ABSTRACT

Title of Dissertation: IRRATIONAL ELLIPSOID EMBEDDINGS AND A CANONICAL GRADING OF EMBEDDED CONTACT HOMOLOGY

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The main content of this thesis is two papers. The first paper defines a canonical $\mathbb{Q}$-grading of the embedded contact homology chain complex under certain conditions. ECH is an invariant of 3-manifolds isomorphic to the Seiberg-Witten Floer cohomology as defined by Kronheimer and Mrowka. The underlying chain complex comes with a relative $\mathbb{Z}/p$-grading, or, an absolute $\mathbb{Z}/p$-grading after fixing a reference generator. However, when $p = 0$, we show that one can actually specify an absolute grading of the chain complex without having to choose a reference generator. Explicitly, that grading set is a subset of $\mathbb{Q}$ isomorphic to $\mathbb{Z}$. As a consequence, $ECH(Y, \lambda)$ is canonically $\mathbb{Q}$-graded whenever $H_1(Y)$ is torsion.

The second paper is an application of ECH capacities to the problem of embedding ellipsoids into irrational ellipsoids. More specifically, we analyze lattice point counting functions to deduce that the existence of an accumulation point of an infinite staircase is impossible for all irrational targets minus a set of irrationals in bijection with $\mathbb{N}$, for which the question has yet to be resolved.

IRRATIONAL ELLIPSOID EMBEDDINGS AND A CANONICAL GRADING OF EMBEDDED CONTACT HOMOLOGY

by

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Contents

Chapter 1

Introduction

All knowledge is simply bringing the essence of life under the laws of reason.

-Leo Tolstoy

1.1 General notions

In all that follows, we will restrict to 3 and 4 dimensions which are the relevant dimensions for this thesis, though these definitions extend to higher dimensions as well. Let Y be a smooth 3-manifold. A *contact structure* ξ on Y is a completely non-integrable 2-plane distribution. The pair (Y, ξ) is called a contact manifold. In the case that λ is a 1-form on Y so that $\xi = \ker \lambda$, we call λ a *contact form*, and denote the contact manifold as the pair (Y, λ). Note for $\ker \lambda$ to be non-integrable it is both necessary and sufficient for $\lambda \wedge d\lambda > 0$. For example, the standard contact structure ξ_{std} on $\mathbb{R}^3$ is given as the kernel of $\lambda = dz - ydx$.

One could choose as a basis for the contact plane at a point (x, y, z) the tangent vectors ∂_y and $y\partial_z + \partial_x$. Figure 1 illustrates the 2-plane field. A fundamental dichotomy of contact structures is the distinction between *tight* and *overtwisted* contact structures: an overtwisted contact structure contains an embedded D^2 whose boundary ∂D^2 is a *Legendrian knot* (see the appendix), while a tight contact structure contains no such disk. A given Y can have many non-equivalent contact structures. For example, Honda [13] proved that the lens space $L(p, q)$ has $\Pi_{i=1}^{k}|(a_i + 1)|$

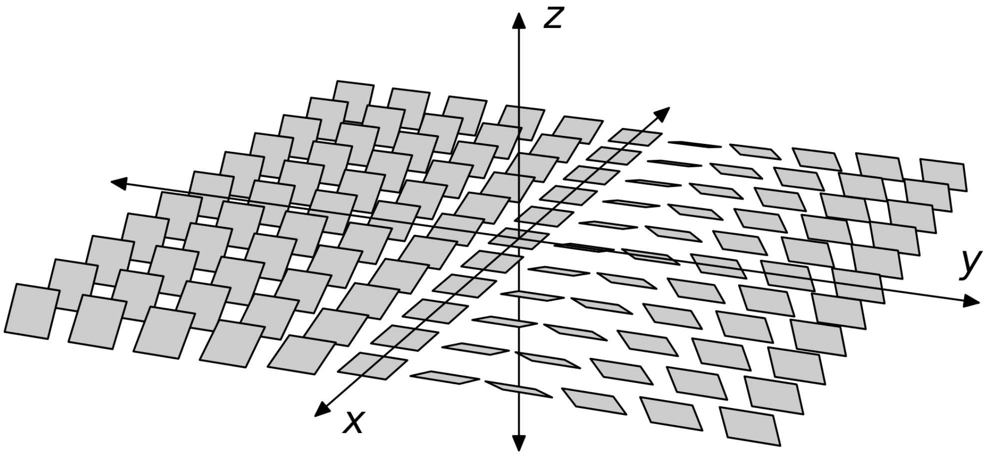

Figure 1.1: Standard contact structure on $\mathbb{R}^3$

distinct tight contact structures, where the $a_i < -1$ are the coefficients in the continued fraction expansion of $-p/q$.

On any (Y, λ) there is a canonical vector field R called the *Reeb vector field*, defined by the equations $d\lambda(R, \cdot) = 0$ and $\lambda(R) = 1$. Questions surrounding the dynamics of R are of central interest to the study of contact manifolds. For example it is known now, due to Taubes [27], that every R has at least one periodic orbit.

The even dimensional counterpart to a contact manifold is a *symplectic manifold*. This is a pair (X, ω) where X is a smooth 4-dimensional manifold and ω is a closed, non-degenerate 2-form on X. That is, $d\omega = 0$ and $\omega \wedge \omega > 0$. The *symplectic volume* of X is defined to be $\int_X \omega \wedge \omega$. Given a smooth function $H : X \to \mathbb{R}$, the *Hamiltonian vector field* of H is defined by the equation

$$-dH = \omega(X_H, \cdot).$$

There are many important relationships between contact and symplectic manifolds. As one example, given any (Y, λ) one can build a symplectic manifold from Y called the *symplectization of* Y whose underlying smooth 4-manifold is $\mathbb{R} \times Y$ and whose symplectic form is given by $\omega = d(e^t\lambda)$, whereby λ in this context we mean the pullback of λ on Y under the projection $\mathbb{R} \times Y \to Y$ and where t is the $\mathbb{R}$ coordinate. On the other hand, symplectic manifolds often contain important hypersurfaces with some induced contact structure, possibly as a boundary. A common example is a generic level set S of a Hamiltonian $H : X \to \mathbb{R}$, with contact form defined by $\iota_v(\omega)$, where v is some vector field transverse to S.

1.2 ECH(Y) and Seiberg-Witten theory

Let (Y, λ) be contact and R its Reeb vector field. Choose an almost complex structure J on the symplectization $\mathbb{R}_t \times Y$ compatible in the sense that $J(\partial_t) = R$. The *embedded contact homology* of (Y, λ, J), or $ECH(Y, \lambda, J)$, is the homology of a chain complex $(ECC(Y, \lambda, J), \partial)$ whose underlying $\mathbb{Z}/2$-module is generated by finite orbit sets $\alpha = \{(m_i, \alpha_i)\}$ where m_i is positive, equal to 1 if α_i is hyperbolic (see [17] for explanation), and each α_i is an embedded Reeb orbit. When a class $\Gamma \in H_1(Y; \mathbb{Z})$ is fixed, the submodule generated by classes $\{(m_i, \alpha_i)\}$ satisfying $\sum_i m_i[\alpha_i] = \Gamma$ is denoted $ECC(Y, \lambda, J, \Gamma)$. The differential counts J-holomorphic curves in $\mathbb{R} \times Y$ asymptotic to two orbits whose ECH index is equal to 1. Hence the differential restricts to $ECC(Y, \lambda, J, \Gamma)$ and we denote its homology by $ECH(Y, \lambda, J, \Gamma)$. Taubes showed that $ECH(Y, \lambda, J)$ is isomorphic to $\widehat{HM}^{-*}(Y)$, the monopole Floer cohomology (or Seiberg-Witten Floer cohomology) of Y. As a consequence, the ECH of Y is in fact a topological invariant, independent of λ and J. Chapter 2 goes into greater detail concerning ECH and monopole Floer theory, and discusses their gradings, which we occasionally suppress from the notation.

1.3 Symplectic embeddings and capacities

The third chapter of this dissertation concerns symplectic embeddings. We say that there is a there is a *symplectic embedding* of (X, ω) into (X', ω') if there exists a smooth embedding $f : X \to X'$ which is also a symplectomorphism, that is, if $f^*\omega' = \omega$. As a shorthand for symplectic embeddings we write $X \hookrightarrow X'$. Note that an immediate consequence is that the symplectic volume of X is no larger than X'; in particular, if X and X' have the same symplectic form (for example as subsets of $\mathbb{R}^4$) then $f : (X, \omega) \to (X', \omega)$ is volume preserving. Therefore symplectic embeddings are volume preserving, but are there more restrictions?

This question owns a great deal of the share of responsibility for spawning interest in symplectic topology. The first milestone was Gromov's nonsqueezing theorem, which states that the symplectic ball $(B^{2n}(r), \omega_0)$, with ω_0 inherited from $\mathbb{R}^{2n}$, embeds into the "cylinder" $(D^2(R) \times \mathbb{R}^{2n-2}, \omega_0)$ if and only if $r \leq R$. Since the target has infinite volume, this answered the question in the positive and introduced the notion of J-holomorphic curves as central objects of study in the symplectic world.

One of the next achievements came by considering embeddings of ellipsoids into a ball. Namely, given $1 \leq a \leq b$ consider in $\mathbb{C}^2$ the region

$$E(a,b) := \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid \frac{\pi |z_1|^2}{a} + \frac{\pi |z_2|^2}{b} \leq 1 \right\}$$

equipped with the restriction of the standard symplectic form ω_0. What are the obstructions to embedding $E(a,b)$ into the ball $B(\mu) := E(\mu, \mu)$ of radius $\sqrt{\mu/\pi} > 0$? For scaling reasons one of the parameters is redundant so we can consider $E(a,1)$. We can codify this question as a function by defining

$$c(a) := \inf\{\mu > 0 \mid E(1,a) \hookrightarrow B(\mu)\}.$$

The volume constraint implies that for all $a > 0$ we must have $c(a) \geq \sqrt{a}$. In [25], McDuff and Schlenk compute $c(a)$, and both the methods and the result itself are quite remarkable. In short, they discovered that $c(a)$ has infinitely many nonsmooth points $(a_n)_{n \geq 1}$, a monotonically increasing sequence whose values a_n are fractions of Fibonacci numbers, converging to τ^4, where τ is the golden ratio. In the intervals away from these nonsmooth points, the graph of $c(a)$ alternates between linear and constant functions, called steps. The function then has finitely many steps between $\tau^4 \approx 6.85$ and $(17/6)^2$, after which $c(a)$ follows the volume curve $\sqrt{a}$. The so-called *Fibonacci staircase* is the first known example of, more generally, an *infinite staircase*. The point τ^4 is called the *accumulation point*.

Very roughly speaking, in [25] the obstructions to an embedding $E(1,a) \hookrightarrow B(\mu)$

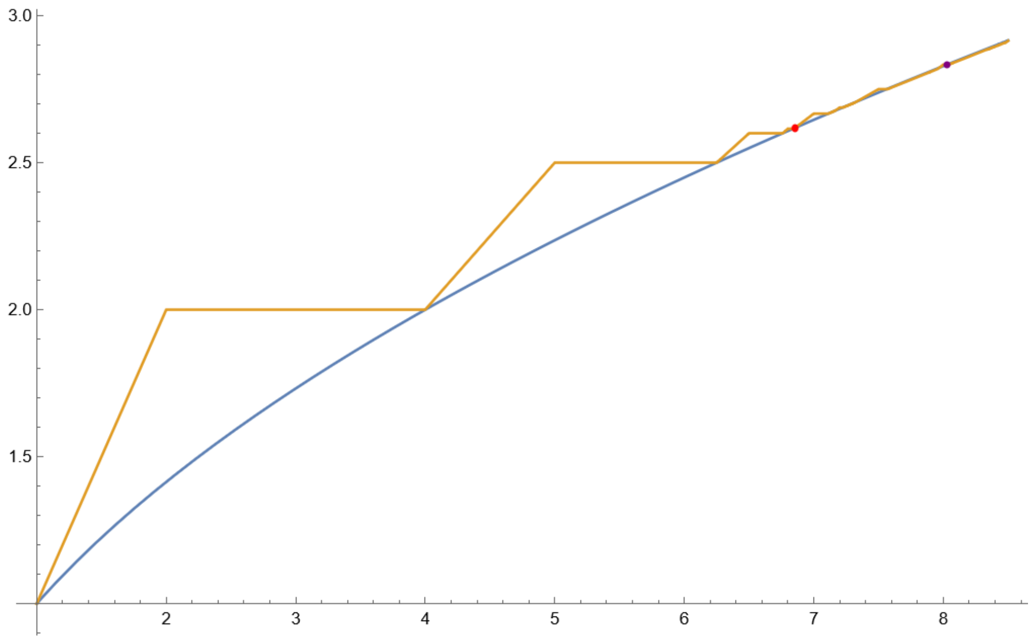

Figure 1.2: A Mathematica plot of the Fibonacci staircase. The blue curve is $\sqrt{a}$, the orange curve $c(a)$; the red point is (τ^4, τ^2), and the purple point $((17/6)^2, 17/6)$.

at rational values $a = p/q$ come from exceptional divisors in the M-fold blowup of $\mathbb{C}P^2$, where M is the length of the weight sequence of p/q, analogous to the continued fraction expansion of p/q. An explicit cohomological description of the symplectic cone $\mathcal{E}_M$ of X_M and Theorem 1.2.2 in [25] lead to Diophantine relations which provide the computational basis for calculating $c(a)$. An alternative route to determining obstructions is through *ECH capacities*, which is the route we follow in Chapter 3.

Let (X, ω) be a compact symplectic 4-manifold with oriented boundary Y, whose symplectic form ω is exact, and suppose there is a 1-form λ on X so that λ restricted to Y is a contact form and $d\lambda = \omega|_Y$. Such a pair (X, ω) is called a *Liouville domain.* Given a Liouville domain (X, ω), there is a sequence of real numbers

$$0 = c_0(X, \omega) < c_1(X, \omega) \leq c_2(X, \omega) \leq \cdots \leq \infty$$

called *ECH capacities* [15]. When (Y, λ) is the boundary of a Liouville domain (X, ω), we define $c_k(Y, \lambda) := c_k(X, \omega)$. The definition of ECH capacities can be extended to more general symplectic spaces, but we will not need this.

Let $Y = \partial E(a,b)$ where for simplicity we will assume that $a/b \notin \mathbb{Q}$. The restriction of the 1-form

$$\lambda = \frac{1}{2}\sum_{i=1}^{2}(x_i dy_i - y_i dx_i)$$

on $\mathbb{R}^4$ serves as the contact form for $\partial E(a,b)$. The Reeb vector field in this case is given by

$$R = \frac{2\pi}{a}\frac{\partial}{\partial\theta_1} + \frac{2\pi}{b}\frac{\partial}{\partial\theta_2}.$$

which has exactly two closed integral curves γ_1 and γ_2 (since $a/b \notin \mathbb{Q}$).

