^2 \partial_x^\alpha \partial_v^\beta f^j, \partial_x^\alpha \partial_v^\beta f^j \right\rangle_{m'} \\
&= - \left\langle \sigma_k e_k \cdot \nabla_v \partial_x^\alpha \partial_v^\beta f^j, \sigma_k e_k \cdot \nabla_v \partial_x^\alpha \partial_v^\beta f^j \right\rangle_{m'} \\
&\quad - \iint_{\mathbb{T}^d \times \mathbb{R}^d} \partial_x^\alpha \partial_v^\beta f^j (\sigma_k e_k \cdot \nabla_v \partial_x^\alpha \partial_v^\beta f^j) \sigma_k e_k \cdot \nabla_v (\langle v \rangle^{m'}) \mathrm{d}v \mathrm{d}x \\
&= - \left\langle \partial_x^\alpha \partial_v^\beta (\sigma_k e_k \cdot \nabla_v f^j), \sigma_k e_k \cdot \nabla_v \partial_x^\alpha \partial_v^\beta f^j \right\rangle_{m'} - \left\langle [\sigma_k e_k \cdot \nabla_v, \partial_x^\alpha \partial_v^\beta] f^j, \sigma_k e_k \cdot \nabla_v \partial_x^\alpha \partial_v^\beta f^j \right\rangle_{m'} \\
&\quad + \frac{1}{2} \iint_{\mathbb{T}^d \times \mathbb{R}^d} |\partial_x^\alpha \partial_v^\beta f^j|^2 (\sigma_k e_k \cdot \nabla_v)^2 (\langle v \rangle^{m'}) \mathrm{d}v \mathrm{d}x \\
&= - \|\sigma_k e_k \cdot \nabla_v (\partial_x^\alpha \partial_v^\beta f^j)\|_{L^2_{m'}}^2 \\
&\quad - \left\langle \partial_x^\alpha \partial_v^\beta (\sigma_k e_k \cdot \nabla_v f^j), [\sigma_k e_k \cdot \nabla_v, \partial_x^\alpha \partial_v^\beta] f^j \right\rangle_{m'} - \left\langle [\sigma_k e_k \cdot \nabla_v, \partial_x^\alpha \partial_v^\beta] f^j, \sigma_k e_k \cdot \nabla_v \partial_x^\alpha \partial_v^\beta f^j \right\rangle_{m'} \\
&\quad + \frac{1}{2} \iint_{\mathbb{T}^d \times \mathbb{R}^d} |\partial_x^\alpha \partial_v^\beta f^j|^2 (\sigma_k e_k \cdot \nabla_v)^2 (\langle v \rangle^{m'}) \mathrm{d}v \mathrm{d}x \\
&= - \|\sigma_k e_k \cdot \nabla_v (\partial_x^\alpha \partial_v^\beta f^j)\|_{L^2_{m'}}^2 && (10.14) \\
&\quad - 2 \left\langle [\sigma_k e_k \cdot \nabla_v, \partial_x^\alpha \partial_v^\beta] f^j, \sigma_k e_k \cdot \nabla_v \partial_x^\alpha \partial_v^\beta f^j \right\rangle_{m'} && (10.15) \\
&\quad - \left\langle [\partial_x^\alpha \partial_v^\beta, \sigma_k e_k \cdot \nabla_v] f^j, [\sigma_k e_k \cdot \nabla_v, \partial_x^\alpha \partial_v^\beta] f^j \right\rangle_{m'} && (10.16) \\
&\quad + \frac{1}{2} \iint_{\mathbb{T}^d \times \mathbb{R}^d} |\partial_x^\alpha \partial_v^\beta f^j|^2 (\sigma_k e_k \cdot \nabla_v)^2 (\langle v \rangle^{m'}) \mathrm{d}v \mathrm{d}x. && (10.17)
\end{aligned}
$$

Thus, we observe that (10.14) cancels the Itô correction (10.13), (10.17) is bounded above by $C\|\partial_x^\alpha \partial_v^\beta f^j\|_{L^2_{m'}}^2$, and (10.16) only contains derivatives of order lower than $|\alpha| + |\beta|$ and is thus bounded above by $C\|f^j\|_{H^{\sigma'}_{m'}}^2$. Next, we turn to (10.15). This term contains (A) terms from the commutator where the total number of derivatives on $f^j$ is strictly less than $|\alpha| + |\beta|$, which can

be treated by integration by parts of the $\sigma_k e_k \cdot \nabla_v$ and are thus bounded above by $C\|f^j\|^2_{H^{\sigma'}_{m'}}$; and (B) a highest order term which we deal with as follows:

$$
\begin{aligned}
&2\sum_{\substack{\alpha'<\alpha\\|\alpha'|=1}} \sigma_k^2 \left\langle \partial_x^{\alpha'} e_k\cdot\nabla_v \partial_x^{\alpha-\alpha_1}\partial_v^\beta f^j, e_k\cdot\nabla_v\partial_x^\alpha\partial_v^\beta f^j\right\rangle_{m'}\\
=&-2\sum_{\substack{\alpha'<\alpha\\|\alpha'|=1}} \sigma_k^2 \left\langle (\partial_x^{\alpha'} e_k\cdot\nabla_v)(e_k\cdot\nabla_v)\partial_x^{\alpha-\alpha'}\partial_v^\beta f^j, \partial_x^\alpha\partial_v^\beta f^j\right\rangle_{m}\\
&-\sum_{\substack{\alpha'<\alpha\\|\alpha|=1}} \sigma_k^2 \iint_{\mathbb{T}^d\times\mathbb{R}^d} \partial_x^\alpha\partial_v^\beta f^j \partial_x^{\alpha'} e_k\cdot\nabla_v\partial_x^{\alpha-\alpha'}\partial_v^\beta f^j e_k\cdot\nabla_v(\langle v\rangle^{m'})\mathrm{d}v\mathrm{d}x\\
=&-\sum_{\substack{\alpha'<\alpha\\|\alpha'|=1}} \sigma_k^2 \left\langle \partial_x^{\alpha'}(e_k\otimes e_k):\nabla_v^2\partial_x^{\alpha-\alpha'}\partial_v^\beta f^j, \partial_x^\alpha\partial_v^\beta f^j\right\rangle_{m'} \qquad (10.18)\\
&-\sum_{\substack{\alpha'<\alpha\\|\alpha|=1}} \sigma_k^2 \iint_{\mathbb{T}^d\times\mathbb{R}^d} \partial_x^\alpha\partial_v^\beta f^j \partial_x^{\alpha'} e_k\cdot\nabla_v\partial_x^{\alpha-\alpha'}\partial_v^\beta f^j e_k\cdot\nabla_v(\langle v\rangle^{m'})\mathrm{d}v\mathrm{d}x. \qquad (10.19)
\end{aligned}
$$

Notice that to obtain the prefactor 1 in (10.18) we used the symmetry of the tensor $e_k\otimes e_k$. Now, (10.19) can be directly bounded by $C\|f^j\|^2_{H^{\sigma'}_{m'}}$, while (10.18) cancels out the highest order term in (10.11). Therefore, we finally conclude using (10.11)–(10.19) that we have

$$
\mathcal{C}_{\alpha,\beta}(f^j)\le C\|f^j\|^2_{H^{\sigma'}_{m'}}\mathrm{d}t.
$$

Next we treat the contribution of the electric field term. It follows from (10.7)-(10.8) in the proof of (10.6) that:

$$
\begin{aligned}
\mathcal{N}_{\alpha,\beta}(f^j)\le& C\theta_R(\|f^{j-1}\|_{H^{s_0}_{m_0}})(\|\varphi_\epsilon * E^{j-1}\|_{W^{1,\infty}}+\|\varphi_\epsilon * E^{j-1}\|_{H^{\sigma'-1}_x})\|f^j\|^2_{H^{\sigma'}_{m'}}\mathrm{d}t\\
\lesssim_{\epsilon,R}&\|f^j\|^2_{H^{\sigma'}_{m'}}\mathrm{d}t.
\end{aligned}
$$

Finally, the martingale contribution is given by

$$\mathcal{M}_{\alpha,\beta}(f^j) = \iint_{\mathbb{T}^d\times\mathbb{R}^d} |\partial_x^\alpha \partial_v^\beta f^j|^2 \mathrm{d}W_t \cdot \nabla_v(\langle v\rangle^{m'})\mathrm{d}v\mathrm{d}x$$
$$- 2\left\langle [\partial_x^\alpha \partial_v^\beta, \mathrm{d}W_t]\nabla_v f^j, \partial_x^\alpha \partial_v^\beta f^j\right\rangle_{m'}. \tag{10.20}$$

We sum (10.10)-(10.20) over $|\alpha| + |\beta| \le \sigma'$ and obtain

$$\mathrm{d}\|f^j\|^2_{H^{\sigma'}_{m'}} \le C\|f^j\|^2_{H^{\sigma'}_{m'}}\mathrm{d}t - 2\nu\|\nabla_v f^j\|^2_{H^{\sigma'}_{m'}}\mathrm{d}t + \sum_{0\le|\alpha|+|\beta|\le\sigma} \mathcal{M}_{\alpha,\beta}(f^j),$$

so integrating in time and using the Burkhölder–Davis–Gundy inequality (see e.g. [33]) (hereinafter abbreviated as BDG) we obtain:

$$\begin{aligned}\mathbf{E}\sup_{t\le T}\|f^j(t)\|^2_{H^{\sigma'}_{m'}} \le& \mathbf{E}\|f_0\|^2_{H^{\sigma'}_{m'}} + C\int_0^T \mathbf{E}\|f^j(t)\|^2_{H^{\sigma'}_{m'}}\mathrm{d}t\\ &+ C\mathbf{E}\left(\int_0^T \|f^j(t)\|^4_{H^{\sigma'}_{m'}}\mathrm{d}t\right)^{\frac{1}{2}}\\ \le& \|f_0\|^2_{H^{\sigma'}_{m'}} + C\int_0^T \mathbf{E}\|f^j(t)\|^2_{H^{\sigma'}_{m'}}\mathrm{d}t\\ &+ \frac{1}{2}\mathbf{E}\sup_{t\le T}\|f^j(t)\|^2_{H^{\sigma'}_{m'}},\end{aligned}$$

where the second line followed from Hölder's inequality. After rearranging and applying Grönwall's inequality we obtain the uniform-in-$j$ estimate:

$$\mathbf{E}\sup_{t'\le T}\|f^j(t)\|^2_{H^{\sigma'}_{m'}} \le C\mathbf{E}\|f_0\|^2_{H^{\sigma'}_{m'}},$$

where the constant $C$ depends on $\epsilon, R, T, m', \sigma'$, but not $f_0$ or $j$. Thus we have obtained (10.2)

for $p = 2$.

Now, we use Itô's formula again, this time for $\|f^j\|^p_{H^{\sigma'}_{m'}}$, with $p > 2$:

$$\begin{aligned}\mathrm{d}\|f^j\|^p_{H^{\sigma'}_{m'}} =& \frac{p}{2}\|f^j\|^{p-2}_{H^{\sigma'}_{m'}}\mathrm{d}\|f\|^2_{H^{\sigma'}_{m'}} \\ &+ \frac{p}{2}\frac{p-2}{2}\|f^j\|^{p-4}_{H^{\sigma'}_{m'}}\sum_k \sum_{|\alpha|+|\beta|\le\sigma'} \left|\left\langle \partial_x^\alpha\partial_v^\beta(\sigma_k e_k\cdot\nabla_v f^j), \partial_x^\alpha\partial_v^\beta f^j\right\rangle_{m'}\right|^2 \mathrm{d}t.\end{aligned}$$

The latter term is treated by a straightforward commutator estimate, and together with the above estimates on $\mathrm{d}\,\|f^j\|^2_{H^{\sigma'}_{m'}}$, we obtain

$$\mathrm{d}\|f^j\|^p_{H^{\sigma'}_{m'}} \le C\|f^j\|^p_{H^{\sigma'}_{m'}}\mathrm{d}t - p\|f^j\|^{p-2}_{H^{\sigma'}_{m'}}\sum_{|\alpha|+|\beta|\le\sigma'}\left\langle \partial_x^\alpha\partial_v^\beta(\nabla_v f^j\cdot \mathrm{d}W_t), \partial_x^\alpha\partial_v^\beta f^j\right\rangle_{m'}.$$

After integrating in time, using the BDG inequality, and applying Hölder's inequality, we have

$$\begin{aligned}\mathbf{E}\sup_{t\le T}\|f^j\|^p_{H^{\sigma'}_{m'}} \le& \mathbf{E}\|f_0^j\|^p_{H^{\sigma'}_{m'}} + C\int_0^T \mathbf{E}\|f^j\|^p_{H^{\sigma'}_{m'}}\mathrm{d}s \\ &+ C\mathbf{E}\left(\int_0^T \|f^j\|^{2p}_{H^{\sigma'}_{m'}}\mathrm{d}s\right)^{\frac{1}{2}} \\ \le& \mathbf{E}\|f_0\|^p_{H^{\sigma'}_{m'}} + C\int_0^T \mathbf{E}\|f^j\|^p_{H^{\sigma'}_{m'}}\mathrm{d}s + \frac{1}{2}\mathbf{E}\sup_{t'\le T}\|f^j\|^p_{H^{\sigma'}_{m'}}.\end{aligned}$$

By rearranging and using Grönwall's lemma, we obtain (10.2).

We now turn to the proof of (10.3). We have:

$$\begin{aligned}\mathbf{E}\left\|f^j\right\|^p_{W^{\alpha,p}([0,T];H^{\sigma'-2}_{m'-1})} \le& C\|f_0\|^p_{H^{\sigma'-2}_{m'-1}} \\ &+ C\mathbf{E}\left\|\int_0^t\left(-v\cdot\nabla_x f^j + \Delta_v f^j + \mathrm{div}_v(f^j v)\right)\mathrm{d}s\right\|^p_{W^{1,p}([0,T];H^{\sigma'-2}_{m'-1})}\end{aligned}$$

$$
\begin{aligned}
&+ C\mathbf{E}\left\| \int_0^t \theta_R \varphi_\epsilon * E^{j-1} \cdot \nabla_v f^j \right\|^p_{W^{1,p}([0,T];H^{\sigma'-2}_{m'-1})} \\
&+ C\mathbf{E}\left\| \frac{1}{2}\int_0^t \sum_k (\sigma_k e_k \cdot \nabla_v)^2 f^j \mathrm{d}s \right\|^p_{W^{1,p}([0,T];H^{\sigma'-2}_{m'-1})} \\
&+ C\mathbf{E}\left\| \int_0^t \nabla_v f^j \cdot \mathrm{d}W_t \right\|^p_{W^{\alpha,p}([0,T];H^{\sigma'-2}_{m'-1})}.
\end{aligned}
$$

The terms that are regular in time are estimated in a straightforward manner using the available regularity:

$$
\begin{aligned}
\mathbf{E}\left\| \int_0^t \left( v \cdot \nabla_x f^j + \mathrm{div}_v(f^j v) \right) \mathrm{d}s \right\|^p_{W^{1,p}([0,T];H^{\sigma'-2}_{m'-1})} &\leq C\mathbf{E} \sup_{t' \leq T} \|f^j(t')\|^p_{H^{\sigma'-1}_{m'}} \\
\mathbf{E}\left\| \int_0^t \Delta_v f^j \mathrm{d}s \right\|^p_{W^{1,p}([0,T];H^{\sigma'-2}_{m'-1})} &\leq C\mathbf{E} \sup_{t' \leq T} \|f^j(t')\|^p_{H^{\sigma'}_{m'-1}} \\
\mathbf{E}\left\| \int_0^t \sum_k (\sigma_k e_k \cdot \nabla_v)^2 f^j \right\|^p_{W^{1,p}([0,T];H^{\sigma'-2}_{m'-1})} &\leq C\mathbf{E} \sup_{t' \leq T} \|f^j(t')\|^p_{H^{\sigma'}_{m'-1}} \\
\mathbf{E}\left\| \int_0^t \theta_R \varphi_\epsilon * E^{j-1} \cdot \nabla_v f^j \mathrm{d}s \right\|^p_{W^{1,p}([0,T];H^{\sigma'-2}_{m'-1})} &\leq C_R\mathbf{E} \sup_{t' \leq T} \|f^j(t')\|^p_{H^{\sigma'-1}_{m'-1}}.
\end{aligned}
$$

The time-regularity is only limited by the stochastic integral, which is estimated by a variant of the BDG inequality adapted to fractional regularity estimates in time (see e.g. [Lemma 2.1; [53]] for a proof), namely

$$
\begin{aligned}
\mathbf{E}\left\| \int_0^t \nabla_v f^j \cdot \mathrm{d}W_t \right\|^p_{W^{\alpha,p}([0,T];H^{\sigma'-2}_{m'-1})} &\leq C\mathbf{E} \int_0^T \left\| \nabla_v f^j(s) \right\|^p_{H^{\sigma'-2}_{m'-1}} ds \\
&\leq C\mathbf{E} \sup_{t' \leq T} \|f^j(t')\|^p_{H^{\sigma'-1}_{m'-1}}.
\end{aligned}
$$

Therefore, using that $W^{1,p}([0,T];H^{\sigma'-2}_{m'-2}) \subset W^{\alpha,p}([0,T];H^{\sigma'-2}_{m'-2})$ continuously and (10.2), we

obtain:

$$\mathbf{E}\|f^j\|^p_{W^{\alpha,p}([0,T];H^{\sigma'-2}_{m'-2})} \leq C_{R,T}\mathbf{E}\|f_0\|^p_{H^{\sigma'}_{m'}}$$

uniformly in $j$, which implies (10.3), completing the proof of Lemma 10.1. □

**Remark 10.4.** By examining the proof above, one can see that one can also treat magnetic fields, due to the special structure of the Lorentz force $v \times B(x)$, which ensures both $\nabla_v \cdot (v \times B) = 0$ and, despite the power of $v$, the estimates do not lose any moments in $v$ as $v \times B$ is orthogonal to $v$ (nor does the $v$ dependence create any issues controlling higher regularity).

We continue the proof of Lemma (9.4). The approximation procedure mixes $f^j$ and $f^{j+1}$ in a way that makes it difficult to apply the usual method of using tightness of the laws in pathspace and applying the Skorohod embedding theorem to construct probabilistically weak solutions which are subsequently upgraded to strong solutions (see e.g. [22, 37, 38, 57]). Instead we will prove that $\{f^j\}_{j=1}^{\infty}$ is Cauchy in a suitable topology. For this we first need the following consequence of Lemma 10.1 and the Borel-Cantelli lemma.

