# ABSTRACT

Title of Dissertation: COHOMOLOGICAL EQUATIONS ON FLAT SURFACES AND SPEED OF WEAK MIXING OF THE CHACON MAP

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We will prove the existence of cohomological equations on the unit tangent bundle on flat surfaces with cone points. We first find the growth rate of the eigenvalues for the Laplacian on the unit tangent bundle of the surface then derive a Cheeger constant-like bound to find solutions for the horizontal Laplacian. This Cheeger constant depends on a Diophantine condition derived from the irrational angles from the cone points. When combined, these results will allow us to solve cohomological equations for the geodesic flow on the surface.

In addition, we consider a non-primitive substitution subshift that is conjugate to the Chacon map. We then derive spectral estimates for a particular subshift and the speed of weak mixing for a class of observables with certain regularity conditions. After, we use these results to find the speed of weak mixing for the Chacon map on the interval and show that this bound is essentially sharp.

# COHOMOLOGICAL EQUATIONS ON FLAT SURFACES AND SPEED OF WEAK MIXING OF THE CHACON MAP

by

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## 1.1 Introduction

The Chacon map [7] is one of the first examples of a transformation that is weakly mixing but not strongly mixing. In fact, this is the 'generic' case. It was shown in [11] that the set of weakly mixing transformations in the space of invertible measure preserving transformations is a dense $G_\delta$ set. On the other hand, in [17] it was proved that the 'general' measure preserving transformation is not mixing. Other examples of systems which are weakly mixing but not mixing have since been found. For example, in [2] it was proved that almost all interval exchange transformations are weakly mixing but are never mixing [12].

Our goal in this paper is to determine the rate of weak mixing for the Chacon map using estimates on the spectral measures. We will use this estimate along with a result from [14] to establish the existence of a set of times for which the Chacon map fails to be strongly mixing. Using a spectral analysis approach to effective weak mixing was used in [3] to find the speed of weak mixing for interval exchange transformations. The same method was also used in [6] and [9] to find estimates for translation flows. The general 'recipe' is to derive Hölder-type bounds on the spectral measures from upper bounds on twisted ergodic sums, which in turn are proved using some sort of quantitative Veech criterion [18] to bound the twisted sums. A general scheme to this approach for the case of substitutions can be found in [5].

The main results in this paper are the following theorems.

[Theorem A]

Let $U$ be the operator on $L^2([0,1])$ defined by $U(f) = f \circ C$, where $C$ is the Chacon map. Let $f \in Lip([0,1])$ have zero expectation and $g \in L^2([0,1])$. Denote $\|f\|_L$ to be the Lipschitz norm of $f$ and $\|g\|_2$ the $L^2$ norm of $g$. Then there is a constant $K_C > 0$ dependent only on the Chacon map such that

$$\frac{1}{N}\sum_{k=0}^{N-1}|\langle U^k f, g\rangle|^2 \leq K_C\|f\|_L\|f\|_2\|g\|_2^2[\log_3 N]^{-\frac{1}{6}} \quad \text{for all } N > 1.$$

We also prove that this estimate is essentially sharp in Section 12. By using the Cauchy-Schwartz inequality for sums with one sequence identically equal to one, we can compare the upper bound from Theorem A with the lower bound found in Theorem B. [Theorem B]

There exists a diverging sequence of $N$ and a constant $C > 0$ dependent only on the Chacon map such that, for each $N$ in the sequence, there is a Lipschitz $f_N$ and a square integrable $g_N$ with the property that

$$\sum_{i=0}^{N-1}\left|\int_X f_N(T^i x)g_N(x)dx - \int_X f_N(x)dx\int_X g_N(x)dx\right| \geq C\frac{N}{\log(N)^2}\cdot\|f_N\|_L^{\frac{1}{2}}\cdot\|f_N\|_2^{\frac{1}{2}}\cdot\|g_N\|_2.$$

Notice that the speed of weak mixing for the Chacon map is logarithmic. Heuristically, this is in part due to the fact that the second highest eigenvalue for the substitution matrix of the Chacon substitution has norm equal to one. In the case of interval exchange transformations of rotation type, it was proved in [3] that the speed of correlation decay is also logarithmic. Contrast this with Section 7 of [3], where it was shown that an interval exchange transformation that is not of rotation type has polynomial decay.

In the case of the Chacon map and rotation type IET's, we are able to bound the renormalization dynamics from the Veech argument (Section 6 of [3] for example) away from the integer lattice with some uniform frequency due to that fact that the second highest eigenvalue has norm 1. In the case of interval exchange transformations, the case is slightly more subtle. Rauzy-Veech induction on the interval exchange transformations gives us a family of substitutions to consider, and if the IET is of rotation type, then we can, in particular, use the rotation-like properties of the family of substitution matrices to gain logarithmic lower bounds. This is explicitly found in Section 7.2 of [3].

For the case of substitutions, we can guess that the speed of weak mixing will be polynomial when the second highest eigenvalue of the substitution matrix is outside the unit circle. For example, in Section 5 of [5] it was shown that in an analogous case the speed of weak mixing is polynomial for suspension flows. In general, there is a close relationship between translation flows, substitutions, and interval exchange transformations. For reference, these relations are highlighted in [3], [5], [9] and [6]. Although the paper shows the speed of weak mixing for the Chacon map, the end of Section 6 describes which technical assumptions are sufficient to extend the proof of effective weak mixing to other substitutions. Apparently, there is a relationship between return words for substitutions and weak mixing that is still being investigated. Knowledge of 'good' return words is almost equivalent to knowledge of the speed of weak mixing for a given substitution given other mild assumptions.

## 1.2 Structure of the Paper

A concise introduction to substitutions and the Chacon map is found in sections 3 and 4 respectively. The method we used to extract quantitative estimates on the substitution system comes from an analysis on the spectral measure of small intervals. The relationship between twisted sums, spectral measure, and the quantitative rate of weak mixing is found in section 5. The work in sections 6 through 8 serve to first find the rate of weak mixing for cylindrical functions defined on arbitrarily large cylinders. Sections 10 and 11 use the fact that Lipschitz functions can be approximated by cylindrical functions to get the final upper bound for the speed of weak mixing. What follows is a proof showing that this bound is essentially sharp. Speed of weak mixing combined with a result from [14] also gives a corollary that bounds the frequency of times for which the Chacon map fails to be mixing.

The Chacon map is constructed by cutting and stacking the unit interval, as outlined in [7]. By tracking which intervals a particular point in the unit interval enters after iterating that point under the action of the Chacon map, we can assign almost every point in the unit interval a code given by its orbit. This coding is conjugate to a primitive substitution subshift. Bufetov and Solomyak showed in [5] that it suffices to bound spectral measures and a Fourier anologue of the Birkhoff sums to find effective weak mixing. In this setting, we can use the techniques from [5] to find the speed of weak mixing for the Chacon substitution subshift. This formulation is outlined in section 5. In other terms, this methodology transfers the problem of effective weak mixing for $L^2$ functions to that of finding upper bounds for the spectral measures and the twisted sums. To find quantitative bounds for the spectral measures, we will need both the speed of ergodicity, provided in [1], and a quantitative Veech argument similar to what is found in [3] and [5].

In this paper we utilize and generalize results from [5], specifically estimates for the spectral measure of a function in terms of the twisted sums, along with quantitative results for rank 1 cylindrical functions. Expanding on these findings, we extend the results to cylindrical functions of arbitrary rank and demonstrate that, in the case of the Chacon map, we can derive explicit bounds for the twisted sums. These bounds, in turn, enable us to estimate the speed of weak mixing. In particular we prove quantitative weak mixing for a primitive substitutions that is conjugate to the Chacon map. We will then transfer these results back to the Chacon map on the unit interval using the map that codes the orbit of each point. This leads to the following theorem, proved in section 10.

We will first prove effective weak mixing for simple functions on the substitution subshift using a Veech argument in sections 6 and 7 along with an estimate for the sped of ergodicity in section 8, then transfer these weak mixing estimates to a larger class of functions that can be approximated by these simple functions. In order to expand our results from simple functions to general observables, we will apply the Chacon substitution to subwords that define the cylinders the simple functions are defined on. We can get the growth rate of the twisted sums on simple functions determined by finite words, then leverage these estimates to get the result for infinite strings in the subshift. This is done by applying the substitution to the finite words iteratively and examining how a spectral variation of simple functions on the substitution subshift grows as we apply the substitution iteratively to words on the subshift. The explicit proof can be found in Section 6.

The substitution matrix $S$ for the Chacon substitution has the crucial property that the second largest eigenvalue has norm equal to one. Our estimate requires us to be able to bound vectors $v$ away from the integer lattice uniformly after iteratively applying the substitution matrix. In particular, we need $\|S^n(\omega \cdot v)\|_{\mathbb{Z}} \geq |\omega|$ for some $\omega \in (0,1)$ for some uniform frequency of $n \in \mathbb{Z}$ with $n \in [0, N]$. This will give us a quantitative Veech estimate, which allows us to follow the type of argument found in [5] or [3] to find effective weak mixing for the Chacon map.

The work from [14] proves that there is an interval $A \subset [0,1]$ such that the number of $n \in [0, N] \cap \mathbb{Z}$ such that $C^{-n}A \cap A = \emptyset$ is bounded below by a power of $\log(N)$. Hence we can create functions $f_A$ supported on $A$ that by definition satisfy $\langle f_A \circ C^n, f_A \rangle = 0$ for a number of times $n$ bounded below in terms of $N$. This allows us to derive lower bounds for effective weak mixing.

## 1.3 Construction of the Chacon Map

The construction of the Chacon map and a proof of its properties can be found in [7]. It was proved that the map is invertible, preserves the Lebesgue measure, and is weakly mixing but not mixing. The Chacon map is created inductively by cutting subintervals out of $[0, 1]$ and mapping them by translation onto each other by some inductive procedure. This mapping can be visually represented by stacking the subintervals in a tower such that each level is mapped onto the one vertically above it by translation. Each step in the procedure uses only part of the unit interval, but eventually every point in $[0, 1]$ will included in the domain of the Chacon map. The inductive scheme is as follows:

Step 1: Take the unit interval and cut it into two pieces so that the left hand interval has length $\frac{2}{3}$ and the right hand interval has length $\frac{1}{3}$. Call these pieces $I_0$ and $I_1$ respectively. Now cut $I_0$ into three equal pieces, labeled first to third from left to right, and also cut an interval of length equal to that of the first interval in $I_0$ from $I_1$. By translation we map the first interval from $I_0$ onto the second, the second interval onto the interval cut from $I_1$, and then map that interval onto the third interval from $I_0$. We can pictorially represent these translations by stacking the range of the translation on top of its domain. The image of each point is then the point vertically above it in the stack.

Step $n$: For the $n$'th step we cut the stack into three equally sized pieces and cut, starting from the left, an interval of equal size from the unused part of $I_1$. By translation we map the top of the left stack to the bottom of the middle stack, send the top of the middle stack to the newly cut piece of equal length from $I_1$, and map this interval onto the bottom of the third stack. Call the map that results $C_n$.

Now define the Chacon map by the pointwise limit $C = \lim_{n\to\infty} C_n$. Notice that for $m < n$ we have that $C_m(x) = C_n(x)$ on the domain of definition for $C_m$. Since this domain of definition for $C_m$ is eventually entire interval (mod sets of measure zero), the pointwise limit is defined almost everywhere.

If we code the orbit of each point $x \in I$ by $x_i = j$ if $C^i(x) \in I_j$ then a pattern emerges in the string $\{x_j\}$. We can see that each word in $\{x_j\}$ is contained in the string given recursively by $S_{n+1} = S_n S_n 1 S_n$ with $S_0 = 0$. This recurrence can be modeled by a substitution, and it is from this model that we will be able to find quantitative estimates for the Chacon map. Let $T$ be the shift map defined on $(\{0,1\}^{\mathbb{Z}}, T)$, and let $\alpha$ be the substitution defined by

$$\alpha(0) = 0010$$
$$\alpha(1) = 1.$$

There is a measurable map $h$ with a measurable inverse from the unit interval to the substitution subshift in $(\{0,1\}^{\mathbb{Z}}, T)$, called $(X, T)$, such that $h \circ C = T \circ h$. This map can be defined explicitly by looking at the code of each $x \in I$ under the action of the Chacon map $(C, I)$. Indeed, define $h : I \to X$ by $h(x)_i = j$ if $C^i(x) \in I_j$ where $j \in \{0,1\}$.

## 1.4 Substitutions

Let $T : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be the shift map on an alphabet $A$. Define $A_w$ to be the set of words with letters in $A$. A substitution is a map $\beta : A \to A_w$ and the substitution dynamical system is the shift map on a subshift defined to be the set of strings with the property that any word in the string is contained in a word of the form $\beta^n(a)$ for some $n \geq 0$ and $a \in A$.

The substitution matrix $S$ is defined such that $S_{i,j}$ is equal to the number of letters $i$ in the word $\beta(j)$. Notice that the length of the word $\beta^n(i)$, denoted by $|\beta^n(i)|$, satisfies

$$|\beta^n(i)| = \langle \hat{1}, S^n e_i \rangle,$$

where $\hat{1}$ is the vector with entries all equal to 1. We say that a substitution is primitive if the matrix $S$ is primitive, meaning there is an $n$ such that $S^n$ is entrywise positive. A substitution subshift is a closed shift invariant set of $A^{\mathbb{N}}$. If the substitution is primitive, then the shift is uniquely ergodic [15].

The substitution given by

$$\alpha(0) = 0010$$
$$\alpha(1) = 1$$

is conjugate to the primitive substitution

$$\beta(0) = 0012$$
$$\beta(1) = 12$$
$$\beta(2) = 012.$$

You can go from $\beta$ to $\alpha$ by replacing every 2 with a 0. The inverse is the function that sends every 0 that appears just after a 1 to a 2. Since primitive substitutions are uniquely ergodic the Chacon map is uniquely ergodic because it is conjugate to the primitive substitution subshift defined by $\beta$.

## 1.5 Spectral Measures and the Speed of Weak Mixing

We will first define quantitatively what it means for a process to "mix" two different sets together. Let $T : X \to X$ be a measurable transformation that preserves the probability measure $\mu$ on $X$.

**Definition 1.5.1.** *A transformation $T : X \to X$ is called weakly mixing if for any $f, g \in L^2(X)$ we have*

$$\frac{1}{N}\sum_{k=0}^{N-1} \left|\int_X f(T^k x)\overline{g(x)}d\mu - \int_X f d\mu \overline{\int_X g d\mu}\right| \to 0 \text{ as } N \to \infty$$

Notice that in the definition of weakly mixing we only demand that the terms inside the sum converge to zero in average. A strengthening of this condition is to demand that the sequence converges to zero.

**Definition 1.5.2.** *A transformation $T : X \to X$ is strongly mixing, or mixing, if for any $f, g \in L^2(X)$*

$$\left|\int_X f(T^k x)\overline{g(x)}d\mu - \int_X f d\mu \overline{\int_X g\, d\mu}\right| \to 0 \text{ as } N \to \infty$$

In this paper we want to find some function $h(N)$ with the property $\frac{h(N)}{N} \to 0$ such that for any measurable functions $f$ and $g$ with certain regularity conditions we have

$$\sum_{k=0}^{N-1} |\int_X f(T^k x)\overline{g(x)}d\mu - \int_X f d\mu \overline{\int_X g d\mu}| = O(h(N)).$$

In our search for a suitable $h(N)$ we will make use of the spectral measure to simplify the sum that appears in the definition of weak mixing.

Let $(X, T, \mu)$ is a measure-preserving transformation. We can associate each $f \in L^2(X)$ with a positive measure on the unit circle, denoted $\sigma_{f,f}$ or $\sigma_f$, that has Fourier coefficients

$$\hat{\sigma}_{f,f}(-k) = \int_0^1 e^{2\pi i k\omega} d\sigma_{f,f}(\omega) = \langle f \circ T^k, f\rangle.$$

The existence of such a measure is guaranteed by Bochner's theorem. Let $U$ be the operator on $L^2(X)$ defined by $U(f) = f \circ T$. Using the spectral measure we can reduce the sum found in the definition of weak mixing. Indeed, for $f$ with zero average we have

$$\begin{aligned} I_N = \sum_{n=0}^{N-1} |\langle U^n(f), f\rangle|^2 & \\ &= \sum_{n=0}^{N-1} \langle U^n(f), f\rangle \int_{R/Z} e^{2\pi i n\omega} d\sigma_f(\omega) \\ &= \int_{R/Z} \langle \sum_{n=0}^{N-1} e^{2\pi i n\omega} U^n(f), f\rangle d\sigma_f(\omega) \\ &= I_\epsilon^- + I_\epsilon^+ \end{aligned}$$

where

$$I_\epsilon^-(N) = \int_{-\epsilon}^{\epsilon} \langle \sum_{n=0}^{N-1} e^{2\pi i n\omega} U^n(f), f\rangle d\sigma_f(\omega) \tag{1.1}$$

and

$$I_\epsilon^+(N) = \int_{\epsilon}^{1-\epsilon} \langle \sum_{n=0}^{N-1} e^{2\pi i n\omega} U^n(f), f\rangle d\sigma_f(\omega). \tag{1.2}$$

The term $\sum_{n=0}^{N-1} e^{2\pi in\omega} U^n(f)(x) = S_N^x(f,\omega)$ is called the twisted sum. We can see from this estimate that we will need to find a bound for the twisted sum in order to find the speed of weak mixing. Sections 6 and 7 provide the machinery to estimate $I_\epsilon^+$ and section 8 will estimate $I_\epsilon^-$.

## 1.6 Estimating the Twisted Sums Using Matrix Products

We say that $w$ is a word in the substitution if it is contained in $\beta^n(a)$ for some $n$ and letter $a \in A$. Let $T$ be the shift map. Let $w = w_1 \dots w_p$ be a word with $w_i \in A$. We define $T^n(w) = w_{n+1} \dots w_p$ for $0 \le n < p$. A cylinder of rank $n$ is defined to be the set of strings such that the first $n$ coordinates are equal to a fixed word of length $n$.