Now recall that $H_1(E(a,b);\mathbb{Z}) = 0$, hence there is only one summand of $ECH(E(a,b),\lambda)$, namely $ECH(Y,\lambda,0)$. One can compute $ECH(E(a,b),\lambda)$ directly as in [17] to obtain

$$ECH_*(E(a,b),\lambda) \cong \begin{cases} \mathbb{Z}/2 & * = 0,2,4,\dots \\ 0 & \text{else} \end{cases}$$

The *action* of an ECH generator α is defined to be $\mathcal{A}(\alpha) = \int_\lambda \alpha$. Hence

$$\mathcal{A}(\gamma_1^m\gamma_2^n) = am + bn.$$

In the specific case of $Y = E(a,b)$, the k-th ECH capacity $c_k(E(a,b),\lambda)$ is the action of the generator in degree $2k$. See [15] for the definition of the U map and the precise definition of $c_k(Y,\lambda)$ to see why this is true. Hence we have

Proposition 1.3.1 *The ECH capacities of $E(a,b)$ are given by*

$$c_k(E(a,b),\lambda) = (am+bn)_k,$$

where $(am+bn)_k$ is the k-th smallest element of the array $(am+bn)_{(m,n)\in\mathbb{Z}^2_{\geq 0}}$.

This can be extended to the case that $a/b \in \mathbb{Q}$.

There is an interpretation of the sequence $(am+bn)_k$ that serves as the starting point for Chapter 3. Let $\mathcal{T}_{a,b}$ denote the right triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(0,a)$, and $(b,0)$. Again for simplicity we assume that $a/b \notin \mathbb{Q}$. If we let t_k denote the value of $(am+bn)_k$, then

$$k+1 = |t_k \cdot \mathcal{T}_{a,b} \cap \mathbb{Z}^2|.$$

That is, if $N_{a,b}(t) := |t \cdot \mathcal{T}_{a,b} \cap \mathbb{Z}^2|$ denotes the lattice point counting function of $\mathcal{T}_{a,b}$ then $N_{a,b}(t_k) = k+1$. Having said this, suppose $N_{c,d}(t)$ is the counting function for some other irrational ellipsoid $E(c,d)$ where $c/d \neq a/b$. If at some point t there is an integer $k+1$ so that $N_{a,b}(t) < k+1 \leq N_{c,d}(t)$ then we must have that $c_k(a,b) > c_k(c,d)$. Hence the capacity $c_k(a,b)$ is an obstruction to $E(a,b) \hookrightarrow E(c,d)$. Moreover, the long term behavior of the counting functions tell us whether there are finitely or infinitely many obstructions. For example, suppose there is a $T > 0$ so that $t \geq T$ implies that $N_{a,b}(t) < N_{c,d}(t)$. Then there are infinitely many such k for which $c_k(a,b)$ is an obstruction to the embedding. If on the other hand we had $N_{a,b}(t) > N_{c,d}(t)$, then at most finitely many capacities that obstruct the embedding.

Again for scaling reasons we can restrict the question to $E(1,a) \hookrightarrow E(1,b)$. In fact, to further eliminate the volume constraint we consider $\sqrt{\frac{b}{a}}E(1,a) \hookrightarrow E(1,b)$. Let $\mathcal{T}_{a,b}$ again stand for the right triangle representing the source $\sqrt{\frac{b}{a}}E(1,a)$ and let $\mathcal{T}_b$ represent the target $E(1,b)$. Now it's generally the case that for a convex polytope P in $\mathbb{R}^2$ containing the origin, the counting function has the form

$$|t \cdot P \cap \mathbb{Z}^2| = \text{area(P)}t^2 + O(t).$$

Since $\text{area}(\mathcal{T}_{a,b}) = \text{area}(\mathcal{T}_b)$ we focus our attention to the $O(t)$ terms of $N_{a,b}(t)$ and $N_{c,d}(t)$.

Remark. When a is rational, $\mathcal{T}_{a,b}$ is an irrational scaling of a rational polytope. Ehrahdt polynomials were extended to the reals by Linke in [21], so in the rational case $N_{a,b}(t)$ is a degree 2 quasi-polynomial, that is, has the form

$$N_{a,b}(t) = \frac{1}{2b}t^2 + Q_1(\mathcal{T}_{a,b}, t)t + Q_0(\mathcal{T}_{a,b}, t),$$

where $Q_i(\mathcal{T}_{a,b}, t)$ for $i = 0, 1$ are periodic functions in t. In general this may not be true.

One can easily show using Weyl's criterion that

$$|t \cdot \mathcal{T}_b \cap \mathbb{Z}^2| = \frac{1}{2b}t^2 + \frac{1}{2}\left(1 + \frac{1}{b}\right)t + O(1).$$

Now fix $a > 1$. If for sufficiently large t one has that

$$\frac{1}{t}\left|N_{a,b}(t) - \frac{1}{2b}t^2\right| \geq \frac{1}{2}\left(1 + \frac{1}{b}\right)$$

then only finitely many ECH capacities can obstruct $\sqrt{\frac{b}{a}}E(1, a) \hookrightarrow E(1, b)$.

This motivates the basic idea behind Chapter 3: the first two propositions prove that a hypothetical accumulation a_0 of an infinite staircase would have to lie on the volume curve and that, in any neighborhood of that accumulation point, there need to be infinitely many ECH capacities that are obstructive in order to actually generate the staircase. The idea behind the calculations that follow is to show, through direct comparison of the lattice point counting function $N_{a,b}(t)$ with the counting function for $E(1, b)$, that in almost all cases whether a_0 is rational or irrational, only finitely many obstructions are actually generated in some small enough neighborhood of a_0. This is done by the lemmas that bound sums of fractional parts, which show up in the lattice point counting functions. The rational case ends up being a bit more subtle than the irrational; to rule out denominators ≥ 2, this is done by using some apriori knowledge about the embedding function, and is done on a case-by-case basis. The $q = 1$ case is not resolved when a_0 is an integer exactly equal to $b + \frac{1}{b} + 2$.

Chapter 2

A canonical $\mathbb{Q}$-grading on embedded contact homology

Author: Matthew Salinger

Abstract: We introduce a canonical $\mathbb{Q}$-grading of $ECH(Y, \lambda, \Gamma)$ when $c_1(\xi)+2PD(\Gamma)$ is torsion in $H^2(Y;\mathbb{Z})$. We then show that this grading is preserved under Taubes' isomorphism with Seiberg-Witten Floer cohomology, as defined by Kronheimer and Mrowka. We also demonstrate how to compute the $\mathbb{Q}$-grading of the empty set generator of $ECH(Y, \xi, 0)$ when Y is a lens space with tight contact structure ξ.

2.1 Introduction

Let (Y, λ) be a closed 3-manifold with contact form λ, Reeb vector field R, and associated contact structure $\xi = \ker \lambda$. The embedded contact homology of Y, or $ECH(Y, \lambda)$, is the homology of a chain complex $ECC(Y, \lambda, J)$ generated by finite orbit sets of the Reeb vector field whose differential is a signed count of J-holomorphic curves in $\mathbb{R} \times Y$, where J is an almost-complex structure on $\mathbb{R} \times Y$ satisfying certain conditions. ECH does not depend on J. The ECH chain complex comes with a relative grading for the subgroups $ECC_*(Y, \lambda, \Gamma)$, consisting of those generators whose underlying orbit sets are in the same homology class Γ. To compute the relative grading between two generators α and β in $ECC_*(Y, \lambda, \Gamma)$, one chooses a relative homology class $Z \in H_2(\mathbb{R} \times Y, \alpha, \beta)$, where $\mathbb{R} \times Y$ is the symplectization of Y, and evaluates $I(Z)$ modulo d, where I is the fundamental *ECH index* and d is the divisibility of the class $c_1(\xi)+2PD(\Gamma) \in H^2(Y;\mathbb{Z})$ modulo torsion. When $c_1(\xi)+2PD(\Gamma)$ is torsion, $d = 0$ and the relative grading is $\mathbb{Z}$-valued. The purpose of this paper is to define a relative $\mathbb{Q}$-grading for a more general class of cobordisms from (Y_+, λ_+) to (Y_-, λ_-) when the appropriate classes are torsion. An important consequence of this is that if (Y, λ) is fillable, then this will define a canonical $\mathbb{Q}$-grading on $ECH_*(Y, \lambda)$.

Definition 2.1.1 *Let $(Y_\pm, \lambda_\pm, \Gamma_\pm)$ be nonempty, closed, contact 3-manifolds so that $c_1(\xi_\pm) + 2PD(\Gamma_\pm)$ is torsion in $H^2(Y_\pm;\mathbb{Z})$. Let (X, ω, J) be a symplectic cobordism from (Y_+, λ_+) to (Y_-, λ_-). If $\alpha_+ \in ECH(Y_+, \lambda_+, \Gamma_+)$ and $\beta_- \in ECH(Y_-, \lambda_-, \Gamma_-)$ so that the inclusions of Γ_+ and Γ_- are equal in $H_1(X;\mathbb{Z})$, then one can assign a rational number between them defined by*

$$gr^{\mathbb{Q}}(\alpha_+, \beta_-) = I(Z) - d(\xi_\pm, Z) \tag{2.1}$$

where Z is any element of $H_2(X, \alpha_+, \beta_-)$, I is the ECH index, and $d(\xi_\pm, Z) := \frac{1}{4}(c_1(X, J) + 2PD(Z))^2 - \iota(X) - \frac{1}{4}\sigma(X)$.

Here $c_1(X, J)$ is the Chern class of (TX, J), $\sigma(X)$ is the signature of the intersection pairing on X, and $\iota(X)$, called the index, is introduced in [KM; sec 2.4.2], and is an integer when both Y_+ and Y_- are nonempty.

Remark. An almost complex cobordism is the minimal structure needed to define (2.1). However in practice symplectic cobordisms are more relevant, since for example this is the structure needed to make any connections with Seiberg-Witten theory.

Now fix a generator $\alpha \in ECC_*(Y, \lambda, \Gamma)$. Let us suppose hypothetically that we have a symplectic cobordism X, one of whose components is S^3 endowed with the standard contact structure, and that the inclusion of Γ into $H_1(X;\mathbb{Z})$ is zero. Let $[\emptyset]$ denote the empty set of Reeb orbits in $ECC(S^3, \lambda_0)$. For the sake of argument suppose that Y_- is the S^3 component. Then one can assign to α the rational number

$gr^{\mathbb{Q}}(\alpha, [\emptyset])$. After filling S^3 with a symplectic ball one obtains a new symplectic cobordism X' from Y to $\emptyset$, and where the choice of $Z \in H_2(X, \alpha, [\emptyset])$ can be identified as an element of $H_2(X', \alpha)$. After filling with a ball, only the topological term $d(\xi, Z)$ changes by $+\frac{1}{2}$. So then one can also assign the same rational number as $gr^{\mathbb{Q}}(\alpha, [\emptyset])$ with respect to a filling $Y \to \emptyset$, provided we compensate for the contribution of $\frac{1}{2}$. One can make a similar argument when the Y_+ component is S^3, in which case we are capping the positive end by $\mathbb{C}P^2$ minus a ball. If that rational number only depends on α and is independent of the filling and choice of $Z \in H_2(X, \alpha)$, then this will canonically grade $ECC_*(Y, \lambda, \Gamma)$ by $\mathbb{Q}$.

There are two special types of fillings that will be of interest to us in the context of symplectic geometry. Given a contact 3-manifold (Y, λ), a symplectic cobordism from $Y_+ = Y$ to $Y_- = \emptyset$ is called a *convex symplectic filling* of Y. Going in reverse, a cobordism from $Y_+ = \emptyset$ to $Y_- = Y$ is a *concave symplectic filling.* We explicitly define this grading in terms of both a convex and concave symplectic fillings.

Definition 2.1.2 *Suppose that* $c_1(\xi) + 2PD(\Gamma)$ *is torsion in* $H^2(Y; \mathbb{Z})$*, and let* $\alpha \in ECC_*(Y, \lambda, \Gamma)$*. The canonical* $\mathbb{Q}$*-grading of* α *is defined as follows. Let* (X, ω, J) *be any concave symplectic filling of* Y*, with an admissible almost complex structure* J*, so that the inclusion of* α *is nullhomologous in* X*. Then*

$$gr^{\mathbb{Q}}(\alpha) = -I(Z) + d(\xi_-, Z) - \frac{1}{2} \tag{2.2}$$

where $Z \in H_2(X, \emptyset, \alpha)$*. If* (X, ω, J) *is a convex symplectic filling, then*

$$gr^{\mathbb{Q}}(\alpha) = I(Z) - d(\xi_+, Z) - \frac{1}{2}, \tag{2.3}$$

where $Z \in H_2(X, \alpha, \emptyset)$*.*

Again, (2.2) just re-expresses $-gr^{\mathbb{Q}}([\emptyset], \alpha)$ in terms of a concave filling, while (2.3) is $gr^{\mathbb{Q}}(\alpha, [\emptyset])$ in terms of a convex filling. The signs and constants are chosen to agree with the usual absolute grading of ECH on S^3. Having said that, in section

2.4 the proof of additivity over cobordism will show that (2.2) and (2.3) are in fact the same rational number. It should be noted at this point that Definition 1.2 could be vacuous, since it is only applicable when such fillings even exist. Nevertheless, one can attach a symplectic 2-handles to a Legendrian approximate to α to obtain obtain a cobordism to a new contact manifold (Y', ξ') in which $[\alpha] = 0$ in H_1 [10]; it is known also that concave symplectic fillings always exist [13]. So by composing a cobordism from (Y', ξ') to (Y, ξ) with a cobordism from the empty set, one can always obtain a concave symplectic filling of Y in which α is nullhomologous. It will be proven in section 2.4 that $gr^{\mathbb{Q}}(\alpha)$ is independent of the chosen cobordism. Consequently, formula (2.2) canonically grades $ECH_*(Y, \lambda, \Gamma)$ by $\mathbb{Q}$.

Readers familiar with Seiberg-Witten theory may notice a somewhat obvious resemblance of these formulas to the $\mathbb{Q}$-grading of monopole Floer groups as constructed in [19]. This is summarized in

Theorem 2.1.3 *Assuming the relevant conditions of definitions* (2.1.1) *or* (2.1.2)*,*

1. *The formula* (2.1) *is additive over cobordisms, and is independent of the choice of cobordism and relative homology class. Consequently* (2.2) *and* (2.3) *are well defined and equal.*

2. *The absolute $\mathbb{Q}$-grading of ECH has the reverse sign of the $\mathbb{Q}$-grading of its image in $\widehat{HM}^{-*}(Y, \mathfrak{s}; \lambda, r)$ under Taubes' isomorphism.*

Here, $\mathfrak{s}$ is the spin^c structure on Y determined by $c_1(\xi) + 2PD(\Gamma)$.