**Lemma 10.5.** *For all $\delta > 0$, $\exists$ a $\mathcal{F}_1$-measurable, almost-surely finite, random constant $C_0$ such that for* all $j \geq 0$ *there holds*

$$\sup_{s<1} \left\|f^j(s)\right\|_{H^{\sigma'}_{m'}} < C_0(\omega,\delta)\left\langle j\right\rangle^{\delta}.$$

*Moreover, $\forall \alpha, n$ there holds,*

$$\mathbf{P}(C_0 > n) \lesssim_{\delta,\alpha} n^{-\alpha}.$$

*Proof.* Recall the uniform in $j$ bound (10.2) for the iterates for $T = 1$:

$$\sup_{j\geq 1} \mathbf{E} \sup_{s\leq 1} \|f^j(s)\|^p_{H^{\sigma'}_{m'}} \leq C_{p,R,\epsilon,M} < \infty.$$

This estimate implies:

$$\mathbf{E} \sum_{j=0}^{\infty} \frac{\sup_{s\leq 1} \|f^j(s)\|^p_{H^{\sigma'}_{m'}}}{\langle j \rangle^{\delta p}} \leq C_{p,R,\epsilon,M},$$

for $p > \frac{1}{\delta}$. Denote by $A_j$ the sets:

$$A_j := \{\omega \in \Omega : \sup_{s\leq 1} \|f^j(s)\|_{H^{\sigma'}_{m'}} > \langle j \rangle^{\delta}\},$$

and note that by Chebyshev's inequality:

$$\sum_{j=0}^{\infty} \mathbf{P}(A_j) \leq \sum_{j=0}^{\infty} \frac{\mathbf{E} \sup_{s\leq 1} \|f^j(s)\|^p_{H^{\sigma'}_{m'}}}{\langle j \rangle^{\delta p}} < \infty.$$

It then follows by the Borel–Cantelli lemma that

$$\mathbf{P}(\limsup_{j\to\infty} A_j) = 0,$$

implying that $\mathbf{P}$-a.s., $\sup_{s\leq 1} \|f^j(s)\|_{H^{\sigma'}_{m'}} > \langle j \rangle^{\delta}$ at most for a finite number of $j$'s. Denote the largest such $j$ by $j_0(\omega)$. We then see that there is a random constant $C_0(\omega, \delta)$ such that

$$\sup_{j\geq 0} \left( \langle j \rangle^{-\delta} \sup_{s\leq 1} \|f^j(s)\|_{H^{\sigma'}_{m'}} \right) < C_0(\omega, \delta)$$

$\mathbf{P}$–almost surely. In particular, we can take:

$$C_0(\omega,\delta) := \inf\left\{ n \in \mathbb{N} : \sup_{j\le j_0(\omega)} \left( \langle j\rangle^{-\delta} \sup_{s\le 1} \|f^j(s)\|_{H^{\sigma'}_{m'}} \right) < n \right\}.$$

To bound the probability that $C_0$ is large, we observe:

$$\begin{aligned}
\mathbf{P}(C_0 > n) \le & \mathbf{P}\left( \sup_{j\ge 0} \left( \langle j\rangle^{-\delta} \sup_{s\le 1} \|f^j(s)\|_{H^{\sigma'}_{m'}} > n \right) \right) \\
& \le \sum_{j=0}^{\infty} \langle j\rangle^{-\delta p} \, \mathbf{E} \sup_{s\le 1} \|f^j(s)\|^p_{H^{\sigma'}_{m'}} n^{-p} \\
& \lesssim n^{-p}.
\end{aligned}$$

This completes the proof of the lemma. □

The next lemma is the crucial convergence estimate.

**Lemma 10.6.** *There exists an increasing sequence of stopping times $\tau_n$ such that $\{f^j\}_{j=1}^{\infty}$ is Cauchy in $L^2_\omega C([0,\tau_n]; H^{s_0}_{m_0})$ and the stopping time*

$$\lim_{n\to\infty} \tau_n = \xi,$$

*is almost-surely greater than* 1.

*Proof.* Define the increasing sequence of stopping times

$$\tau_n = \inf\left\{ t : \exists j : \left\| f^j(t) \right\|_{H^{s_0+1}_{m_0}} > n \left\langle j \right\rangle^{\delta} \right\}.$$

Note that by Lemma 10.5 there holds

$$\begin{aligned}\mathbf{P}(\tau_n \geq 1) &= \mathbf{P}\left(\sup_{j\geq 0}\langle j\rangle^{-\delta}\sup_{t<1}\left\|f^j(t)\right\|_{H^{s_0+1}_{m_0}} < n\right)\\ &\geq \mathbf{P}(C_0 < n)\\ &\geq 1-\mathbf{P}(C_0 > n)\\ &\geq 1-\mathcal{O}(n^{-\alpha}).\end{aligned}$$

Therefore, $\lim_{n\to\infty}\mathbf{P}(\tau_n > 1) = 1$ and so if we define

$$\xi = \lim_{n\to\infty}\tau_n,$$

then $\xi$ is almost-surely greater than or equal to $1$.

Let $\delta \in (0, 1/6)$ be fixed arbitrary. We will show by induction that $\exists K_0 > 0$ (deterministic constant depending on $\delta$) such that for all $j \geq 1$, there holds

$$\mathbf{E}\sup_{s<t\wedge\tau_n}\left\|f^j - f^{j-1}\right\|^2_{H^{s_0}_{m_0}} \leq \frac{(K_0 n^4 t)^j j^{4\delta j}}{j!}. \tag{10.21}$$

First consider the case $j = 1$. The calculation of $\mathrm{d}\left\|\partial_x^\alpha\partial_v^\beta(f^1 - f^0)\right\|^2_{L^2_{m_0}}$ is the same in Lemma 10.1 except for the nonlinear terms. That is, for $|\alpha| + |\beta| \leq s_0$ we have for some constant $C > 0$

$$\begin{aligned}\mathrm{d}\|\partial_x^\alpha\partial_v^\beta(f^1 - f^0)\|^2_{L^2_{m_0}} \leq\ & C\left\|f^1 - f^0\right\|^2_{H^{s_0}_{m_0}}\mathrm{d}t\\ &- 2\left\langle\theta_R(\left\|f^0\right\|_{H^{s_0}_{m_0}})\partial_x^\alpha\partial_v^\beta(\varphi_\epsilon * E^0\cdot\nabla_v f^1), \partial_x^\alpha\partial_v^\beta(f^1 - f^0)\right\rangle_{m_0}\mathrm{d}t\\ &- 2\left\langle\partial_x^\alpha\partial_v^\beta(\nabla_v(f^1 - f^0)\cdot\mathrm{d}W_t), \partial_x^\alpha\partial_v^\beta(f^1 - f^0)\right\rangle_{m_0}.\end{aligned}$$

For the nonlinear term we note that by (10.6) we have, recalling the definition of $\tau_n$

$$\left|\left\langle \theta_R(\left\|f^0\right\|_{H^{s_0}_{m_0}})\partial_x^\alpha\partial_v^\beta(\varphi_\epsilon * E^0\cdot\nabla_v f^1), \partial_x^\alpha\partial_v^\beta(f^1-f^0)\right\rangle_{m_0}\right| \lesssim_{\epsilon,R} n\left\|f^1-f^0\right\|_{H^{s_0}_{m_0}}.$$

Integrating in time and using the BDG inequality as above, we obtain (note that $f^1$ and $f^0$ have the same initial data),

$$\begin{aligned}\mathbf{E}\sup_{s<t\wedge\tau_n}\|f^1(s)-f^0(s)\|^2_{H^{s_0}_{m_0}} \leq & C\int_0^{t\wedge\tau_n}\mathbf{E}\|f^1(s)-f^0(s)\|^2_{H^{s_0}_{m_0}}\,\mathrm{d}s\\ &+Cn^2t+C\mathbf{E}\left(\int_0^{t\wedge\tau_n}\|f^1(s)-f^0(s)\|^4_{H^{s_0}_{m_0}}\,\mathrm{d}s\right)^{\frac12}\\ \leq & Cn^2t+C\int_0^{t\wedge\tau_n}\mathbf{E}\|f^1(s)-f^0(s)\|^2_{H^{s_0}_{m_0}}\,\mathrm{d}s\\ &+\frac12\mathbf{E}\sup_{s\leq t\wedge\tau_n}\|f^1(s)-f^0(s)\|^2_{H^{s_0}_{m_0}},\end{aligned}$$

Therefore, Grönwall's inequality verifies (10.21) for $j=1$ and some large $K_0$.

Next consider the inductive step. Hence, suppose that (10.21) holds for $j$ and we wish to verify that it holds for $j+1$. As above, for some constant $C>0$

$$\begin{aligned}\mathrm{d}\|\partial_x^\alpha\partial_v^\beta(f^{j+1}-f^j)\|^2_{L^2_{m_0}} \leq & C\left\|f^{j+1}-f^j\right\|^2_{H^{s_0}_{m_0}}\,\mathrm{d}t\\ &-2\left\langle\theta_R(\left\|f^j\right\|_{H^{s_0}_{m_0}})\partial_x^\alpha\partial_v^\beta(\varphi_\epsilon * E^j\cdot\nabla_v f^{j+1}),\partial_x^\alpha\partial_v^\beta(f^{j+1}-f^j)\right\rangle_{m_0}\mathrm{d}t\\ &+2\left\langle\theta_R(\left\|f^{j-1}\right\|_{H^{s_0}_{m_0}})\partial_x^\alpha\partial_v^\beta(\varphi_\epsilon * E^{j-1}\cdot\nabla_v f^{j}),\partial_x^\alpha\partial_v^\beta(f^{j+1}-f^j)\right\rangle_{m_0}\mathrm{d}t\\ &-2\left\langle\partial_x^\alpha\partial_v^\beta(\nabla_v(f^{j+1}-f^j)\cdot\mathrm{d}W_t),\partial_x^\alpha\partial_v^\beta(f^{j+1}-f^j)\right\rangle_{m_0}.\end{aligned}$$

The nonlinearity separates into several natural terms, namely

$$
\begin{aligned}
(*) = &-2\left\langle \theta_R(\left\|f^j\right\|_{H^{s_0}_{m_0}})\partial_x^\alpha\partial_v^\beta(\varphi_\epsilon * E^j\cdot(\nabla_v f^{j+1}-\nabla_v f^j)),\partial_x^\alpha\partial_v^\beta(f^{j+1}-f^j)\right\rangle_{m_0}\mathrm{d}t\\
&-2\left\langle \left(\theta_R(\left\|f^j\right\|_{H^{s_0}_{m_0}})-\theta_R(\left\|f^{j-1}\right\|_{H^{s_0}_{m_0}})\right)\partial_x^\alpha\partial_v^\beta(\varphi_\epsilon * E^j\cdot\nabla_v f^j),\partial_x^\alpha\partial_v^\beta(f^{j+1}-f^j)\right\rangle_{m_0}\mathrm{d}t\\
&-2\left\langle \theta_R(\left\|f^{j-1}\right\|_{H^{s_0}_{m_0}})\partial_x^\alpha\partial_v^\beta((\varphi_\epsilon * E^j-\varphi_\epsilon * E^{j-1})\cdot\nabla_v f^j),\partial_x^\alpha\partial_v^\beta(f^{j+1}-f^j)\right\rangle_{m_0}\mathrm{d}t\\
&=\mathcal{N}_1+\mathcal{N}_2+\mathcal{N}_3.
\end{aligned}
$$

The term $\mathcal{N}_1$ is treated via (10.6) in the same manner as in Lemma 10.1, giving

$$
\mathcal{N}_1\lesssim_{R,\epsilon}\left\|f^{j+1}-f^j\right\|^2_{H^{s_0}_{m_0}}.
$$

The terms $\mathcal{N}_2,\mathcal{N}_3$ however are different. The term $\mathcal{N}_3$ is estimated via the following for $t<\tau_n$:

$$
\begin{aligned}
\mathcal{N}_3&\lesssim_R\left\|f^j\right\|_{H^{s_0+1}_{m_0}}\left\|f^j-f^{j-1}\right\|_{H^{s_0}_{m_0}}\left\|f^{j+1}-f^j\right\|_{H^{s_0}_{m_0}}\\
&\lesssim nj^\delta\left\|f^j-f^{j-1}\right\|_{H^{s_0}_{m_0}}\left\|f^{j+1}-f^j\right\|_{H^{s_0}_{m_0}}.
\end{aligned}
$$

The term $\mathcal{N}_2$ requires a control on the difference $\theta_R(\|f^j\|_{H^{s_0}_{m_0}})-\theta_R(\|f^{j-1}\|_{H^{s_0}_{m_0}})$:

$$
\begin{aligned}
&\theta_R(\left\|f^j\right\|_{H^{s_0}_{m_0}})-\theta_R(\left\|f^{j-1}\right\|_{H^{s_0}_{m_0}})\\
&\qquad=\int_0^1\theta_R'(z\left\|f^j\right\|_{H^{s_0}_{m_0}}+(1-z)\left\|f^{j-1}\right\|_{H^{s_0}_{m_0}})(\left\|f^j\right\|_{H^{s_0}_{m_0}}-\left\|f^{j-1}\right\|_{H^{s_0}_{m_0}})\mathrm{d}z.
\end{aligned}
$$

Therefore, for $t < \tau_n$

$$\begin{aligned}\mathcal{N}_2 &\lesssim \left| \left\| f^j \right\|_{H^{s_0}_{m_0}} - \left\| f^{j-1} \right\|_{H^{s_0}_{m_0}} \right| \left\| f^j \right\|_{L^2_{m_0}} \left\| f^j \right\|_{H^{s_0+1}_{m_0}} \left\| f^{j+1} - f^j \right\|_{H^{s_0}_{m_0}} \\ &\lesssim \left\| f^j - f^{j-1} \right\|_{H^{s_0}_{m_0}} \left\| f^j \right\|_{L^2_{m_0}} \left\| f^j \right\|_{H^{s_0+1}_{m_0}} \left\| f^{j+1} - f^j \right\|_{H^{s_0}_{m_0}} \\ &\lesssim n^2 j^{2\delta} \left\| f^j - f^{j-1} \right\|_{H^{s_0}_{m_0}} \left\| f^{j+1} - f^j \right\|_{H^{s_0}_{m_0}} .\end{aligned}$$

Integrating in time and using the BDG inequality as above, we obtain (noting that $f^{j+1}$ and $f^j$ have the same initial data) for $t < \tau_n$:

$$\begin{aligned}\mathbf{E} \sup_{s<t} \|f^{j+1}(s) - f^j(s)\|^2_{H^{s_0}_{m_0}} \leq & C \int_0^t \mathbf{E} \|f^{j+1}(s) - f^j(s)\|^2_{H^{s_0}_{m_0}} \, \mathrm{d}s \\ & + C_0 n^4 j^{4\delta} \mathbf{E} \int_0^t \left\| f^j(s) - f^{j-1}(s) \right\|^2_{H^{s_0}_{m_0}} \, \mathrm{d}s \\ & + \frac{1}{2} \mathbf{E} \sup_{s \leq t} \|f^{j+1}(s) - f^j(s)\|^2_{H^{s_0}_{m_0}}\end{aligned}$$

By the inductive hypothesis

$$\begin{aligned}C n^4 j^{4\delta} \mathbf{E} \int_0^t \left\| f^j(s) - f^{j-1}(s) \right\|^2_{H^{s_0}_{m_0}} \, \mathrm{d}s &\leq C_0 n^4 j^{4\delta} \int_0^t \frac{(C_0 n^4 s)^j j^{4\delta j}}{j!} \mathrm{d}s \\ &\leq \frac{(C_0 n^4 t)^{j+1} j^{4\delta(j+1)}}{(j+1)!},\end{aligned}$$

and so we have verfied (10.21).

Finally, we show that (10.21) implies that $\{f^j\}$ is Cauchy in $L^2_\omega L^2_t([0, \tau_n]; H^{s_0}_{m_0})$. Indeed,

let $k < \ell$ and

$$\mathbf{E} \sup_{s<t\wedge\tau_n} \left\|f^\ell - f^k\right\|^2_{H^{s_0}_{m_0}} \leq \sum_{j=k}^{\ell} \frac{(K_0 n^4 t)^j j^{4\delta j}}{j!}.$$

Hence, if we choose $k > (2C_0 n^4 t)^{1/\delta}$, then

$$\mathbf{E} \sup_{s<t\wedge\tau_n} \left\|f^\ell - f^k\right\|^2_{H^{s_0}_{m_0}} \leq \sum_{j=k}^{\ell} \frac{1}{2^{-\delta j}} \frac{k^{j\delta} j^{4\delta j}}{j!} \leq \sum_{j=k}^{\ell} \frac{1}{2^{-\delta j}} \frac{j^{5\delta j}}{j!}.$$

By Stirling's formula we have the following uniformly in $j$ (using $5\delta < 1$),

$$\frac{j^{5\delta j}}{j!} \lesssim_\delta 1,$$

therefore

$$\mathbf{E} \sup_{s<t\wedge\tau_n} \left\|f^\ell - f^k\right\|^2_{H^{s_0}_{m_0}} \lesssim \sum_{j=k}^{\ell} \frac{1}{2^{-\delta j}} \lesssim \frac{1}{2^{-\delta k}}.$$

We conclude that the sequence is Cauchy as claimed in the lemma. □

**Lemma 10.7.** *For each $n$, the iterates $\{f^j\}_{j=1}^\infty$ converge uniformly in $H^{s_0}_{m_0}$ on compact subintervals of $[0, \tau_n]$ to a strong pathwise solution of the SPDE (9.2) on the set $\{\tau_n > 0\} \subset \Omega$.*

*Proof.* Consider only $\omega \in \{\tau_n > 0\} \subset \Omega$. Let $f$ be the limiting process of the $f^j$ in $L^2_\omega C([0, \tau_n]; H^{s_0}_{m_0})$ - whose existence is guaranteed by Lemma 10.6. We will show that each term in (10.1) converges

to the corresponding term in (9.2). The convergence of the linear terms is straightforward:

$$\mathbf{E}\sup_{t\le T\wedge\tau_n}\left\|\int_0^t v\cdot\nabla_x(f^{j+1}-f)\mathrm{d}s\right\|^2_{H^{s_0-1}_{m_0-1}}\lesssim_{m_0} T^2\mathbf{E}\sup_{t\le T\wedge\tau_n}\|f^{j+1}-f\|^2_{H^{s_0}_{m_0}}\to 0,$$

$$\mathbf{E}\sup_{t\le T\wedge\tau_n}\left\|\int_0^t \Delta_v(f^{j+1}-f)\mathrm{d}s\right\|^2_{H^{s_0-2}_{m_0}}\lesssim T^2\mathbf{E}\sup_{t\le T\wedge\tau_n}\|f^{j+1}-f\|^2_{H^{s_0}_{m_0}}\to 0,$$

$$\mathbf{E}\sup_{t\le T\wedge\tau_n}\left\|\int_0^t \mathrm{div}_v(f^{j+1}v-fv)\mathrm{d}s\right\|^2_{H^{s_0-1}_{m_0-1}}\lesssim_{m_0} T^2\mathbf{E}\sup_{t\le T\wedge\tau_n}\|f^{j+1}-f\|^2_{H^{s_0}_{m_0}}\to 0,$$

$$\mathbf{E}\sup_{t\le T\wedge\tau_n}\left\|\int_0^t \sum_k(\sigma_k e_k\cdot\nabla_v)^2(f^{j+1}-f)\mathrm{d}s\right\|^2_{H^{s_0-2}_{m_0}}\lesssim T^2\mathbf{E}\sup_{t\le T\wedge\tau_n}\|f^{j+1}-f\|^2_{H^{s_0}_{m_0}}\to 0.$$