For $x \in X$, define $x[0,k] = x_0 \dots x_k$. A cylinder of rank $n$ is defined to be the set of $x \in X$ such that $x[0, n-1] = w$ for some word $w$ in the substitution with length $n$. Let $J_n$ be the number of cylinders of length $n$ in the shift invariant substitution space $X$. We will enumerate each cylinder of rank $n$ and index them with $k$. The notation $[k,n]$ means the $k$'th cylinder of rank $n$ and $1_{[k,n]}$ is the characteristic function on $[k,n]$. Since $\beta$ is a primitive substitution we have $J_n \le c'n$ for some $c' > 0$ that depends only on the substitution [15]. If a function $f : X \to \mathbb{R}$ is of the form

$$f(x) = \sum_{k=1}^{J_n} r_k 1_{[k,n]}(x) \quad \text{for } r_k \in \mathbb{R}$$

then we say that $f$ is a cylindrical function of rank $n$. Note that $n$ is the length of the cylinder $[k,n]$.

The following results in this section follow the strategy used in section 3 of [5]. We will alter that method slightly to prove resutls for cylindrical functions of arbitrarily large rank. Let $v$ be a word of length equal to or greater than $n$.

**Definition 1.6.1.** *For a word $v$ in the substitution and $\omega \in [0,1)$ let*

$$\phi_{[k,n]}(v,\omega) = \sum_{j=0}^{|v|-n} 1_{[k,n]}(T^j v)e^{-2\pi i\omega j}.$$

There is a formula for $\phi_{[k,n]}(vw,\omega)$ in terms of $\phi_{[k,n]}(v,\omega)$ and $\phi_{[k,n]}(w,\omega)$ that is similar to what is found in [5] for rank one cylindrical functions. One notable difference is that the indicator function $1_{[k,n]}$ will eventually process words that are parts of both $v$ and $w$ after they have been shifted a sufficient number of times. The following term will account for this.

**Definition 1.6.2.** *Let $v = v_1 \cdots v_p$ and $w = w_1 \cdots w_q$ be words in the substitution with $v_i, w_i \in A$ and $p, q \geq n > 1$. Define*

$$H(v,w,\omega,n,k) = \sum_{j=1}^{n-1} 1_{[k,n]}(v_{p-j+1}\cdots v_p w_1 \cdots w_{n-j})e^{-2\pi i\omega j}.$$

The $\omega$, $n$ and $k$ will often be fixed and sometimes dropped.

**Lemma 1.6.3.** *Let $v$ and $w$ be words in the substitution and $\omega \in [0,1)$. Then*

$$\phi_{[k,n]}(vw,\omega) = \phi_{[k,n]}(v,\omega) + e^{-2\pi i\omega|v|}\phi_{[k,n]}(w,\omega) + e^{-2\pi i\omega(|v|-n+1)}H(v,w,\omega,n,k)$$

*Proof.* The proof can be seen by simply writing out the definition of the terms on the right hand side. □

For the remainder of the section, our objective will be to find a bound for entries in an associated matrix product using an inductive process in order to bound the $\phi_{[k,n]}$ .

**Definition 1.6.4.** *Let $1, \ldots, p \in A$ enumerate the letters in the alphabet $A$ and let $v^t$ denote the transpose of the vector $v$. For $\omega \in [0,1)$ we define*

$$\Psi_m^{[k,n]}(\omega) = [\phi_{[k,n]}(\beta^m(1),\omega), \cdots, \phi_{[k,n]}(\beta^m(p),\omega)]^t$$

*where $[k,n]$ is the k'th cylinder of rank n.*

As in the case of rank one cylinders, if we can estimate the growth of $\phi_{[k,n]}(\beta^m(i), \omega)$ then we can obtain bounds for the twisted Birkhoff sums for the cylindrical function $f$. This is due to the fact that the words $\beta^m(i)$ are in some sense dense in the substitution subshift.

Remark: A technicality to keep in mind is we need the length of the words $\beta^m(i)$ to be greater than $n$, the length of the cylinder. The Perron-Frobenius theorem tells us that there are constants $c'$ and $c$ such that $c'\theta^m \leq |\beta^m(i)| \leq c\theta^m$ for each $i \in A$, where $\theta$ is the Perron-Frobenius eigenvalue. Hence we will assume that the fixed length of the cylinder $n$ satisfies $n \leq c'\theta^m \leq |\beta^m(i)|$.

In parallel with [5] we begin our estimates by rewriting the relationship in Lemma 1.6.3 as a matrix product with some error term.

**Definition 1.6.5.** *Let $\omega \in [0, 1)$ and let the length of the substitution cylinders be a fixed $n$. Define*

$$\Pi_m(\omega) := [\Psi_m^{[1,n]}(\omega) \cdots \Psi_m^{[J_n,n]}(\omega)].$$

**Lemma 1.6.6.** *There is a matrix $M_{m-1}(\omega)$ and a vector $E_{m-1}(\omega)$ with $|E_{m-1}| \leq Kn$ entry-wise such that for $K' + m \geq \log_\theta(n)$ and $\omega \in [0, 1)$ we have the following relation:*

$$\Pi_m(\omega) = M_{m-1}(\omega)\Pi_{m-1}(\omega) + E_{m-1}(\omega).$$

*Proof.* The result will follow after looking at the expansion of $\phi_{[k,n]}(\beta^m(b), \omega)$.

Let $\beta(b) = u_1^{(b)} \cdots u_{k_b}^{(b)}$ with $u_i^{(b)} \in A$.

$$\phi_{[k,n]}(\beta^m(b), \omega) = \sum_{j=1}^{k_b} \exp\left[\left[-2\pi i \left(|\beta^{m-1}(u_1^{(b)})\omega| + \cdots + |\beta^{m-1}(u_{j-1}^{(b)})|\right)\omega\right] \phi_{[k,n]}(\beta^{m-1}(u_j^{(b)}), \omega)$$
$$+ \sum_{j=1}^{k_b-1} e^{i\alpha_j} H\left(\beta^m(u_j^{(b)}), \beta^m(u_{j+1}^{(b)})\right),$$

where the $\alpha_j \in \mathbb{R}$ is calculated as in Lemma 1.6.3 and $H$ is from Definition 1.6.2. This implies the following definition for $M_{m-1}(\omega)$:

$$M_{m-1}(\omega)_{b,c} = \sum_{j \le |\beta(b)| : u_j^{(b)} = c} \exp\left[-2\pi i \left(|\beta^{m-1}(u_1^{(b)})| + \cdots + |\beta^{m-1}(u_{j-1}^{(b)})|\right)\omega\right].$$

Now define

$$E_m = \left[\sum_{j=1}^{k_1-1} e^{i\alpha_j} H\left(\beta^m(u_j^{(1)}), \beta^m(u_{j+1}^{(1)})\right), \cdots, \sum_{j=1}^{k_p-1} e^{i\alpha_j} H\left(\beta^m(u_j^{(p)}), \beta^m(u_{j+1}^{(p)})\right)\right]^t$$

where $p$ is the size of the alphabet for the substitution. □

Hence for $m + K' \geq \log_\theta(n)$ we can use induction on Lemma 1.6.6 to get the following formula.

$$\Pi_m(\omega) = \left[\prod_{j=n+1}^{m} M_j(\omega)\right] \Pi_n(\omega) + \sum_{k=n+1}^{m} M_k(\omega) E_{k-1}(\omega). \tag{1.3}$$

To proceed with bounding $\Pi_m(\omega)$ we need to bound both the $E_{m-1}$ and the product of the $M_k$. The majority of the rest of this section will be dedicated towards finding a bound on the product of the $M_j(\omega)$.

**Theorem 1.6.7.** *There is a constant $0 < c' < 1$ dependent only on the substitution such that*

$$\left[\prod_{j=n+1}^{m} M_j(\omega)\right] \hat{1} \leq \prod_{k=n+1}^{m} (1 - c'\|\omega\|^2)(S^t \hat{1})^{m-n}.$$

*entrywise.*

Since $\omega$ will be fixed, we occasionally omit it in our notation. Our proof will depend on a series of quantitative lemmas. Furthermore, inequalities comparing matrices and vectors are understood to be entrywise. We will also denote $|A|$ to be the matrix with entries equal to the absolute value of those from $A$.

**Lemma 1.6.8.** *The entries of* $\Pi_n(\omega)$ *are bounded above by* $c\theta^n$*, where* $c > 0$ *is dependent only on the substitution.*

*Proof.* The matrix $\Pi_n(\omega)$ has entries equal to

$$\phi_{[k,n]}(v,\omega) = \sum_{j=0}^{|v|-n} 1_{[k,n]}(T^j v)e^{-2\pi i\omega j}$$

with $v = \beta^n(i)$. Hence we have the entrywise bound

$$|\phi_{[k,n]}(v,\omega)| \leq |v| - n \leq c\theta^n$$

for $\Pi_n(\omega)$.

□

**Lemma 1.6.9.** $|\sum_{k=n}^{m} M_k(\omega)E_{k-1}(\omega)| \leq mnC_S$ *where* $C_S > 0$ *depends only on the substitution.*

*Proof.* This follows from the fact that $|M_k(\omega)| \leq S^t$, where $S$ is the substitution matrix, and $|E_k(\omega)| \leq n$. Now set $C_S = \|S^t\|$

□

**Definition 1.6.10.** *Let* $v$ *be an* $m$*-dimensional real vector. Define*

$$\|v\|_{\mathbb{Z}} = \max_i \|v_i\|_{\mathbb{Z}}$$

*to be the maximum of the distances between each coordinate* $v_i$ *to the integers.*

We will sometimes abbreviate the notation to $\|v\|$. Note that in the one dimensional case this reduces to just the distance of the point to the integers.

Our proof will need the existence of words $v_i$ such that for each $b \in A$ there is a word $v_i$ and a letter $c_i$ such that $v_i$ starts with $c_i$ and $v_i c_i$ is contained in $\beta(b)$. We will call the $v_i$ return words. Hence for each $b$ we can define $p_i^{(b)}$ and $q_i^{(b)}$ so that $\beta(b) = p_i^{(b)} v_i c_i q_i^{(b)}$. Note that the $M_n(\omega)$ have entries that are trigonometric polynomials with coefficients equal to 1 with terms not more than the corresponding integer entry in the transpose of the substitution matrix $S^t$. Using the definition of $M_n$ we have that the terms $e^{-2\pi i|p_i^{(b)}|\omega}$ and $e^{-2\pi i|p_i^{(b)}v_i|\omega}$ are both contained in the entry $M_n(\omega)_{b,c_i}$. By excluding them through subtraction and then adding the absolute value of their sum we get

$$|M_n(\omega)_{b,c_i}|\leq (S^t)_{b,c_i} - 2 + |e^{2\pi i\omega|\beta^n(v_i)|} + 1|.$$

The inequality

$$|1 + e^{2\pi it}|\leq 2 - \tfrac{1}{2}\|t\|^2$$

then implies

$$|M_n(\omega)_{b,c_i}|\leq (S^t)_{b,c_i} - \tfrac{1}{2}\|\omega|\beta^n(v_i)|\|^2$$

for each $i$.

From [5] we get, for $\hat{x} = (x_1, \ldots, x_n) > 0$ a positive vector and $v$ a fixed return word that starts with the letter $c \in A$,

$$\begin{aligned}
(|M_k|\hat{x})_b &= \sum_{j=1}^{p}|M_k(b,j)|x_j \\
&\leq \sum_{j=1}^{p} S^t(b,j)x_j - \frac{1}{2}\|\omega|\beta^k(v)|\|^2 x_c \\
&\leq (1 - c(\hat{x})\|\omega|\beta^k(v)|\|^2)\sum_{j=1}^{p} S^t(b,j)x_j \\
&= (1 - c(\hat{x})\|\omega|\beta^k(v)|\|^2)(S^t\hat{x})_b,
\end{aligned} \tag{1.4}$$

where $c(\hat{x})$ is defined by the equation

$$c(\hat{x}) = \frac{x_c}{2m\max_j (S^t)_{b,j}\max_j x_j}. \tag{1.5}$$

We can use this formula inductively to get a bound for a product of the $M_k(\omega)$ . During each step of the induction we can pick a suitable return word $v_k$ to control the dynamics of the quantity $\|\omega|\beta^k(v)|\|$. The choice of $v_k$ will be determined in the next few paragraphs.

Note that equation (1.4) implies that entrywise

$$|M_{k+1}M_k|\hat{x} \leq |M_{k+1}|\Big((1 - c(\hat{x})\|\omega|\beta^k(v)|\|^2)S^t\hat{x}\Big). \tag{1.6}$$

If we use this along with the substitution $\hat{x} \to S^t\hat{x}$ we get the entrywise inequality

$$M_{k+1}M_k\hat{x} \leq \Big(1 - c(\hat{x})\|\omega|\beta^k(v)|\|^2)(1 - c(S^t\hat{x})\|\omega|\beta^k(v)|\|^2)\Big)(S^t)^2\hat{x}. \tag{1.7}$$

Now set $\hat{x} = \hat{1}$. The Perron-Frobenius theorem implies that $c = \inf_n\{c((S^t)^n\hat{1})\}$ is positive. Using the above calculations iteratively we obtain the bound

$$|\prod_{j=n+1}^{m} M_j(\omega)|\hat{1} \leq \prod_{k=n+1}^{m}(1 - c\|\omega|\beta^k(v_k)|\|^2)(S^t\hat{1})^{m-n}. \tag{1.8}$$

The following is found in [6]. Since the substitution is primitive, the dynamical system $X_\beta$ is equal to $X_{\beta^n}$ for each $n \geq 1$. Recall that return words were defined after Definition 1.6.10. It is possible that there are no return words when examining $\beta$, however, we can pass to $\beta^n$ and obtain the existence of return words if needed. In the case of the Chacon substitution we note that all of $v_1 = 12$, $v_2 = 012$ and $v_3 = 01201$ are return words for $\beta^3$. If $v$ is a word in the substitution we define the population vector $l(v)$ to be equal to the vector whose $i$'th entry is equal to number of the letter $i$ in $v$. Notice that the population vectors for these return words generate $\mathbb{Z}^3$.

This implies that there are $a_{i,j} \in \mathbb{Z}$ such that $\sum_{i=1}^{k} a_{j,k}l(v_k) = e_j$. Hence

$$\begin{aligned}\|\hat{x}\| = \max_i\|x_i\|_{\mathbb{Z}} &= \max_i\|\langle\sum_{j=1}^{k} a_{i,j}l(v_j), \hat{x}\rangle\|_{\mathbb{Z}} \\ &\leq \max_{i\leq m}\sum_{j=1}^{k}|a_{i,j}|\cdot\max_{j\leq k}|\langle l(v_j), \hat{x}\rangle|\end{aligned} \tag{1.9}$$

This shows that there is a $C > 1$ independent of $x$ such that

$$C^{-1}\|\hat{x}\| \leq \max_{j\leq k} |\langle l(v_j), \hat{x}\rangle| \tag{1.10}$$

The Chacon map is conjugate to the substitution given by

$$0 \to 0012$$

$$1 \to 12$$

$$2 \to 012$$

The substitution matrix $S$ is

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and $b = [-1, 1, 1]^t$ is an eigenvector for $S$ with eigenvalue 1. Now define

$$\hat{x}_k(\omega) := \omega(S^t)^k \hat{1}. \tag{1.11}$$

**Lemma 1.6.11.** *There is an $\alpha \in (0,1)$ such that for all $k \in \mathbb{Z}^+$ and all $\omega \in [0,1)$ we have $\|\hat{x}_k(\omega)\|_{\mathbb{Z}} \geq \alpha\|\omega\|_{\mathbb{Z}}$*

*Proof.* If not then there are sequences $k_j$ and $\omega_j > 0$ such that

$$\|\hat{x}_{k_j}(\omega)\| \leq \frac{1}{j}\|\omega_j\|.$$

Note that $\langle \hat{x}_{k_j}(\omega), b\rangle = \omega$ since $\langle b, \hat{1}\rangle = 1$ and $Sb = b$. This implies that

$$\|\omega_j\|_{\mathbb{Z}} = \|\langle \hat{x}_{k_j}(\omega), b, \rangle\|_{\mathbb{Z}} \leq \left(\sum_i |b_i|\right) \|x_{k_j}(\omega)\|_{\mathbb{Z}} \leq \frac{1}{j}\left(\sum_i |b_i|\right)\|\omega_j\|_{\mathbb{Z}}$$

for all $j$, a contradiction. ∎

It follows that

$$\|\omega(S^t)^k\hat{1}\|_{\mathbb{Z}}= \|\hat{x}_k(\omega)\|_{\mathbb{Z}}\geq \alpha\|\omega\|_{\mathbb{Z}} \tag{1.12}$$

for some $1 > \alpha > 0$.

Thus we obtain

$$\prod_{k=n+1}^{m}(1-c(x_k)\|\omega|\beta^k(v_k)|\|^2)(S^t\hat{1})^{m-n} \leq \prod_{k=n+1}^{m}(1-c'\|\omega\|^2)(S^t\hat{1})^{m-n}. \tag{1.13}$$

where $v_k \in \{12, 012, 01201\}$ is chosen to maximize $|\langle l(v_k), \hat{x}_k\rangle|$ and $c' = \alpha \cdot \inf_i\{c(x_i)\} > 0$ by the Perron-Frobenius theorem. This completes the proof of Theorem 1.6.7.