One last point of interest is in computing the $\mathbb{Q}$-grading of the empty set generator $[\emptyset] \in ECC(Y, \lambda, 0)$. In this paper we will compute some examples for convex fillings. In fact, let (X, ω) be a convex filling of Y. Since α is the empty set generator, $Z \in H_2(X; \mathbb{Z})$ which means that

$$I(Z) = \langle c_1(X, J), Z \rangle + Z \cdot Z. \tag{2.4}$$

Since $Z \in H_2(X;\mathbb{Z})$ the cup square term can be expanded and one gets

$$d(\xi, Z) = I(Z) + d(\xi) + \frac{1}{2}b_1(Y). \tag{2.5}$$

where $d(\xi) = \frac{1}{4}(c_1(X,J)^2 - 3\sigma(X) - 2\chi(X))$. [1] So then we have the following

Proposition 2.1.4 *The $\mathbb{Q}$-grading of $[\emptyset]$, the empty set generator of $ECH(Y, \lambda, 0)$, is*

$$gr^{\mathbb{Q}}([\emptyset]) = -d(\xi) - \frac{1}{2}b_1(Y) - \frac{1}{2}, \tag{2.6}$$

with respect to a convex symplectic filling, and where $d(\xi) = \frac{1}{4}(c_1(X,J)^2 - 3\sigma(X) - 2\chi(X))$.

An analogous formula for a concave filling can be derived in the same way. As an immediate check, when Y is the boundary of an irrational ellipsoid equipped with the standard contact structure and X is (D^4, ω_{std}), one gets $d(\xi) = -\frac{1}{2}$ so (2.6) yields $gr^{\mathbb{Q}}([\emptyset]) = 0$ as expected. More generally in section 2.3 we demonstrate the application of (2.6) to a lens spaces $Y = L(p,q)$ equipped with any tight contact structure ξ.

2.2 Preliminaries

2.2.1 Embedded contact homology

Let (Y, λ) be a closed 3-manifold with contact form λ. The contact requirement means that $\lambda \wedge d\lambda > 0$ is a volume form on Y. The 1-form λ determines a canonical vector field R called the *Reeb vector field*, defined by the equations

$$d\lambda(R, \cdot) = 0, \qquad \lambda(R) = 1.$$

A *Reeb orbit* is a closed integral curve of the Reeb vector field, namely a map

[1] When X is compact and $b^+(X) \geq 1$, the integer $d(\xi, Z)$ is also the dimension of the moduli space of solutions to the 4-dimensional Seiberg-Witten equations with respect to the spin^c structure associated to $c_1(X) + 2PD(Z)$ [19, Thm. 1.4.4].

$\gamma : \mathbb{R}/T\mathbb{Z} \to Y$ for some $T > 0$ so that $\gamma'(t) = R(\gamma(t))$. The orbit γ is said to be *nondegenerate* when the linearized return map P_γ, a symplectic automorphism of $(\xi_{\gamma(0)}, d\lambda)$, does not have 1 as an eigenvalue. If all Reeb orbits are nondegenerate, then λ is said to be nondegenerate. When $T > 0$ is minimal, γ is said to be *simple*. Every orbit is a multiple cover of a simple orbit. From now on a finite collection of orbits will be denoted $\alpha = \{(\alpha_i, m_i)\}$, where the pair (α_i, m_i) designates a simple Reeb orbit α_i along with a multiplicity m_i. When α_i is hyperbolic, m_i is required to be 1.

Fix $\Gamma \in H_1(Y)$. Then the embedded contact homology $ECH_*(Y, \lambda, \Gamma, J)$ is the homology of a chain complex $ECC_*(Y, \lambda, \Gamma, J)$ generated by orbit sets $\alpha = \{(\alpha_i, m_i)\}$ whose total homology class $\sum_i m_i[\alpha_i]$ is Γ. The differential $\langle \partial\alpha, \beta\rangle$ is a mod 2 count of punctured J-holomorphic curves in $\mathbb{R} \times Y$ asymptotic to α and β at the $\pm\infty$ ends and whose relative ECH index is equal to 1. Here, J is an almost complex structure on $\mathbb{R} \times Y$ rotating the contact planes ξ positively with respect to $d\lambda$ and satisfying $J(\partial_s) = R$. Almost complex structures on $\mathbb{R} \times Y$ satisfying these properties are said to be *admissible*. It turns out that ECH is independent of the chosen J, which is why one can write $ECH_*(Y, \lambda, \Gamma)$ for the homology groups. Apriori $ECC_*(Y, \lambda, \Gamma)$ is a $\mathbb{Z}/2$-module relatively graded by $\mathbb{Z}/d$, where d stands for the divisibility of $c_1(\xi) + 2PD(\Gamma)$ in $H^2(Y, \mathbb{Z})$ modulo torsion. When $c_1(\xi) + 2PD(\Gamma)$ is torsion, $d = 0$ so the relative grading is $\mathbb{Z}$-valued. The highly nontrivial fact that $\partial^2 = 0$ is discussed in [17, Sec. 5.4].

2.2.2 Symplectic cobordisms

To explain a little bit about where the relative grading on ECH comes from and set the stage for explaining definition (2.1.1), we review the definition of the ECH index. In this context, a *symplectic cobordism* from (Y_+, λ_+) to (Y_-, λ_-)[2] will mean a smooth 4-manifold

$$X = E_- \cup_{Y_-} \overline{X} \cup_{Y_+} E_+ \tag{2.7}$$

[2]The convention in other sources (including [19]) is to say *from Y_- to Y_+*. We adopt the opposite convention throughout this paper, following [14].

where $E_+ := [0,\infty) \times Y_+$ and $E_- := (-\infty, 0] \times Y_-$ and where $\overline{X}$ is a compact symplectic 4-manifold with boundary $-Y_- \cup Y_+$. The signs of the boundary components are determined by a *Liouville vector field* on $\bar{X}$. This is a smooth vector field v satisfying $\mathcal{L}_v\omega = \omega$ which points into Y_+ and out of Y_-. The positive boundary component Y_+ is called the *convex end* while Y_- is called the *concave end*.

Consider two generators $\alpha_+ \in ECC(Y_+, \lambda_+)$ and $\beta_- \in ECC(Y_-, \lambda_-)$ with equal class in $H_1(X;\mathbb{Z})$. The set $H_2(X, \alpha_+, \beta_-)$ consists of equivalence classes of 2-chains with boundary

$$\partial Z = \sum_i m_i^+ \{1\} \times \alpha_i^+ - \sum_j m_j^- \{-1\} \times \beta_j^-$$

modulo boundaries of 3-chains. The set $H_2(X, \alpha_+, \beta_-)$ is affine over $H_2(X;\mathbb{Z})$. Whenever $(Y_+, \lambda_+) = (Y_-, \lambda_-)$ and $\overline{X}$ is empty, the cobordism is just the usual symplectization $\mathbb{R} \times Y$.

Fix a trivialization of τ of ξ over the Reeb orbits of α and β. The *ECH index* is defined by the formula

$$I(Z) = c_\tau(Z) + Q_\tau(Z) + \mu_\tau(\alpha) - \mu_\tau(\beta) \tag{2.8}$$

where $c_\tau(Z)$ is a signed count of zeros of a generic section of ξ restricted to a representative of Z, $Q_\tau(Z)$ is the self intersection term, and $CZ_\tau(\alpha, \beta)$ is a Conley-Zehnder term. Each of the terms in (2.8) depends on τ, but their sum does not. See [17] for more details. In the special case that $\bar{X} = \emptyset$ so that the cobordism is just $\mathbb{R} \times Y$, then this is equivalent to choosing a Z in $H_2(Y, \alpha, \beta)$. A crucial feature of the ECH index in this case is that if Z' is any other element of $H_2(Y, \alpha, \beta)$, then

$$I(Z) - I(Z') = \langle Z - Z', c_1(\xi) + 2PD(\Gamma) \rangle \tag{2.9}$$

hence the relative index defined by $I(\alpha, \beta) := I(Z)$ is well defined as an element of $\mathbb{Z}/d(c_1(\xi) + 2PD(\Gamma))$. When $c_1(\xi) + 2PD(\Gamma)$ is torsion the right hand side vanishes. The analog to that statement in a cobordism is an expression on the right hand side that vanishes on closed 4-manifolds.

Remark. The relative grading on $ECC_*(Y, \lambda, \Gamma)$ can be refined to an absolute grading by homotopy classes of 2-plane fields (regardless of torsion). Let $\mathcal{P}(Y)$ denote the set of homotopy classes of 2-plane fields on Y, and let $\mathcal{P}(Y, c)$ denote the subset consisting of those 2-plane fields whose first Chern class is $c \in H^2(Y, \mathbb{Z})$. There is a transitive $\mathbb{Z}$-action on this set whose stabilizer is $d(c)$, meaning that $\mathcal{P}(Y, c)$ and $\mathbb{Z}/d(c)$ are isomorphic as $\mathbb{Z}$ sets, see [14, sec. 3.1] for example. In [14, sec. 3], Hutchings describes how to construct a homotopy class of 2-plane fields out of a given $\alpha \in ECC(Y, \lambda, \Gamma)$. This construction is canonical, hence there is a decomposition

$$ECC_*(Y, \lambda, \Gamma) = \bigoplus_{\mathfrak{p}} ECC_{\mathfrak{p}}(Y, \lambda, \Gamma)$$

where $\mathfrak{p}$ ranges over $\mathcal{P}(Y, c(\xi) + 2PD(\Gamma))$. Once again, when $c_1(\xi) + 2PD(\Gamma)$ is torsion, the set $\mathcal{P}(Y, c(\xi) + 2PD(\Gamma))$ is affine over $\mathbb{Z}$.

2.2.3 Spin^c structures

Here we review spin^c structures in dimensions 3 and 4. All of what follows can be found in various resources including [19], Hutching and Taubes' introduction [18], or Francesco Lin's wonderfully concise introductory notes [20]. The convention adopted in this paper is to denote pairs consisting of a spin^c-connection and spinor as (B, ψ) in dimension 3 and (A, Ψ) in dimension 4.

Spin^c in dimension 3

Let Y be a closed, oriented, Riemannian 3-manifold. The data for a spin^c structure on Y consists of a unitary rank 2 vector bundle $\mathbb{S}$ and a bundle map

$$\rho : TY \to \mathrm{End}(\mathbb{S})$$

which identifies TY isometrically with the subbundle $\mathfrak{su}(\mathbb{S})$ of traceless, skew-adjoint endomorphisms equipped with the inner product $\frac{1}{2}\,\mathrm{tr}(a^* b)$. The map ρ respects orientation in the sense that

$$\rho(e_1)\rho(e_2)\rho(e_3) = 1$$

where the e_i form an orthonormal frame TY. This means that one can always find an orthonormal basis $e_0, \ldots, e_2$ for which $\rho(e_i) = \sigma_i$, where σ_i's denote Pauli spin matrices. On any oriented 3-manifold, spin^c structures exist since TY is parallelizable. Clifford multiplication is extended to forms by the rule

$$\rho(\alpha \wedge \beta) = \frac{1}{2}(\rho(\alpha)\rho(\beta) + (-1)^{\deg\alpha \deg\beta}\rho(\beta)\rho(\alpha)). \tag{2.10}$$

On the space of spin^c structures $\mathrm{Spin}^c(Y)$ there is an action by $H^2(Y, \mathbb{Z})$: tensor $\mathbb{S}$ by a complex line bundle L and update the Clifford multiplication by

$$\rho_L = 1_L \otimes \rho.$$

Since line bundles are characterized by first Chern class, this a well defined action by $H^2(Y, \mathbb{Z})$. The action is free and transitive [19, sec. 1.1].

Let $C(Y, \mathfrak{s})$ denote the configuration space of pairs (B, ψ) consisting of a spin^c connection B and spinor ψ. The *gauge group* $\mathcal{G}(Y, \mathfrak{s}) = \{Y \to S^1\}$ is the group of automorphisms of $\mathfrak{s}$ defined by

$$(B, \psi) \mapsto (B - u^{-1}du, u\psi).$$

This action is free except when $\psi \equiv 0$. In this case the stabilizer of the action is S^1. This is a major source of technical difficulty in defining Morse homology on the configuration space since quotienting by the S^1-action ruins the manifold structure. A configuration with $\psi \equiv 0$ is said to be *reducible*, and *irreducible* otherwise.

The relevant notion of differentiation for spinors begins with a choice of *spin^c-connection.* This is a unitary connection B on $\mathbb{S}$ which is parallel to ρ. In particular the covariant derivative of B satisfies

$$\nabla_B(\rho(v)\Psi) = \rho(\nabla v)\Psi + \rho(v)\nabla_B\Psi,$$

where v is any smooth vector field on Y and ∇ is the usual Levi-Civita covariant derivative. The space of spin^c-connections is affine over $\Omega^1(Y, i\mathbb{R})$. Given a spin^c-connection B, the connection induced on $\det(\mathbb{S})$ is denoted B^t. Associated to a spin^c connection is the *Dirac operator* $D_A : \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$. This is a first order elliptic operator. Consequently, if Y is compact then D_A is Fredholm.

Spin^c in dimension 4

Let X be an oriented Riemannian 4-manifold. A spin^c structure on X is again a pair (S, ρ), where this time S is a rank 4 Hermitian vector bundle over X and Clifford multiplication is a bundle map

$$\rho : TX \to \mathrm{End}(S)$$

which identifies each fiber of TX isometrically with $\mathfrak{su}(4)$ given the usual metric $\frac{1}{2}\mathrm{tr}(a\bar{b})$. In particular, this means that at each fiber in TX there is some orthonormal basis $e_0, \ldots, e_3$ so that

$$\rho(e_0) = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}, \quad \rho(e_i) = \begin{pmatrix} 0 & -\sigma_i^* \\ \sigma_i & 0 \end{pmatrix},$$

where the σ_i again denote Pauli spin matrices. Clifford multiplication is extended to forms by the same rule as (2.10), and one can check that Clifford multiplication by the Riemannian volume form $\mathrm{vol}_X = e_0 \wedge e_1 \wedge e_2 \wedge e_3$ satisfies

$$\rho(\mathrm{vol}_X) = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}$$

hence there is a splitting

$$S = S^+ \oplus S^-$$

where S^+ and S^- are defined to be the -1 and +1 eigenspaces of $\rho(\text{vol}_X)$, respectively. Clifford multiplication changes sign in the sense that $\rho(e) : S^+ \to S^-$ and vice versa for any $e \in TX$.

On a Riemannian 4-manifold X a spin^c connection A is defined in the same way, and the space of spin^c connections is affine over $\Omega^1(X, i\mathbb{R})$. Having chosen a spin^c connection A, the *Dirac operator* D_A is defined to be the composition

$$\Gamma(S^+) \xrightarrow{\nabla_A} \Gamma(T^*X \otimes S^+) \xrightarrow{\rho} \Gamma(S^-)$$

where ∇_A is the covariant derivative of A and $\rho : T^*X \otimes S^+ \to S^-$ means $\rho(\alpha \otimes \psi) = \rho(\alpha)\psi$. It should be pointed out that there is no analogous decomposition in dimension 3.