For the nonlinear electric field terms, we have:

$$\mathbf{E}\sup_{t\le T\wedge\tau_n}\left\|\int_0^t(\theta_R^j\varphi_\epsilon * E^j\cdot\nabla_v f^j-\theta_R\varphi_\epsilon * E\cdot\nabla_v f)\mathrm{d}s\right\|_{H^{s_0-1}_{m_0}}\lesssim \mathcal{N}_1+\mathcal{N}_2+\mathcal{N}_3,$$

where:

$$\mathcal{N}_1:=\mathbf{E}\sup_{t\le T\wedge\tau_n}\left\|\int_0^t(\theta_R^j-\theta_R)\varphi_\epsilon * E^j\cdot\nabla_v f^j\mathrm{d}s\right\|_{H^{s_0-1}_{m_0}},$$

$$\mathcal{N}_2:=\mathbf{E}\sup_{t\le T\wedge\tau_n}\left\|\int_0^t\theta_R\varphi\varphi_\epsilon * (E^j-E)\cdot\nabla_v f^j\mathrm{d}s\right\|_{H^{s_0-1}_{m_0}},$$

$$\mathcal{N}_3:=\mathbf{E}\sup_{t\le T\wedge\tau_n}\left\|\int_0^t\theta_R\varphi_\epsilon * E\cdot\nabla_v(f^j-f)\mathrm{d}s\right\|_{H^{s_0-1}_{m_0}}.$$

These terms are estimated as follows:

$$\begin{aligned}\mathcal{N}_1\lesssim_R&\mathbf{E}\sup_{t\le T\wedge\tau_n}\int_0^t\left|\|f^j\|_{H^{s_0}_{m_0}}-\|f\|_{H^{s_0}_{m_0}}\right|\|f^j\|^2_{H^{s_0}_{m_0}}\mathrm{d}s\\ \lesssim&\mathbf{E}\sup_{t\le T\wedge\tau_n}\left(\|f^j-f\|_{L^2([0,t];H^{s_0}_{m_0})}\|f^j\|^2_{L^4([0,T];H^{s_0}_{m_0})}\right)\end{aligned}$$

$$\to 0,$$

$$\mathcal{N}_2 \lesssim \mathbf{E} \sup_{t\le T\wedge\tau_n} \left( \left\| f^j - f \right\|_{L^2([0,t];H^{s_0}_{m_0})} \|f^j\|_{L^2([0,t];H^{s_0}_{m_0})} \right) \to 0,$$

$$\mathcal{N}_3 \lesssim \mathbf{E} \sup_{t\le T\wedge\tau_n} \left( \|f\|_{L^2([0,t];H^{s_0}_{m_0})} \|f^j - f\|_{H^{s_0}_{m_0}} \right) \to 0.$$

Lastly, for the martingale terms we use the BDG inequality:

$$\begin{aligned}
\mathbf{E} \sup_{t\le T\wedge\tau_n} \left\| \int_0^t \sum_k \sigma_k e_k \cdot \nabla_v (f^{j+1} - f) \cdot \mathrm{d}W_s^k \right\|^2_{H^{s_0-1}_{m_0}} \lesssim & \mathbf{E} \int_0^T \|f^{j+1} - f\|^2_{H^{s_0}_{m_0}} \,\mathrm{d}s \\
\lesssim & T^2 \mathbf{E} \sup_{t\le T\wedge\tau_n} \|f^{j+1} - f\|^2_{H^{s_0}_{m_0}} \\
\to & 0.
\end{aligned}$$

Combining the above, we see that $f$ is a solution of (9.2). □

**Corollary 10.8.** *There exists a global, strong pathwise solution of the SPDE* (9.2) *such that* $\forall p \in [2,\infty)$, $f \in L^p_\omega C_{t,loc} H^{\sigma'-3}_{m'-2} \cap L^p_\omega L^\infty_{t,loc} H^{\sigma'}_{m'}$.

*Proof.* By sending $n \to \infty$ and using that $\tau_n$ is a non-decreasing sequence such that $\lim_{n\to\infty} \mathbb{P}(\tau_n > 1) = 1$ we see that almost-surely, $\{f^j\}_{j=1}^\infty$ converges uniformly in $H^{s_0}_{m_0}$ on compact subintervals of $[0,1)$ to a limiting function $f \in C_t([0,1); H^{s_0}_{m_0})$. By Sobolev interpolation, and uniform boundedness in $L^\infty_{t,\mathrm{loc}} H^{\sigma'}_{m'}$, we obtain similar uniform convergence in $H^{s''}_{m''}$ for all $0 \le s'' < s'$ and $m'' < m'$. At the same time, the uniform bounds on $\{f^j\}$ from Lemma 10.1 imply that

$\forall p \in [2,\infty)$, $f \in L^p_\omega C_t H^{\sigma'-3}_{m'-2} \cap L^p_\omega L^\infty_{t,loc} H^{\sigma'}_{m'}$ by the lower semicontinuity of weak convergence. By Lemma 10.7, the limiting function $f$ is also a solution of (9.2). Now, we simply iterate the construction starting at $t = 1/2, 3/2, ...$ to obtain the existence of a global solution satisfying the desired bounds. □

The following lemma proves uniqueness of solutions to (9.2), thus completing the proof of Lemma 9.4.

**Lemma 10.9.** *Let $f, f'$ be two global pathwise solutions to (9.2) on the same stochastic basis with $f(0) = f'(0) = f_0$ for some $\mathcal{F}_0$-measurable $f_0$ with $\mathbf{E}\|f_0\|^p_{H^{\sigma'}_{m'}} < \infty$ for some $p > 2$ and such that for all $\epsilon > 0$, $f, f' \in L^{p-\epsilon}_\omega C_t H^{\sigma'-3}_{m'-2} \cap L^{p-}_\omega L^\infty_{t,loc} H^{\sigma'}_{m'}$. Then $f, f'$ are indistinguishable, that is:*

$$\mathbf{P}\left(f(t) = f'(t) \text{ for all } 0 \le t\right) = 1.$$

*Proof.* This is proved by an energy estimate on $\|f - f'\|^2_{H^{\sigma'-1}_{m'-1}}$. Similarly to the proof of Lemma 10.1, for $|\alpha| + |\beta| \le \sigma' - 1$ we have:

$$\begin{aligned}
\mathrm{d}\|\partial_x^\alpha \partial_v^\beta (f - f')\|^2_{L^2_{m'-1}} \le & C\|f - f\|^2_{H^{\sigma'-1}_{m'-1}} \mathrm{d}t \\
& - 2\left\langle \partial_x^\alpha \partial_v^\beta (\theta_R \varphi_\epsilon * E \cdot \nabla_v f - \theta'_R \varphi_\epsilon * E' \cdot \nabla_v f'), \partial_x^\alpha \partial_v^\beta (f - f')\right\rangle_{m'-1} \mathrm{d}t \\
& - 2\int_0^t \left\langle \partial_x^\alpha \partial_v^\beta (\nabla_v (f - f') \cdot \mathrm{d}W_t), \partial_x^\alpha \partial_v^\beta (f - f')\right\rangle_{m'-1}. \qquad (10.22)
\end{aligned}$$

We split the electric field contributions as:

$$\left\langle \partial_x^\alpha \partial_v^\beta (\theta_R \varphi_\epsilon * E \cdot \nabla_v f - \theta'_R \varphi_\epsilon * E' \cdot \nabla_v f'), \partial_x^\alpha \partial_v^\beta (f - f')\right\rangle_{m'-1} = \mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_3,$$

where:

$$\mathcal{N}_1 := \left\langle (\theta_R - \theta'_R)\partial_x^\alpha \partial_v^\beta (\varphi_\epsilon * E \cdot \nabla_v f), \partial_x^\alpha \partial_v^\beta (f - f') \right\rangle_{m'-1},$$

$$\mathcal{N}_2 := \left\langle \theta'_R \partial_x^\alpha \partial_v^\beta (\varphi_\epsilon * (E - E') \cdot \nabla_v f), \partial_x^\alpha \partial_v^\beta (f - f') \right\rangle_{m'-1},$$

$$\mathcal{N}_3 := \left\langle \theta'_R \partial_x^\alpha \partial_v^\beta (\varphi_\epsilon * E' \cdot \nabla_v (f - f')), \partial_x^\alpha \partial_v^\beta (f - f') \right\rangle_{m'-1}.$$

These are estimated as follows:

$$|\mathcal{N}_1| \lesssim_R \|f\|^2_{H^{\sigma'}_{m'}} \|f - f'\|^2_{H^{\sigma'-1}_{m'-1}}, \tag{10.23}$$

$$|\mathcal{N}_2| \lesssim \|f\|_{H^{\sigma'}_{m'}} \|f - f\|^2_{H^{\sigma'-1}_{m'-1}}, \tag{10.24}$$

$$|\mathcal{N}_3| \lesssim_R \|f - f'\|^2_{H^{\sigma'-1}_{m'-1}}, \tag{10.25}$$

where in (10.23) we used the mean value theorem for $\theta_R$ and (10.5) , in (10.24) we used (10.5), and in (10.25) we used (10.6) - in addition to Lemma 10.2 for each electric field.

Now, fix $K > 0$. Since $f, f' \in L^{p-}_\omega L^\infty_{t,loc} H^{\sigma'}_{m'}$, the stopping time:

$$\xi_K = \inf\{t \geq 0 : \sup_{s \leq t} \|f(s)\|_{H^{\sigma'}_{m'}} + \sup_{s \leq t} \|f'(s)\|_{H^{\sigma'}_{m'}} > K\} \wedge \tau \wedge \tau'$$

is almost surely finite. Even though it is not clear that $\xi_K$ is almost surely positive in general, for almost every $\omega \in \Omega$ there exists $K > 0$ such that $\xi_K > 0$, and in addition $\xi_K \to \tau \wedge \tau'$ $\mathbf{P}$–a.s. as $K \to \infty$. With this in mind, we fix $T > 0$ and use (10.23)-(10.25) and the BDG inequality in

(10.22), to obtain:

$$\begin{aligned}\mathbf{E}\sup_{s\leq t\wedge\xi_K}\|f(s)-f'(s)\|^2_{H^{\sigma'-1}_{m'-1}}\lesssim&\mathbf{E}\int_0^t\sup_{s\leq s'\wedge\xi_K}\|f(s)-f'(s)\|^2_{H^{\sigma'-1}_{m'-1}}\mathrm{d}s'\\&+\mathbf{E}\left(\int_0^t\sup_{s\leq s'\wedge\xi_K}\|f(s)-f'(s)\|^4_{H^{\sigma'-1}_{m'-1}}\mathrm{d}s'\right)^{\frac{1}{2}}\\\leq&C\mathbf{E}\int_0^t\sup_{s\leq s'\wedge\xi_K}\|f(s)-f'(s)\|^2_{H^{\sigma'-1}_{m'-1}}\mathrm{d}s'\\&+\frac{1}{2}\mathbf{E}\sup_{s\leq t\wedge\xi_K}\|f(s)\|^2_{H^{\sigma'-1}_{m'-1}},\end{aligned}$$

for all $t\leq T$, whereby the usual rearrangement and Grönwall's lemma give:

$$\mathbf{E}\sup_{s\leq T\wedge\xi_K}\|f(s)-f'(s)\|^2_{H^{\sigma'-1}_{m'-1}}=0.$$

Taking $K\to\infty$ and then $T\to\infty$, the conclusion follows. □

## 10.2 Proof of Lemma 9.1

Next, we want to pass to the limit $\epsilon\to 0$, for which we need uniform-in-$\epsilon$ estimates similar to those of Lemma 10.1, but this time for a family $\{f_\epsilon\}_{\epsilon>0}$ of solutions to (9.2). Note that since $f_\epsilon\in L^\infty_{t,loc}H^{\sigma'}_{m'}\cap C_tH^{\sigma'-3}_{m'-2}$, the highest norm in which we know $f_\epsilon$ is continuous is $C_tH^{\sigma'-1}_{m'-1}$ - and thus we use this as the base for our estimates.

**Lemma 10.10.** *Let $f$ be a solution of (9.2). For $\alpha\in(0,\frac{1}{2})$, $p\geq 2$, we have the uniform in $\epsilon$ estimates:*

$$\mathbf{E}\sup_{t\leq T}\|f(t)\|^p_{H^{\sigma'-1}_{m'-1}}\lesssim_{p,T,R,f_0}1 \tag{10.26}$$

*and*

$$\mathbf{E}\|f\|^p_{W^{\alpha,p}([0,T];H^{\sigma'-3}_{m'-2})} \lesssim_{p,T,R,f_0} 1. \tag{10.27}$$

*Proof.* The proof proceeds by induction in the number of derivatives[5]. The inductive hypothesis is that for $s > d/2$ derivatives on a solution $f$ of (9.2), we have:

$$\mathbf{E}\sup_{t'\leq T}\|f\|^p_{H^s_{m'-1}} \lesssim_{p,R,T,f_0} 1.$$

We show that this implies the same estimate for $s+1$. Begin by using Itô's formula on $\|\partial_x^\alpha\partial_v^\beta f\|^2_{L^2_{m'-1}}$ for $|\alpha|+|\beta| = s+1$, where similarly to (10.9) we obtain:

$$\begin{aligned}
\mathrm{d}\|\partial_x^\alpha\partial_v^\beta f\|^2_{L^2_{m'-1}} =& -2\left\langle\partial_x^\alpha\partial_v^\beta(v\cdot\nabla_x f),\partial_x^\alpha\partial_v^\beta f\right\rangle_{m'-1}\mathrm{d}t\\
&+2\left\langle\Delta_v\partial_x^\alpha\partial_v^\beta f,\partial_x^\alpha\partial_v^\beta f\right\rangle_{m'-1}\mathrm{d}t\\
&+2\left\langle\partial_x^\alpha\partial_v^\beta(\mathrm{div}_v(fv)),\partial_x^\alpha\partial_v^\beta f\right\rangle_{m'-1}\mathrm{d}t\\
&-2\left\langle\theta_R(\|f\|_{H^{s_0}_{m_0}})\partial_x^\alpha\partial_v^\beta(\varphi_\epsilon * E\cdot\nabla_v f),\partial_x^\alpha\partial_v^\beta f\right\rangle_{m'-1}\mathrm{d}t\\
&-2\left\langle\partial_x^\alpha\partial_v^\beta(\nabla_v f\cdot\mathrm{d}W_t),\partial_x^\alpha\partial_v^\beta f\right\rangle_{m'-1}\\
&+\sum_k\left\langle\partial_x^\alpha\partial_v^\beta[(\sigma_k e_k\cdot\nabla_v)^2 f],\partial_x^\alpha\partial_v^\beta f\right\rangle_{m'-1}\mathrm{d}t\\
&+\sum_k\|\partial_x^\alpha\partial_v^\beta(\sigma_k e_k\cdot\nabla_v f)\|^2_{L^2_{m'-1}}\\
=&\mathcal{T}_{\alpha,\beta}(f)+\mathcal{D}_{\alpha,\beta}(f)+\mathcal{F}_{\alpha,\beta}(f)+\mathcal{N}_{\alpha,\beta}(f)+\mathcal{M}_{\alpha,\beta}(f)\\
&+\mathcal{C}_{\alpha,\beta}(f).
\end{aligned}\tag{10.28}$$

The linear terms are treated as in the proof of Lemma 10.1, and the only term that requires new

[5] see for instance [81] for similar inductive energy estimates for the relativistic Vlasov–Maxwell system.

attention is $\mathcal{N}_{\alpha,\beta}(f)$. By the classical Gagliardo-Nirenberg inequality (see e.g. [Proposition A.3 [102]] we have:

$$
\begin{aligned}
&\left|\left\langle \partial_x^\alpha \partial_v^\beta(\varphi_\epsilon * E\cdot\nabla_v f), \partial_x^\alpha\partial_v^\beta f\right\rangle_{m'-1}\right| \\
\leq & C\|\varphi_\epsilon * E\|_{W^{1,\infty}}\|f\|^2_{H^{s+1}_{m'-1}}\,\mathrm{d}t \\
&+\sum_{\substack{\gamma<\alpha\\ |\alpha-\gamma|\geq 2}} \|\partial_x^{\alpha-\gamma}\varphi_\epsilon * E\|_{L_x^{2\frac{s}{|\alpha-\gamma|-1}}}\|\nabla_v\partial_v^\beta\partial_x^\gamma f\|_{L^2_{v,m}L_x^{2\frac{s}{|\beta|+|\gamma|+1}}}\|\partial_x^\alpha\partial_v^\beta f\|_{L^2_{m'-1}}\mathrm{d}t \\
\leq & C\|\varphi_\epsilon * E\|_{W^{1,\infty}}\|f\|^2_{H^{s+1}_{m'-1}}\,\mathrm{d}t \\
&+C\sum_{\substack{\gamma<\alpha\\ |\alpha-\gamma|\geq 2}} \|\nabla_x\varphi_\epsilon * E\|_{L_x^\infty}^{\frac{|\beta|+|\gamma|+1}{s}}\|\varphi_\epsilon * E\|_{H_x^{s+1}}^{\frac{|\alpha-\gamma|-1}{s}}\|\nabla_v\partial_v^\beta f\|^{1-\eta}_{L^2_{m'-1}}\|f\|^{1+\eta}_{L^2_{m'-1}}\mathrm{d}t, \qquad (10.29)
\end{aligned}
$$

where for fixed $\gamma$, the interpolation parameter $\eta$ is given by:

$$
\begin{aligned}
\eta =& \frac{|\gamma|}{|\alpha|-1}+\frac{d}{|\alpha|-1}\left(\frac{1}{2}-\frac{|\beta|+|\gamma|+1}{2s}\right)\\
=& \frac{|\gamma|}{|\alpha|-1}+\frac{d}{|\alpha|-1}\cdot\frac{|\alpha-\gamma|-1}{2s}
\end{aligned}
$$

and thus $\eta<1$ provided $s>\frac{d}{2}$. By Young's inequality and (10.29) it follows that

$$
\mathcal{N}_{\alpha,\beta}\leq C\left(1+\|f\|^p_{H^s_{m'-1}}\right)\mathrm{d}t+CR\|f\|^2_{H^{s+1}_{m'-1}}\,\mathrm{d}t.
$$

Plugging this back into (10.28), and using the same procedure as in the proof of Lemma 10.1, we obtain:

$$
\mathrm{d}\|f\|^2_{H^{s+1}_{m'-1}}\leq C(1+R)\|f\|^2_{H^{s+1}_{m'-1}}\,\mathrm{d}t+\sum_{|\alpha|+|\beta|=s+1}\mathcal{M}_{\alpha,\beta}(f)+\mathcal{M}_{0,0}(f)
$$

$$+ C(1 + \|f\|^p_{H^s_{m'-1}})\mathrm{d}t.$$

We again integrate in time and apply the BDG inequality as in the proof of (10.2) for $p = 2$, where the only difference is the term $\|f\|^p_{H^s_{m'-1}}$, which is now controlled by the inductive hypothesis, and we get:

$$\mathbf{E} \sup_{t \leq T} \|f\|^2_{H^{s+1}_{m'-1}} \lesssim_{R,p,T,f_0} 1.$$

With this in hand, we can directly transfer the proof of (10.2) for $p > 2$ and obtain (10.26). Then the same argument as the proof of (10.3) (i.e. using the variant of BDG from [53, Lemma 2.1]) gives (10.27).