□

**Theorem 1.6.12.** *There is an independent constant $k > 0$ such that for each $i$ and $j$*

$$(e_i)^t|\prod_{j=n+1}^{m} M_j(\omega)|e_j \leq k\prod_{j=n+1}^{m}(1-c'\|\omega\|^2)\theta^{m-n}$$

*where $\{e_k\}$ is the standard basis.*

*Proof.* This follows from Theorem 5.3 and the inequality $\langle \hat{e}_j, (S^t)^q\rangle = |\beta^q(j)|\leq k\theta^q$, where $k > 0$ depends only on the substitution. □

**Corollary 1.6.13.** *Let $m \leq K\log_\theta N$ for some constant $K > 1$ dependent only on the substitution. Then, entrywise, there are consants $c''$ and $C_S$ dependent only on the substitution such that*

$$|\Pi_m(\omega)|\leq C_S N^{1-c''\|\omega\|^2} + C_S mn.$$

*Proof.* Choose $n$ so that $n \leq \frac{1}{2}\log_\theta N$ Since the entries in the product are less than one, and since $m - n - 1 \geq \frac{1}{2}\log_\theta N$ for large enough $N$, we have

$$\prod_{k=n+1}^{m}(1-c'\|\omega\|^2) \leq ((1-c'\|\omega\|^2))^{\frac{1}{2}\log_\theta N} = N^{\frac{1}{2}\log_\theta(1-c'\|\omega\|^2)} \leq N^{-c''\|\omega\|^2}. \tag{1.14}$$

The result then follows from Lemma 1.3, Lemma 1.6.9, Theorem 1.6.7, and Theorem 1.6.12.

□

The ingredients specific to the Chacon map to prove Theorem 1.6.7 include the existence of a set of return words whose population vectors span $\mathbb{Z}^n$. We also used the property that the eigenvector for the substitution matrix corresponding to eigenvalue 1 has non-trivial projection onto the vector with ones in all entries and that the substitution is primitive. These assumptions are sufficient to extend the quantitative Veech criterion, or Theorem 1.6.7, to other substitutions that satisfy the above conditions.

## 1.7 Estimates involving arbitrary strings

In the previous section we found estimates for the twisted sums on words of the form $\beta^k(b)$ for some $b \in A$. We will use this result to find bounds for the growth of twisted sums on arbitrary strings. What is found in this section is partially a generalization of the techniques found in [5]. Let $[k,n]$ be the $k$'th cylinder of rank $n$. We will consider cylindrical functions $f = \sum_k r_k 1_{[k,n]}$. Set

$$\phi_f(v,\omega) = \textstyle\sum_k r_k \phi_{[k,n]}(v,\omega)$$

The following theorem is the building block of the paper. Later we will see that cylindrical functions are able to approximate Lipschitz functions sufficiently enough to transfer the following bound.

**Theorem 1.7.1.** *Let $n$ be the rank of the cylidnrical function $f$ and let $x[0, N-1]$ be the word comprised of the first $N$ terms of the string $x$. Let $n \leq \frac{1}{2}\log_\theta N$ and $m \leq C\log_\theta N$ for some $C > 0$ large enough. Then we have the following bound for the twisted sum:*

$$|\phi_f(x[0, N-1], \omega)| \leq 2L \cdot C'_S \cdot n \cdot \|f\|_L \Big(N^{1-c'\|\omega\|^2} + 6(\log_\theta N)^2 + 2\log_\theta N + 1\Big)$$

The main idea behind the proof is that the words $\beta^k(b)$ form the building blocks of the strings $x \in X$. The following lemma, found in [16], gives an explicit statement.

**Lemma 1.7.2.** *Let $x \in X$ and $N \geq 1$. Then*

$$x[0, N-1] = u_0\beta(u_1)\cdots\beta^m(u_m)\beta^m(v_m)\cdots\beta(v_1)v_0$$

*where $m \geq 0$ and the $u_i, v_i$, are respectively proper prefixes and suffixes of the words $\beta(b)$ for $b \in A$.*

The words $u_i$ and $v_i$ may be empty but at least one of $u_n, v_n$ is non-empty. We will also need the following lemma, which relates the $N$ in $x[0, N-1]$ with the $m$ in the prefix-suffix decomposition. This next result is a corollary of the above lemma and the Perron-Frobenius theorem.

**Lemma 1.7.3.** *For all $b \in A$ there are constants $c, c' > 0$ such that*

$$c\theta^m \leq \min_{b\in A}|\beta^m(b)| \leq N \leq 2\max_{b\in A}|\beta^{m+1}(b)| \leq 2c'\theta^{m+1}$$

We can now decompose the generalized twisted sum for $x[0, N-1]$ by using Lemma 1.6.3 and Lemma 1.7.2.

$$
\begin{aligned}
|\phi_f(x[0,N-1],\omega)| \leq & \sum_{j=0}^{m}\left(|\phi_f(\beta^j(u_j)|+|\phi_f(\beta^j(v_j)|\right) \\
& + \|f\|_L \sum_{j=0}^{m-1} H(\beta^j(u_j),\beta^{j+1}(u_{j+1})) \\
& + \|f\|_L \sum_{j=0}^{m-1} H(\beta^{m-j}(v_{m-j}),\beta^{m-j-1}(v_{m-j-1})) \\
& + \|f\|_L H(\beta^m(u_m),\beta^m(v_m))
\end{aligned}
$$

If the rank of the cylindrical function $f$ is $n$ then the second and the third sums satisfy

$$
\|f\|_L \sum_{j=0}^{m-1} |H\left(\beta^j(u_j),\beta^{j+1}(u_{j+1})\right)| \leq Kmn\|f\|_L \tag{1.15}
$$

for $K>0$. We now only need to find a bound for the quantity

$$
\sum_{j=0}^{m}(|\phi_f(\beta^j(u_j),\omega)|+|\phi_f(\beta^j(v_j),\omega)|).
$$

Each word of length $n$ is contained in exactly one cylinder of the corresponding length. It is also true that the $H(v,w)$ collects at most $n$-many shifts of the concatenation of $v$ and $w$. This implies that, in particular, we only collect at most $n$-many non-zero terms of the form $r_k 1_{[k,n]}(T^q(vw))$ from the $H(v,w)$. Lemma 1.6.3 then implies that

$$
|\phi_f(vw,\omega)| \leq |\phi_f(v,\omega)|+|\phi_f(w,\omega)|+\|f\|_\infty n. \tag{1.16}
$$

Now set $L=\max_{b\in A}|\beta(b)|$. The above inequality and Lemma 1.7.2 implies

$$
|\phi_f(\beta^j(u_j)| \leq L\left(\max_{b\in A}|\phi_f(\beta^j(b),\omega)|\right)+nL\|f\|_\infty. \tag{1.17}
$$

Here we used the fact that $u_j$ and $v_j$ are prefixes and suffixes of the words $\beta(b)$ for $b\in A$. Since we are considering a primitive substitution, the number of cylinders of length $n$, called $J_n$, is bounded above by $C'n$ for some $C'>0$ dependent only on the substitution [15]. If we set

$$\phi_f(v,\omega) = \textstyle\sum_k r_k \phi_{[k,n]}(v,\omega).$$

then each $|r_k| \leq \|f\|_L$. Thus for $j$ such that $|\beta^j(i)| \geq n$ we have

$$|\phi_f(\beta^j(b),\omega)| = |\sum_{k=1}^{J_n} r_k \phi_{[k,n]}(\beta^j(b),\omega)|$$

$$\leq C' n \|f\|_L \max_{k \leq J_n, i \in A} |\phi_{[k,n]}(\beta^j(i),\omega)|$$

We will now use a lemma that is analogous to what is found in [5] for rank $n$ cylindrical functions.

**Lemma 1.7.4.** *Let $\beta$ be a primitive substitution on $A$, and let $\theta$ be the Perron-Frobenius eigenvalue of the substitution matrix $S$. Take a cylindrical function $f = \sum r_k 1_{[k,n]}$ and a number $\omega \in [0,1)$, and suppose there exists a sequence $\{F_\omega(n)\}_{n\geq 0}$ satisfying*

$$\tfrac{F_\omega(n)}{\theta'} \leq F_\omega(n+1) \leq F_\omega(n), \quad n \geq 0$$

*with $1 < \theta' < \theta$, such that*

$$|\phi_f(\beta^m(b),\omega)| \leq n|\beta^m(b)| F_\omega(m) + mn.$$

*Then*

$$|\phi_f(x[0,N-1],\omega)| \leq \|f\|_L \Big(\tfrac{C_1}{\theta - \theta'} n N F_\omega(\lfloor \log_\theta N - C_2 \rfloor) + 6m^2 n + 1\Big)$$

*Proof.* The estimates (1.15) - (1.18) imply the first line in the inequality below, and the second line follows from the hypothesis in the lemma.

$$
\begin{aligned}
|\phi_f(x[0,N-1],\omega)| &\leq n\|f\|_L\left[K\sum_{j=0}^{l}\max_{b\in A}|\beta^j(b)|F_\omega(j)+(2l+1)\right] \\
&\leq nK\|f\|_L\left[\sum_{j=0}^{l}\theta^j(\theta')^{l-j}F_\omega(l)\right] \\
&< K'\|f\|_L\Big(n\theta^l F_\omega(l)+6m^2n\Big)
\end{aligned}
\tag{1.18}
$$

Since $c\theta^l \leq N \leq c'\theta^{l+1}$ the result follows. □

We can now complete the proof for Theorem 1.7.1.

*Proof.* If we consider both equation (1.18) and Corollary 1.6.13 then

$$
|\phi_f(\beta^j(b),\omega)|\leq k'n|\beta^j(b)|\prod_{k=n+1}^{m}(1-c'\|\omega\|^2)+mn. \tag{1.19}
$$

Set $F_\omega(m)=\prod_{k=n+1}^{m}(1-c'\|\omega\|^2)$ and let $c'<\frac{\theta-1}{\theta+1}$ (we can make $c'$ as small as needed). If $\theta'=\frac{1+\theta}{2}$ then

$$
|\sum_{j=0}^{m}\Big(|\phi_f(\beta^j(u_j),\omega)|+|\phi_f(\beta^j(v_j),\omega)|\Big)|\leq C'_S n\|f\|_L\Big(N^{1-c'\|\omega\|^2}+6m^2+1\Big) \tag{1.20}
$$

The estimate on $m$ in the hypothesis completes the proof of Theorem 1.7.1. □

## 1.8 Speed of Ergodicity for Cylindrical Functions

In order to estimate the quantity $I_\epsilon^-$ from equation (1.1) we need to determine the behavior of the spectral measure $\sigma_f$ near 0. To accomplish this we are going to bound the Birkhoff sum of the cylindrical function $f$ and use the fact that bounds for this quantity imply bounds for the measure of small sets. The main result of this section is the following theorem.

**Theorem 1.8.1.** *Let* $f = \sum_{k=1}^{J_n} r_k 1_{[k,n]}$ *be a rank* $n$ *cylindrical function with*

$$0 = \int_X f d\mu = \sum_{k=1}^{J_n} r_k \mu([k,n])$$

*and let* $\|f\|_\infty$ *be the maximum of the absolute value of* $f$*. Then the Birkhoff sums of* $f$ *have the following bound for some* $C'' > 0$ *dependent only on the substitution:*

$$|\sum_{j=0}^{N-1} f(T^j(x)| \leq nC''\|f\|_\infty (\log_\theta N)^2.$$

*Proof.* We will need the following result, which is theorem 3 from [1].

**Theorem 1.8.2.** *Let* $A^n(X)$ *denote the allowable words of length* $n$ *in the substitution dynamical system* $(X,T)$*. Then we have*

$$D_N(X) = \sup_{x\in X} \sup_{w\in A^n(X)} |\sum_{k=0}^{N-1} \left(1_{[w]}(T^k(x) - N\mu([w])\right) | \leq C[\log_\theta N]^2$$

uniformly in $n$ where $C$ depends only on the substitution.

□

Recall that the number of words of length $n$ in $X$, denoted by $J_n$, satisfies $J_n \leq C'n$. Since $f$ has zero average the lemma implies that

$$\begin{aligned}
|\sum_{j=0}^{N-1} f(T^j x)| &= |\sum_{j=0}^{N-1} \left( \sum_k r_k 1_{[k,n]}(T^j x) - N r_k \mu([k,n]) \right) | \\
&\leq J_n \|f\|_\infty \max_{1\leq k\leq J_n} |\sum_{j=0}^{N-1} 1_{[k,n]} T^j(x) - N\mu([k,n])| \\
&\leq C'n\|f\|_\infty D_N(X) \\
&\leq C''n\|f\|_\infty [\log_\theta N]^2
\end{aligned}$$

This implies that the speed of ergodicity for cylindrical functions with average zero of rank $n$ satisfies

$$|\sum_{k=0}^{N-1} f(T^k x)| \leq nC''\|f\|_\infty (\log_\theta N)^2$$

where $C''$ is independent of $N$ and $x$ and depends only on the substitution.

□

## 1.9 Speed of Weak Mixing for Particular Cylindrical Functions

Some of the analytic techniques from this section are borrowed from [3]. The framework has been slightly adapted for the case of cylindrical functions of arbitrarily large rank.

**Theorem 1.9.1.** *If $f$ is a cylindrical function of rank $n = \lfloor (\log_\theta N)^{\frac{1}{6}} \rfloor$ that has zero average and if $g \in L^2(X)$, then*

$$\tfrac{1}{N} \sum_{k=0}^{N-1} |\langle U^k(f), g\rangle|^2 \leq K_S \|f\|_L \|f\|_2 \|g\|_2^2 [\log_\theta N]^{-\frac{1}{6}}$$

*Proof.* Observe that

$$\begin{aligned} I = \sum_{n=0}^{N-1} |\langle U^n(f), g\rangle|^2 &= \sum_{n=0}^{N-1} \langle U^n(f), g\rangle \int_{R/Z} e^{2\pi i n\omega} d\sigma_{f,g}(\omega) \\ &= \int_{R/Z} \langle \sum_{n=0}^{N-1} e^{2\pi i n\omega} U^n(f), f\rangle d\sigma_{f,g}(\omega) \\ &= I_\epsilon^- + I_\epsilon^+ \end{aligned}$$

where

$$I_\epsilon^- = \int_{-\epsilon}^{\epsilon} \langle \sum_{n=0}^{N-1} e^{2\pi i n\omega} U^n(f), g\rangle d\sigma_{f,g}(\omega)$$

and

$$I_\epsilon^+ = \int_{\epsilon}^{1-\epsilon} \langle \sum_{n=0}^{N-1} e^{2\pi i n\omega} U^n(f), g\rangle d\sigma_{f,g}(\omega).$$

We are now going to use bounds on the twisted sums to control $I_\epsilon^+$ and speed of ergodicity estimates for $I_\epsilon^-$. The upper bound from Theorem 1.7.1 implies

$$|S_N^x(f,\omega)| \leq C_S' \cdot (\log_\theta N)^{\frac{1}{6}} \cdot \|f\|_L \left(N^{1-c'\epsilon^2} + (\log_\theta N)^2\right). \tag{1.21}$$

This along with the inequalities

$$|\sigma_{f,g}|(B)|\leq \sqrt{\sigma_{f,f}(B)}\sqrt{\sigma_{g,g}(B)} \leq \|f\|_2\|g\|_2$$

for measureable $B$ implies that

$$\begin{aligned}|I_\epsilon^+|&\leq \|S_N^x(f,\omega)\|_\infty\|g\|_2^2\|f\|_2\\ &\leq C_s'(\log_\theta N)^{\frac{1}{6}}\left(N^{1-c'\epsilon^2}+(\log_\theta N)^2\right)\|f\|_L\|f\|_2\|g\|_2^2\end{aligned}$$

We will now bound $I_\epsilon^-$. We need the following lemma from [5] to relate the size of the measure of small $\epsilon$-balls with bounds for the twisted sums.

**Lemma 1.9.2.** *If* $G_N(f,\omega)=N^{-1}\int_X|S_N^x(f,\omega)|^2d\mu(x)$ *and* $r=\lfloor(2N)^{-1}\rfloor$ *then*

$$\sigma_f(B(\omega,r))\leq \tfrac{\pi^2}{4N}G_N(f,\omega)$$

□

In the following estimate we set $\omega=0$ and use the above lemma along with equation (1.21) to get

$$\sigma_f(B(0,\epsilon))\leq K_1\|f\|_L^2(2\epsilon)^2[\log_\theta(\tfrac{1}{2\epsilon})]^{4+\frac{1}{3}}.$$

It follows that

$$\begin{aligned}I_\epsilon^-&\leq \sigma_{f,g}(B(0,\epsilon))N\|g\|_2\|f\|_2\\ &\leq K_2(2\epsilon)[\log_\theta(\frac{1}{2\epsilon})]^{\frac{13}{6}}N\|f\|_L\|f\|_2\|g\|_2^2.\end{aligned}$$

To relate the upper bounds on $I_\epsilon^+$ and $I_\epsilon^-$, we will now write the quantity $N(2\epsilon)[\log_\theta(\frac{1}{2\epsilon})]^{\frac{13}{6}}$ as a power of $N$ and compare it with the quantity $N^{1-c\epsilon^2}$ on the right hand side of $I_\epsilon^+$.

If we let $u = \frac{1}{2\epsilon}$ and transform $u$ by the the increasing unbounded function

$$u \to \frac{4u^2}{c'}\left(\ln(u) - \frac{13}{6}\ln\log_\theta u\right)$$

then for sufficiently large $N$ our choice of the new $u$ is equal to $\ln N$. The powers are then equal and

$$I \leq C_s\|g\|_2^2\|f\|_L\|f\|_2\Big(N^{1-c'\epsilon^2} + (\log_\theta N)^{\frac{1}{6}}N^{1-c'\epsilon^2} + (\log_\theta N)^{\frac{13}{6}}\Big).$$

Since

$$\frac{4}{c'}u^2\ln u < \ln N < u^3,$$

for large enough $u$ we have

$$N^{-\frac{c'}{4u^2}} < u^{-1} < [\ln N]^{-\frac{1}{3}}.$$

Thus

$$I \leq K''\|f\|_2\|g\|_2^2\|f\|_L\left([\ln N]^{-\frac{1}{3}}[\log_\theta N]^{\frac{1}{6}}N + \frac{N}{(\log_\theta N)^{\frac{1}{3}}}\right) \leq K'''\frac{N}{(\log_\theta N)^{\frac{1}{3}}}\|f\|_2\|g\|_2^2\|f\|_L.$$

We have therefore proved that

$$\frac{1}{N}\sum_{k=0}^{N-1}|\langle U^k(f), g\rangle|^2 \leq K_S\|f\|_L\|f\|_2\|g\|_2^2[\log_\theta N]^{-\frac{1}{6}}$$

where $K_S > 0$ is a constant that depends only on the substitution. This completes the proof of Theorem 1.9.1.