2.2.4 Monopole Floer homology

Basic setup

Let (Y, g) be an oriented Riemannian 3-manifold. Fix a spin^c structure $\mathfrak{s}$ on Y, a reference connection B_0 on $\det(\mathbb{S})$ so that $B - B_0$ is an imaginary valued 1-form for any other $B \in \text{Conn}(\mathbb{S})$, and a real closed 2-form μ. *Monopole Floer homology* is the Morse homology of a chain complex generated by critical points of the *perturbed Chern-Simons-Dirac functional* $\mathcal{L}$. The functional $\mathcal{L}$ is a real valued function on $C(Y, \mathfrak{s})$ defined by

$$\mathcal{L}(B, \psi) = -\frac{1}{8} \int_Y (B^t - B_0^t) \wedge (F_{B^t} + F_{B_0^t} - 2i\mu) + \frac{1}{2} \int_Y \langle D_B\psi, \psi \rangle d\text{vol}_Y.$$

Critical points of $\mathcal{L}$ are solutions to grad $\mathcal{L} = 0$, namely solutions to the (3-dimensional) *perturbed Seiberg-Witten equations*

$$\frac{1}{2}\langle\rho(\cdot)\psi,\psi\rangle + \star(i\mu - F_B) = 0,$$
$$D_B\psi = 0,$$

where $\star$ denotes the Hodge star operator. Solutions to these equations are called *monopoles* and are gauge invariant, meaning that if (B,ψ) is a solution, then so is $u\cdot(B,\psi) = (B - u^{-1}du, u\psi)$.

As in finite dimensional Morse theory one would like to study solutions to the gradient flow equations

$$\frac{d}{dt}(B(t),\psi(t)) = -\text{grad } \mathcal{L}(B(t),\psi(t)) \tag{2.11}$$

See, for example, [4, sec. 3.3] for a summary of the relevant metrics. The PDE's derived from (2.11) are the 4-dimensional Seiberg-Witten equations, but we will not need them here. There are considerable technical difficulties in constructing the homology groups. For example the quotient of the configuration space by the gauge action fails to be a smooth manifold at reducible points. See [20, sec. 3] for a summary. Roughly speaking, Kronheimer and Mrowka's solution to this is to perform Morse homology on a configuration space in which the reducible points are blown-up. The resulting *monopole Floer groups* are denoted

$$\check{HM}_*(Y,\mathfrak{s}), \quad \widehat{HM}_*(Y,\mathfrak{s}), \quad \overline{HM}_*(Y,\mathfrak{s}).$$

These are, respectively, the "HM-to", "HM-from", and "HM-bar" groups. Summing over all spin^c structures on Y one obtains diffeomorphism invariants $\check{HM}_*(Y)$, $\widehat{HM}_*(Y)$, and $\overline{HM}_*(Y)$.

The monopole Floer groups are relatively $\mathbb{Z}/d$-graded. The relative grading between two critical points $\mathfrak{a}$ and $\mathfrak{b}$ in the blown-up configuration space is obtained by computing a spectral flow, and is the expected dimension of the moduli space of trajectories connecting $\mathfrak{a}$ and $\mathfrak{b}$ [20, sec. 4]. Here d denotes the divisibiliy of the first Chern class of $\mathfrak{s}$. This relative grading can also be lifted to an absolute $\mathbb{Z}/d$-grading by homotopy classes of 2-plane fields $\mathcal{P}(Y)$ exploiting a bijection between $\mathcal{P}(Y)$ and pairs $(\mathfrak{s}, \Phi)$, where Φ represents the homotopy class of a unit spinor. See [19, ch. 28] for more details.

The Seiberg-Witten index in a cobordism

A *spin^c cobordism* from $(Y_+, \mathfrak{s}_+)$ to $(Y_-, \mathfrak{s}_-)$ is a smooth oriented 4-manifold $(X, \mathfrak{s}_X)$ with boundary $Y_+ \cup -Y_-$ with a spin^c structure restricting to $\mathfrak{s}_\pm$ on the boundary components. On the boundary, Clifford multiplication by the outward normal vector naturally identifies S^+ and S^- and induces the Dirac operators on $Y_\pm$, see [19, sec. 4.5]. Following the notation in [4, sec. 3.4], let $\mathfrak{c}_\pm \in C(Y_\pm, \mathfrak{s}_\pm)$ satisfy the 3-dimensional Seiberg-Witten equations and let $\mathfrak{c}$ denote the combined configuration on the disjoint union $Y_+ \cup -Y_-$. Let $B(X, \mathfrak{c}, \mathfrak{s}_X)$ denote the moduli space of gauge equivalence classes of configurations (A, Ψ) in the spin^c structure $\mathfrak{s}_X$ that limit to $\mathfrak{c}$ on the cylindrical ends. The union over all spin^c structures is denoted $B(X, \mathfrak{c})$. Then define $M(X, \mathfrak{c}, \mathfrak{s}_X)$ to be those configurations in $B(X, \mathfrak{c}, \mathfrak{s}_X)$ satisfying the 4-dimensional Seiberg-Witten equations, while denoting the union over spin^c extensions as $M(X, \mathfrak{c})$. The space $\pi_0(B(X, \mathfrak{c}))$ is affine over $H^2(X, \partial X; \mathbb{Z})$. Having chosen a $z \in \pi_0(B(X, \mathfrak{c}))$, the *Seiberg-Witten index $gr_z(X, \mathfrak{c})$* is defined to be the dimension of the moduli space of solutions $M(X, \mathfrak{c})$ in the component z. If (A, Ψ) is a chosen lift of the gauge equivalence class of an element of $B(X, \mathfrak{c})$, then $gr_z(X, \mathfrak{c})$ is equal to the index of the Fredholm operator $D^X_{A,\Psi}$, the linearization of the 4-dimensional Seiberg-Witten equations with gauge fixing term.

Taubes' perturbation

Important for us is the following version of Seiberg-Witten Floer homology, defined by Taubes, that sees the contact form on (Y, λ). First, let g be a Riemannian metric on (Y, λ) defined so that $|\lambda| = 1$ and $d\lambda = 2 \star \lambda$. Let J be the admissible almost-complex structure on the symplectization $(\mathbb{R}\times Y, d(e^s\lambda))$ defined by

$$g(v,w)=\frac{1}{2}d\lambda(v,Jw)$$

for any vectors $v,w\in\xi$. Having defined J, regard ξ as a Hermitian line bundle, denoted K^{-1}, where K is the canonical line bundle on $\mathbb{R}\times Y$. Associated to ξ there is a natural spin^c structure $\mathfrak{s}_\xi$ determined by

$$\mathbb{S}=\underline{\mathbb{C}}\oplus K^{-1}$$

where $\underline{\mathbb{C}}$ is the trivial complex line bundle. Any other spin^c structure $\mathfrak{s}$ is obtained from $\mathfrak{s}_\xi$ by tensoring with a complex line bundle E to obtain $\mathbb{S}=E\oplus K^{-1}E$. Taubes defines the following $\mathbb{R}_{\geq 1}$-family of perturbed Seiberg-Witten equations:

$$\begin{aligned}\star F_B = r(\langle\rho(\cdot)\psi,\psi\rangle - i\lambda) + i\varpi\\ D_B\psi = 0,\end{aligned}$$

where B is now a connection on E; see [16, sec. 10]. One should think of $c_1(E)\in H^2(Y;\mathbb{Z})$ as dual to $\Gamma\in H_1(Y;\mathbb{Z})$ in the ECH picture. In fact, what Taubes constructed was an isomorphism of graded chain complexes

$$ECC_*^L(Y,\lambda,\Gamma)\to \widehat{CM}_L^{-*}(Y,\mathfrak{s};\lambda,r).$$

For very large r, the direct limit determines an isomorphism between $ECH(Y,\lambda,\Gamma)$ and $\widehat{HM}^{-*}(Y,\mathfrak{s})$ where $c_1(E)$ is Poincare dual to Γ.

The absolute $\mathbb{Q}$-grading on monopole Floer groups

In this section we discuss the $\mathbb{Q}$-grading on Seiberg-Witten theory while also introducing the topological terms in equation (2.1). To begin, let X be any oriented 4-manifold with non-empty boundary. There is a bilinear pairing

$$H^2(X,\partial X;\mathbb{Q})\times H^2(X;\mathbb{Q})\to\mathbb{Q}$$

given by evaluating the cup-product on the fundamental class $[X, \partial X]$. Let $I_2(X)$ denote the image of $H^2(X, \partial X; \mathbb{Z})$ in $H^2(X; \mathbb{Z})$ under the restriction map. If $c \in H^2(X; \mathbb{Z})$ so that $c|_{\partial X}$ is torsion, then c lies in the image $I_2(X)$ and has a well defined cup square by choosing a lift $\tilde{c} \in H^2(X, \partial X; \mathbb{Q})$ and defining c^2 to be $(\tilde{c} \smile \tilde{c})[X, \partial X]$. This determines a nondegenerate pairing on $I_2(X)$. Letting $\sigma(X)$ denote the signature of this pairing, the characteristic number of X is defined to be

$$\iota(X) = \frac{1}{2}(\chi(X) + \sigma(X) + b_1(Y_+) - b_1(Y_-)). \tag{2.12}$$

If X is a cobordism and neither Y_+ nor Y_- is empty, $\iota(X)$ is an integer, as opposed to a half integer. Having said all of this we have the following

Definition 2.2.1 *Let $\mathfrak{s}$ be a torsion spin^c structure on Y, and let $[\mathfrak{a}]$ be a corresponding critical point. Then let X be any cobordism from Y to S^3 over which $\mathfrak{s}$ extends.*[3] *Let z be a X-path from $[\mathfrak{a}_0]$ to $[\mathfrak{a}]$. Then the rational number $gr^{\mathbb{Q}}([\mathfrak{a}])$ is defined by the formula*

$$gr^{\mathbb{Q}}([\mathfrak{a}]) = -gr_z([\mathfrak{a}_0], X, [\mathfrak{a}]) + \frac{1}{4}c(S^+)^2 - \iota(X) - \frac{1}{4}\sigma(X) \tag{2.13}$$

where $\iota(X)$ is the characteristic number, $\sigma(X)$ the signature of X, and S^+ is the spinor bundle with spin^c structure associated to the choice of z.

Here gr_z is the Seiberg-Witten index. Analogous formulas to (2.2) and (2.3) can be given by composing with a cobordism to or from the empty set.

2.3 Some calculations on lens spaces

In this section we will consider calculating $gr^{\mathbb{Q}}([\emptyset])$ when Y is a lens space. For p, q relatively prime integers, $L(p, q)$ is obtained by rational $-\frac{p}{q}$-surgery on an unknot in S^3. Alternatively, a description of this surgery that will be useful for our purposes is the following. Let $-\frac{p}{q} = [a_1, \ldots, a_\ell]$ denote the continued fraction expansion of

[3] In [19, def 28.3.1] this is stated as "from S^3 to Y".

$-\frac{p}{q}$, where ℓ refers to the length of the continued fraction and $a_i < -1$ for $1 \leq i \leq \ell$. That is,

$$-\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \frac{1}{a_\ell}}}.$$

Then $-\frac{p}{q}$-surgery on an unknot is equivalent to integral surgery along the chain of unknots L given in Figure 1, each of whose components is given the framing $-a_i$. Moreover, if the unknots are placed in Legendrian position with $tb(L_i) - 1 = a_i$ then integral (-1)-Legendrian surgery yields $(L(p,q), \xi)$, where ξ is a tight contact structure on $L(p,q)$. Beautifully, the tight contact structures on $L(p,q)$ that one can obtain from Legendrian surgery correspond precisely to the possible stabilizations of the Legendrian chain. Since the Thurston-Bennequin number of each component is fixed at $a_i + 1$, the possible stabilizations of the ith component correspond to the number of ways to arrange the "zig-zags", which is just $|a_i + 1|$ [13].

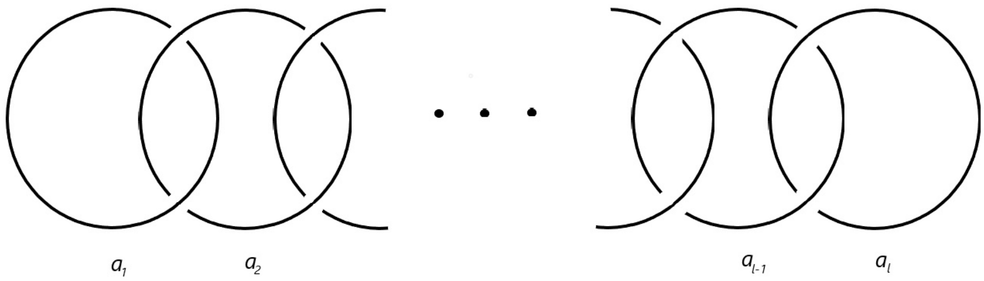

Figure 2.1: Chain of unknots in S^3

Now assume that each component of the chain of unknots is put into Legendrian position, and denote the resulting Legendrian link as $\mathcal{L} = \bigcup_{i=1}^{\ell} L_i$. For each $1 \leq i \leq \ell$, attach a symplectic 2-handle with framing a_i. As a result, one obtains a symplectic filling (X, ω) with contact boundary $(L(p,q), \xi)$. To compute the grading of the empty set generator with respect to the filling X, by Corollary 1.4, we need only compute $d(\xi)$.