The last thing that remains is to demonstrate the inductive base of the preceding scheme. Here this is done by first estimating the $H^2_{m'}$ norm of $f$. This is sufficient to start the inductive scheme above in $1 \leq d \leq 3$ as $2 > d/2$. As the linear terms are always controlled in the same way, we only focus on the electric field contributions. As always, we have:

$$\left|\left\langle \partial_x^\alpha \partial_v^\beta (E \cdot \nabla_v f), \partial_x^\alpha \partial_v^\beta f \right\rangle_{m'}\right| \leq C\|E\|_{W^{1,\infty}} \|f\|^2_{H^2_{m'}} + \sum_{\substack{\gamma < \alpha \\ |\alpha - \gamma| \geq 2}} \left\langle \partial_x^{\alpha-\gamma} E \cdot \nabla_v \partial_v^\beta \partial_x^\gamma f, \partial_x^\alpha \partial_v^\beta f \right\rangle_{m'},$$

but since only two derivatives are acting on $f$ at this point, the terms in the summation are only present when $|\alpha| = 2$ and $|\beta| = |\gamma| = 0$. Let $q = 4$ in $d = 1, 2$, and for $d \geq 3$ let $q$ be arbitrary such that $2 < q < \frac{2d}{d-2}$. Then by Hölder's inequality and Sobolev embeddings we have

$$|\langle \partial_x^\alpha E \cdot \nabla_v f, \partial_x^\alpha f \rangle_{m'}| \leq \|\nabla_x^2 E\|_{L_x^{\frac{q}{q-1}}} \|\nabla_v f\|_{L^2_{v,m} L^q_x} \|f\|_{H^2_{m'}}$$

$$\lesssim \|\nabla_x^2 E\|_{H_x^{\frac{d(q-2)}{2q}}} \|f\|_{H^2_{m'}}^2$$

$$\lesssim \|f\|_{H^{s_0}_{m_0}} \|f\|_{H^2_{m'}}^2, \tag{10.30}$$

where we have used that $s_0 > \frac{d}{2} + 1 > \frac{d(q-2)}{2q}$ and the embedding $H_x^1 \subset L_x^q$ which holds for all $d \geq 1$ due to our choice of $q$. From this point on the procedure is the same as in the inductive step. We plug (10.30) into (10.28) for $\alpha, \beta$ with $|\alpha| + |\beta| = 2$, sum over all such $\alpha, \beta$ as well as the case when $\alpha = \beta = 0$, integrate in time, apply the BDG inequality and Grönwall's lemma and obtain:

$$\mathbf{E} \sup_{t \leq T} \|f(t)\|_{H^2_{m'}}^2 \lesssim_{T,R,f_0} 1.$$

Then applying the same argument as in the proof of (10.2) for $p > 2$, we also obtain for $p > 2$:

$$\mathbf{E} \sup_{t \leq T} \|f(t)\|_{H^2_{m'}}^p \lesssim_{p,T,R,f_0} 1.$$

This provides the inductive base and therefore the proof of the lemma is complete for $1 \leq d \leq 3$. □

For solutions to (9.2), it is unclear how to prove $\{f^\epsilon\}_{\epsilon > 0}$ forms a Cauchy sequence as $\epsilon \to 0$. Instead, we employ a standard procedure based on the Skorokhod embedding theorem (see e.g. [67]) to produce probabilistically weak (called *martingale*) solutions on a new stochastic basis, and then upgrade them to probabilistically strong using the Gyöngy–Krylov lemma from [59] (see Lemma 10.12 below). We let $(\epsilon_n)_{n=1}^\infty$ be a decreasing sequence of positive numbers with $\epsilon_n \to 0$ as $n \to \infty$ and define the corresponding sequence $f_n := f^{\epsilon_n}$ of solutions to (9.2), which we have shown satisfy the uniform bounds (10.26) and (10.27). For $\alpha \in (0, \frac{1}{2})$ and $p > 2$ such

that $\alpha p > 1$, we define the pathspace

$$\mathbf{X} := W^{\alpha,p}_{loc}([0,\infty); H^{\sigma'-3}_{m'-2}) \cap L^{\infty}_{loc}([0,\infty); H^{\sigma'-1}_{m'-1}).$$

Recall that since $\alpha p > 1$ and $H^{\sigma'-3}_{m'-2} \subset H^{\sigma'-4}_{m'-3}$ compactly, from [53, Theorem 2.2], we have:

$$W^{\alpha,p}_{loc}([0,\infty); H^{\sigma'-3}_{m'-2}) \subset C_t H^{\sigma'-4}_{m'-3}.$$

By the uniform estimates (10.26) and (10.27), the laws $\nu^n := \mathcal{L}(f_n)$ are bounded in probability in $\mathbf{X}$, and thus they are *tight* in the smaller pathspace

$$\mathbf{X}_c := C([0,\infty); H^{\sigma'-4}_{m'-3}) \cap L^{\infty}_{loc}([0,\infty); H^{\sigma'-1}_{m'-1}).$$

Note that the tightness in $L^{\infty}_{loc}([0,\infty); H^{\sigma'-1}_{m'-1})$ is in the weak-$\star$ topology. We now use this to obtain a martingale solution to (9.1) in high regularity.

**Proposition 10.11.** *Let $\mu_0$ be a probability measure on $H^{\sigma'}_{m'}$ so that $\int_{H^{\sigma'}_{m'}} \|f\|^p_{H^{\sigma'}_{m'}} \mathrm{d}\mu_0(f)$ for some $p > 2$. Then there exists a stochastic basis $\tilde{S} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde{\mathbf{P}})$ and a predictable process*

$$\tilde{f} \in L^{p-}_{\omega} C_t H^{\sigma'-4}_{m'-3} \cap L^{p-}_{\omega} L^{\infty}_{t,loc} H^{\sigma'-1}_{m'-1}$$

*such that $\mathcal{L}(\tilde{f}(0)) = \mu_0$ and $\tilde{f}$ solves (9.1) in the sense that, there is a sequence of i.i.d Brownian motions $\left\{\widetilde{W}^{(k)}_t\right\}$ such that the following equality holds $\mathbf{P}$–a.s. in $C([0,\infty); H^{\sigma'-3}_{m'-2})$*

$$\tilde{f}(t) = f_0 + \int_0^t \left(-v \cdot \nabla_x \tilde{f}(s) - \theta_R(\|f\|_{H^{s_0}_{m_0}}) \tilde{E}(s) \cdot \nabla_v \tilde{f}(s) + \nu \mathcal{L} \tilde{f}(s)\right) \mathrm{d}s - \int_0^t \nabla_v \tilde{f}(s) \circ \mathrm{d}\widetilde{W}_s,$$

*with*

$$\tilde{E} = \nabla_x(-\Delta_x)^{-1}\left(\int_{\mathbb{R}^d} \tilde{f}(t,\cdot,v)\mathrm{d}v - 1\right).$$

*Proof.* Let $\mu^n = \mathcal{L}(f_n, \mathcal{W})$ in $\mathbf{X}_c \times C([0,\infty);\mathfrak{U}_0)$. The sequence $(\mu^n)_{n=1}^{\infty}$ is tight by the uniform estimates (10.26) and (10.27) combined with the fact that its projection onto $C([0,\infty);\mathfrak{U}_0)$ is the same for each $n$. By Prokhorov's theorem, $(\mu^n)_{n=1}^{\infty}$ has a weakly convergent subsequence - reindexed to $\mu^n$. By Skorokhod's embedding theorem, there exists a new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$ and on it random elements $(\tilde{f}_n, \tilde{\mathcal{W}}_n)$ with laws $\mu^n$ which converge $\tilde{\mathbf{P}}$–a.s. to some limit $(\tilde{f}, \tilde{\mathcal{W}})$ in the product topology of $\mathbf{X}_c \times C([0,\infty);\mathfrak{U}_0)$. Then by a variation of the mollification technique employed in the proof of [14, Equation 4.17] [6] the random elements $(\tilde{f}_n, \tilde{\mathcal{W}}_n)$ satisfy (9.2) just like $(f_n, \mathcal{W})$, but in the new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$:

$$\tilde{f}_n(t) + \int_0^t \left(v\cdot\nabla_x\tilde{f}_n + \tilde{E}_n\cdot\nabla_v\tilde{f}_n - \frac{1}{2}\sum_k(\sigma_k e_k\cdot\nabla_v)^2\tilde{f}_n - \nu\mathcal{L}\tilde{f}_n\right)\mathrm{d}s + \int_0^t \nabla_v\tilde{f}_n\cdot\mathrm{d}\tilde{W}_t^n = \tilde{f}_n(0),$$

where we have denoted by $\tilde{W}_t^n$ the external electric field corresponding to $\tilde{\mathcal{W}}_n$; that is to say, if:

$$\tilde{\mathcal{W}}_n = \sum_k g_k\tilde{W}_k^n,$$

then $\tilde{W}^n$ is simply given by:

$$\tilde{W}^n = \sum_k \sigma_k e_k\tilde{W}_k^n.$$

[6] See also [22, Proposition 3.2, (iii)] for an application to the primitive equations where the noise is present as a stochastic transport term.

The passage to the limit $n \to \infty$ in the SPDEs (9.2) satisfied by $(\tilde{f}_n, \tilde{\mathcal{W}}_n)$ to obtain that the limit $(\tilde{f}, \tilde{\mathcal{W}})$ solves (9.1) can be carried out by combining the convergences $\tilde{f}_n \to \tilde{f}$ and $\tilde{\mathcal{W}}_n \to \tilde{\mathcal{W}}$ with [37, Lemma 2.1], so we omit the proof for technicalities. We simply note that the presence of the transport noise does not cause any additional difficulties in our setting. □

As our goal is to construct solutions in $\mathcal{S}$, we need to "upgrade" the martingale solutions of the preceding lemma to probabilistically strong solutions. With the above in mind, we now state the Gyöngy–Krylov lemma from [59], which will allow us to combine the tightness of $(f_n)_{n=1}^{\infty}$ with the pathwise uniqueness of the limit (Lemma 10.13 below) to show that in fact (9.1) has (unique) global solutions on the original stochastic basis $\mathcal{S}$.

**Lemma 10.12** (Gyöngy–Krylov)**.** *Let $(Y_n)_{n=1}^{\infty}$ be a sequence of $X$-valued random variables, where $X$ is a complete separable metric space. Then $(Y_n)_n$ converges in probability if and only if for every two subsequences $Y_{n_k}, Y_{l_k}$ the joint sequence $(Y_{n_k}, Y_{l_k})$ has a subsequence $(Y_{n_{k'}}, Y_{l_{k'}})$ whose laws converge weakly to a probability measure $\nu$ supported on the diagonal of $X \times X$:*

$$\nu\left(\{(x,y) \in X \times X : x = y\}\right) = 1.$$

With this lemma at hand, we now set to prove pathwise uniqueness of solutions to (9.1), which is the content of the following:

**Lemma 10.13.** *Let $f, f'$ be global solutions to (9.1) on the same stochastic basis with $f(0) = f'(0) = f_0$ almost surely, where $\mathbf{E}\|f_0\|_{H^{\sigma'}_{m'}}^p < \infty$ for some $p > 2$ and such that $f, f' \in L^{p-}_{\omega} C_t H^{\sigma'-4}_{m'-3} \cap L^{p-}_{\omega} L^{\infty}_{t,loc} H^{\sigma'-1}_{m'-1}$. Then $f, f'$ are indistinguishable, that is:*

$$\mathbf{P}\left(f(t) = f'(t) \text{ for all } t \geq 0\right) = 1. \tag{10.31}$$

*Proof.* First of all, notice that since $f, f' \in L^\infty_{t,loc} H^{\sigma'-1}_{m'-1}$, for $K > 0$ the stopping times

$$\xi_K := \inf\{t > 0 : \|f\|^2_{H^{\sigma'-1}_{m'-1}} + \|f'\|^2_{H^{\sigma'-1}_{m'-1}} \geq K\}$$

are well defined and satisfy $\xi_K \to \infty$ as $K \to \infty$, $\mathbf{P}$–almost surely. We now perform an energy estimate on $H^{s_0}_{m_0}$. We use Itô's formula on the quantity $\|\partial^\alpha_x \partial^\beta_v (f - f')\|^2_{L^2_{m_0}}$, for $|\alpha| + |\beta| = s_0$:

$$\begin{aligned}
\mathrm{d}\|\partial^\alpha_x \partial^\beta_v (f - f')\|^2_{L^2_{m'}} =& -2\left\langle \partial^\alpha_x \partial^\beta_v (v \cdot \nabla_x (f - f')), \partial^\alpha_x \partial^\beta_v (f - f')\right\rangle_{m_0} \mathrm{d}t \\
&+ 2\left\langle \Delta_v \partial^\alpha_x \partial^\beta_v (f - f'), \partial^\alpha_x \partial^\beta_v (f - f')\right\rangle_{m_0} \mathrm{d}t \\
&+ 2\left\langle \partial^\alpha_x \partial^\beta_v \operatorname{div}_v ((f - f')v), (f - f')\right\rangle_{m_0} \mathrm{d}t \\
&- 2\left\langle \partial^\alpha_x \partial^\beta_v (\theta_R E^f \cdot \nabla_v f - \theta'_R E^{f'} \cdot \nabla_v f'), \partial^\alpha_x \partial^\beta_v (f - f')\right\rangle_{m_0} \mathrm{d}t \\
&- 2\left\langle \partial^\alpha_x \partial^\beta_v (\nabla_v (f - f') \cdot \mathrm{d}W_t), \partial^\alpha_x \partial^\beta_v (f - f')\right\rangle_{m_0} \\
&+ \sum_k \left\langle \partial^\alpha_x \partial^\beta_v ((\sigma_k e_k \cdot \nabla_v)^2 (f - f')), \partial^\alpha_x \partial^\beta_v (f - f')\right\rangle_{m_0} \mathrm{d}t \\
&- \sum_k \|\partial^\alpha_x \partial^\beta_v (\sigma_k e_k \cdot \nabla_v (f - f'))\|^2_{L^2_{m_0}} \mathrm{d}t.
\end{aligned}$$

Clearly, all terms except those involving the electric fields can be estimated as in the proof of (10.2), so we only examine the electric field term:

$$\begin{aligned}
&\left|\left\langle \partial^\alpha_x \partial^\beta_v (\theta_R E^f \cdot \nabla_v f - \theta'_R E^{f'} \cdot \nabla_v f'), \partial^\alpha_x \partial^\beta_v (f - f')\right\rangle_{m_0}\right| \\
\leq& \left|(\theta_R - \theta'_R)\left\langle \partial^\alpha_x \partial^\beta_v (E^f \cdot \nabla_v f), \partial^\alpha_x \partial^\beta_v (f - f')\right\rangle_{m_0}\right| \\
&+ \left|\theta_{R'}\left\langle \partial^\alpha_x \partial^\beta_v ((E^f - E^{f'}) \cdot \nabla_v f'), \partial^\alpha_x \partial^\beta_v (f - f')\right\rangle_{m_0}\right| \\
&+ \left|\theta_{R'}\left\langle \partial^\alpha_x \partial^\beta_v (E^{f'} \cdot \nabla_v (f - f')), \partial^\alpha_x \partial^\beta_v (f - f')\right\rangle_{m_0}\right|
\end{aligned}$$

$$
\begin{aligned}
\leq & C_R \left| \|f\|_{H^{s_0}_{m_0}} - \|f'\|_{H^{s_0}_{m_0}} \right| \cdot \|E^f\|_{H^{s_0}_x} \|f - f'\|_{H^{s_0}_{m_0}} \\
& + C\|E^f - E^{f'}\|_{H^{s_0}} \|\nabla_v f\|_{H^{s_0}_{m_0}} \|f - f'\|_{H^{s_0}_{m_0}} \\
& + CR\|f - f\|^2_{H^{s_0}_{m_0}} \\
\lesssim & _{R,K} \|f - f'\|^2_{H^{s_0}_{m_0}}
\end{aligned}
$$

for $t \leq \xi_K$, where we have used the mean value theorem on $\theta_R$ and Lemma 10.3. Thus, arguing similarly to the proof of Lemma 10.9 (i.e. by the BDG and Grönwall inequalities), we have:

$$
\mathbf{E} \sup_{t' \leq t \wedge \xi_K \wedge T} \|f - f'\|^2_{H^{s_0}_{m_0}} \leq C_{R,K,T} \|f(0) - f'(0)\|^2_{H^{s_0}_{m_0}} = 0.
$$

Since $\xi_K \to \infty$ as $K \to \infty$, the monotone convergence theorem implies that for all $T > 0$ we have:

$$
\mathbf{E} \sup_{t \leq T} \|f(t) - f'(t)\|^2_{H^{s_0}_{m_0}} = 0,
$$

which implies (10.31) since $T$ is arbitrary. □

We now have everything we need to (subsequentially) pass to the limit $\epsilon \to 0$ in the original stochastic basis.