□

## 1.10 Speed of Weak Mixing on the Substitution Subshift

A bounded function $f : X \to \mathbb{R}$ is weakly Lipschitz if there is a constant $C > 0$ such that for any cylinder $[i, n]$, if $x, y \in [i, n]$ we have

$$|f(x) - f(y)| \leq C\mu([i, n]).$$

Define $\|f\|_L = C_f + \|f\|_\infty$, where $C_f$ is the infimum of the $C$ as in the definition for weakly Lipschitz. We will find the speed of weak mixing for weakly Lipschitz functions by approximating their Birkhoff sums with rank $n$ cylindrical functions.

**Lemma 1.10.1.** *If $f$ is weakly Lipschitz with zero average then there exists cylindrical function $g_n$ of rank $n$ such that*

$$|f(y) - g_n(y)| = |f(y) - r_i| \leq Cn^{-1}\|f\|_L,$$

*where $C > 0$ depends only on the substitution.*

*Proof.* Define $g_n = \sum_{k=1}^{J_n} r_k 1_{[k,n]}$ where $r_k = \frac{1}{\mu([k,n])}\int_{[k,n]} f d\mu$. Since $f$ has zero average so does $g_n$. Each $y \in X$ is contained in exactly one cylinder $y \in [i,n]$. Thus, since $f$ is weakly Lipschitz,

$$|f(y) - g_n(y)| = |f(y) - r_i| \leq \|f\|_L \mu([k,n]). \tag{1.22}$$

There is some $C > 0$ such that $\mu([i,n]) \leq \frac{C}{n}$ uniformly in $n$ and $i$. A proof for a similar lower bound can be found in Theorem 10.1, and the same technique can be used to establish this upper bound.

□

**Theorem 1.10.2.** *If $f : X \to \mathbb{R}$ is weakly Lipschitz with zero average, and if $g \in L^2(X)$ then*

$$\frac{1}{N}\sum_{n=0}^{N-1} |\langle U^n(f), g\rangle|^2 \leq C_S\|f\|_L\|f\|_2\|g\|_2^2[\log_\theta N]^{-\frac{1}{6}}$$

*where $C_S$ is dependent only on the substitution.*

*Proof.* Let $f_n$ be a cylindrical function that satisfies the condition from Lemma 1.10.1. Observe:

$$
\begin{aligned}
N^{-1}\sum_{k=0}^{N-1}|\langle U^k f,g\rangle|^2 \\
&\le N^{-1}\sum_{k=0}^{N-1}|\langle U^k(f-f_n),g\rangle|^2+2|\langle U^k(f-f_n),g\rangle||\langle U^k f_n,g\rangle|+|\langle U^k f_n,g\rangle|^2 \\
&\le 2C_S\frac{C}{n}\|f\|_L\|f\|_2\|g\|_2^2+K_S\frac{C}{n}\|f\|_L\|f\|_2\|g\|_2^2+\|f\|_L\|f\|_2\|g\|_2^2[\log_\theta N]^{-\frac{1}{6}}.
\end{aligned}
$$

The first two terms on the right hand side come from Lemma 1.10.1, and the last one follows from Theorem 1.9.1. Recall that in this case $n=\lfloor(\log_\theta N)^{\frac{1}{6}}\rfloor$. Hence

$$
\sum_{k=0}^{N-1}|\langle U^k f,g\rangle|^2\le C_S N\|f\|_L\|f\|_2\|g\|_2^2[\log_\theta N]^{-\frac{1}{6}}.
$$

□

## 1.11 Speed of Weak Mixing for the Chacon map on the Interval

The map that codes the orbit of the Chacon map into the substitution subshift has an inverse that transforms in a nice enough way that we can transfer the quantitative results from the Chacon substitution to the Chacon map on the interval. This is the content of Theorem 1.11.1.

**Theorem 1.11.1.** *The inverse of the code $h: X\to I$ for the Chacon map is weakly Lipschitz.*

*Proof.* Let $x,y\in[i,n]$, and let $w$ be the length $n$ word such that $1_{[i,n]}(w)=1$. Choose $m$ so that $3^{m-1}\le n<3^m$. The fixed point of the substitution is generated by the inductive formula $S_{k+1}=S_kS_k1S_k$ with $S_0=0$. Since the orbit of the fixed point is dense, and since the heights of the towers at the $k$'th step of the iteration is $h_k=\frac{3^{k+1}-1}{2}$, we have that $w$ is contained in the word $S_mS_m1S_m=S_{m+1}$. We can see from the sequence $S_{m+1}$ that if $h(x)$ and $h(y)$ are at different vertical levels of the tower then the code for one of the strings $x$

or $y$ will hit the letter 1 in $S_mS_m1S_m$ before the other in not more than twice the height of $S_m$ plus 1 steps in the orbit under the action of the Chacon map. Since $3^{m-1} \leq n$, the points $h(x)$ and $h(y)$ must be in the same vertical level of the tower created during step $(m-2)$ of the construction. Otherwise they would have a different code in less than $n$ steps, contradicting the fact that they are in a rank $n$ cylinder. The length of the vertical levels in the tower is equal to $l_k = \frac{2}{3^k}$. By unique ergodicity, the measure of the cylinders of length $n$ is equal to its frequency in the fixed point of the substitution. Since $n \leq 3^m$, and since the fixed point is dense, we can see that each block $S_{m+2}$ contains at least one instance of the word $w$. Hence the frequency of $w$ is not less than the frequency of $S_{m+2}$. We can see that there are 3 occurrences of $S_{m+2}$ per string of length $h_{m+3}$. Hence the frequency of $S_{m+2}$ is bounded below by $\frac{C}{3^m}$. Since $n \leq 3^m$, there is a $C' > 0$ such that $l_{m-2} \leq C'\mu([i,n])$. Thus

$$|h(x) - h(y)| \leq l_{m-2} \leq C\mu([i,n]).$$

□

**Theorem 1.11.2.** *The speed of weak mixing for the Chacon map on the unit interval is the same as that in Theorem 1.10.2.*

*Proof.* Let $C$ be the Chacon map on the interval and let $\sigma$ be the shift on the Chacon substitution. The code $h$ satisfies $C \circ h = h \circ \sigma$. If $\mu$ is the invariant measure on the uniquely ergodic substitution subshift, then the pullback measure $\mu(h^{-1}(A))$ is invariant under the action of the Chacon map. Since $(I, C, L)$ is uniquely ergodic, $\mu(h^{-1}(A) = L(A)$. It follows that for Lipschitz functions $f, g \in Lip(I)$,

$$\langle f \circ C^k, g\rangle_L = \langle f \circ C^k \circ h, g \circ h\rangle_\mu = \langle f \circ h \circ \sigma^k, g \circ h\rangle_\mu.$$

Since $f$ and $g$ are Lipschitz with $h$ weakly Lipschitz, $f \circ h$ and $g \circ h$ are both weakly Lipschitz. If we use the formula

$$|\sigma_{f,g}|(B) \leq \sqrt{\sigma_{f,f}(B)}\sqrt{\sigma_{g,g}(B)}$$

along with Theorem 1.10.2

$$\tfrac{1}{N}\sum_{n=0}^{N-1}|\langle U^n(f), g\rangle|^2 \leq K_S\|f\|_L\|f\|_2\|g\|_2[\log_\theta N]^{-\frac{1}{6}}$$

where $K_S > 0$ depends only on the substitution.

□

## 1.12 Lower Bounds

We will show that our upper bound for the speed of weak mixing is essentially sharp in that it cannot be improved to be better than logarithmic. Let $A_k = [0, 2 \cdot 3^{-(k+1)}]$ and set $E_k = \{n \in \mathbf{N} : \mu(A_k \cap T^{-n}A_k) = 0\}$. From [14] we have the constraints $C_k = (2\cdot(h_k-3)! \cdot(\log(3)^{h_k})^{-1}$ and $t+4 < h_k$ , where $h_k = (3^{k+1}-1)/2$. Since $A_k = [0, 2\cdot 3^{-(k+1)}]$ we have $\mu(A_k) \geq \frac{1}{4h_k}$. Now set $t = h_k - 5$ and pick $N$ and $h_k$ so that $\frac{1}{2}\cdot\log(N) > h_k > \frac{1}{4}\cdot\log(N)$. From Theorem 7.2 of [14] we have

**Lemma 1.12.1.** *$E_k \cap [0, N] \geq C\log(N)^t$, where $C > 0$ and $t > 0$.*

Let $g = 1_{A_k}$ be the characteristic function on $A_k$. Now choose $f_N$ to be positive, continuously differentiable, and supported on $A_{N,k}$ with the condition that

$$\log(N)|\int_0^1 f_N(x)dx| \geq \frac{1}{100}\left(\max_{[0,1]}|f'_N| + \|f_N\|_\infty\right). \tag{1.23}$$

This is possible since $\log(N)$ is comparable to the measure of $\mu(A_k)^{-1}$ and $f_N$ is supported on $A_k$. Note that if $n \in E_k$ then $\int_0^1 f_N(T^n x)g dx = 0$. This implies the following:

$$
\begin{aligned}
\sum_{i=0}^{N-1} & \left| \int_x f(T^i x) g dx - \int_X f dx \int_X g dx \right| \\
& \geq \sum_{i \in E_k} \left| \int_x f(T^i x) g dx - \int_X f dx \int_X g dx \right| \\
& = \sum_{i \in E_k} \left| \int_X f dx \int_X g dx \right| \\
& \geq C_k \log(N)^t \|g\|_2 \mu(A_k)^{\frac{1}{2}} \left| \int_X f dx \right|
\end{aligned}
$$

Here $t+4 < h_k$ and $C_k = (2 \cdot (h_k - 3)! \cdot (\log(3)^{h_k})^{-1}$ and $t+4 < h_k$ , where $h_k = (3^{k+1}-1)/2$ [14].

**Lemma 1.12.2.** *We can choose $t$ and $k$ so that*

$$\cdot C_k \log(N)^t \mu(A_k)^{\frac{1}{2}} \geq C \cdot \tfrac{N}{\log(N)}$$

*for some $C > 0$ independent of the parameters.*

*Proof.* Sterling's formula gives the estimate $n! \leq e n^{n+\frac{1}{2}} e^{-n}$. Hence

$$
\begin{aligned}
\log & \left( C_k \log(N)^t \mu(A_k)^{\frac{1}{2}} \right) \\
& \geq h_k - (h_k + \frac{1}{2}) \log(h_k) - 1 - h_k \log\log(3) \\
& + t \cdot \log\log(N) - \frac{1}{2} \log(4 \cdot h_k).
\end{aligned}
$$

This is equivalent to

$$O(h_k) + t \cdot \log\log(N) - h_k \log(h_k).$$

Since $\frac{1}{2} \cdot \log(N) > h_k > \frac{1}{4} \cdot \log(N)$, our choice of $t$ dictates that that the above quantity is bounded below by

$$C\left(\log(N) - \log\log(N)\right). \tag{1.24}$$

for some $C > 0$ independent of $k, N, f$ and $g$. $\square$

The Lipschitz norm of a function is the sum of its supremum norm and the smallest $C_f$ such that $|f(x) - f(y)| \leq C_f|x - y|$. Hence Lemma 12.2 and equation (12.2) gives us

$$\begin{aligned}\sum_{i=0}^{N-1}\left|\int_X f(T^i x)g(x)dx - \int_X f(x)dx\int_X g(x)dx\right| \\ &\geq C\frac{N}{\log(N)^2}\|f_N\|_L\|g_N\| \\ &\geq C\frac{N}{\log(N)^2}\cdot\|f_N\|_L^{\frac{1}{2}}\cdot\|f_N\|_2^{\frac{1}{2}}\cdot\|g_N\|_2\end{aligned}$$

with $g_N = 1_{A_{N,k}}$ and $f_N$ chosen as above.

## 1.13 Existence of Exceptional Sets

Recall that a measure preserving transformation $T$ is weakly mixing if and only if for all measurable $A$ and $B$ there is an exceptional set $J_{A,B} \subset \mathbf{N}$ such that $\frac{1}{n}|J_{A,B} \cap [0.n]| \to 0$ and for $n \notin J_{A,B}$,

$$\lim_{n\to\infty}\mu(T^{-n}(A) \cap B) \to \mu(A)\mu(B).$$

The following lemma from [14] gives us a way to bound the density of such an exceptional set $J_{f,g}$ for Lipschitz $f$ and square integrable $g$ in terms of the speed of weak mixing.

**Lemma 1.13.1.** *Let $(a_n)$ be a decreasing sequence of non-negative numbers. Suppose that*

$$\tfrac{1}{N}\sum_{j=0}^{N-1} a_j \leq b_N$$

*for all* $N \in \mathbf{N}$ *and* $b_N \to 0$ *as* $N \to \infty$*. Let* $c_N$ *decrease and converge to zero. Then there is a set* $J \subset \mathbf{N}$ *such that* $\frac{c_N}{N b_N}|[0,N] \cap J|$ *converges to zero, and* $a_n \to 0$ *for* $n \notin J$*.*

□

In the case of the Chacon map we look to Theorem 1.10.2. If we set $b_N = C\|f\|_2\|f\|_L\|g\|_2^2 \log(N)]^{-\frac{1}{6}}$ and let $a_j = |\int_X f(T^j x)g(x)d\mu(x)|^2$ then lemma 12.1 with $c_N = C\|f\|_2\|f\|_L\|g\|_2^2[\log_3(N)]^{-\frac{1}{3}}$ implies the following.

**Theorem 1.13.2.** *Let* $f$ *be Lipschitz with zero average and* $g \in L^2([0,1])$*. There is some* $J_{f,g}$ *such that* $\langle f(T^k), g\rangle \to 0$ *for* $k \notin J_{f,g}$ *and* $\frac{1}{N}|J_{f,g} \cap [0,N]| \leq [\log_3(N)]^{-\frac{1}{6}}$

# Chapter 2

# Cohomological Equations on Flat Surfaces

## 2.1 Introduction

Let the pair $(S, R)$ be a flat surface together with a flat metric $R$ that has finitely many cone points. We will consider the geodesic flow of $R$, which is euclidean away from each cone point. The geodesic flow $\phi_t^X$ acts on the tangent bundle $TS$ by the formula $\phi_t^X(v) = \gamma_v(t)$, where $\gamma_v(t)$ is the unit tangent vector to the geodesic with initial condition equal to $v$. Let $\{p_1, \ldots, p_\sigma\} := \Sigma \subset S$ be the cone points for the metric $R$ on $S$ and let the cone angle for $p_i$ be denoted by $2\pi(\alpha_i + 1)$.

The vector field $X$ acts on differentiable functions $u$ by

$$\frac{d}{dt}|_{t=0} u(\phi_t^X(p, v)),$$

We can define an orthogonal geodesic flow with generator $Y$ that acts on $T_1S$ in a similar way. Notice that the volume form on $T_1S$ is invariant under the flow generated by both $X$ and $Y$. This and the differential property of the vector fields $X$ and $Y$ gives us

$$0 = \int_{T_1S} X(uv)dV = \int_{T_1S} X(u)vdV + \int_{T_1S} uX(v)dV. \tag{2.1}$$

Each vector in $T_1S$ can be rotated by an angle $\theta$. Locally this rotational flow along $\theta$ gives a generator $\Theta$ that acts on differentiable functions $f$ locally at the point $(p, v) \in T_1S$

by

$$\Theta|_{(p,v)}\ f = \frac{d}{d\theta}|_{\theta=0} f(p, R_\theta v).$$

Define $E_n = \{f \in L^2(T_1S) : \Theta f = inf\}$. We will show later that $L^2(T_1S) = \bigoplus_{n\in\mathbb{Z}} E_n$. Define the horizontal (sometimes called partial) Laplacian on by

$$H = -(X^2 + Y^2).$$

If at least one of the cone angles $2\pi(\alpha_i + 1)$ is irrational, then a simultaneous Diophantine condition implies that

$$d(n\alpha_1, \mathbb{Z}) + \cdots + d(n\alpha_m) \geq \frac{C}{n^\gamma}. \tag{2.2}$$

We will use a Cheeger constant argument to show the following theorem.

**Theorem 2.1.1.** *Assuming the simultaneous Diophantine condition holds for the cone angles, there is a constant $C > 0$ and $\gamma > 2$ such that the smallest eigenvalue $\lambda_n(H)$ of $-H$ when restricted to $E_n$ is bounded below as*

$$\lambda_n(H) \geq \frac{C}{n^\gamma}.$$

Since the spaces $E_n$ decompose $L^2(T_1S)$, we can use the regularity of functions $f$ in $\Theta$ to get a sufficient enough decay of the Fourier coefficients to find solutions $u$ to the equation $-Hu = f$. This decay combined with the Cheeger estimate yields the following result.