Let Σ_i denote the image of $D_i^2 \times 0$ after the handle attachment. Denote its relative homology class by $[\Sigma_i] \in H_2(X, \partial X; \mathbb{Z})$. By proposition 8.2.4 of [26] $\langle c_1(X, J), [\Sigma_i] \rangle =$

rot(K_i) which implies that

$$c_1^2(X,J) = r^T Q_X r \tag{2.14}$$

where $r \in \mathbb{Z}^\ell$ is the integer vector consisting of rotation numbers of the components L_i and Q_X is the intersection form on X. The intersection pairing is defined on all of $H_2(X, \partial X; \mathbb{Z})$ since $H_1(L(p,q)) = \mathbb{Z}/p$ is torsion. In fact, $[\Sigma_i] \cdot [\Sigma_j] = lk(K_i, K_j)$ hence the matrix representing the intersection form in that basis is the same as the linking matrix of $\mathcal{L}$. That is

$$Q_X = \begin{pmatrix} a_1 & 1 & \dots & 0 \\ 1 & \ddots & \vdots & \vdots \\ \vdots & \cdots & a_{\ell-1} & 1 \\ 0 & \cdots & 1 & a_\ell \end{pmatrix}. \tag{2.15}$$

Since we've only attached 2-handles, b_1 and b_3 both equal 0, and according to (2.15), $b_2 = b_2^-$. So it follows that $\chi = 1 - \sigma$. But $b_2 = \ell$ hence $\sigma(X) = -\ell$, so after simplifying one gets

$$d(\xi) = \frac{1}{4}(r^T Q_X r + \ell - 2). \tag{2.16}$$

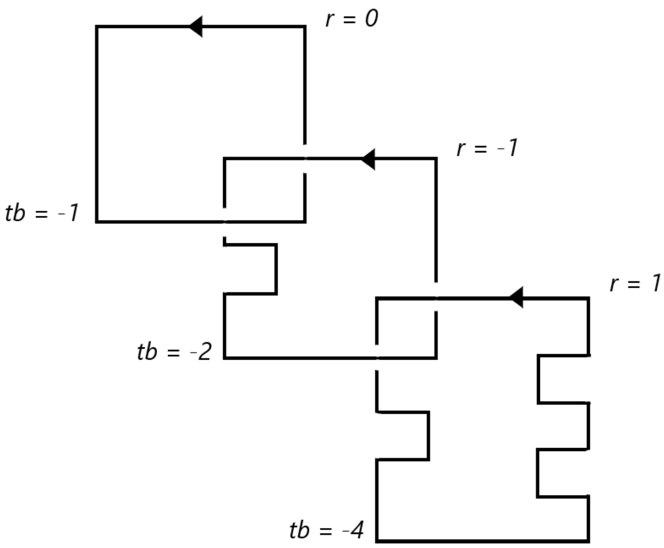

Figure 2.2: front diagram of 3-chain in a Legendrian position

As an example, consider $L(23,14)$, where $-\frac{23}{14} = [-2,-3,-5]$, and let us put L_1, L_2 and L_3 into Legendrian position as illustrated by the front diagrams in Figure 4. For reference, the Thurston-Bennequin number is $\mathrm{tb}(L_i) = w(L_i) - c(L_i)$, where the *writhe* $w(L_i)$ is a signed count of crossings (which is 0, since L_i is unknotted), and $c(L_i)$ is the number of upper right hand corners. The rotation number $\mathrm{rot}(L_i)$ is equal to the number of "up" pegs minus the number of "down" pegs. Let ξ denote the tight contact structure on $L(23,14)$ that resulting from the surgery. Now then $r = (0,-1,1)$ and

$$Q_X = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -4 \end{pmatrix}$$

so from equation (2.14) it follows that $c_1^2(W,J) = -9$. Hence $d(\xi) = -2$ and

$$gr^{\mathbb{Q}}([\emptyset]) = \frac{3}{2}.$$

2.4 Proofs

2.4.1 Proof of Theorem 2.1.3

1. Independence of and additivity over cobordisms of $gr^{\mathbb{Q}}$

First, the the ECH index and the terms ι and χ are already additive over cobordisms, so the only thing to check is the additivity of the cup-square term. Suppressing some notation for clarity, let X be the symplectic cobordism obtained by composing two cobordisms X_1 and X_2 and let $c_i := c_1(X_i, J_i) + 2PD(Z_i)$ for $i = 1,2$. By definition each c_i lifts to an element of $H^2(X_i, \partial X_i; \mathbb{Z})$, which we'll also denote c_i. Since each c_1 and c_2 have disjoint supports, it follows that $(PD(c_1)+PD(c_2))^2 = PD(c_1)^2 + PD(c_2)^2$. Hence (1) is additive.

To prove the independence of (2.1) with respect to symplectic cobordism, we will show that (2.1) is independent of almost complex structure. This is enough since (2.1) does not depend directly on the symplectic structure. So let (X_1, J_1)

and (X_2, J_2) be two almost complex cobordisms from (Y_+, ξ_+) to (Y_-, ξ_-). Let $(-X_2, -J_2)$ denote the 4-manifold with the opposite orientation to X_2 induced by $-J_2$. Compose X_1 with $-X_2$ to obtain a closed 4-manifold (X, J). Now let $Z \in H_2(X; \mathbb{Z})$. Then by equation (2.4) for $I(Z)$,

$$d(Z) = I(Z) + \frac{1}{4}(c_1^2(X, J) - 3\sigma(X) - 2\chi(X)). \tag{2.17}$$

But $c_1^2 - 3\sigma - 2\chi$ vanishes for closed, almost-complex 4-manifolds by Hirzebruch's signature theorem [26, sec. 6.2], so we are left with

$$d(Z) = I(Z)$$

hence $gr^{\mathbb{Q}} = 0$.

2. Agreement of absolute $\mathbb{Q}$-grading with monopole Floer $\mathbb{Q}$-grading

To proof of this is almost an immediate consequence of Theorem 5.1 of [4]. The basic idea behind Theorem 5.1 is to construct an explicit class in $H_2(X, \alpha, \beta)$ using the data provided by the symplectic form on X and a generically chosen pair (A, Ψ) in the space of configurations on X limiting to c_α and c_β on the ends. This homology class is extracted from the S^+ component of Ψ: after observing a complex splitting of S^+, it is shown in [4, sec. 5] that the zero locus C of one of the complex components of Ψ bounds α and β and that the evaluation of the ECH index on $[C] \in H_2(X, \alpha, \beta)$ is equal to the Seiberg-Witten index $gr_z(X, \mathfrak{c})$, where $\mathfrak{c}$ is the configuration determined by c_α and c_β. We provide a rough sketch of the basic setup of Theorem 5.1. See [4] for some of the finer details.

So let (X, ω, J) be a symplectic cobordism from (Y_+, λ_+) to (Y_-, λ_-) with J an ω-compatible almost-complex structure, and let $Y = Y_+ \cup -Y_-$. Let $L > 0$ and let α and β be generators of $ECC_*^L(Y_\pm, \lambda_\pm, \Gamma_\pm)$. Let c_α and c_β be the respective images of α and β in $\widehat{CM}_L^{-*}(Y, \mathfrak{s}_\xi + PD(\Gamma); \lambda, r)$ under Taubes' bijection (17). Let $\mathfrak{s}_+$ and $\mathfrak{s}_-$ be the spin^c structures on Y_+ and Y_- determined by α and β, an let $\mathfrak{s}$ denote the induced spin^c structure on Y.

Now choose a representative (A, Ψ) in the gauge equivalence class of an element in $\mathcal{B}(X, \mathfrak{c})$. This determines a spin^c structure $\mathfrak{s}_{A,\Psi}$ on X. As usual let S^+ and S^- denote the -1 and +1 eigenspaces of $\rho(\text{vol}_X)$. Clifford multiplication by the symplectic form induces further a splitting of S^+ into $-2i$ and $+2i$ eigenspaces (locally write $\rho(\omega)$ in a Darboux basis and apply the rule in (11)). The addition of the almost-complex structure J and the resulting decomposition of $\bigwedge T^*X \otimes \mathbb{C}$ into Dolbeaux forms allows one to write this splitting as

$$S^+ = E \oplus K^{-1}E \tag{2.18}$$

where K^{-1} is the inverse of the determinant line bundle on X and $c_1(E)$ is the first Chern class of $\mathfrak{s}_{A,\Psi}$. See Lemma 4.3 in [18] for more details. Having said this, one can write

$$\Psi = (a, b). \tag{2.19}$$

Assuming (A, Ψ) is chosen so that a intersects the zero section of E transversely, this determines a surface $C_{A,\Psi}$ in $\bar{X}$. Composing with cobordisms to the orbit sets α and β in the cylindrical ends, this induces a homology class $Z_{A,\Psi}$ which leads to the following

Theorem 2.4.1 *Let $z \in \pi_0(\mathcal{B}(X, \mathfrak{c}))$ and represent z by a configuration (A, Ψ) over X. The integer $gr_z(X, \mathfrak{c})$ is equal to $I(Z_{A,\Psi})$.*

Now we're in a position to prove part 2 of Theorem 1.3. For compatability we will assume we are working with a cobordism in the direction provided by Definition 5, equivalent to a convex symplectic filling. There is no loss of generality since Theorem 6 is not sensitive to direction, so we could carry out the same proof by making the appropriate changes to the Seiberg-Witten $\mathbb{Q}$-grading for a cobordism from S^3 to Y.

So let (X, ω, J) be a symplectic cobordism from (S^3, λ_0) to (Y, λ) with ω admissible almost-complex structure J. Let $\mathfrak{s}$ denote the spin^c structure for which $c(\mathfrak{s}) = c_1(\xi) + 2PD(\Gamma)$, and let $\mathfrak{s}_X$ denote any spin^c extension of $\mathfrak{s}$. Since $c_1(X) = c_1(K^{-1})$, by (2.18) this implies that

$$c_1(S^+) = c_1(X, J) + 2c_1(E). \tag{2.20}$$

Choose a $z \in \pi_0(\mathcal{B}(W, \mathfrak{c}))$ and choose a representative (A, Ψ) from the gauge equivalence class of a configuration in the component z. Let $Z_{A,\Psi} \in H_2(W, \alpha, \emptyset)$ and note that $[\mathfrak{a}_0]$ is the image of $[\emptyset] \in ECC(S^3, \lambda_0, 0)$ under Tabues' bijection. To prove the theorem, after cancelling constants we reduce to showing that

$$I(Z_{A,\Psi}) - \frac{1}{4}(c_1(X) + 2PD(Z_{A,\Psi}))^2 = gr_z([\mathfrak{a}_0], X, c_\alpha) - \frac{1}{4}c_1(S^+)^2. \tag{2.21}$$

It follows by Theorem 6 that $I(Z_{A,\Psi}) = gr_z([\mathfrak{a}_0], X, c_\alpha)$. And recall that $C_{A,\Psi}$ is a generic zero locus of a section of E, hence $Z_{A,\Psi}$ is Poincare dual to $c_1(E)$ so by (2.21) we are done.

■

2.4.2 Proof that (2) = (3)

This is an immediate consequence of additivity since (2.2) is equal to $-gr^{\mathbb{Q}}([\emptyset], \alpha)$ and (2.3) equals $gr^{\mathbb{Q}}(\alpha, [\emptyset])$.

■

Chapter 3

The infinite staircase problem for irrational ellipsoids

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Abstract: We continue the study of infinite staircases for the ellipsoid embedding function of a convex toric domain. Previous work on this problem has required that the target is finite type, in other words that the target has a finite negative weight expansion. Here we study the case where the target is an irrational ellipsoid, perhaps the simplest case that is not of finite type. We show that in almost all cases, the embedding function does not admit an infinite staircase. More precisely, we show that if the target is an irrational ellipsoid of eccentricity b, there is no infinite staircase, except possibly if $b + 1/b$ is a positive integer.

3.1 Introduction

Recall the *symplectic ellipsoid*

$$E(a,b) = \left\{ \frac{\pi}{a}|z_1|^2 + \frac{\pi}{b}|z_2|^2 < 1 \right\} \subset \mathbb{C}^2,$$

and fix a number $b \geq 1$. Consider the *ellipsoid embedding function*

$$c_b(a) := min\{\lambda \mid E(1,a) \to \lambda E(1,b)\},$$

defined for $a \geq 1$; here the arrow means that the map is a symplectic embedding. This is a continuous function [25, 8]. We say that $c_b(a)$ *has an infinite staircase* if it has infinitely many *singular points*, that is, non-smooth points. For example, when $b = 1$, the pioneering work of McDuff-Schlenk [25] implies that c_b has an infinite staircase.

Theorem 3.1.1 *If b is irrational and $b + 1/b$ is not an integer, then the function $c_b(a)$ does not have an infinite staircase*

Remark 3.1.2 *It seems likely that the assumption that $b+1/b$ is not an integer can be dropped, in other words that there is no infinite staircase in these cases. However, our current methods do not seem strong enough to show this. Our methods do show that in the case where $b + 1/b$ is an integer, then if there is an infinite staircases, it must accumulate at $b + 1/b + 2$.*

For context, a series of previous works [25, 7, 30, 8, 3, 6, 1, 22, 23] have studied the question of when the ellipsoid embedding function, for a convex toric domain, has an infinite staircase. A tool in many of these works is the "staircase obstruction", defined by whether or not there is a full filling of the target by an $E(1, a_0)$, where $a_0 \geq 1$ is the solution of the quadratic equation

$$x^2 - \left(\frac{per^2}{vol} - 2\right) x + 1 = 0. \tag{3.1}$$

Here, *per* denotes the "affine perimeter" of the target, see [8], and *vol* denotes the volume. When the target is of finite type, [8] proves that if there is an infinite staircase, the staircase obstruction must vanish i.e. a full filling by $E(1, a_0)$ must exist. However, at the time of writing, it is not clear whether or not this remains true without the finite type assumption. If it were true, then one could show much more simply that $E(1,b)$ for irrational b never has an infinite staircase, since the solutions[1] of (3.1) in this case are b and $1/b$.

Since we do not know whether or not the finite type assumption for the staircase obstruction can be dropped, we take here a more ad-hoc approach, although it is certainly our hope that our methods can be generalized to more targets. For example, it may well be the case that the staircase obstruction still is an obstruction without the finite type hypothesis, and this could be an interesting topic for future study.

As a final point, much previous work on this problem has at some points involved studying the symplectic cone on blow-ups of $\mathbb{C}P^2$. Here, we take a more combinatorial approach, studying instead some arithmetic of ECH capacities, which are known to be sharp for this problem. One reason for this is that an irrational ellipsoid does not correspond directly to a finite blow-up of $\mathbb{C}P^2$. In addition, even on a heuristic level, some equations that are key for establishing the "staircase obstruction", in particular [8, Eq. 4.8] involve the length of the weight expansion as a parameter, and it is not currently known what the appropriate replacement of this parameter should be when the length is infinite.

3.2 Accumulation points of infinite staircases and obstructions

An observation from [8] that remains true here is that if a is sufficiently large, then $c_b(a) = \sqrt{a/b}$, in other words there are no obstructions to finding an embedding aside from the volume. In particular, if c_b has an infinite staircase, then the nonsmooth points of c_b must accumulate at finite values a_0 called *accumulation points*.

In this section, we prove some results about accumulation points that we will need. The results are similar to some parts of [8, Step 3, Thm. 1.11]; however, the arguments there were proved under a finite type hypothesis that is not assumed

[1]The fact that the solutions of (3.1) are b and $1/b$ in the case of an irrational ellipsoid was first pointed out to the first named author on this work by Holm.

here. Since we are not assuming this, we need to give different arguments.

3.2.1 Recollections

Recall the *ECH capacities* $c_k(E(a,b))$, defined in [15] and computed for integer $k \geq 0$ to be the $(k+1)^{st}$ smallest number in the matrix $(ma+nb)_{m,n \in \mathbb{Z}_{\geq 0}}$. A fundamental fact [24] is that the ellipsoid embedding problem is completely encoded by the combinatorics of this sequence: there is a symplectic embedding $E(a,b) \to E(c,d)$ if and only if

$$c_k(E(a,b)) \leq c_k(E(c,d))$$

for all k. In particular, as a consequence

$$c_b(a) = sup_k \frac{c_k(E(1,a))}{c_k(E(1,b))}, \tag{3.2}$$

which is central to our approach.