*Proof of Lemma 9.1.* We define the joint laws $\mu^{n,l} = \mathcal{L}(f_n, f_l, \mathcal{W})$, Similarly to the discussion in Lemma 10.11, for any sequence $\mu^{n_k,l_k}$ with $n_k, l_k \to \infty$ as $k \to \infty$, by Prokhorov's theorem the estimates (10.26) and (10.27) (and the fact that $\{\mathcal{W}\}$ is a singleton) provide a weakly convergent subsequence of probability measures in $\mathbf{X}_c \times \mathbf{X}_c \times C([0,\infty); \mathfrak{U}_0)$, which we still denote (after relabelling) by $\mu^{n_k,l_k}$, and we denote its limit by $\mu$. By the Skorokhod embedding theorem, we can construct a new stochastic basis again denoted by $\tilde{\mathcal{S}} = (\tilde{\Omega}, \tilde{F}, \tilde{\mathbf{P}})$ and on it a sequence of

random elements $(\tilde{f}_{n_k}, \tilde{f}_{l_k}, \tilde{\mathcal{W}}^k)$ and $(\tilde{f}, \tilde{\tilde{f}}, \tilde{\mathcal{W}})$ such that $\mathcal{L}(\tilde{f}_{n_k}, \tilde{f}_{l_k}, \tilde{\mathcal{W}}^k) = \mu^{n_k, l_k}$, $\mathcal{L}(\tilde{f}, \tilde{\tilde{f}}, \tilde{\mathcal{W}})$ and:

$$(\tilde{f}_{n_k}, \tilde{f}_{l_k}, \tilde{\mathcal{W}}^k) \to (\tilde{f}, \tilde{\tilde{f}}, \tilde{\mathcal{W}}) \quad \text{in } \mathbf{X}_c \times \mathbf{X}_c \times C([0,\infty); \mathfrak{U}_0), \quad \tilde{\mathbf{P}}\text{–a.s..}$$

As in Lemma 10.11, $(\tilde{f}_{n_k}, \tilde{\mathcal{W}}^k)$ and $(\tilde{f}_{l_k}, \tilde{\mathcal{W}}^k)$ satisfy the SPDE (9.2) in the new stochastic basis (by the method of [14, Section 4.3.4]), so we can pass to the limit $k \to \infty$ in all the terms of (9.2) for $(\tilde{f}_{n_k}, \tilde{\mathcal{W}}^k)$ and $(\tilde{f}_{l_k}, \tilde{\mathcal{W}}^k)$ (using [37, Lemma 2.1] for the stochastic integrals) to show that $(\tilde{f}, \tilde{\mathcal{W}})$ $(\tilde{\tilde{f}}, \tilde{\mathcal{W}})$ are solutions to (9.1) on the new stochastic basis. Since $\mathbf{P}(f_{n_k}(0) = f_{l_k}(0)) = 1$, we also have $\tilde{\mathbf{P}}(\tilde{f}_{n_k}(0) = \tilde{f}_{l_k}(0)) = 1$, and thus in the limit $k \to \infty$ we obtain $\tilde{\mathbf{P}}(\tilde{f}(0) = \tilde{\tilde{f}}(0)) = 1$. Therefore, by Lemma 10.13, $\tilde{f}$ and $\tilde{\tilde{f}}$ are indistinguishable. This means that the measure $\bar{\mu}$, defined as the projection of $\mu$ onto the first two components $\mathbf{X}_c \times \mathbf{X}_c$, is in fact supported on the diagonal of $\mathbf{X}_c \times \mathbf{X}_c$. Thus, by Lemma 10.12, a subsequence of $(f_n)$ converges in probability in the original stochastic basis $\mathcal{S}$ in the topology of $\mathbf{X}_c'$ to a limiting process $f$ which solves (9.1). This concludes the proof of Lemma 9.1. □

## 11 Proof of the main theorem

In this section, with the results of Section 10.2 at hand, we prove the main result of the paper, Theorem 8.3. At first, we consider initial data $f_0$ satisfying $\|f_0\|_{H^\sigma_m} \leq M < \infty$ almost surely, for a fixed deterministic $M$. This assumption can be removed at the end by a cutting argument similar to that of Lemma 9.2. We treat the initial data $f_0$ with a sequence of regularization

and velocity cutoff operators $\mathcal{R}^n$, obtaining a sequence of regularized data

$$f_0^n := \mathcal{R}^n f_0,$$

defined as

$$\mathcal{R}^n f := \theta_n(v) n^{2d} \eta\left(\frac{\cdot}{n}\right) *_{x,v} f,$$

where $\eta \in C_c^\infty(\mathbb{R}^{2d})$ satisfies $\eta(x,v) \geq 0$ and $\iint \eta(x,v) \mathrm{d}v\mathrm{d}x = 1$. We note the following properties of this regularization. The proofs are standard and are omitted for brevity.

**Lemma 11.1.** *Let $s' \geq s \geq 0$ be integers and $m' \geq m \geq 0$ be arbitrary. Then,*

*(i) The regularization operators $\mathcal{R}^n$ are uniformly bounded on $H_m^\sigma$*

$$\sup_{n\geq 1} \|\mathcal{R}^n f\|_{H_m^\sigma} \lesssim_{m,\sigma} \|f\|_{H_m^\sigma}.$$

*(ii) The regularization operators satisfy: for $f \in H_m^\sigma$*

$$\|\mathcal{R}^n f\|_{H_{m'}^{\sigma'}} \lesssim n^{\sigma'-\sigma} n^{(m'-m)/2} \|f\|_{H_m^\sigma}.$$

*(iii) The regularization operators converge in the following senses: for $f \in H_m^\sigma$ there holds*

$$\begin{aligned} &\lim_{n\to\infty} \|\mathcal{R}^n f - f\|_{H_m^\sigma} = 0 \\ &\lim_{n\to\infty} n \|\mathcal{R}^n f - f\|_{H_m^{\sigma-1}} = 0. \end{aligned} \tag{11.1}$$

In the previous section, we showed that each of the $f_0^n$ generates a maximal solution $f^n$

of (8.1) in $C([0,\tau_n);H^\sigma_m)\cap L^\infty_{\text{loc}}([0,\tau_n);H^{\sigma'-1}_{m'-1})$, where $\tau_n$ is the maximal time of existence of $f^n$. We now show that the sequence $(f^n)_{n=1}^\infty$ of approximate solutions has a strongly convergent subsequence.

We start by defining the stopping times:

$$\tau_n^T := \inf\left\{t\geq 0:\ \|f^n(t)\|_{H^\sigma_m} > \|f_0^n\|_{H^\sigma_m}+2\right\}\wedge T,$$

$$\tau_{n,l}^T = \tau_n^T\wedge\tau_k^T$$

The following is similar to [82, Lemma 37] or [57, Lemma 7.1]:

**Lemma 11.2.** *Let $\tau_n$ be a sequence of stopping times and suppose that a sequence of predictable processes $f^n\in C([0,\tau_n];H^\sigma_m)$ satisfy:*

$$\lim_{n\to\infty}\sup_{l\geq n}\mathbf{E}\sup_{t'\leq\tau^T_{n,l}}\|f^n-f^l\|^2_{H^\sigma_m}=0, \tag{11.2}$$

$$\lim_{\epsilon\to 0}\sup_{n\geq 1}\mathbf{P}\left[\sup_{t\leq\tau_n^T\wedge\epsilon}\|f^n\|_{H^\sigma_m}>\|f_0^n\|_{H^\sigma_m}+1\right]=0. \tag{11.3}$$

*Then, there exists a stopping time $\tau$ with $\mathbf{P}(0<\tau\leq T)=1$, a predictable process $f\in C([0,\tau];H^\sigma_m)$, and a subsequence $(f^{n_j})_{j=1}^\infty$ of $(f^n)_{n=1}^\infty$ such that*

$$\sup_{t\leq\tau}\|f^{n_j}(t)-f(t)\|_{H^\sigma_m}\to 0 \text{ as } j\to\infty,\ \mathbf{P}\text{–a.s.}$$

*and*

$$\sup_{t\leq\tau}\|f(t)\|_{H^\sigma_m}\leq 2+\sup_n\|f_0^n\|_{H^\sigma_m},\ \mathbf{P}\text{–a.s.}$$

We now verify that the regularized solutions $\{f^n\}_{n\geq 1}$ satisfy the conditions of Lemma 11.2.

**Lemma 11.3.** *The solutions $(f^n)_{n=1}^{\infty}$ generated by the regularized data $(f_0^n)_{n=1}^{\infty}$ satisfy (11.2) and (11.3).*

*Proof.* We begin with proving that the Cauchy property (11.2) holds for the sequence $f^n$ of solutions to (9.1) with initial data $f_0^n$. This is done via an energy estimate with some similarities with the uniqueness and convergence proofs in Section 10.2 with one significant difference. For $t \leq \tau_{n,l}^T$, we have:

$$\begin{aligned}
\mathrm{d}\|\partial_x^\alpha \partial_v^\beta (f^n - f^l)\|_{L^2_m}^2 =& -2\left\langle \partial_x^\alpha \partial_v^\beta (v\cdot\nabla_x (f^n - f^l)), \partial_x^\alpha \partial_v^\beta (f^n - f^l)\right\rangle_m \mathrm{d}t \\
&+ 2\left\langle \Delta_v (\partial_x^\alpha \partial_v^\beta (f^n - f^l)), \partial_x^\alpha \partial_v^\beta (f^n - f^l)\right\rangle_m \mathrm{d}t \\
&+ 2\left\langle \partial_x^\alpha \partial_v^\beta \operatorname{div}_v ((f^n - f^l)v), \partial_x^\alpha \partial_v^\beta (f^n - f^l)\right\rangle_m \mathrm{d}t \\
&- 2\left\langle \partial_x^\alpha \partial_v^\beta (E^n\cdot\nabla_v f^n - E^l\cdot\nabla_v f^l), \partial_x^\alpha \partial_v^\beta (f^n - f^l)\right\rangle_m \mathrm{d}t \\
&- 2\left\langle \partial_x^\alpha \partial_v^\beta (\nabla_v (f^n - f^l)\cdot \mathrm{d}W_t), \partial_x^\alpha \partial_v^\beta (f^n - f^l)\right\rangle_m \\
&+ \sum_k \left\langle \partial_x^\alpha \partial_v^\beta (\sigma_k e_k\cdot\nabla_v)^2 (f^n - f^l), \partial_x^\alpha \partial_v^\beta (f^n - f^l)\right\rangle_m \mathrm{d}t \\
&+ \sum_k \|\partial_x^\alpha \partial_v^\beta (\sigma_k e_k\cdot\nabla_v (f^n - f^l))\|_{L^2_m}^2 \mathrm{d}t.
\end{aligned}$$

We will control the above for $|\alpha| + |\beta| = \sigma$. Of course, the linear terms are treated in the same way as in the estimates of Section 10.2. We now explain how the electric field terms are to be estimated. For $t \leq \tau_{n,l}$, we have:

$$\begin{aligned}
&\left|\left\langle \partial_x^\alpha \partial_v^\beta (E^n\cdot\nabla_v f^n - E^l\cdot\nabla_v f^l), \partial_x^\alpha \partial_v^\beta (f^n - f^l)\right\rangle_m\right| \\
\leq &\left|\left\langle \partial_x^\alpha \partial_v^\beta ((E^n - E^l)\cdot\nabla_v f^n), \partial_x^\alpha \partial_v^\beta (f^n - f^l)\right\rangle_m\right|
\end{aligned}$$

$$
\begin{aligned}
&+ \left|\left\langle \partial_x^\alpha \partial_v^\beta ((E^l \cdot \nabla_v (f^n - f^l)), \partial_x^\alpha \partial_v^\beta (f^n - f^l) \right\rangle_m\right| \\
\leq & C \|f^n - f^l\|_{H_m^{\sigma-1}} \|f^n\|_{H_m^{\sigma+1}} \|f^n - f^l\|_{H_m^\sigma} + C \|f^l\|_{H_m^{\sigma-1}} \|f^n - f^l\|_{H_m^\sigma}^2 \\
\leq & C \|f^n - f^l\|_{H_m^\sigma}^2 + \|f^n - f^l\|_{H_m^{\sigma-1}}^2 \|f^n\|_{H_m^{\sigma+1}}^2, \qquad (11.4)
\end{aligned}
$$

where we have used Lemma 10.3. Combining our estimates from the previous section with (11.4) and the fact that we are taking $t \leq \tau_{j,k}^T$, we have:

$$
\begin{aligned}
\mathrm{d}\left(\|f^n - f^l\|_{H_m^\sigma}^2\right) \leq & C \|f^n - f^l\|_{H_m^\sigma}^2 \mathrm{d}t + C \|f^n - f^l\|_{H_m^{\sigma-1}}^2 \|f^n\|_{H_m^{\sigma+1}}^2 \mathrm{d}t \\
& - 2 \sum_{|\alpha|+|\beta|=\sigma} \left\langle \partial_x^\alpha \partial_v^\beta (\nabla_v (f^n - f^l) \cdot \mathrm{d}W_t), \partial_x^\alpha \partial_v^\beta (f^n - f^l) \right\rangle_m \\
& - 2 \left\langle \nabla_v (f^n - f^l) \cdot \mathrm{d}W_t, f^n - f^l \right\rangle_m . \qquad (11.5)
\end{aligned}
$$

In what follows we denote

$$
\begin{aligned}
\mathcal{M}_\sigma (f^n - f^l) := & - 2 \sum_{|\alpha|+|\beta|=\sigma} \left\langle \partial_x^\alpha \partial_v^\beta (\nabla_v (f^n - f^l) \cdot \mathrm{d}W_t), \partial_x^\alpha \partial_v^\beta (f^n - f^l) \right\rangle_m \\
& - 2 \left\langle \nabla_v (f^n - f^l) \cdot \mathrm{d}W_t, f^n - f^l \right\rangle_m . \qquad (11.6)
\end{aligned}
$$

The estimate (11.5) would close similarly to before (i.e. by using BDG and Grönwall's inequalities), save for the fact that we do not a priori know that the term $\|f^n - f^l\|_{H_m^{\sigma-1}}^2 \|f^n\|_{H_m^{\sigma+1}}^2$ is in $L_\omega^1 L_t^1$, so we now estimate it separately[7] (compare to [57, Lemma 7.2]). The stochastic product

[7] This loss of probabilistic moments was addressed by the cutoff in the approximation scheme of Section 10.2.

rule gives:

$$\begin{aligned}\mathrm{d}(\|f^n - f^l\|^2_{H^{\sigma-1}_m}\|f^n\|^2_{H^{\sigma+1}_m}) =&\|f^n\|^2_{H^{\sigma+1}_m}\mathrm{d}\|f^n - f^l\|_{H^{\sigma-1}_m} + \|f^n - f^l\|^2_{H^{\sigma-1}_m}\mathrm{d}\|f^n\|^2_{H^{\sigma+1}_m} \\ &+ (\mathrm{d}\|f^n - f^l\|^2_{H^{\sigma-1}_m})(\mathrm{d}\|f^n\|^2_{H^{\sigma+1}_m}). \end{aligned} \tag{11.7}$$

The correction term in (11.7) is:

$$(\mathrm{d}\|f^n - f^l\|^2_{H^{\sigma-1}_m})(\mathrm{d}\|f^n\|^2_{H^{\sigma+1}_m}) = \sum_k B_{k,\sigma-1}(f^n - f^l)B_{k,\sigma+1}(f^n)$$

where:

$$B_{k,s}(h) := \langle \sigma_k e_k \cdot \nabla_v(h), h\rangle_m + \sum_{|\alpha|+|\beta|=s} \left\langle \partial_x^\alpha \partial_v^\beta(\sigma_k e_k \cdot \nabla_v h), \partial_x^\alpha \partial_v^\beta h\right\rangle_m .$$

Note that $B_{k,s}(h) \leq C\sigma_k \|e^k\|_{W^{s,\infty}}\|h\|^2_{H^s_m}$, so:

$$\mathrm{d}(\|f^n - f^l\|^2_{H^{\sigma-1}_m})\mathrm{d}(\|f^n\|^2_{H^{\sigma-1}_m}) \leq C\|f^n - f^l\|^2_{H^{\sigma-1}_m}\|f^n\|^2_{H^{\sigma+1}_m}\mathrm{d}t. \tag{11.8}$$

For the main terms of (11.7) we have similar estimates as before. For the difference $f^n - f^l$ and for $t \leq \tau_{n,l}$ we have (recalling the definition (11.6)):

$$\begin{aligned}&\mathrm{d}\|f^n - f^l\|^2_{H^{\sigma-1}_m} \\ &\leq C\|f^n - f^l\|^2_{H^{\sigma-1}_m}\mathrm{d}t + C\|f^n - f^l\|^2_{H^{\sigma-1}_m}\|f^n\|^2_{H^\sigma_m}\mathrm{d}t \\ &- 2\sum_{|\alpha|+|\beta|=\sigma-1} \left\langle \partial_x^\alpha \partial_v^\beta \left[\nabla_v(f^n - f^l)\cdot \mathrm{d}W_t\right], \partial_x^\alpha \partial_v^\beta(f^n - f^l)\right\rangle_m\end{aligned}$$

$$
\begin{aligned}
&- 2\left\langle \nabla_v (f^n - f^l)\cdot \mathrm{d}W_t, f^n - f^l\right\rangle_m \\
\leq & C\|f^n - f^l\|^2_{H^{\sigma-1}_m}\mathrm{d}t + \mathcal{M}_{\sigma-1}(f^n - f^l),
\end{aligned} \tag{11.9}
$$

where we used (11.4) for $\sigma - 1$ instead of $\sigma$ derivatives and the definition of the stopping time $\tau_n^T$. Similarly, for the norm of $f^n$ and for $t \leq \tau_{n,l}$ we have:

$$
\begin{aligned}
\mathrm{d}\|f^n\|^2_{H^{\sigma+1}_m} \leq & C\|f^n\|^2_{H^{\sigma+1}_m}\mathrm{d}t \\
&+ C\|f^n\|_{H^{\sigma}_m}\|f^n\|^2_{H^{\sigma+1}_m}\mathrm{d}t \\
&- 2\sum_{|\alpha|+|\beta|=\sigma+1}\left\langle \partial_x^\alpha \partial_v^\beta(\nabla_v f^n\cdot \mathrm{d}W_t), \partial_x^\alpha\partial_v^\beta f^n\right\rangle_m \\
&- 2\left\langle \nabla_v f^n\cdot \mathrm{d}W_t, f^n\right\rangle_m \\
\leq & C\|f^n\|^2_{H^{\sigma+1}_m}\mathrm{d}t + \mathcal{M}_{\sigma+1}(f^n),
\end{aligned} \tag{11.10}
$$

where we again used the definition of the stopping time $\tau_n^T$. Now, plugging (11.8), (11.9) (11.10) into (11.7), we obtain:

$$
\begin{aligned}
&\mathrm{d}(\|f^n - f^l\|^2_{H^{\sigma-1}_m}\|f^n\|^2_{H^{\sigma+1}_m}) \leq C\|f^n - f^l\|^2_{H^{\sigma-1}_m}\|f^n\|^2_{H^{\sigma+1}_m}\mathrm{d}t \\
&+ \mathcal{M}_{\sigma+1}(f^n)\|f^n - f^l\|^2_{H^{\sigma-1}_m} + \mathcal{M}_{\sigma-1}(f^n - f^l)\|f^n\|^2_{H^{\sigma+1}_m},
\end{aligned} \tag{11.11}
$$