**Theorem 2.1.2.** *Suppose the simultaneous Diophantine estimate 2.6.1 holds, and let $t > \gamma + 1$. Then for $f$ with $L^2$ bounded derivatives up to order $t$ and with $\int_{T_1S} f dV = 0$, there is a solution $u$ to the cohomological equation*

$$-Hu = f.$$

**Definition 2.1.3.** *The vector fields $X$ and $Y$ are defined for all points on $T_1(S - \Sigma)$, and* *on that set $[X,Y]=0$. Hence the Frobenius theorem implies that there is a foliation $\mathcal{F}_H$ in $T_1(S-\Sigma)$ with tangent space equal to the span of $X$ and $Y$. We will call $\mathcal{F}_H$ the horizontal foliation. Equivalently, the horizontal foliation is locally the set of points $(q,w)$ near $(p,v)\in T_1S$ that are parallel transports of $(p,v)$ in the flat metric on $S-\Sigma$.*

The irrationality of the cone angles also impacts the dynamics of the horizontal foliations on $T_1(S-\Sigma)$.

**Theorem 2.1.4.** *Suppose at least one of the cone angles $2\pi(\alpha_i+1)$ is irrational. If $u\in L^2(T_1S)$ is constant on the horizontal foliation, then $u$ is constant.*

This paper also contains quantitative estimates for the growth rate of the eigenvalues for the Laplacian

$$-\Delta = X^2+Y^2+\Theta^2.$$

**Theorem 2.1.5.** *Let $N_\Delta(\Lambda)$ be the number of eigenvalues for $-\Delta$ less than or equal to $\Lambda$. The growth rate for $N_\Delta(\Lambda)$ can be bounded above in terms of $\Lambda$ and a constant $C>0$ dependent on the metric as*

$$N_\Delta(\Lambda)\ \leq\ C\Lambda^{\frac{3}{2}}.$$

## 2.2 Geometry and Analysis on Flat Surfaces

When we set the coordinates to be $z=x+iy$ at any point, the metric and area form on $S$ are locally

$$R=|\phi(z)|\Big(dx^2+dy^2\Big)^{\frac{1}{2}},\ \omega=|\phi(z)|^2dx\wedge dy. \tag{2.3}$$

Around a regular point, we can choose coordinates such that $\phi(z)=1$. If we are considering the metric around a cone point with angle $2\pi(\alpha+1)$, then we can write $\phi(z)=z^\alpha$. Hence, if we make the change of coordinates $z = x + iy$, we have

$$R = |z|^{\alpha}\left(dx^2 + dy^2\right)^{\frac{1}{2}}. \tag{2.4}$$

Let $X$ be the generator of the geodesic flow, and let $Y$ be the generator for the orthogonal geodesic flow. Define $\partial^+ = X + iY$ and $\partial^- = X - iY$. At the cone points we can locally write

$$\begin{aligned} \partial^+ &= e^{-i\theta}\bar{z}^{-\alpha}\frac{\partial}{\partial \bar{z}} \\ \partial^- &= e^{i\theta}z^{-\alpha}\frac{\partial}{\partial z}. \end{aligned} \tag{2.5}$$

We say a function is smooth at the cone point if it is smooth in the coordinates $\psi(z,\theta) = z^{\alpha+1}e^{-i\theta}$ and the conjugate of $\psi$. Define the set $C^{\infty}_{\Sigma}(T_1(S))$ to be the set of functions on $T_1(S)$ that are smooth at both the cone points and at the regular points.

For $(s,v) \in \mathbb{R}^+ \times \mathbb{N}$, define the norm

$$|f|_{s,v} = \sum_{i+j\leq s}\sum_{l\leq v}\|X^iY^j\Theta^l f\|^2_{L^2}.$$

Define $H^{s,v}(T_1(S))$ to be the $L^2$ Sobolev space on $T_1S$ to be the closure in $|\cdot|_{s,v}$ of the following set:

$$H^{s,v}(T_1S) = \overline{\{f \in C^{\infty}_{\Sigma}(T_1(S)) : \sum_{i+j\leq s}\sum_{l\leq v}\|X^iY^j\Theta^l f\|^2_{L^2} < \infty\}}. \tag{2.6}$$

We also define $H^{-s,-v}(T_1(S))$ to be the dual space to $H^{s,v}(T_1(S))$. Define the (fractional) Friedrich's inner product by

$$\langle u, v\rangle_s := \sum_{k=1}^{\infty}(1+\lambda_k)^s\langle u, e_k\rangle\langle e_k, v\rangle, \tag{2.7}$$

and let $\|u\|_s := \sqrt{\langle u, u\rangle_s}$.

**Lemma 2.2.1.** *For $0 < r < s$ there are positive constants $C_r$ and $C_s > 0$ such that for every $u \in H^{s,t}(T_1S)$ the following inequality holds*

$$C_r \|u\|_{r+t} \leq |u|_{r,t} \leq \|u\|_{s+t}$$

At a regular point with respect to coordinates such that $R = |dz| = (dx^2 + dy^2)^{\frac{1}{2}}$ on the base of the unit tangent bundle and at $\theta$ in the fiber, we can write

$$\begin{aligned} X &= \cos(\theta)\partial_x + \sin(\theta)\partial_y \\ Y &= \sin(\theta)\partial_x - cos(\theta)\partial_y \\ \Theta &= \partial_\theta. \end{aligned} \tag{2.8}$$

In this case we can see that

$$\begin{aligned} [X, Y] &= 0 \\ [\Theta, X] &= Y \\ [\Theta, Y] &= -X. \end{aligned} \tag{2.9}$$

We restrict the domain of both $X$ and $Y$ to the space $H^{1,0}(T_1S)$. We would like these vector fields to be skew adjoint on a domain as large as possible such that the operators are skew adjoint on this extended domain. The Friedrichs extension of the laplacian $-\Delta$ gives us such an extension. Indeed, the volume form on $T_1S$ is flow invariant with respect to vector fields $X$, and $Y$, hence they are anti-symmetric in the $L^2$ norm on the dense set $H^{1,0}(T_1S)$. Hence there is an essentially skew-symmetric and unique extension of $X$ and $Y$ on the domain $H^{1,0}(T_1S) \subset L^2(T_1S)$.

The set of trajectories which hit a cone point is a countable union of dimension two subspaces. Hence it has zero volume. Indeed, since the metric is flat, for each $p \in S$ there is one straight line segment in the direction $v_k$ that points toward $p_k \in \Sigma$. The length of the geodesic segment $(0, l_{p,k})$ between $p$ and $p_k$ gives an embedding $\tau_k : (0, l_{p,k}) \to S$. Now define

$$C(\tau_k) = \{(\tau_k(s), \tau_k'(s) : s \in (0, l_{p,k}))\} \subset T_1 S.$$

The $C(\tau_k)$ are continuously parametrized by the starting point $\tau_k(0)$ and there are finitely many $k$ to choose from for each starting point. Hence the set of points in $T_1 S$ that give initial conditions for geodesics that end up at a cone point is a finite union of 2 dimensional surfaces. Hence it has volume zero.

Let

$$E_k = \{f \in L^2(T_1(S)) : \Theta f = ikf\}. \tag{2.10}$$

**Lemma 2.2.2.** $L^2(T_1(S)) = \bigoplus_k E_k$

*Proof.* Let $\phi_t^\Theta$ be the flow given by rotation of a vector $V \in T_1(S)$. Define the function $\pi_n : L^2(T_1(S)) \to L^2(T_1(S))$ by

$$\pi_n(f) = \int_0^{2\pi} e^{-int}(f \circ \phi_t^\Theta) dt \tag{2.11}$$

Suppose that $f \in L^2(T_1(S))$ and let $\pi_n(f) = \int_0^{2\pi} e^{-int} f \circ \phi_t^\Theta dt$ and consider a finitely supported partition of unity $\{\psi_\alpha : \alpha \in I\}$ on charts that cover $T_1(S)$. Locally $\Theta = \frac{d}{d\theta}$. Hence integration by parts implies

$$\begin{aligned} \Theta\pi_n(\psi_\alpha f) &= \frac{d}{d\theta}\int_0^{2\pi} e^{-int}((\psi_\alpha f) \circ \phi_t^\Theta) dt \\ &= \int_0^{2\pi} e^{-int}\frac{d}{d\theta}((\psi_\alpha f) \circ \phi_t^\Theta) dt \\ &= in\pi_n(\psi_\alpha f) + \pi_n(\Theta\psi_\alpha). \end{aligned} \tag{2.12}$$

Summing over $\alpha$ yields $\Theta\pi_n(f) = in\pi_n(f)$.

Thus $\pi_k$ is the projection onto the eigenspace $E_k$. Now we will show that $\sum_{k\in\mathbb{Z}} \pi_k = 1$. Suppose that $g$ is smooth and let $\psi_\alpha$ be a family of bump functions supported on coordinate charts that cover $T_1S$. Then $g = \sum_{\alpha} \psi_\alpha g$ and integration by parts locally gives

$$|\pi_n(\psi_\alpha g)| = |n|^{-m} |\pi_n(\Theta_m(\psi_\alpha g))| \leq \frac{C_m}{|n|^m} \tag{2.13}$$

Thus, it suffices to show that the terms in the right-hand side of the above equation tends to zero locally for any given coordinate chart as $N \to \infty$. This, however, follows from a fiber-wise Fourier expansion and classical Fourier analysis.

□

If $f \in H^{s,v}(T_1S)$ then

$$\begin{aligned} &\sum_{n=0}^{\infty} (1+n^{2v}) \langle \pi_n(f), \pi_n(f) \rangle \\ &= \langle f, f \rangle + \langle \Theta^v f, \Theta^v f \rangle \\ &\leq |f|_{s,v}^2 < \infty. \end{aligned} \tag{2.14}$$

Thus there is some constant $C$ independent of $n$ such that

$$\|\pi_n(f)\|_{L^2} \leq C \left(1+n^{2v}\right)^{-\frac{1}{2}} \leq C n^{-v} \tag{2.15}$$

**Lemma 2.2.3.** *We want to show that* $\partial^+ : E_n \cap H^{1,0}(T_1S) \to E_{n-1}$ *and* $\partial^- : E_n \cap H^{1,0}(T_1S) \to E_{n+1}$.

*Proof.* Let $f \in E_n \cap H^{1,0}(T_1 S)$. Using the previous commutator calculations we get

$$\begin{aligned}\Theta(\partial^+ f) =& \Theta X f + i\Theta Y f \\ &= Yf + X\Theta f - iXf + iY\Theta f \\ &= Yf + inXf - iXf + inYf \\ &= i(n-1)(Xf + iYf) \\ &= i(n-1)\partial^+ f.\end{aligned} \tag{2.16}$$

The proof for $\partial^-$ is similar. □

If $u \in E_n$ satisfies $-\Delta u = \lambda u$ then

$$\begin{aligned}-(X^2 + Y^2 + \Theta^2)u = \lambda u \\ -Hu = (\lambda - n^2)u\end{aligned} \tag{2.17}$$

Let $N_\Delta(\Lambda)$ denote the number of eigenvalues for $\Delta$ less than $\Lambda$. Then the Weyl asymptotics for $\Delta$ obey the following rule:

$$\frac{N_\Delta(\Lambda)}{\Lambda^{\frac{3}{2}}} = O(1). \tag{2.18}$$

The next few sections will offer a proof.

## 2.3 Local Weyl Asymptotics for Regular Points

Write the manifold $T_1(S) = \bigcup_{i=1}^{n} \overline{U_i}$, where $\overline{U_i}$ is a compact coordinate chart. If $U_i$ does not contain a cone point then $\overline{U_i}$ is diffeomorphic to $\overline{A_i} \times S^1$ with $A_i \subset \mathbb{R}^2$ a small rectangle.

Let $\Sigma$ denote the cone points of the flat surface $S$. In this section, we will find the Weyl asymptotics for the distribution of eigenvalues for the full Laplacian on the space $T_1(S)$ in a neighborhood around a regular point, or a neighborhood that does not contain a point in $\Sigma$.

If we consider the case where $U_i$ does not contain a cone point, we have the following result.

**Theorem 2.3.1.** *Let $N_i(\Lambda)$ be the number of eigenvalues for the Laplacian $-\Delta = X^2+Y^2+\Theta^2$ on $U_i \subset T_1(S)$ less than $\Lambda$. Then $N_i(\Lambda) \leq (2\pi/2\epsilon)^{-2} 8\Lambda^{\frac{3}{2}}$.*

Let $H = -(X^2+Y^2)$ and let $u$ be a solution to

$$-\Delta u = (H + \Theta^2)u = \lambda u. \tag{2.19}$$

We can reduce the asymptotics of the eigenvalues of $\Delta$ to the study of the asymptotics of $H$. Indeed, we will first decompose $u$ according to 2.2.2. In local coordinates $(z,\theta)$ on $T_1S$ we have $\Theta = \frac{\partial}{\partial\theta}$. If $v \in E_m$, then it is by definition a solution to the differential equation $\frac{\partial}{\partial\theta}v = imv$. Hence we necessarily have $v(z,\theta) = v_m(z)e^{im\theta}$. Consider now the following decomposition of $u$ according to its projections onto the $E_n$ splitting:

$$u(z,\theta) = \sum_{n=0}^{\infty} u_n(z)e^{i\theta n}.$$

After plugging this form into $-\Delta = \lambda u$ and equating similar components we get

$$e^{in\theta} \cdot H(u_n) - n^2 e^{in\theta} u_n = -\lambda u_n e^{in\theta}. \tag{2.20}$$

After simplification we arrive at

$$Hu_n = \mu_n u_n \tag{2.21}$$

with $\mu_n = -\lambda + n^2$. Hence we have the following reduction.

**Lemma 2.3.2.** *If $u$ solves $-\Delta u = \lambda u$ on $A_i \times S^1$ with Neumann boundary condition $\nabla u \cdot \nu\,|_{\partial A_i \times S^1} = 0$, then its projection $u_n$ onto $E_n$ satisfies $Hu_n = \mu_n u_n$ on $A_i \times S^1$.*

Thus it suffices to find the growth rate of the eigenvalues of $-H$ when restricted to $E_n$. Let $v$ solve $-Hv = \mu v$ with Neumann data on $A_i \times S^1$, and denote by $u(z,\theta) = v_{n_0}(z)\, e^{in_0\theta}$ its projection onto the eigenspace $E_{n_0}$ (with $v_{n_0} : A_i \to \mathbb{R}$). Restrict to the square $R = [-\epsilon, \epsilon]^2$ and impose Neumann boundary condition $\partial_\nu u \mid_{\partial R} = 0$. The Neumann eigenfunctions on $R$ are the products

$$\cos(n\pi x/\epsilon)\,\cos(m\pi y/\epsilon), \qquad n, m \in \mathbb{N} \cup \{0\},$$

with eigenvalues $(\pi/\epsilon)^2(n^2 + m^2)$. Expanding $u$ in this basis,

$$u(x, y, \theta) = \sum_{n,m \geq 0} c_{n,m}\, \cos\Big(\frac{n\pi}{\epsilon} x\Big)\cos\Big(\frac{m\pi}{\epsilon} y\Big)\, e^{in_0\theta},$$

and inserting into $-Hu = \mu u$ gives

$$(\frac{\pi}{\epsilon})^2(n^2 + m^2)\, c_{n,m} = \mu\, c_{n,m}, \quad \forall n, m.$$

Hence, for every non–zero coefficient,

$$\mu = \Big(\frac{\pi}{\epsilon}\Big)^2 (n^2 + m^2) \tag{2.22}$$

The number of lattice points that satisfy this condition is proportional to the number of integer pairs contained in a square with sides $2\sqrt{\Lambda}$, which is not greater than the area of this square. Hence when restricted to $E_n$, $N_H(\Lambda) \leq C\Lambda$. Now for the proof of 2.3.1.

*Proof.* Let $N_i(\Lambda)$ be the number of eigenvalues for $\Delta$ less than $\Lambda$ in the case where $\Delta$ is restricted to the set $U_i$, which is diffeomorphic to $A_i \times S^1$ on $T_1 S$. The above calculations show that

$$\begin{aligned}\#\{\lambda \in spec(\Delta) : \lambda = \mu_k + k^2 \leq \Lambda\} &\leq \#\{(n, m, k) \in \mathbb{Z}^3 : (2\pi/\epsilon)^2(n^2 + m^2 + k^2) \leq \Lambda\} \\ &\leq (2\pi/2\epsilon)^{-2} vol(cube_3(2\sqrt{\Lambda})) = (2\pi/2\epsilon)^{-2} 8\Lambda^{\frac{3}{2}}.\end{aligned} \tag{2.23}$$

Hence if $R$ is a small rectangle on $R \times S^1$,

$$N_i(\Lambda) = \#\{\lambda \in spec(\Delta) : \lambda = \mu + k^2 \leq \Lambda\} \leq (2\pi/2\epsilon)^{-2} 8\Lambda^{\frac{3}{2}}. \tag{2.24}$$

□

## 2.4 Local Weyl Asymptotics for Cone Points

In the case of the cone point, the metric has the form 2.4 with $z = x + iy$. Switching to polar coordinates we set

$$\begin{aligned} x &= r\cos(\phi) \\ y &= r\sin(\phi), \end{aligned} \tag{2.25}$$

which gives the metric the following form:

$$R = r^\alpha \left(dr^2 + r^2 d\phi^2\right)^{\frac{1}{2}}. \tag{2.26}$$

Setting

$$\rho = r^{\alpha+1}/(\alpha+1) \tag{2.27}$$

gives

$$\begin{aligned} R &= \left(d(r^{\alpha+1}/(\alpha+1))^2 + (\alpha+1)^2\rho^2 d\phi^2\right)^{\frac{1}{2}} \\ &= \left(d\rho^2 + (\alpha+1)^2\rho^2 d\phi^2\right)^{\frac{1}{2}}. \end{aligned} \tag{2.28}$$

Using Cartan's formula and the definition of divergence in terms of the Lie derivative of the volume form, one can show that in local coordinates the partial Laplacian takes the form

$$Hu = \frac{1}{\sqrt{\det(g_{ij})}} \partial_k (\sqrt{\det(g_{ij})} g^{kl} \partial_l u). \tag{2.29}$$

Using the metric $R^2$, we have the following formula for the Laplace equation on $U_p$:

$$Hu = (\partial_{\rho\rho} + \rho^{-1}\partial_\rho + (c\rho)^{-2}\partial_{\phi\phi})u, \tag{2.30}$$

where $c = \alpha + 1$.