3.2.2 ECH capacities and infinite staircases

The following proposition is fundamental to our approach. Call a capacity c_k *obstructive* on an interval I if there is some $x \in I$ with $c_b(x) = \frac{c_k(E(1,x))}{c_k(E(1,b))} > \sqrt{\frac{x}{b}}$.

Proposition 3.2.1 *Let a_0 be an accumulation point. Then, for every open interval I containing a_0, there are infinitely many c_k that are obstructive on I.*

Proof Assume the opposite. Then, there is some interval I containing a_0 such that only finitely many c_k are obstructive on I. Let S be the set of k, such that c_k is obstructive on I; this is a finite set. Then, on I, we have

$$c_b|_I = max\left(sup_{k \in S} \frac{c_k(E(1,x))}{c_k(E(1,b))}, \sqrt{\frac{x}{b}} \right),$$

by (3.2). Since S is finite, this is the supremum of finitely many functions. To prove the proposition, it then suffices to show that each of these functions has only finitely many singular points.

To prove this, we will prove that the function $c_k(E(1,x))$ has only finitely many singular points for $1 \leq x$. To prove this, we consider the functions $f_{m,n} = m + n \cdot x$, defined for nonnegative integers m, n; the function $c_k(E(1,x))$ is exactly the $(k+1)^{st}$ smallest value of $f_{m,n}(x)$. We know that $c_k(E(1,x)) \leq k$. Hence, any $f_{m,n}$ satisfying $f_{m,n}(x) = c_k(E(1,x))$ must satisfy

$$m \leq k, \quad n \leq k.$$

It therefore follows that, for all x, $c_k(E(1,x))$ is the $(k+1)^{st}$ smallest value among the finitely many values $\{f_{m,n}(x) | m, n \leq k\}$. Since each $f_{m,n}$ is a linear function of x, in particular has no singular points at all, it now follows that $c_k(E(1,x))$ has only finitely many singular values, hence the result.

3.2.3 The volume obstruction

We now want to prove that any accumulation point a_0 must lie on the volume obstruction.

Proposition 3.2.2 *Let a_0 be an accumulation point of an infinite staircase. Then $c_b(a_0) = \sqrt{a_0/b}$.*

Proof

Assume that it does not. That is, $c_b(a_0) > \sqrt{a_0/b}$, or equivalently $\frac{c_b(a_0)}{\sqrt{a_0/b}} > 1$.

Step 1. Preliminaries. We first enlarge this inequality to an entire interval and fix an explicit lower bound for the difference between these quantities. More precisely, by continuity of c_b, given this inequality, we can find $1 > \epsilon > 0$ such that

$$\frac{c_b(x)}{\sqrt{x/b}} > 1 + m \tag{3.3}$$

for some $m > 0$, if $x \in [a_0 - \epsilon, a_0 + \epsilon]$. We know that $a_0 > b$, so we can assume in addition that $a_0 - \epsilon \geq b$.

Step 2. The Buse-Hind estimate. The idea is now to show that, by (3.3), one can show that only finitely many c_i are obstructive on $[a_0 - \epsilon, a_0 + \epsilon]$, which contradicts Proposition 3.2.1, since a_0 is an accumulation point. For this, the following estimate on the number of lattice points in a triangle is very useful; the purpose of this step is to state this estimate. The estimate says that the number of lattice points is approximately given by the area of the triangle, with an explicit bound on the error in the estimate.

To state the estimate, define $R_{a,b}(T) = sup\{k+1 | c_k(E(a,b)) \leq T\}$. This is the number of lattice points in the right triangle with legs on the axes and area $\frac{1}{2ab}$. The following estimate is proved in [2, Prop. 2.5]

$$|R_{a,b}(T) - \frac{1}{2ab}T^2| \leq \left(\frac{1}{2a} + \frac{1}{b}\right) T + \frac{b}{8a} + 1. \tag{3.4}$$

Step 3. Completion of the proof. With these preliminaries out of the way, we can now complete the proof. By Proposition 3.2.1, to establish a contradiction it suffices to show that only finitely many c_i are obstructive on the interval $[a_0 - \epsilon, a_0 + \epsilon]$ from Step 1. Let c_i be obstructive, say at the point y'. Then, by (3.3), the definition of being obstructive, and continuity of c_i there is some irrational y in $[a_0 - \epsilon, a_0 + \epsilon]$ with $\frac{c_i(E(1,y))}{c_i(E(1,b))} \geq (1+m)\sqrt{y/b}$. Hence

$$c_i(E(1,y)) \geq (1+m) c_i(E(\sqrt{y/b}, \sqrt{yb})). \tag{3.5}$$

Now let $T = c_i(E(1,y))$. We seek a bound on T, in order to get the desired bound on i.

We now derive the needed bound on T. Before going through the relevant calculations, we state here the underlying combinatorial idea. The inequality (3.5) relates capacities of two ellipsoids with strictly different volume. We will show that it can not hold for infinitely many i, because the corresponding lattice point counts are well enough approximated by the volume, and the volumes are different. A little bit of care is required to get a “uniform” estimate, that is an estimate that does not depend on the choice of $y \in [a_0 - \epsilon, a_0 + \epsilon]$.

Let us now provide the details. The estimate we require is not too sensitive, so in the following, we often simplify the algebraic expressions by applying further simple estimates (e.g. $\frac{b}{8a} \leq \frac{b}{a}$) without commentary.

To get the bound on T, note first that $R_{1,y}(T) = i + 1$, because y is irrational, and $R_{(1+m)\sqrt{y/b},(1+m)\sqrt{yb}}(T) \geq i + 1$, by (3.5). It follows that

$$R_{(1+m)\sqrt{y/b},(1+m)\sqrt{yb}}(T) \geq R_{1,y}(T).$$

Hence, by the area estimate (3.4)

$$\frac{1}{2(1+m)^2 y} T^2 \geq \frac{1}{2y} T^2 - 4T - 2y - 2.$$

Multiplying through by y gives

$$\frac{1}{2(1+m)^2} T^2 \geq \frac{1}{2} T^2 - 4Ty - 2y^2 - 2y.$$

We know that $y \leq a_0 + \epsilon \leq 2a_0$ hence

$$\frac{1}{2}(1 - \frac{1}{(1+m)^2})T^2 \leq 8Ta_0 + 8a_0^2 + 8a_0 \leq 8T(a_0 + 8a_0^2 + 8a_0),$$

since $T \geq 1$. Hence, we get the explicit upper bound on T:

$$T \leq \frac{8(a_0 + 8a_0^2 + 8a_0)}{1 - \frac{1}{1+m^2}}. \tag{3.6}$$

Now, to get a corresponding bound on i, first note that $i = R_{1,y}(T) - 1$ from above. Now by (3.4)

$$R_{1,y}(T) \leq \frac{1}{2y} T^2 + 2T + y + 1.$$

Hence, the explicit upper bound on T, together with the bound $y \leq 2a_0$, gives an explicit upper bound on i, which completes the proof in view of Proposition 3.2.1.

3.3 Computation of the lattice points

As already seen in the proof of Proposition 3.2.2, and as is well-known, ECH capacities of ellipsoids can profitably be encoded in terms of certain lattice point counts.

Here is the relevant count for our purposes. We want to understand the number of integer lattice points $N_{a,b}(t)$ in the triangle $\mathcal{T}_{a,b}(t)$ with vertices

$$(0,0), \quad (t\sqrt{1/(ab)},0), \quad (0,t\sqrt{a/b}).$$

Here, $b>1$ is irrational, $a>b$ and $t>0$ is real. The motivation is that this is the triangle corresponding to the lattice point count for the ECH capacities of an ellipsoid $\sqrt{\frac{b}{a}}E(1,a)$, which is normalized to have the same volume as an $E(1,b)$ target, but with the additional parameter a. More precisely,

$$\#\{k \mid c_k\left((\sqrt{b/a})E(1,a)\right) \leq t\} = \#\{\mathbb{Z}^2 \cap \mathcal{T}_{a,b}(t)\}. \tag{3.7}$$

To state our formula for this lattice point count, define $f := 1-\{\cdot\}$, where $\{\cdot\}$ denotes the fractional part.

Lemma 3.3.1 $N_{a,b}(t) = \frac{1}{2}t^2/b+\frac{1}{2}\sqrt{\frac{a}{b}}t+\sum_{m=0}^{\lfloor t\sqrt{1/(ab)}\rfloor} f(t\sqrt{a/b}-am)+\frac{a}{2}(\{t\sqrt{1/(ab)}\}-\{t\sqrt{1/(ab)}\}^2)$.

Proof

The proof uses a similar idea to the proof of [5, Lem. 2.3]; however, that proof was solely about the subleading asymptotics, under an irrationality hypothesis, and here we give a fuller formula with no such assumption.

We count in vertical slices to obtain the formula

$$N_{a,b}(t) = \sum_{m=0}^{\lfloor t\sqrt{1/(ab)}\rfloor} (t\sqrt{a/b}-am) + \sum_{m=0}^{\lfloor t\sqrt{1/(ab)}\rfloor} f(t\sqrt{a/b}-am). \tag{3.8}$$

We can evaluate the first sum explicitly, which gives

$$(\lfloor t\sqrt{1/(ab)}\rfloor + 1)t\sqrt{a/b} - a\frac{\lfloor t\sqrt{1/(ab)}\rfloor\lfloor t\sqrt{1/(ab)}+1\rfloor}{2}$$

which we rewrite as

$$\frac{1}{2}t^2/b + \frac{1}{2}t\sqrt{a/b} + \frac{a}{2}(\{t\sqrt{1/(ab)}\} - \{t\sqrt{1/(ab)}\}^2).$$

This establishes the lemma.

Corollary 3.3.2 $N_{b,b}(t) = \frac{1}{2}t^2/b + \frac{1}{2}(1+1/b)t + o(t)$.

Proof

By Weyl's criteria,

$$\sum_{m=0}^{\lfloor t\sqrt{1/(ab)}\rfloor} \{t - bm\} = \frac{1}{2}\lfloor t\sqrt{1/(ab)}\rfloor + o(t),$$

hence the corollary follows from Lemma 3.3.1.

Remark 3.3.3 *When attempting to apply Lemma 3.3.1, usually the hardest term is the* $\sum f(\cdot)$. *Much of the rest of our arguments involve some estimates on this term.*

3.4 Lattice point estimates

In this section, we prove some important estimates regarding lattice point counts. In view of the formula (3.8), the crux of the issue is estimating sums of the form

$$\sum_{m=0}^{N}\{\beta - m\alpha\},$$

which is the focus of this section. In connection with above, we are interested in the case $\alpha = a$ and $\beta = \sqrt{a/b}$. For these estimates, we distinguish two cases, namely the case where a is irrational and the case where a is rational.

3.4.1 The irrational case

Here is the bound we need in the irrational case.

Lemma 3.4.1 *Fix an irrational* a_0 *and fix* $\epsilon > 0$*. Then, there exists a* $\delta > 0$ *and a positive number* N_0 *such that for* $N \geq N_0$*,*

$$\sum_{m=0}^{N} \{\beta - m\alpha\} \leq \left(\frac{1}{2} + \epsilon\right) N,$$

for any β*, as long as* α *is irrational and satisfies* $|\alpha - a_0| < \delta$*.*

The proof follows from several other lemmas, which we now introduce.

To state the first lemma, write

$$|\alpha - \frac{p}{q}| < \frac{1}{q^2},$$

with $p \neq 0$ and (p, q) relatively prime; this can be done, for infinitely many different q, by Dirichlet's theorem, since α is irrational.

Here is a bound that is rather coarse, but strong enough for what we need at this stage.

Lemma 3.4.2 $\sum_{m=0}^{q-1} \{\beta - m\alpha\} \leq \frac{q-1}{2} + 3$

Proof

We write

$$\frac{k-1}{q} \leq \beta \leq \frac{k}{q}.$$

Then

$$-m\frac{p}{q} - \frac{1}{q} < -m\alpha < -m\frac{p}{q} + \frac{1}{q},$$

hence

$$\frac{(k-m-1)p}{q} - \frac{1}{q} < \beta - m\alpha < \frac{(k-m)p}{q} + \frac{1}{q}. \tag{3.9}$$

Now as m ranges from 0 to $q-1$, $\{\frac{(k-m)p}{q}\}$ ranges across the set $S = \{\frac{0}{q}, \ldots, \frac{q-1}{q}\}$. When

$$\{\frac{(k-m)p}{q}\} = \frac{i}{q}, \quad 2 \le i \le q-1,$$

it follows from (3.9) that

$$\{\beta - m\alpha\} \le \frac{i}{q} + \frac{1}{q}.$$

In the cases $i \in \{0, 1\}$, we have the trivial bound

$$\{\beta - m\alpha\} \le \frac{i}{q} + 1.$$

Summing these equations over $i \in S$, we get the upper bound claimed by the lemma.

We can use this bound to get the following estimate. Set $N = q\ell + x$ with $x < q$ and assume that $N\alpha \le \beta$.

Lemma 3.4.3 $\sum_{m=0}^{N}\{\beta - m\alpha\} \le \frac{q\ell}{2} + 2.5\ell + q$

Proof

We write

$$\sum_{m=0}^{N}\{\beta - m\alpha\} = \sum_{m=0}^{q-1}\{\beta - m\alpha\} + \sum_{m=0}^{q-1}\{(\beta - q\alpha) - m\alpha\} + \ldots + \sum_{m=0}^{q-1}\{(\beta - (q-1)\alpha) - m\alpha\} + \delta,$$

where δ is trivially bounded by q. We then bound the other q terms by using the previous lemma.

We can now prove the lemma from the beginning of this section.

Proof [Proof of Lemma 3.4.1]

Note that the previous lemma gives the bound

$$\sum_{m=0}^{N}\{\beta - m\alpha\} \le \frac{N}{2} + \frac{2.5}{q}N + q.$$

Now, according to Dirichlet's theorem, for any N_0' there exists p/q with $1 \le q \le N_0$ such that

$$|\alpha - \frac{p}{q}| < \frac{1}{qN_0'}. \tag{3.10}$$

We now choose δ' so small that $\frac{2.5}{q} < \epsilon/4$, for any $\frac{p}{q}$ satisfying

$$|a_0 - \frac{p}{q}| < \delta'. \tag{3.11}$$

Set $\delta = \delta'/2$ and set $N_0' = \lceil \frac{4}{\delta} \rceil$. Then, if $|\alpha - a_0| < \delta$, then any $\frac{p}{q}$ satisfying (3.10) satisfies (3.11). We now apply Dirichlet's theorem with N_0' to get a rational $\frac{p}{q}$ satisfying (3.10) with $q \leq N_0'$. We now choose N_0 large enough that $q \leq \frac{\epsilon}{4} N$ whenever $N \geq N_0$.

3.4.2 The rational case

In the rational case, the bound in Lemma 3.4.1 is no longer true, see for example the proof of Lemma 3.5.6 below, but we have a similar, although somewhat weaker, statement.