Integrating (11.11) in time and using the BDG inequality, we obtain:

$$
\begin{aligned}
\mathbf{E}\sup_{t'\leq t\wedge\tau_{n,l}}\left(\|f^n - f^l\|^2_{H^{\sigma-1}_m}\|f^n\|^2_{H^{\sigma+1}_m}\right) \leq & \mathbf{E}\left(\|f_0^n - f_0^l\|^2_{H^{\sigma-1}_m}\|f_0^n\|^2_{H^{\sigma+1}_m}\right) \\
&+ C\mathbf{E}\int_0^t \sup_{t'\leq s\wedge\tau_{n,l}}\|f^n - f^l\|^2_{H^{\sigma-1}_m}\|f^n\|^2_{H^{\sigma+1}_m}\mathrm{d}s
\end{aligned}
$$

$$+ C\mathbf{E}\left(\int_0^t \|f^n - f^l\|_{H_m^{\sigma-1}}^4 \|f^n\|_{H_m^{\sigma+1}}^4 \mathrm{d}s\right)^{1/2},$$

so after rearranging and using Grönwall's inequality as done previously in e.g. the proof of Lemma 10.1, we get:

$$\mathbf{E}\sup_{t'\leq t\wedge\tau_{n,l}}\left(\|f^n - f^l\|_{H_m^{\sigma-1}}^2 \|f^n\|_{H_m^{\sigma+1}}^2\right) \leq C\mathbf{E}\left(\|f_0^n - f_0^l\|_{H_m^{\sigma-1}}^2 \|f_0^n\|_{H_m^{\sigma+1}}^2\right). \tag{11.12}$$

Now returning to (11.5), integrating in time, using the BDG inequality, plugging in (11.12), we obtain:

$$\begin{aligned}\mathbf{E}\sup_{t'\leq t\wedge\tau_{n,l}}\|f^n - f^l\|_{H_m^\sigma}^2 \leq& \mathbf{E}\|f_0^n - f_0^l\|_{H_m^\sigma}^2 \\ &+ C\int_0^t \mathbf{E}\sup_{t'\leq s\wedge\tau_{n,l}}\|f^n - f^l\|_{H_m^\sigma}^2 \mathrm{d}s \\ &+ C\mathbf{E}\left(\int_0^{t\wedge\tau_{n,l}}\|f^n - f^l\|_{H_m^\sigma}^4 \mathrm{d}s\right)^{\frac{1}{2}} \\ &+ C\mathbf{E}\left(\|f_0^n - f_0^l\|_{H_m^{\sigma-1}}^2 \|f_0^n\|_{H_m^{\sigma+1}}^2\right).\end{aligned}$$

Therefore we have

$$\begin{aligned}\mathbf{E}\sup_{t'\leq \tau_{n,l}}\|f^n - f^l\|_{H_m^\sigma}^2 \leq& C\mathbf{E}\|f_0^n - f_0^l\|_{H_m^\sigma}^2 \\ &+ C\mathbf{E}\left(\|f_0^n - f_0^l\|_{H_m^{\sigma-1}}^2 \|f_0^n\|_{H_m^{\sigma+1}}^2\right).\end{aligned} \tag{11.13}$$

Then (11.2) follows from (11.13) and Lemma 11.1 (in particular, note (11.1)).

Next, we move to the proof of (11.3). By Itô's formula, we have:

$$\begin{aligned}\mathrm{d}\|f^n\|^2_{H^\sigma_m} \leq & C\|f^n\|^2_{H^\sigma_m}\mathrm{d}t + \|E^n\|^2_{H^\sigma}\mathrm{d}t \\ & - 2\sum_{|\alpha|+|\beta|=\sigma}\left\langle \partial_x^\alpha\partial_v^\beta(\nabla_v f^n\cdot \mathrm{d}W_t), \partial_x^\alpha\partial_v^\beta f^n\right\rangle_m \\ & - 2\left\langle \nabla_v f^n\cdot \mathrm{d}W_t, f^n\right\rangle_m. \end{aligned} \tag{11.14}$$

Let us denote

$$\mathcal{M}_\sigma(f^n) := 2\sum_{|\alpha|+|\beta|=\sigma}\left\langle \partial_x^\alpha\partial_v^\beta(\nabla_v f^n)\cdot \mathrm{d}W_t, \partial_x^\alpha\partial_v^\beta f^n\right\rangle_m + 2\left\langle \nabla_v f^n\cdot \mathrm{d}W_t, f^n\right\rangle_m$$

so that for $t\leq \tau_n^T\wedge\epsilon$, after integrating in time, (11.14) gives:

$$\|f^n(t)\|^2_{H^\sigma_m} \leq \|f_0^n\|^2_{H^\sigma_m} + C\int_0^t \|f^n(s)\|^2_{H^\sigma_m}\mathrm{d}s + \int_0^t \mathcal{M}^n(s).$$

Therefore, by Chebyshev's inequality for the usual deterministic integral and Doob's ienquality for the martingale, we have

$$\begin{aligned}\mathbf{P}(\sup_{t'\leq\tau_n^T\wedge\epsilon}\|f^n\|^2_{H^\sigma_m} > \|f_0^n\|^2_{H^\sigma_m}+1) \leq & \mathbf{P}\left(C\int_0^{\tau_n^T\wedge\epsilon}\|f^n(s)\|^2_{H^\sigma_m}\mathrm{d}s > \frac{1}{2}\right) \\ & + \mathbf{P}\left(\sup_{t'\leq\tau_n^T\wedge\epsilon}\left|\int_0^{t'}\mathcal{M}^n(s)\right| > \frac{1}{2}\right) \\ \leq & C\mathbf{E}\int_0^{\tau_n^T\wedge\epsilon}\|f^n(s)\|^2_{H^\sigma_m}\mathrm{d}s \\ & + C\mathbf{E}\int_0^{\tau_n^T\wedge\epsilon}\|f^n(s)\|^4_{H^\sigma_m}\mathrm{d}s \\ \leq & C\epsilon\mathbf{E}\|f_0^n\|^2_{H^\sigma_m}\end{aligned}$$

$$\leq CM\epsilon;$$

note we also used that $t \leq \tau_n^T$ implies $\|f^n(t)\|_{H_m^\sigma} \leq C$ for a constant $C > 0$ that depends on the size of the initial data $f_0$ *uniformly* in $n$, since $\|f_0^n\|_{H_m^\sigma} \leq C\|f_0\|_{H_m^\sigma}$ independently of $n$. Taking $\epsilon \to 0$, we obtain (11.3). □

Combining Lemmas 11.2 and 11.3, we obtain the existence of a local strong solutions to (8.1) when $\|f_0\|_{H_m^\sigma} \leq M < \infty$ almost surely. A splitting of the general random initial condition similar to the one in Lemma 9.2 can now provide a local solution whenever $f_0$ is $\mathcal{F}_0$-measurable with $\|f_0\|_{H_m^\sigma} < \infty$ **P**–a.s.. Specifically, since $f_0 = \sum_{M=0}^\infty \mathbb{1}_{M \leq \|f_0\|_{H_m^\sigma} < M+1} f_0$, each component $f_{0,M} := \mathbb{1}_{M \leq \|f_0\|_{H_m^\sigma} < M+1} f_0$ generates a local strong solution $(f_M, \tau_M)$ to (8.1) and we re-construct the full $f$ and $\tau$ using

$$f = \sum_{M=0}^\infty \mathbb{1}_{M \leq \|f_0\|_{H_m^\sigma} < M+1} f_M,$$

and

$$\tau = \sum_{M=0}^\infty \mathbb{1}_{M \leq \|f_0\|_{H_m^\sigma} < M+1} \tau_M.$$

This completes the proof of Theorem 8.3.

## 12 Hypoelliptic regularization

Theorem 8.9 follows by a priori regularization estimates of (9.1), specifically, it suffices to prove that solutions to (9.1) are almost-surely $C_{x,v}^\infty$ for $t > 0$.

We first prove that if $f_0 \in H_m^\sigma$, then the solution lies in $f(t) \in H_m^{\sigma+1}$ for $t > 0$ (with

size depending only on the $H^\sigma$ norm of the initial condition). As mentioned in Section 8, this hypoelliptic regularization is proved using a time-weighted variation of the classical hypocoercive energy functional for the kinetic Fokker–Planck equation (see [111]). For the linear case, a related hypoelliptic regularization estimate can be found in [36]. Taking the standard energy from [111] and scaling derivatives with the powers of $t$ expected from known hypoelliptic regularization estimates (alternatively, one can deduce them from scaling arguments; see e.g. [16]) we have

$$\mathcal{E}_1[t,f] = \|f(t)\|_{L^2_m}^2 + at\,\|\nabla_v f(t)\|_{L^2_m}^2 + bt^2\,\langle \nabla_v f(t), \nabla_x f(t)\rangle_m + ct^3\,\|\nabla_x f(t)\|_{L^2_m}^2 .$$

For $H^\sigma_m$ estimates we hence define

$$\mathcal{E}_\sigma[t,f] := \sum_{0\le q\le \sigma} \mathcal{E}_{1,m}[t, \nabla_x^{\sigma-q}\nabla_v^q f].$$

The constants are chosen (indepedent of $\sigma$) such that $0 < c < b < a$ and $b^2 < \sqrt{ac}$ so that

$$\mathcal{E}_\sigma \approx \|f\|_{H^\sigma_m} + t\,\|\nabla_v f\|_{H^\sigma_m}^2 + t^3\,\|\nabla_x f\|_{H^\sigma_m}^2 .$$

The parameters $a, b, c$ are chosen more specifically to satisfy: for some sufficiently small $\varepsilon \ll 1$ we require

$$\begin{cases} 1 \ge \frac{a}{\varepsilon} \ge \frac{b}{\varepsilon^2} \ge \frac{c}{\varepsilon^3} \\ a \le \varepsilon\sqrt{1\cdot b}, \quad b \le \varepsilon\sqrt{a\cdot c}. \end{cases} \tag{12.1}$$

We recall the proof that such $a, b, c$ exist in Lemma 12.2 below. Note that these conditions imply $bt^2 \le \varepsilon at + \varepsilon ct^3$, a fact we use repeatedly below.

Hence, an estimate on $\mathcal{E}_\sigma$ in terms of $\|f_0\|_{H^\sigma_m}$ implies the desired regularization estimates (along with some more quantitative information that we will not directly use here). The main result of this Section is the following.

**Proposition 12.1.** *Let $f_0$ be a $\mathcal{F}_0$-measurable initial data and suppose that for all $p < \infty$, $\exists M_p > 0$ such that for some $\sigma > \frac{d}{2} + 1$ we have:*

$$\mathbf{E}\,\|f_0\|^p_{H^\sigma_m} < M_p. \tag{12.2}$$

*Let $R < \infty$, let $f$ be the unique pathwise solution of (9.1).*

*Then $\exists T > 0$ depending only on $\sigma$ such that there holds for all $p < \infty$,*

$$\mathbf{E}\left(\sup_{0<t<T} \mathcal{E}_\sigma[t, f(t)]\right)^p \leq C(R, p, M_2, M_3, \ldots).$$

*Therefore, almost surely $f(t) \in H^{\sigma+1}_m$ for all $0 < t < T$.*

Proposition 12.1 implies a corresponding instantaneous regularization for the maximal pathwise solution of (8.1). Once the above proposition is proved, one may simply iterate it, observing that for all $\delta > 0$, $f(\delta)$ is an $\mathcal{F}_\delta$-measurable random variable with

$$\mathbf{E}\,\|f(\delta)\|^p_{H^{\sigma+1}_m} < \infty.$$

Therefore, we may apply Proposition 12.1 to the initial data $f(\delta)$ with $\sigma \mapsto \sigma + 1$. Finally, similar to the proof of Lemma 9.2, a simple cutting procedure can be applied to remove the moment constraint on the initial condition. Hence, to prove Theorem 8.9, it suffices to prove

Proposition 12.1.

*Proof.* For notational simplicity, we will take $\nu = 1$ but the same arguments (up to a suitable rescaling of the coefficients $a, b, c$) apply for any $\nu > 0$. Define the dissipation rate:

$$\mathbb{D}_\sigma(t, f(t)) := \sum_{|\alpha|+|\beta|\leq\sigma} \Big( \left\| \nabla_v \partial_x^\alpha \partial_v^\beta f(t) \right\|_{L^2_m}^2 + at \left\| \nabla_v^2 \partial_x^\alpha \partial_v^\beta f(t) \right\|_{L^2_m}^2 + \frac{b}{2} t^2 \left\| \nabla_x \partial_x^\alpha \partial_v^\beta f(t) \right\|_{L^2_m}^2 + ct^3 \left\| \nabla_v \nabla_x \partial_x^\alpha \partial_v^\beta f(t) \right\|_{L^2_m}^2 \Big),$$

which we show arises from $\mathrm{d}\mathcal{E}_\sigma$. Note that this is almost the same as the contribution from $\mathrm{d}\mathcal{E}_\sigma$ that arises when the time derivative lands on the powers of $t$ in front of the higher-order terms. In order to reduce some of the notation in the ensuing calculation, we use $B(h, g)$ to denote an $L^2_m$-bounded bilinear form, the exact form of which is irrelevant, i.e, a form which is linear in both arguments and such that for any $h, g \in L^2_m$

$$\left\| B(h, g) \right\|_{L^2_m} \lesssim \left\| h \right\|_{L^2_m} \left\| g \right\|_{L^2_m}.$$

The main step of the proof is to calculate the following

$$\mathrm{d}\mathcal{E}_\sigma = \sum_{|\alpha|+|\beta|\leq\sigma} \mathrm{d} \left\| \partial_x^\alpha \partial_v^\beta f(t) \right\|_{L^2_m}^2 + \mathrm{d} \left( at \left\| \nabla_v \partial_x^\alpha \partial_v^\beta f(t) \right\|_{L^2_m}^2 \right)$$

$$+ \sum_{|\alpha|+|\beta|\leq\sigma} \mathrm{d} \left( bt^2 \left\langle \nabla_v \partial_x^\alpha \partial_v^\beta f(t), \nabla_x \partial_x^\alpha \partial_v^\beta f(t) \right\rangle_m \right) + \mathrm{d} \left( ct^3 \left\| \nabla_x \partial_x^\alpha \partial_v^\beta f(t) \right\|_{L^2_m}^2 \right).$$

As in the proof of the various bounds in Sections 10.2–11, we have:

$$\begin{aligned}
\mathrm{d}\left\|\partial_x^\alpha\partial_v^\beta f\right\|_{L^2_m}^2 =& -2\left\langle\partial_x^\alpha\partial_v^\beta(v\cdot\nabla_x f),\partial_x^\alpha\partial_v^\beta f\right\rangle_m\mathrm{d}t\\
&+2\left\langle\Delta_v\partial_x^\alpha\partial_v^\beta f,\partial_x^\alpha\partial_v^\beta f\right\rangle_m\mathrm{d}t\\
&+2\left\langle\partial_x^\alpha\partial_v^\beta\operatorname{div}_v(fv),\partial_x^\alpha\partial_v^\beta f\right\rangle_m\mathrm{d}t\\
&-2\left\langle\partial_x^\alpha\partial_v^\beta(\theta_R(\|f\|_{H^{s_0}_{m_0}})E\cdot\nabla_v f),\partial_x^\alpha\partial_v^\beta f\right\rangle_m\mathrm{d}t\\
&-2\left\langle\partial_x^\alpha\partial_v^\beta(\nabla_v f\cdot\mathrm{d}W_t),\partial_x^\alpha\partial_v^\beta f\right\rangle_m\\
&+\sum_k\left\langle\partial_x^\alpha\partial_v^\beta((\sigma_k e_k\cdot\nabla_v)^2 f),\partial_x^\alpha\partial_v^\beta f\right\rangle_m\mathrm{d}t\\
&-\sum_k\left\|\partial_x^\alpha\partial_v^\beta(\sigma_k e_k\cdot\nabla_v f)\right\|_{L^2_m}^2\mathrm{d}t.
\end{aligned}\tag{12.3}$$

This formula, and its straightforward variations, are then used to expand most of the terms of $\mathrm{d}\mathcal{E}_\sigma$, with the exception of the cross-terms (i.e. those multiplied by $b$). For the cross-terms we instead have

$$\begin{aligned}
&\mathrm{d}\left\langle\partial_x^\alpha\partial_v^{\beta+e_j}f,\partial_x^{\alpha+e_j}\partial_v^\beta f\right\rangle_m=\\
&-\left(\left\langle\partial_x^\alpha\partial_v^{\beta+e_j}(v\cdot\nabla_x f),\partial_x^{\alpha+e_j}\partial_v^\beta f\right\rangle_m+\left\langle\partial_x^\alpha\partial_v^{\beta+e_j}f,\partial_x^{\alpha+e_j}\partial_v^\beta(v\cdot\nabla_x f)\right\rangle_m\right)\mathrm{d}t\\
&-\left(\left\langle\theta_R\partial_x^\alpha\partial_v^{\beta+e_j}(E\cdot\nabla_v f),\partial_x^{\alpha+e_j}\partial_v^\beta f\right\rangle_m+\left\langle\partial_x^\alpha\partial_v^{\beta+e_j}f,\theta_R\partial_x^{\alpha+e_j}\partial_v^\beta(E\cdot\nabla_v f)\right\rangle_m\right)\mathrm{d}t\\
&+\left(\left\langle\partial_x^\alpha\partial_v^{\beta+e_j}(\Delta_v f),\partial_x^{\alpha+e_j}\partial_v^\beta f\right\rangle_m+\left\langle\partial_x^\alpha\partial_v^{\beta+e_j}f,\partial_x^{\alpha+e_j}\partial_v^\beta(\Delta_v f)\right\rangle_m\right)\mathrm{d}t\\
&+\left(\left\langle\partial_x^\alpha\partial_v^{\beta+e_j}(\operatorname{div}_v(fv)),\partial_x^{\alpha+e_j}\partial_v^\beta f\right\rangle_m+\left\langle\partial_x^\alpha\partial_v^{\beta+e_j}f,\partial_x^{\alpha+e_j}\partial_v^\beta(\operatorname{div}_v(fv))\right\rangle_m\right)\mathrm{d}t\\
&-\left\langle\partial_x^\alpha\partial_v^{\beta+e_j}(\nabla_v f\cdot\mathrm{d}W_t),\partial_x^{\alpha+e_j}\partial_v^\beta f\right\rangle_m-\left\langle\partial_x^\alpha\partial_v^{\beta+e_j}f,\partial_x^{\alpha+e_j}\partial_v^\beta(\nabla_v f\cdot\mathrm{d}W_t)\right\rangle_m\\
&+\frac{1}{2}\sum_k\left\langle\partial_x^\alpha\partial_v^{\beta+e_j}(\sigma_k e_k\cdot\nabla_v)^2 f,\partial_x^{\alpha+e_j}\partial_v^\beta f\right\rangle_m\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_k \left\langle \partial_x^\alpha \partial_v^{\beta+e_j} f, \partial_x^{\alpha+e_j}\partial_v^\beta (\sigma_k e_k\cdot\nabla_v)^2 f\right\rangle_m \mathrm{d}t \\
&+\sum \left\langle \partial_x^\alpha \partial_v^{\beta+e_j} (\sigma_k e_k\cdot\nabla_v f), \partial_x^{\alpha+e_j}\partial_v^\beta (\sigma_k e_k\cdot\nabla_v f)\right\rangle_m \mathrm{d}t \\
:=&\mathcal{T}_{c,\alpha,\beta,j}(f)+\mathcal{N}_{c,\alpha,\beta,j}(f)+\mathcal{D}_{c,\alpha,\beta,j}(f)+\mathcal{F}_{c,\alpha,\beta,j}(f)+\mathcal{M}_{c,\alpha,\beta,j}(f)+\mathcal{C}_{c,\alpha,\beta,j}(f),
\end{aligned}
$$

where we abbreviated $\theta_R := \theta_R(\|f\|_{H^{s_0}_{m_0}})$ and $\mathcal{T}_{c,\alpha,\beta,j}, \mathcal{N}_{c,\alpha,\beta,j}, \mathcal{D}_{c,\alpha,\beta,j}, \mathcal{F}_{c,\alpha,\beta,j}, \mathcal{M}_{c,\alpha,\beta,j}, \mathcal{C}_{c,\alpha,\beta,j}$ indicate transport, nonlinear (electric field), dissipation, friction, martingale, and (Itô) correction contributions (which incorporate the last three terms) to the cross-terms, respectively.