The fiber angle $\theta$ undergoes the twist

$$(\phi, \theta) \sim (\phi + 2\pi, \theta + 2\pi(\alpha + 1)).$$

Hence when restricted to $E_n$ a Fourier series expansion for a solution $u$ of the partial Laplace equation

$$-Hu = \lambda u \tag{2.31}$$

has the form $u_n(\rho, \phi, \theta) = \sum_k u_k(\rho) e^{ik\phi} e^{in[-(\alpha+1)\phi+\theta]}$. Hence equating like local components in the equation $-Hu = \lambda u$ gives

$$\left(\partial_{\rho\rho} + \rho^{-1}\partial_\rho + (-\left(\frac{k - n(\alpha+1)}{\alpha+1}\right)^2 \rho^{-2} + \lambda)\right) u_k(\rho) = 0. \tag{2.32}$$

A Bessel function $J_\nu$ of the first kind is a solution to the ordinary differential equation

$$x^2 z'' + xz' + (x^2 - \nu^2)z = 0. \tag{2.33}$$

We can transform 2.32 into this form by setting

$$x = \sqrt{\lambda}\rho,$$

$$z(x) = u_k(\rho),$$

$$\nu(n, k) = \frac{k - n(\alpha + 1)}{\alpha + 1}.$$

Thus a solution to 2.31 when we restrict to $E_n$ is

$$u_n(r,\phi,\theta) = \sum_{k=0}^{\infty} a_{n,k} J_{\nu(n,k)}(\sqrt{\lambda} r^{\alpha+1}/(\alpha+1)) e^{i(k-(\alpha+1)n)\phi} e^{in\theta}. \tag{2.34}$$

On the geodesic ball $U_i \subset T_1S$ of radius $\delta$ we can enforce a Neumann boundary condition by demanding that $J'_{\nu(n,k)}(\sqrt{\lambda}\delta) = 0$. Hence the asymptotics for admissible $\lambda$ subordinate to a Neumann boundary condition is equivalent to finding the asymptotics for the $m$'th largest zero $j'_{\nu,m}$ of $J'_{\nu,m}$. Let $j_{\nu,m}$ be the $m$'th largest zero of $J_{\nu,m}$.

Theorem 1.3 from [13] states that

$$j_{\nu,m} < j'_{\nu,m+1}. \tag{2.35}$$

Section 3 of [19] gives the following bound:

$$|j_{\nu,k} - (k + \frac{\nu}{2} - \frac{1}{4})\pi| \leq \frac{0.9|4\nu^2 - 1|}{\pi(k + \frac{\nu}{2} - 0.314)}. \tag{2.36}$$

Hence

$$(m + \frac{\nu(n,k)}{2} - \frac{1}{4})\pi - \frac{0.9|4\nu(n,k)^2 - 1|}{\pi(m + \frac{\nu(n,k)}{2} - 0.314)} < j_{\nu(n,k),(m+1)}. \tag{2.37}$$

The index $n$ is the restriction to $E_n$, $m$ indexes the size of the eigenvalues for 2.32 and $k$ is the $k$'th term in the Fourier expansion 2.34 after restricting to $E_n$. The relationship 2.17 implies that the eigenvalues $\mu_{n,m,k}$ of the Laplacian $-\Delta$ when restricted to $E_n$ satisfy

$$\pi^2 \left(m + \frac{\nu(n,k)}{2} - \frac{1}{4}\right)^2 \left(1 - \frac{0.9|4\nu(n,k)^2 - 1|}{\pi^2(m + \frac{\nu(n,k)}{2} - 0.314)^2}\right)^2 + n^2 < \delta^2 \mu_{n,m}. \tag{2.38}$$

Hence

$$\frac{\pi^2}{420}(m^2 + \nu(n,k)^2) + n^2 < \delta \mu_{n,m,k} \tag{2.39}$$

with $\nu(n,k) = |(1+\alpha)^{-1}k - n|$. Now $0 \leq \mu_{n,m,k} \leq \Lambda$ implies that

$$\nu(n,k)^2 \leq C^{-1}\Lambda - m^2 - n^2 \leq C^{-1}\Lambda$$

with

$$n^2 + m^2 \leq C^{-1}\Lambda.$$

The number of $k$ that enables $\nu(n,k)$ satisfy the first inequality is bounded above by $C'\sqrt{\Lambda}$ if we fix a particular pair of $n$ and $m$ that satisfy the second inequality. The number of integer pairs in $n$ and $m$ that can satisfy the second equality is bounded above by $K\Lambda$. Hence

$$N_\Delta(\Lambda) = \{\mu_{n,m,k} \in \text{Spectrum}(-\Delta) : \mu_{n,m,k} \leq \Lambda\} \leq C'K\Lambda^{\frac{3}{2}} \tag{2.40}$$

## 2.5 Comparison Theorem

This section follows some of the arguments from [8]. Cover $T_1S$ with a finite number of compact sets $U_i$ such that the $U_i$ have pairwise disjoint interior. We write

$$T_1S \ = \ \bigcup_{i=1}^{t} U_i,$$

We will also assume that the $U_i$ are diffeomorphic to coordinate sets $A_i$ via functions $g_i : A_i \to U_i$. Since there are finitely many $g_i$, we may assume that for all $i$ there are constants $R, r > 0$ such that

$$R > \text{Det}(g_i) > r > 0$$

We may also assume that the shape of $A_i$ is the same as that from the previous sections depending on if $A_i$ contains a cone point or not.

Let $-\Delta = X^2 + Y^2 + \Theta^2$ be the Laplace operator on $T_1S$. We define $N_\Delta(\Lambda)$ to be the number of eigenvalues of $\Delta$ that are less than or equal to $\Lambda$, with corresponding eigenfunctions defined on the entire space $T_1S$. Locally, for each $i$, let $N_i(\Lambda)$ denote the number of eigenvalues of $-\Delta$ less than or equal to $\Lambda$ when restricted to $A_i$, subject to Neumann boundary conditions. We claim:

**Theorem 2.5.1.** *For each* $\Lambda \geq 0$,

$$N_\Delta(\Lambda) \ \leq \ \sum_{i=1}^{t} N_i(\frac{R}{r}\Lambda) \leq C\Lambda^{\frac{3}{2}}.$$

Define

$$P(u,v) = \langle Xu, Xv\rangle + \langle Yu, Yv\rangle + \langle \Theta u, \Theta v\rangle \tag{2.41}$$

*Proof.* Organize the local eigenvalues for the Neumann problem to $-\Delta u = \lambda u$ on all $A_m$ for $m \in I$ as

$$0 \ \leq \ \lambda_1 \ \leq \ \lambda_2 \ \leq \ \cdots,$$

repeated with multiplicity and let $\{\phi_i\}$ be the corresponding eigenfunctions when restricted to their respective sets $A_j$. Extend $\phi_j \circ g_j^{-1}$ to the entire space by setting their value equal to zero outside of $U_i$. Similarly, let

$$0 \ \leq \ \mu_1 \ \leq \ \mu_2 \ \leq \ \cdots$$

be the global eigenvalues of $-\Delta$ on $T_1S$, with corresponding eigenfunctions $\{\psi_j\}$.

Choose a non-trivial linear combination $f = \sum_{j=1}^{l} \alpha_j \psi_j$ which is orthogonal to $\{\phi_j : j \leq l-1\}$. The orthogonality of the $\{\psi_i \circ g_i^{-1}\}$ and Green's identity

$$P(u,u) = \int_{T_1S} |\nabla u|^2 dV' = -\int_{T_1S} u(\Delta u) dV' \tag{2.42}$$

implies

$$P(f,f) \ = \ \sum_{j=1}^{l} \mu_j \, \alpha_j^2.$$

When restricted to the subdomain $A_k$ we can expand $f = \sum_{i=1}^{\infty} a_i^{(k)} \phi_i^{(k)}$, where the set $\{\phi_i^{(k)} : i \in \mathbb{N}\}$ is an orthogonal basis for $L^2(A_i)$ of eigenfunctions for $-\Delta$ with Neumann data. The assumption that $f \in \{\phi_1 \circ g_1^{-1}, \phi_2 \circ g_2^{-1}, \ldots, \phi_l \circ g_l^{-1}\}^{\perp}$ implies that

$$\int_{A_i \times S^1} -f(\Delta f) dV \geq \lambda_{l+1} \int_{A_i \times S^1} f^2 dV$$

on each $A_i$.

Therefore

$$\begin{aligned} R\mu_{l+1} \sum_{i=1}^{t} \int_{A_i \times S^1} (f \circ g_i)^2 dV' &\geq \mu_{l+1} \int_{T_1 S} f^2 dV \geq P(f, f) \\ &= \sum_{i=1}^{t} \int_{U_i} (-\Delta f) f dV \\ &\geq r\lambda_{l+1} \sum_{i=1}^{t} \int_{A_i \times S^1} (f \circ g_i)^2 dV'. \end{aligned} \tag{2.43}$$

Hence $\mu_l \leq \Lambda \implies \frac{r}{R}\lambda_l \leq \Lambda$. It follows that $\#\{k : \mu_k \leq \Lambda\} \leq \#\{k : \lambda_k \leq \frac{R}{r}\Lambda\}$.

Hence, at most $\sum_{i=1}^{t} N_i(\frac{r}{R}\Lambda)$ of the global eigenvalues $\mu_k$ are less than or equal to $\Lambda$. Equations 2.24 and 2.40 then imply

$$N_\Delta(\Lambda) \ \leq \ \sum_{i=1}^{t} N_i(\frac{R}{r}\Lambda) \leq C''\Lambda^{\frac{3}{2}}. \tag{2.44}$$

□

We will need the following lemma later

**Lemma 2.5.2.** *Let $L : V \to \mathbb{R}$ be a linear operator on the vector space $V$ with eigenvalues $\lambda_k \in \mathbb{R}$ ordered by size, and suppose that the counting functions for the eigenvalues of $L$ satisfies $N_L(\Lambda) \leq C\Lambda^p$ for some $p > 0$. Then the sum*

$$\sum_{k=0}^{\infty}\left(\frac{1}{1+\lambda_k}\right)^r \tag{2.45}$$

*converges for $p < r$.*

*Proof.* Note that

$$\begin{aligned}\sum_{k=0}^{\infty}\left(\frac{1}{1+\lambda_k}\right)^r = \int_0^{\infty}\frac{1}{(1+t)^r}dN(t) = \Big[(1+t)^{-r}N(t)\Big]_0^{\infty} + \int_0^{\infty} N(t)(1+t)^{-r-1}dt \\ \leq c\lim_{t\to\infty}(1+t)^{-r}t^p + cr\int_0^{\infty} t^p(1+t)^{-r-1}dt.\end{aligned} \tag{2.46}$$

Hence the limit converges provided $p < r$. □

## 2.6 Cheeger Constant

This section follows, in part, a proof given in [4]. The formula for the Laplacian of $f : T_1S \to \mathbb{R}$ in local coordinates is

$$\Delta f = \frac{1}{\sqrt{\det(g_{ij})}}\partial_k(\sqrt{\det(g_{ij})}g^{kl}(\partial_l f)). \tag{2.47}$$

If we want $X^2 + Y^2 + \epsilon\Theta^2 = \Delta_\epsilon$, then $g_{xx} = g_{yy} = 1$ and $g_{\theta\theta} = \frac{1}{\epsilon}$ with other terms zero gives an appropriate metric $g_\epsilon$.

Let $\phi_n \in E_n$ be a solution to $-\Delta_\epsilon\phi_n = \lambda_n(\epsilon)\phi_n$. Since $-H$ has real eigenvalues, and since on $E_n$ the equation $-\Delta_\epsilon\phi_n = \lambda_n(\epsilon)\phi_n$ implies $-H\phi_n = (\lambda_n(\epsilon) - \epsilon n^2)\phi_n$, $\lambda_n(\epsilon)$ is also real. Define $\lambda_n(H)$ to be the least eigenvalue of $H$ on $E_n$. Note that $\lambda_n(\epsilon)$ decreases to $\lambda_n(H) = \lambda_n(0)$ as $\epsilon \to 0$.

Now set $u = \phi_n + \bar{\phi_n} \in E_n + E_{-n}$. Then $u$ is real-valued and

$$-\Delta_\epsilon(u) = -(\Delta_\epsilon\phi_n + \Delta_\epsilon\bar{\phi_n}) = \lambda_n(\epsilon)\phi_n + \overline{\lambda_n(\epsilon)\phi_n} = \lambda_n(\epsilon)u.$$

Consider the splitting

$$
\begin{aligned}
M_1 &= u^{-1}((-\infty, t)), \\
M_2 &= u^{-1}((t, \infty)), \\
Y(t) &= u^{-1}(t).
\end{aligned} \tag{2.48}
$$

Define $h_n(\epsilon)(u)$ to be the infimum of the ratio

$$
\frac{\text{Area}_\epsilon(Y(t))}{\min\{\text{Vol}_\epsilon(M_1), \text{Vol}_\epsilon(M_2)\}}. \tag{2.49}
$$

over $t$. We now define $h_n(\epsilon)$ to be the infimum of the $h_n(\epsilon)(u)$ over $u$ such that $u = \phi_{n,m} + \overline{\phi_{n,m}}$ with $\phi_{n,m} \in E_n$ an eigenfunction of $-\Delta$. Let $\lambda_n(H)$ denote the smallest eigenvalue for $H$ when restricted to $E_n$. This section aims to prove a Cheeger constant-like bound as follows.

**Theorem 2.6.1.** *Assuming the simultaneous Diophantine 2.2 condition holds for the cone angles, there is a constant $C > 0$ and $\gamma > 2$ such that the smallest eigenvalue $\lambda_n(H)$ of $-H$ when restricted to $E_n$ is bounded below as*

$$
\lambda_n(H) \geq \frac{C}{n^\gamma}.
$$

We will first prove the following inequality, then bound $h_n(\epsilon)$ below.

**Theorem 2.6.2.** *The least eigenvalue $\lambda_n(\epsilon)$ of $-\Delta_\epsilon$ restricted to $E_n$ on $T_1S$ is bounded below as*

$$
\lambda_n(\epsilon) \geq \left(\frac{h_n(\epsilon)}{2}\right)^2.
$$

*Proof.* In the following proof we will use the metric $g_\epsilon$ for our integral calculations, including gradient, area and volume. Let $u = \phi_{n,m} + \overline{\phi_{n,m}} \in E_n \oplus E_{-n}$. For regular values $t \in \mathbb{R}$ of

$u^2$ let $M_t = \{x \in T_1S : |u(x)|^2 > t\}$ and define $V(t) = \mathrm{Vol}(M_t)$. By possibly replacing $u^2$ with $-u^2$ we can assume that $\mathrm{Vol}(M_2) > \mathrm{Vol}(M_1)$. The Rayleigh quotient characterization of eigenvalues says that if $\lambda_n(\epsilon)$ is the smallest eigenvalue for

$$\lambda_n(\epsilon) = \inf_{\{w \in E_n\}} \frac{\int_{T_1S} |\nabla w|^2}{\int_{T_1S} w^2}. \tag{2.50}$$

We will find a lower bound for this ratio and use the metric $g_\epsilon$ in the following integral calculations. Observe that

$$\frac{\int_{M_1} |\nabla u|^2 dV}{\int_{M_1} u^2 dV} = \frac{\left(\int_{M_1} u^2 dV\right)\left(\int_{M_1} |\nabla u|^2 dV\right)}{\left(\int_{M_1} u^2 dV\right)^2}. \tag{2.51}$$

The Cauchy- Schwartz inequality implies

$$\frac{\left(\int_{M_1} u^2 dV\right)\left(\int_{M_1} |\nabla u|^2 dV\right)}{\left(\int_{M_1} u^2 dV\right)^2} \geq \frac{\left(\int_{M_1} |u||\nabla u| dV\right)^2}{\left(\int_{M_1} u^2 dV\right)^2}. \tag{2.52}$$

The quantity on the right hand side is equal to

$$\frac{\left(\int_{M_1} |u||\nabla u| dV\right)^2}{\left(\int_{M_1} u^2 dV\right)^2} = \frac{1}{4} \frac{\left(\int_{M_1} |\nabla u^2| dV\right)^2}{\left(\int_{M_1} u^2 dV\right)^2}. \tag{2.53}$$

Since up to a set of measure zero $M_1 \cup M_2 = T_1S$, it suffices to show that

$$\int_{M_1} |\nabla u^2| dV \geq h_n(\epsilon) \int_{M_1} u^2 dV. \tag{2.54}$$

Set $w = u^2$. The following formula hold for any Riemannian metric, and in particular they will be applied to the metric $g_\epsilon$.

**Lemma 2.6.3.** *If $w$ is Lipschitz on on open set $\Omega \subset \mathbb{R}^n$ and if $g \in L^1(\Omega)$ then the following* *relation holds:*

$$\int_\Omega g(x)|\nabla w|dV = \int_{-\infty}^{\infty} \left( \int_{\{w=t\}} g(x) dS_t \right) dt \tag{2.55}$$

Let $V(t_0) = \mathrm{Vol}(M_1)$. Setting $g \equiv 1$ in the above formula gives

$$\begin{aligned}
\int_{M_1} |\nabla w| &= \int_0^{t_0} \left( \int_{\{w=t\}} dS_t \right) dt \\
&= \int_0^{t_0} \mathrm{Area}(w^{-1}(t)) dt \\
&\geq h_n(\epsilon) \int_0^{t_0} \min\left(V(t), 1 - V(t)\right) dt \\
&= h_n(\epsilon) \int_0^{t_0} V(t) dt
\end{aligned} \tag{2.56}$$

By Sard's theorem, the set of regular values for $w$ is a closed, measure zero set of $\mathbb{R}$. Hence we can discard a negligible set $N$ in $\mathbb{R}$ and assume that for $t \in \mathbb{R} - N$, $|\nabla u^2| \neq 0$ on $M_t$. The following holds for any Riemannian metric, it will in particular be applied to $g_\epsilon$ for the integral calculations below.