Lemma 3.4.4 *Fix a rational $a_0 = \frac{p}{q}$ and fix $\epsilon > 0$. Then, there exists a $\delta > 0$ and a positive number N_0 such that for $N \geq N_0$,*

$$\sum_{m=0}^{N} \{\beta - m\alpha\} \leq \left(\frac{1}{2}(1 + \frac{1}{q}) + \epsilon \right) N,$$

for any β, as long as α is irrational and satisfies $|\alpha - a_0| < \delta$.

Proof

Let $U_i = [i/q, (i+1)/q]$ for $0 \leq i \leq q-1$. Assume that $\alpha < p/q$ and let $\delta_\alpha = |\alpha - \frac{p}{q}|$. Assume also that $\delta < 1/(nq^2)$, hence $\delta_\alpha < 1/(nq^2)$, where n is a large positive number to be fixed later in the proof. For any real number x let $[x]_q \in [0, \frac{1}{q})$ denote the representative of x in the additive group $\mathbb{Z}$ modulo $(1/q)\mathbb{Z}$. That is, $x = \frac{k}{q} + [x]_q$ for some integer k. Let $b_m := \{\beta - m\alpha\}$.

Step 1. A single cycle of length q. We consider first the case where $N = q$. We distinguish two possible cases that can occur.

1. Case 1: Each U_i for $0 \leq i \leq q-1$ contains precisely one element of (b_m).

2. Case 2: Some U_i contains multiple elements of (b_m).

We write down upper bounds for each case. In case (1), we have

$$\sum_{m=0}^{q-1} b_m \leq \sum_{i=0}^{q-1} \#(U_i \cap (b_m))\left(\frac{i+1}{q}\right) = \sum_{i=0}^{q-1}\left(\frac{i+1}{q}\right) = \frac{q+1}{2}.$$

In case (2), we take as upper bound the coarse bound

$$\sum_{m=0}^{q-1} b_m \leq q.$$

Step 2. Many full cycles of length q. We next seek to estimate $\sum_{m=0}^{N_0'-1} b_m$, where $N_0' := kq$; this is the heart of the estimate. This step collects some first estimates in that regard. Let $Q_j = \sum_{m=jq}^{m=q(j+1)-1} b_m$, for $0 \leq j \leq k-1$. For fixed j, any b_m for $jq \leq m \leq (j+1)q - 1$ has the form

$$\{(\beta - jq\alpha) - m\alpha\},$$

for $0 \leq m < q$, and so the upper bounds from the above two cases in the previous step apply for bounding Q_j, with β in the previous step replaced by $(\beta - jq\alpha)$; for future reference we denote $\beta_j := \beta - jq\alpha$. With that understood, let S denote the set consisting of $j \in \{0, 1, \ldots, k-1\}$ for which case 2 in the previous step applies to those b_m counted by Q_j. Then by the previous step, we have the bound

$$\begin{aligned}\sum_{m=0}^{N_0'-1} b_m = \sum_{j=0}^{k-1} Q_j &= \sum_{j \notin S} Q_j + \sum_{j \in S} Q_j \\ &\leq \frac{k(q+1)}{2} + |S|q.\end{aligned}$$

Hence we obtain

$$\sum_{m=0}^{N_0'-1} b_m \leq \frac{k(q+1)}{2} + |S|q. \tag{3.12}$$

Step 3. Case two rarely occurs. We now wish to bound $|S|$, which is the goal of this step. To do this, we need to clarify those conditions under which Case 2 from Step 1 occurs; the outcome of this step will be that Case 2 occurs "rarely" when δ is sufficiently small, with a precise quantitative estimate given in (3.13) below.

Consider a sum of elements b_m, with $0 \le m \le (q-1)$. Write $\beta = i_0/q + [\beta]_q$. As m ranges from 0 to $q-1$, $i_0 - mp$ ranges over all equivalence classes, modulo q, exactly once, and we seek to understand which U_k contains $\{\beta - m\alpha\}$. Let $i = i_0 - mp \pmod q$. We distinguish three subcases, based on the value of $[\beta]_q - m(\alpha - a_0)$. In case 1, $-1/q < [\beta]_q - m(\alpha - a_0) < 0$, in which case $[\beta - m\alpha] \in U_{i-1}$; in case 2, $0 \le [\beta]_q - m(\alpha - a_0) < 1/q$, in which case $[\beta - m\alpha] \in U_i$; in case 3, $1/q \le [\beta]_q - m(\alpha - a_0) < 2/q$, in which case $[\beta - m\alpha] \in U_{i+1}$. Note that these cases exhaust all possibilities for the value of $[\beta]_q - m(\alpha - a_0)$, given our bound on δ_α.

To better keep track of these subcases, introduce the interval $\beta - t(\alpha - \alpha_0)$, for $0 \le t \le q$. This is an interval $I \subset \mathbb{R}$ of length $q\delta_\alpha < 1/q$. Then, by the analysis in the previous paragraph, Case 2 will occur for all $0 \le m \le q$ as long as I does not contain a lattice point of the form $\frac{1}{q}\mathbb{Z}$.

Having analyzed the situation for the case where m ranges from 0 to $q-1$, which corresponds to Q_0, and which applies to all Q_j by replacing β with β_j, we return to bounding $|S|$.

For fixed j, the intervals I_j given by $\beta_j - t(\alpha - \alpha_0) \subset \mathbb{R}$ for $0 \le t \le q$ all have length $q\delta_\alpha$; and two consecutive intervals I_j and I_{j+1} connect at β_{j+1}. By the above analysis, if $j \in S$, then I_j must contain a lattice point of the form $\frac{1}{q}\mathbb{Z}$.

In particular, as a coarse bound, the size of S is no more than twice the number of $\frac{1}{q}\mathbb{Z}$ lattice points in $T_k := \cup_{j=0}^{k-1} I_j$, since any lattice point can be contained in at most two distinct I_j. We therefore have $|S| \le 2\#\{\frac{1}{q} \cap T_k\}$. On the other hand, if T_k contains ℓ lattice points, then its length must be at least $\ell - 1$, so we get

$$\mathit{length}(T_k) = kq\delta_\alpha \ge \#\{\frac{1}{q} \cap T_k\} - 1 \ge |S|/2 - 1.$$

In particular, we have obtained the key bound:

$$|S| \leq 2kq\delta_\alpha + 2. \tag{3.13}$$

Step 4. Completion of the proof. We now put this all together to complete the proof of the Lemma. If we have general $N = N_0' + \ell$, with $0 \leq \ell < q$ and $N_0' = kq$, then we can write $\sum_{m=0}^{N} b_m \leq (\sum_{m=0}^{N_0'} b_m) + q$. By (3.12) and (3.13) above, we have

$$\sum_{m=0}^{N_0'} b_m + q \leq \frac{N_0'}{2}(1 + \frac{1}{q}) + 2kq^2\delta_\alpha + 3q \leq \frac{N}{2}(1 + \frac{1}{q}) + 2q\delta_\alpha N + N(3q/N).$$

Taking n in the definition of δ sufficiently small then ensures that $2q\delta_\alpha < \epsilon/2$, hence the lemma.

3.5 Putting it all together

We now prove Theorem 3.1.1. If an infinite staircase exists, then as stated before it must have at least one accumulation point a_0. The idea, then, is to show that such an accumulation point can not exist. The proof consists of two parts.

3.5.1 The large values of a

The first part of the proof of Theorem 3.1.1 is the following bound, which follows from combining the estimates from above.

Proposition 3.5.1 *Let a satisfy*

$$\sqrt{\frac{a}{b}} + (1 - \frac{1}{q})\sqrt{\frac{1}{ab}} > 1 + \frac{1}{b}, \tag{3.14}$$

where here $a = p/q$ in the rational case, and we formally interpret $q = \infty$ (that is, $\frac{1}{q} = 0$) in the irrational case. Then a can not be an accumulation point.

Proof

Assume that a_0 satisfies (3.14) and choose $\epsilon > 0$ such that

$$\sqrt{\frac{a_0}{b}} + (1 - \frac{1}{q})\sqrt{\frac{1}{a_0 b}} - \epsilon > 1 + \frac{1}{b}. \tag{3.15}$$

By combining Corollary 3.3.2 with the bound from Lemma 3.4.1 in the case a_0 irrational, or Lemma 3.4.4 in the case a_0 rational, we can then obtain an interval I around a_0 such that for any irrational a in this interval

$$N_{a,b}(t) \geq \frac{1}{2}t^2/b + \frac{1}{2}\left(\sqrt{\frac{a}{b}} + (1 - \frac{1}{q})\sqrt{\frac{1}{ab}} - \epsilon/2\right)t,$$

for sufficiently large t. We can also choose this interval so that

$$\frac{1}{2}(\sqrt{\frac{a}{b}} + (1 - \frac{1}{q})\sqrt{\frac{1}{ab}} - \epsilon/2) \geq \frac{1}{2}(\sqrt{\frac{a_0}{b}} + (1 - \frac{1}{q})\sqrt{\frac{1}{a_0 b}} - \epsilon).$$

By (3.15) and the first item in Corollary 3.3.2, it now follows that if t_0 is sufficiently large and $t \geq t_0$ then we must have

$$N_{a,b}(t) \geq N_{b,b}(t).$$

for all irrational a in I. It now follows from (3.7) that if $c_k(1,b) = t$ for some $t \geq t_0$, then $c_k(\sqrt{\frac{b}{a}}E(1,a)) \leq c_k(E(1,b))$. Hence, there are only finitely many c_k that are obstructive at irrational points in I. It then follows, since being obstructive is an open condition, that there are only finitely many c_k that are obstructive in I. It then follows from Proposition 3.2.1 that there can not be an infinite staircase at a_0.

3.5.2 The small values of a

In view of Proposition 3.5.1, to complete the proof of Theorem 3.1.1, we just have to consider points a satisfying

$$\sqrt{\frac{a}{b}} + (1 - \frac{1}{q})\sqrt{\frac{1}{ab}} \leq 1 + \frac{1}{b}. \tag{3.16}$$

This will be our standing assumption in what follows.

A basic consideration, that will be helpful in what follows, is that the left hand side of (3.16), for fixed (b, q), is an increasing function of a. Indeed, its partial derivative, with respect to a, is

$$\frac{a - \frac{q-1}{q}}{2ba\sqrt{\frac{a}{b}}},$$

which is positive for $a \geq 1$.

The irrational values

The analysis is simplest when a_0 is irrational.

Lemma 3.5.2 *If a_0 is irrational and satisfies* (3.16)*, then a_0 can not be an accumulation point.*

Proof

In this case $\frac{1}{q} = 0$, so as the left hand side of (3.16) is an increasing function of a, the only such a_0 is b itself. However, it is known [6, Lem. 2.9] that $c_b(a) = 1$ for $1 \leq a \leq b$ and[2] $c_b(a) = \frac{a}{b}$ for $b \leq a \leq \lceil b \rceil$. In particular, there can not be an infinite staircase at a_0.

Rational numbers with denominator at least six

As proved above, the left hand side of (3.16) is an increasing function of a. On the other hand, we have the following.

Lemma 3.5.3 *Let $b \leq a_0 \leq \lfloor b \rfloor + 2$. Then a_0 is not an accumulation point of c_b.*

Proof

Equipped with the new Proposition 3.2.2, the proof then proceeds along the lines of some related results in [6].

[2] In [6, Lem. 2.9], it is not actually stated that $c_b(a) = a/b$ for $b \leq a \leq \lceil b \rceil$; only the lower bound is stated. However, the matching upper bound follows by scaling the domain as in the proof of the subscaling property in [6, Sec. 2.6].

To elaborate, by Proposition 3.2.2, if there is an infinite staircase at a_0, then $c_b(a_0) = \sqrt{a_0/b}$. Now, by [6, Lem. 2.9], if a_0 is any point satisfying $b \le a_0 \le \lfloor b \rfloor + 2$ and $c_b(a_0) = \sqrt{a_0/b}$, then for x sufficiently close to a_0, we have

$$c_b(x) \ge c_b(a_0)$$

if $x \le a_0$ and

$$c_b(x) \ge (x/a_0)c_b(a_0)$$

if $x \ge a_0$. Thus, by the Monotonicity and Subscaling properties as in [6, Sec. 2.6], if $c_b(a_0) = \sqrt{a_0/b}$, then in fact equality must hold in the above inequalities. However, this is a contradiction, since then a_0 would not be a limit point of an infinite staircase.

In view of the fact that $\lfloor b \rfloor + 2 \ge b+1$, the above lemma, together with the fact that the left hand side of (3.16) is an increasing function of a, motivates solving the equation

$$\sqrt{\frac{b+1}{b}} + (1 - \frac{1}{q})\sqrt{\frac{1}{(b+1)b}} = 1 + \frac{1}{b}, \tag{3.17}$$

as a function of (b, q). What falls out of this is the following.

Lemma 3.5.4 *Let $a_0 = p/q$ be a rational number in lowest terms, with $q \ge 6$. Then a_0 can not be an accumulation point of an infinite staircase for c_b.*

Proof

The equation (3.17) has a unique solution x_b for $x = (1 - \frac{1}{q})$ as a function of b, hence a unique solution q_b for q.

We claim that the solution q_b is a decreasing function of b; it is equivalent to show that x_b is a decreasing function of b. To see this, note that:

$$\frac{d}{db}x_b = \frac{1}{2b\sqrt{b+b^2}}\left(-1 + b + (2 - 2\sqrt{\frac{b+1}{b}})b^2\right).$$

We claim that the right hand side of the above expression has no zeros for $b > 0$; on the other hand, for $b = 1$, the right hand side evaluates to $\frac{2-2\sqrt{2}}{2\sqrt{2}} < 0$, and so the right hand side is always negative given the claim about no zeros. To see why this claim about no zeros is true, note that the numerator of the right hand side of the above expression is

$$-\frac{y(y+2)}{(y+1)^2},$$

where $y := \sqrt{\frac{b+1}{b}}$. This has just two roots for y, neither of which correspond to positive values of b.

Now, since $q_1 = 3+2\sqrt{2} < 6$, we therefore learn in view of this monotonicity that if $q \geq 6$, then either $a_0 > (b+1)$, in which case $a = a_0$ does not satisfy (3.16) and so there is no infinite staircase at a_0; or, $a_0 \leq (b+1)$, in which case Lemma 3.5.3 applies to show that there is not an infinite staircase at a_0.

Rational numbers with denominators between two and six

For rational numbers with denominators between 2 and 5, we need a somewhat more ad-hoc approach.

Lemma 3.5.5 *Let $a = p/q$ in lowest terms, with $2 \leq q \leq 5$. Then a can not be an accumulation point for c_b.*

Proof

We argue case by case, although the general arguments are quite similar from one case to the next.