## Linear, deterministic contributions:

First, we collect the contributions of the linear terms, namely those that arise from the $v\cdot\nabla_x$ free transport and the Fokker–Planck operator. The main effect of these terms is to introduce the dissipation $\mathbb{D}_\sigma$. The calculation is standard (see [111]) and so we omit most of the details. We define the total contribution of the linear terms of the SPDE for $f$ to $\mathrm{d}\mathcal{E}$ by:

$$
\begin{aligned}
Lin(\alpha,\beta) :=\ & a\left\|\nabla_v\partial_x^\alpha\partial_v^\beta f\right\|_m^2 + 2bt\left\langle \nabla_v\partial_x^\alpha\partial_v^\beta f, \nabla_x\partial_x^\alpha\partial_v^\beta f\right\rangle_m + 3ct^2\left\|\nabla_x\partial_x^\alpha\partial_v^\beta f\right\|_{L^2_m}^2 \\
&- 2\left\langle \partial_x^\alpha\partial_v^\beta(v\cdot\nabla_x f), \partial_x^\alpha\partial_v^\beta f\right\rangle_m + 2\left\langle \Delta_v\partial_x^\alpha\partial_v^\beta f, \partial_x^\alpha\partial_v^\beta f\right\rangle_m \\
&+ 2\left\langle \partial_x^\alpha\partial_v^\beta \operatorname{div}_v(fv), \partial_x^\alpha\partial_v^\beta f\right\rangle_m \\
&- 2at\left\langle \nabla_v\partial_x^\alpha\partial_v^\beta(v\cdot\nabla_x f), \nabla_v\partial_x^\alpha\partial_v^\beta f\right\rangle_m + 2at\left\langle \nabla_v\partial_x^\alpha\partial_v^\beta\Delta_v f, \nabla_v\partial_x^\alpha\partial_v^\beta f\right\rangle_m \\
&+ 2at\left\langle \partial_x^\alpha\partial_v^\beta\nabla_v\operatorname{div}_v(fv), \nabla_v\partial_x^\alpha\partial_v^\beta f\right\rangle_m \\
&- bt^2\left(\sum_j \mathcal{T}_{c,\alpha,\beta,j}+\mathcal{D}_{c,\alpha,\beta,j}+\mathcal{F}_{c,\alpha,\beta,j}\right) \\
&- 2ct^3\left\langle \nabla_x\partial_x^\alpha\partial_v^\beta(v\cdot\nabla_x f), \nabla_x\partial_x^\alpha\partial_v^\beta f\right\rangle_m
\end{aligned}
$$

$$+ 2ct^3 \left\langle \Delta \partial_x^{\alpha+e_j} \partial_v^\beta f, \partial_x^{\alpha+e_j} \partial_v^\beta f \right\rangle_m + 2ct^3 \left\langle \partial_x^{\alpha+e_j} \partial_v^\beta (\mathrm{div}_v(fv)), \partial_x^{\alpha+e_j} \partial_v^\beta f \right\rangle \quad (12.4)$$

For the rest of this proof, we denote $p := |\alpha|, q := |\beta|$. By integration by parts we may write

$$\begin{aligned}
\mathcal{T}_{c,\alpha,\beta,j}(f) &= - \left\langle \partial_x^{\alpha+e_j} \partial_v^\beta (v \cdot \nabla_x f), \partial_x^\alpha \partial_v^{\beta+e_j} f \right\rangle_m \mathrm{d}t - \left\langle \partial_x^{\alpha+e_j} \partial_v^\beta f, \partial_x^\alpha \partial_v^{\beta+e_j} (v \cdot \nabla_x f) \right\rangle_m \mathrm{d}t \\
&= - \left\| \partial_x^{\alpha+e_j} \partial_v^\beta f \right\|_{L^2_m}^2 \mathrm{d}t \\
&\quad - \left\langle v \cdot \nabla_x (\partial_x^{\alpha+e_j} \partial_v^\beta f), \partial_x^\alpha \partial_v^{\beta+e_j} f \right\rangle_m \mathrm{d}t - \left\langle \partial_x^{\alpha+e_j} \partial_v^\beta f, v \cdot \nabla_x (\partial_x^\alpha \partial_v^{\beta+e_j} f) \right\rangle_m \\
&\quad + \sum_{\substack{\beta' < \beta \\ |\beta'|=1}} \binom{\beta}{\beta'} \left( \left\langle \partial_x^{\alpha+e_j+\beta'} \partial_v^{\beta-\beta'} f, \partial_x^\alpha \partial_v^{\beta+e_j} f \right\rangle_m + \left\langle \partial_x^{\alpha+e_j} \partial_v^\beta f, \partial_x^{\alpha+\beta'} \partial_v^{\beta-\beta'+e_j} f \right\rangle_m \right) \mathrm{d}t \\
&= \begin{cases} - \left\| \partial_x^{\alpha+e_j} f \right\|_{L^2_m}^2 \mathrm{d}t, & \text{if } \beta = 0 \\ - \left\| \partial_x^{\alpha+e_j} \partial_v^\beta f \right\|_{L^2_m}^2 \mathrm{d}t + \sum_{q'=q-1}^q B(\nabla_x^{p+1} \nabla_v^{q'} f, \nabla_x^{p+1} \nabla_v^q f) \mathrm{d}t & \text{if } |\beta| > 0, \end{cases}
\end{aligned} \quad (12.5)$$

where recall from above that $B(\cdot, \cdot)$ denotes an $L^2_m$ bounded bilinear form, the exact form of which is not relevant. The dissipation term is more easily treated, yielding

$$\begin{aligned}
\mathcal{D}_{c,\alpha,\beta}(f) &= \left\langle \partial_x^\alpha \partial_v^{\beta+e_j} \Delta_v f, \partial_x^{\alpha+e_j} \partial_v^\beta f \right\rangle_m \mathrm{d}t + \left\langle \partial_x^\alpha \partial_v^{\beta+e_j} f, \partial_x^{\alpha+e_j} \partial_v^\beta \Delta_v f \right\rangle_m \mathrm{d}t \\
&= -2 \left\langle \partial_x^\alpha \partial_v^{\beta+e_j} \nabla_v f, \partial_x^{\alpha+e_j} \partial_v^\beta \nabla_v f \right\rangle_m \mathrm{d}t \\
&\quad + B(\nabla_x^p \nabla_v^{q+2} f, \nabla_x^{p+1} \nabla_v^q f) \mathrm{d}t + B(\nabla_x^p \nabla_v^{q+1} f, \nabla_x^{p+1} \nabla_v^{q+1} f) \mathrm{d}t.
\end{aligned} \quad (12.6)$$

The friction term can be re-arranged as follows

$$\mathcal{F}_{c,\alpha,\beta}(f) = \left\langle \partial_x^\alpha \partial_v^{\beta+e_j} (\mathrm{div}_v(fv)), \partial_x^{\alpha+e_j} \partial_v^\beta f \right\rangle_m \mathrm{d}t + \left\langle \partial_x^\alpha \partial_v^{\beta+e_j} f, \partial_x^{\alpha+e_j} \partial_v^\beta (\mathrm{div}_v(fv)) \right\rangle_m \mathrm{d}t$$

$$
\begin{aligned}
&= \left\langle \mathrm{div}_v(\partial_x^{\alpha+e_j}\partial_v^{\beta} f v), \partial_x^{\alpha}\partial_v^{\beta+e_j} f \right\rangle_m \mathrm{d}t + \left\langle \partial_x^{\alpha+e_j}\partial_v^{\beta} f, \mathrm{div}_v(\partial_x^{\alpha}\partial_v^{\beta+e_j} f v) \right\rangle_m \mathrm{d}t \\
&\quad + \sum_{\substack{\beta'<\beta \\ |\beta'|=1}} \left\langle \partial_x^{\alpha+e_j}\partial_v^{\beta+\beta'} f, \partial_x^{\alpha}\partial_v^{\beta+e_j} f \right\rangle_m + \sum_{\substack{\beta'<\beta \\ |\beta'|=1}} \left\langle \partial_x^{\alpha+e_j}\partial_v^{\beta} f, \partial_x^{\alpha}\partial_v^{\beta+\beta'+e_j} f \right\rangle_m \mathrm{d}t \\
&= 2\left\langle \mathrm{div}_v(\partial_x^{\alpha+e_j}\partial_v^{\beta} f \partial_x^{\alpha}\partial_v^{\beta+e_j} f v), 1 \right\rangle_m \mathrm{d}t - \left\langle v \cdot \nabla_v(\partial_x^{\alpha+e_j}\partial_v^{\beta} f \partial_x^{\alpha}\partial_v^{\beta+e_j} f), 1 \right\rangle_m \mathrm{d}t \\
&\quad + \sum_{\substack{\beta'<\beta \\ |\beta'|=1}} \left\langle \partial_x^{\alpha+e_j}\partial_v^{\beta+\beta'} f, \partial_x^{\alpha}\partial_v^{\beta+e_j} f \right\rangle_m + \sum_{\substack{\beta'<\beta \\ |\beta'|=1}} \left\langle \partial_x^{\alpha+e_j}\partial_v^{\beta} f, \partial_x^{\alpha}\partial_v^{\beta+\beta'+e_j} f \right\rangle_m \mathrm{d}t. \quad (12.7)
\end{aligned}
$$

The fundamental structure of the hypocoercive norm $\mathcal{E}$ is that the $\mathcal{T}$ term gives rise to the $\nabla_x$ dissipation term that would otherwise be missing from the dissipation of a kinetic equation. That is, from (12.5), we obtain:

$$
\begin{aligned}
bt^2\mathcal{T}_{c,\alpha,\beta,j}(f) + bt^2 \left\| \partial_x^{\alpha+e_j}\partial_v^{\beta} f \right\|_{L^2_m}^2 \mathrm{d}t \leq & bt^2 B\left(\nabla_v(\nabla_x^{p+1}\nabla_v^{q-1} f), \nabla_v(\nabla_x^{p+1}\nabla_v^{q-1} f)\right) \mathrm{d}t \\
& + bt^2 B\left(\nabla_x^{p+1}\nabla_v^{q-1} f, \nabla_v(\nabla_x^{p+1}\nabla_v^{q-1} f)\right) \mathrm{d}t \\
\lesssim & \frac{b}{a} t\mathbb{D}_\sigma \mathrm{d}t + bt^2\mathcal{E}_\sigma \\
\lesssim & \varepsilon t\mathbb{D}_\sigma \mathrm{d}t + bt^2\mathcal{E}_\sigma \mathrm{d}t, \quad (12.8)
\end{aligned}
$$

and similarly, from (12.6) and (12.7):

$$
\begin{aligned}
bt^2\mathcal{D}_{c,\alpha,\beta,j} + bt^2\mathcal{F}_{c,\alpha,\beta,j} &\lesssim \left(\frac{b}{a} + \frac{ba}{c}\right) t\mathbb{D}_\sigma \mathrm{d}t + \mathcal{E}_\sigma \mathrm{d}t \\
&\lesssim t\mathbb{D}_\sigma \mathrm{d}t + \mathcal{E}_\sigma \mathrm{d}t. \quad (12.9)
\end{aligned}
$$

Putting together the negative definite terms that arise from $\mathcal{T}_{c,\alpha,\beta,j}$ and those in (12.4), we obtain for short $t$ and for some $C > 0$:

$$Lin(\alpha,\beta) \leq -(2 - Ct)\,\mathbb{D}_\sigma + C\mathcal{E}_\sigma.$$

In fact this is somewhat sub-optimal, as the second term on the right-hand side above can be taken in weaker norms. However, such refinements will be irrelevant here as we are only interested in short time regularization.

## Nonlinear contributions:

Next, we collect the contributions of the nonlinear electric field. Namely,

$$\begin{aligned}
NL(\alpha,\beta) := & -2\left\langle \partial_x^\alpha\partial_v^\beta(\theta_R(\|f\|_{H^{s_0}_{m_0}})E\cdot\nabla_v f), \partial_x^\alpha\partial_v^\beta f\right\rangle_m \\
& - 2at\left\langle \partial_x^\alpha\partial_v^\beta\nabla_v(\theta_R(\|f\|_{H^{s_0}_{m_0}})E\cdot\nabla_v f), \partial_x^\alpha\partial_v^\beta\nabla_v f\right\rangle_m \\
& - bt^2\left(\left\langle \theta_R\partial_x^\alpha\partial_v^{\beta+e_j}(E\cdot\nabla_v f), \partial_x^{\alpha+e_j}\partial_v^\beta f\right\rangle_m + \left\langle \partial_x^\alpha\partial_v^{\beta+e_j} f, \theta_R\partial_x^{\alpha+e_j}\partial_v^\beta(E\cdot\nabla_v f)\right\rangle_m\right) \\
& - 2ct^3\left\langle \partial_x^\alpha\partial_v^\beta\nabla_x(\theta_R(\|f\|_{H^{s_0}_{m_0}})E\cdot\nabla_v f), \partial_x^\alpha\partial_v^\beta\nabla_x f\right\rangle_m .
\end{aligned}$$

We first analyze the electric field's contribution to (12.3), similarly to the various nonlinear estimates of Section 10.2:

$$\begin{aligned}
& \left|\left\langle \partial_x^\alpha\partial_v^\beta(E\cdot\nabla_v f), \partial_x^\alpha\partial_v^\beta f\right\rangle_m\right| \\
& \lesssim \sum_{\substack{\gamma<\alpha \\ |\alpha-\gamma|\geq 2}} \left|\left\langle \partial_x^{\alpha-\gamma}E\cdot\nabla_v\partial_v^\beta\partial_x^\gamma f, \partial_x^\alpha\partial_v^\beta f\right\rangle_m\right| \mathrm{d}t + \|E\|_{W^{1,\infty}}\|f\|^2_{H^\sigma_m}
\end{aligned}$$

$$\leq \sum_{\substack{\gamma<\alpha \\ |\alpha-\gamma|\geq 2}} \left\|\nabla_x^{|\alpha-\gamma|}E\right\|_{L_x^{2\frac{\sigma-1}{|\alpha-\gamma|-1}}} \left\|\nabla_v\partial_v^\beta\partial_x^\gamma f\right\|_{L^2_{v,m}L_x^{2\frac{\sigma-1}{|\beta|+|\gamma|}}} \left\|\partial_x^\alpha\partial_v^\beta f\right\|_{L^2_m}$$

$$+ \|E\|_{W^{1,\infty}} \|f\|^2_{H^\sigma_m}$$

$$\lesssim \sum_{\substack{\gamma<\alpha \\ |\alpha-\gamma|\geq 2}} \|E\|^{1-\theta}_{W^{1,\infty}} \|E\|^\theta_{H^\sigma} \left\|\nabla_v\partial_v^\beta f\right\|^{1-\eta}_{L^2_m} \|f\|^{1+\eta}_{H^\sigma_m} + \|E\|_{W^{1,\infty}} \|f\|^2_{H^\sigma_m}, \tag{12.10}$$

where for each $\alpha, \beta, \gamma$, the interpolation index $\eta$ is given as:

$$\begin{aligned}\eta =& \frac{|\gamma|}{|\alpha|-1} + \frac{d}{|\alpha|-1}\left(\frac{1}{2} - \frac{|\beta|+|\gamma|}{2(\sigma-1)}\right) \\ =& \frac{|\gamma|}{|\alpha|-1} + \frac{d}{|\alpha|-1}\cdot\frac{|\alpha-\gamma|-1}{2(\sigma-1)},\end{aligned}$$

Note that in the above, $\eta < 1$ since $\frac{d}{2(\sigma-1)} < 1$. Therefore,

$$\begin{aligned}\|E\|^{1-\theta}_{W^{1,\infty}} \|E\|^\theta_{H^\sigma} \left\|\nabla_v\partial_v^\beta f\right\|^{1-\eta}_{L^2_m} \|f\|^{1+\eta}_{H^\sigma_m} &\lesssim \|f\|^2_{H^\sigma_m} + \left(\|E\|^{1-\theta}_{W^{1,\infty}} \|E\|^\theta_{H^\sigma}\right)^{\frac{2}{1-\eta}} \|f\|^2_{H^{\sigma-1}_m} \\ &\lesssim \|f\|^2_{H^\sigma_m} + \|f\|^{\frac{4-2\eta}{1-\eta}}_{H^{\sigma-1}_m}.\end{aligned}$$

Note that in this term, it was not necessary to make use of the dissipation, as the first term in the final inequality above is controlled by at most $H^\sigma_m$ and the second term, which is derived from the factor $\|E\|^{1-\theta}_{W^{1,\eta}} \|E\|^\theta_{H^\sigma} \|\nabla_v\partial_v^\beta f\|^{1-\eta}_{L^2_m}$, contains at most $\sigma-1$ derivatives (since $|\beta| \leq \sigma-2$ whenever $|\alpha-\gamma| \geq 2$).