**Lemma 2.6.4.** *For regular values $t \in \mathbb{R}$*

$$-V'(t) = \int_{\{w=t\}} |\nabla w|^{-1} dS_t. \tag{2.57}$$

Since $V(t)$ is strictly decreasing, $V'(s)$ exists almost everywhere and is integrable. In particular, the above lemma shows that the function $g(x) = \frac{w}{|\nabla w|}$ is $L^1$ on $M_1$.

We now return to 2.56. Using the above formula and integration by parts gives us

$$\begin{aligned}
\int_0^{t_0} V(t) dt &= [tV(t)]_0^{t_0} - \int_0^{t_0} tV'(t) dt \\
&\geq \int_0^{t_0} t \left( \int_{\{w=t\}} \frac{1}{|\nabla w|} dS_t \right) dt \\
&= \int_0^{t_0} \left( \int_{\{w=t\}} \frac{w}{|\nabla w|} dS_t \right) dt
\end{aligned} \tag{2.58}$$

Using the co-area formula on the right-hand side with $g(x) = \frac{w}{|\nabla w|}$ and $w = u^2$ then gives

$$\int_0^{t_0} V(t)dt \geq \int_{M_1} u^2 dV \tag{2.59}$$

Hence

$$\int_{M_1} |\nabla u^2| dV \geq h_n(\epsilon) \int_{M_1} u^2 dV \tag{2.60}$$

The same proof will give the above integral inequality for $M_2$. Hence the result. □

We now need to bound $h_n(\epsilon)$ below. Let $\Sigma = \{p_1, \dots p_\sigma\}$ denote the cone points of $S$, and let the cone angle at $p_i$ be $2\pi(\alpha_i + 1)$.

**Theorem 2.6.5.** *The ratio $h_n(\epsilon)$ defined in 2.49 is bounded below as*

$$h_n(\epsilon) \geq C \sum_{i=1}^{|\Sigma|} d(n\alpha_i, \mathbb{Z}),$$

*where $C > 0$ is a constant dependent on the metric.*

*Proof.* Recall that the metric associated with $\Delta_\epsilon$ is

$$g_\epsilon = dx^2 + dy^2 + \frac{1}{\epsilon} d\theta^2.$$

**Lemma 2.6.6.** *Let $\pi : T_1 S \to S$ be the projection map. Then $\pi : Y(t) \to S - \Sigma$ is an $n$-sheeted covering map.*

*Proof.* Near a point in $T_1 S - \Sigma$ the function $u = \phi_n + \overline{\phi_n} \in E_n + E_{-n}$ has the form

$$u(x, y, \phi) = a_n(x, y) \sin(n\theta) + b_n(x, y) \cos(n\theta). \tag{2.61}$$

If

$$t - a_n(x, y) \sin(n\theta) + b_n(x, y) \cos(n\theta) \tag{2.62}$$

has at least one solution $(x_0, y_0, \theta_0)$ then it has at has exactly $n$ solutions of then form $(x_0, y_0, \theta_0 + \frac{2\pi k}{n})$ for $0 \leq k \leq n-1$ for fixed $\pi(x_0, y_0) \in S$. Indeed, since the above local equation has exactly $n$ solutions in $\theta$ for $t = 0$, and since the number of points in the preimage of regular value is constant on the manifold, the size of the set $\{u = t\}$ has exactly $n$ points or is empty if $t$ is a regular value. The implicit function theorem implies that there are $n$-many functions $\theta_j(x, y) : U_j \to \mathbb{R}$ with $U_j \subset Y_n$ such that $u(x, y, \theta_j(x, y)) = t$. Hence $\pi(x, y, \theta_j(x, y)) = (x, y)$ is a local diffeomorphism from $U_j$ to $S - \Sigma$. $\square$

The 2-dimenisonal surface $Y(t)$ has an area element on $Y(t)$ with respect to $g_\epsilon$ that is of the square root of the form

$$\frac{1}{\epsilon}|dy \wedge d\theta|^2 + \frac{1}{\epsilon}|dx \wedge d\theta|^2 + |dx \wedge dy|^2. \tag{2.63}$$

Around a cone point $p_i$ we write

$$x = r\cos\phi,$$

$$y = r\sin\phi.$$

Hence

$$dx = \cos\phi dr - r\sin\phi d\phi,$$

$$dy = \sin\phi dr + r\cos\phi d\phi$$

and

$$|dx \wedge d\theta|^2 + |dy \wedge d\theta|^2 = |dr \wedge d\theta|^2 + r^2|d\theta \wedge d\phi|^2 \geq |dr \wedge d\theta|^2.$$

Around each cone point $p_i \in \Sigma$ consider pairwise disjoint balls $B_i = B(p_i, r_0) \subset S$ of small radius $r_0 > 0$. Let $U_i = Y(t) \cap \pi^{-1}(B(p_i, r_0))$. The above calculation implies that

$$\begin{aligned} \text{Area}_\epsilon(U_i) &= \int_{U_i} \sqrt{\frac{1}{\epsilon}|dy \wedge d\theta|^2 + \frac{1}{\epsilon}|dx \wedge d\theta|^2 + |dx \wedge dy|^2} \geq \frac{1}{\sqrt{\epsilon}} \int_{U_i} |dr \wedge d\theta| \\ &\geq \frac{1}{\sqrt{\epsilon}} \left| \int_{U_i} dr \wedge d\theta \right| = \frac{1}{\sqrt{\epsilon}} \left| \int_{\partial U_i} r d\theta \right| = \frac{r_0}{\sqrt{\epsilon}} \left| \int_{\partial U_i} d\theta \right|. \end{aligned} \tag{2.64}$$

The boundary $\partial U_i$ is the lift of the circle of radius $r_0$ on the $n$-fold covering map $\pi$. Specifically, denote the base curve that travels around the cone point $\gamma : [0,1] \to S$. Let $v(s) = (r\cos\phi, r\sin\phi, \theta)$ be the parallel transport of a unit vector with $\theta(1) - \theta(0) = 2\pi(1+\alpha_i)$. The lift of $\gamma(s)$ is $\hat{\gamma}(s) = (\gamma(s), v(s)) \in Y(t)$. Since $\pi$ is an $n$-fold covering map, the angle that is traversed around the boundary $\partial U_i$ is

$$n(\theta(1) - \theta(0)) \mod 2\pi = 2\pi n\alpha_i.$$

Hence the integral on the right hand side is bounded below by $2\pi r_0 d(n\alpha_i, \mathbb{Z})$.

**Lemma 2.6.7.** *For Lebesgue almost every irrational $r \in \mathbb{R}$ there are constants $\gamma > 2$ and $K > 0$ such that $|r - k/n| \geq Kn^{-\gamma}$.*

Cover each point in the preimage of $\pi^{-1}(\Sigma)$ with a geodesic ball of radius $r_0$ and set $U_i = \pi^{-1}B(p_i, r_0) \cap Y(t)$. Then

$$\text{Area}_\epsilon(Y(t)) = \int_{Y(t)-\cup U_i} dS_\epsilon + \sum_i \int_{U_i} dS_\epsilon \geq \sum_i \int_{U_i} dS_\epsilon. \tag{2.65}$$

The volume form for the metric $g_\epsilon$ is of the form $\frac{1}{\sqrt{\epsilon}} dx \wedge dy \wedge d\theta$. Thus

$$\begin{aligned} \frac{\text{Area}_\epsilon(Y(t))}{\min\{\text{Vol}_\epsilon(M_1(t)), \text{Vol}_\epsilon(M_2(t))\}} &\geq \text{Area}(Y(t)) \\ &\geq C \sum_i d(n\alpha_i, \mathbb{Z}) \\ &\geq Cn^{-\gamma} \end{aligned} \tag{2.66}$$

Hence there is some $C > 0$ and $\gamma > 2$ such that

$$h_n(\epsilon) \geq \frac{C}{n^\gamma} \tag{2.67}$$

Since $h_n(\epsilon) \searrow h_n(0) = h_n(H)$, we have $h_n(H) \geq \frac{C}{n^\gamma}$.

□

**Definition 2.6.8.** *Recall that the vector fields $X$ and $Y$ are defined for all points on $T_1(S - \Sigma)$, and on that set $[X, Y] = 0$. Hence the Frobenius theorem implies that there is a foliation $\mathcal{F}_H$ in $T_1(S - \Sigma)$ with tangent space equal to the span of $X$ and $Y$. We will call $\mathcal{F}_H$ the horizontal foliation. Equivalently, the horizontal foliation is locally the set of points $(q, w)$ near $(p, v) \in T_1S$ that are parallel transports of $(p, v)$ in the flat metric on $S - \Sigma$.*

**Theorem 2.6.9.** *Suppose at least one of the cone angles $2\pi(\alpha_i + 1)$ is irrational. If $u \in L^2(T_1S)$ is constant on the horizontal foliation, then $u$ is constant.*

*Proof.* Let $\phi_t^X : T_1S \to T_1S$ denote the geodesic flow generated by $X$ on $T_1S$. Note that this flow is volume preserving on $T_1S$. Let $g \in H^{1,0}(T_1S)$. Then, by hypothesis,

$$\begin{aligned} 0 &= \frac{d}{dt}\langle u \circ \phi_t^X, g\rangle_{L^2} \\ &= \frac{d}{dt}\langle u, g \circ \phi_{-t}^X\rangle_{L^2} \\ &= -\langle u, Xg\rangle_{L^2} \end{aligned} \tag{2.68}$$

Similarly, $0 = \langle u, Yg\rangle$ for all $g \in H^{1,0}(T_1S)$. Hence $\partial^+ u = \partial^- u = 0$ weakly. Now expand $u = \sum_n u_n$ in terms of the eigenfunctions of $\Theta$.

Recall that

$$\partial^+ : E_n \cap H^{1,0}(T_1S) \to E_{n-1}, \quad \partial^- : E_n \cap H^{1,0}(T_1S) \to E_{n+1}.$$

Now let $v \in H^{1,0}(T_1S)$ and decompose $v = \sum_n v_n$ in terms of the eigenfunctions of $\Theta$. Then by definition,

$$\langle \partial^+ u_n, v\rangle = -\langle u_n, \partial^- v\rangle = -\langle u_n, \partial^- v_{n-1}\rangle = -\langle u, \partial^- v_{n-1}\rangle = \langle \partial^+ u, v_{n-1}\rangle = 0$$

Hence $\partial^+ u_n = 0$ weakly. Similarly, $\partial^- u_n = 0$ weakly. It follows that $u_n$ is holomoprhic and anti-holomorphic along the horizontal foliation, hence it is constant on the horizontal foliation. Now take a path $\gamma : [0, 1] \to \mathcal{F}_H \subset T_1S$ that projects to a loop $\gamma' : [0, 1] \to S$

that travels around the cone point with an irrational angle. In this case, the end points of $\gamma$ belong to a fiber above a point $(r, \phi) \in S$. In local coordinates,

$$u_n(r, \phi, \theta) = f_n(r, \phi)e^{in\theta}.$$

Since $u_n$ is constant on the path $\gamma$,

$$f_n(r, \phi)e^{in\theta} = f_n(r, \phi)e^{in(\alpha_i+\theta)}.$$

Since $\alpha_i$ is irrational, and since the starting point of the loop $\gamma$ is arbitrary, $u_n = 0$ on $T_1S$. Hence $u_n = 0$ for $n \neq 0$. Hence $u_n$ is constant with respect to the projection of the geodesic flow $X$ and its orthogonal flow $Y$, hence it has all directional derivatives on $S$ equal to zero, therefore it is constant.

□

As a corollary of the Cheeger constant estimate, we can show that the partial Laplacian $H$ has solutions $-Hu = f$ if $f$ is sufficiently regular in $\Theta$.

**Theorem 2.6.10.** *Suppose the simultaneous Diophantine estimate 2.6.1 holds, and let $t \geq \gamma$. Then for $f \in H^{s,t}(T_1S)$ with zero average, there is a solution $u \in H^{s,t-\gamma}(T_1S)$ to the equation*

$$-Hu = f.$$

*Proof.* The eigenfunctions in $E_0$ of $\Theta$ are locally pullbacks of functions on the surface $S$. Therefore the kernel of the operator $H$ on $E_0$ is the kernel of the flat Laplacian on $S$. These functions are harmonic on $S$ and hence constant. Moreover, $H$ has a spectral gap on $E_0$ and the inverse of the eigenvalues of $H$ on $E_0$ are bounded above by a constant $K > 0$. The Cheeger estimate implies that all eigenvaleus are non-zero on $E_n$ for $n \neq 0$. Hence $f$

perpendicular to constants implies that, in particular, we can write

$$f = \sum_{n,k} \langle f, \phi_{n,k} \rangle \phi_{n,k}$$

with $\lambda_{n,k} \neq 0$, where $-H\phi_{n,m} = \lambda_{n,m}\phi_{n,m}$ with $\phi_{n,m} \in E_n$. Note that $u = \sum_{n,k} \lambda_{n,k}^{-1} \langle f, \phi_{n,k} \rangle \phi_{n,k}$ is a solution to $-Hu = f$. Using the previous estimates for the Cheeger constants for $H$ gives

$$\begin{aligned} \|u\|_{L^2}^2 &= \sum_{n,k,n\neq 0} |\lambda_{n,k}|^{-2} |\langle f, \phi_{n,k} \rangle|^2 + K \sum_k |\lambda_{0,k}|^{-2} |\langle f, \phi_{0,k} \rangle|^2 \\ &\leq \sum_{n,k,n\neq 0} C^{-2} n^{2\gamma} |\langle f, \phi_{n,k} \rangle|^2 + K \sum_k |\lambda_{0,k}|^{-2} |\langle f, \phi_{0,k} \rangle|^2 \\ &\leq \sum_{n,\ n\neq 0} C^{-2} n^{2\gamma} \|\pi_n(f)\|_{L^2}^2 + K \|\pi_0(f)\|_{L^2}^2 \\ &\leq C' |f|_{0,t} = C' |Hu|_{0,t}. \end{aligned} \tag{2.69}$$

Lemma 2.15 implies that if $f \in H^{s,t}(T_1S)$ for $t \geq \gamma + 1$ then the Fourier coefficients of $f$ decay fast enough to guarantee convergence of the right-hand sum. Hence $u \in L^2(T_1S)$. Note that the right hand side of the equation is bounded above by $C'|f|_{0,t} = C'|Hu|_{0,t}$. Hence the above calculation implies

$$\|u\|_{L^2} \leq C' |Hu|_{0,t}.$$

Since $[H, Y] = [H, X] = [H, \Theta] = 0$, and since $f \in H^{s,t}(T_1S)$ with $-Hu = f$, the above regularity estimate implies that $u \in H^{s,t-\gamma}(T_1S)$. □

## 2.7 Resolvent Estimates

Note that

$$
\begin{aligned}
X_\theta &:= \cos(\theta)X + \sin(\theta)Y \\
&= \frac{1}{2}\left[e^{i\theta}(X - iY) + e^{-i\theta}(X + iY)\right] \\
&= \frac{1}{2}\left[e^{i\theta}\partial^- + e^{-i\theta}\partial^+\right].
\end{aligned}
\tag{2.70}
$$

This leads to the formal formula

$$
X_\theta = \frac{e^{-i\theta}}{2}\left(\partial^+(\partial^-)^{-1} + e^{2i\theta}\right).\partial^- \tag{2.71}
$$

Our goal for the rest of the section is to make sense of the formal operator $\partial^+(\partial^-)^{-1}$. To begin, notice that $L^2(T_1S) = \overline{R(\partial^+)} \oplus M^-$, where $M^\pm$ is the kernel of $\partial^\pm$. Likewise, we also have $L^2(T_1S) = \overline{R(\partial^-)} \oplus M^+$. Define $R^\pm$ to be the range of the operators $\partial^\pm$. To attain an isometry $U : L^2(T_1S) \to L^2(T_1S)$, it suffices to produce partial isometries $U' : R^- \to R^+$ and $J : M^+ \to M^-$.

On $R^-$ consider the assignment $U' : \partial^- u \to \partial^+ u$. We now need to give a map $J : M^+ \to M^-$. For this we decompose the kernel into the sum of $E_n = \{u \in L^2 : \Theta u = inu\}$ intersected with $M^+$. Let $J'$ be the conjugation map.

**Lemma 2.7.1.** *$J' : E_n \cap M^+ \to E_{-n} \cap M^-$ is a conjugate linear isometry from $M^+ \to M^-$*

*Proof.* This follows from the definition. □

Since there exists a conjugate linear isometry from $M^+$ to $M^-$, there also exists a linear isometry $J$ from $M^+$ to $M^-$ Define $U' : R^- \to R^+$ by

$$
U'(\partial^- u) = \partial^+ u \tag{2.72}
$$

Let $\pi^-$ be the projection onto $R^-$ and define

$$
U_J = U'\pi_- + J(1 - \pi_-). \tag{2.73}
$$

Then $U$ is a linear isometry on $L^2(T_1S)$.

The quantity 2.71 can now be rewritten as

$$X_\theta = \frac{e^{-i\theta}}{2}\left(U_J + e^{2i\theta}\right)\partial^- \tag{2.74}$$

Define the resolvent of the unitary operator $U$ by

$$R_U(z) := (U - zI)^{-1} \tag{2.75}$$

for $|z|< 1$, where $U : H \to H$ is a unitary operator on a separable Hilbert space. The spectral theorem [20] implies that

$$\langle R_U(z)u, v\rangle = \int_0^{2\pi}(z - e^{it})^{-1}d(E_U(t)u, v) \tag{2.76}$$

for all $|z|< 1$ and is also holomorphic in $z$. let $f$ be an $L^2$ function on the unit disk $D^2$ and let $\Omega_\alpha(\theta) \subset D^2$ be the set of points inside the cone with aperature $0 < \alpha < 1$ whose vertex is at angle $\theta \in S_1$ on the boundary of $D^2$. Define the non-tangential maximal function

$$N_\alpha(f)(\theta) := \sup\{|f(z)|: z \in \Omega_\alpha(\theta)\}. \tag{2.77}$$

We will define $N_\alpha(u, v)$ to be the non-tangential maximal function for $f(z) = \langle R_U(z)u, v\rangle$.