Let us first consider the case $q = 2$. In this case, one first observes that, since a in this case would have to be an integer with denominator 2 that by Lemma 3.5.3 is at least $\lfloor b \rfloor + 2$, we have $a \geq b + 1.5$. Now, if $b > \sqrt{3}$, then

$$\sqrt{\frac{b+1.5}{b}} + \frac{1}{2}\sqrt{\frac{1}{(b+1.5)(b)}} > 1 + \frac{1}{b},$$

since solving the corresponding equality (that is, the equality given by setting the left and right hand sides of the above inequality equal to each other) gives that the only solution to that equality is $b = \sqrt{3}$, and so then checking for example

$b = 2$ gives a strict inequality as asserted, hence a does not satisfy (3.16) and so there is no infinite staircase by Proposition 3.5.1. It remains to consider the case $1 \leq b \leq \sqrt{3} < 2$. In this case, one then has the improvement in view of Lemma 3.5.3, and the fact that a has denominator 2, that $a \geq \lfloor b \rfloor + 2.5 = b + (2 - b) + 1.5 = 3.5$ On the other hand, for $b \leq 2$, we have

$$\sqrt{\frac{3.5}{b}} + \frac{1}{2}\sqrt{\frac{1}{(3.5)(b)}} > 1 + \frac{1}{b},$$

since solving the corresponding equality gives that the solutions to the equality are precisely $\frac{9 \pm 4\sqrt{2}}{7}$, and so it suffices to check the inequality for $b = 1$. Hence, as above, a does not satisfy (3.16) and so there is no infinite staircase. This completes the analysis in the case $q = 2$.

The rest of the cases are similar so we are slightly more brief.

In the case $q = 4$, we have

$$\sqrt{\frac{b + 1.25}{b}} + \frac{3}{4}\sqrt{\frac{1}{(b + 1.25)(b)}} > 1 + \frac{1}{b},$$

for $b > 1$: indeed, the corresponding equality has a unique solution at $b = 1$, and so one can check the desired inequality by checking, for example, the case where $b = 2$. Hence, by combining Lemma 3.5.3 and Proposition 3.5.1 there is no infinite staircase.

Similarly, in the case $q = 5$, we have

$$\sqrt{\frac{b + 1.2}{b}} + \frac{4}{5}\sqrt{\frac{1}{(b + 1.2)(b)}} > 1 + \frac{1}{b},$$

for $b \geq 1$, since the corresponding equality has a unique solution at $b = \frac{\sqrt{105} - 3}{8}$, and one can then check for example at $b = 1$ that strict inequality holds. So, a staircase can not occur by the same analysis as above.

Finally, as for the case $q = 3$, as long as $b \geq 2$, we have

$$\sqrt{\frac{b + 4/3}{b}} + \frac{2}{3}\sqrt{\frac{1}{(b + 4/3)(b)}} > 1 + \frac{1}{b},$$

since the corresponding equality has a unique root at $\frac{\sqrt{33}-1}{4}$, and one can then check at for example $b = 2$ that strict inequality holds. In the case $b < 2$, we have the improvement $a \geq b + 4/3 + (2 - b) = 2 + 4/3$, similarly to in the case of denominator 2. Then, since for $1 \leq b \leq 2$, we have

$$\sqrt{\frac{2 + 4/3}{b}} + \frac{2}{3}\sqrt{\frac{1}{(2 + 4/3)(b)}} > 1 + \frac{1}{b},$$

since the corresponding equality has solutions at $\frac{7 \pm 2\sqrt{6}}{5}$, and one can then check that strict inequality holds at for example $b = 1$. Hence, there is no infinite staircase in this case as well, and the result now follows.

3.5.3 The case of integers

When a_0 is an integer, we use a different set of arguments; at the time of writing, these are not strong enough to handle the case where $b + 1/b$ is an integer, which is why our result in Theorem 3.1.1 has this assumption.
More precisely, we prove here the following.

Lemma 3.5.6 *Let $1 \leq a_0 < b + 1/b + 2$ be an integer. Then a_0 is not an accumulation point for c_b.*

In view of the above work handling all other cases for a_0, this proves Theorem 3.1.1, since if $a_0 > b + 1/b + 2 = (b + 1)^2/b$, then a_0 does not satisfy (3.16) and we can then apply Proposition 3.5.1.
Proof
[Proof of Lemma 3.5.6]
By Proposition 3.2.2, if a_0 is an accumulation point, then $c_b(a_0) = \sqrt{a_0/b}$. However, we will show that under the assumptions on a_0 in the lemma, c_b is actually obstructed

at a_0, in other words is not equal to the volume obstruction.

To see this, it suffices by (3.7) to find an integer t with $N_{a,b}(t) < N_{1,b}(t)$. To produce such a t, we use the formulas for $N_{a,b}$ from §3.3. We know that $\sqrt{a/b}$ is irrational, since a is rational and b is irrational. There are infinitely many rational approximations to $\sqrt{a/b}$; choose an upper convergent p/q. This means that

$$|\sqrt{a/b} - p/q| < \frac{1}{q^2}, \quad p/q > \sqrt{a/b}.$$

Now take $t = q$. Then, since a is an integer,

$$f(t\sqrt{a/b} - am) < \frac{1}{q},$$

and so the final summand in the formula for $N_{a,b}(q)$ in (3.8) is no more than

$$(\lfloor q\sqrt{1/(ab)} + 1\rfloor)\frac{1}{q}.$$

Now this stays bounded as q gets large. On the other hand, the equation

$$a < b + 2 + \frac{1}{b^2}.$$

implies that

$$\sqrt{a/b} < 1 + \frac{1}{b}.$$

Thus in view of Corollary 3.3.2, there are in fact infinitely many integer t with $N_{a,b}(t) < N_{1,b}(t)$.

$$\sqrt{a/b} < 1 + \frac{1}{b},$$

Remark 3.5.7 *The above argument in fact shows that at such an a_0, there are infinitely many obstructive ECH capacities.*

This is important to keep in mind when attempting to form heuristics about infinite staircases. For example, it follows that, for any N one can find an irrational number *close to* a_0 *with an obstructive ECH capacity of index at least* N*. One might have hoped that the subleading asymptotics of the ECH capacities at such an irrational number might preclude this from occurring, since for such an irrational number they will eventually imply that at high enough index the capacities can no longer be obstructive. This "subleading asymptotic argument" has previously been mentioned as a heuristic for finding accumulation points; however, this remark shows that in practice trying to implement this seems subtle due to non-uniformity of the subleading estimates as one varies the domain.*

Appendix A

Surgery

This section is a cursory summary of the basic notions for symplectic handlebody attachments that are used to compute the $\mathbb{Q}$-grading of the empty set generator of a lens spaces. The essential details for this construction are taken directly from [26] and [9], and are presented solely for the convenience of the reader.

A.1 Legendrian knots and their invariants

Let (Y, ξ) be a contact 3-manifold. An oriented knot $L \subset Y$ is said to be *Legendrian* if it is tangent to the contact distribution ξ, that is, if $TL \subset \xi$. A *framing* of L is a choice of trivialization of the normal bundle νL with $L \times \mathbb{R}^2$. For a general knot K there may not be a canonical choice of framing (unless it is nullhomologous, in which case one has a canonical trivialization induced by a Seifert surface), but in the case of a Legendrian knot L, there is a canonical choice of framing. Indeed, since the contact planes of ξ intersect νL transversely, a trivialization is obtained by choosing a nonvanishing section of the real line bundle $\ell = \xi \cap \nu$. In other words we can also think of a framing as a nonvanishing vector field along L, transverse to TL and contained in ξ. The framing given by such a vector field is called the *Thurston-Bennequin framing.*

Now if L is nullhomologous, then we get an invariant (up to orientation) of L called the *Thurston-Bennequin number*, denoted $tb(L)$. This is obtained as follows:

let v be a nonvanishing section of ℓ. Then let L' denote a pushoff of L in the direction of v. Then $tb(L)$ is defined to be the linking number $lk(L, L')$. Intuitively, $tb(L)$ measures how many times ξ twists around L.

The second invariant of a nullhomologous L is the *rotation number*, $r(L)$. Let Σ denote a Seifert surface with boundary L. The contact distribution ξ restricted to Σ is trivial, so choose a trivialization $\xi|_\Sigma \cong \Sigma \times \mathbb{R}^2$. This trivialization induces a trivialization $\xi|_L \cong L \times \mathbb{R}^2$. Now let v be any vector field tangent to L in the same direction as the orientation on L. Using this trivialization, we can think of v as a path of nonzero vectors around the origin in $\mathbb{R}^2$ parameterized by L. As such it has a winding number, and this defines $r(L)$. This depends on the choice of orientation of L.

Suppose L is a C^1-embedded Legendrian knot in $(\mathbb{R}^3_{y>0}, \xi_{std})$ and let $\Pi : \mathbb{R}^3 \to \mathbb{R}^2$ denote the "front projection" map, sending $(x, y, z) \mapsto (x, z)$. Recall that $TL \subset \xi$. Consequently, if $S^1 \to L : t \mapsto (x(t), y(t), z(t))$ is a parameterization of L, then

$$z'(t) - x'(t)y(t) = 0.$$

Hence $z'(t_0) = 0$ if and only if $x'(t_0) = 0$ at a point $t_0 \in S^1$. Roughly speaking, the consequence of this is that the image of L under Π has cusps instead of vertical tangencies; see [Etnyre] for a more careful discussion of this. For our purposes there are convenient formulas for $tb(L)$ and $r(L)$ that one can read off from the front projection of L: namely

$$tb(L) = \text{writhe}(\Pi(L)) - \frac{1}{2}(\#\text{cusps})$$

and

$$r(L) = \frac{1}{2}(D - U), \tag{A.1}$$

where D is the number of "down" cusps and U is the number of "up" cusps, which of course depends on the orientation of L. An equivalent description of the rotation number that will be important for us is that it is also the obstruction to extending a nonzero vector field v along L to ξ_Σ, if Σ is a surface bounding L. In addition, for our purposes we only consider Legendrian unknots, which have no writhe so we omit discussing the term writhe$(\Pi(L))$. That is, for a Legendrian unknot L,

$$tb(L) = -\frac{1}{2}(\#\text{cusps}). \tag{A.2}$$

An important operation on Legendrian knots is *stabilization*, which is to add a zig-zag to the front projection of L. The knot can be stabilized either positively or negatively, denoted $S_\pm(L)$. It follows from forumulas (A.1) and (A.2) that

$$tb(S_\pm(L)) = tb(L) - 1 \quad \text{and} \quad r(S_\pm(L)) = r(L) \pm 1.$$

A.2 Symplectic handle attachments

We will only focus on surgery in 3 and 4 dimensions. Let Y be a smooth manifold 3-manifold. The basic data involved in topological surgery on Y is the following:

- A knot $K \subset Y$
- A framing of K, that is, a choice of trivialization $\varphi : \nu S^1 \to S^1 \times D^2$
- The gluing map $f := \tilde{\varphi}|_{\partial(S^1 \times D^2)}$, an orientation reversing diffeomorphism of T^2

The gluing map f determines the diffeomorphism type of the resulting surgered manifold Y'. The diffeomorphism type of Y' is only distinguished by the isotopy type of $f : T^2 \to T^2$, and this is determined precisely by the induced isomorphism on homology $f_* : H_1(T^2;\mathbb{Z}) \to H_1(T^2;\mathbb{Z})$. In fact, f_* is specified by the image $f_*([pt \times \partial D^2]) \in H_1(T^2;\mathbb{Z}) \cong \mathbb{Z}^2$, which a primitive class. Therefore the manifold resulting from surgery along K is fully specified by a pair (p, q) of relatively prime integers. Reversing the orientation of the knot changes the sign of both p and q which means that p/q is independent of the orientation of K. In particular, when

$Y = S^3$ and $K \subset S^3$ is the unknot, $-p/q$-surgery on K is called a lens space and is denoted $L(p,q)$. See [26, 2.2] for more details.

Here is another description of $-p/q$ surgery on an unknot $K \subset S^3$ that will be useful for us. Let $[a_0, \ldots, a_k]$ denote the continued fraction expansion of p/q, where $a_i \geq 2$ for $0 \leq i \leq k$. Then $-p/q$ surgery is equivalent to integral surgery along a chain of unknots $\bigcup K_i$ where each K_i has framing $-a_i$. This is easy to see by iterating "slam dunks". With this in mind, one can also realize $L(p,q)$ as the boundary of a $(k+1)$-handlebody: attach k four dimensional 2-handles to D^4 along the chain of unknots on the boundary $\partial D^4 = S^3$ with framings given the continued fraction coefficients a_i, for $0 \leq i \leq k$. This yields a 4-manifold W with $\partial W = L(p,q)$.

The importance of the handlebody description is that the construction works in the symplectic category: in fact, we can realize $L(p,q)$ with *any* tight contact structure ξ as the ω-convex boundary of a symplectic 4-manifold obtained by attaching symplectic 2-handles, also known as Weinstein handles. To illustrate, the standard 4-dimensional symplectic 2-handle is taken to be the region in $\mathbb{C}^2$ bounded by the the hypersurface

$$\left\{x_1^2 + x_2^2 - \frac{1}{2}(y_1^2 + y_2^2) = -1\right\} \cup \left\{x_1^2 + x_2^2 - \frac{\epsilon}{6}(y_1^2 + y_2^2) = \frac{\epsilon}{2}\right\}$$

(the interior of this region is in fact a Stein domain). Now the boundary is a contact hypersurface in $\mathbb{C}^2$ with contact form $\alpha = \iota_v(\omega_0)$, where ω_0 is the standard symplectic form on $\mathbb{C}^2$ and v is the gradient of $x_1^2 + x_2^2 - \frac{1}{2}(y_1^2 + y_2^2)$, and, importantly, both the attaching and belt circles

$$S = \{x_1 = x_2 = 0,\ y_1^2 + y_2^2 = 2\}$$
$$B = \{y_1 = y_2 = 0,\ x_1^2 + x_2^2 = \epsilon/2\}$$

are Legendrian with respect to α. Roughly speaking, the contact neighborhood theorem provides the gluing contactomorphism between a normal neighborhood of

S and the target Legendrian knot $L \subset Y$, where Y is the ω-convex boundary of some symplectic manifold X. See Chapter 7 of [26] for the definitions and technical propositions that make this work. The framing of the 2-handle H relative to the canonical Legendrian framing of L is -1, leading to the following

Theorem A.2.1 *Suppose that (X, ω) is a symplectic 4-manifold with ω-convex boundary ∂X and $L \subset \partial X$ is a Legendrian curve with respect to the induced contact structure. If we attach a 2-handle H with framing (-1) with respect to its canonical contact framing to ∂X along L then ω extends to $X \cup H$.*

Remark: There is a construction involving ω-concave boundaries, in which case one glues along the belt circle B rather than S.

In particular, when $(X, \omega) = (D^4, \omega_0)$ and v is the radial vector field on $\mathbb{R}^4$, then attaching symplectic 2-handles to a chain of Legendrian unknots $\cup_i L_i$ with framings $tb(L_i) - 1$ results in a symplectic (actually Stein) filling X of the resulting lens space. The tight contact structure on the lens space is determined by the positioning of the zig-zags on the components L_i.

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