In a similar manner, the corresponding "second term" in the electric field contributions to the $\partial_x^{\alpha+e_j}\partial_v^\beta f$ and $\partial_x^\alpha\partial_v^{\beta+e_j} f$ terms of the energy will contain at most $\sigma$ derivatives. This means that all in all, for short $t$ we can bound the electric field contributions to (12.3), as well as those

to the higher order terms in the definition of $\mathcal{E}_\sigma$, from (12.10):

$$\left|\left\langle \partial_x^\alpha \partial_v^\beta (E\cdot\nabla_v f), \partial_x^\alpha \partial_v^\beta f\right\rangle_m\right| \lesssim \|f\|^p_{H^{\sigma-1}_m} + \mathcal{E}_\sigma \tag{12.11}$$

$$at\left|\left\langle \partial_x^\alpha \partial_v^{\beta+e_j} (E\cdot\nabla_v f), \partial_x^\alpha \partial_v^{\beta+e_j} f\right\rangle_m\right| \lesssim \|f\|^p_{H^{\sigma}_m} + \mathcal{E}_\sigma$$

$$ct^3\left|\left\langle \partial_x^{\alpha+e_j} \partial_v^{\beta} (E\cdot\nabla_v f), \partial_x^{\alpha+e_j} \partial_v^{\beta} f\right\rangle_m\right| \lesssim \|f\|^p_{H^{\sigma}_m} + \mathcal{E}_\sigma, \tag{12.12}$$

for some $p > 2$ fixed. Note that the high power of $\|f\|_{H^\sigma_m}$ is a priori controlled (in $L^1_\omega L^\infty_{t,loc}$) by the finite $p$-th moment assumptions (12.2).

We now move to estimating the electric field contribution to the cross term. First, we integrate by parts for convenience, in order to "symmetrize" $\mathcal{N}_{c,\alpha,\beta,j}$ up to a lower order term:

$$\begin{aligned}
\mathcal{N}_{c,\alpha,\beta,j} =& \left\langle \partial_x^{\alpha+e_j} \partial_v^{\beta} (E\cdot\nabla_v f), \partial_x^{\alpha} \partial_v^{\beta+e_j} f\right\rangle_m + \left\langle \partial_x^{\alpha+e_j} \partial_v^{\beta} f, \partial_x^{\alpha} \partial_v^{\beta+e_j} (E\cdot\nabla_v f)\right\rangle_m \\
=& -\left\langle \partial_x^{\alpha} \partial_v^{\beta} (E\cdot\nabla_v f), \partial_x^{\alpha+e_j} \partial_v^{\beta+e_j} f\right\rangle_m + \left\langle \partial_x^{\alpha+e_j} \partial_v^{\beta} f, \partial_x^{\alpha} \partial_v^{\beta+e_j} (E\cdot\nabla_v f)\right\rangle_m \\
=& \quad 2\left\langle \partial_x^{\alpha+e_j} \partial_v^{\beta} f, \partial_x^{\alpha} \partial_v^{\beta+e_j} (E\cdot\nabla_v f)\right\rangle_m \\
&+ \iint \partial_x^\alpha \partial_v^\beta (E\cdot\nabla_v f) \partial_x^{\alpha+e_j} \partial_v^\beta f \partial_{v^j}(\langle v\rangle^m) \mathrm{d}v\mathrm{d}x.
\end{aligned} \tag{12.13}$$

We analyze the first term in the final equality above:

$$\begin{aligned}
&\left|\left\langle \partial_x^{\alpha} \partial_v^{\beta+e_j} (E\cdot\nabla_v f) \partial_x^{\alpha+e_j} \partial_v^{\beta} f\right\rangle_m\right| \\
\lesssim& \sum_{\gamma\le\alpha} \left|\left\langle \partial_x^{\alpha-\gamma} E\cdot\nabla_v \partial_v^{\beta+e_j} \partial_x^\gamma f, \partial_x^{\alpha+e_j} \partial_v^\beta f\right\rangle_m\right| \\
\le& \|E\|_{L^\infty_x} \left\|\nabla_v^{q+2}\nabla_x^p f\right\|_{L^2_m} \left\|\nabla_v^q \nabla_x^{p+1} f\right\|_{L^2_m} \\
&+ \|\nabla_x E\|_{L^\infty_x} \left\|\nabla_v^{q+2}\nabla_x^{p-1} f\right\|_{L^2_m} \left\|\nabla_x^{p+1} \nabla_v^q f\right\|_{L^2_m}
\end{aligned}$$

$$+ \sum_{\substack{\gamma<\alpha \\ |\alpha-\gamma|\geq 2}} \left\|\nabla_x^{|\alpha-\gamma|}E\right\|_{L_x^{2\frac{\sigma}{|\alpha-\gamma|-1}}} \left\|\nabla_v^{q+2}\nabla_x^{|\gamma|}f\right\|_{L^2_{v,m}L_x^{2\frac{\sigma}{|\beta|+|\gamma|}}} \left\|\nabla_v^q\nabla_x^{p+1}f\right\|_{L^2_m}. \tag{12.14}$$

To estimate each term in the summation above, we again interpolate as in the proof of Lemma 10.10;

$$\left\|\nabla_x^{|\alpha-\gamma|}E\right\|_{L_x^{2\frac{\sigma}{|\alpha-\gamma|-1}}} \lesssim \|\nabla_x E\|_{L_x^\infty}^{1-\frac{|\alpha-\gamma|-1}{\sigma}} \left\|\nabla_x^{\sigma+1}E\right\|_{L_x^2}^{\frac{|\alpha-\gamma|-1}{\sigma}}, \tag{12.15}$$

$$\left\|\nabla_v^{q+2}\nabla_x^{|\gamma|}f\right\|_{L^2_{v,m}L_x^{2\frac{\sigma}{|\beta|+|\gamma|}}} \lesssim \left\|\nabla_v^{q+2}f\right\|_{L^2_m}^{1-\eta} \left\|\nabla_v^{q+2}\nabla_x^p f\right\|_{L^2_m}^{\eta}, \tag{12.16}$$

where

$$\begin{aligned}\eta =& \frac{|\gamma|}{|\alpha|} + \frac{d}{|\alpha|}\left(\frac{1}{2} - \frac{|\beta|+|\gamma|}{2\sigma}\right)\\ =& \frac{|\gamma|}{|\alpha|} + \frac{d}{|\alpha|}\frac{|\alpha-\gamma|}{2\sigma},\end{aligned}$$

which again satisfies $\eta < 1$ since $\sigma > d/2$. With these exponents, plugging (12.15)-(12.16) into (12.14) we get:

$$\begin{aligned}&\left|\left\langle \partial_x^\alpha \partial_v^{\beta+e_j}(E\cdot\nabla_v f), \partial_x^{\alpha+e_j}\partial_v^\beta f\right\rangle_m\right|\\ \lesssim& \|E\|_{L_x^\infty}\left\|\nabla_v^{q+2}\nabla_x^p f\right\|_{L^2_m}\left\|\nabla_v^q\nabla_x^{p+1}f\right\|_{L^2_m} + \|\nabla_x E\|_{L_x^\infty}\left\|\nabla_v^{q+2}\nabla_x^{p-1}f\right\|_{L^2_m}\left\|\nabla_x^{p+1}\nabla_v^q f\right\|_{L^2_m}\\ &+ \sum_{\substack{\gamma<\alpha \\ |\alpha-\gamma|\geq 2}} \|\nabla_x E\|_{L_x^\infty}^{\frac{|\beta|+|\gamma|+1}{\sigma}} \left\|\nabla_x^{\sigma+1}E\right\|_{L_x^2}^{\frac{|\alpha-\gamma|-1}{\sigma}} \left\|\nabla_v^{q+2}f\right\|_{L^2_m}^{1-\eta}\left\|\nabla_v^{q+2}\nabla_x^p f\right\|_{L^2_m}^{\eta}\left\|\nabla_v^q\nabla_x^{p+1}f\right\|_{L^2_m}.\end{aligned} \tag{12.17}$$

The lower order term in (12.13) produces a less significant contribution - as it contains a smaller number of derivatives - and hence we omit the treatment for brevity. Therefore, since $|\alpha-\gamma| \geq 2$ implies $q+2 \leq \sigma$, we obtain from (12.17) that $\exists C(\varepsilon) > 0$ such that

$$bt^2 \mathcal{N}_{c,\alpha,\beta,j} \leq \frac{\varepsilon}{10}\mathbb{D}_\sigma \mathrm{d}t + C(\varepsilon)\mathcal{E}_\sigma \mathrm{d}t + C(\varepsilon)\left\|f(t)\right\|_{H^\sigma_m}^p \mathrm{d}t. \tag{12.18}$$

This completes the required estimates on the electric field.

## Itô corrections:

Next, let us analyze the contributions of the corrections to the cross term. This equals:

$$\begin{aligned}
\mathcal{C}_{c,\alpha,\beta,j}(f) =& \frac{1}{2}\sum_k \Big( \left\langle \partial_x^\alpha \partial_v^{\beta+e_j}(\sigma_k e_k \cdot \nabla_v)^2 f, \partial_x^{\alpha+e_j}\partial_v^\beta f \right\rangle_m \\
&+ \left\langle \partial_x^\alpha \partial_v^{\beta+e_j} f, \partial_x^{\alpha+e_j}\partial_v^\beta (\sigma_k e_k \cdot \nabla_v)^2 f \right\rangle_m \Big) \mathrm{d}t \\
&+ \sum_k \left\langle \partial_x^\alpha \partial_v^{\beta+e_j}(\sigma_k e_k \cdot \nabla_v f), \partial_x^{\alpha+e_j}\partial_v^\beta (\sigma_k e_k \cdot \nabla_v f) \right\rangle_m \mathrm{d}t \\
=& \sum_{p' \leq p} \Big( B(\nabla_x^{p'}\nabla_v^{q+2} f, \nabla_x^{p+1}\nabla_v^{q+1} f) + B(\nabla_x^{p+1}\nabla_v^{q+2} f, \nabla_x^{p'}\nabla_v^{q+1} f) \Big) \mathrm{d}t \\
&+ \sum_{p' \leq p} \Big( B(\nabla_x^{p'}\nabla_v^{q+2} f, \nabla_x^{p+1}\nabla_v^{q} f) + B(\nabla_x^{p+1}\nabla_v^{q+1} f, \nabla_x^{p'}\nabla_v^{q+1} f) \Big) \mathrm{d}t \\
&+ \sum_{p' \leq p} \sum_{p'' \leq p+1} B(\nabla_x^{p'}\nabla_v^{q+2} f, \nabla_x^{p''}\nabla_v^{q+1} f) \mathrm{d}t.
\end{aligned}$$

As mentioned above, here $B$ denotes a bilinear form which is bounded on $L^2_m \times L^2_m$, the exact form of which is irrelevant. It follows that

$$bt^2 \mathcal{C}_{c,\alpha,\beta,j} \lesssim (\varepsilon + t)\mathbb{D}_\sigma + \mathcal{E}_\sigma.$$

As in (12.3), from similar calculations to those in the proof of (10.2) in Lemma 10.1 for $p = 2$, we have

$$\sum_k \left\langle \partial_x^\alpha \partial_v^\beta ((\sigma_k e_k \cdot \nabla_v)^2 f), \partial_x^\alpha \partial_v^\beta f \right\rangle_m - \sum_k \left\| \partial_x^\alpha \partial_v^\beta (\sigma_k e_k \cdot \nabla_v f) \right\|_{L^2_m}^2 \lesssim \mathcal{E}_\sigma$$

$$at \sum_k \left\{ \left\langle \nabla_v \partial_x^\alpha \partial_v^\beta ((\sigma_k e_k \cdot \nabla_v)^2 f), \nabla_v \partial_x^\alpha \partial_v^\beta f \right\rangle_m - \sum_k \left\| \nabla_v \partial_x^\alpha \partial_v^\beta (\sigma_k e_k \cdot \nabla_v f) \right\|_{L^2_m}^2 \right\} \lesssim \mathcal{E}_\sigma$$

$$ct^3 \sum_k \left\{ \left\langle \nabla_x \partial_x^\alpha \partial_v^\beta ((\sigma_k e_k \cdot \nabla_v)^2 f), \nabla_x \partial_x^\alpha \partial_v^\beta f \right\rangle_m - \sum_k \left\| \nabla_x \partial_x^\alpha \partial_v^\beta (\sigma_k e_k \cdot \nabla_v f) \right\|_{L^2_m}^2 \right\} \lesssim \mathcal{E}_\sigma.$$

This completes the necessary estimates on the Itô correction terms.

## Final estimate:

For $t$ sufficiently small, combining the estimates on the linear terms of (12.3) from the above arguments with (12.8), (12.9), (12.18) and (12.11)-(12.12), we ultimately obtain

$$\mathrm{d}\mathcal{E}_\sigma \leq C\mathcal{E}_\sigma \mathrm{d}t - (2 - C\varepsilon)\mathbb{D}_\sigma \mathrm{d}t + \mathcal{M}_\sigma + C \left\| f \right\|_{H^\sigma_m}^p \mathrm{d}t,$$

where $\mathcal{M}_\sigma$ denotes all of the martingale terms:

$$\begin{aligned}
\mathcal{M}_\sigma = & -2 \left\langle \partial_x^\alpha \partial_v^\beta (\nabla_v f \cdot \mathrm{d}W_t), \partial_x^\alpha \partial_v^\beta f \right\rangle_m \\
& - 2at \sum_{j=1}^d \left\langle \partial_x^\alpha \partial_v^{\beta+e_j} (\nabla_v f \cdot \mathrm{d}W_t), \partial_x^\alpha \partial_v^{\beta+e_j} f \right\rangle_m \\
& - bt^2 \sum_{j=1}^d \left\langle \partial_x^\alpha \partial_v^{\beta+e_j} (\nabla_v f \cdot \mathrm{d}W_t), \partial_x^{\alpha+e_j} \partial_v^\beta f \right\rangle_m \\
& - bt^2 \sum_{j=1}^d \left\langle \partial_x^\alpha \partial_v^{\beta+e_j} f, \partial_x^{\alpha+e_j} \partial_v^\beta (\nabla_v f \cdot \mathrm{d}W_t) \right\rangle_m
\end{aligned}$$

$$- 2ct^3 \sum_{j=1}^{d} \left\langle \partial_x^{\alpha+e_j} \partial_v^{\beta} (\nabla_v f \cdot \mathrm{d}W_t), \partial_x^{\alpha+e_j} \partial_v^{\beta} f \right\rangle_m .$$

Therefore, integrating in time and using the BDG inequality, we obtain

$$\begin{aligned}
\mathbf{E} \sup_{t' \le \tau} \mathcal{E}_\sigma(t') + (2 - C\varepsilon)\mathbf{E} \int_0^\tau \mathbb{D}_\sigma(s)\mathrm{d}s \le & \mathbf{E} \left\| f_0 \right\|_{H_m^\sigma}^2 + C\mathbf{E} \int_0^\tau \mathcal{E}_\sigma(s)\mathrm{d}s + C\mathbf{E} \int_0^\tau \left\| f(s) \right\|_{H_m^\sigma}^p \mathrm{d}s \\
& + C\mathbf{E} \left( \int_0^\tau (\mathcal{E}_\sigma(s) + \varepsilon \sqrt{\mathcal{E}_\sigma(s)\mathbb{D}_\sigma(s)})^2 \mathrm{d}s \right)^{\frac{1}{2}} \\
\le & \mathbf{E} \left\| f_0 \right\|_{H_m^\sigma}^2 + C\mathbf{E} \int_0^\tau \mathcal{E}_\sigma(s)\mathrm{d}s + C\mathbf{E} \int_0^\tau \left\| f(s) \right\|_{H_m^\sigma}^p \mathrm{d}s \\
& + C\mathbf{E} \sup_{t' \le \tau} \mathcal{E}^{\frac{1}{2}}(t') \left( \int_0^\tau (\mathcal{E}_\sigma(s)^{\frac{1}{2}} + \varepsilon \mathbb{D}_\sigma(s)^{\frac{1}{2}})^2 \mathrm{d}s \right)^{\frac{1}{2}} \\
\le & \mathbf{E} \left\| f_0 \right\|_{H_m^\sigma}^2 + C\mathbf{E} \int_0^\tau \mathcal{E}_\sigma(s)\mathrm{d}s + C\mathbf{E} \int_0^\tau \left\| f(s) \right\|_{H_m^\sigma}^p \mathrm{d}s \\
& + \frac{1}{2}\mathbf{E} \sup_{t' \le \tau} \mathcal{E}_\sigma(t') + C\varepsilon^2 \mathbf{E} \int_0^\tau \mathbb{D}_\sigma(s)\mathrm{d}s.
\end{aligned}$$

Rearranging terms and applying Grönwall's inequality, we obtain

$$\mathbf{E} \sup_{t' \le \tau} \mathcal{E}_\sigma(t') \lesssim 1.$$

As is standard (and as in the proofs of the $p > 2$ estimates (10.2) and (10.26) in Section 10.2), a straightforward variation of the above argument extends to prove

$$\mathbf{E} \left( \sup_{t' \le \tau} \mathcal{E}_\sigma(t') \right)^p \lesssim 1,$$

completing the desired estimates. □

For the reader's convenience, we include the elementary proof that such constants $a, b, c$ as

prescribed in (12.1) do indeed exist. A more general situation is treated in [110, Lemma A.16].

**Lemma 12.2.** *Let $\varepsilon > 0$. Then there exist constants $a, b, c$ such that (12.1) holds:*

$$\begin{cases} 1 \geq \frac{a}{\varepsilon} \geq \frac{b}{\varepsilon^2} \geq \frac{c}{\varepsilon^3} \\ a \leq \varepsilon\sqrt{1 \cdot b}, \quad b \leq \varepsilon\sqrt{a \cdot c} \end{cases}$$

*Proof.* Let $\vartheta > 0$, and pick $m_1, m_2, m_3 > 0$ such that

$$m_1 = 1, \quad m_2 \in (1, 2), \quad m_3 \in (m_2, 2m_2 - 1)$$

and set:

$$a = \vartheta^{m_1} = \vartheta, \quad b = \vartheta^{m_2}, \quad c = \vartheta^{m_3}.$$

The proof of the lemma is concluded by picking $\vartheta$ sufficiently small. □

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