**Lemma 2.7.2.** *([8]) Let $R_U(z)$ denote the resolvent of a unitary operator $U : H \to H$ on a Hilbert space $H$. Then the non-tangential maximal functions $N_\alpha(u, v)$ belong to $L^p$, for any $0 < p < 1$ and $\alpha < 1$, and there exists constants $A_\alpha$, $A_{\alpha,p} > 0$ with $A_{\alpha,p} \to \infty$ as $p \to 1$, such that*

$$|N_\alpha(u, v)|_p \leq A_\alpha\|\langle R_U(\cdot)u, v\rangle\|_p \leq A_{\alpha,p}\|u\|_H\|v\|_H \tag{2.78}$$

## 2.8 Distributional Solutions

This section closely follows the work from [10]. Let $|\cdot|_{s,t}$ be the $(s,t)$ Sobolev norm defined on the space from formula (2.6).

**Definition 2.8.1.** *A distribution $u \in \bar{H}^{-r,-t}(T_1S)$ will be called a (distributional) solution of $X_\theta u = f$ for $f \in \bar{H}^{-s,-t}(T_1S)$ if*

$$\langle u, X_\theta v\rangle = -\langle f, v\rangle \tag{2.79}$$

*for all $v \in H^{r+1,t}(T_1S) \cap \bar{H}^{s,t}(T_1S)$*

Define the (fractional) Friedrich's inner product by

$$\langle u, v\rangle_s := \sum_{k=1}^{\infty}(1+\lambda_k)^s \langle u, e_k\rangle\langle e_k, v\rangle, \tag{2.80}$$

and let $\|u\|_s := \sqrt{\langle u, u\rangle_s}$. Recall also the definition of the Sobolev norm from formula (2.2). Throughout this section and the rest of the paper we will fix the regularity $t$ in the $\Theta$ derivatives to be such that $t > \gamma + 1$ as in Theorem (2.6.10). Note that $\gamma > 2$ as well.

**Theorem 2.8.2.** *Let $t > \gamma + 1$ as in Theorem (2.6.10) to satisfy the simultaneous Diophantine condition. Let $r > 3$ and $p \in (0,1)$ be such that $pr > 3$. There exists a bounded linear operator*

$$W : H^{1,t}(T_1S) \to L^p(S^1, \overline{H}^{-r,-t}(T_1S)) \tag{2.81}$$

*such that the following holds. Define $X_\theta = \cos(\theta)X + \sin(\theta)Y$. For any $f \in H^{1,t}(T_1S)$ with zero average, there exists a full measure set $\mathcal{F}_r(f) \subset S^1$ such that for $\theta \in \mathcal{F}_r(f)$ there is $u_\theta := W(f)(\theta) \in \overline{H}^{-r,-t}(T_1S)$ that is a distributional solution for $X_\theta u_\theta = f$ . In addition there exists a constant $B(p,r) > 0$ such that, for all $f \in H^{-1,-t}(T_1S)$,*

$$|W(f)|_p := \left(\int_{S^1} \|W(f)(\theta)\|^p_{-r-t} d\theta\right)^{\frac{1}{p}} \le B|f|_{-1,-t} \tag{2.82}$$

The above theorem depends on the following a priori estimate.

**Lemma 2.8.3.** *Let $\gamma$ be as in the Diophantine condition for 2.6.10 and let $t \geq \gamma$. Let $r > 2\gamma + 2$ and $p \in (0,1)$ be such that $p(r - 2\gamma - 2) > 3$. For any $f \in H^{1,\gamma}(T_1S)$ perpendicular to constants there is a measurable function $A(f) := A(f,p,r) \in L^p(S^1)$ such that for all $v \in H^{r+1,t}(T_1S)$ we have*

$$|\langle f, v\rangle| \leq A(f)(\theta)\|X_\theta v\|_{r+t}.$$

*In addition, there exists a constant $B := B(p,r) > 0$ such that for every $f \in H^{1,t}(T_1S)$ with zero average,*

$$|A(f)|_p \leq B|f|_{1,\gamma}$$

We will use the resolvent estimates and Weyl asymptotics along with the invertibility of $-H$ to derive the stability bound. Let $f \in H^{1,\gamma}(T_1S)$ be perpendicular to constants.

Suppose $t$ is chosen large enough to satisfy the Diophantine condition from Theorem (2.6.10) with $t \geq \gamma$.

*Proof.* Let $v \in H^{r+1,t}(T_1S)$. By Theorem 2.6.10, $-H$ is invertible on the set of functions on $H^{r+1,t}(T_1S)$ with zero average. Since $f$ has zero average, there is a $u \in H^{1,0}(T_1S)$ such that $-Hu = f$. Now set $\partial^{\pm}u = F^{\pm}$. Then $F^{\pm} \in H^{0,0}(T_1S) = L^2(T_1S)$.

Let $\Phi^{\pm}$ be the linear functional on $R^{\pm} \subset H^{r,t}(T_1S)$ defined by $\partial^{\pm}v \to -\langle f, v\rangle$ and $\partial^{\pm}w_n \to u$ for $w_n \in H^{r+1,t}(T_1S)$. Since $\partial^{\pm}w_n \to u$, the Cauchy-Schwartz inequality $\langle f, w_n - w_m\rangle$ implies that the real-valued sequence $\{\Phi^{\pm}(\partial^{\pm}w_n)\}$ is Cauchy. By completeness, $\Phi^{\pm}(u) := \lim \Phi(w_n)$ is well-defined. It follows that $\Phi^{\pm}$ can be extended to the closure of $R^{\pm}$. Since $\Phi^{\pm}$ is a bounded linear functional on a closed space, the Hahn-Banach theorem states that $\Phi^{\pm}$ can be extended to all of $L^2(T_1S)$. Now define $M^{\mp}$ to be the orthogonal complement of $\overline{R^{\pm}}$. In particular, if $P$ is the orthogonal projection onto $\overline{R^{\pm}}$ we define the extension of $\Phi_0^{\pm}$ for $u \in L^2(T_1S)$ to be $\Phi^{\pm}(Pu)$.

Notice also that $f = \partial^{\pm}F^{\pm}$ implies $\|F\|_{L^2} \leq |f|_{1,\gamma}$

Recall the definition of $U_J$ from formula (2.73) and $R_U$ from formula (2.75). Observe that

$$X_\theta = \{e^{-i\theta}\partial^+ + e^{i\theta}\partial^-\}/2 = \frac{e^{-i\theta}}{2}\{U_J + e^{2i\theta}\}\partial^- = \frac{e^{i\theta}}{2}\{U_J^{-1} + e^{-2i\theta}\}\partial^+. \tag{2.83}$$

This implies that

$$\langle \partial^{\pm}v, F^{\pm}\rangle = 2e^{\mp i\theta}\langle R_J^{\pm}(z)X_\theta v, F^{\pm}\rangle - (z + e^{\mp 2i\theta})\langle R_J^{\pm}(z)\partial^{\pm}v, F^{\pm}\rangle. \tag{2.84}$$

Let $\lambda_k$ denote the $k$'th eigenvalue of the full Laplacian $-\Delta$. Let $w \in H^{r+1,t}(T_1S)$. Define $R_k^{\pm}(z) := \langle R_J^{\pm}(z)e_k, F^{\pm}\rangle$. We have

$$\langle R_J^{\pm}(z)w, F^{\pm}\rangle = \sum_{k=0}^{\infty}\langle w, e_k\rangle R_k^{\pm}(z). \tag{2.85}$$

The Cauchy-Schwartz inequality and the definition of the Friedrichs norm implies

$$|\langle R_J^{\pm}(z)w, F^{\pm}\rangle| \leq \|w\|_r \left(\sum_{k=0}^{\infty}\frac{|R_k^{\pm}(z)|^2}{(1+\lambda_k)^r}\right)^{\frac{1}{2}}. \tag{2.86}$$

Let $N_k^{\pm}(\theta)$ denote the associated non-tangential maximum function for $R_k^{\pm}(z)$ and define

$$N^{\pm}(\theta) := \left(\sum_{k=0}^{\infty}\frac{|N_k^{\pm}(\theta)|^2}{(1+\lambda_k)^r}\right)^{\frac{1}{2}}. \tag{2.87}$$

By Lemma 2.7.2, for fixed aperture $\alpha \in (0,1)$

$$|N_k^{\pm}|_p \leq A_{\alpha,p}\|e_k\|_{L^2}\|F^{\pm}\|_{L^2} \leq A_{\alpha,p}|f|_{1,\gamma}. \tag{2.88}$$

Define $N^{\pm}(w) : S^1 \to \mathbb{R}$ to be the maximal tangential function for the holomorphic function $\langle R_J^{\pm}(z)w, F^{\pm}\rangle$ as in formula (2.77).

Lemma 2.7.2 implies that for all $w \in H^{r,t}(T_1S)$

$$N^{\pm}(w)(\theta) \le N^{\pm}(\theta)\|w\|_r. \tag{2.89}$$

By a triangular inequality on $L^{p/2}(T_1S)$ and Lemma 2.7.2 we have that for some $0 < p < 1$,

$$|N^{\pm}|_p^p \le (A_p)^p \left(\sum_{k=0}^{\infty} \frac{1}{(1+\lambda_k)^{pr/2}}\right) |f|_{1,\gamma}^p, \tag{2.90}$$

where $\lambda_k$ are the eigenvalues for $-\Delta$. By Lemma 2.5.2 for the Weyl asymptotics, the sum in the above formula converges. If we let $z \to -e^{\pm 2i\theta}$ then formulas 2.84 and 2.89 imply that

$$|\langle f, w\rangle| = |\langle \partial^{\pm} w, F^{\pm}\rangle| \le N^{\pm}(\pi \mp 2\theta)\|X_\theta w\|_r. \tag{2.91}$$

Hence we have a priori estimates for all $\theta \in S^1$ such that $|N^{\pm}(\pi \mp 2\theta)| < \infty$. Equation 2.90 shows that this happens for almost every $\theta$. Setting

$$B(f,p,r) := (A_p)^p \left(\sum_{k=0}^{\infty} \frac{1}{(1+\lambda_k)^{pr/2}}\right) |f|_{1,\gamma}^p$$

and

$$A(f) = N^{\pm}(\pi \mp 2\theta)$$

completes the proof of Lemma 2.8.3. □

Now for the proof of Theorem 2.8.2. The stability bound from Lemma 2.8.3 implies that the linear map $X_\theta u \to -\langle f, u\rangle$ can be extended by continuity to the closure of the range of the operator $X_\theta$ on the space $H^{r,t}(T_1S)$. Let $W(f)(\theta)$ be defined as the unique linear extension equal to zero on the orthogonal complement of the range of $X_\theta$. The map $W(f)(\theta)$

is a distributional solution to the cohomological equation $X_\theta u = f$ with

$$\|W(f)(\theta)\|_{-r} \leq A(f).$$

By Lemma 2.8.3, the $L^p$ norm of $W(f)(\theta)$ satisfies

$$|W(f)(\theta)|_p := \int_{S^1} \|W(f)(\theta)\|_{-r}^p d\theta \leq B|f|_{1,\gamma}.$$

**Theorem 2.8.4.** *Let $t$ be large enough to satisfy the Diophantine condition as in Theorem 2.6.10. For any $r > 0$ and for all zero-average $f \in H^{r,t}(T_1 S)$, the cohomological equation $X_\theta u = f$ has a distributional solution $u \in H^{-r+1,-t+\gamma}(T_1 S)$.*

Let $\{e_k\}$ be the basis for the Laplacian $-\Delta$ on $T_1 S$. If $u_k(\theta)$ is a solution to $X_\theta u = e_k$, then we have the formal (distributional) solution $u_\theta = \sum_{k \geq 1} \langle f, e_k \rangle u_k(\theta)$ to the cohomological equation $X_\theta u = f$. The following inequalities show that this in fact is a solution.

*Proof.* Using a Holder and triangle inequality we get

$$\|u(\theta)\|_{-r} \leq \left( \sum_{k \geq 1} \frac{\|u_k(\theta)\|_{-r}^2}{(1+\lambda_k)^{r-1}} \right)^{\frac{1}{2}} \|f\|_{r-1}. \tag{2.92}$$

Note that by Theorem 2.8.2 and 2.2.1

$$\left( \int_{S^1} \|u_k(\theta)\|_{-r}^p d\theta \right)^{\frac{1}{p}} \leq B|e_k|_{1,\gamma} \leq B'(1+\lambda_k)^{1+\gamma} \tag{2.93}$$

Integrating $u_\theta$ with respect to $\theta$ and applying the two previous bounds yields

$$\int_{S^1} \|u(\theta)\|_{-r}^p d\theta \leq C_r^p \left( \sum_{k \geq 1} \frac{(1+\lambda_k)^{p\gamma+p}}{(1+\lambda_k)^{pr/2}} \right) \|f\|_{r-1}^p. \tag{2.94}$$

Lemma 2.5.2 and equation (2.44) implies that the series on the right hand side converges and hence that $X_\theta u_\theta = f$ is a solution for almost all $\theta \in S^1$. ☐

## 2.9 Properties of the Cokernel

Throughout this section $S$ is a flat surface with $\Sigma = \{z_1, \ldots, z_n\}$ the set of cone points with cone angles $2\pi(\alpha_i + 1)$ with $\alpha_i + 1 \notin \mathbb{Q}$. Recall

$$\partial^+ := X + iY, \qquad \partial^- := X - iY.$$

Along the foliations generated by a rotation of $\phi$ around a cone point with angle $2\pi(\alpha_i+1)$ we have

$$\begin{aligned} \partial^+ &= \bar{z}^{-\alpha_i} \frac{\partial}{\partial \bar{z}} e^{-i\theta} \\ \partial^- &= z^{-\alpha_i} \frac{\partial}{\partial z} e^{i\theta}. \end{aligned} \tag{2.95}$$

A $k$ - differential $q_k$ is a function on $(p, v) \in T_1 S$ that locally has the form

$$q_k(p, v) = f(p) v^k.$$

Here we identify $v = e^{ik\theta}$. If $g \in E_n \cap H^{0,1}(T_1 S)$ then

$$g(z, \theta) = g_n(z) e^{in\theta}.$$

Hence there is an identification between $k$-differentials and functions in $E_k \cap H^{0,1}(T_1 S)$. The main theorem of this section is as follows. The operator $\partial^+$ can act on the dual Sobolev spaces

$$\partial^+ : H^{-r,-s}(T_1 S) \to H^{-r+1,-s}(T_1 S)$$

by $U \to \partial^+ U$.

Let $m \in ker(\partial^+) \cap H^{1,t}(T_1 S) \hookrightarrow L^2(T_1 S)$ and let $\hat{m}(v) = \langle m, v \rangle_{L^2}$. Let $D_X = \{D \in$

$H^{-s,-t}(T_1S) : XD = 0\}$. Since the partial Laplacian $H$ is invertible, the space $\partial^+ D_X \subset H^{-s+1,-t}(T_1S)$ is closed. Theorem 2.8.2 implies that if $m$ is perpendicular to constants then there is a solution $U$ to $XU = \hat{m}$. This relationship gives us an assignment $F : \ker(\partial^+) \to H^{-s,-t}(T_1S)/\partial^+ D_X$ by $F(m) = \partial^+ U + \langle \partial^+ D_X \rangle$. An $X$-invariant distribution is a distribution $D$ such that $XD = 0$, or $D(Xf) = 0$ for all $f \in H^{s,t}(T_1S)$. Note also that $F(m)$ is $X$-invariant since

$$XF(m) = X\partial^+ U = \partial^+(XU) = \partial^+ m = 0.$$

It follows that the map $F$ sends the distributional kernel of $\partial^+$ into the space of $X$-invariant distributions that vanish on constant functions modulo $\partial^+ D_X$.

**Lemma 2.9.1.** *$F$ is onto the set of $X$-invariant distributions in the sense that if $XD = 0$ with $D \in H^{-r,-t}(T_1S)$ vanishing on constant functions, then there is an $m \in \ker \partial^+ \cap H^{1,0}(T_1S)$ and $U \in H^{-r+1,-t}(T_1S)$ with $XU = \hat{m}$ such that $\partial^+ U = D$*

*Proof.* Let $D \in H^{-r,-t}(T_1S)$ be $X$-invariant and vanishing on constant functions. Let $v \in H^{r,t}(T_1S)$ have zero average and define $U$ on the range of $\partial^+$ so that

$$U(\partial^+ v) = -D(v).$$

Then

$$|U(\partial^+ v)| = |D(v)| \leq |D|_{-r,-t} \cdot |v|_{r,t}.$$

Lemma 2.6.10 implies that

$$|v|_{r,s} \leq C|Hv|_{r-2,t} \leq C|\partial^+ v|_{r-1,t}$$

It follows that

$$|U(\partial^+ v)| \leq C|D|_{-r,-t}|\partial^+ v|_{r-1,t}$$

hence $U$ is continuous with respect to the norm $|\cdot|_{r-1,t}$ in the range of $\partial^+$. It can be extended by continuity to the closure of the range and by orthogonality to the whole space to give a solution $U \in H^{-r+1,-t}(T_1 S)$ to $\partial^+ U = D$.

Define $m := XU$. Then

$$\partial^+ m = \partial^+ XU = X(\partial^+ U) = XD = 0.$$

Hence $m$ is in the distributional kernel of $\partial^+$ and, by definition, $\partial^+ U = D$. $\square$

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